Reading Group on Stochastic Modelling

https://matteoiacopini.github.io/stochmodgroup/index.html

Meetings, Discussants and Material

Discussed paper:

A Structural Model of Dense Network Formation

Mele (2017)

Econometrica

Introduction

Literature

Related works

- similar paper [7]
- extensions [14], [3], [11]

Model

- potential games [13]
- network formation models [1]
- ERGM theory [8], [18]

Asymptotics

- many networks asymptotics [17]
- large network asymptotics [9], [15], [12]
  - graph limits [2]
  - large deviations [6]
  - variational methods for the exponential family [20]

Estimation

- ERGM estimation [18], [5]

Introduction

Strategic models of network formation provide a framework to interpret the observed network as the equilibrium of a (potential) game.

Estimation and identification of strategic models is challenging

1) multiple equilibria $\Rightarrow$ links generate externalities not fully accounted for by agents
2) curse dimensionality $\Rightarrow$ # network configs grows exponentially with # agents
3) data on single graph $\Rightarrow$ only one network snapshot is observable

Proposed model of network formation

- combines features from the strategic and random network formation literature
- players' utilities depend on payoffs from direct links and link externalities (e.g., reciprocity, indirect friends, popularity, . . .)
- network formation is dynamic: each period, a player meets another one and decides whether to form a new link, keep an existing link, or do nothing
- process generates a sequence of directed dense graphs

Model of Network Formation

Model of Network Formation

Setup

- n agents, with characteristics $X_i \in \mathbb{R}^A, \;\; \forall i \in \mathcal{I} := \{1, \dots, n\}$
- discrete time $t \in \mathbb{N}$
- directed, binary network $G \in \mathcal{G}$, realisations each time g^t

Definition 1 (Individual utility function).

Let $u_{ij}^{\theta_u} = u(X_i, X_j | \theta_u)$, $m_{ij}^{\theta_m} = m(X_i, X_j | \theta_m)$, $v_{ij}^{\theta_v} = v(X_i, X_j | \theta_v)$, $w_{ij}^{\theta_w} = w(X_i, X_j | \theta_w)$ where $\boldsymbol{\theta} = (\theta_u, \theta_m, \theta_v, \theta_w)' \in \mathbb{R}^4$ are parameters.

The **utility of agent** i from network g is given by the sum of four components

$$U_i(g, X | \boldsymbol{\theta}) = \underbrace{\sum_{j=1}^{n} g_{ij} u_{ij}^{\theta_u}}_{\text{direct links}} + \underbrace{\sum_{j=1}^{n} g_{ij} g_{ji} m_{ij}^{\theta_m}}_{\text{mutual links}} + \underbrace{\sum_{j=1}^{n} g_{ij} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} g_{jk} v_{ik}^{\theta_v}}_{\text{indirect links}} + \underbrace{\sum_{j=1}^{n} g_{ij} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} g_{ki} w_{kj}^{\theta_w}}_{\text{popularity}} .$$

▶ "Markovian" only indirect links are valuable and are perfect substitutes (no utility from two-links-away contacts)

Potential Game

Definition 2 (Potential Game).

A game is said to be a **Potential Game** if the incentive of all players to change their strategy (here: link formation choice) can be expressed using a single global function called the **potential function** $\mathcal{Q} : \mathcal{G} \times \mathcal{X} \to \mathbb{R}$ such that:

$$\mathcal{Q}(g_{ij}, g_{-ij}, X) - \mathcal{Q}(g'_{ij}, g_{-ij}, X) = U_i(g_{ij}, g_{-ij}, X) - U_i(g'_{ij}, g_{-ij}, X), \quad \forall i, j \; \forall g_{-ij}$$

Remark 1.

The **potential function** is useful for:

- analyse equilibrium properties of games,
  set of pure Nash equilibria corresponds to the local optima of potential function; existence of profitable deviations performed using the potential, instead of checking each player's possible deviation
- study convergence and finite-time convergence of iterated game towards a Nash equilibrium

Assumption 1. (Preferences)

$$m_{ij}^{\theta_m} = m(X_i, X_j|\theta_m) = m(X_j, X_i|\theta_m) = m_{ji}^{\theta_m} \quad \forall i, j \in \mathcal{I} \times \mathcal{I}$$
$$w_{kj}^{\theta_v} = w(X_k, X_j|\theta_v) = v(X_k, X_j|\theta_v) = v_{kj}^{\theta_v} \quad \forall k, j \in \mathcal{I} \times \mathcal{I}$$

- first is necessary for identification of the utility from indirect links and popularity;
- second makes another agent i internalise the externality she creates.

Proposition 1 (Existence Potential Function).

