Short Essay on Game Theory

Exercise on Friendship Networks

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Consider n individuals who derive utility from social interaction. A link between two individuals is interpreted as a friendship relation. Individuals like more friends and their utility is increasing in the time they are able to spend with each friend. Each person has a fixed amount of total time available, and allocates this time equally among their different friends. Finally, each relationship involves an investment of time and resources. These considerations lead us to write the payoff to individual i from a network g as follows:

$$Y_i(g) = (d_i(g))^{\frac{1}{2}} + \sum_{k \in N_i(g)} \frac{1}{d_k(g)}, \tag{1}$$

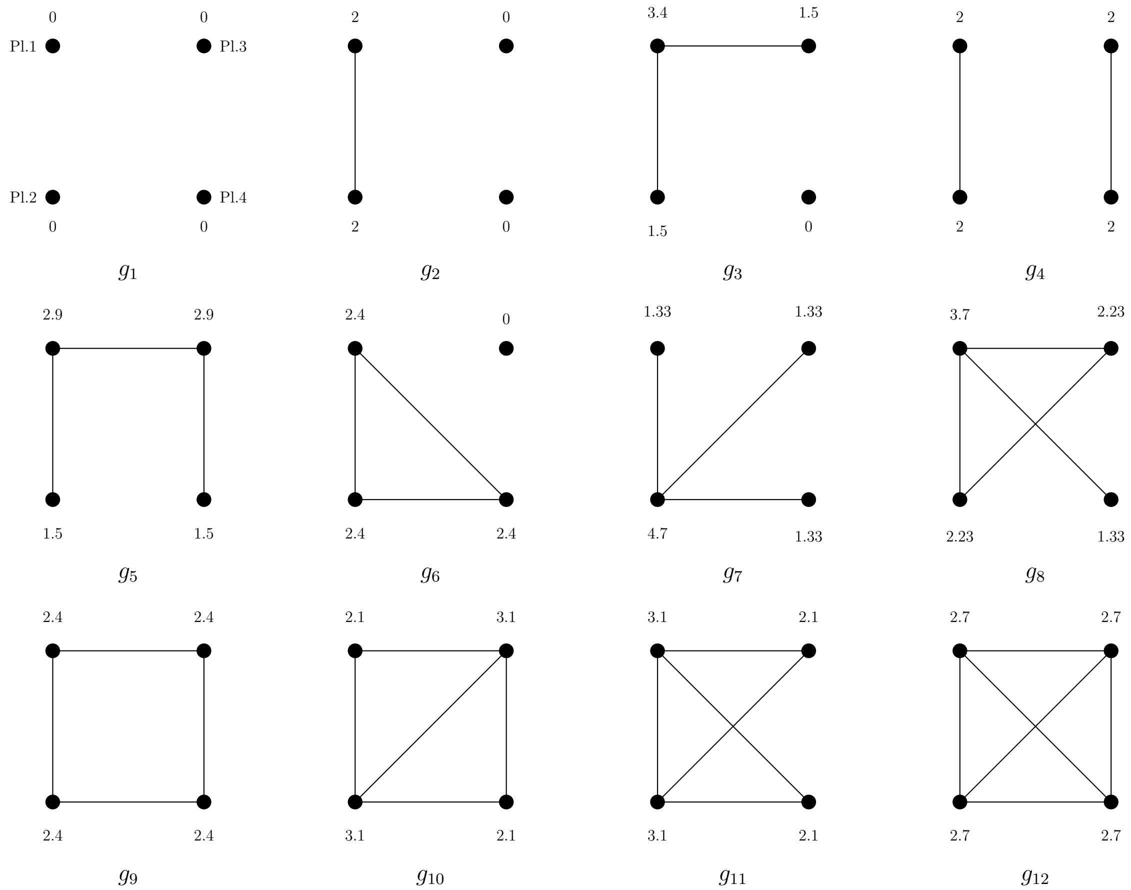
where $N_i(g)$ is the set of player i's neighbors in g and $d_i(g) = \#N_i(g)$ is the number of links player i has in g.

Proposition 1

Take $n = 4$. Therefore, the complete network is the unique strongly efficient network, but it is not uniquely Pareto Efficient.

