Combinatorics

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1 Counting Organized Data - The Twelvefold Way

Goal - Count the number of functions between multi-sets (Some elements may be identical).

- How many surjective functions exist between $\{1, 2, 3, 4, 5\}$ to $\{a, b, c\}$?

We could try to list them all, but there should be a way to take advantage of the structure. Given that we are considering a multi-set, lets generalize to: $\{1, 2, 3, 4, 5\} \to \{\cdot, \cdot, \cdot\}$ Where each dot can be thought of as a bucket. It doesnt matter necessarily which bucket a number is being placed in, but how many and which went in each of the 3 baskets.

1.1 The Twelvefold Way

There turns out to be 12 major classifications of mappings between sets. The number of such functions for each classification is a very useful tool when counting. Let us map $N \to X$, with $|N| = n, |X| = x$.

	Functions	Injective	Surjective
Distinguishable Domain and Codomain	x^n	$x^{\underline{n}}$	$x!\left\{{n \atop x}\right\}$
Indistinguishable Domain, Dist. Codomain	$\binom{n+x-1}{x-1}$	$\binom{x}{n}$	
Dist. Domain, Indist. Codomain	$\sum_{k=1}^{x} \left\{{n \atop k}\right\}$	$[n \leq x]$	$\left\{{n \atop x}\right\}$
Indist. Domain and Codomain		$[n \leq x]$	

Distinguishable Functions

The top-left entry is x^n. This is because in identifying a function of this type, you have x choices of a preimage for n choices of an image.

Distinguishable and Injective Functions

Very similar to non-injective functions, but every output must be unique. Thus having x choices for your first element to map, then $x - 1$, n times gives $x(x - 1) \cdots (x - n + 1)$ choices in all. This is the falling factorial function $x^{\underline{n}}$

Injective and Indistinguishable Functions

The notion of an indistinguishable domain or codomain is that if you mess around with order or specificity, it makes no difference. So we should look at the ways you can permute either of the sets. Let S_N denote the group of permutations on N. Let X^N denote the set of all functions $F : N \to X$.

Define a group action $S_N \circlearrowright X^N$ by $(\sigma \cdot f)(n) = f(\sigma^{-1}(n))$. Notice that this is in fact an action as associativity holds:

$$((\sigma \circ \tau) \cdot f)(n) = f(\tau^{-1}(\sigma^{-1}(n))) = (\sigma \cdot (\tau \cdot f))(n)$$

And $\mathrm{Id} \in S_N$ works as an identity in this setting as well. Lets prove that $S_N \circlearrowright \mathrm{Inj}(X^N)$ is a free action. If $f \in X^N$ is injective and $\sigma \cdot f = f$, then $(\sigma \cdot f)(n) = f(n)$.

$$f(\sigma^{-1}(n)) = f(n)$$
$$n = \sigma(n)$$
$$\sigma = \mathrm{Id}$$

Thus $\mathrm{Stab}_{S_N}(f) = \{Id\}$ and the action is free. By the Orbit-Stabilizer theorem,

$$\left|\mathrm{Inj}(X^N)/S_N\right| = \frac{x^{\underline{n}}}{n!}$$

This can be rewritten as $\binom{x}{n}$.

Finally, for injective functions with an indistinguishable codomain, it really doesnt matter where you map N, as long as you can without overlap. Thus, for the last two rows, we have the condition that the codomain has enough room for $X : [n \leq x]$

2 Surjective functions

Start with a distinguishable domain N and indistinguishable X. The order in which you group N to go to X does not matter, just that there are x nonempty groups that then map to different baskets of X. Since the entirety of N must be included as the domain, we know that we must partition N into x partitions, which is given by this notation $\left\{ n \atop x \right\}$. These are called stirling numbers of the second kind.

If we have a distinguishable codomain once again, we can just multiply by our options of order: $x!$. Thus the number of surjective functions between sets is $x!\left\{ n \atop x \right\}$.

Carrying over to all functions from dist N to indist X, not just surjective ones, is equivalent to partitioning the set in a number of different ways, from one element in X to all elements in X. Therefore we sum the ways to partition the domain:

$$\sum_{k=1}^{n} \left\{ n \atop k \right\}$$

3 Indistinguishable Domain and Codomain

This row has the least amount of structure. The injective functions were simple to think about, but regular functions and surjective functions have very little structure, and thus are hard to give a closed form for. In a similar way to the previous row, if we can find the number of surjective functions onto x elements, we can simply sum this value up for all possible sizes of a function's image from 1 to x to count how many functions there are in total.

3.1 Partitions

A *partition* of $n \in \mathbb{N}$ is a decreasing sequence $a_1, \cdots, a_k \in \mathbb{N}$ such that $\sum_{i=0}^{k} a_i = n$.

If we consider surjective mappings from n to x indistinguishable elements, the only way to identify a unique function is by the sizes of the subsets in its preimage partition. For $k = x$ elements, we identify that these functions have a bijection with the partitions of n into x elements, because all n must be grouped into one of the preimages and there are x elements that the preimages are generated from.