



SAPIENZA  
UNIVERSITÀ DI ROMA

# MARKOV DECISION PROCESSES

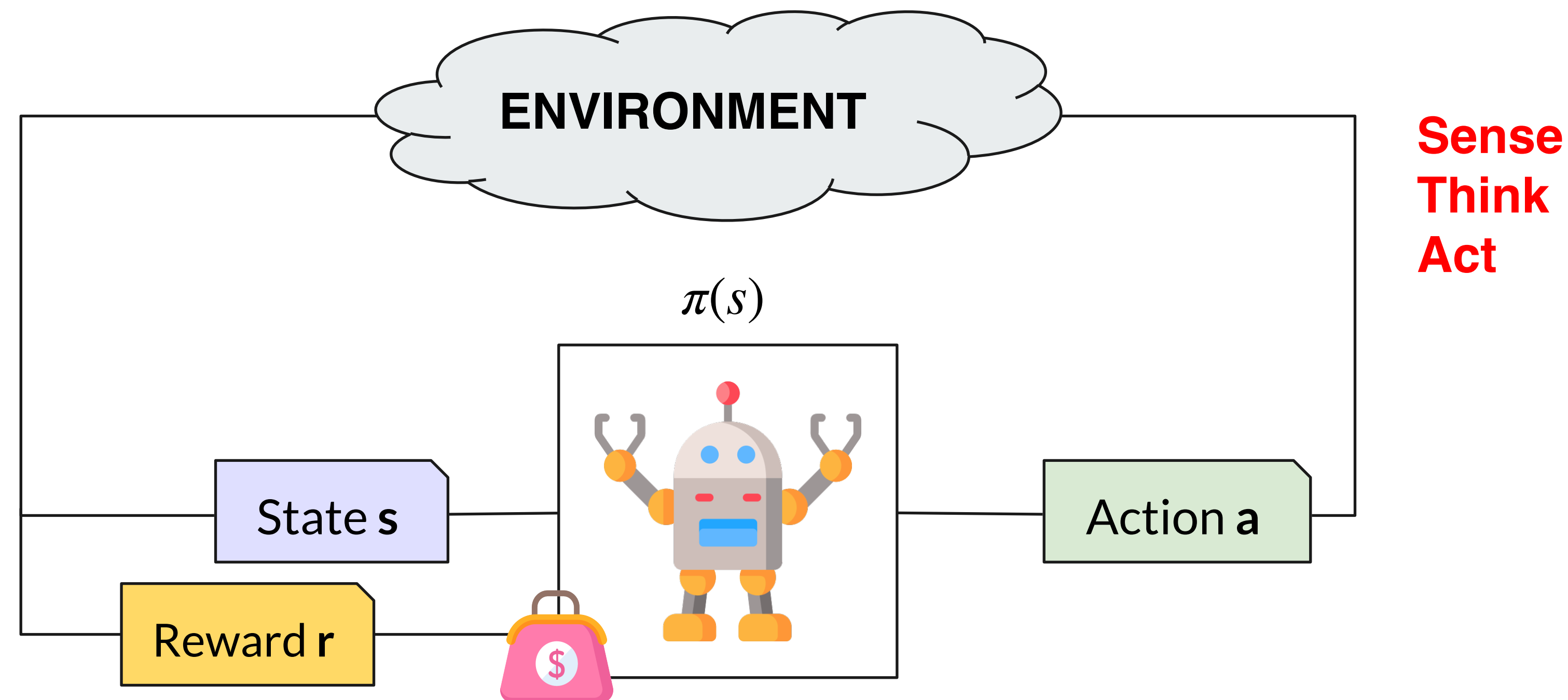
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COMPUTER NETWORK PERFORMANCE 14-12-2020

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# Reinforcement Learning

- Branch of automatic learning (just like Supervised Learning or Unsupervised Learning)
- An **agent** wants to learn an **optimal policy** (how to act) in an **environment**, maximising the cumulative **reward** signals it gets from the environment





# Reinforcement Learning an Example

- A self-driving car (the agent) drives autonomously without going off-road:
  - ◉ **States:** raw pixel inputs cameras mounted on the car (radars...)
  - ◉ **Actions:** steer right, steer left, do nothing
  - ◉ **Rewards:** a function of the distance of the car from the border lines





# Reward Maximisation

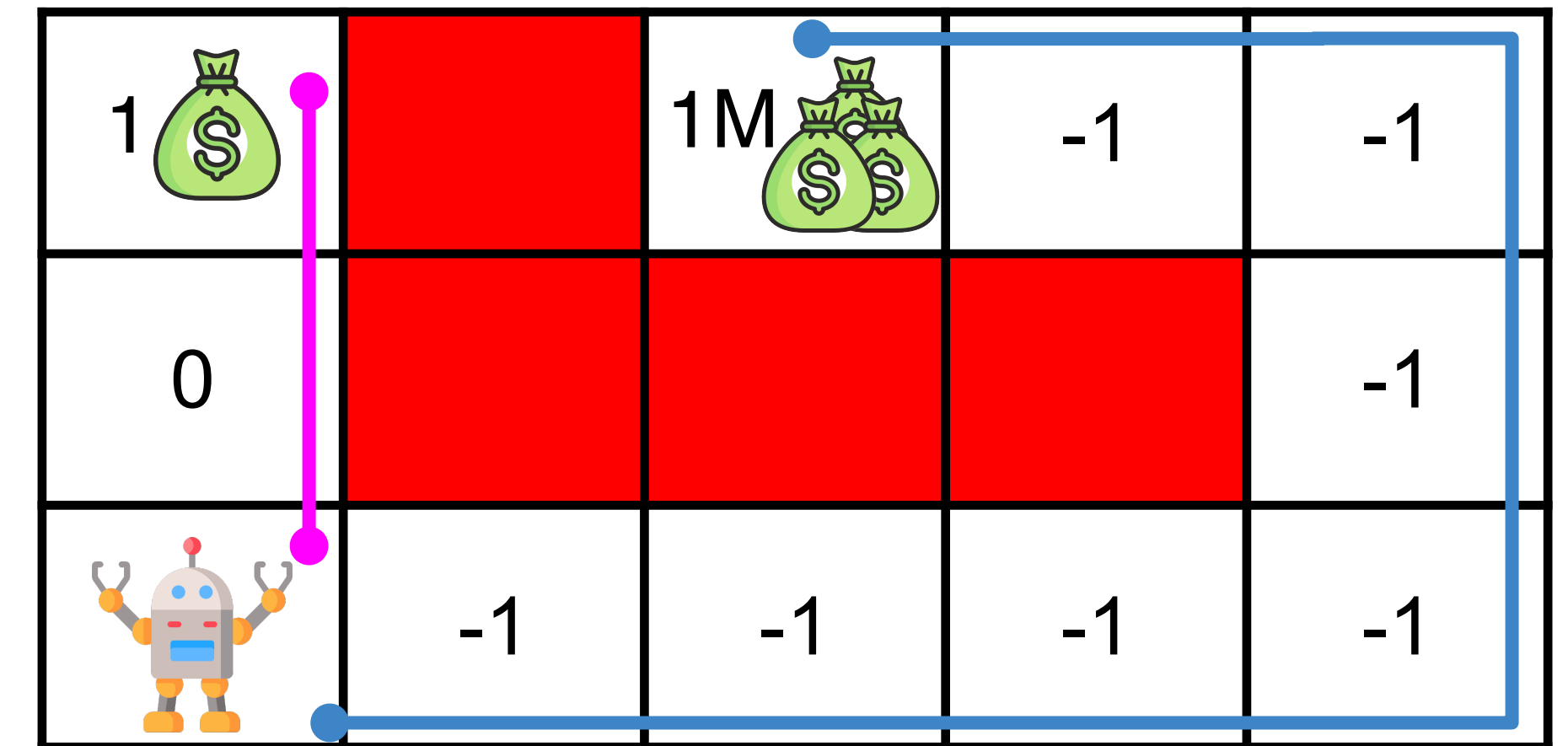
- Three ways to optimise the sum of future rewards:

**Infinite horizon**  $\sum_{i=0}^{\infty} r_i$

*P1: 0, 1, 0, 1 ...  $\rightarrow \infty$*   
*P2: -1, -1, -1, -1, -1, -1, -1, -1, 1M, -1, 1M ...  $\rightarrow \infty$*

**Finite horizon**  $\sum_{i=0}^n r_i$

*P1 (n = 3): 0, 1, 0  $\rightarrow 1\$$*   
*P2 (n = 3): -1, -1, -1  $\rightarrow -3\$$*



- Consider:** if it takes 4 years to get 1\$, that dollar is less valuable than if it takes 1 year (**inflation**)!
- Discounted rewards**  $0 < \gamma \leq 1$  with parameter  $\gamma = 0.95$  means that each reward in the future values 5% less than the immediate reward