Proof Proposition 1

Consider a situation where four players can form links. The utilities they obtained from different network configurations are as follows. For the empty network $g^{\emptyset}$, $Y_i(g^{\emptyset}) = 0$ since $d_i(g) = 0$ and $N_i(g) = \emptyset \ \forall i \in N$. For the complete network g^N, $Y_i(g^N) = 2.7$ since $Y_i(g^N) = (3)^{\frac{1}{2}} + (\frac{1}{3} + \frac{1}{3} + \frac{1}{3})$. For a line network g^{L4} with four players, $Y_i(g^{L4}) = (d_i(g^{L4}))^{\frac{1}{2}} + \sum_{k \in N_i(g^{L4})} \frac{1}{d_k(g^{L4})} \ \forall i \in N$. For a line network g^{L3} with three players, $Y_i(g^{L3}) = (d_i(g^{L3}))^{\frac{1}{2}} + \sum_{k \in N_i(g^{L3})} \frac{1}{d_k(g^{L3})} \ \forall i \in N(g^{L3})$, while $Y_j(g^{L3}) = 0 \ \forall j \in N \setminus N(g^{L3})$. For all other networks g, $Y_i(g) = 2$ since $Y_i(g) = (1)^{\frac{1}{2}} + (\frac{1}{1})$. The figure below gives some of the network configurations.



By definition, a network g is Pareto Efficient relative to the value function v and the allocation rule Y if there does not exist any $g^{'} \in \mathcal{G}$ (i.e.,) such that $Y_i(g^{'}, v) \geq Y_i(g, v)$ for all i with at least one strict inequality. Therefore, in this case, it is possible to observe that the complete network g^N (g_{12}) is Pareto Efficient since there is no other network where all the players can obtain at least the same utility they reach in g^N. We can easily observe this statement does not hold for g_1 (e.g., $Y_i(g_{12}, v) \geq Y_i(g_1, v)\ \forall i \in N$); for g_2 (e.g., $Y_i(g_{12}, v) \geq Y_i(g_2, v)\ \forall i \in N$); for g_3 (e.g., $Y_i(g_8, v) \geq Y_i(g_3, v)\ \forall i \in N$); for g_4 (e.g., $Y_i(g_{12}, v) \geq Y_i(g_4, v)\ \forall i \in N$); for g_5 (e.g., $Y_i(g_{11}^{\{1,2\},\{1,3\},\{2,3\},\{3,4\}}, v) \geq Y_i(g_5, v)\ \forall i \in N$) (notice that $g_{11}^{\{1,2\},\{1,3\},\{2,3\},\{3,4\}}$ denotes the chain with links $\{1,2\},\{1,3\},\{2,3\},\{3,4\}$); for g_6 (e.g., $Y_i(g_{12}, v) \geq Y_i(g_6, v)\ \forall i \in N$); for g_9 (e.g., $Y_i(g_{12}, v) \geq Y_i(g_9, v)\ \forall i \in N$).
On the other hand, we can observe that g_7 is Pareto Efficient since player 1, player 3, and player 4 can all increase their own utility by adding links (so, Y_i is higher in all the chains where each 1, 3, and 4, have more than $\#N_i(g) > 1$ since the allocation rule is increasing in $d_i(g)$), but indirect links lowers player 2 utility (the allocation rule is decreasing in $d_k(g)$). The same results applies to the different configurations of g_7 (i.e, $g_7^{\{1,2\},\{2,3\},\{2,4\}}$, $g_7^{\{1,2\},\{1,3\},\{1,4\}}$, $g_7^{\{1,3\},\{2,3\},\{3,4\}}$, $g_7^{\{1,4\},\{2,4\},\{3,4\}}$). Similarly, networks g_8 (i.e., $g_8^{\{1,2\},\{1,3\},\{2,3\},\{1,4\}}$, $g_8^{\{1,3\},\{3,4\},\{2,3\},\{1,4\}}$, $g_8^{\{1,4\},\{3,4\},\{2,4\},\{2,3\}}$, $g_8^{\{1,2\},\{1,4\},\{2,4\},\{2,3\}}$), g_{10} (i.e., $g_{10}^{\{1,2\},\{1,3\},\{3,4\},\{2,3\}}$, $g_{10}^{\{1,2\},\{1,3\},\{3,4\},\{1,4\}}$), and g_{11} (i.e., $g_{11}^{\{1,2\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}}$, $g_{11}^{\{1,2\},\{1,3\},\{3,4\},\{1,4\},\{2,3\}}$, $g_{11}^{\{1,2\},\{2,4\},\{3,4\},\{1,4\},\{2,3\}}$, $g_{11}^{\{1,3\},\{2,4\},\{3,4\},\{1,4\},\{2,3\}}$) are Pareto Efficient.
By definition, a network $g \in \mathcal{G}$ is said strongly efficient relative to v if $v(g) \geq v(g^{'})$ for all $g^{'} \in \mathcal{G}$. Notice that $v(g) = \sum_i Y_i(g, v)$. So, in order to find the strongly efficient networks, it is sufficient to sum each player's utility for each network. Without loss of generality, we see that: $v(g_1) = 0$; $v(g_2) = 4$; $v(g_3) = 6.4$; $v(g_4) = 8$; $v(g_5) = 8.8$; $v(g_6) = 7.2$; $v(g_7) = 8.7$; $v(g_8) = 9.5$; $v(g_9) = 9.6$; $v(g_{10}) = 10.4$; $v(g_{11}) = 10.4$; $v(g_{12}) = 10.8$.