Under Assumption 1, the deterministic components of the incentives of any player in any state of the network are summarized by a **potential function** $\mathcal{Q} : \mathcal{G} \times \mathcal{X} \to \mathbb{R}$ and the network game is a Potential Game

$$\mathcal{Q}(g, X|\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} u_{ij}^{\theta_u} + \sum_{i=1}^{n}\sum_{j>i}^{n} g_{ij} g_{ji} m_{ij}^{\theta_m} + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\sum_{\substack{k=1 \\ k\neq i,j}}^{n} g_{ij} g_{jk} v_{ik}^{\theta_v}$$

$\mathcal{Q}$ is an aggregate function summarising: (i) state of network; (ii) deterministic incentives of players in each state.

Reading Group on Stochastic Modelling

A brief overview of:

Incomplete Simultaneous Discrete Response Model with Multiple Equilibria

Tamer (2003)

Review of Economic Studies

In relation to:

A Structural Model of Dense Network Formation

Mele (2017)

Econometrica

Mele (2017) footnote n.17 p.830

"The **second part of the assumption 1 [see below]** *is an identification restriction, that guarantees the model's* **coherency** *in the sense of Tamer (2003)."*

Individual i **values his popularity** effect as much as k **values the indirect link to** j through any "bridging" individual:

$$\underbrace{w_{kj}^{\theta_v} = w(X_k, X_j|\theta_v)}_{\substack{\textbf{indirect link:}\\ \text{utility for } k\neq i,j\\ \text{for indirectly linking with } j\\ \text{through any } i\neq k,j\\ \text{k}\rightarrow\text{i}\rightarrow\text{j}}} = \underbrace{v(X_k, X_j|\theta_v) = v_{kj}^{\theta_v}}_{\substack{\textbf{popularity:}\\ \text{utility for any } i\neq k,j\\ \text{for bridging } k \text{ to } j\\ \text{k}\rightarrow\text{i}\rightarrow\text{j}}} \quad \forall k, j \in \mathcal{I} \times \mathcal{I}$$

Under assumption 1.2 any individual $i \in \mathcal{I}$ **internalizes all the externalities generated by his links**:

- The **popularity** component of $U_i(g, \mathbf{X}|\Theta)$ is **equal to** the sum, over all $j \in \mathcal{I} - i$, of the utility of **indirect links of** j **passing through** i, which are the indirect links that can be influenced by i's link-formation **decisions**;

Summary of Tamer (2003)

Modelling framework - **2x2 entry-game with perfect information**

two players ($i \in \{-1, 1\}$), action set ($y_i \in \{0, 1\}$) and externalities δ_i. The payoff π_i of player i is defined as:

$$\pi_i := y_i(x_i\beta_i + y_{-i}\delta_i + u_i)$$

Where:

- $\mathbf{y} = (y_{-1}, y_1)$ are response variables;
- $\mathbf{x} = (x_{-1}, x_1) \in \mathcal{R}^d$ are observable exogenous variables;
- $\mathbf{u} = (u_{-1}, u_1)$ are random variables unobservable to the econometricians;
- $\beta = (\beta_{-1}, \beta_1, \delta_{-1}, \delta_1)$ are parameters of interest;

Distinction of model identification issues - **Incoherency Vs Incompleteness**

1− **incoherent model**: hasn't a well-defined reduced form, or, is logically inconsistent. For example:
if externalities δ_{-1} and δ_1 are both negative, the above model gives
$Pr[(0,0)|x] + Pr[(0,1)|x] + Pr[(1,0)|x] + Pr[(1,1)|x] > 1$

2− **incomplete model**: the relationship from input variables (x_is and u_is) to responses (y_is) is a **correspondence** and not a function. For example:
if δ_is are both negative, $\exists$ a non-empty region of $\mathbf{u}$'s *support for which the model predicts a non-unique outcome* $(1,0)$ OR $(0,1)$

Contribution and findings of Tamer (2003)

- **Identifies sufficient conditions for parameter point identification (when externalities have same sign);**
- **Develops a technique for semi-parametric maximum (quasi)likelihood (SML) estimation:** by "replacing" $Pr[(y_{-1}, y_1)|x]$ for outcomes (0,1) and/or (1,0), with local approximations of the the empirical relative frequencies of these outcomes as a function of exogenous variables;

Why Assumption 1.2 relevant for identification in Mele (2017)?

