1 Starting Point

In the Louise notes we arrived to:

$$P(\mathbf{t},\mathbf{t}^*|\mathbf{x}_0^*,\mathbf{s}^*) = P(\mathbf{t}|\mathbf{x}_0^*,\mathbf{s}^*)P(\mathbf{t}^*|\mathbf{x}_0^*,\mathbf{s}^*) = \\ \propto \prod_{i=1}^{N} \psi_i(t_i,t_{\partial i})\xi_i(t_i,t_i^*)\psi_i^*(t_i^*|t_{\partial i}^*)$$

I changed the notation a little bit just to make the problem appear more symmetrical. Nonetheless, this is exactly what obtained Louise, i.e. a product of three terms. The first is the distribution of the inferred trajectory. The second is the *observation* term that couples the planted and the inferred trajectory and the third is the deterministic constraint on the planted trajectory. So, more precisely, let us define the ψ term, the ξ term and the ψ^* term as Louise did.

$$\psi_i(t_i,t_{\partial i}) = \gamma(t_i)\left(\prod_{t=0}^{t_i-2}\prod_{j\in\partial i}(1-\lambda\theta(t-t_j))\right)\left(1-\phi(t_i)\prod_{j\in\partial i}(1-\lambda\theta(t_i-1-t_j))\right)$$

with

$$\phi(t_i) = \begin{cases} 0 & t_i = 0, T+1 \\ 1 & 0 < t_i < T+1 \end{cases}$$

Then we define the observation term:

$$\xi(t_i,t_i^*) = \delta\left(x_i^T(t_i), x_i^{*T}(t_i^*)\right)$$

Finally we define the deterministic term. **Notice** that in this last term we have dropped the explicit dependence over $\mathbf{x}_0^*,\mathbf{s}^*$ because they play here the role of noise (the equivalent of J_{ij} in the Ising Model, so they are present but they don't play the role of dynamical variables). So the term is:

$$\psi_i^*(t_i^*|t_{\partial i}^*) = \begin{cases} 1 & t_i^* = \delta_{x_i^{*0},S}\min_{j\in\partial i}(t_j^*+s_{ji}^*) \\ 0 & \text{else} \end{cases}$$

2 Passage to copies

Here I would like, in order to keep the treatment symmetrical, to introduce copies both for the planted and the inferred trajectory. This should work. So

$$T_{ij} = T_{ji} = (t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, t_j^{*(i)})$$

This is the variable node structure. We can also attach the leaf (t_i,t_i^*) to each factor node. This is free and only symbolical, but is nice. The factor node structure is:

$$\Psi_i(t_i,t_i^*,\{T_{ij}\}_{j\in\partial i}) = \psi_i(t_i,t_{\partial i}^{(i)})\xi_i(t_i,t_i^*)\psi_i^*(t_i^*|t_{\partial i}^{*(i)})\prod_{j\in\partial i}\delta(t_i,t_i^{(j)})\delta(t_i^*,t_i^{*(j)})$$

3 Messages

Now we write BP equations:

$$\nu_{\psi_i \to j}(T_{ij}) = \sum_{t_i, t_i^*} \sum_{\{T_{ki}\}_{k\in\partial i\setminus j}} \Psi_i(t_i, t_i^*, \{T_{ik}\}_{k\in\partial i}) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki})$$

4 Simplifications

We first sum over t_i, t_i^* and all the $t_i^{(k)}, t_i^{*(k)}$. This is easy beacause there is the constraint. So we impose that all the copies are equal to $t_i^{(j)}, t_i^{(j)*}$, which are inside the r.h.s.

$$\begin{aligned}
\nu_{\psi_i \to j}(T_{ij}) = & \sum_{\left\{t_k^{(i)}, t_k^{*(i)}\right\}_{k\in\partial i\setminus j}} \psi_i(t_i^{(j)}, t_{\partial i}^{(i)}) \xi_i(t_i^{(j)}, t_i^{*(j)}) \psi_i^*(t_i^{*(j)} | t_{\partial i}^{*(i)}) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) = \\
= & \, \xi_i(t_i^{(j)}, t_i^{*(j)}) \sum_{\left\{t_k^{(i)}, t_k^{*(i)}\right\}_{k\in\partial i\setminus j}} \psi_i(t_i^{(j)}, t_{\partial i}^{(i)}) \psi_i^*(t_i^{*(j)} | t_{\partial i}^{*(i)}) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki})
\end{aligned} \tag{1}$$

Note that is clear that every time $t_i^{(k)}$related to i must be equal to $t_i^{(j)}$ and the same is for planted times. We can remove the copy index to simlify notation whenever we want.

4.1 Rewriting the factor node

The is to simplify the BP message by expressing both ψ^* and ψ as sums of products over ∂i. If we do so we can then exchange the sum $\sum_{\left\{t_j^{(i)}, t_j^{*(i)}\right\}_{k\in\partial i\setminus j}}$ with the product $\prod_{k\in\partial i\setminus j}$ appearing in (1). We start by writing ψ^* because we can use the trick already developed in Sib:

$$\begin{aligned}
\psi_i^*(t_i^* | t_{\partial i}^*) = & \begin{cases} 1 & t_i^* = \delta_{x_i^{*0}, S} \min_{j\in\partial i}(t_j^* + s_{ji}^*) \\ 0 & \text{else} \end{cases} = \\
& = \delta_{x_i^{*0}, I} \delta_{t_i^*, 0} + \delta_{x_i^{*0}, S} \prod_{k\in\partial i} 1\left(t_i^* \le t_k^* + s_{ki}^*\right) - \delta_{x_i^{*0}, S} \prod_{k\in\partial i} 1\left(t_i^* < t_k^* + s_{ki}^*\right)
\end{aligned}$$

