Exponential Random Variables

Matteo Paz, Dylan Rupel

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1 Formulation

Definition 1 *A nonnegative random variable X is* <u>memoryless</u> *if*

$$P\{X > s+t | X > t\} = P\{X > s\}$$

Lets work with this. We know how to rewrite this conditional probability:

$$\frac{P\{X > s+t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

Because $X > s+t$ implies that $X > t$ already,

$$P\{X > s+t\} = P\{X > s\}P\{X > t\}$$

Define $\overline{F}(x) = P\{X > x\}$. Then a memoryless random variable gives:

$$\overline{F}(s+t) = \overline{F}(s)\overline{F}(t)$$

Proposition 1 *Let $g(x)$ be a continuous nonnegative function which satisfies $g(s+t) = g(s)g(t)$ for $s, t \in \mathbb{R}_{\geq 0}$. Then $g(x)$ is an exponential function.*

Proof

$$g(1) = g\left(n \cdot \frac{1}{n}\right) = g\left(\frac{1}{n}\right)^n \implies g\left(\frac{1}{n}\right) = g(1)^{\frac{1}{n}}$$

$$g(n) = g(1)^n$$

For all $n \in \mathbb{Q}$. However, any real number is the limit of a sequence of rational numbers. Let $x \in \mathbb{R}$ and (x_i) a sequence of rationals converging to x. Using the continuity of $g(x)$ to move the limit around...

$$g(x) = g\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} g(x_i) = \lim_{i\to\infty} g(1)^{x_i}$$

$$= (g(1))^{\lim_{i\to\infty} x_i} = g(1)^x$$

We can express this directly as

$$g(x) = e^{-\lambda x}$$

where $\lambda = -\ln(g(1))\square$

Finally then, we can express $\overline{F}(x) = e^{-\lambda x}$ for some $\lambda \in \mathbb{R}$. From this, we know the CDF is $F(x) = 1 - e^{-\lambda x}$. The probability distribution function is then the derivative, giving:

$$f(x) = \frac{d}{dx}F(x) = \lambda e^{-\lambda x}$$