1. **externalities are "paired"**: each **indirect-link** effect has a **corresponding popularity effect** with same sign, value and parameter;
2. **number of parameters of the model is reduced**: from 4 $(\theta_u, \theta_m, \theta_w, \theta_v)$ to 3 $(\theta_u, \theta_m, \theta_v)$. Condition necessary for model completeness (?);
3. guarantees that the system of **conditional linking probabilities** implied by the model **generate a proper joint distribution** of the network matrix;
4. can **use** the potential function $\mathcal{Q}$ **to construct the network as a best-response potential game**. Via **sequential** link-formation **decisions** the game converges through an **improvement path** to a **Pure Strategy Nash Equilibrium** network with $Pr = 1$;

Network formation process

Stochastic best-response dynamics, generating a Markov chain of graphs:

- for each t, randomly chosen player i meets j according to meeting technology
- *meeting process* is a stochastic sequence $\mathbf{m} = \{m^1, ..., m^t\}_t$ supported on $\mathcal{I} \times \mathcal{I}$, with realisations $m^t = ij = \{i, j\}$ whose probability is

$$\mathbb{P}(m^t = ij | g^{t-1}, X) = \rho(g^{t-1}, X_i, X_j)$$

Assumption 2. (Meeting process)

The meeting probability between i, j does not depend on the existence of a link between them, and each meeting has a positive probability of occurring, that is

$$\rho(g^{t-1}, X_i, X_j) = \rho(g^{t-1}_{-ij}, X_i, X_j) > 0 \qquad \forall ij$$

▶ guarantees any equilibrium network can be reached with positive probability

Idea: Assumption 2 makes the Markov chain irreducible

▶ identification: if ρ depends on link g_{ij} $\Rightarrow$ prevents closed form likelihood

Players' rules

- conditional on meeting $m^t = ij$, player i updates link g_{ij} to maximise her utility
- existing network g_{-ij}^{t-1} is taken as given
- complete information: everybody known each others' attributes and whole network
- myopia: agents not account for effects of their linking strategy on future evolution of network

Assumption 3. (Idiosyncratic shocks)

Idiosyncratic shock on individual preferences: $\varepsilon_{ij,t} \overset{iid}{\sim} EV_1(\varepsilon_{ij,t})$ Type I extreme value distribution, *iid* among links and across time

$\Rightarrow$ > 0 proba moving out from any state $\rightarrow$ eliminates absorbing states

- link established if and only if

$$U_i(g_{ij}^t = 1, g_{-ij}^{t-1}, X|\boldsymbol{\theta}) + \varepsilon_{1t} > U_i(g_{ij}^t = 0, g_{-ij}^{t-1}, X|\boldsymbol{\theta}) + \varepsilon_{0t}$$

Process generates a Markov chain of networks:

✓ transition proba determined by: (i) meeting process, (ii) agents' linking choices

✓ irreducible (from Assumption 2), aperiodic (from Assumption 3)

Equilibrium

Remark 2.

Any change in utility for any agent is equivalent to change in potential $\mathcal{Q}$. So, any deviation from Nash (equilibrium) network must decrease the potential.

Thus, the Nash network is a local maximizer of the potential function over the set of networks that differ from the current network for at most one link.

Theorem 1 (Uniqueness and Characterisation of Stationary Equilibrium).

The network formation game, under Assumptions 1–3, converges to a **unique** stationary distribution

$$\pi(g, X|\boldsymbol{\theta}) = \frac{\exp\{\mathcal{Q}(g, X|\boldsymbol{\theta})\}}{\sum_{\omega \in \mathcal{G}} \exp\{\mathcal{Q}(\omega, X|\boldsymbol{\theta})\}} \tag{1}$$

Comments

- existence and uniqueness come from irreducibility and aperiodicity of Markov chain
- closed form stationary $\pi(g, X|\boldsymbol{\theta})$ corresponds to the likelihood of observing a specific network configuration in the long run
- estimation: uniqueness avoids multiple equilibria
  $\Rightarrow$ unique stationary = unique likelihood

$\Rightarrow$ can estimate $\boldsymbol{\theta}$ with only one network, assumed drawn from stationary equilibrium

- $\pi(g, X|\boldsymbol{\theta})$ coincides with likelihood of ERGM (Exponential Random Graph Model), where probability observing a network is proportional to exponential of linear combination of network statistics

Corollary 2.1.

Let Assumptions 1–3 hold. If the utility functions are linear in parameters, the stationary distribution $\pi(g, X|\boldsymbol{\theta})$ describes an ERGM, with $\mathbf{t}(g, X)$ a vector of canonical statistics

$$\pi(g, X|\boldsymbol{\theta}) = \frac{\exp\{\boldsymbol{\theta}'\mathbf{t}(g, X)\}}{\sum_{\omega \in \mathcal{G}} \exp\{\boldsymbol{\theta}'\mathbf{t}(\omega, X)\}} \tag{2}$$

Model without shocks

Proposition 2 (Model Without Shocks: Equilibria and Long Run).

Consider the model without idiosyncratic preference shocks. Under Assumptions 1-2:

(i) there exists at least one pure-strategy Nash equilibrium network.

(ii) the set $\mathcal{NE}(\mathcal{G}, X, U)$ of all pure-strategy Nash equilibria of the network formation game is completely characterized by the local maxima of the potential function:

$$\mathcal{NE}(\mathcal{G}, X, U) = \left\{ g^* : g^* = \arg \max_{g \in \mathcal{N}(g^*)} Q(g, X) \right\}.$$

(iii) any pure-strategy Nash equilibrium is an absorbing state.

