The synthesis of complex audio spectra by means of Frequency Modulation

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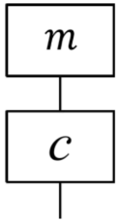
Fundamentals of Frequency Modulation synthesis

Frequency Modulation (*FM*) synthesis is a technic that allows to generate complex audio spectra by varying the instantaneous frequency of a carrier wave (c) according to a modulating wave (m), such that the rate at which the carrier varies is the frequency of the modulating wave. The resulting signal can be written as follows:

$$\boxed{x(t) = \sin(\omega_c t + I \sin \omega_m t) = \sum_{n=-\infty}^{\infty} J_n(I) \sin[(\omega_c + n\omega_m)t\,]}$$

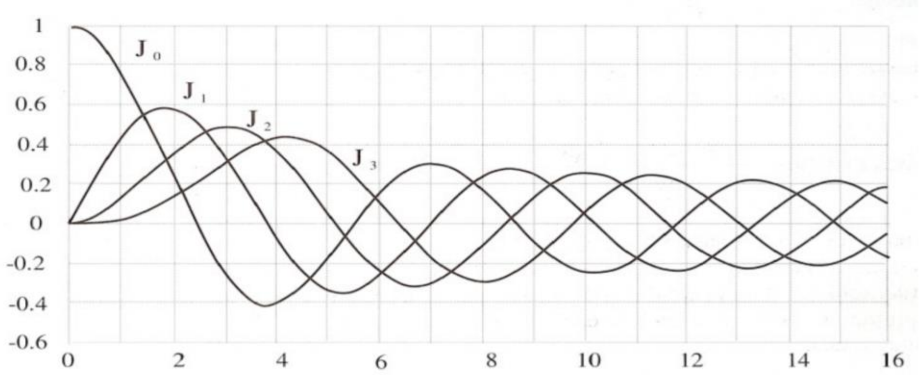
- $\omega_c = 2\pi f_c \Longrightarrow$ Carrier frequency
- $\omega_m = 2\pi f_m \Longrightarrow$ Modulating frequency
- $I \Longrightarrow$ Modulation index



Naturally, the negative components of the spectrum reflect into the positive frequency domain with a change of sign (inversion of phase) and add algebraically to the corresponding positive components. It may so happen that this process causes the creation of new spectral components. The following spectra show only the absolute value of the Fourier components in the positive frequency domain.

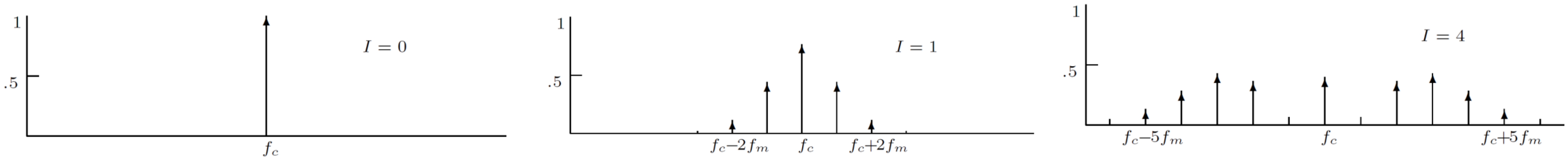
$$\boxed{J_n(I) = \left(\frac{I}{2}\right)^n \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\,(n+h)!} \left(\frac{I}{2}\right)^{2h}}$$

$\Rightarrow$ Bessel function of the first kind and *n*th order



Simple FM

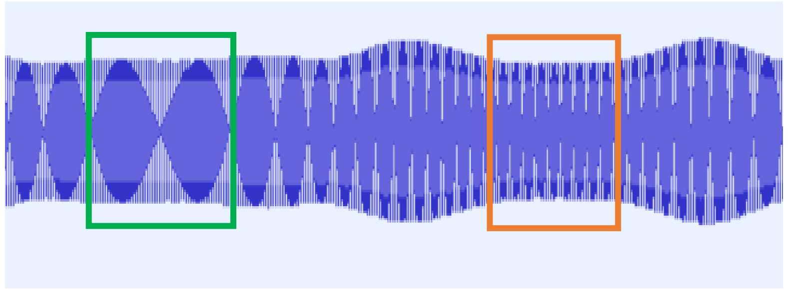
Below is illustrated an example that shows how the spectrum of the carrier becomes more and more complex as the modulation index increases.



The maximum instantaneous frequency deviation of the carrier wave around its central value f_c is equal to:

$$\boxed{\Delta f = I \times f_m}$$

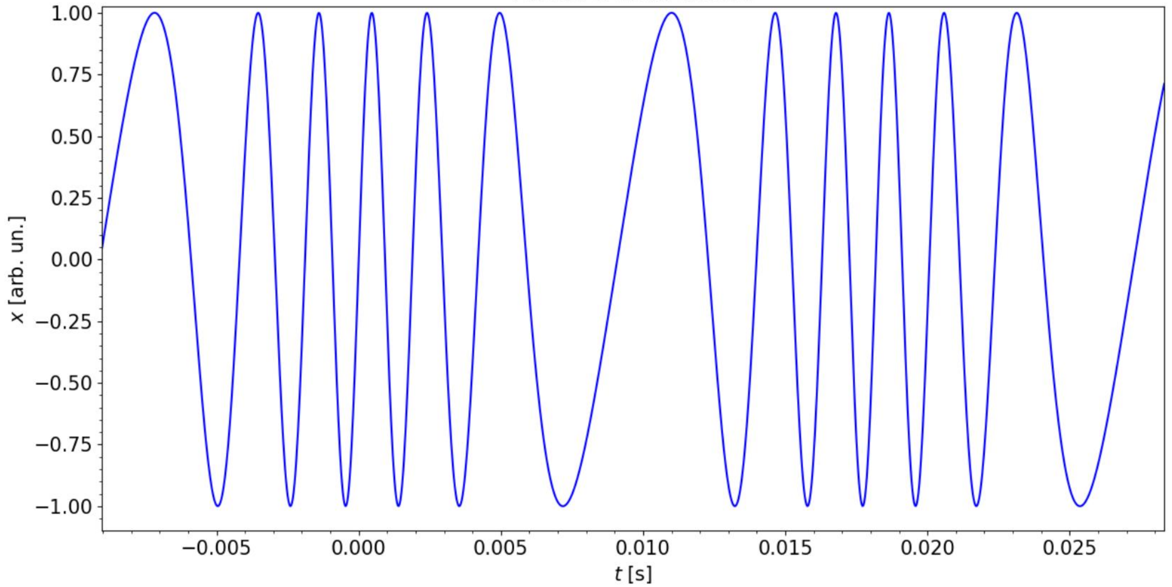
For f_c =330 Hz, f_m= 1 Hz, I=13 the effect of the modulation is similar to a *vibrato* with Δf=13 Hz.