So for the ψ_i^* term we did it! It is now the sum of products. Now it is time for ψ_i:

$$\begin{aligned}
\psi_i(t_i, t_{\partial i}) &= \gamma(t_i)\left(\prod_{t=0}^{t_i-2}\prod_{j\in\partial i}(1-\lambda\theta(t-t_j))\right)\left(1-\phi(t_i)\prod_{j\in\partial i}(1-\lambda\theta(t_i-1-t_j))\right) = \\
&= \gamma(t_i)\left(\prod_{t=0}^{t_i-2}\prod_{j\in\partial i}(1-\lambda\theta(t-t_j)) - \phi(t_i)\prod_{t=0}^{t_i-1}\prod_{j\in\partial i}(1-\lambda\theta(t-t_j))\right) = \\
&= \gamma(t_i)\left(\prod_{j\in\partial i}\prod_{t=t_j}^{t_i-2}(1-\lambda) - \phi(t_i)\prod_{j\in\partial i}\prod_{t=t_j}^{t_i-1}(1-\lambda)\right) = \\
&= \gamma(t_i)\left(\prod_{j\in\partial i}(1-\lambda)^{(t_i-t_j-1)\theta(t_i-t_j-1)} - \phi(t_i)\prod_{j\in\partial i}(1-\lambda)^{(t_i-t_j)\theta(t_i-t_j)}\right) = \\
&=: \gamma(t_i)\left(\prod_{j\in\partial i}a(t_i-t_j-1) - \phi(t_i)\prod_{j\in\partial i}a(t_i-t_j)\right)
\end{aligned}$$

where we defined:

$$a(t) := (1-\lambda)^{t\theta(t)}$$

Now also the ψ term is the sum of products. Now we multiply ψ with ψ^*. We end up with 6 terms:

$$\begin{aligned}
&\psi_i^*(t_i^*|t_{\partial i}^*)\psi_i(t_i, t_{\partial i}) = \\
&\qquad = \gamma(t_i)\delta_{x_i^{*0},I}\delta_{t_i^*,0}\prod_{k\in\partial i}a(t_i-t_k-1)+ \\
&\qquad + \gamma(t_i)\delta_{x_i^{*0},S}\prod_{k\in\partial i}a(t_i-t_k-1)1\left(t_i^* \le t_k^*+s_{ki}^*\right) \\
&\qquad - \gamma(t_i)\delta_{x_i^{*0},S}\prod_{k\in\partial i}a(t_i-t_k-1)1\left(t_i^* < t_k^*+s_{ki}^*\right) \\
&\qquad - \gamma(t_i)\phi(t_i)\delta_{x_i^{*0},I}\delta_{t_i^*,0}\prod_{k\in\partial i}a(t_i-t_k) \\
&\qquad - \gamma(t_i)\delta_{x_i^{*0},S}\phi(t_i)\prod_{k\in\partial i}a(t_i-t_k)1\left(t_i^* \le t_k^*+s_{ki}^*\right) \\
&\qquad + \gamma(t_i)\delta_{x_i^{*0},S}\phi(t_i)\prod_{k\in\partial i}a(t_i-t_k)1\left(t_i^* < t_k^*+s_{ki}^*\right)
\end{aligned}$$

Now we substitute this form into the (1) in order to simplify the sums:

$$
\begin{aligned}
\nu_{\psi_i \to j}(T_{ij}) = \gamma(t_i)\xi_i(t_i, t_i^*)\times \\
&\times \sum_{\left\{t_k, t_k^*\right\}_{k\in\partial i\setminus j}} \Bigg(a(t_i - t_j - 1)\delta_{x_i^{*0}, I}\delta_{t_i^*, 0} \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k - 1) + \\
&+ \delta_{x_i^{*0}, S} a(t_i - t_j - 1) 1\left(t_i^* \leq t_j^* + s_{ji}^*\right) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k - 1) 1\left(t_i^* \leq t_k^* + s_{ki}^*\right) + \\
&- \delta_{x_i^{*0}, S} a(t_i - t_j - 1) 1\left(t_i^* < t_j^* + s_{ji}^*\right) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k - 1) 1\left(t_i^* < t_k^* + s_{ki}^*\right) + \\
&- \phi(t_i) a(t_i - t_j)\delta_{x_i^{*0}, I}\delta_{t_i^*, 0} \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k) + \\
&- \delta_{x_i^{*0}, S} \phi(t_i) a(t_i - t_j) 1\left(t_i^* \leq t_j^* + s_{ji}^*\right) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k) 1\left(t_i^* \leq t_k^* + s_{ki}^*\right) + \\
&+ \delta_{x_i^{*0}, S} \phi(t_i) a(t_i - t_j) 1\left(t_i^* < t_j^* + s_{ji}^*\right) \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i - t_k) 1\left(t_i^* < t_k^* + s_{ki}^*\right) \Bigg).
\end{aligned}
$$

Now we can, term by term, exchanging the sums and the products. For example, we take the following term of the 6 appearing:

$$
\sum_{\left\{t_k^{(i)}, t_k^{*(i)}\right\}_{k\in\partial i\setminus j}} \prod_{k\in\partial i\setminus j} \nu_{k\to\psi_i}(T_{ki}) a(t_i^{(j)} - t_k^{(i)}) 1\left(t_i^* \leq t_k^* + s_{ki}^*\right)
$$

This term becomes:

$$
\prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}, t_k^{*(i)}} \nu_{k\to\psi_i}(T_{ki}) a(t_i^{(j)} - t_k^{(i)}) 1\left(t_i^* \leq t_k^* + s_{ki}^*\right)
$$

Let us put the indicator function in the sum:

$$
\begin{aligned}
\prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}, t_k^{*(i)}} \nu_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)}) 1\left(t_i^* \leq t_k^* + s_{ki}^*\right) = \\
= \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}} \sum_{t_k^{*(i)} \geq t_i^{(j)*} - s_{ki}^*} \nu_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)})
\end{aligned}
$$

The other five terms can be treated in the same way. Now we want to sum over planted and inferred times.