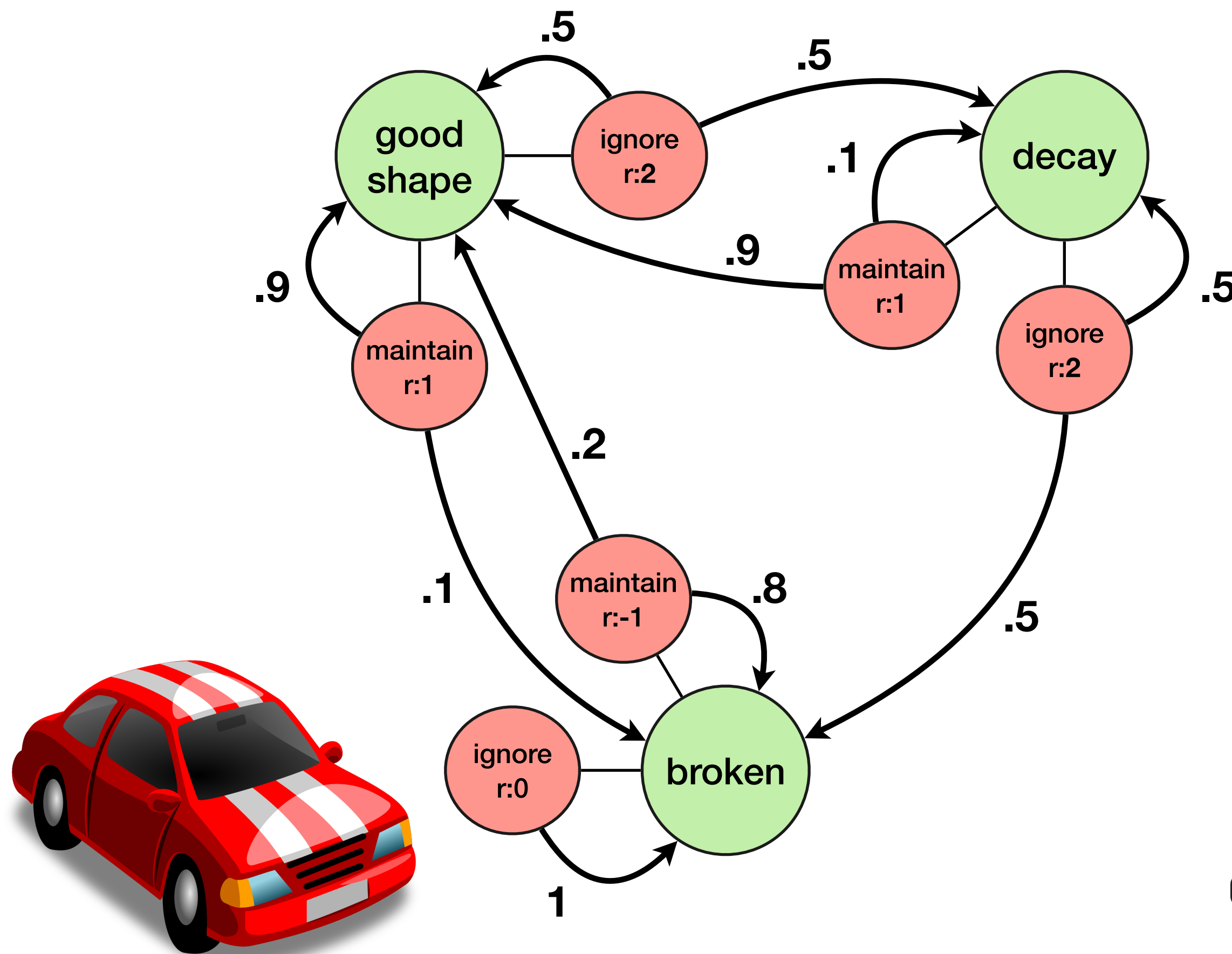
**Discounted reward**  $\sum_{i=0}^{\infty} \gamma^i r_i = r_0 + \gamma r_1 + \gamma^2 r_2 + \dots$

# Markov Decision Processes

- A Markov Decision Process is a 4 tuple **MDP** =  $(S, A, T, R)$ 
  - ⦿  $S$  : set of **states**
  - ⦿  $A$  : set of **actions**
  - ⦿  $T : \{S \times A \times S\} \rightarrow [0,1]$  : **transition probability** function s.t.  $\sum_{s' \in S} T[s, a, s'] = 1 \quad \forall s \in S, a \in A$
  - ⦿  $R : \{S \times A\} \rightarrow \mathbb{R}$  : **reward function**
- The **policy** is a function  $\pi : S \rightarrow A$  specifying what action to take in each state
- The goal is to find  $\pi^*$ , a policy that maximises cumulative discounted reward  $\sum_{i=0}^{\infty} \gamma^i r_i$

# Markov Decision Processes

- A Markov Decision Process is a 4 tuple  $\text{MDP} = (S, A, T, R)$



$S = \{ \text{good shape, decay, broken} \}$

$A = \{ \text{maintain, ignore} \}$

$$R = \begin{matrix} & \text{m.} & \text{i.} \\ \begin{bmatrix} +1 & +2 \\ +1 & +2 \\ -1 & 0 \end{bmatrix} & \begin{matrix} \text{good shape} \\ \text{decay} \\ \text{broken} \end{matrix} \end{matrix}$$

$$T = \begin{matrix} & \text{m.} & & \text{i.} \\ \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix} & \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} & \begin{matrix} \text{good shape} \\ \text{decay} \\ \text{broken} \end{matrix} \end{matrix}$$

$\pi = [ \text{ignore, ignore, maintain} ]$



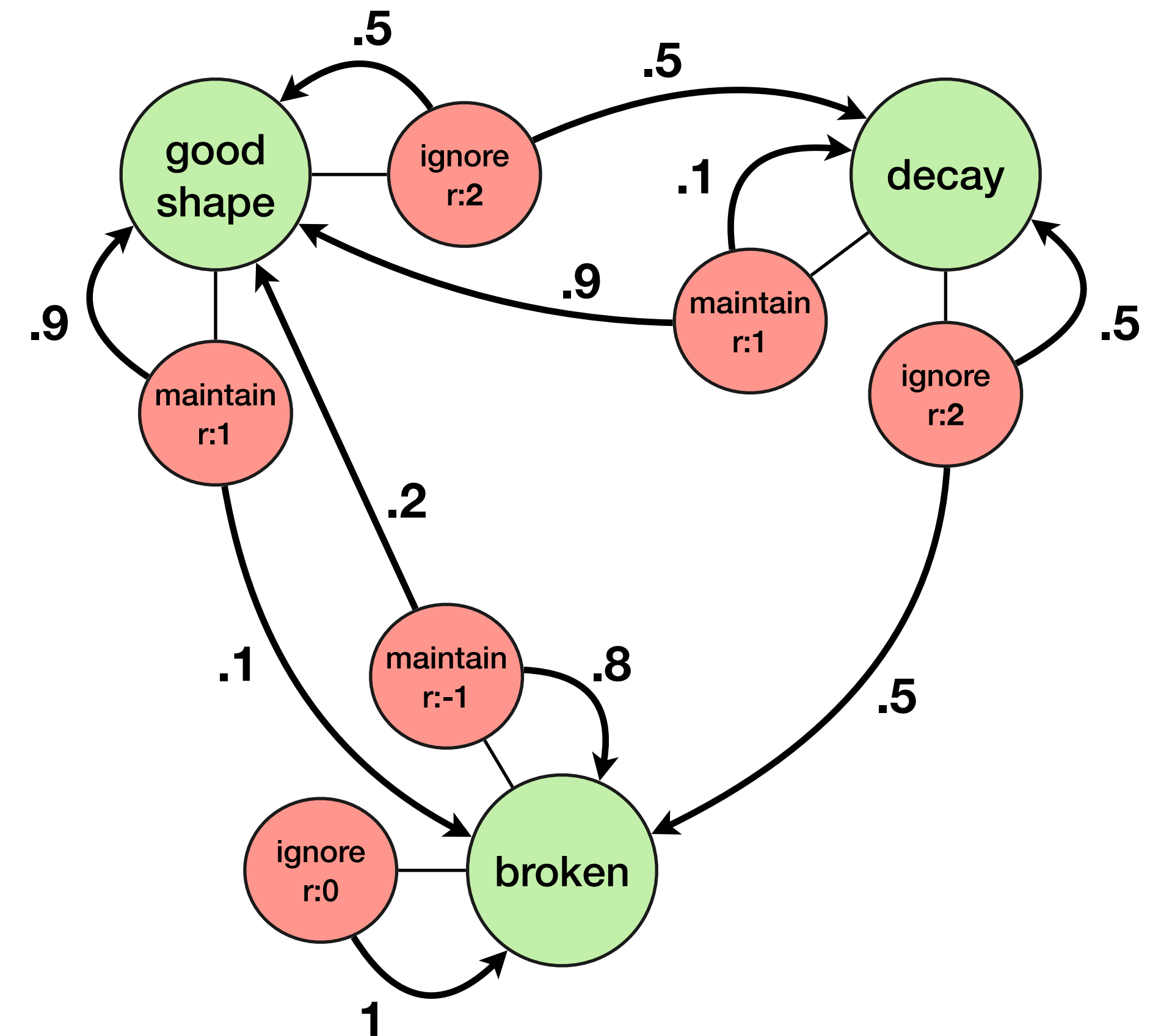
# Markov Decision Processes

## Process evolution

- At time step  $t = 0$ , given policy  $\pi$  and initial state  $s_0$
- For  $t = 0$  until done:
  - Agent select  $a_t \leftarrow \pi[s_t]$
  - Get target state  $s_{t+1} \sim T[s_t, a_t, *]$
  - Get reward  $r_t \leftarrow R[s_t, a_t]$
  - Agent collects **experience tuple**  $(s_t, a_t, s_{t+1}, r_t)$

$\pi = [\text{ignore}, \text{ignore}, \text{maintain}]$   $s_0 = \text{good shape}$

$$T = \begin{matrix} & \text{m.} & & \text{i.} \\ \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix} & \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} & R = \begin{matrix} & \text{m.} & \text{i.} \\ \begin{bmatrix} +1 & +2 \\ +1 & +2 \\ -1 & 0 \end{bmatrix} & \begin{matrix} \text{good shape} \\ \text{decay} \\ \text{broken} \end{matrix} \end{matrix}$$



