# The Singular Value Decomposition

Matteo Paz, Dylan Rupel

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First, we need a matrix. Let $A \in \mathrm{Mat}_n(\mathbb{C})$ be Hermetian. Therefore, we can find an *orthonormal basis of eigenvectors*. There is a small issue with this, what if we want this for a non-square matrix?
**Goal: Be able to do the same thing if $A$ is not square.**

Assume that $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is an orthonormal basis for $\mathbb{C}^n$, and $A$ not necessarily hermitian. What properties must there be of $\mathbf{w}_i$ if $A\mathbf{w}_1, \ldots, A\mathbf{w}_n$ is still an orthogonal set?

$$\langle A\mathbf{w}_i, A\mathbf{w}_j \rangle = 0$$
$$\langle A^H A\mathbf{w}_i, \mathbf{w}_j \rangle = 0$$

This would work if $\mathbf{w}_i$ was an eigenvector for $A^H A$. Let $B = A^H A$. Notice that $B$ is hermetian.

$$B^H = (A^H A)^H = \overline{(\overline{A}^T A)}^T = (A^T \overline{A})^T = A^H A = B$$

Then we would just want an orthonormal set of eigenvectors of $B$, which since $B$ is hermetian, does exist. Thus, set those original $\mathbf{w}_1, \ldots, \mathbf{w}_n$ to the orthonormal set of eigenvectors for $B$, with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. This would imply that $A\mathbf{w}_1, \ldots, A\mathbf{w}_n$ is again orthogonal. But would it be orthonormal?

$$|A\mathbf{w}_i| = \sqrt{\langle A\mathbf{w}_i, A\mathbf{w}_i \rangle}$$
$$|A\mathbf{w}_i| = \sqrt{\langle A^H A\mathbf{w}_i, \mathbf{w}_i \rangle}$$
$$\begin{aligned} |A\mathbf{w}_i| &= \sqrt{\langle B\mathbf{w}_i, \mathbf{w}_i \rangle} \\ &= \sqrt{\lambda_i \langle \mathbf{w}_i, \mathbf{w}_i \rangle} \\ &= \sqrt{\lambda_i} \end{aligned}$$

Thus the magnitude of each basis vector after application of $A$ is the square root of its eigenvalue under $A^H A$. These magnitudes are the **singular values**. We denote $\sigma_i = \sqrt{\lambda_i}$.

Define $\mathbf{v}_i = \frac{A\mathbf{w}_i}{\sigma_i}$ and let $\mathbf{v}_{n+1}, \ldots, \mathbf{v}_m$ be an orthonormal basis for $\mathrm{Nul}(A^T)$, which is perpendicular to the image of $A$. Set

$$U = \begin{bmatrix} \mathbf{w}_1 | & \cdots & | \mathbf{w}_n \end{bmatrix}$$

$$V = \begin{bmatrix} \mathbf{v}_1 | & \cdots & | \mathbf{v}_m \end{bmatrix}$$

Now, notice that $AU = \begin{bmatrix} A\mathbf{w}_1 \cdots A\mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \sigma_1 \mathbf{v}_1 \cdots \sigma_n \mathbf{v}_n \end{bmatrix} = V \begin{bmatrix} \text{diag}(\sigma_1, \ldots, \sigma_n) \\ \mathbf{0} \end{bmatrix}$

Let $S = \begin{bmatrix} \text{diag}(\sigma_1, \ldots, \sigma_n) \\ \mathbf{0} \end{bmatrix}$. Then we have $AU = VS$.

$$A = VSU^H \tag{1}$$

Now, we have a factorization where:

1. $V$ is an orthonormal basis for $\mathbb{C}^m$, where the first $n$ columns are the orthonormal results of multiplication by $B = A^H A$.
2. $S$ is a modified diagonal matrix of the singular values of $A$
3. $U$ is the orthonormal eigenbasis generated by $B = A^H A$

This is the singular value decomposition.