Reading Group on Stochastic Modelling

A Structural Model of Dense Network Formation

Mele (2017)

Econometrica

Introduction

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2) curse dimensionality ⇒ # network configs grows exponentially with # agents
3) data on single graph ⇒ only one network snapshot is observable

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2) curse dimensionality $\Rightarrow$ # network configs grows exponentially with # agents
3) data on single graph $\Rightarrow$ only one network snapshot is observable

Proposed model of network formation

- combines features from the strategic and random network formation literature
- players' utilities depend on payoffs from direct links and link externalities (e.g., reciprocity, indirect friends, popularity, . . .)
- network formation is dynamic: each period, a player meets another one and decides whether to form a new link, keep an existing link, or do nothing
- process generates a sequence of directed dense graphs

Model of Network Formation

Setup

- n agents, with characteristics $X_i \in \mathbb{R}^A \;\; \forall i \in \mathcal{I} := \{1, \ldots, n\}$
- discrete time $t \in \mathbb{N}$
- directed, binary network $G \in \mathcal{G}$, realisations each time g^t
- utility of network $=$ sum utility from links, each having four components

$$U_i(g, X|\boldsymbol{\theta}) = \underbrace{\sum_{j=1}^{n} g_{ij} u_{ij}^{\theta_u}}_{\text{direct links}} + \underbrace{\sum_{j=1}^{n} g_{ij} g_{ji} m_{ij}^{\theta_m}}_{\text{mutual links}} + \underbrace{\sum_{j=1}^{n} g_{ij} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} g_{jk} v_{ik}^{\theta_v}}_{\text{indirect links}} + \underbrace{\sum_{j=1}^{n} g_{ij} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} g_{ki} w_{kj}^{\theta_w}}_{\text{popularity}}$$

where $u_{ij}^{\theta_u} := u(X_i, X_j|\theta_u)$, $m_{ij}^{\theta_m} := m(X_i, X_j|\theta_m)$, $v_{ij}^{\theta_v} := v(X_i, X_j|\theta_v)$, $w_{ij}^{\theta_w} := w(X_i, X_j|\theta_w)$

Assumption 1. (Preferences)

$$m_{ij}^{\theta_m} = m(X_i, X_j|\theta_m) = m(X_j, X_i|\theta_m) = m_{ji}^{\theta_m} \quad \forall i, j \in \mathcal{I} \times \mathcal{I}$$
$$w_{ij}^{\theta_v} = w(X_k, X_j|\theta_v) = v(X_k, X_j|\theta_v) = v_{ji}^{\theta_v} \quad \forall k, j \in \mathcal{I} \times \mathcal{I}$$

- first is necessary for identification of the utility from indirect links and popularity;
- second makes another agent i internalise the externality she creates.

Proposition 1 (Existence Potential Function).

Under Assumption 1, the deterministic components of the incentives of any player in any state of the network are summarized by a **potential function** $\mathcal{Q} : \mathcal{G} \times \mathcal{X} \to \mathbb{R}$ *and the network game is a* Potential Game

$$\mathcal{Q}(g, X|\boldsymbol{\theta}) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} u_{ij}^{\theta_u} + \sum_{i=1}^{n} \sum_{j>1}^{n} g_{ij} g_{ji} m_{ij}^{\theta_m} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq 1}}^{n} \sum_{\substack{k=1 \\ k\neq i,j}}^{n} g_{ij} g_{jk} v_{ik}^{\theta_v}$$

$\mathcal{Q}$ is an aggregate function summarising: (i) state of network; (ii) deterministic incentives of players in each state.

Network formation process

Process follows a **stochastic best-response dynamics**, that generates a Markov chain of networks

- for each t, randomly chosen player i meets j according to meeting technology
- *meeting process* is a stochastic sequence $\mathbf{m} = \{m^t\}_t$ supported on $\mathcal{I} \times \mathcal{I}$, with realisations $m^t = ij = \{i, j\}$

$$\mathbb{P}(m^t = ij | g^{t-1}, X) = \rho(g^{t-1}, X_i, X_j)$$

Assumption 2. (Meeting process)

The meeting probability between i, j does not depend on the existence of a link between them, and each meeting has a positive probability of occurring, that is

$$\rho(g^{t-1}, X_i, X_j) = \rho(g^{t-1}_{-ij}, X_i, X_j) > 0 \qquad \forall ij$$