4.2 Summation of the planted times

Let us define:

$$
\sigma_{ji}^* := 1+\mathrm{sign}(t_j^* + s_{ji}^* - t_i^*)
$$

We substitute this definition and the manipulations described before in the BP equation. We have:

$$
\begin{aligned}
&\nu_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, t_j^{*(i)}) = \\
&= \gamma(t_i^{(j)}) \xi_i(t_i^{(j)}, t_i^{*(j)}) \times \\
&\times \Bigg(a(t_i^{(j)} - t_j^{(i)} - 1) \delta_{x_i^{*0}, I} \delta_{t_i^{*(j)}, 0} \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}, t_k^{*(i)}} \nu_{k \to \psi_i}(T_{ki}) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&+ \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 1, 2\right) \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}} \sum_{t_k^{*(i)} \geq t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&- \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 2\right) \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}} \sum_{t_k^{*(i)} > t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&- \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) \delta_{x_i^{*0}, I} \delta_{t_i^*, 0} \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}, t_k^{*(i)}} \nu_{k \to \psi_i}(T_{ki}) a(t_i^{(j)} - t_k^{(i)}) + \\
&- \delta_{x_i^{*0}, S} \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 1, 2\right) \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}} \sum_{t_k^{*(i)} \geq t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)}) + \\
&+ \delta_{x_i^{*0}, S} \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 2\right) \prod_{k \in \partial i \setminus j} \sum_{t_k^{(i)}} \sum_{t_k^{*(i)} > t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) a(t_i^{(j)} - t_k^{(i)}) \Bigg).
\end{aligned}
$$

We can see that time $t_j^{*(i)}$ appears only inside the indicator function and on the messages. We can also notice that these messages only appeared summed over t_k^*. We want, therefore, to define a new message which is the sum over t_k^* of the old message. Let us define:

$$
\sum_{t_k^{*(i)}} \nu_{k \to \psi_i}(T_{ki}) := \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 0) + \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 1) + \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 2) \tag{2}
$$

$$
\sum_{t_k^{*(i)} \geq t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) =: \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 1) + \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 2)
$$

$$
\sum_{t_k^{*(i)} > t_i^{(j)*} - s_{ki}^*} \nu_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, t_k^{*(i)}) =: \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 2)
$$

The $\tilde{\nu}$ are the new messages. With this definition the BP equation becomes:

$$
\begin{aligned}
&\tilde{\nu}_{\psi_i\to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, \sigma_{ji}^*) = \\
&= \gamma(t_i^{(j)})\xi_i(t_i^{(j)}, t_i^{*(j)})\times \\
&\times \Bigg(a(t_i^{(j)} - t_j^{(i)} - 1)\delta_{x_i^{*0},I}\delta_{t_i^{*(j)},0} \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)},\sigma_{ki}^*} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, \sigma_{ki}^*) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&+ \delta_{x_i^{*0},S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 1,2\right) \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}} \sum_{\sigma_{ki}^*=1}^{2} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, \sigma_{ki}^*) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&- \delta_{x_i^{*0},S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 2\right) \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 2) a(t_i^{(j)} - t_k^{(i)} - 1) + \\
&- \phi(t_i) a(t_i^{(j)} - t_j^{(i)})\delta_{x_i^{*0},I}\delta_{t_i^*,0} \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)},\sigma_{ki}^*} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, \sigma_{ki}^*) a(t_i^{(j)} - t_k^{(i)}) + \\
&- \delta_{x_i^{*0},S}\phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 1,2\right) \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}} \sum_{\sigma_{ki}^*=1}^{2} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, \sigma_{ki}^*) a(t_i^{(j)} - t_k^{(i)}) + \\
&+ \delta_{x_i^{*0},S}\phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 2\right) \prod_{k\in\partial i\setminus j} \sum_{t_k^{(i)}} \tilde{\nu}_{k\to\psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, 2) a(t_i^{(j)} - t_k^{(i)}) \Bigg).
\end{aligned}
$$

This equation relates the new function-to-variable messages $\tilde{\nu}_{\psi_i\to j}$ with the new variable-to-function messages $\tilde{\nu}_{i\to\psi_j}$. Now we want to find the BP equation that relates variable-to-function messages with function-to-variable messages. This is simple because we just need to write the definition (2) for each value of $\sigma^* = 2, 1, 0$:

$$
\tilde{\nu}_{k\to\psi_i}(t_i^{(k)}, t_k^{(i)}, t_i^{*(k)}, \sigma_{ki}^*) := \begin{cases} \sum_{t_k^{*(i)} < t_i^{(k)*} - s_{ki}^*} \tilde{\nu}_{\psi_k\to i}(t_k^{(i)}, t_i^{(k)}, t_k^{*(i)}, 2) & \sigma_{ki}^* = 0 \\ \tilde{\nu}_{\psi_k\to i}(t_k^{(i)}, t_i^{(k)}, t_i^{*(k)} - s_{ki}^*, 2) & \sigma_{ki}^* = 1 \\ \sum_{t_k^{*(i)} > t_i^{(k)*} - s_{ki}^*} \tilde{\nu}_{\psi_k\to i}\left(t_k^{(i)}, t_i^{(k)}, t_k^{*(i)}, 1 + \operatorname{sign}\left(t_i^{*(k)} - t_k^{*(i)} + s_{ik}^*\right)\right) & \sigma_{ki}^* = 2 \end{cases}
$$

