Differential Forms

Dylan Rupel, Matteo Paz

May 8, 2024

Let there be a function $f : \mathbb{R}^n \to \mathbb{R}$. What is its gradient?:

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right\rangle$$

The gradient thus gives us a notion of the rate of change of the output f, per unit moved along each of the x_i **basis** vectors.[1]

However, lets say we want to measure the rate of change along some vector v of our choice. A formulation using the traditional definition of a derivative gives:

$$D_v f|_x = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

Which actually turns out to be equivalent to

$$\nabla f(x) \cdot v$$

Thus, the gradient lets us evaluate the rate of change for any movement in the entire space.

Differential Forms

Consider some $f : \mathbb{R}^n \to \mathbb{R}$ and some path parametrized by s: $x(s) : \mathbb{R} \to \mathbb{R}^n$. By composition, we have $f(x(s)) : \mathbb{R} \to \mathbb{R}$. Lets say we want to find the derivative of this: $\frac{df}{ds}$, notice we yield a single-dimension derivative rather than partials, this is significant:

$$\frac{df}{ds} = \nabla f(x(s)) \cdot x'(s)$$

$$\left\langle \frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{dx_1}{ds} \cdots \frac{dx_n}{ds} \right\rangle$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x_1}\frac{dx_1}{ds} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{ds}$$

[1] Typically, the basis vectors are taken to be the standard euclidean basis of 1s and 0s, however the definition is exactly the same for any orthonormal basis.

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots \frac{\partial f}{\partial x_n} dx_n$$

Notice this looks like a linear combination of dx_i. If we take these dx_i to be formal basis vectors of a vector space spanned by them, then df is a member of this space, where the gradient ∇f encodes its coefficients. Call this space Ω_1.

k-Form

Let Ω_1 denote the vector space over $\mathcal{C}_\infty(\mathbb{R}^n)$ spanned by $dx_1 \cdots dx_n$.