- guarantees any equilibrium network can be reached with positive probability
- identification: allow ρ to depend on *current* link g_{ij} prevents closed form likelihood

Utility

Players' rules

- conditional on meeting $m^t = ij$, player i updates link g_{ij} to maximise her utility
- existing network g_{-ij}^{t-1} is taken as given
- complete information: everybody known each others' attributes and whole network
- myopia: agents not account for effects of their linking strategy on future evolution of network
- idiosyncratic shock on individual preferences: $\varepsilon \sim EV_1(\varepsilon)$ Type I extreme value distribution, *iid* among links and across time
- link established if and only if

$$U_i(g_{ij}^t = 1, g_{-ij}^{t-1}, X|\boldsymbol{\theta}) + \varepsilon_{1t} > U_i(g_{ij}^t = 0, g_{-ij}^{t-1}, X|\boldsymbol{\theta}) + \varepsilon_{0t}$$

Process generates a Markov chain of networks:

✓ transition proba determined by: (i) meeting process, (ii) agents' linking choices

✓ irreducible, aperiodic

Equilibrium

Remark 1.

Any change in utility for any agent is equivalent to change in potential $\mathcal{Q}$. So, any deviation from Nash (equilibrium) network must decrease the potential.

Thus, the Nash network is a local maximizer of the potential function over the set of networks that differ from the current network for at most one link.

Theorem 1 (Uniqueness and Characterisation of Stationary Equilibrium).

The network formation game, under Assumptions 1–3, converges to a **unique** *stationary distribution*

$$\pi(g, X|\boldsymbol{\theta}) = \frac{\exp\{\mathcal{Q}(g, X|\boldsymbol{\theta})\}}{\sum_{\omega \in \mathcal{G}} \exp\{\mathcal{Q}(\omega, X|\boldsymbol{\theta})\}} \tag{1}$$

Comments

- existence and uniqueness come from irreducibility and aperiodicity of Markov chain
- uniqueness crucial for estimation (avoid multiple equilibria)

- closed form stationary $\pi(g, X|\boldsymbol{\theta})$ corresponds to the likelihood of observing a specific network configuration in the long run
  $\Longrightarrow$ can estimate $\boldsymbol{\theta}$ with only one network, assuming it is drawn from the stationary equilibrium

- $\pi(g, X|\boldsymbol{\theta})$ coincides with likelihood of ERGM (Exponential Random Graph Model), where probability observing a network is proportional to exponential of linear combination of network statistics

Corollary 1.1.

Let Assumptions 1–3 hold. If the utility functions are linear in parameters, the stationary distribution $\pi(g, X|\boldsymbol{\theta})$ describes an ERGM, with $\mathbf{t}(g, X)$ a vector of canonical statistics

$$\pi(g, X|\boldsymbol{\theta}) = \frac{\exp\{\boldsymbol{\theta}'\mathbf{t}(g, X)\}}{\sum_{\omega \in \mathcal{G}} \exp\{\boldsymbol{\theta}'\mathbf{t}(\omega, X)\}} \tag{2}$$

Extensions

Utility functions

☞ possible to include additional utility components, as long as possible to find restrictions on payoffs that guarantee the existence of a potential function

Undirected networks

☞ possible to extend existence results, characterisation of equilibrium, relation with ERGM and asymptotic results to undirected networks

Sparsity

☞ model with negative linking externalities is compatible with a certain degree of sparsity

Estimation and Identification

Likelihood function

$$\boxed{L(g, X|\boldsymbol{\theta}) = \pi(g, X|\boldsymbol{\theta}) = \frac{\mathcal{Q}(g, X|\boldsymbol{\theta})}{\sum_{\omega \in \mathcal{G}} \mathcal{Q}(\omega, X|\boldsymbol{\theta})} = \frac{\mathcal{Q}(g, X|\boldsymbol{\theta})}{c(\mathcal{G}, X, \boldsymbol{\theta})}}$$

whose normalizing constant $c(\mathcal{G}, X, \boldsymbol{\theta})$ is intractable since it sums $2^{n(n-1)}$ terms.