We have summed over the planted trajectory. Therefore, now messages are functions which can take $3T^3$ values. We want to simplify this point again by summing over the inferred time

4.3 Summation over the inferred trajectory:

We define:

$$
\sum_{t_k} \tilde{\nu}_{k\to\psi_i}(t_i, t_k, t_i^*, \sigma_{ki}^*) a(t_i - t_k - c) =: \mu_{k\to\psi_i}(t_i, c, t_i^*, \sigma_{ki}^*)
$$

with $c = 0, 1$. So:

$$\begin{aligned}
&\tilde{\nu}_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, \sigma_{ji}^*) = \\
&= \gamma(t_i^{(j)}) \xi_i(t_i^{(j)}, t_i^{*(j)}) \times \\
&\times \Bigg(a(t_i^{(j)} - t_j^{(i)} - 1) \delta_{x_i^{*0}, I} \delta_{t_i^{*(j)}, 0} \prod_{k \in \partial i \setminus j} \sum_{\sigma_{ki}^*} \mu_{k \to \psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, \sigma_{ki}^*) + \\
&+ \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) \mathbb{1}(\sigma_{ji} = 1, 2) \prod_{k \in \partial i \setminus j} \sum_{\sigma_{ki}^* = 1}^{2} \mu_{k \to \psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, \sigma_{ki}^*) + \qquad (3) \\
&- \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) \mathbb{1}(\sigma_{ji} = 2) \prod_{k \in \partial i \setminus j} \mu_{k \to \psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, 2) + \\
&- \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) \delta_{x_i^{*0}, I} \delta_{t_i^*, 0} \prod_{k \in \partial i \setminus j} \sum_{\sigma_{ki}^*} \mu_{k \to \psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, \sigma_{ki}^*) + \\
&- \delta_{x_i^{*0}, S} \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) \mathbb{1}(\sigma_{ji} = 1, 2) \prod_{k \in \partial i \setminus j} \sum_{\sigma_{ki}^* = 1}^{2} \mu_{k \to \psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, \sigma_{ki}^*) + \\
&+ \delta_{x_i^{*0}, S} \phi(t_i) a(t_i^{(j)} - t_j^{(i)}) \mathbb{1}(\sigma_{ji} = 2) \prod_{k \in \partial i \setminus j} \mu_{k \to \psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, 2) \Bigg).
\end{aligned}$$

So again we are moving from a type of message $\tilde{\nu}$ to another type: μ. As we did before, we now want to close the equations by writing the variable-to-function message. We define:

$$\begin{aligned}
&\mu_{k \to \psi_i}(t_i^{(k)}, c, t_i^{*(k)}, \sigma_{ki}^*) := \sum_{t_k^{(i)}} \tilde{\nu}_{k \to \psi_i}(t_i^{(j)}, t_k^{(i)}, t_i^{*(j)}, \sigma_{ki}^*) a(t_i - t_k - c) = \\
&= \sum_{t_k^{(i)}} a(t_i - t_k - c) \begin{cases} \sum_{t_k^{*(i)} < t_i^{(k)*} - s_{ki}^*} \tilde{\nu}_{\psi_k \to i}(t_k^{(i)}, t_i^{(k)}, t_k^{*(i)}, 2) & \sigma_{ki}^* = 0 \\ \tilde{\nu}_{\psi_k \to i}(t_k^{(i)}, t_i^{(k)}, t_i^{*(k)} - s_{ki}^*, 2) & \sigma_{ki}^* = 1 \\ \sum_{t_k^{*(i)} > t_i^{(k)*} - s_{ki}^*} \tilde{\nu}_{\psi_k \to i}\left(t_k^{(i)}, t_i^{(k)}, t_k^{*(i)}, 1 + \mathrm{sign}\left(t_i^{*(k)} - t_k^{*(i)} + s_{ik}^*\right)\right) & \sigma_{ki}^* = 2 \end{cases}
\end{aligned}$$

So up to now we have an equation for μ which is the variable to function message and an equation for $\tilde{\nu}$ which is the function to variable message. Substituting the form of $\tilde{\nu}$ inside the μ expression gives us a closed form of the equations. Everithing is closed.

5 Implementation

Let us try to work on these equations in order to have a structure to implement. We refer to the figure. If we want the function $\mu_{i \to \psi_j}$ we need the functions $\{\mu_{k \to \psi_i}\}_{k \in \partial i \setminus j}$. To do so it is a good idea to rewrite here the messages with the index coherent to the figure:

$$\mu_{i\to\psi_j}(t_j^{(i)}, c, t_j^{*(i)}, \sigma_{ij}^*) := \sum_{t_i^{(j)}} \tilde{\nu}_{i\to\psi_j}(t_j^{(i)}, t_i^{(j)}, t_j^{*(i)}, \sigma_{ij}^*) a(t_j - t_i^{(j)} - c) =$$

$$= \sum_{t_i^{(j)}} a(t_j - t_i^{(j)} - c) \begin{cases} \sum_{t_i^{*(j)} < t_j^{(i)*} - s_{ij}^*} \tilde{\nu}_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, 2) & \sigma_{ki}^* = 0 \\ \tilde{\nu}_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_j^{*(i)} - s_{ij}^*, 2) & \sigma_{ki}^* = 1 \\ \sum_{t_i^{*(j)} > t_j^{(i)*} - s_{ij}^*} \tilde{\nu}_{\psi_i \to j}\left(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, 1 + \mathrm{sign}\left(t_j^{*(i)} - t_i^{*(j)} + s_{ji}^*\right)\right) & \sigma_{ki}^* = 2 \end{cases}$$

So now we just rewrite the $\tilde{\nu}$ equation and we have here all we need to implement! As the following formula suggests, everything is coherent with the figure. We have therefore that to have $\mu_{i\to\psi_j}$ we need *only* the $\{\mu_{k\to\psi_i}\}_{k\in\partial i\setminus j}$ combined in a proper way.