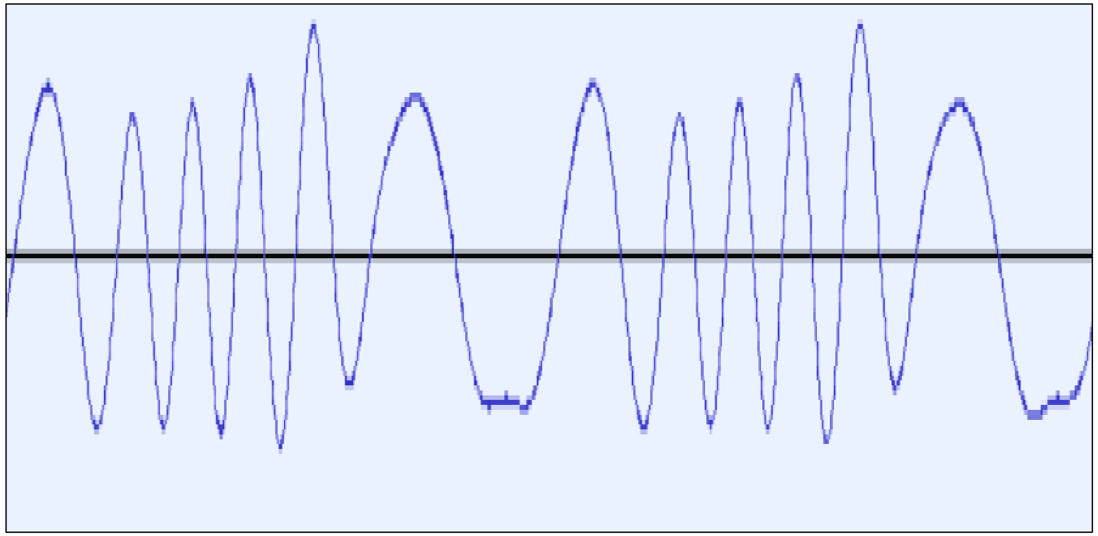
Example of FM

Example of a signal of frequency f_c =330 Hz modulated by another wave of frequency f_m= 55 Hz with I=3.9.