(iv) as $t \to \infty$, the network converges to one of the Nash networks with probability 1.

Extensions

Utility functions
☞ possible to include additional utility components, as long as possible to find restrictions on payoffs that guarantee the existence of a potential function

Undirected networks
☞ possible to extend existence results, characterisation of equilibrium, relation with ERGM and asymptotic results to undirected networks

Sparsity
☞ model with negative linking externalities is compatible with a certain degree of sparsity

Estimation and Identification

Estimation and Identification

Likelihood function

$$L(g,X|\boldsymbol{\theta}) = \pi(g,X|\boldsymbol{\theta}) = \frac{\mathcal{Q}(g,X|\boldsymbol{\theta})}{\sum_{\omega\in\mathcal{G}} \mathcal{Q}(\omega,X|\boldsymbol{\theta})} = \frac{\mathcal{Q}(g,X|\boldsymbol{\theta})}{c(\mathcal{G},X,\boldsymbol{\theta})}$$

whose normalizing constant $c(\mathcal{G},X,\boldsymbol{\theta})$ is intractable since it sums $2^{n(n-1)}$ terms.

✗ standard ML infeasible
✗ MCMC with standard MH step infeasible (ratio of normalizing constants)

Estimation Algorithm

ERGM literature $\Rightarrow$ approximate $c(\mathcal{G}, X, \boldsymbol{\theta})$ via MCMC (for fixed $\boldsymbol{\theta}_0$)

Algorithm 1 Metropolis-Hastings for Network Simulations

procedure $\textsc{MH\_NetSim}(\boldsymbol{\theta}_0, g_0, R)$
  for $r = 1, \ldots, R$ **do**
    1) propose network $g' \sim q_g(g'|g^{(r)})$
    2) accept network g' with probability

$$\alpha(g^{(r)}, g') = \min\left\{1, \frac{\exp\{\mathcal{Q}(g', X|\boldsymbol{\theta}_0)\}}{\exp\{\mathcal{Q}(g^{(r)}, X|\boldsymbol{\theta}_0)\}} \frac{q_g(g^{(r)}|g')}{q_g(g'|g^{(r)})}\right\}$$

  end for
  return sequence of R networks $\{g^{(r)}\}_r$
end procedure

✓ not requires $c(\mathcal{G}, X, \boldsymbol{\theta})$

✗ *slow convergence*
✗ *local sampler* at each iteration, update link g_{ij} according to $\alpha(\cdot, \cdot)$
✗ *degeneracy* problem: large probability mass on few networks

Estimation Algorithm

ERGM literature $\Rightarrow$ approximate $c(\mathcal{G}, X, \boldsymbol{\theta})$ via MCMC (for fixed $\boldsymbol{\theta}_0$)

Algorithm 1 Metropolis-Hastings for Network Simulations

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5: **end for**
6: **return** sequence of R networks $\{g^{(r)}\}_r$
7: **end procedure**

▶ how to choose $q_g(\cdot|g^{(r)})$?

Asymptotic behaviour

Classes of asymptotics for networks

1) *many networks* $\Rightarrow$ same players, growing number of networks
2) $\boxed{\textit{large networks}}$ $\Rightarrow$ growing number of players, same network

- Hp: homogeneous players (i.e. $X_i = X_j, \forall i, j$)
- potential function re-scaled by $n^{\nu(H)}$, with $\nu(H)$ # players in each utility term
- example **re-scaled likelihood**, with $\mathcal{T}(g) = \alpha t(H_1, g) + \beta t(H_2, g)$ re-scaled potential and $\psi_n = n^{-2} \log(c(\alpha, \beta, \mathcal{G}_n))$ log-normalising constant

$$\pi_n(g|\alpha,\beta) = \frac{\exp\left\{n^2\left[\alpha\frac{\sum_{i=1}^n\sum_{j=1}^n g_{ij}}{n^2} + \beta\frac{\sum_{i=1}^n\sum_{j=1}^n\sum_{k\neq i}^n g_{ij}g_{jk}}{n^3}\right]\right\}}{c(\alpha,\beta,\mathcal{G}_n)}$$

$$= \exp\left\{n^2\left[\mathcal{T}(g) - \psi_n\right]\right\} \tag{3}$$

Asymptotic behaviour

Theorem 2 (Nonnegative Link Externalities).