Therefore, the complete network is the unique strongly efficient network.

Proposition 2

Take $n = 4$. The unique pairwise stable network is the complete network, which is also the unique strongly stable network.

Proof Proposition 2

By definition, a network $g \in \mathcal{G}$ is said pairwise stable with respect to the value function v and the allocation rule Y if:

- for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$, and
- for all $ij \notin g$, $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$

We can trivially observe that the unique pairwise stable network is the complete network (g_{12}) since no player has incentive to delete any link. As a matter of fact, the allocation rule is increasing in the number of links, so that any players has an incentive to form the maximum number of links. A player She would end up worse off after having cut one of her three links (consider g_{11} and g_{10}). In any partial network, the player who has less than three links does want to link with another player to form a star network. Moreover, the players who are linked have no incentives to cut their links to move to a shorter line network. The empty network g_1 is not pairwise stable because two players have incentives to link to each other (see g_2 and g_4). The star network g_7 is not pairwise stable since the peripheral players have incentives to add the missing links to form the complete network. This result is supported from the fact that the network utility function is pairwise monotonic (I proved this characteristic in the last point of the current exercise). Exact pairwise monotonicity implies that strongly efficient networks are pairwise stable. Moreover, the complete network is also the unique strongly stable network. To prove this, we can make use of the following proposition (Jackson and van den Nouweland (2005); Grandjean et al. (2011)):

- Take any u such that (**i**) $u_i(g) = u_j(g)$ for all $i, j \in S \in \Pi(g)$ and (**ii**) $u_i(g) = u_i(h)$ with $h \in C(g)$ and $i \in N(h)$. Then, the set of strongly efficient networks is the set of strongly stable networks if and only if u is top convex.

In particular, a network utility function u is top convex if some strongly efficient network maximizes the per-capita sum of utilities among players. Let $\rho(u, S) = max_{g \subseteq g^S} \sum_{i \in S} \frac{u_i(g)}{\#S}$. The network utility function u is top convex if $\rho(u, N) \geq \rho(u, S)$ for all $S \subseteq N$. It is quite trivial to show that the utility function of the exercise is top convex. Consider $S = N = 4$. The network $g \subseteq g^{S=4}$ that maximizes the per-capita sum of utilities among players is indeed the complete network, such that $\rho(u, N) = 2.7$. Consider now $S = 3$ and $S = 2$. The network $g \subseteq g^3$ that maximizes the per-capita sum of utilities among players is g_6, but $\rho(u, S = 3) = 2.4$. On the other hand, $\rho(u, S = 2) = 2$. This prove that the u is top convex and, therefore, the complete network g^N (which is the strongly efficient network) must be the unique strongly stable network.