Expect waveform

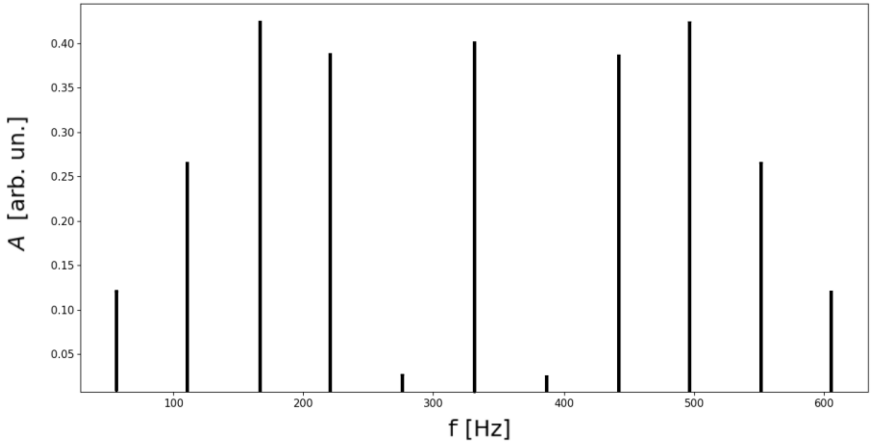


Measured waveform

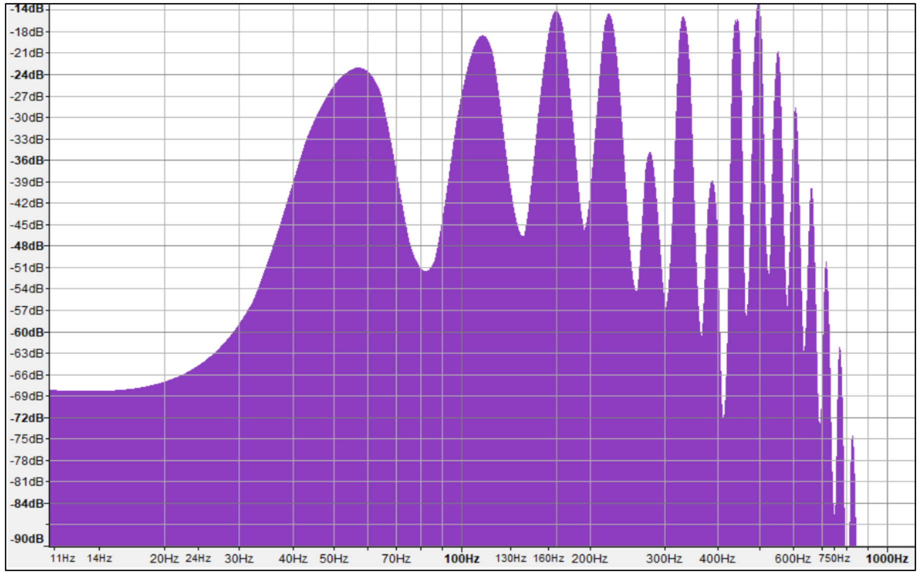


Example FM

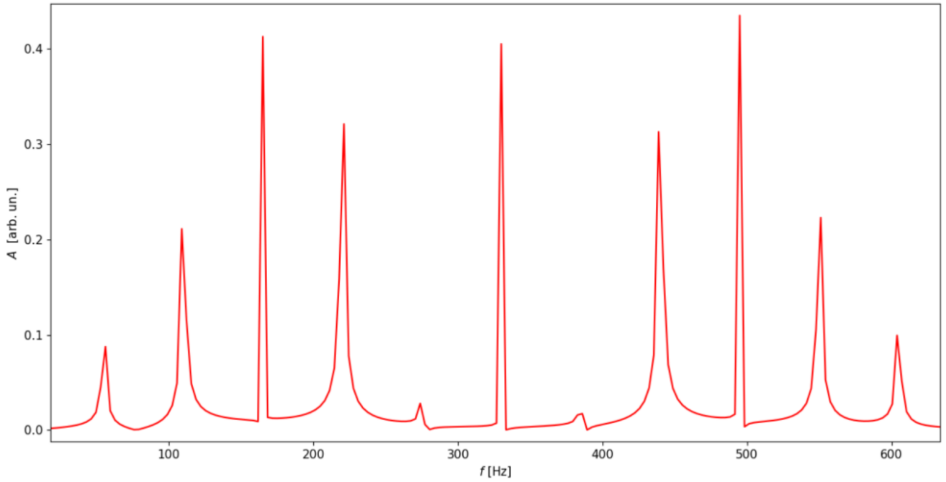
Expect spectrum



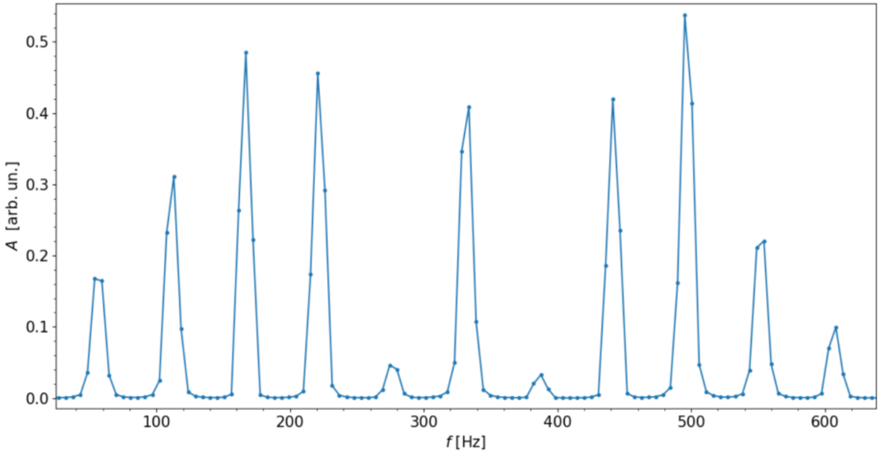
Measured spectrum



FFT of the expected waveform



FFT of the measured waveform



Harmonic spectrum

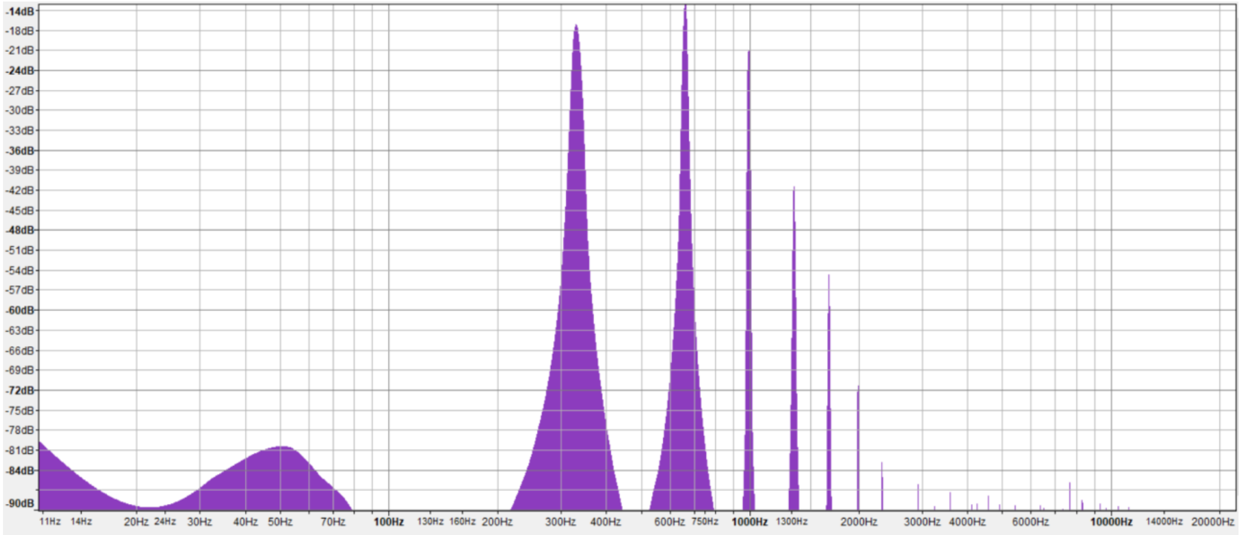
If f is the examined pitch we can write: $f_c = c \cdot f \, , f_m = m \cdot f$

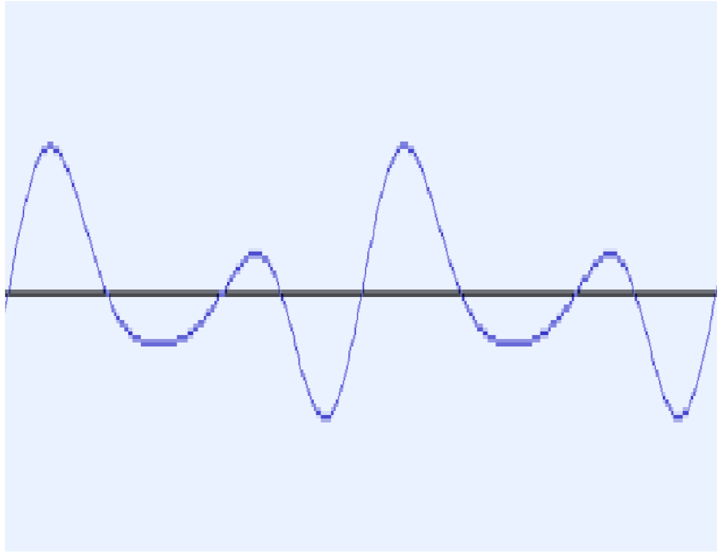
If $c\!:\!m = N_1\!:\!N_2$ (N_1, N_2 integer numbers) $\Longrightarrow$The spectrum is harmonic

In the following examples f is always equal to 333Hz.

The spectrum below contains all the harmonics multiple of the fundamental frequency (333 Hz).

