The Exterior Algebra

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Definition 1 *Let V be a vector space over $\mathbb{R}$.*
Define $\wedge^2 V$ as the vector space spanned by elements $v \wedge w$ for $v, w \in V$ subject to the properties:

1. $c(v \wedge w) = (cv) \wedge w = v \wedge (cw)$
2. $v \wedge w = -w \wedge v$
3. $v \wedge (w + x) = v \wedge w + v \wedge x$

A notable property of this vector space is that wedges of parallel vectors and only parallel vectors are zero:

$$\lambda v = w$$
$$v \wedge w = -w \wedge v$$
$$\lambda(v \wedge v) = -\lambda(v \wedge v)$$
$$2\lambda(v \wedge v) = 0$$
$$v \wedge (\lambda v) = 0$$
$$v \wedge w = 0$$

0.1 Bases

Fix a basis $e_1, \ldots, e_n$ of V. Now, write $v = v_1 e_1 + \cdots + v_n e_n$, and $w = w_1 e_1 + \cdots + w_n e_n$. Consider $v \wedge w$:

$$(v_1 e_1 + \cdots + v_n e_n) \wedge (w_1 e_1 + \cdots + w_n e_n)$$

The distribution and collecting of terms is left as an exercise. However, the ultimate result is that:

$$v \wedge w = \sum_{i<j} (v_i w_j - w_j v_i) e_i \wedge e_j$$

By inspection, the wedge $v \wedge w$ then encodes all determinants of all 2×2 minors of the matrix:

$$\begin{bmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_n & w_n \end{bmatrix}$$

Through its coefficients in the basis. This fact is general for any $\wedge^k V$.

0.2 General Exterior Powers

Definition 2 $\wedge^k V$ *is the vector space over* $\mathbb{R}$ *spanned by* $v_1 \wedge \cdots \wedge v_k$ *for* $v_i \in V$ *subject to the following:*

1. $c(v_1 \wedge \cdots \wedge v_k) = v_1 \wedge \cdots \wedge cv_i \wedge \cdots v_k$
2. *Any odd permutation* $\tau \in S_k$ *with any* $(v_1 \wedge \cdots \wedge v_k) \in \wedge^k V$ *yields* $v_1 \wedge \cdots \wedge v_k = -v_{\tau(1)} \wedge \cdots \wedge v_{\tau(k)}$
3. $v_1 \wedge \cdots \wedge (v_i + w_i) \wedge \cdots \wedge v_k = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k + v_1 \wedge \cdots \wedge w_i \wedge \cdots \wedge v_k$