Proposition 3

Take $n = 4$. The set of pairwise Nash stable networks coincides with the set of pairwise stable networks, that is the singleton $\{g^N\}$.

Proof Proposition 3

By definition, a strategy profile σ is a pairwise Nash Equilibrium of Myerson's linking game if and only if, for each player i, each strategy $\sigma_i^{'} \neq \sigma_i$, $u_i(g(\sigma)) \geq u_i(g(\sigma_i^{'}, \sigma_{-i}))$, and there does not exist a pair of players i and j such that $u_i(g(\sigma)+ij) \geq u_i(g(\sigma))$, $u_j(g(\sigma)+ij) \geq u_j(g(\sigma))$ with strict inequality for one of the players. Concretely, Pairwise Nash stability requires that a network is immune both to a formation of a new link by any two players and to the deletion of any number of links by any player. Calvo-Armengol and Ilkilic (2009) provide conditions on the network link marginal utilities such that the sets of pairwise and pairwise Nash stable networks coincide. In particular, this happens if and only if u is α-submodular on the set of pairwise stable networks, for some $\alpha \geq 0$.

- The network utility function u is α-submodular in own current links on $G \subseteq \mathcal{G}$ if and only if $u_i(g) - u_i(g - l) \geq \alpha \sum_{ij \in l}(u_i(g) - u_i(g - ij))$ for all $g \in G$, $i \in N$, and $l \subseteq \{jk \in g | j = i \text{ or } k = i\}$.

This condition applies only to marginal utilities from existing links, and it imposes that the marginal benefits from a group l of links already in the network are higher than the sum of marginal benefits of each single link in l scaled by α. So, it is sufficient to prove the α-submodularity of the network utility analysed. Let $ij \in g$. In our network utility, $d_{ik}(g) \leq d_{ik}(g - ij)$ for all $k \in N$. Let:

$$\Delta_{ij}(g) = \{k \in N : d_{ik}(g) < d_{ik}(g - ij)\} \tag{2}$$

In particular, $j \in \Delta_{ij}(g)$. Then:

$$m_{ij}u_i(g) = (1)^{\frac{1}{2}} + \sum_{k \in \Delta_{ij}(g)} \left[\frac{1}{d_{ik}(g)} - \frac{1}{d_{ik}(g - ij)}\right] \tag{3}$$

Let $il \in g$, where $l \neq j$. Define $\Delta_{ij,il}(g) = \{k \in N : d_{ik}(g) < d_{ik}(g - ij - il)\}$. So:

$$m_{ij,il}u_i(g) = (2)^{\frac{1}{2}} + \sum_{k \in \Delta_{ij}(g)} \left[\frac{1}{d_{ik}(g)} - \frac{1}{d_{ik}(g - ij - il)}\right] \tag{4}$$

We can trivially observe that $\left[\frac{1}{d_{ik}(g)} - \frac{1}{d_{ik}(g-ij-il)}\right] \geq \left[\frac{1}{d_{ik}(g)} - \frac{1}{d_{ik}(g-ij)}\right]$ as $d_{ik}(g - ij) \leq d_{ik}(g - ij - il)$. Similarly for $\left[\frac{1}{d_{ik}(g)} - \frac{1}{d_{ik}(g-il)}\right]$. At the same time, we observe that the $(d_i)^{\frac{1}{2}}$ component is not 1-convex. All in all, α needs to be equal to $\sqrt{\frac{n}{4(n-1)}}$ in order to satisfy α-submodularity condition. We can see our network utility is $\sqrt{\frac{n}{4(n-1)}}$ -convex. As a matter of fact, we find $\alpha = \sqrt{\frac{n}{4(n-1)}}$ as interior solution by rearranging $(n)^{\frac{1}{2}} \geq \alpha \cdot ((n-1)^{\frac{1}{2}} + (n-1)^{\frac{1}{2}})$.

Consequently, the set of pairwise Nash stable networks coincides with the set of pairwise stable networks, that is the singleton $\{g^N\}$. And, recalling again the previous result of top convexity, the complete network is also the unique strong Nash stable network. It is important to remark that the strongly stable networks correspond exactly to the strong Nash equilibria of the network formation game suggested by Myerson (1991). The equivalence holds for the corresponding definition of strong Nash equilibrium which requires that there are no deviations by a coalition that make all members weakly better off and some strictly better off (Jackson and van den Nouweland (2005)).