$$\begin{aligned}
&\tilde{\nu}_{\psi_i\to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, \sigma_{ji}^*) = \\
&= \gamma(t_i^{(j)})\xi_i(t_i^{(j)}, t_i^{*(j)})\times \\
&\times \Bigg(a(t_i^{(j)} - t_j^{(i)} - 1)\delta_{x_i^{*0}, I}\delta_{t_i^{*(j)}, 0} \prod_{k\in\partial i\setminus j} \sum_{\sigma_{ki}^*} \mu_{k\to\psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, \sigma_{ki}^*) + \\
&+ \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 1, 2\right) \prod_{k\in\partial i\setminus j} \sum_{\sigma_{ki}^*=1}^{2} \mu_{k\to\psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, \sigma_{ki}^*) + \\
&- \delta_{x_i^{*0}, S} a(t_i^{(j)} - t_j^{(i)} - 1) 1\left(\sigma_{ji}^* = 2\right) \prod_{k\in\partial i\setminus j} \mu_{k\to\psi_i}(t_i^{(j)}, 1, t_i^{*(j)}, 2) + \\
&- \phi(t_i) a(t_i^{(j)} - t_j^{(i)})\delta_{x_i^{*0}, I}\delta_{t_i^*, 0} \prod_{k\in\partial i\setminus j} \sum_{\sigma_{ki}^*} \mu_{k\to\psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, \sigma_{ki}^*) + \\
&- \delta_{x_i^{*0}, S}\phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 1, 2\right) \prod_{k\in\partial i\setminus j} \sum_{\sigma_{ki}^*=1}^{2} \mu_{k\to\psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, \sigma_{ki}^*) + \\
&+ \delta_{x_i^{*0}, S}\phi(t_i) a(t_i^{(j)} - t_j^{(i)}) 1\left(\sigma_{ji}^* = 2\right) \prod_{k\in\partial i\setminus j} \mu_{k\to\psi_i}(t_i^{(j)}, 0, t_i^{*(j)}, 2) \Bigg)
\end{aligned}$$

The algorithm now is straightforward:

1. We have a population of μ. Once the quenched disordered is fixed we extract a number of messages μ from the population which play the role of the $\{\mu_{k\to\psi_i}\}_{k\in\partial i\setminus j}$.
2. We calculate the associated message $\tilde{\nu}_{\psi_i\to j}$.
3. Finally we can have the message $\mu_{i\to\psi_j}$.

Let us make an estimate of execution time and memory usage of each passage.

1. In order to store the messages μ we have to store their values and μ is a function of two times and a σ and a c. Therefore, each μ can take $6T^2$

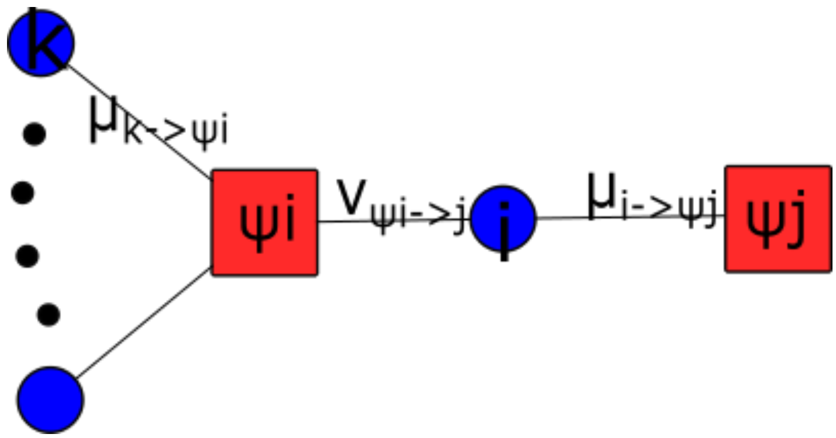

Figure 1: The messages on the graph. We close the scheme by epressing the $\mu_{i\to\psi_j}$ in function of $\{\mu_{k\to\psi_i}\}_{k\in\partial i\setminus j}$

values. We will need to keep in memory $6NT^2$ numbers where N is the populations size

2. Each message $\tilde{\nu}$ can take $3T^3$ values and we must calculate only one function $\tilde{\nu}$ which plays the role of $\tilde{\nu}_{\psi_i\to j}$. The calculation of $\tilde{\nu}$ for each of its values does not scale with time (we just need to sum over 6 terms each of one is a product of sums over 3 terms. So we have definitely to do less than $6\,(|\partial i|-1)$ sums and than we have to make 6 times a product and 6 times the evaluation of an exponential. We can say therefore that we have a term that scales as $|\partial i|T^3$. We need to keep in **temporary** memory all these $\propto T^3$ numbers.

3. The μ to be calculated is the sum over couple of times. So, since each μ is characterized by providing $6T^2$ values and each μ is calculated with a time $\propto T^2$, then in principle we need a time $t \sim T^4$ to calculate each μ. However, we can pre-calculate the cumulants of $\tilde{\nu}$ with respect to planted time as soon as we finish passage 2. This should reduce the scaling to T^3.

If $T = 100$, $|\partial i| = 10$, $N = 1000$ then we have to do 10^7 operations for each iteration of pop-dynamic and we should keep in memory $\approx 10^8$ numbers.