- $c\!:\!m = 1\!:\!1$, I=1.5



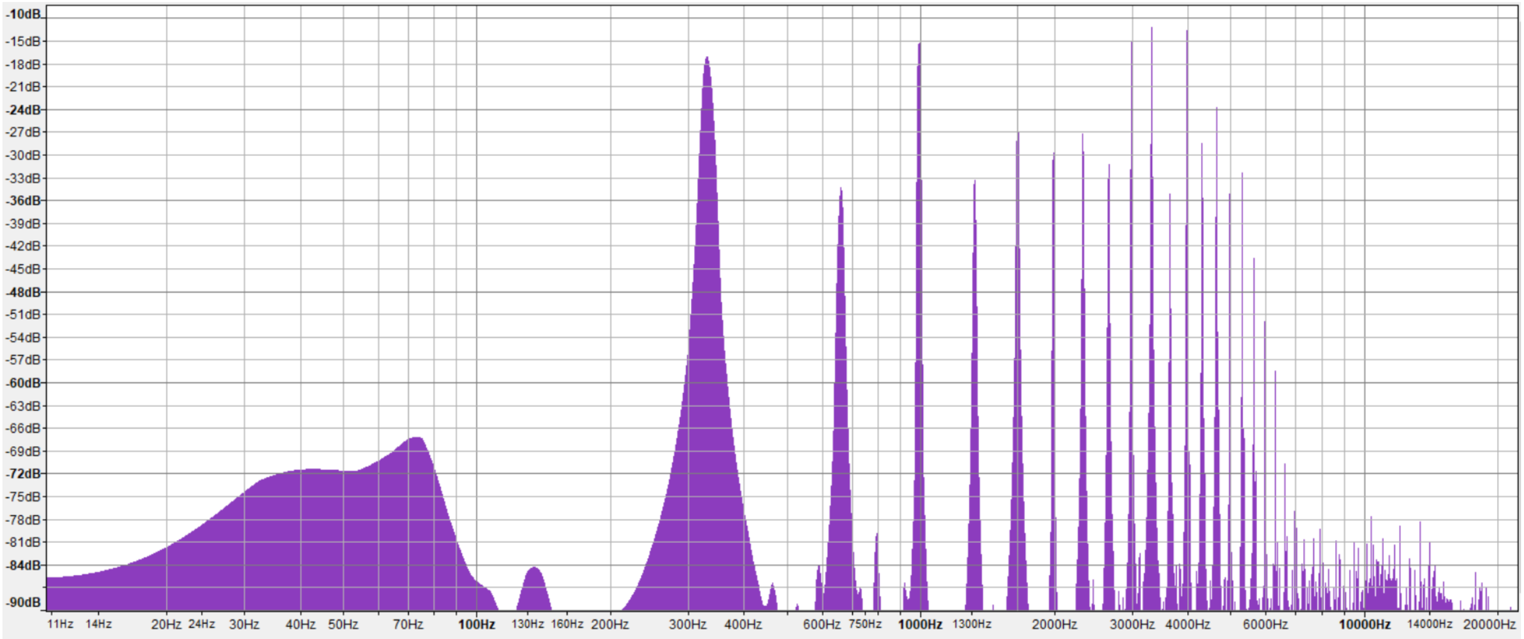


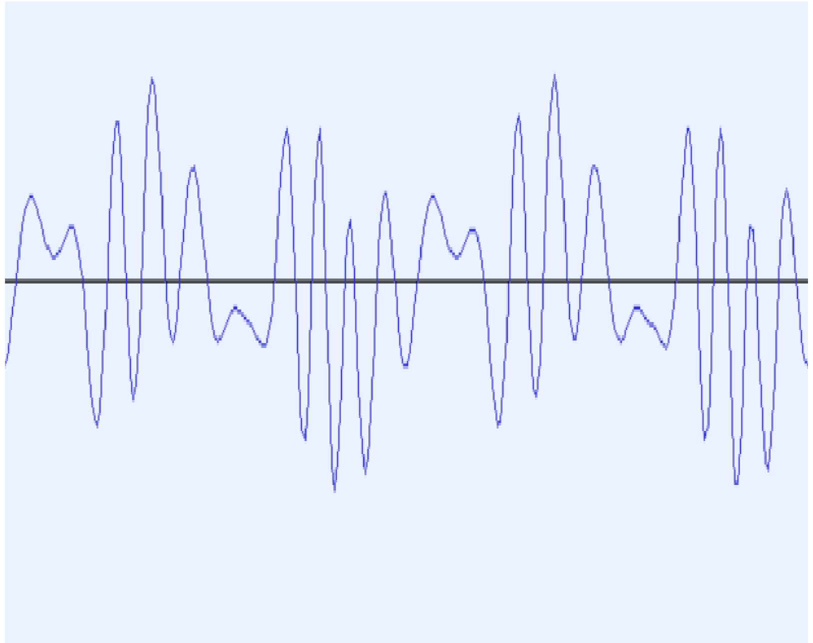
Dependence of the spectrum on the various paramenters

Variation of *I*

Compared to the previous example, we can see how increasing the modulation index causes the harmonics in the spectrum to rise, hence the more complex waveform.

