Homeomorphisms

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Definition 1 *A continuous bijective function $f : X \to Y$ is a* <u>*homeomorphism*</u> *if f^{-1} is also continuous. X and Y are homeomorphic if there exists a homeomorphism between them.*

Example 1 Consider the function $e^x : \mathbb{R} \to \mathbb{R}_{\geq 0}$. This is continuous, surjective (onto positive reals), and injective. Equally the inverse $\ln(x)$ is continuous injective and surjective. Thus $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ are homeomorphic. Thus they are homeomorphic topological spaces.

Example 2 Consider the spaces $[0, 2\pi) \subset \mathbb{R}$ and $S' \in \mathbb{C}$ (the complex unit circle). We have the function $f(x) = e^{ix}$ between them. If our open sets in S' are just arcs from angle a to b, then our preimage is (a, b) in $[0, 2\pi)$. If the arc contains 0, we just take the disjoint union $[0, a) \sqcup (b, 2\pi)$. These are open sets from f^{-1}, thus f is continuous. However, f^{-1} is not continuous, because $[0, a)$ is open on $[0, 2\pi)$, but under the action of $f = {f^{-1}}^{-1}$, it generates the non-open arc containing its endpoint 0. Thus we cannot say that $[0, 2\pi)$ and S' are homeomorphic.