Model (3) with nonnegative link externalities $\beta \geq 0$ exhibits the following behaviour

1) asymptotic normalizing constant ψ solves

$$\psi = \lim_{n\to\infty} \psi_n = \max_{\mu\in[0,1]} \left\{ \alpha\mu + \beta\mu^2 - \mu\log(\mu) - (1-\mu)\log(1-\mu) \right\} \tag{4}$$

2) networks generated by the model are indistinguishable from directed Erdös–Rényi graph with linking probability μ^*, defined as follows:

   (a) if the maximization (4) has a unique solution, then μ^* satisfies $2\beta\mu(1-mu) < 1$ for almost all $\alpha \in \mathbb{R}$ and $\beta \geq 0$, and solves

$$\mu = \frac{\exp\{\alpha + 2\beta\mu\}}{1 + \exp\{\alpha + 2\beta\mu\}} \tag{5}$$

   (b) if the maximization (4) has two solutions, then μ^* picked randomly from same proba distribution over μ_1^* and μ_2^*, such that $\mu_1^* < 0.5 < \mu_2^*$, and both solve (5) and satisfy $2\beta\mu(1-\mu) < 1$.

Comments on Theorem 2

- consistent estimator of log-normalising constant – analogue of variational representation of the discrete exponential family
- $\beta \geq 0 \Rightarrow$ realisations using (α, β) indistinguishable from those using $(\alpha', 0) = (\log(\mu^*/(1-\mu^*)), 0)$, that is from Erdös-Rényi model

Corollary 2.2.

When $\beta \geq 0$, the **externality cannot be identified**.

Corollary 2.3.

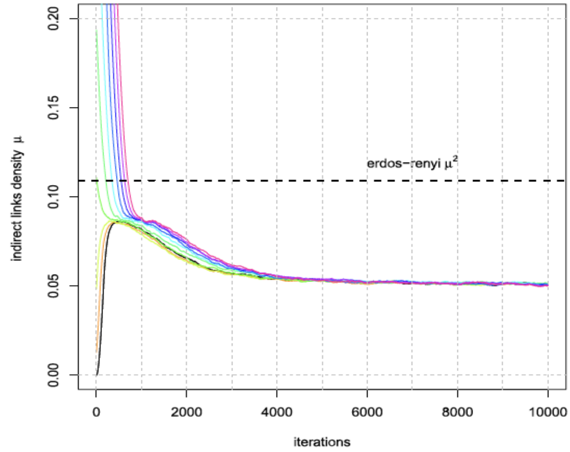
When $\beta \geq 0$, Algorithm 1 is not necessary since Erdös-Rényi graphs can be simulated using Bernoulli draws.

Asymptotic behaviour

Theorem 3 (Negative Link Externalities).

If $\beta < 0$ and sufficiently large in magnitude, model (3) is asymptotically different from a directed Erdös-Rényi model.

✓ sparser graphs than Erdös-Rényi



Asymptotic behaviour

Theorem 3 (Negative Link Externalities).

If $\beta < 0$ and sufficiently large in magnitude, model (3) is asymptotically different from a directed Erdös-Rényi model.

- how much "sufficiently large" magnitude?

- how to know it, since we must estimate β?

Asymptotic behaviour

Consider an additional utility component (cyclic triangles):

$$T(g) = \alpha t(H_1, g) + \beta t(H_2, g) + \gamma t(H_3, g), \qquad t(H_3, g) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i}^{n} g_{ij} g_{jk} g_{ki} \quad (6)$$

Theorem 4.

Consider model (6) as $n \to \infty$

1) If $\beta \geq 0$ and $\gamma \geq 0$, then the asymptotic normalising constant ψ solves

$$\psi = \lim_{n \to \infty} \psi_n = \max_{\mu \in [0,1]} \left\{ \alpha\mu + \beta\mu^2 + \gamma\mu^3 - \mu \log(\mu) - (1 - \mu) \log(1 - \mu) \right\} \quad (7)$$

   and model is asymptotically indistinguishable from directed Erdös-Rényi graph, with μ^* maximising (7). If the maximisation problem has multiple solutions, then μ^* picked randomly from some distribution on maximisers.

2) If at least **one** externality is negative (i.e. $\beta < 0$ or $\gamma < 0$) and sufficiently large, then model (6) not converge asymptotically to directed Erdös-Rényi graph and externalities can be identified.

Summary of Asymptotics

Remark 3.

Homogeneous players ($X_i = X_j$, $\forall i, j$):

(a) positive externalities

- asymptotically indistinguishable from Erdös-Rényi graph
- externalities not identified
- can approximate likelihood of model via likelihood of Erdös-Rényi graph

(b) at least one externality negative and large

- asymptotically sparser than Erdös-Rényi graph
- externalities identified

Heterogeneous players ($\exists i, j$ such that $X_i \neq X_j$):

- no results
- preliminary study in Mele & Zhu (2017) - working paper

Sampler Convergence

Theorem 5 (Convergence of Local Sampler with Nonnegative Externalities).

Model (6), with probability of meeting $\rho_{ij} = 1/(n(n-1))$. Fix $\gamma \geq 0$. Then, in the case of nonnegative externalities $\beta \geq 0$, there exists a V-shaped region of the parameter space delimited by functions $S_\gamma(\phi_1(\alpha)), S_\gamma(\phi_2(\alpha))$ such that

1) if (α, β) belongs to the V-shaped region, then model converges to stationarity in e^{Cn^2} steps, $C > 0$. This results holds for any local sampler.
2) otherwise, model converges in $Cn^2 \log(n)$ steps, $C > 0$.

