Paths

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Definition 1 *A path in a topological space Y is a continuous mapping from $[0,1] \to Y$.*

Intuitively, a path is just a line or curve in your topological space. We define it to be homeomorphic to the real interval $[0,1]$. If we have two paths p_1 and p_2, we can also concatenate or merge them naturally, given that the endpoint $p_1(1) = p_2(0)$:

$$p_1 * p_2 = \begin{cases} p_1(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ p_2(2t) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

Using this we can define a natural equivalence relation, in which all points which are path-connected are considered as a group:

Proposition 1 *$x_1 \sim x_2$ if there exists a path $p : [0,1] \to X$ with $p(0) = x_1$ and $p(1) = x_2$ gives an equivalence relation*

Proof.

1. Reflexive: For $x \in X$, the path $p(t) = x$ for all $t \in [0,1]$ satisfies.
2. Symmetric: If $x \sim x'$ then there is a path $p : x \to x'$. If we define $p'(t) = 1 - p$, then $p'(0) = x'$ and $p'(1) = x$
3. Transitive: If $x \sim x'$ and $x' \sim x''$, then simply concatenate the respective paths with the aforementioned construction so that $p * p' : x \to x''$

Thus, path-connectedness is an equivalence relation.