$t = 0 : (\text{good shape}, \text{ignore}, \text{good shape}, 2)$

$t = 1 : (\text{good shape}, \text{ignore}, \text{decay}, 2)$

$t = 2 : (\text{decay}, \text{ignore}, \text{broken}, 2)$

$t = 3 : (\text{broken}, \text{maintain}, \text{broken}, -1) \dots$

$R = 2 + 2 + 2 - 1 \dots$

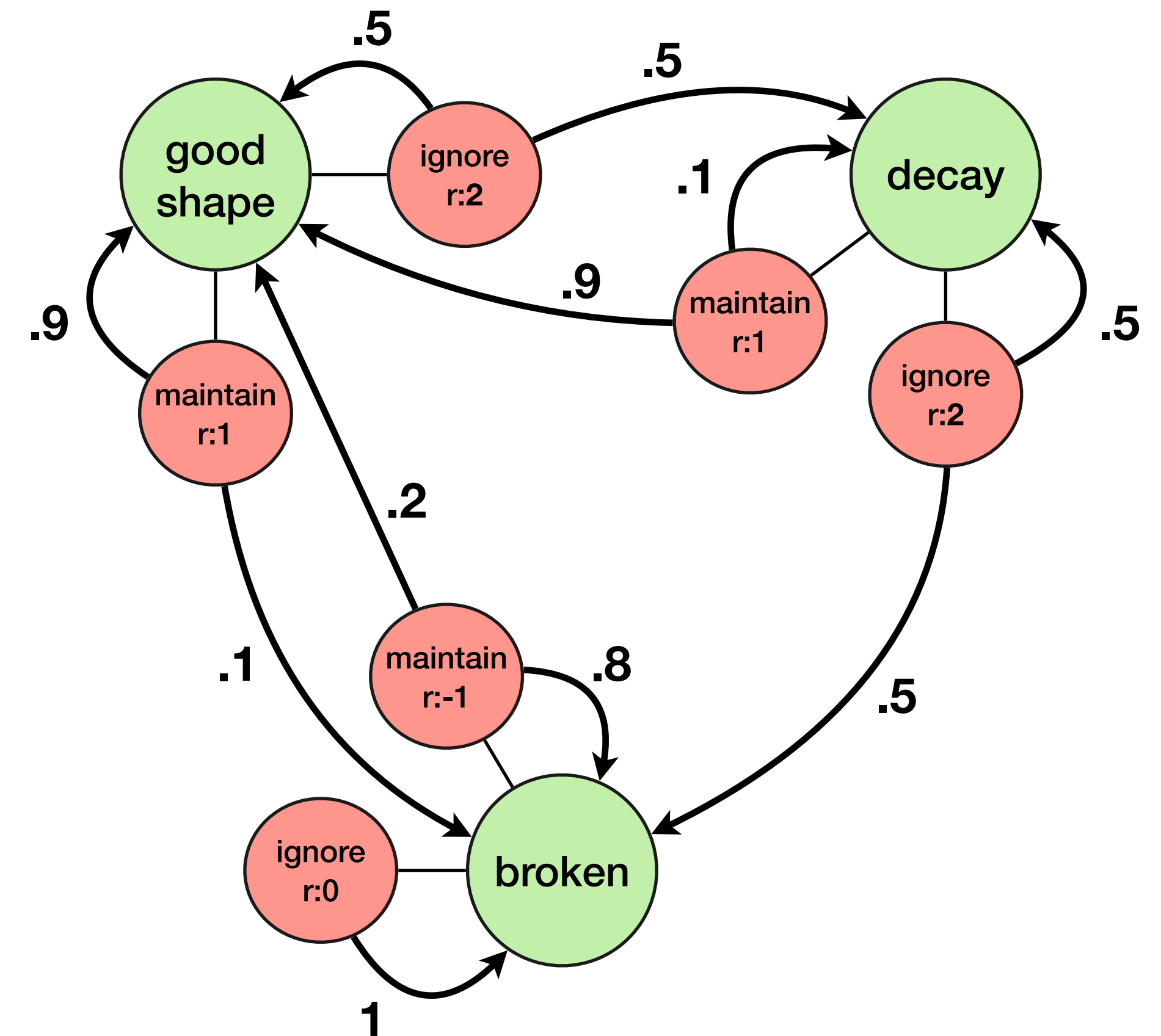
# Markov Decision Processes

## Process evolution

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$\pi = [\text{ignore}, \text{ignore}, \text{maintain}]$   $s_0 = \text{good shape}$

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$$T_{\pi} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$



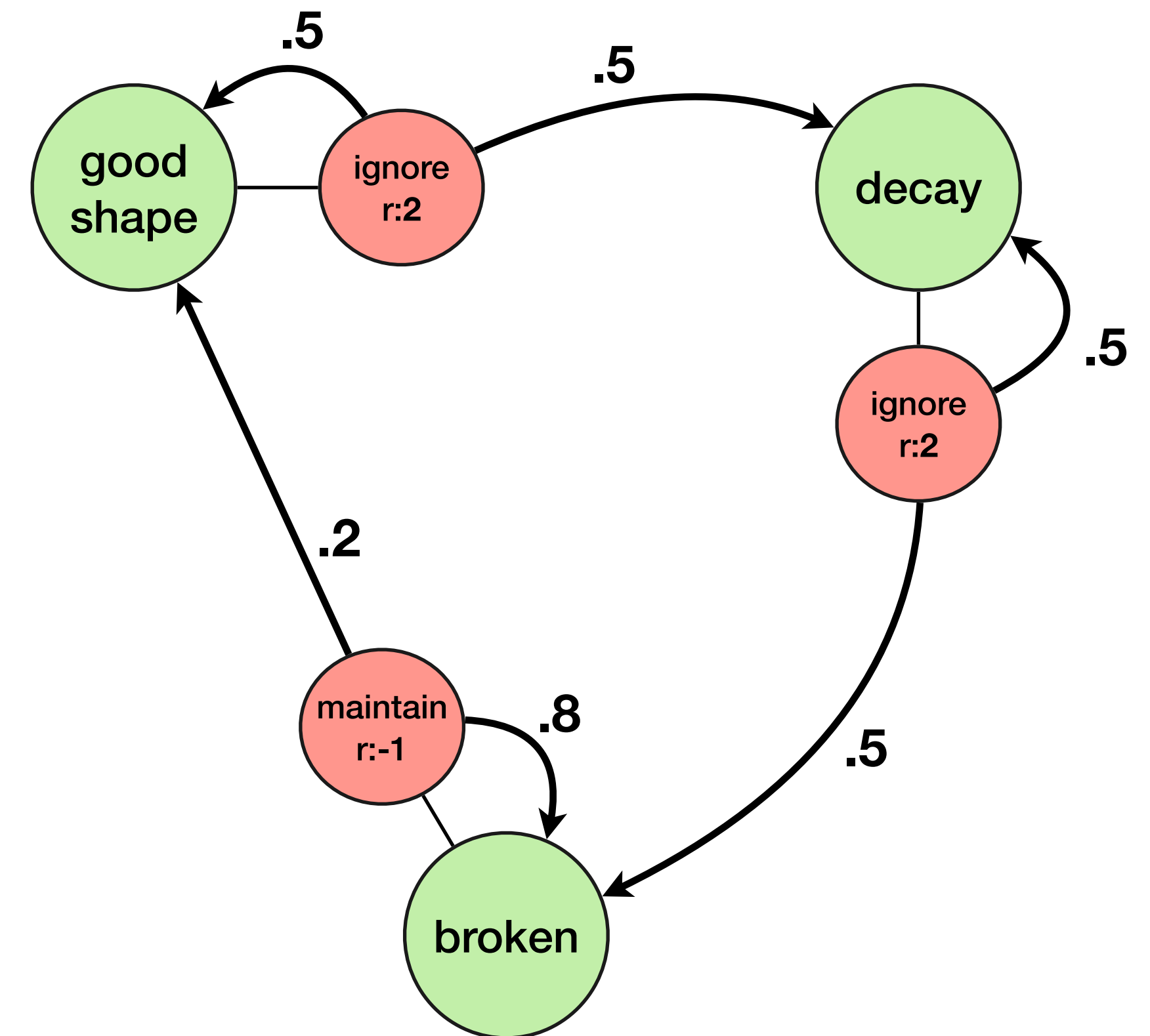
# Markov Decision Processes

## Process evolution

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- For  $t = 0$  until done:
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  - Agent collects **experience tuple**  $(s_t, a_t, s_{t+1}, r_t)$

$\pi = [\text{ignore}, \text{ignore}, \text{maintain}]$   $s_0 = \text{good shape}$

$$T = \begin{matrix} & \text{m.} & & \text{i.} \\ \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix} & \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} & R = \begin{matrix} \text{m.} & \text{i.} \\ \begin{bmatrix} +1 & +2 \\ +1 & +2 \\ -1 & 0 \end{bmatrix} & \begin{matrix} \text{good shape} \\ \text{decay} \\ \text{broken} \end{matrix} \end{matrix}$$



$$T_{\pi} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0.8 \end{bmatrix}$$

# Markov Decision Processes

- For every MDP there exists an optimal policy (*proven*)
- The optimal policy is the one that **maximises the expected cumulative discounted reward**

$$\pi^* = \arg \max_{\pi} \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \mid \pi \right]$$

## How good is a state?

The **value function** at state  $s$  is the expected cumulative reward from following the policy from state  $s$ . It obeys the Bellman equation, relating the value function to itself as:

$$V^{\pi}(s) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \mid s_0 = s, \pi \right] = \underbrace{R[s, \pi[s]]}_{\text{immediate reward}} + \underbrace{\sum_{s' \in S} T[s, \pi[s], s'] \cdot \gamma \cdot V^{\pi}(s')}_{\text{expected cumulative discounted reward}}$$

**Optimal  
value function**

$$V^*(s) = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V^*(s')$$

# Value Iteration Algorithm

- set  $V[s] = 0 \ \forall s \in S$
- while true:
  - ⊙ for each  $s \in S$  :
    - \*  $V[s] = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$
  - ⊙ exit if convergence (small change from previous and actual  $V[s] \ \forall s$ )

**Optimal policy!**  $\pi^*[s] = \arg \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$



# Value Iteration Algorithm

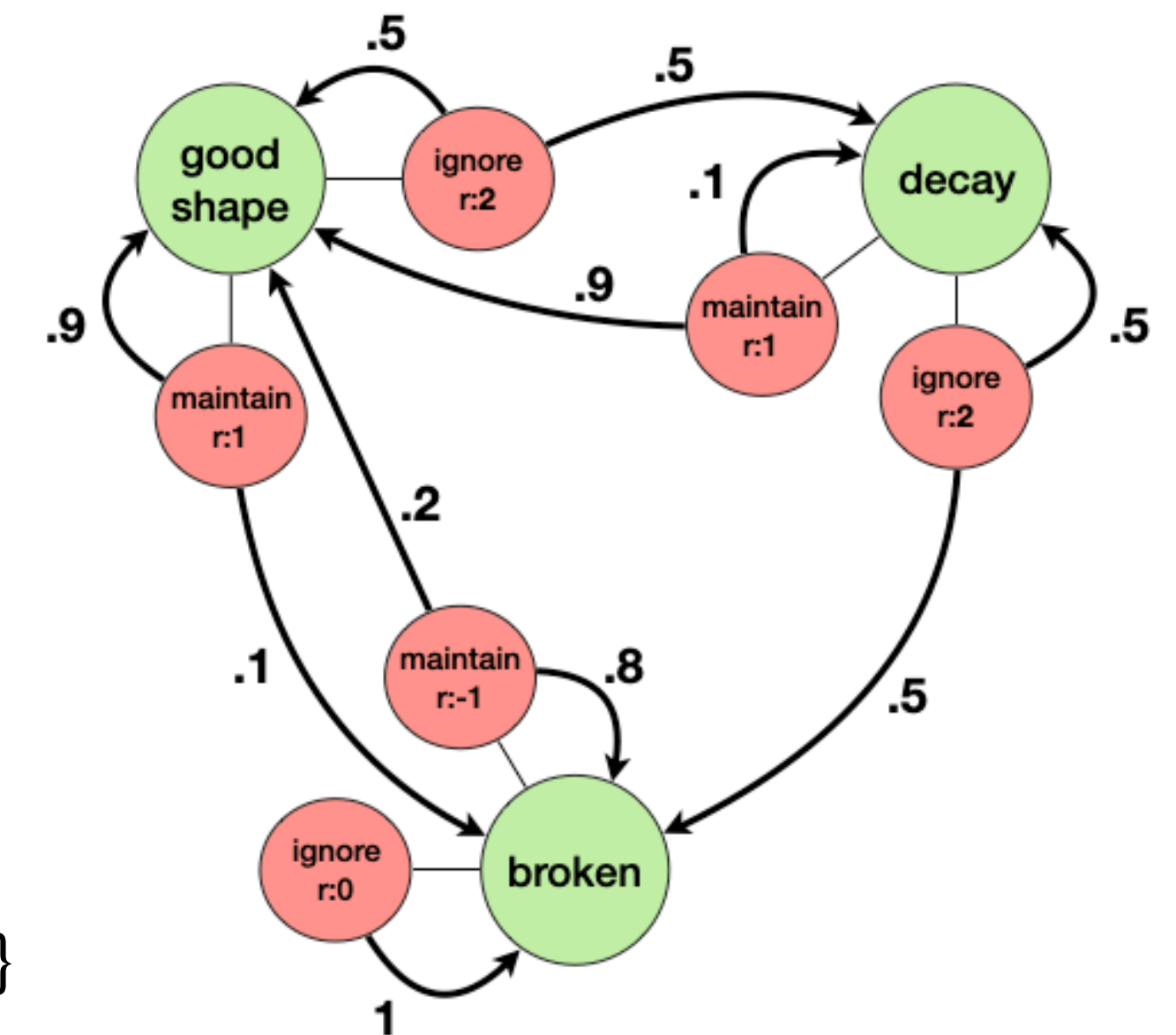
$$T = \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix} \quad \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} +1 & +2 \\ +1 & +2 \\ -1 & 0 \end{bmatrix} \quad \begin{array}{l} \text{good shape} \\ \text{decay} \\ \text{broken} \end{array}$$

