Continuous Functions

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def $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if for any $\varepsilon > 0$, $\exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

def A metric space (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ with properties:

1. Symmetric $d(x, x') = d(x', x)$
2. Identity $d(x, x') = 0 \iff x = x'$
3. Triangular $d(x, x') + d(x', x'') \geq d(x, x'')$

def Let $(X, d), (Y, p)$ be metric spaces. A function $f : X \to Y$ is continuous if for all $\varepsilon > 0$, $\exists \delta > 0$ such that $d(x, x') < \delta$ implies that $p(f(x), f(x')) < \varepsilon$.

Example: Consider $(\mathbb{R}, |\cdot - \cdot|)$, the number line with the standard euclidean metric, and $(\mathbb{C}, d)$, defined by $d(e^{it}, e^{it'}) = |t - t'|$.

Prove that $t \in \mathbb{R} \to \cos(t) + i \sin(t)$ is a continuous function.
Pf: Choose $\varepsilon > 0$ and $\delta = \varepsilon$. Let $|t - t'| < \delta$. Then $f(t), f(t')$ are $e^{it}, e^{it'}$. $d(e^{it}, e^{it'}) = |t - t'| < \delta = \varepsilon$. Thus f is continuous.

Now, we can rephrase this into purely topological terms and get rid of metric spaces:
Thm: $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for any open $V \subseteq Y$.
Proof: Assume f is cont. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open and f is cont., we know that there exists a neighborhood $B \subseteq V$ containing $f(x)$, which corresponds to a neighborhood $C \subset f^{-1}(V)$ containing x. Thus $f^{-1}(V)$ is an open set.
Assume $f^{-1}(V)$ is open. Let $x \in X$ and choose any open neighborhood B around $f(x)$. Then $f^{-1}(B)$ is open as well, with $x \in f^{-1}(B)$. Thus since its open, there exists a smaller neighborhood $C \subset f^{-1}(B) \subset X$. Thus if we choose neighborhood C around x, we can guarantee that the image is inside of neighborhood B in the image. Thus f is continuous.

Suppose (Y, p) is a metric space and $f : X \to Y$ is any function from a set X to Y. Our goal is to define a metric d on X such that f is continuous.

Define $d : X \times X \to \mathbb{R}$ by $d(x_1, x_2) = p(f(x_1), f(x_2))$. Essentially we just map the x points to the codomain and ask what their distance is there, leveraging the fact we already know p. <u>Check:</u>

1. $d(x, x') \sim p \geq 0 \checkmark$
2. $d(x, x') = p(f(x), f(x')) = p(f(x'), f(x)) = d(x', x) \checkmark$
3. $d(x, x') = 0 \implies p(f(x), f(x')) = 0 \implies f(x) = f'(x)$?
4. $d(x, x') + d(x', x'') = p(f(x), f(x')) + p(f(x'), f(x'')) \geq p(f(x), f(x'')) = d(x, x'') \checkmark$

Therefore, if we want this metric to be possible, we either need to have an injective function, or work on an equivalence relation $x \sim x' \iff f(x) = f(x')$ so that property 3 is satisfied.