5.1 Implementing $\tilde{\nu}$ function

In input we have $|\partial i| - 1$ functions of μ tipe. We call them $\{\mu_k\}_{k=1,\ldots,d-1}$ where $d := |\partial i|$. The idea is that we want to calculate:

$$\nu(t_i, t_j, \tau_i, \sigma)$$

I have changed the notation eliminating the tilde and renaming the variable so to make them more suitable for a code. The ingredients needed are the:

$$\mu_k(t_i, c_k, \tau_i, \sigma_k)$$

Before entering the calculations we have to extract the disorder. If the degree is fixed to d then we have to extract the zero-patient variables and the delays. By looking at BP equations we see that we need to extract:

$$\left(x_i^{*0}, s_{ji}^*, s_{ij}^*\right)$$

where x_i^{*0} is a BERNOULLI distribution and s is extracted according to $P(s) = (1-\lambda)^s\lambda$. The variable x_i^{0*} decides the form of ν: if $x_i^{0*} = I$:

$$\begin{aligned}
&\nu(t_i, t_j, 0, \sigma) = \\
&= \gamma(t_i)\xi_i(t_i, 0)\times \\
&\times \left(a(t_i - t_j - 1)\prod_{k=1}^{d-1}\sum_{\sigma_k}\mu_k(t_i, 1, 0, \sigma_k) + \right. \\
&\left. - \phi(t_i)a(t_i - t_j)\prod_{k=1}^{d-1}\sum_{\sigma_k^*}\mu_k(t_i, 0, 0, \sigma_k)\right)
\end{aligned}$$

So in the case in which it is a zero patient is very simple. The function is nonzero only for $\tau_i = 0$ and it is constant in σ and t_j. If instead $x_i^{0*} = S$:

$$\begin{aligned}
&\nu(t_i, t_j, \tau_i, \sigma) = \\
&= \gamma(t_i)\xi_i(t_i, \tau_i)\times \\
&\times \left(a(t_i - t_j - 1)1\left(\sigma = 1, 2\right)\prod_{k=1}^{d-1}\sum_{\sigma_k=1}^{2}\mu_k(t_i, 1, \tau_i, \sigma_k) + \right. \\
&- a(t_i - t_j - 1)1\left(\sigma = 2\right)\prod_{k=1}^{d-1}\mu_k(t_i, 1, \tau_i, 2) + \\
&- \phi(t_i)a(t_i - t_j)1\left(\sigma = 1, 2\right)\prod_{k=1}^{d-1}\sum_{\sigma_k=1}^{2}\mu_k(t_i, 0, \tau_i, \sigma_k) + \\
&\left. + \phi(t_i)a(t_i - t_j)1\left(\sigma = 2\right)\prod_{k=1}^{d-1}\mu_k(t_i, 0, \tau_i, 2)\right)
\end{aligned}$$

And we also observe that, always in the case $x_i^{0*} = S$:

$$\nu(t_i, t_j, \tau_i, 0) = 0$$

$$
\begin{aligned}
&\nu(t_i, t_j, \tau_i, 1) = \\
&= \gamma(t_i)\xi_i(t_i, \tau_i)\times \\
&\times \left(a(t_i - t_j - 1) \prod_{k=1}^{d-1} \sum_{\sigma_k=1}^{2} \mu_k(t_i, 1, \tau_i, \sigma_k) + \right. \\
&\left. - \phi(t_i) a(t_i - t_j) \prod_{k=1}^{d-1} \sum_{\sigma_k=1}^{2} \mu_k(t_i, 0, \tau_i, \sigma_k) \right)
\end{aligned}
$$

$$
\begin{aligned}
&\nu(t_i, t_j, \tau_i, 2) = \\
&= \nu(t_i, t_j, \tau_i, 1) + \gamma(t_i)\xi_i(t_i, \tau_i)\times \\
&\times \left(-a(t_i - t_j - 1) \prod_{k=1}^{d-1} \mu_k(t_i, 1, \tau_i, 2) + \right. \\
&\left. + \phi(t_i) a(t_i - t_j) \prod_{k=1}^{d-1} \mu_k(t_i, 0, \tau_i, 2) \right)
\end{aligned}
$$

5.2 Implementing the μ function

When the quantity ν is calculated for all of its arguments we take it and obtain the μ:

$$
\begin{aligned}
&\mu(t_j, c, \tau_j, \sigma) := \\
&= \sum_{t_i} a(t_j - t_i - c) \begin{cases} \sum_{\tau_i < \tau_j - s^*_{ij}} \nu(t_i, t_j, \tau_i, 2) & \sigma = 0 \\ \nu(t_i, t_j, \tau_j - s^*_{ij}, 2) & \sigma = 1 \\ \sum_{\tau_i > \tau_j - s^*_{ij}} \nu\left(t_i, t_j, \tau_i, 1 + \mathrm{sign}\left(\tau_j - \tau_i + s^*_{ji}\right)\right) & \sigma = 2 \end{cases}
\end{aligned}
$$

We define :

$$
\Sigma(t_i, t_j, k, \sigma) := \sum_{\tau_i=0}^{k} \nu(t_i, t_j, \tau_i, \sigma)
$$

and:

$$
\sigma^*_{ji} := 1 + \mathrm{sign}\left(\tau_j - \tau_i + s^*_{ji}\right)
$$

Let us study the term:

$$
\sum_{\tau_i=\tau_j - s^*_{ij}+1}^{T+1} \nu\left(t_i, t_j, \tau_i, 1 + \mathrm{sign}\left(\tau_j - \tau_i + s^*_{ji}\right)\right) =
$$

$$
\begin{aligned}
&\sum_{\tau_i=\tau_j - s^*_{ij}+1}^{\tau_j + s^*_{ji}-1} \nu\left(t_i, t_j, \tau_i, 2\right) + \nu\left(t_i, t_j, \tau_j + s^*_{ji}, 1\right) + \sum_{\tau_i=\tau_j + s^*_{ji}+1}^{T+1} \nu\left(t_i, t_j, \tau_i, 0\right) = \\
&\Sigma(t_i, t_j, \tau_j + s^*_{ji} - 1, 2) - \Sigma(t_i, t_j, \tau_j - s^*_{ij}, 2) + \nu\left(t_i, t_j, \tau_j + s^*_{ji}, 1\right) + \\
&+ \Sigma(t_i, t_j, T+1, 0) - \Sigma(t_i, t_j, \tau_j + s^*_{ji}, 0) =: \Gamma(t_i, t_j, \tau_j)
\end{aligned}
$$