$$\gamma = 0.9 \quad \text{maintain} = a_0 \quad \text{ignore} = a_1$$

- set  $V[s] = 0 \forall s \in S$
  - while true:
    - ⊙ for each  $s \in S$  :
      - \*  $V[s] = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$
    - ⊙ exit if convergence (small change from previous and actual  $V[s] \forall s$ )
- Optimal policy!**  $\pi^*[s] = \arg \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$

**While  
iter 1**

- $V = [0, 0, 0]$   
state 0 – good shape:  $V[0] = \max\{ a_0 : 1 + (0.9 \cdot \gamma \cdot 0) + (0 \cdot \gamma \cdot 0) + (0.1 \cdot \gamma \cdot 0);$   
 $a_1 : 2 + (0.5 \cdot \gamma \cdot 0) + (0.5 \cdot \gamma \cdot 0) + (0 \cdot \gamma \cdot 0) \} = \max\{1, 2\}$
- $V = [2, 0, 0]$   
state 1 – decay:  $V[1] = \max\{ a_0 : 1 + (0.9 \cdot \gamma \cdot 2) + (0.1 \cdot \gamma \cdot 0) + (0 \cdot \gamma \cdot 0);$   
 $a_1 : 2 + (0 \cdot \gamma \cdot 2) + (0.5 \cdot \gamma \cdot 0) + (0.5 \cdot \gamma \cdot 0) \} = \max\{2.62, 2\}$
- $V = [2, 2.62, 0]$   
state 2 – broken:  $V[2] = \max\{ a_0 : -1 + (0.2 \cdot \gamma \cdot 2) + (0 \cdot \gamma \cdot 2.68) + (0.8 \cdot \gamma \cdot 0);$   
 $a_1 : 0 + (0 \cdot \gamma \cdot 2) + (0 \cdot \gamma \cdot 2.68) + (1 \cdot \gamma \cdot 0) \} = \max\{-0.64, 0\}$



# Value Iteration Algorithm

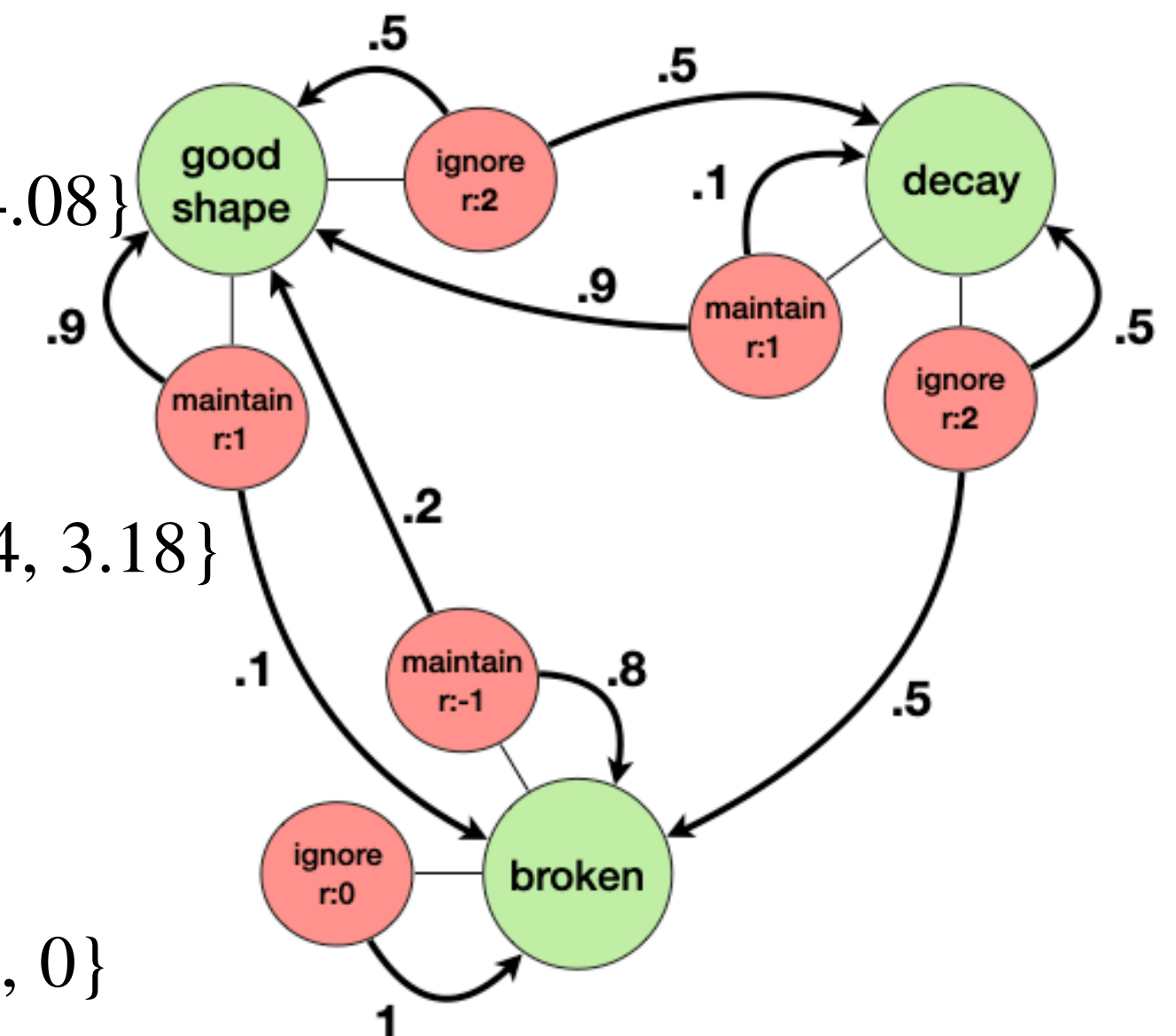
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$$\gamma = 0.9 \quad \text{maintain} = a_0 \quad \text{ignore} = a_1$$

- set  $V[s] = 0 \forall s \in S$
  - while true:
    - ⊙ for each  $s \in S$ :
      - \*  $V[s] = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$
    - ⊙ exit if convergence (small change from previous and actual  $V[s] \forall s$ )
- Optimal policy!**  $\pi^*[s] = \arg \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$

**While  
iter 2**

- $V = [2, 2.62, 0]$   
state 0 – good shape:  $V[0] = \max\{ a_0 : 1 + (0.9 \cdot \gamma \cdot 2) + (0 \cdot \gamma \cdot 2.62) + (0.1 \cdot \gamma \cdot 0);$   
 $a_1 : 2 + (0.5 \cdot \gamma \cdot 2) + (0.5 \cdot \gamma \cdot 2.62) + (0 \cdot \gamma \cdot 0) \} = \max\{2.62, 4.08\}$
- $V = [4.08, 2.62, 0]$   
state 1 – decay:  $V[1] = \max\{ a_0 : 1 + (0.9 \cdot \gamma \cdot 4.08) + (0.1 \cdot \gamma \cdot 2.62) + (0 \cdot \gamma \cdot 0);$   
 $a_1 : 2 + (0 \cdot \gamma \cdot 4.08) + (0.5 \cdot \gamma \cdot 2.62) + (0.5 \cdot \gamma \cdot 0) \} = \max\{4.54, 3.18\}$
- $V = [4.08, 4.54, 0]$   
state 2 – broken:  $V[2] = \max\{ a_0 : -1 + (0.2 \cdot \gamma \cdot 4.08) + (0 \cdot \gamma \cdot 4.54) + (0.8 \cdot \gamma \cdot 0);$   
 $a_1 : 0 + (0 \cdot \gamma \cdot 4.08) + (0 \cdot \gamma \cdot 4.54) + (1 \cdot \gamma \cdot 0) \} = \max\{-0.27, 0\}$



# Value Iteration Algorithm

$$T = \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} R = \begin{bmatrix} +1 & +2 \\ +1 & +2 \\ -1 & 0 \end{bmatrix} \begin{matrix} \text{good shape} \\ \text{decay} \\ \text{broken} \end{matrix}$$

$$\gamma = 0.9 \quad \text{maintain} = a_0 \quad \text{ignore} = a_1$$

- set  $V[s] = 0 \forall s \in S$
  - while true:
    - ⊙ for each  $s \in S$  :
      - \*  $V[s] = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$
    - ⊙ exit if convergence (small change from previous and actual  $V[s] \forall s$ )
- Optimal policy!**  $\pi^*[s] = \arg \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$