- $c{:}m = 1{:}1$, I=13



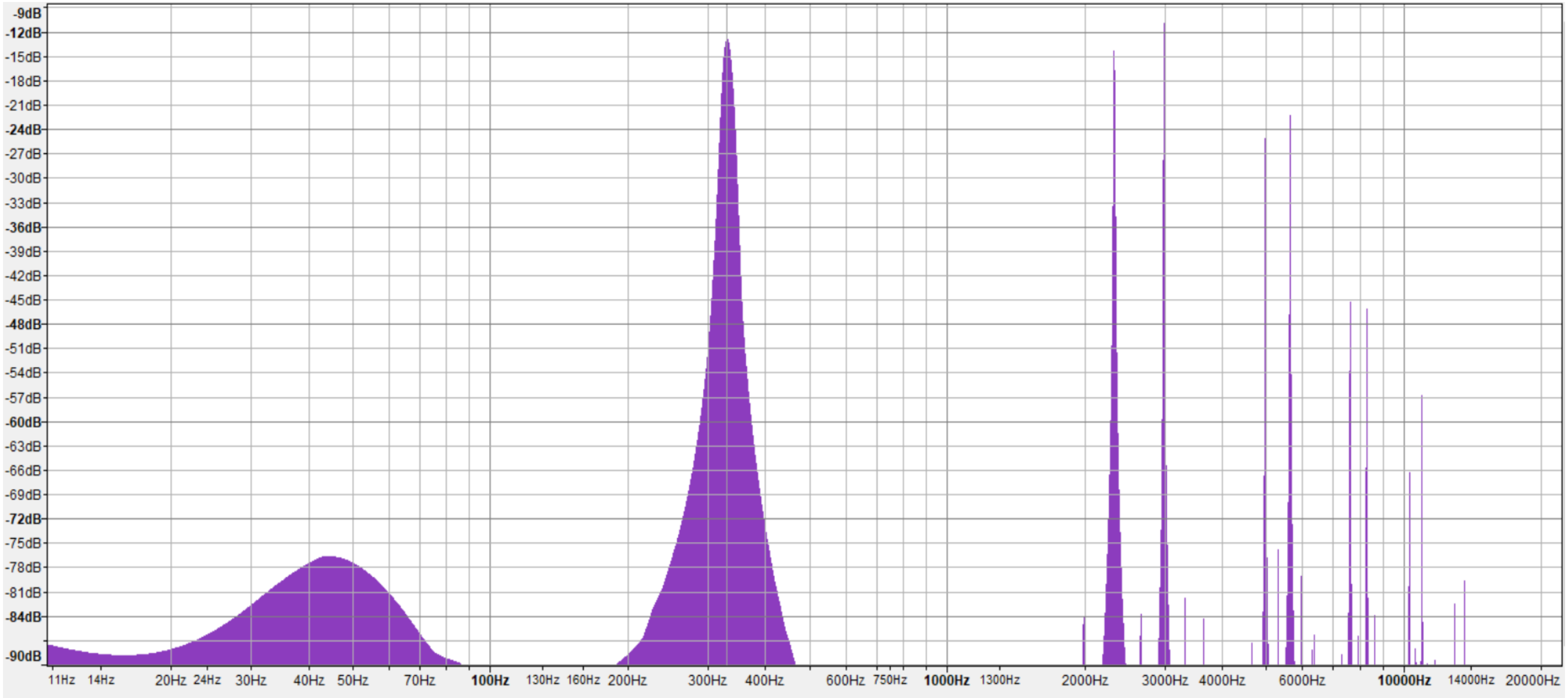


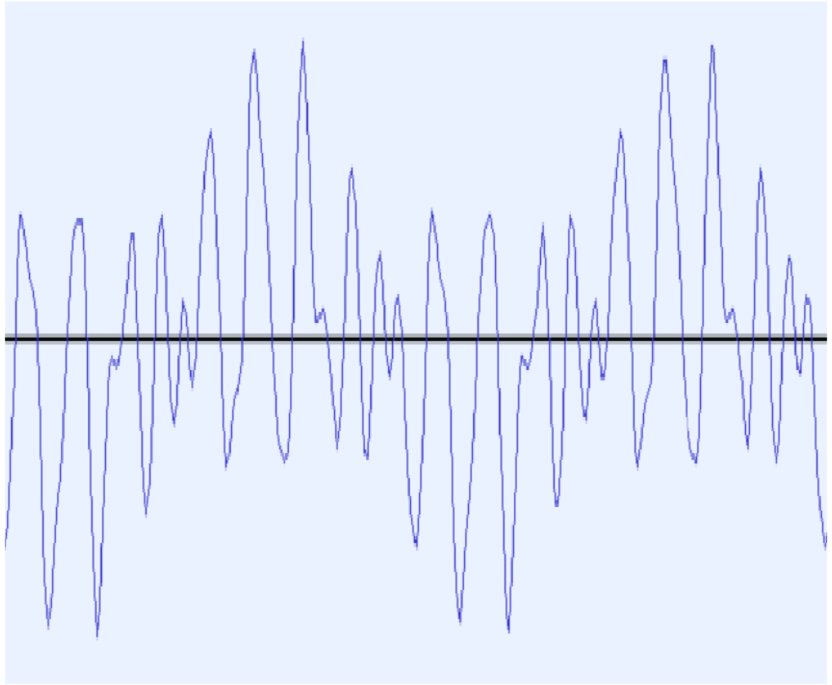
Dependence of the spectrum and the timbre on the various paramenters

Variation of m

In this example we can see how the spectrum contains the 7th harmonic of the fundamental frequency, at 2300 Hz, as well. This is due to the fact that we must take into account also the negative components of the spectrum, reflected into the positive frequency domain (i.e. $|1-8|=7$).