Intuition:

(1a) in the V-shaped region problem (7) has 2 *local maxima*, the sampler spend exponential time at one of them (i.e. probability e^{-Cn^2} to escape from local max)

(1b) increasing $\gamma \implies$ increase area of exponentially slow convergence

(2a) when convergence is quadratic $\implies$ sampler feasible for $n < 500$

(2b) this happens when model is indistinguishable from directed Erdös-Rényi graph

Simulation and Estimation

Simulation and Estimation in Finite Networks

Posterior inference via approx version of exchange algorithm of MGM06 [16]

- double Metropolis-Hastings step to avoid computing $c(\mathcal{G}, X, \boldsymbol{\theta})$
- data augmentation via auxiliary network g'
- higher $R \implies$ better approximation of posterior, but higher rejection rate

Algorithm 2 Approximate Exchange Algorithm

1: **procedure** $\mathrm{AEA}(\boldsymbol{\theta}, g, M, R)$
2: **for** $m = 1, \ldots, M$ **do**
3: 1) propose parameter $\boldsymbol{\theta}' \sim q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$
4: 2) run Algorithm 1 for R iterations using $\boldsymbol{\theta}'$. Keep last simulated network g'
5: 3) accept parameter $\boldsymbol{\theta}'$ with probability

$$\alpha(\boldsymbol{\theta}, \boldsymbol{\theta}', g', g) = \min\left\{1, \frac{\exp\{\mathcal{Q}(g', X|\boldsymbol{\theta})\}}{\exp\{\mathcal{Q}(g, X|\boldsymbol{\theta})\}} \frac{p(\boldsymbol{\theta}')}{p(\boldsymbol{\theta})} \frac{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}|\boldsymbol{\theta}')}{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}'|\boldsymbol{\theta})} \frac{\exp\left\{\mathcal{Q}(g, X|\boldsymbol{\theta}')\right\}}{\exp\left\{\mathcal{Q}(g', X|\boldsymbol{\theta}')\right\}}\right\}$$

6: **end for**
7: **return** sequence of M parameters $\{\boldsymbol{\theta}^{(m)}\}_m$
8: **end procedure**

Simulation and Estimation in Finite Networks

Algorithm 2 Approximate Exchange Algorithm

procedure $\mathrm{AEA}(\boldsymbol{\theta}, g, M, R)$
    for $m = 1, \ldots, M$ **do**
        1) propose parameter $\boldsymbol{\theta}' \sim q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$
        2) run Algorithm 1 for R iterations using $\boldsymbol{\theta}'$. Keep last simulated network g'
        3) accept parameter $\boldsymbol{\theta}'$ with probability

$$\alpha(\boldsymbol{\theta}, \boldsymbol{\theta}', g', g) = \min\left\{1, \frac{\exp\{\mathcal{Q}(g', X|\boldsymbol{\theta})\}}{\exp\{\mathcal{Q}(g, X|\boldsymbol{\theta})\}} \frac{p(\boldsymbol{\theta}')}{p(\boldsymbol{\theta})} \frac{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}|\boldsymbol{\theta}')}{q_{\boldsymbol{\theta}}(\boldsymbol{\theta}'|\boldsymbol{\theta})} \frac{\exp\left\{\mathcal{Q}(g, X|\boldsymbol{\theta}')\right\}}{\exp\left\{\mathcal{Q}(g', X|\boldsymbol{\theta}')\right\}}\right\}$$

    end for
    return sequence of M parameters $\{\boldsymbol{\theta}^{(m)}\}_m$
end procedure

▶ what prior distribution $p(\boldsymbol{\theta})$?

▶ what proposal distribution $q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$?

Simulation results

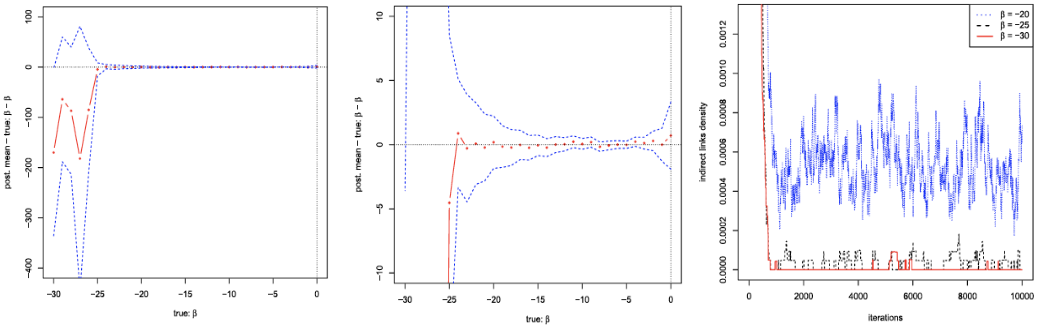


Figure: *Left*: Estimates of $\beta < 0$, with 95% credibility intervals (*middle*: zoom-in). *Right*: indirect links density.