**Converged at iter. 50! Now it is time to find the optimal policy**

- $V = [16.69, 15.95, 7.15]$ 
  - state 0 – good shape:  $\pi^*[0] = \arg \max \{ a_0 : 1 + (0.9 \cdot \gamma \cdot 16.69) + (0 \cdot \gamma \cdot 15.95) + (0.1 \cdot \gamma \cdot 7.15);$   
 $a_1 : 2 + (0.5 \cdot \gamma \cdot 16.69) + (0.5 \cdot \gamma \cdot 15.95) + (0 \cdot \gamma \cdot 7.15) \} = \arg \max \{ 15.16, 16.68 \} = 1$  **ignore**
  - state 1 – decay:  $\pi^*[1] = \arg \max \{ a_0 : 1 + (0.9 \cdot \gamma \cdot 16.69) + (0.1 \cdot \gamma \cdot 15.95) + (0 \cdot \gamma \cdot 7.15);$   
 $a_1 : 2 + (0 \cdot \gamma \cdot 16.69) + (0.5 \cdot \gamma \cdot 15.95) + (0.5 \cdot \gamma \cdot 7.15) \} = \arg \max \{ 15.95, 12.4 \} = 0$  **maintain**
  - state 2 – broken:  $\pi^*[2] = \arg \max \{ a_0 : -1 + (0.2 \cdot \gamma \cdot 16.69) + (0 \cdot \gamma \cdot 15.95) + (0.8 \cdot \gamma \cdot 7.15);$   
 $a_1 : 0 + (0 \cdot \gamma \cdot 16.69) + (0 \cdot \gamma \cdot 15.95) + (1 \cdot \gamma \cdot 7.15) \} = \arg \max \{ 7.15, 6.44 \} = 0$  **maintain**

$$\pi^* = [\text{ignore}, \text{maintain}, \text{maintain}]$$



# Policy Iteration Algorithm

- Value iteration is slow  $\mathcal{O}(S^2A)$

## Initialisation

- **set**  $V[s]$  random value and  $\pi[s]$  random action  $\forall s \in S$

## Policy evaluation

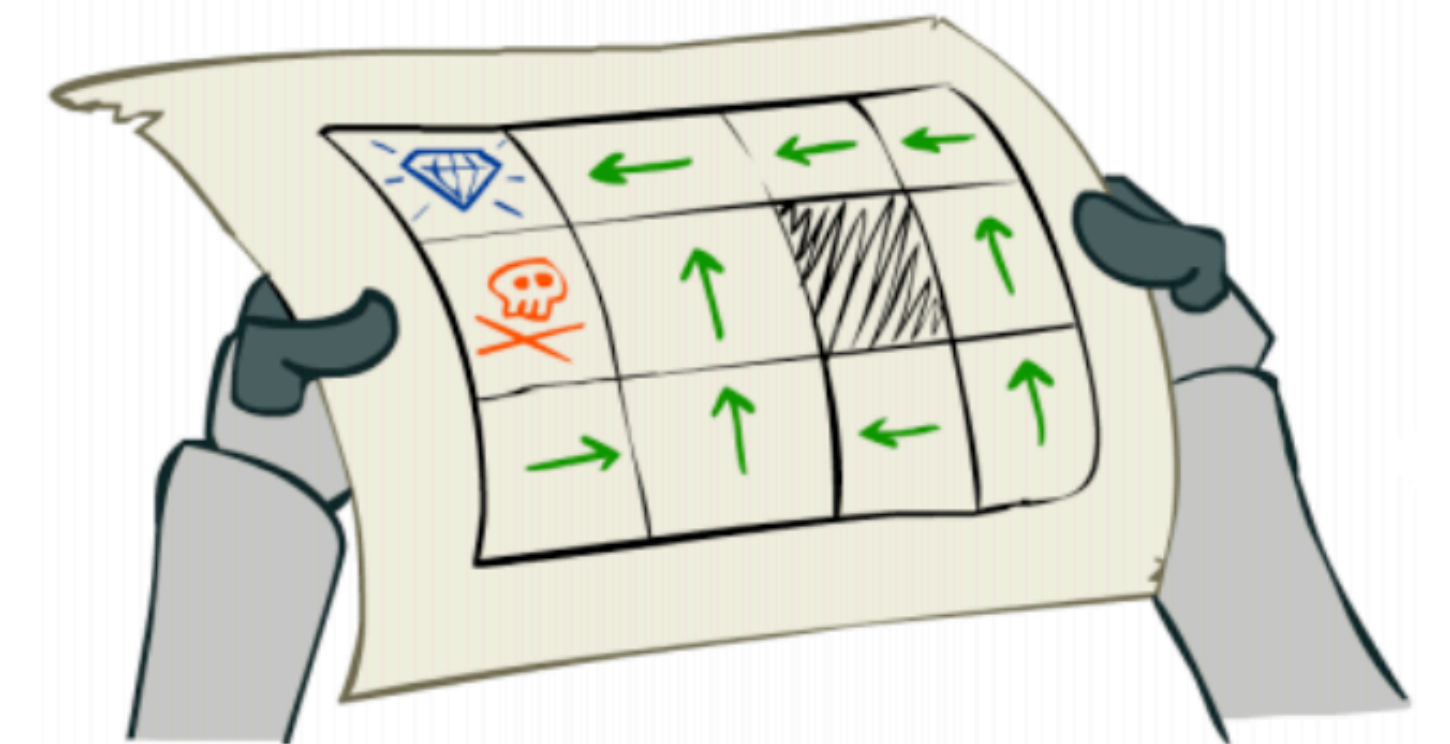
- **while** true:
  - ⊙ **for** each  $s \in S$  :
    - \*  $V[s] = R[s, \pi[s]] + \sum_{s' \in S} T[s, \pi[s], s'] \cdot \gamma \cdot V[s']$
  - ⊙ **exit** if convergence (small change from previous and actual  $V[s] \forall s$ )

## Policy update

- **for** each  $s \in S$  :
  - \*  $\pi^*[s] = \arg \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V[s']$
- **if** the policy did not change after the last loop, return it; else go back to **policy evaluation**

# Two Reinforcement Learning Ways

- What we have seen so far was **model based reinforcement learning**, use Value Iteration and Policy Iteration algorithms for finding an optimal policy
- What if you **don't have full knowledge of the environment** and you want to discover the optimal policy anyways. You **don't know T and R**, at least not completely
- Another reinforcement learning class of problems is called **model free reinforcement learning**
- A robot discovering an *unknown* environment and learning what to do, from the *unknown* feedback it gets from it
- That's when **Q-learning** comes in



# Q-Learning

- Remember value function:

**Optimal  
value function**

$$V^*(s) = \max_{a \in A} R[s, a] + \sum_{s' \in S} T[s, a, s'] \cdot \gamma \cdot V^*(s')$$

- Let  $Q$  be a function mapping a **state** and an **action** to the quality of that pair, it satisfies the Bellman equation as:

$$Q^*(s, a) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \middle| s_0 = s, a_0 = a \right] = \mathbb{E} \left[ r + \gamma \max_{a' \in A} Q^*(s', a') \middle| s_0 = s, a_0 = a \right]$$

**Optimal  
policy**

$$\pi^*(s) = \arg \max_{a \in A} Q^*(s, a)$$



# Q-Learning Algorithm

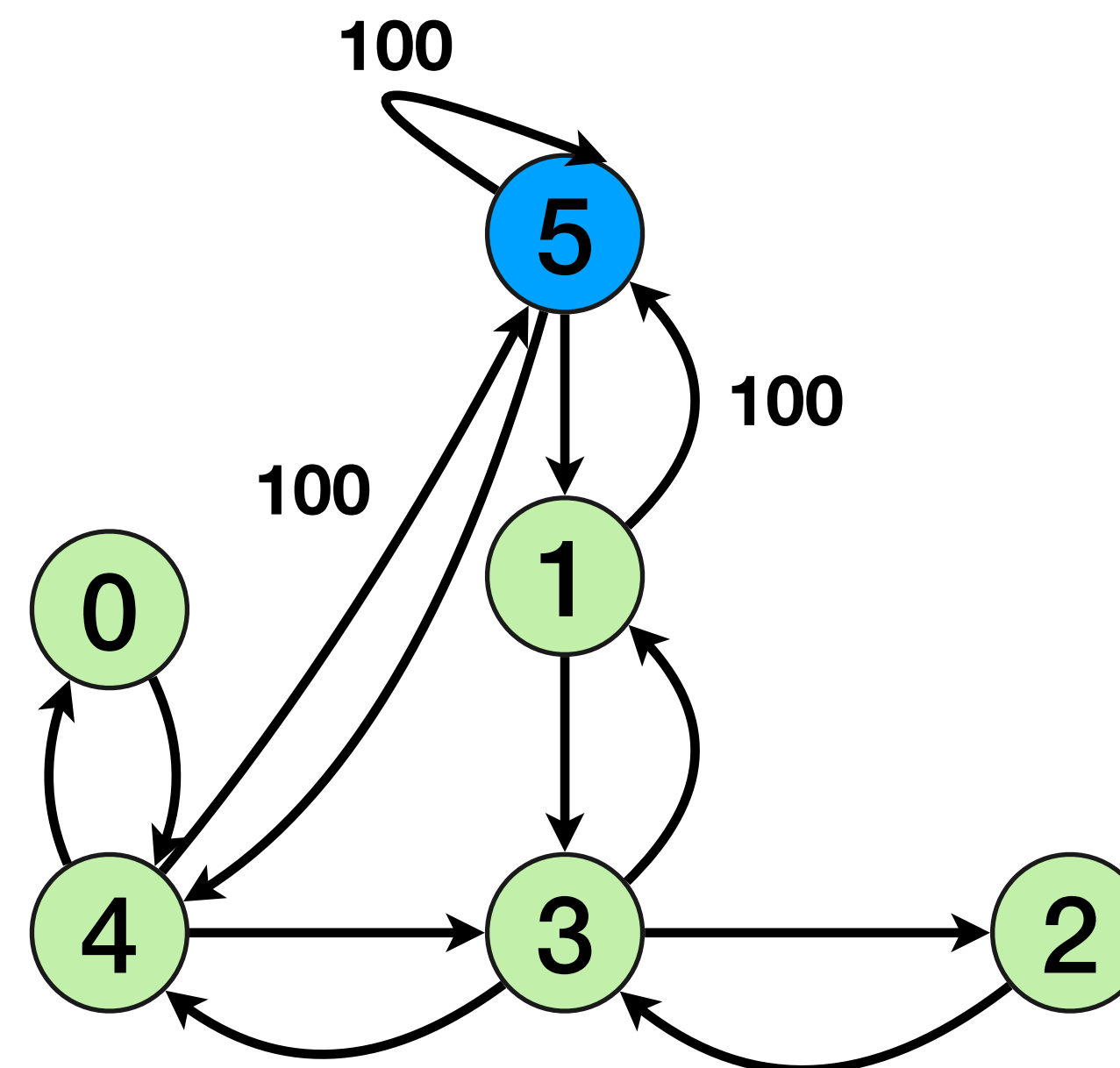
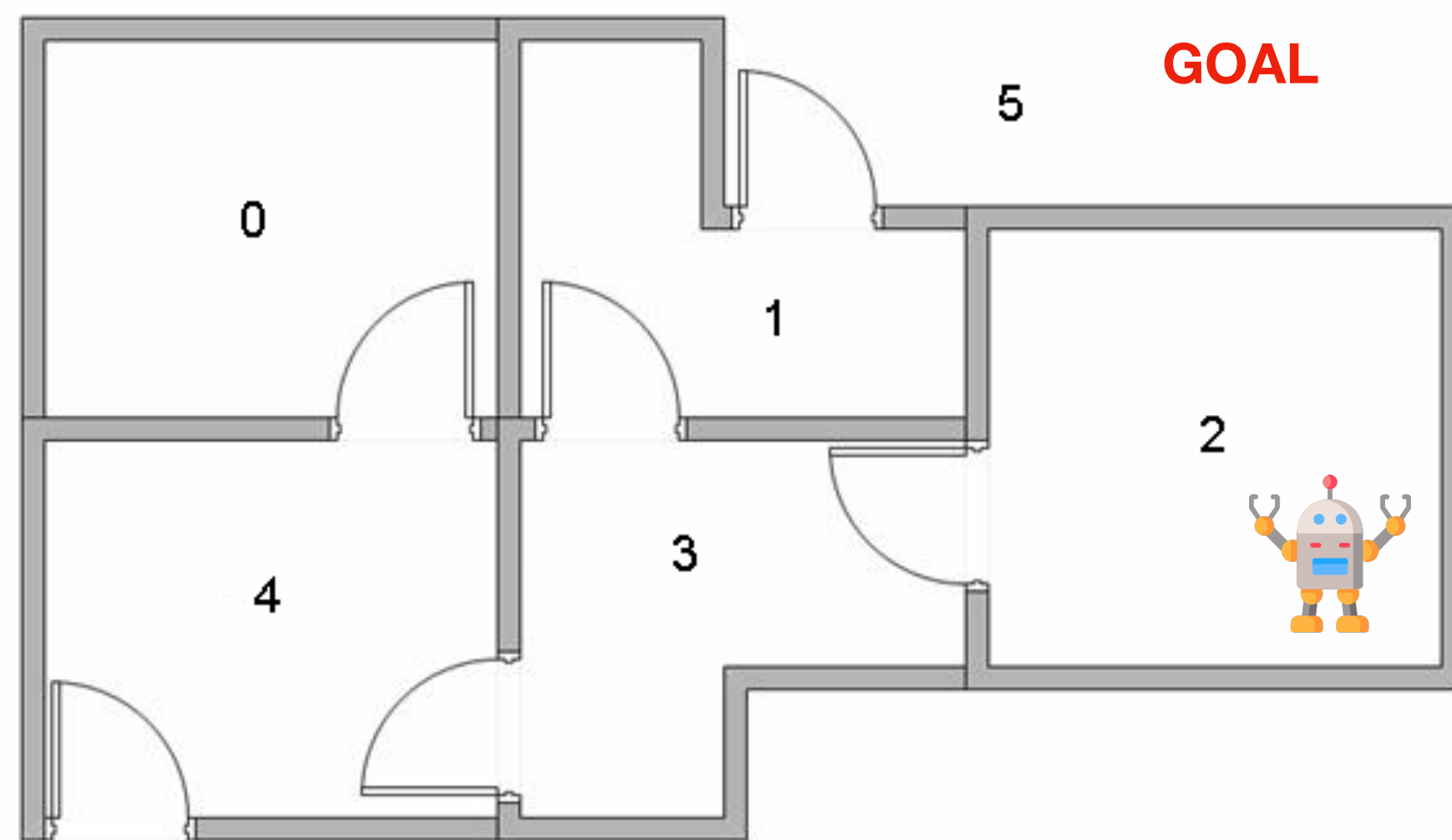
- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve, until  $s$  is end, as:
    - \* choose action *initially random* with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged

