Notions of Derivative in $\mathbb{R}^3$

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Gradient

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle$. For notational convenience we denote $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Somehow, by applying ∇, we map from functions to vector fields, where every point in the space is assigned not a value but a vector.

$$f \xrightarrow{nabla} F$$

$$\text{functions} \xrightarrow{nabla} \text{vector fields}$$

Consider the level curve for $f(x) = c$ and any gradient $\nabla f|_x$. Any movement with a component tangent to the level curve produces no change to the value of the function. Thus, $\nabla f|_x$ is orthogonal to the level surface. $(\nabla f|_x)^\perp$ is used to find the tangent space to the level curve.

Vector Fields

Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.

- Cross with ∇:

$$\nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$$

$$= \langle \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \rangle$$

  This is curl, where the direction of the vector is the axis of rotation and the magnitude is the rate of rotation.

- Dot with ∇:

$$\nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

  This is divergence, where the direction of the vector is the direction of the flow and the magnitude is the rate of flow.