Proposition 4
Take $n = 4$. The vNM farsighted stable set is the complete network.

Proof Proposition 4
Firstly, it is important to define the farsighted improving paths. In particular, a farsighted improving path is a sequence of networks that can emerge when players add or delete links based on the improvement the end network offers relative to the current network. For a given network g, let $F(g) = \{g^{'} \in \mathcal{G} | g \to g^{'}\}$, the notation $g \to g^{'}$ implies that it exists a farsighted improving path from g to $g^{'}$. Without loss of generality (since results applies also to different configurations), we can observe the following improving paths:

- $F(g_1) = \{g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}, g_{12}\}$
- $F(g_2) = \{g_3, g_5, g_8, g_{10}, g_{12}\}$
- $F(g_3) = \{g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}, g_{12}\}$
- $F(g_4) = \{g_5, g_8, g_9, g_{10}, g_{12}\}$
- $F(g_5) = \{g_8, g_9, g_{10}, g_{12}\}$
- $F(g_6) = \{g_8, g_{11}, g_{12}\}$
- $F(g_7) = \{g_8, g_{11}, g_{12}\}$
- $F(g_8) = \{g_{11}, g_{12}\}$
- $F(g_9) = \{g_{10}, g_{12}\}$
- $F(g_{10}) = \{g_{12}\}$
- $F(g_{11}) = \{g_{12}\}$
- $F(g_{12}) = \emptyset$

Consider the set of networks $G \subseteq \mathcal{G}$. It is pairwise farsightedly stable if (Herings et al. (2009)):

(i) $\forall g \in G$,

  (ia) $\forall ij \notin g$ such that $g + ij \notin G$, $\exists g' \in F(g + ij) \cap G$ such that $(u_i(g'), u_j(g')) = (u_i(g), u_j(g))$ or $u_i(g') < u_i(g)$ or $u_j(g') < u_j(g)$,

  (ib) $\forall ij \in g$ such that $g - ij \notin G$, $\exists g', g'' \in F(g - ij) \cap G$ such that $u_i(g') \leq u_i(g)$ and $u_j(g'') \leq u_j(g)$.

(ii) $\forall g' \in \mathcal{G} \setminus G$, $F(g') \cap G \neq \emptyset$.

(iii) $\nexists G' \subseteq \mathcal{G}$ such that G' satisfies conditions (ia), (ib), and (ii).

In our configuration, $\{g_{12}\}$ is pairwise farsightedly stable. Since $g_{12} \in \bigcap_{g \in \mathcal{G} \setminus \{g_{12}\}} F(g)$, condition (ii) of the definition is satisfied. In addition, condition (i) is also satisfied, since any deviation from g_{12} may lead back to g_{12}. The set $\{g_{12}\}$ is clearly minimal, so condition (iii) is satisfied too. Since $F(g_{12}) = \emptyset$, condition (ii) implies that g_{12} belongs to any pairwise farsightedly stable set. Using condition (iii) it follows that $\{g_{12}\}$ is the only pairwise farsightedly stable set. Following Herings et al. (2009), we can also conclude that $G = \{g_{12}\}$ is also the unique pairwise farsightedly stable set (since the set $\{g\}$ is the unique pairwise farsightedly stable set if and only if for every $g' \in \mathcal{G} \setminus \{g\}$ we have $g \in F(g')$ and $F(g) = \emptyset$).
The set G is a von Neumann - Morgenstern farsightedly stable set if:

(i) $\forall g \in G,\ F(g) \cap G = \emptyset$ and

(ii) $\forall g' \in \mathcal{G} \setminus G,\ F(g') \cap G \neq \emptyset$.

By corollary, the set {g} is a pairwise farsightedly stable set if and only if it is a von Neumann-Morgenstern farsightedly stable set. Therefore, we can conclude that the vNM farsighted stable set is the complete network. This result is indeed consistent with the following: the set of strongly efficient networks is the unique pairwise farsightedly stable set if and only if u is top convex (Jackson and van den Nouweland (2005); Grandjean et al. (2011)).

