Normal Distributions

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1 Basic Formulation

1.1 An Experiment - Assumptions

Picture a dartboard in the xy-plane with the bullseye at the origin. Let (X, Y) be the location where the dart lands. Let's make some assumptions.

1. **Rotational invariance**: The *deviation* from the origin only depends of the distance from the origin and not on the direction.
2. **Distribution**: X and Y are independent random variables.

1.2 What kind of function is this?

Let $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ denote the probability density function. Assumption (1) implies that $\tilde{f}(x, y) = g(x^2 + y^2)$ for some function g, with the squared sum representing the invariance under rotation. Assumption (2) $\tilde{f}(x, y) \propto \tilde{f}(x, 0) f(0, y)$ as independent random events must multiply. If we reframe into the perspective of the single-variable function g, we have

$$g(x^2 + y^2) = g(x^2)g(y^2)$$

The kinds of functions that have this property are exponentials, so we can say:

$$g(t) = Ae^{-Bt} \qquad A, B \in \mathbb{R}_{>0}$$

With the added factor of -1 because we are aiming for the center, so the probability should be greatest at low distances from the center, rather than the opposite. We can decide on the value of A and B by forcing the entire volume of the probability to 1, as it should be:

$$1 = \iint_{\mathbb{R}^2} \tilde{f}(x, y) dxdy = \iint_{\mathbb{R}^2} g(x^2 + y^2) dxdy$$

1.3 Using the Jacobean

To simplify this integral, we should respect the rotational symmetry and rewrite in terms of polar coordinates, r and θ.

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

It's not possible to provide thorough justification for this particular step, but a useful tool in this integration is the wedge operation $\wedge$. The key properties are:

- We can replace $dxdy$ with $dx \wedge dy$ and vice versa without issue.
- It is skew-symmetric: $dx \wedge dy = -(dy \wedge dx)$
- $dx \wedge dx = 0$ (consequence of the previous property)

Lets use the wedge operator to replace $dxdy$ in the integral:

$$\begin{aligned}
dx \wedge dy &= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\
&= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\
&= r\left(\cos^2(\theta) + \sin^2(\theta)\right) dr \wedge d\theta \\
&= rdr \wedge d\theta
\end{aligned}$$

Now we have our substitution, so rewriting our integral with new bounds $0 \leq \theta \leq 2\pi$ and $r \geq 0$ we have

$$\int_0^{2\pi} \int_0^{\infty} g(r^2) \;\; rdrd\theta$$

Lets perform the calculation:

$$\begin{aligned}
&= 2\pi \int_0^{\infty} Are^{-Br^2} dr \\
&= \left[\frac{-2\pi A}{2B} e^{-Br^2}\right]_0^{\infty} \\
&= \frac{\pi A}{B} = 1
\end{aligned}$$

So $A = \frac{\pi}{B}$ and finally we have the probability distribution based on a single parameter B:

$$\tilde{f}(x, y) = \frac{B}{\pi} e^{-B(x^2+y^2)}$$

By assumption 2, we can factor this. Naturally we could have:

$$f(x) = \sqrt{\frac{B}{\pi}} e^{-Bx^2}$$

We don't necessarily yet know that this is a probability distribution. The proof of this is contained already in these notes, and the explicit procedure is left to the reader. *Hint: Use the fact that the only nonzero idempotent is 1.* However, once we prove that it is, this is the general form of the normal distribution.

1.4 Expectation

Lets compute $E[X]$ for random variable X with probability density function $f(x) = \sqrt{\frac{B}{\pi}} e^{-Bx^2}$

$$\int_{\mathbb{R}} x f(x) dx = \int_{-\infty}^{\infty} \sqrt{\frac{B}{\pi}} x e^{-Bx^2} dx$$
$$\left[-\frac{1}{2\sqrt{B\pi}} e^{-Bx^2} \right]_{-\infty}^{\infty}$$
$$E[X] = 0$$

1.5 Variance

Lets find the variance for this general normal distribution, based on the parameter B. First we need to find the second moment $E[X^2]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-\infty}^{\infty} x^2 \sqrt{\frac{B}{\pi}} e^{-Bx^2} dx$$

Using integration by parts, choose $u = x \quad dv = \sqrt{\frac{B}{\pi}} x e^{-Bx^2}$. Then $du = dx \quad v = -\frac{1}{2\sqrt{B\pi}} e^{-Bx^2}$

$$uv - v \int du = \left[-1 \frac{1}{2\sqrt{B\pi}} x e^{-Bx^2} \right]_{-\infty}^{\infty} + \frac{1}{2\sqrt{B\pi}} \int_{-\infty}^{\infty} e^{-Bx^2} dx$$

The integral on the right must be $\sqrt{\frac{\pi}{B}}$ since we already found $f(x)$ to be a probability density function.

$$0 + \frac{1}{2\sqrt{B\pi}} \cdot \sqrt{\frac{\pi}{B}}$$
$$E[X^2] = \frac{1}{2B}$$

Finally

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$
$$\mathrm{Var}(X) = \frac{1}{2B}$$

If you want to reparametrize the normal distribution, controlling for the standard deviation σ, you could thus use $B = \frac{1}{2\sigma^2}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$$

If you want to modify the mean to be at $x = \mu$, Then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

2 Shifting

Proposition. Let X be normal with mean μ and variance σ^2, then $aX+b$ is also normal, with mean $a\mu+b$ and variance $a^2\sigma^2$.

Proof. Variance:

$$\mathrm{Var}(aX+b) = E[(aX+b)^2] - E[aX+b]^2$$

$$E[a^2X^2 + 2abX + b^2] - E[aX+b]^2$$

$$a^2E[X^2] + 2abE[X] + b^2 - (aE[X]+b)^2$$

$$a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2$$

$$a^2(E[X^2] - E[X]^2) = a^2\sigma^2 \checkmark$$

Mean:

$$E[aX+b] = aE[X] + b = a\mu + b\checkmark$$

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