**Optimal policy**  $\pi^*(s) = \arg \max_{a \in A} Q(s, a)$

- success depends on *exploration* and experience replay

# Q-Learning Example

- There are 5 rooms (0 to 4) in a building connected by doors. Doors 1 and 4 lead into the building from the outside. Let us teach to an agent the **best policy to go out**.



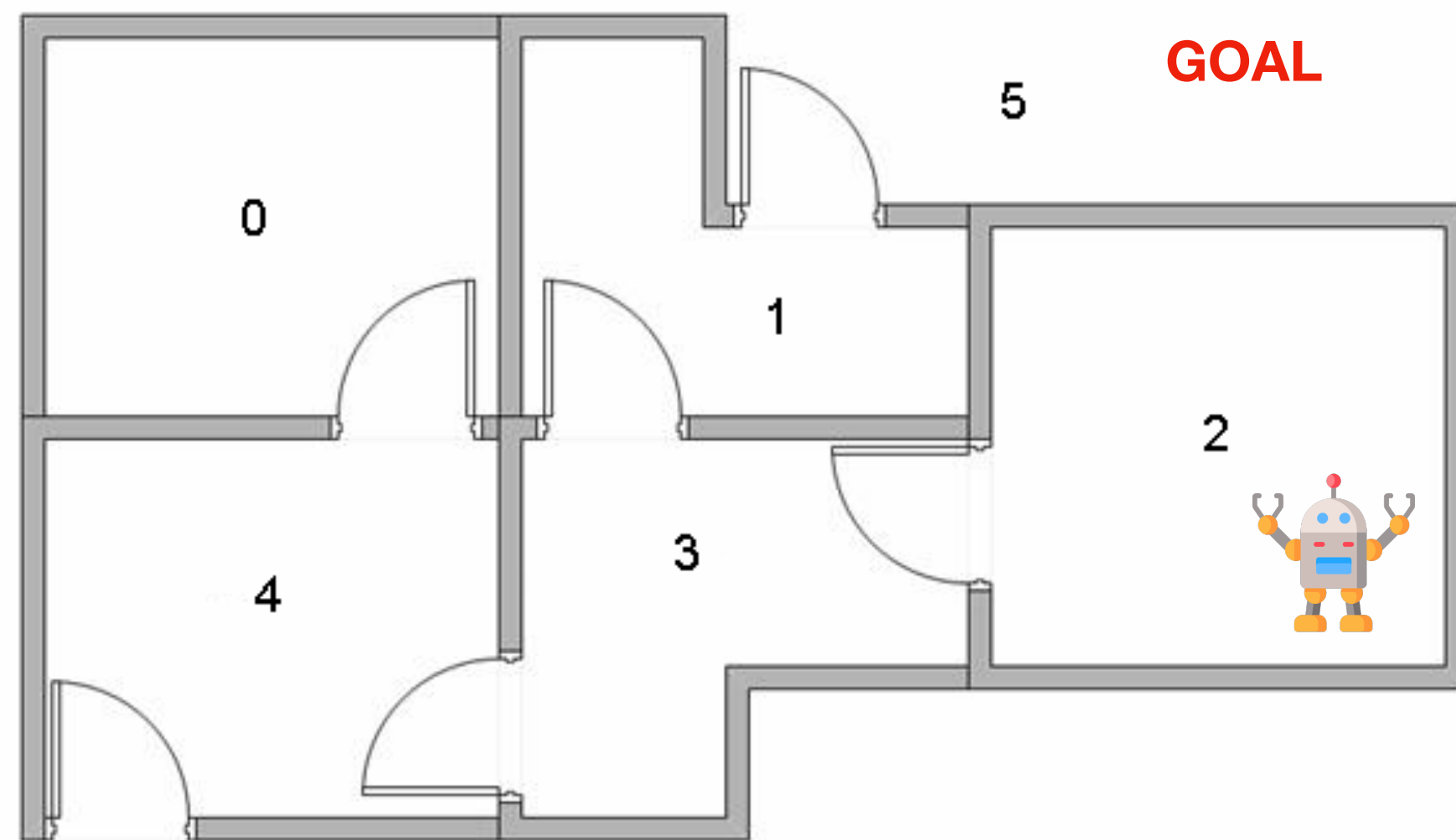
Rewards map

State	Action					
	0	1	2	3	4	5
0	-1	-1	-1	-1	0	-1
1	-1	-1	-1	0	-1	100
2	-1	-1	-1	0	-1	-1
3	-1	0	0	-1	0	-1
4	0	-1	-1	0	-1	100
5	-1	0	-1	-1	0	100

- R is unknown to the agent, it just perceives those feedbacks online. -1 represent null rewards.

# Q-Learning Example

- Exploit a Q table as representing the memory of what the agent has learned through experience. The **rows** of matrix Q represent **the current state** of the agent, and the **columns** represent the **possible actions** leading to the next state



$$R = \begin{array}{c} \text{State} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array} \begin{array}{c} \text{Action} \\ \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \end{array} \begin{bmatrix} -1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & 100 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -1 & 100 \\ -1 & 0 & -1 & -1 & 0 & 100 \end{bmatrix}$$

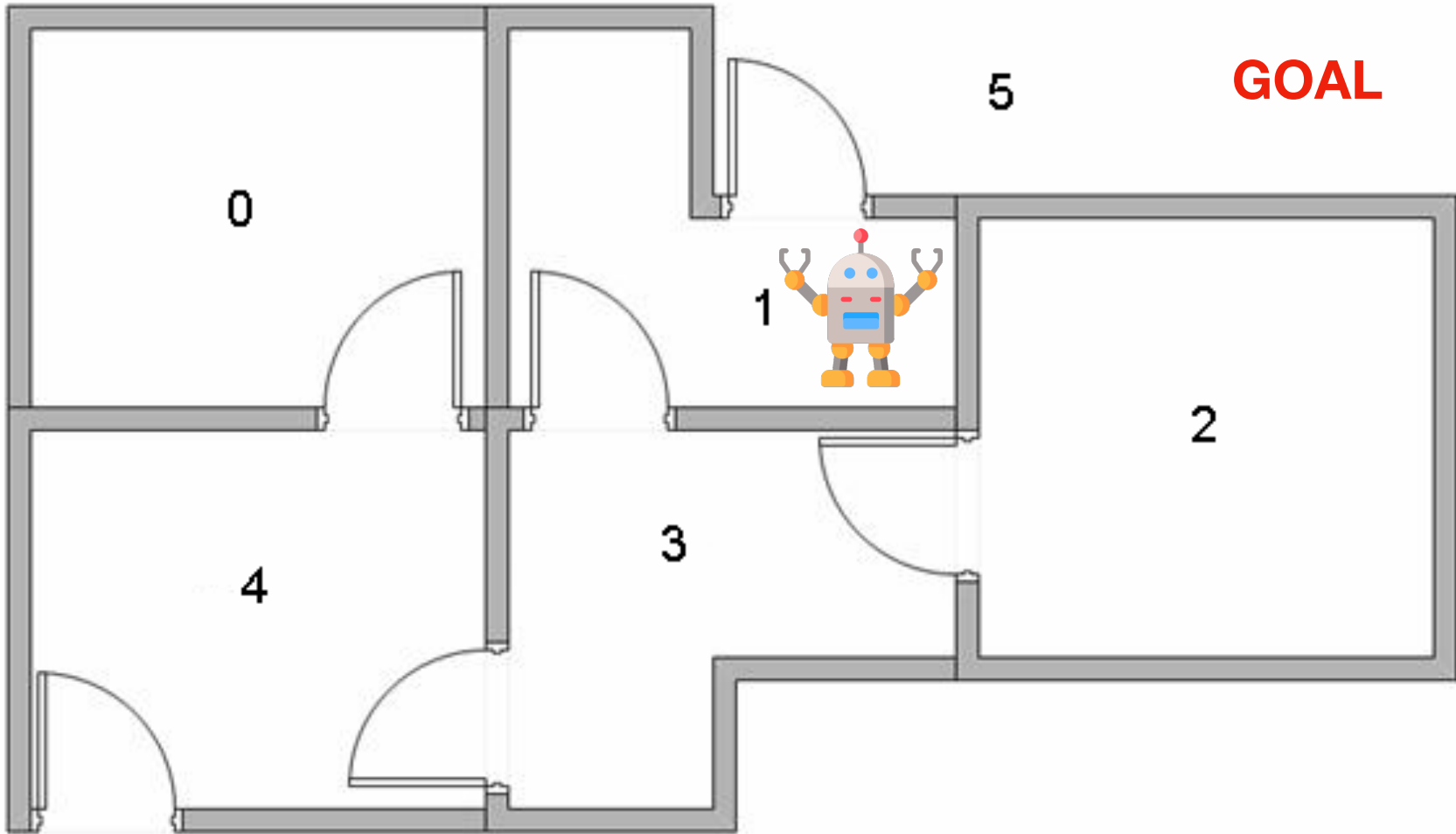
$$Q = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We'll start by setting the value of the learning parameter  $\gamma = 0.8, \alpha = 1$