✗ standard ML infeasible

✗ MCMC with standard MH step infeasible (ratio of normalizing constants)

Estimation Algorithm

ERGM literature $\Rightarrow$ approximate $c(\mathcal{G}, X, \boldsymbol{\theta})$ via MCMC (for fixed $\boldsymbol{\theta}_0$)

Algorithm 1 Metropolis-Hastings for Network Simulations

procedure $\textsc{MH\_NetSim}(\boldsymbol{\theta}_0, g_0, R)$
  for $r = 1, \ldots, R$ **do**
    1) propose network $g' \sim q_g(g'|g^{(r)})$
    2) accept network g' with probability

$$\alpha(g^{(r)}, g') = \min\left\{1, \frac{\exp\{\mathcal{Q}(g', X|\boldsymbol{\theta}_0)\}}{\exp\{\mathcal{Q}(g^{(r)}, X|\boldsymbol{\theta}_0)\}} \frac{q_g(g^{(r)}|g')}{q_g(g'|g^{(r)})}\right\}$$

  end for
  return sequence of R networks $\{g^{(r)}\}_r$
end procedure

✓ not requires $c(\mathcal{G}, X, \boldsymbol{\theta})$

✗ *slow convergence*
✗ *local sampler* at each iteration, update link g_{ij} according to $\alpha(\cdot, \cdot)$
✗ *degeneracy* problem: large probability mass on few networks

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 end for
 return sequence of R networks $\{g^{(r)}\}_r$
end procedure

▶ how to choose $q_g(\cdot|g^{(r)})$?

Asymptotic behaviour

Classes of asymptotics for networks

1) *many networks* $\Rightarrow$ same players, growing number of networks
2) *large networks* $\Rightarrow$ growing number of players, same network

- Hp: homogeneous players (i.e. $X_i = X_j, \forall i, j$)
- potential function re-scaled by $n^{\nu(H)}$, with $\nu(H)$ # players in each utility term
- example **re-scaled likelihood**, with $\mathcal{T}(g) = \alpha t(H_1, g) + \beta t(H_2, g)$ re-scaled potential and $\psi_n = n^{-2} \log(c(\alpha, \beta, \mathcal{G}_n))$ log-normalising constant

$$\pi_n(g|\alpha, \beta) = \frac{\exp\left\{ n^2 \left[\alpha \frac{\sum_{i=1}^n \sum_{j=1}^n g_{ij}}{n^2} + \beta \frac{\sum_{i=1}^n \sum_{j=1}^n \sum_{k \neq i}^n g_{ij} g_{jk}}{n^3} \right] \right\}}{c(\alpha, \beta, \mathcal{G}_n)} = \exp\left\{ n^2 \left[\mathcal{T}(g) - \psi_n \right] \right\} \tag{3}$$

Asymptotic behaviour

Theorem 2 (Nonnegative Link Externalities).

Model (3) *with nonnegative link externalities* $\beta \geq 0$ *exhibits the following behaviour*

1) *asymptotic normalizing constant* ψ *solves*

$$\psi = \lim_{n \to \infty} \psi_n = \max_{\mu \in [0,1]} \left\{ \alpha\mu + \beta\mu^2 - \mu \log(\mu) - (1-\mu)\log(1-\mu) \right\} \quad (4)$$

2) *networks generated by the model are indistinguishable from* directed Erdös–Rényi *graph with linking probability* μ^**, defined as follows:*
   (a) *if the maximization* (4) *has a unique solution, then* μ^* *satisfies* $2\beta\mu(1-mu) < 1$ *for almost all* $\alpha \in \mathbb{R}$ *and* $\beta \geq 0$*, and solves*

$$\mu = \frac{\exp\{\alpha + 2\beta\mu\}}{1 + \exp\{\alpha + 2\beta\mu\}} \quad (5)$$

   (b) *if the maximization* (4) *has two solutions, then* μ^* *picked randomly from same proba distribution over* μ_1^* *and* μ_2^**, such that* $\mu_1^* < 0.5 < \mu_2^*$*, and both solve* (5) *and satisfy* $2\beta\mu(1-\mu) < 1$.

Comments on Theorem 2

- consistent estimator of log-normalising constant – analogue of variational representation of the discrete exponential family
- $\beta \geq 0 \Rightarrow$ realisations using (α, β) indistinguishable from those using $(\alpha', 0) = (\log(\mu^*/(1-\mu^*)), 0)$, that is from Erdös-Rényi model

Corollary 2.1.

When $\beta \geq 0$, the **externality cannot be identified**.

Corollary 2.2.

