Integration

Dylan Rupel, Matteo Paz

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Introduction

Classically, we define the integral in one dimension as a limit of a riemann sum:

$$\int f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1} f(x_i)\Delta x \tag{1}$$

This is essentially a mapping from a function and a 1D interval, to a 2D measure of area under a curve. This is done by taking $\Delta x \sim dx$ as the width, and $f(x)$ as the height, then multiplying them.

In general if we have a function $f : \mathbb{R}^n \to \mathbb{R}$, we can define the integral over some n-dimensional domain D as a limit of a sum:

$$\int_D f dx_1 \cdots dx_k = \lim_{\Delta x_j \to 0} \sum_{i=1} f \Delta x_1 \cdots \Delta x_k \tag{2}$$

Thus, we express the measure of a $k+1$ dimensional volume over a k dimensional domain.

Differential Forms and Integrals

Consider the k-form $\omega \in \Omega_k(\mathbb{R}^n) = \wedge^k \Omega_1(\mathbb{R}^n)$. Thus ω has dimension $\binom{n}{k}$, and can be written as:

$$\omega = \sum_{I=\{i_1 < \cdots < i_k\}} \omega_I dx_I$$

Where $dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and ω_I are the coefficients of the form, in $\mathcal{C}^\infty(\mathbb{R}^n)$. For our case, we will only be considering ω which are compactly supported. This means that ω (equivalently all ω_I) only takes a nonzero value in a closed and bounded subset of $\mathbb{R}^n$.

Integration of Differential Forms

Goal: Define the integral $\int_D \omega$ for a compactly supported k-form ω over a k-dimensional domain $D \subset \mathbb{R}^n$.