# Q-Learning Example

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- while true:
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    - \* choose action *initially random* with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



		Action					
		0	1	2	3	4	5
$Q =$	0	0	0	0	0	0	0
	1	0	0	0	0	0	0
	2	0	0	0	0	0	0
	3	0	0	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	0

		Action					
		0	1	2	3	4	5
$R =$	0	-1	-1	-1	-1	0	-1
	1	-1	-1	-1	0	-1	100
	2	-1	-1	-1	0	-1	-1
	3	-1	0	0	-1	0	-1
	4	0	-1	-1	0	-1	100
	5	-1	0	-1	-1	0	100

# Q-Learning Example

- While  
iter 1  
(epoch)
- state room 1 (chosen randomly)

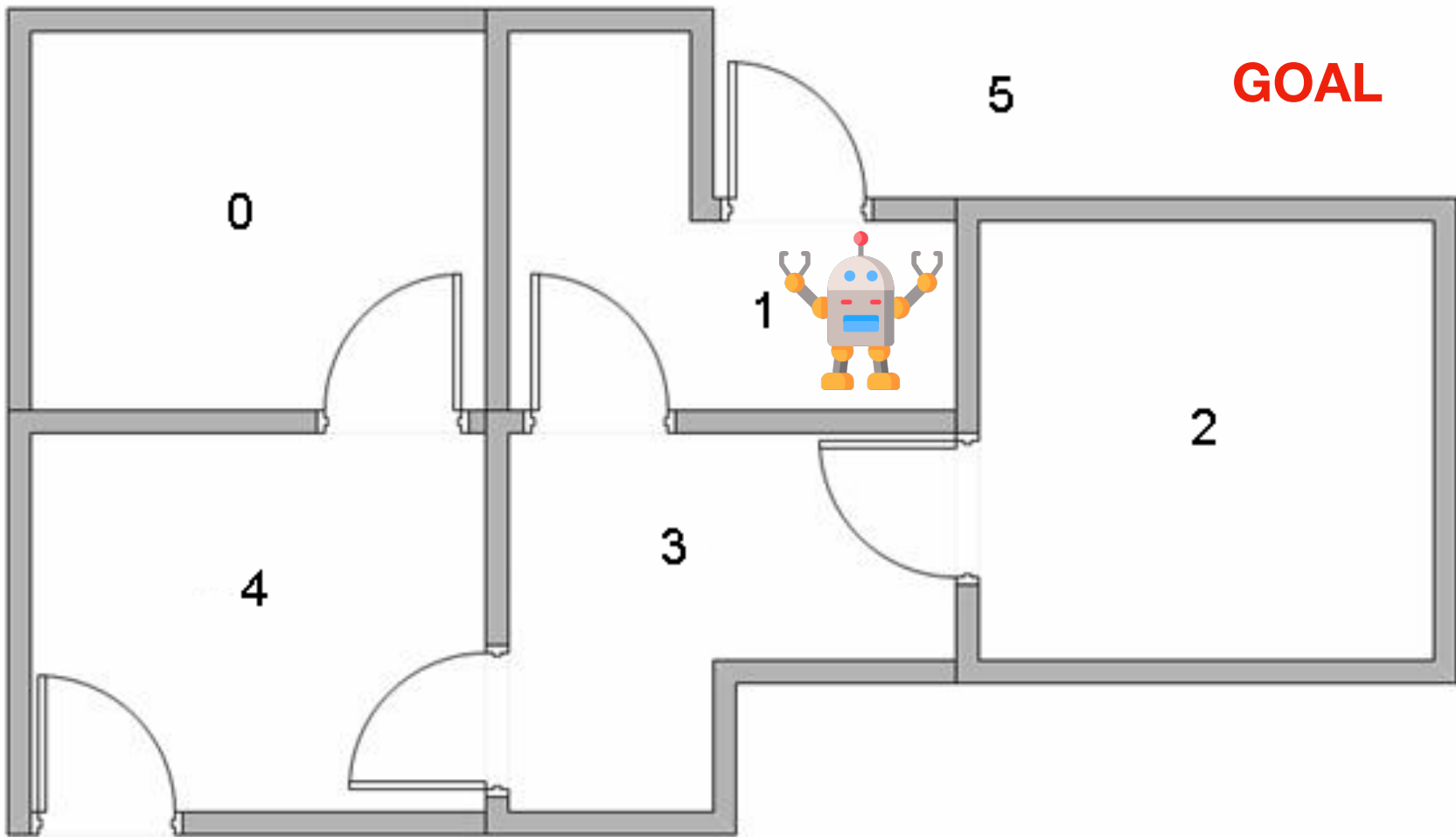
- choose action randomly from go to room {5, 3} -> 5

- received reward 100

-  $Q[1,5] = R[1,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4], Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$

-  $s = 5$

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



		Action					
		0	1	2	3	4	5
Q =	0	0	0	0	0	0	0
	1	0	0	0	0	0	100
	2	0	0	0	0	0	0
	3	0	0	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	0

		Action					
		0	1	2	3	4	5
R =	0	-1	-1	-1	-1	0	-1
	1	-1	-1	-1	0	-1	100
	2	-1	-1	-1	0	-1	-1
	3	-1	0	0	-1	0	-1
	4	0	-1	-1	0	-1	100
	5	-1	0	-1	-1	0	100

# Q-Learning Example

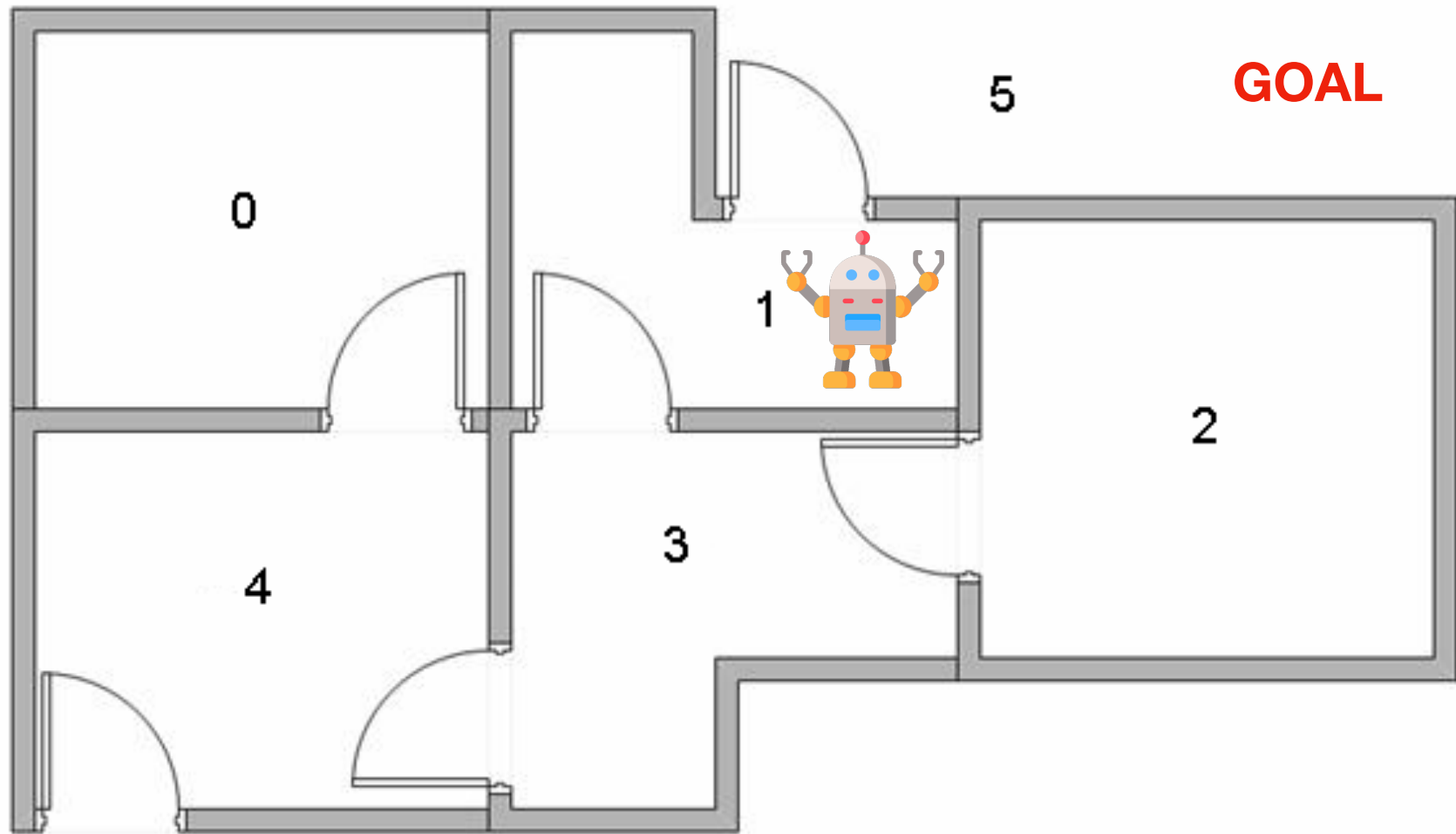
- While  
iter 1  
(epoch)
- state room 1 (chosen randomly)