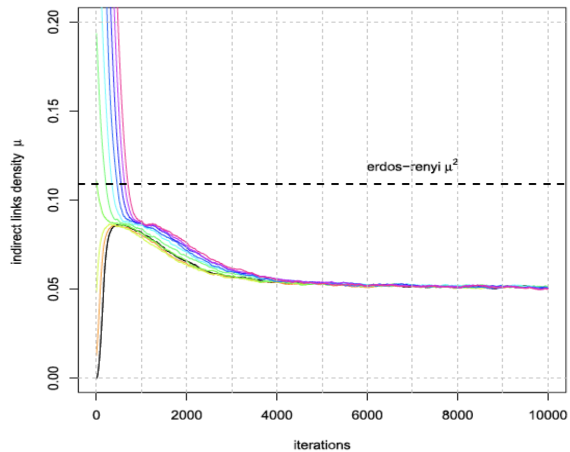
When $\beta \geq 0$, Algorithm 1 is not necessary since Erdös-Rényi graphs can be simulated using Bernoulli draws.

Asymptotic behaviour

Theorem 3 (Negative Link Externalities).

If $\beta < 0$ and sufficiently large in magnitude, model (3) is asymptotically different from a directed Erdös-Rényi model.

✓ sparser graphs than Erdös-Rényi



Asymptotic behaviour

Theorem 3 (Negative Link Externalities).

If $\beta < 0$ and sufficiently large in magnitude, model (3) is asymptotically different from a directed Erdös-Rényi model.

- how much "sufficiently large" magnitude?

- how to know it, since we must estimate β?

Asymptotic behaviour

Consider an additional utility component (cyclic triangles):

$$T(g) = \alpha t(H_1, g) + \beta t(H_2, g) + \gamma t(H_3, g), \qquad t(H_3, g) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i}^{n} g_{ij} g_{jk} g_{ki} \tag{6}$$

Theorem 4.

Consider model (6) as $n \to \infty$

1) *If $\beta \geq 0$ and $\gamma \geq 0$, then the asymptotic normalising constant ψ solves*

$$\psi = \lim_{n \to \infty} \psi_n = \max_{\mu \in [0,1]} \left\{ \alpha\mu + \beta\mu^2 + \gamma\mu^3 - \mu \log(\mu) - (1-\mu)\log(1-\mu) \right\} \tag{7}$$

   and model is asymptotically indistinguishable from directed Erdös-Rényi graph, with μ^* maximising (7). If the maximisation problem has multiple solutions, then μ^* picked randomly from some distribution on maximisers.

2) *If at least* **one** *externality is negative (i.e. $\beta < 0$ or $\gamma < 0$) and sufficiently large, then model (6) not converge asymptotically to directed Erdös-Rényi graph and externalities can be identified.*

Summary of Asymptotics

Remark 2.

Homogeneous players ($X_i = X_j, \forall i, j$):

(a) positive externalities
- asymptotically indistinguishable from Erdös-Rényi graph
- externalities not identified
- can approximate likelihood of model via likelihood of Erdös-Rényi graph

(b) at least one externality negative and large
- asymptotically sparser than Erdös-Rényi graph
- externalities identified

Heterogeneous players ($\exists i, j$ such that $X_i \neq X_j$):

- no results
- preliminary study in Mele & Zhu (2017) - working paper

Sampler Convergence

Theorem 5 (Convergence of Local Sampler with Nonnegative Externalities).

Model (6)*, with probability of meeting* $\rho_{ij} = 1/(n(n-1))$*. Fix* $\gamma \geq 0$*. Then, in the case of nonnegative externalities* $\beta \geq 0$*, there exists a V-shaped region of the parameter space delimited by functions* $S_\gamma(\phi_1(\alpha)), S_\gamma(\phi_2(\alpha))$ *such that*

1) *if* (α, β) *belongs to the V-shaped region, then model converges to stationarity in* e^{Cn^2} *steps,* $C > 0$*. This results holds for any local sampler.*
2) *otherwise, model converges in* $Cn^2 \log(n)$ *steps,* $C > 0$*.*

Intuition:

(1a) in the V-shaped region problem (7) has 2 *local maxima*, the sampler spend exponential time at one of them (i.e. probability e^{-Cn^2} to escape from local max)

(1b) increasing $\gamma \implies$ increase area of exponentially slow convergence

(2a) when convergence is quadratic $\implies$ sampler feasible for $n < 500$

(2b) this happens when model is indistinguishable from directed Erdös-Rényi graph

Simulation and Estimation in Finite Networks

Posterior inference via approx version of exchange algorithm of Murray et al (2006)