- $\beta \geq 0 \implies$ Erdös-Rényi case, not identified
- $\beta < 0 \implies$ identified
- $\beta \ll 0 \implies$ estimation impossible: # indirect links close to 0

Simulation results

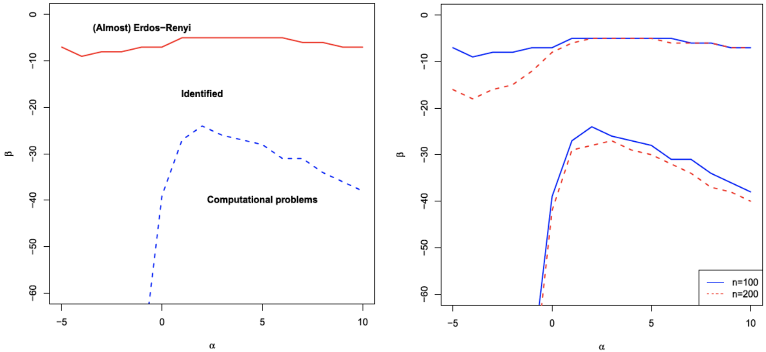


Figure: *Left*: approximate regions of identified parameters, for $n = 100$. *Right*: comparison of regions for $n = 100$, $n = 200$.

Remark 4.

Regions of identified parameters (α, β) vary with n, the number of players.

Simulation results

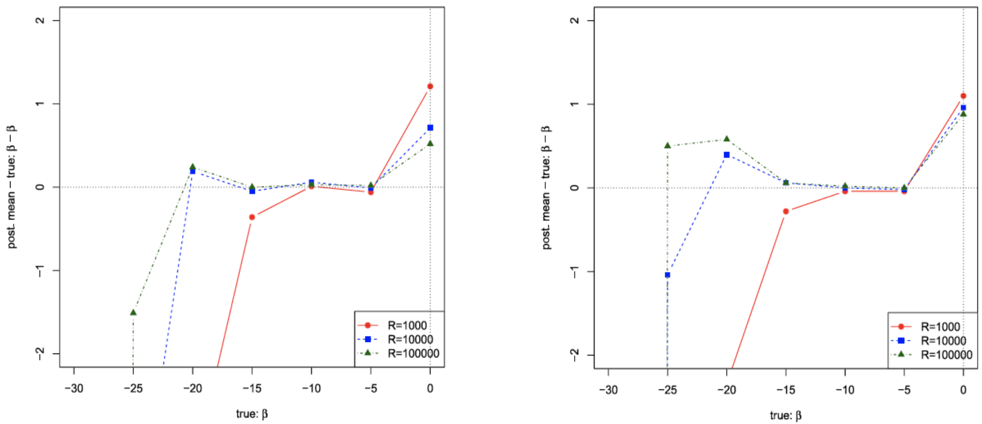


Figure: Difference between posterior estimates and true, for varying number of network simulations R: $n = 100$ (*left*) and $n = 200$ (*right*).

Simulation results

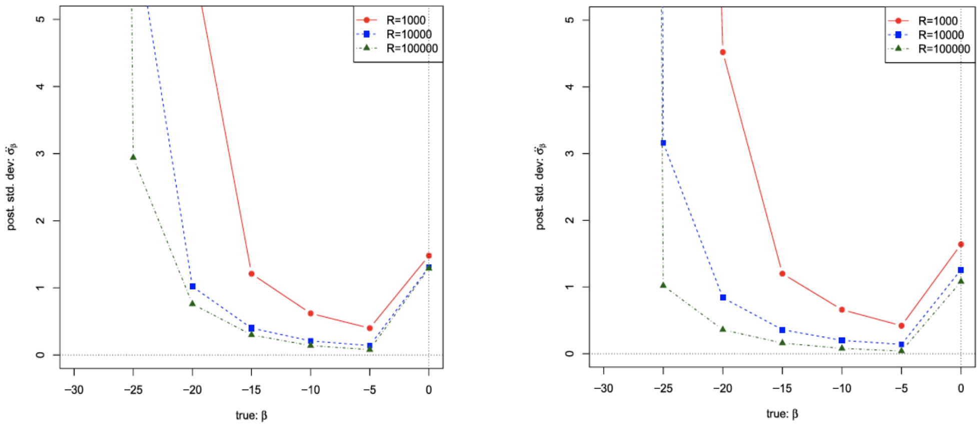


Figure: Posterior standard deviation for varying number of network simulations R: $n = 100$ (*left*) and $n = 200$ (*right*).

