The Quotient Topology

Matteo Paz, Dylan Rupel

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1 The Quotient Topology and Examples

Definition 1 *Let X be a topological space and suppose $\sim$ is an equivalence relation on X. Then there is a natural surjective function $f : X \to X/\sim$ onto the set of equivalence classes $[x]_\sim$. The quotient topology is the finest topology for which f is continuous. This can be achieved exactly by defining $U \subseteq X/\sim$ open if and only if $f^{-1}(U) = \{x \in X : [x] \in U\}$ is open in X.*

Example 1 *Consider the unit sphere S_3 in $\mathbb{R}^3$, and the disjoint union of two unit discs S_2' in $\mathbb{R}^2$, after identifying the boundaries of the disc with $\sim$.*

If we take the intersection of an open ball in $\mathbb{R}^3$ with S_3, we get a 2 dimensional ball on the surface of S_3, which are exactly the open sets of S_2' - 2D balls. By identifying the boundaries of the two discs, we connect the equator of the sphere. Thus, these spaces are homeomorphic, and S_3 is the quotient topology for $S_2' \sqcup S_2'/\sim$

Example 2 *Consider $[0,1]/\sim$ where $0 \sim 1$.*

This essentially wraps the interval $[0,1]$ into a circle of circumference 1. Open intervals on the image about 0 become half open in the domain: $[0,a) \sqcup (b,1]$.

Example 3 *Consider $\mathbb{R}$. Take $\mathbb{Z}$ to be one equivalence class, identifying them all together, and leaving all non integers alone.*

Similar to example 2, this creates a circle for every interval $[z, z+1]$. However, each circle does not identify except at its point z. Thus it creates a wedge of infinitely many circles, all meeting only at 0. Any open set about that 0 point will include an open interval about every point in $\mathbb{Z}$ in the preimage.

2 Quotient of a Topological Space Under a Group Action

Definition 2 *An action of a group G on a set X is $* : G \times X \to X$ such that*

1. *$g.(h.x) = (gh).x$ for all $g, h \in G$ and $x \in X$*

2. *$e.x = x$ for all $x \in X$*

Definition 3 *A group G is a topological group if it has a topology such that with g inG, the maps $\phi_g(h) = gh$ and $\rho_g(h) = hg$ are continuous.*

Lets see some topological groups.

1. To start, any discrete / finite group is a topological group. With the discrete topology, any function is continuous, because all preimages are open: $(\mathbb{Z}, +), (S_n, \circ), D_n, (\mathbb{Z}_n, +)$

2. $GL_n(\mathbb{R}), GL_n(\mathbb{C})$

3. Any subgroup of a topological group

Definition 4 *Let G be a topological group and X a topological space. Then G has a continuous action on X if there exists $* : G \times X \to X$ so that $\phi_g : X \to X$ is a continuous function.*