- double Metropolis-Hastings step to avoid computing $c(\mathcal{G}, X, \boldsymbol{\theta})$
- data augmentation via auxiliary network g'
- higher $R \implies$ better approximation of posterior, but higher rejection rate

Algorithm 2 Approximate Exchange Algorithm

1: **procedure** $\mathrm{AEA}(\boldsymbol{\theta}, g, M, R)$
2: **for** $m = 1, \ldots, M$ **do**
3: 1) propose parameter $\boldsymbol{\theta}' \sim q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$
4: 2) run Algorithm 1 for R iterations using $\boldsymbol{\theta}'$. Keep last simulated network g'
5: 3) accept parameter $\boldsymbol{\theta}'$ with probability

$$\alpha(\boldsymbol{\theta}, \boldsymbol{\theta}', g', g) = \min\left\{1, \frac{\exp\{\mathcal{Q}(g', X|\boldsymbol{\theta})\}}{\exp\{\mathcal{Q}(g, X|\boldsymbol{\theta})\}} \frac{p(\boldsymbol{\theta}')}{p(\boldsymbol{\theta})} \frac{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}|\boldsymbol{\theta}')}{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}'|\boldsymbol{\theta})} \frac{\exp\left\{\mathcal{Q}(g, X|\boldsymbol{\theta}')\right\}}{\exp\left\{\mathcal{Q}(g', X|\boldsymbol{\theta}')\right\}}\right\}$$

6: **end for**
7: **return** sequence of M parameters $\{\boldsymbol{\theta}^{(m)}\}_m$
8: **end procedure**

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▶ what prior distribution $p(\boldsymbol{\theta})$?

▶ what proposal distribution $q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$?

Simulation results

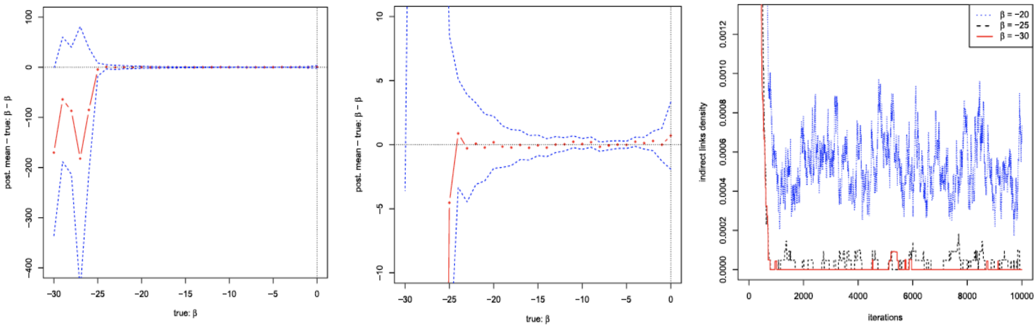

Figure: *Left*: Estimates of $\beta < 0$, with 95% credibility intervals (*middle*: zoom-in). *Right*: indirect links density.

- $\beta \geq 0 \implies$ Erdös-Rényi case, not identified
- $\beta < 0 \implies$ identified
- $\beta \ll 0 \implies$ estimation impossible: # indirect links close to 0

Simulation results

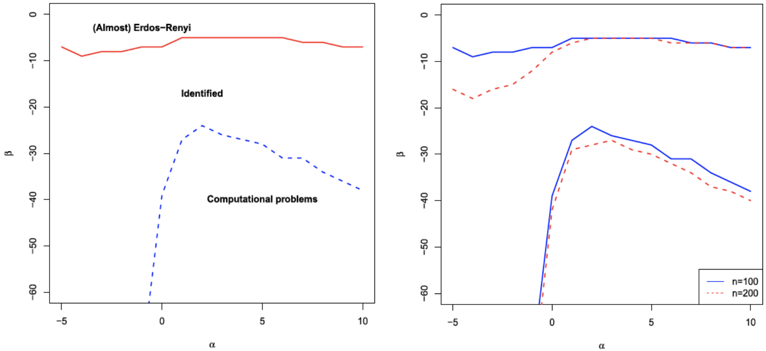

Figure: *Left*: approximate regions of identified parameters, for $n = 100$.
Right: comparison of regions for $n = 100$, $n = 200$.

Remark 3.

Regions of identified parameters (α, β) vary with n, the number of players.