we have, substituting the temporary variable Γ (just to make the equation not too long):

$$\mu(t_j, c, \tau_j, \sigma) :=$$
$$= \sum_{t_i} a(t_j - t_i - c) \begin{cases} \Sigma(t_i, t_j, \tau_j - s^*_{ij} - 1, 2) & \sigma = 0 \\ \nu(t_i, t_j, \tau_j - s^*_{ij}, 2) & \sigma = 1 \\ \Gamma(t_i, t_j, \tau_j) & \sigma = 2 \end{cases}$$

6 Marginals and observables

Let us suppose that the equations converge. Then, we have a population of μ from which we want to recover the original marginals. We are somehow fortunate because the function $\tilde{\nu}$ that we calculate in the algorithm is actually almost everything we need to obtain the marginals. In fact, the original message ν and the reparametrized message $\tilde{\nu}$ are related by:

$$\nu_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, t_j^{*(i)}) = \tilde{\nu}_{\psi_i \to j}(t_i^{(j)}, t_j^{(i)}, t_i^{*(j)}, 1 + \operatorname{sign}\left(t_j^{*(i)} - t_i^{*(j)} + s^*_{ji}\right))$$

Let us find then the marginal we want. We start from:

$$p_i(t_i, t_j, t_i^*, t_j^*) \propto \nu_{\psi_i \to j}(t_i, t_j, t_i^*, t_j^*) \nu_{\psi_j \to i}(t_j, t_i, t_j^*, t_i^*) =$$
$$= \tilde{\nu}_{\psi_i \to j}(t_i, t_j, t_i^*, 1 + \operatorname{sign}\left(t_j^* - t_i^* + s^*_{ji}\right)) \tilde{\nu}_{\psi_j \to i}(t_j, t_i, t_j^*, 1 + \operatorname{sign}\left(t_i^* - t_j^* + s^*_{ij}\right))$$

We sum over t_j^*:

$$p_i(t_i, t_j, \tau_i) \propto \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 0) \sum_{\tau_j = 0}^{\tau_i - s^*_{ji} - 1} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, 2) +$$
$$+ \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 1) \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_i - s^*_{ji}, 2) +$$
$$+ \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 2) \sum_{\tau_j = \tau_i - s^*_{ji} + 1}^{T+1} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, 1 + \operatorname{sign}\left(\tau_i - \tau_j + s^*_{ij}\right))$$

This is a T^3 vector so it is not bigger than other objects we have to calculate. For t_i and t_j fixed, we can precalculate

$$\Sigma(t_j, t_i, k, \sigma) := \sum_{\tau_j = 0}^{k} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, \sigma)$$

Algorithm 1 RS Epidemble Population Dynamics

1. Pre-calculate $a(t) = (1-\lambda)^{t\theta(t)}$ for $-T-2 \le t \le T+1$
2. Extract $\left(x_i^{*0}, s_{ji}^*, s_{ij}^*\right)$ and $d-1$ μ's from the population.
3. Initialize all $\nu = 0$
4. **if** $x_i^{*0} = S$
   (a) **Loop** over τ_i and t_i
   - **if** $\xi_i(t_i, \tau_i) == 0$
     i. $\nu = 0$
     ii. **continue**
   - Calculate the $\prod_k$ of the μ_k's (or sums over μ_k). They are just four numbers that we store as $m_1, \ldots, m_4$
   - **Loop** over t_j
     i. $\nu(t_i, t_j, \tau_i, 1) = \gamma(t_i)\left[a(t_i - t_j - 1)m_1 - \phi(t_i)a(t_i - t_j)m_2\right]$
     ii. $\nu(t_i, t_j, \tau_i, 2) = \nu(t_i, t_j, \tau_i, 1) + \gamma(t_i)\left[\phi(t_i)a(t_i - t_j)m_4 - a(t_i - t_j - 1)m_3\right]$
5. **else if** $x_i^{*0} = I$
   (a) **Loop** over t_j and t_i
   - **if** $\xi_i(t_i, \tau_i) == 0$
     i. $\nu = 0$
     ii. **continue**
   - Calculate the ν and this value is the same for all σ
6. Pre-Calculate $\Sigma(t_i, t_j, k, \sigma) := \sum_{\tau_i=0}^{k} \nu(t_i, t_j, \tau_i, \sigma)$ for $k = 0, \ldots, T+1$
7. **Loop** over t_j
   (a) **Loop** over τ_j
   - Set $\mu(t_j, c, \tau_j, \sigma) = 0$ for every σ and c
   - **Loop** over t_i
     i. Calculate $\Gamma(t_i, t_j, \tau_j)$ and save it as Γ
     ii. **Loop** over c
       - $\mu(t_j, c, \tau_j, 0) += a(t_j - t_i - c)\Sigma(t_i, t_j, \tau_j - s_{ij}^* - 1, 2)$
       - $\mu(t_j, c, \tau_j, 1) += a(t_j - t_i - c)\nu(t_i, t_j, \tau_j - s_{ij}^*, 2)$
       - $\mu(t_j, c, \tau_j, 2) += a(t_j - t_i - c)\Gamma$
8. Update μ to the population.