- $c\!:\!m=1\!:\!8$, I=1.5



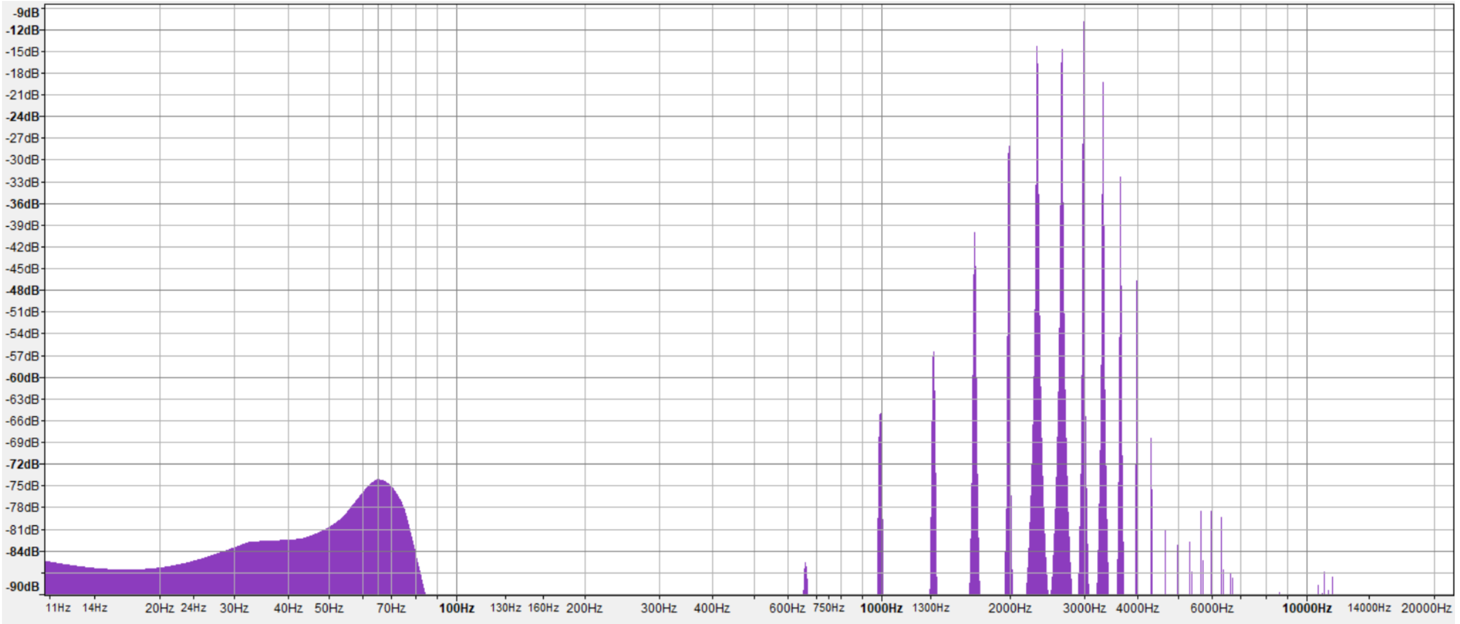


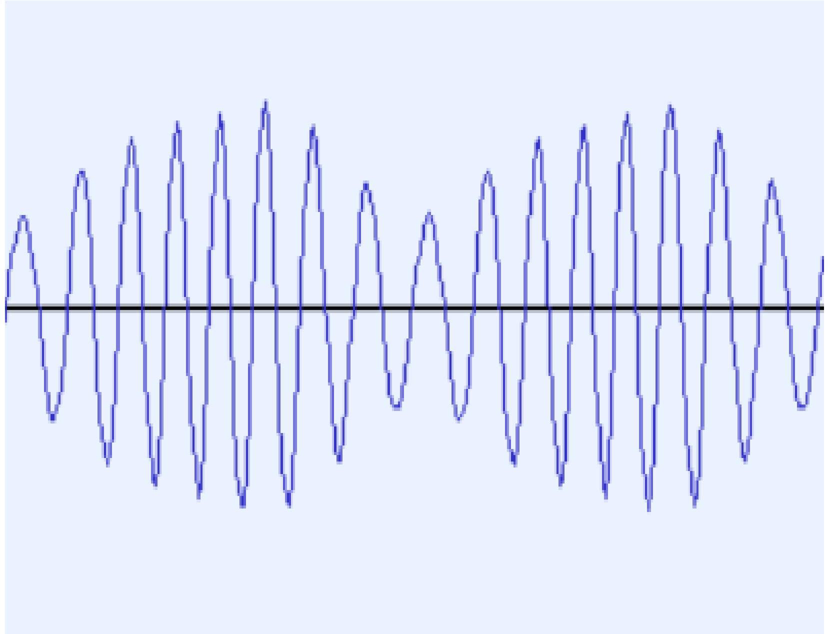
Dependence of the spectrum on the various paramenters

Variation of *c*

By increasing the frequency of the carrier wave, the spectrum moves towards higher frequencies. The components on the left side of the central harmonic are the frequencies that in the previous examples belonged in the negative domain and were reflected. However, in this case, the left components are not negative and can be observed even without the reflection process, hence the symmetric spectrum.

- $c\!:\!m = 8\!:\!1$, I=1.5

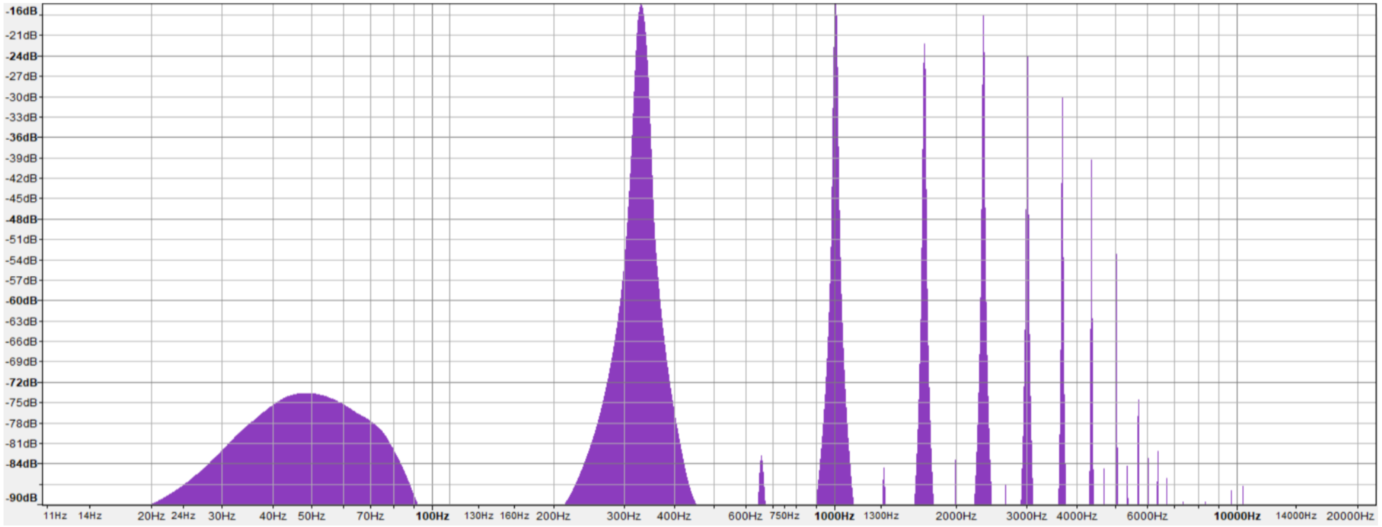




Inharmonic spectrum

If $c : m \neq N_1 : N_2 \implies$ The spectrum is inharmonic

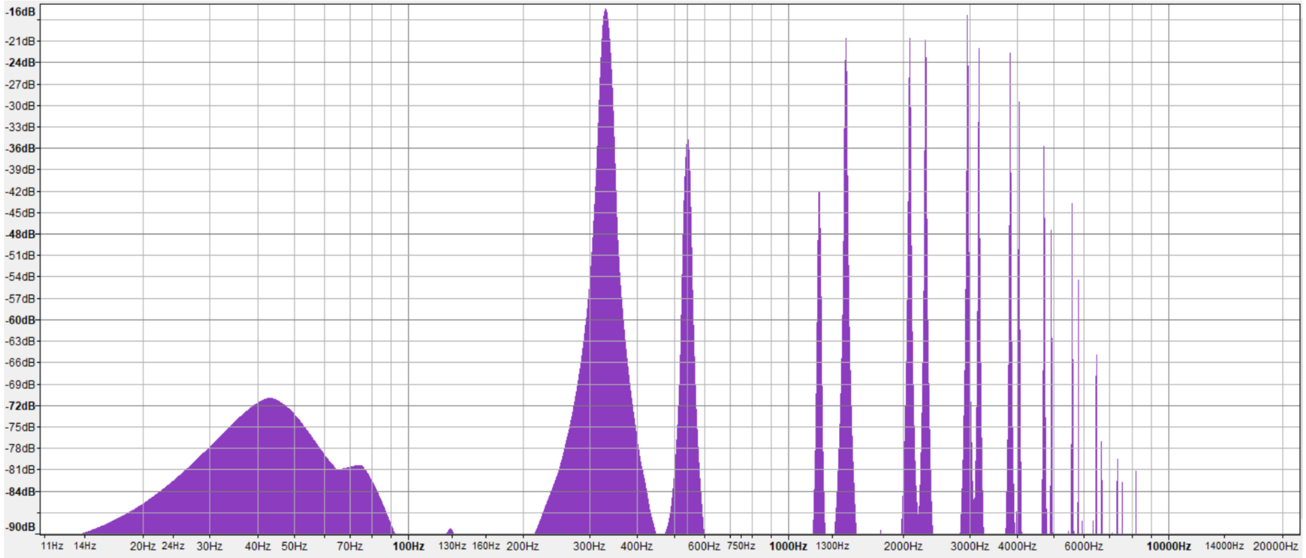
- $c : m = 1 : 2.02$, I=4



Increasing the inharmonicity, we can see that the spectrum ceases to contain harmonics of the fundamental frequency and instead many other frequencies not multiple of the fundamental one appear.