Simulation results

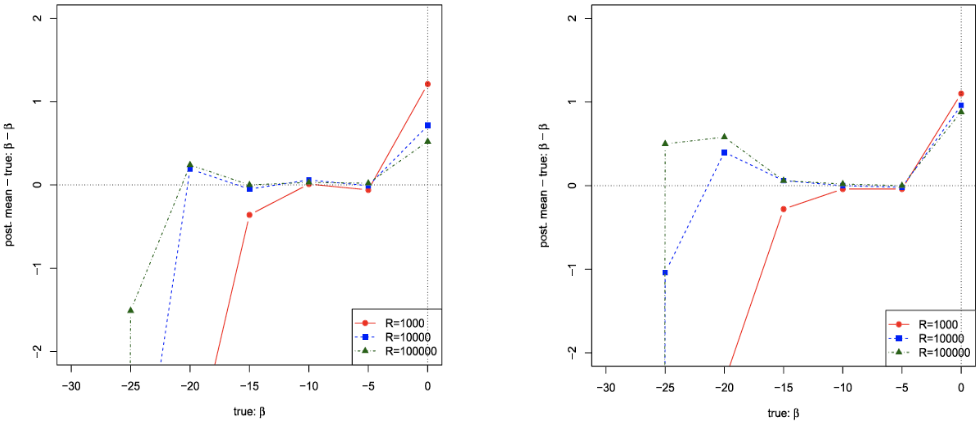

Figure: Difference between posterior estimates and true, for varying number of network simulations R: $n = 100$ (*left*) and $n = 200$ (*right*).

Simulation results

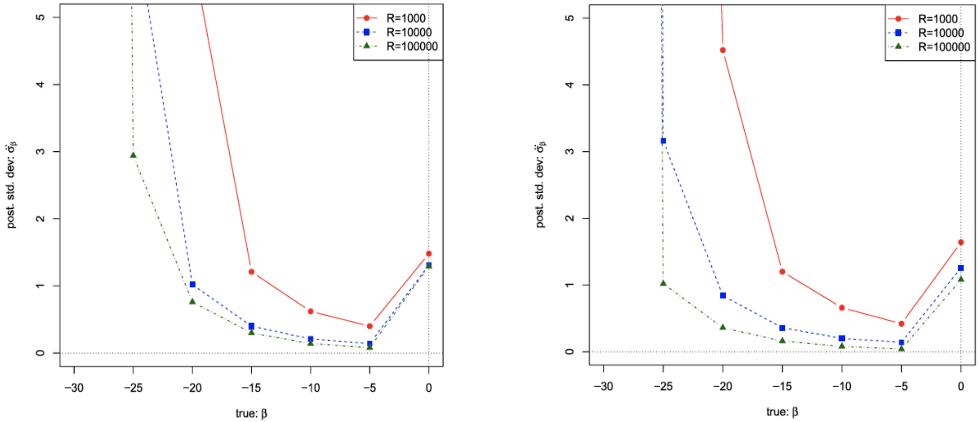

Figure: Posterior standard deviation for varying number of network simulations R: $n = 100$ (*left*) and $n = 200$ (*right*).

Simulation results

- $R = 1000 \implies$ imprecise estimates
- no significant difference between $R = 10,000$ and $R = 100,000 \implies$ suggest rule-of-thumb $R = 10,000$
- cost of increasing network simulations $\implies$ almost linear $\mathcal{O}(R)$
- results suggest convergence is almost quadratic $\mathcal{O}(n^2)$ in this area of parameter space

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- results suggest convergence is almost quadratic $\mathcal{O}(n^2)$ in this area of parameter space

- what was the computing time?

- what about real data applications?

Conclusions

The paper in a nutshell:

- network formation model, combining strategic and random networks features
- payoffs depend on links: direct + indirect (externalities)
- homogeneous players meet sequentially at random, myopically updating links
- network formation process is a potential game and converges to ERGM, generating directed dense networks
- identification: only if at least 1 externality negative and sufficiently large
- standard estimation for ERGMs exponentially slow $\Rightarrow$ Bayesian MCMC (almost quadratic time)

Conclusions

Unclear points and questions:

- ☞ theoretical quantification of "sufficiently large" (negative) magnitude of β?
- ☞ choice of prior for parameters $p(\boldsymbol{\theta})$?
- ☞ choice of proposal for network $q_g(\cdot|g)$?
- ☞ choice of proposal for parameters $q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$?
- ☞ duration of computing time in simulations?
- ☞ real data applications?

Thanks for your attention!

Any question?