for $\sigma = 0, 1, 2$. And finally we can express the marginal as:

$$\begin{aligned}
p_i(t_i, t_j, \tau_i) \propto\ & \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 0) \sum_{\tau_j=0}^{\tau_i - s^*_{ji} - 1} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, 2) + \\
& + \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 1) \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_i - s^*_{ji}, 2) + \\
& + \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 2) \sum_{\tau_j = \tau_i - s^*_{ji} + 1}^{\tau_i + s^*_{ij} - 1} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, 2) \\
& + \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 2) \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_i + s^*_{ij}, 1) \\
& + \tilde{\nu}_{\psi_i \to j}(t_i, t_j, \tau_i, 2) \sum_{\tau_j = \tau_i + s^*_{ij} + 1}^{T+1} \tilde{\nu}_{\psi_j \to i}(t_j, t_i, \tau_j, 0)
\end{aligned}$$

Once the marginal $p(t_i, t_j, \tau_i)$ is obtained, we can easily trace over t_j in order to get $p_i(t_i, \tau_i)$. In the algorithmic language:

$$\begin{aligned}
p_i(t_i, t_j, \tau_i) \propto\ & \nu_1(t_i, t_j, \tau_i, 0) \sum_{\tau_j=0}^{\tau_i - s^*_{ji} - 1} \nu_2(t_j, t_i, \tau_j, 2) + \\
& + \nu_1(t_i, t_j, \tau_i, 1) \nu_2(t_j, t_i, \tau_i - s^*_{ji}, 2) + \\
& + \nu_1(t_i, t_j, \tau_i, 2) \sum_{\tau_j = \tau_i - s^*_{ji} + 1}^{\tau_i + s^*_{ij} - 1} \nu_2(t_j, t_i, \tau_j, 2) \\
& + \nu_1(t_i, t_j, \tau_i, 2) \nu_2(t_j, t_i, \tau_i + s^*_{ij}, 1) \\
& + \nu_1(t_i, t_j, \tau_i, 2) \sum_{\tau_j = \tau_i + s^*_{ij} + 1}^{T+1} \nu_2(t_j, t_i, \tau_j, 0)
\end{aligned}$$

So:

$$\begin{aligned}
p_i(t_i, t_j, \tau_i) \propto\ & \nu_1(t_i, t_j, \tau_i, 0) \Sigma(t_j, t_i, \tau_i - s^*_{ji} - 1, 2) + \\
& + \nu_1(t_i, t_j, \tau_i, 1) \nu_2(t_j, t_i, \tau_i - s^*_{ji}, 2) + \\
& + \nu_1(t_i, t_j, \tau_i, 2) \left(\Sigma(t_j, t_i, \tau_i + s^*_{ij} - 1, 2) - \Sigma(t_j, t_i, \tau_i - s^*_{ji}, 2) \right) \\
& + \nu_1(t_i, t_j, \tau_i, 2) \nu_2(t_j, t_i, \tau_i + s^*_{ij}, 1) \\
& + \nu_1(t_i, t_j, \tau_i, 2) \left(\Sigma(t_j, t_i, T+1, 0) - \Sigma(t_j, t_i, \tau_i + s^*_{ij}, 0) \right)
\end{aligned}$$

Algorithm 2 Calculation of the marginals

1. Extract $2d-2$ μ's from the population.
2. Divide the μ's into two groups of $d-1$ functions and use each group to calculate a ν
3. We have ν_1 and ν_2.
4. Cumulate one of the two ν's, for example ν_2, into Σ
5. **loop** over t_i, t_j, τ_i
   (a) Calculate $\Gamma(t_i, t_j, \tau_i) := \Sigma(t_j, t_i, \tau_i + s^*_{ij} - 1, 2) - \Sigma(t_j, t_i, \tau_i - s^*_{ji}, 2) + \nu_2(t_j, t_i, \tau_i + s^*_{ij}, 1) + \Sigma(t_j, t_i, T+1, 0) - \Sigma(t_j, t_i, \tau_i + s^*_{ij}, 0)$
   (b) Update $p_i(t_i, t_j, \tau_i) = \nu_1(t_i, t_j, \tau_i, 0)\Sigma(t_j, t_i, \tau_i - s^*_{ji} - 1, 2) + \nu_1(t_i, t_j, \tau_i, 1)\nu_2(t_j, t_i, \tau_i - s^*_{ji}, 2) + \nu_1(t_i, t_j, \tau_i, 2)\Gamma$
6. Marginalize over t_j and normalize
7. Save p_i as an element of the population of marginals.

7 A limiting case

Let us say that the observation constraint is: $\tau_i = t_i$.

$$\mu(t_j, c, \tau_j, \sigma) :=$$

$$= \sum_{t_i} a(t_j - t_i - c) \begin{cases} \nu(t_i, t_j, t_i, 2)\mathbb{I}[t_i < \tau_j - s^*_{ij}] & \sigma = 0 \\ \nu(t_i, t_j, t_i, 2)\mathbb{I}[t_i = \tau_j - s^*_{ij}] & \sigma = 1 \\ \nu\left(t_i, t_j, t_i, 1 + \text{sign}\left(\tau_j - t_i + s^*_{ji}\right)\right)\mathbb{I}[t_i > \tau_j - s^*_{ij}] & \sigma = 2 \end{cases}$$

$$= \begin{cases} \sum_{t_i < \tau_j - s^*_{ij}} a(t_j - t_i - c)\nu(t_i, t_j, t_i, 2) & \sigma = 0 \\ a(t_j - t_i - c)\nu(t_i, t_j, t_i, 2)\delta[t_i, \tau_j - s^*_{ij}] & \sigma = 1 \\ \sum_{t_i > \tau_j - s^*_{ij}} a(t_j - t_i - c)\nu\left(t_i, t_j, t_i, 1 + \text{sign}\left(\tau_j - t_i + s^*_{ji}\right)\right) & \sigma = 2 \end{cases}$$