- $c : m = 1 : 2.64$, I=4

The spectrum on the right shows the “split” of the Fourier components which were originally formed by the algebraical sum of the positive and negative frequencies that had the same absolute value. The fact that the ratio between *c* and *m* is not equal to the ratio of two integers causes the reflected negative components to fall right next to, and not on top of, the positive ones resulting in an inharmonic spectrum.

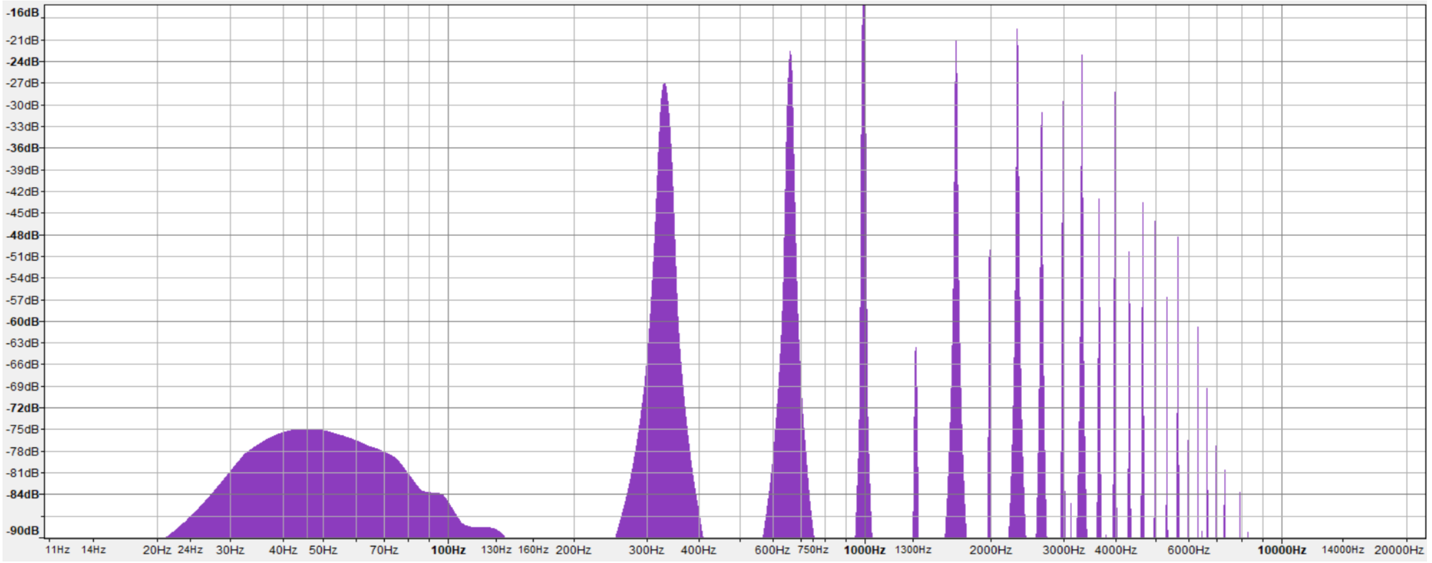


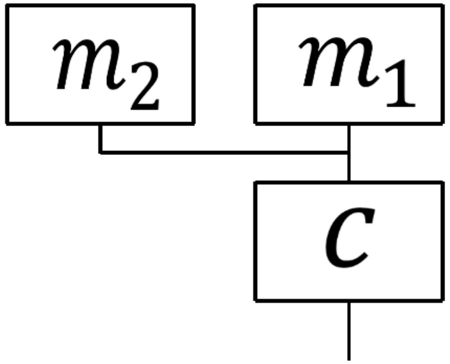
Parallel modulators

In order to increase the complexity of the spectrum we can use two modulating waves acting on the same carrier. The resulting spectrum contains the sum of the carrier frequency with the linear combination of the modulating frequencies, while the relative amplitudes are equal to the product of the respective Bessel functions.

$$x(t) = \sin(\omega_c t + I_1 \sin \omega_{m_1} t + I_2 \sin \omega_{m_2} t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} J_{n_1}(I_1)\, J_{n_2}(I_2) \sin(\omega_c + n_1\omega_{m_1} + n_2\omega_{m_2})t$$

