Paths

Matteo Paz, Dylan Rupel

February 27th, 2024

Definition 1 *A path in a topological space Y is a continuous mapping from $[0,1] \to Y$.*

Intuitively, a path is just a line or curve in your topological space. We define it to be homeomorphic to the real interval $[0,1]$. If we have two paths p_1 and p_2, we can also concatenate or merge them naturally, given that the endpoint $p_1(1) = p_2(0)$:

$$p_1 * p_2 = \begin{cases} p_1(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ p_2(2t) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

Using this we can define a natural equivalence relation, in which all points which are path-connected are considered as a group:

Proposition 1 *$x_1 \sim x_2$ if there exists a path $p : [0,1] \to X$ with $p(0) = x_1$ and $p(1) = x_2$ gives an equivalence relation*

Proof.

1. Reflexive: For $x \in X$, the path $p(t) = x$ for all $t \in [0,1]$ satisfies.
2. Symmetric: If $x \sim x'$ then there is a path $p : x \to x'$. If we define $p'(t) = 1 - p$, then $p'(0) = x'$ and $p'(1) = x$
3. Transitive: If $x \sim x'$ and $x' \sim x''$, then simply concatenate the respective paths with the aforementioned construction so that $p * p' : x \to x''$

Thus, path-connectedness is an equivalence relation. How does this relate to regular connectedness though?

Proposition 2 *If X is path connected, then X is connected.*

Proof.

Assume X is disconnected and path connected, thus $X = A \sqcup B$ for disjoint open sets $A, B \subseteq X$. Let $x \in A$ and $y \in B$. Let there be a path $p : x \to y$. Since paths are continuous functions, $p^{-1}(A) \sqcup p^{-1}(B) = [0,1]$ with the preimages being open. However, the interval $[0,1]$ is connected. Thus we reach a contradiction, and X must then be connected. $\square$

So, we can conclude that the condition of path-connectedness is a stronger statement than connetedness. However, we used a hidden lemma in that last proof:

Proposition 3 $[0,1]$ *is connected.*

Proof.
Assume that $[0,1]$ is disconnected and is equal to the disjoint union of open sets A, B. WLOG assume that 1 lies in B. Let $x = \sup(A)$. Any interval around x would contain points greater than $\sup(A)$. Therefore x is not an open point for A, and $x \notin A$. Also, any interval around x would contain points less than x, which are thus in A. Therefore x is not an open point for B and $x \not{i}nB$. Thus $\sup(A)$ is in X but not included in $A \sqcup B$. This is a contradiction, so $[0,1]$ is connected.

Definition 2 *A topological space X is locally path connected if every point has an open neighborhood which is path connected.*

Theorem 1 *Let X be a locally path connected topological space. Then X is path connected if and only if X is connected.*

Proof.
$\implies$
Already shown that path connectedness implies connected.

$\impliedby$ Using the equivalence relation for paths earlier, we call such a space a *path component*, a set where all elements are reachable by path. Let A be a path component. Let $a \in A$. Then since X is locally path connected, there is an open neighborhood $a \in U$ that is path connected. Thus there exists a path from any point in A to a, then from a to any point in U. So all of U is reachable from all of A. Thus $U \subseteq A$. Thus A is open.

If there were more than one path component, e.g. any points were not reachable by path, then this would give an open partition of X, and X could not be connected. Thus locally path connected and connected implies path connectedness. $\square$