Proposition 5

Take $n = 4$. The set $\{g_{12}\}$ is the unique myopic-farsighted stable set, regardless of whether players are myopic or farsighted.

Proof Proposition 5

By definition, a set of networks is a myopic-farsighted stable set if (internal stability) there is no myopic-farsighted improving path between networks within the set and (external stability) there is a myopic-farsighted improving path from any network outside the set to some network within the set.
A myopic-farsighted improving path is a sequence of networks that can emerge when farsighted players form or delete links based on the improvement the end network offers relative to the current network while myopic players form or delete links based on the improvement the resulting network offers relative to the current network. Formally, a myopic-farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either:

(i) $g_{k+1} = g_k - ij$ for some ij such that $U_i(g_{k+1}) > U_i(g_k)$ and $i \in M$ or $U_j(g_K) > U_j(g_k)$ and $j \in F$; or

(ii) $g_{k+1} = g_k + ij$ for some ij such that $U_i(g_{k+1}) > U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$ if $i, j \in M$, or $U_i(g_K) > U_i(g_k)$ and $U_j(g_K) \geq U_j(g_k)$ if $i, j \in F$, or $U_i(g_{k+1}) \geq U_i(g_k)$ and $U_j(g_{k+1}) \geq U_j(g_k)$ (with one inequality holding strictly) if $i \in M, j \in F$.

If there exists a myopic-farsighted improving path from a network g to a network g', then we write $g \rightarrow g'$. The set of all networks that can be reached from a network $g \in \mathcal{G}$ by a myopic-farsighted improving path is denoted by $\phi(g)$, $\phi(g) = \{g' \in \mathcal{G} \mid g \rightarrow g'\}$.
When all players are farsighted, the notion of myopic-farsighted improving path reverts to Herings et al. (2009) notion of farsighted improving path, and the myopic-farsighted

stable set is simply the farsighted stable set as denoted in Herings et al. (2009) or in Ray and Vohra (2015) (therefore, in our case $G = \{g_{12}\}$ is the myopic-farsighted stable set when all players are farsighted).
When all players are myopic, the notion of myopic-farsighted improving path reverts to Jackson and Watts (2002) notion of (myopic) improving path. Firstly consider that a set of networks C, form a cycle if for any $g \in C$ and $g' \in C$ there exists a (myopic) improving path connecting g to g'. A cycle C is a closed cycle if no network in C lies on a (myopic) improving path leading to a network that is not in C. Let P be the set of pairwise stable networks.

- Suppose that all players are myopic, $\# M = n$. Let $C^1, \ldots, C^r$ be the set of closed cycles. A set of networks $\mathcal{G} \subseteq \mathcal{G}$ is a myopic-farsighted stable set if and only if $\mathcal{G} = P \cup \{g^1, \ldots, g^r\}$ with $g^k \in C^k$ for $k = 1, \ldots, r$.

In our case, the singleton including the complete network g_{12} is the unique myopic-farsighted stable set when all players are myopic since $\phi(g_{12}) = \emptyset$ and $g_{12} \in \phi(g)$ for all g. This conclusion is also supported by the fact that Jackson and Watts (2002) provide a condition on the network utility function that rules out cycles (so $C^1, \ldots, C^r = \emptyset$ and $\mathcal{G} = P$, the set of pairwise stable networks). In particular, we have that the network utility function u is exact pairwise monotonic if g' defeats g if and only if $\sum_{i \in N} u_i(g') > \sum_{i \in N} u_i(g)$ and g' is adjacent to g. This is verified in our case since $v(g_1) = 0 < v(g_2) = 4 < v(g_3) = 6.4 < v(g_5) = 8.8 < v(g_9) = 9.6 < v(g_{10}) = 10.4 < v(g_{12}) = 10.8$ (without loss of generality, I took into consideration just some examples).
Consequently, in case we have players 1 and 2 myopic while players 3 and 4 farsighted (or even players 2,3, and 4 farsighted, while only 1 myopic), the set $\{g_{12}\}$ is the unique myopic-farsighted stable set since it satisfies both the external and internal stability conditions.

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