- $c = 5, m_1 = 2, m_2 = 5, I_1 = 1,\ I_2 = 0.6$



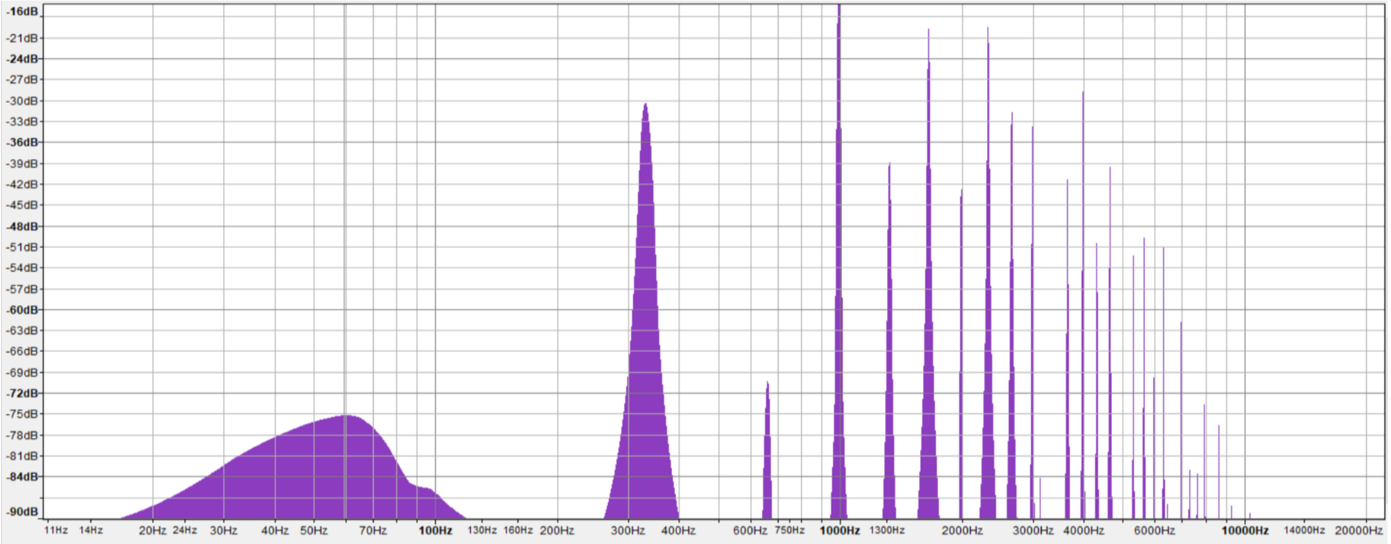


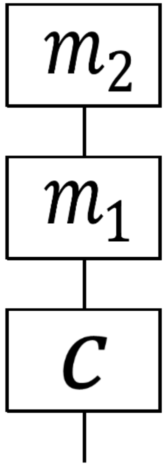
Cascade modulators

In this configuration the modulating wave is itself modulated by another signal and the resulting spectrum is similar to the latter but with different amplitudes. For instance, the component at 3330 Hz is now absent since it corresponds to the case in which n_1=0 and $n_2 = 1$.

$$x(t) = \sin(\omega_c t + I_1 \sin(\omega_{m_1} t + I_2 \sin \omega_{m_2} t)) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} J_{n_1}(I_1)\, J_{n_2}(n_1 I_2) \sin(\omega_c + n_1\omega_{m_1} + n_2\omega_{m_2})t$$

- $c = 5, m_1 = 2, m_2 = 5, I_1 = 1, I_2 = 0.6$

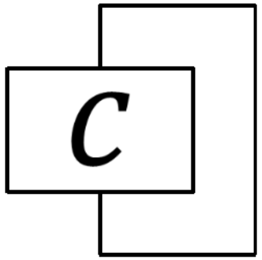




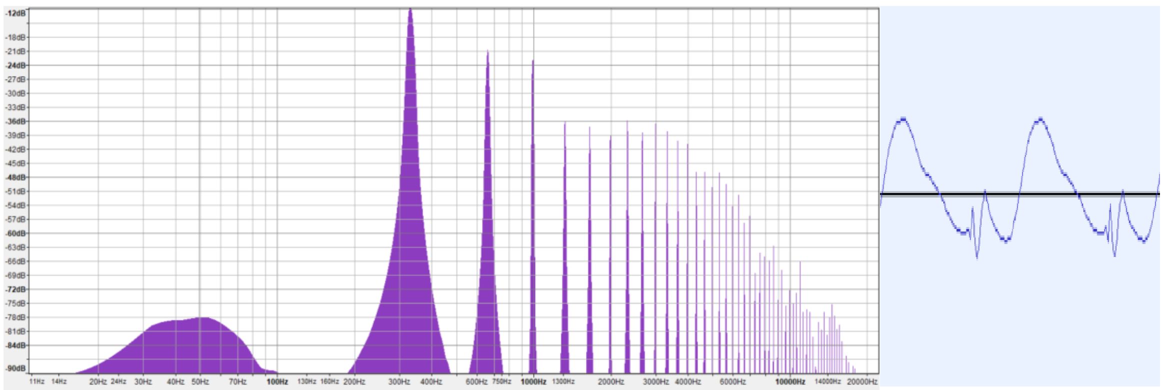
Feedback

This last configuration can be obtained with just one oscillator by means of a feedback mechanism; namely the carrier wave is used also as the modulating signal operating on itself. The outcome is an unstable system which produces high frequency noise for large values of the modulation index.

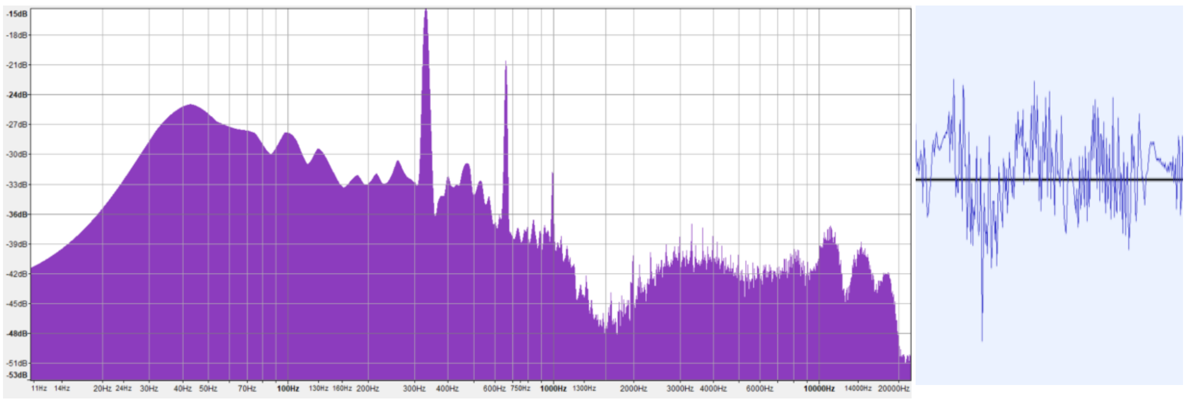
$$\boxed{x(t) = \sin(\omega_c t + I x(t)) = \sum_{n=1}^{\infty} \frac{2\, J_n(nI)}{nI} \sin(n\omega_c t)}$$



- $c = m = 1, I = 1$



- $c = m = 1, I = 13$



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- John Chowning, David Bristow, *"FM Theory & Applications - By Musicians For Musicians"*, Tokyo, Yamaha, 1986, ISBN 4-636-17482-8.

- Dave Benson, *"Music: A Mathematical Offering"*, Cambridge University Press, 2006, ISBN 9780511811722.

- J. Chowning, *"The Synthesis of Complex Audio Spectra by Means of Frequency Modulation"*, in *Journal of the Audio Engineering Society*, vol. 21, n. 7, 1973.

- Roads, Curtis, *"The Computer Music Tutorial"*, MIT Press, 1996, ISBN 978-0-262-68082-0.

Programs used for the making of the various spectra

- Dexed: https://www.kvraudio.com/product/dexed-by-digital-suburban
- Audacity: https://www.audacityteam.org
- Python: https://www.python.org