Simulation results

- $R = 1000 \implies$ imprecise estimates
- no significant difference between $R = 10,000$ and $R = 100,000 \implies$ suggest rule-of-thumb $R = 10,000$
- cost of increasing network simulations $\implies$ almost linear $\mathcal{O}(R)$
- results suggest convergence is almost quadratic $\mathcal{O}(n^2)$ in this area of parameter space

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- what was the computing time?

- what about real data applications?

Conclusions

Conclusions

The paper in a nutshell:

- network formation model, combining strategic and random networks features
- payoffs depend on links: direct + indirect (externalities)
- homogeneous players meet sequentially at random, myopically updating links
- network formation process is a potential game and converges to ERGM, generating directed dense networks
- identification: only if at least 1 externality negative and sufficiently large
- standard estimation for ERGMs exponentially slow $\Rightarrow$ Bayesian MCMC (almost quadratic time)

Conclusions

Unclear points and questions:

- ☞ theoretical quantification of "sufficiently large" (negative) magnitude of β?
- ☞ choice of prior for parameters $p(\boldsymbol{\theta})$?
- ☞ choice of proposal for network $q_g(\cdot|g)$?
- ☞ choice of proposal for parameters $q_{\boldsymbol{\theta}}(\cdot|\boldsymbol{\theta})$?
- ☞ duration of computing time in simulations?
- ☞ real data applications?

Thanks for your attention!

Any question?

References

References i

Jackson, M. O. (2008), Social and Economics Networks. Princeton

Lovasz, L. (2012), Large Networks and Graph Limits. American Mathematical Society Colloquium Publications, Vol. 60. American Mathematical Society

Badev, A. (2017), "Discrete Games in Endogenous Networks: Equilibria and Policy", Workin Paper. Available at arXiv https://arxiv.org/abs/1705.03137v1

Besag, J. (1974), "Spatial Interaction and the Statistical Analysis of Lattice Systems", Journal of the Royal Statistical Society. Series B (Methodological), 36(2):192–236.

Caimo, A., & N. Friel (2011), "Bayesian Inference for Exponential Random Graph Models", Social Networks, 33(1):41–55

Chatteriee, S., & S. R. S. Varadhan (2011), "The Large Deviation Principle for the Erdös-Rényi Random Graph" European Journal of Combinatorics, 32(7):1000–1017. Homomorphisms and Limits

References ii

- Christakis, N., J. Fowler, G. W. Imbens, & K. Kalyanaraman (2010), "An Empirical Model for Strategic Network Formation" Harvard University
- Frank, O., & D. Strauss (1986), "Markov Graphs", Journal of the American Statistical Association, 81:832–842
- Graham, B. (2017), "An Econometric Model of Network FormationWith Degree Heterogeneity", Econometrica, 85(4):1033-1063
- Gouriéroux, J., Lafont, J. J. and Monfort, A. (1981), "Coherency Conditions in Simultaneous Linear Equation Models with Endogenous Switching Regimes", Econometrica, 48(3):675–696.
- Hsieh, C.-S., & L.-F. Lee (2016), "A Social Interactions Model With Endogenous Friendship Formation and Selectivity", Journal of Applied Econometrics, 31(2):301–319
- Leung, M. (2018), "A weak law for moments of pairwise-stable networks", Working Paper. Available at SSRN: https://ssrn.com/abstract=2663685

References iii

- Monderer, D., & L. Shapley (1996), "Potential Games", Games and Economic Behavior, 14, 124–143.
- Mele, A., & L. Zhu (2017), "Approximate Variational Estimation for a Model of Network Formation", Working Paper. Available at arXiv: https://arxiv.org/abs/1702.00308
- Menzel, K. (2017), "Strategic Network Formation With Many Agents", Working Papers, NYU
- Murray, I. A., Z. Ghahramani, & D. J. C. MacKay (2006), "MCMC for Doubly-Intractable Distributions", Uncertainty in Artificial Intelligence
- Miyauchi, Y. (2016), "Structural Estimation of a Pairwise Stable Network Formation With Nonnegative Externality", Journal of Econometrics, 195(2):224–235
- Snijders, T. A. B. (2002), "Markov Chain Monte Carlo Estimation of Exponential Random Graph Models", Journal of Social Structure, 3(2)

References iv

- Tamer, E. (2003), "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria", The Review of Economic Studies, 70:147–165.
- Wainwright, M. J., & M. Jordan (2008), "Graphical Models, Exponential Families, and Variational Inference", Foundations and Trends® in Machine Learning, 1(1–2):1–305

Meetings, Discussants and Material

Links: Mele (2017) paper; Mele (2017) supplements;

1^{st} session: 11/10/2018 - Matteo Iacopini - Main slides on Mele (2017);

2^{nd} session: 18/10/2018 - Carlo Santagiustina - Sup. slides on Tamer (2003);