  - choose action randomly from go to room {5, 3} -> 5
  - received reward 100
  - $Q[1,5] = R[1,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4], Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$
  - $s = 5$

state room 5

  - choose action randomly from go to room {1, 4, 5} -> 5
  - received reward 100
  - $Q[5,5] = R[5,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4] + Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$
  - $s = 1$

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



		Action					
		0	1	2	3	4	5
Q =	0	0	0	0	0	0	0
	1	0	0	0	0	0	100
	2	0	0	0	0	0	0
	3	0	0	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	100

		Action					
		0	1	2	3	4	5
R =	0	-1	-1	-1	-1	0	-1
	1	-1	-1	-1	0	-1	100
	2	-1	-1	-1	0	-1	-1
	3	-1	0	0	-1	0	-1
	4	0	-1	-1	0	-1	100
	5	-1	0	-1	-1	0	100

# Q-Learning Example

- While  
iter 1  
(epoch)
- state room 1 (chosen randomly)

- choose action randomly from go to room {5, 3} -> 5

- received reward 100

$Q[1,5] = R[1,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4], Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$

$s = 5$
- state room 5

- choose action randomly from go to room {1, 4, 5} -> 5

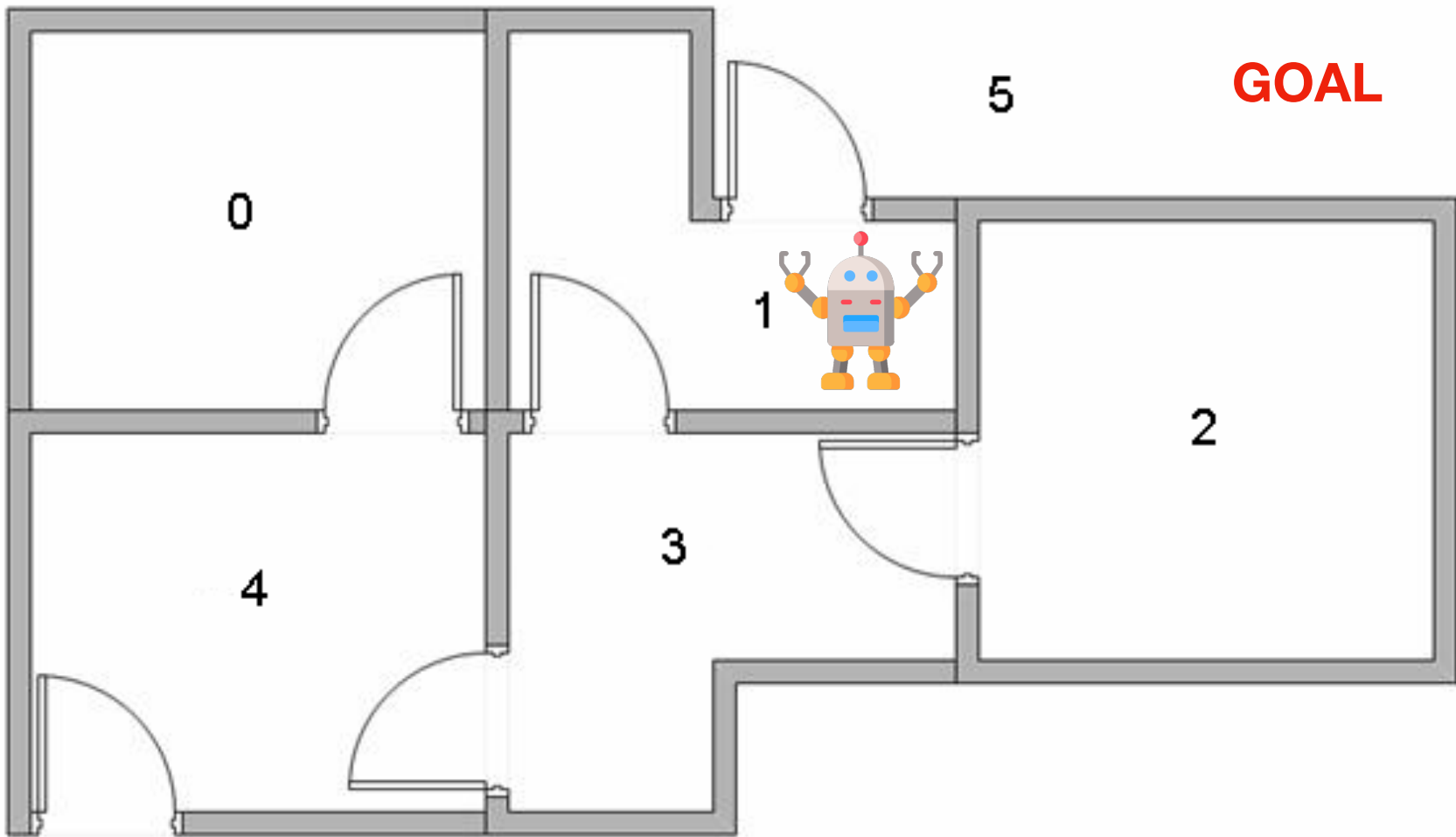
- received reward 100

$Q[5,5] = R[5,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4] + Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$

$s = 1$

State 5 is a “done state” thus a new epoch should start

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



	Action					
	0	1	2	3	4	5
State 0	0	0	0	0	0	0
State 1	0	0	0	0	0	100
State 2	0	0	0	0	0	0
State 3	0	0	0	0	0	0
State 4	0	0	0	0	0	0
State 5	0	0	0	0	0	100

	Action					
	0	1	2	3	4	5
State 0	-1	-1	-1	-1	0	-1
State 1	-1	-1	-1	0	-1	100
State 2	-1	-1	-1	0	-1	-1
State 3	-1	0	0	-1	0	-1
State 4	0	-1	-1	0	-1	100
State 5	-1	0	-1	-1	0	100



# Q-Learning Example

**While iter 1 (epoch)** state room 1 (chosen randomly)

- choose action randomly from go to room {5, 3} -> 5
- received reward 100
- $Q[1,5] = R[1,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4], Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$
- $s = 5$

state room 5

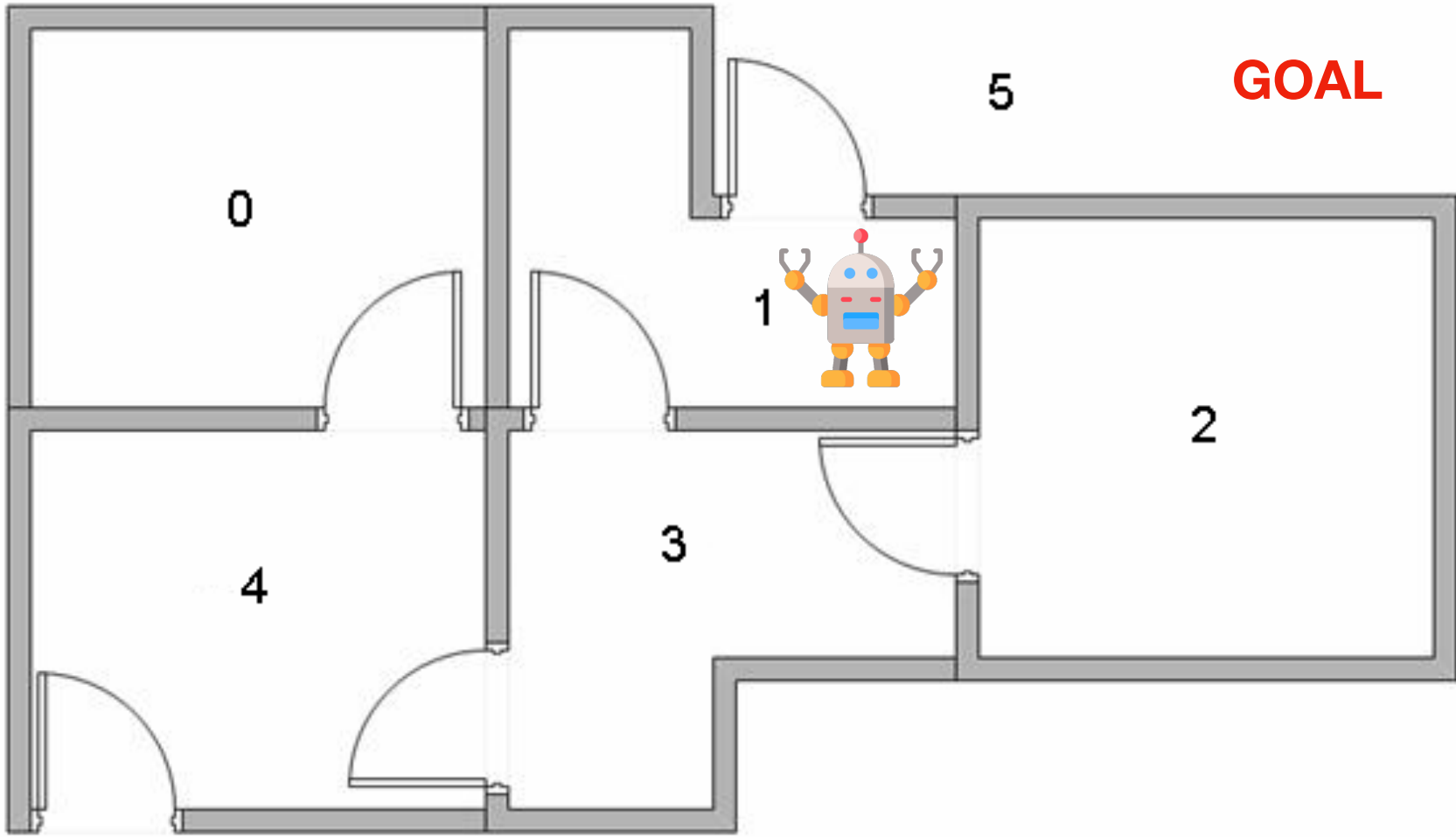
- choose action randomly from go to room {1, 4, 5} -> 5
- received reward 100
- $Q[5,5] = R[5,5] + 0.8 \cdot \max\{Q[5,1], Q[5,4] + Q[5,5]\} = 100 + 0.8 \cdot 0 = 100$
- $s = 1$

State 5 is a “done state” thus a new epoch should start

**While iter 2 (epoch)** state room 3 (chosen randomly)

- choose action randomly from go to room {1, 2, 4} -> 1
- received reward 0
- $Q[3,1] = R[3,1] + 0.8 \cdot \max\{Q[1,3], Q[1,5]\} = 0 + 0.8 \cdot 100 = 80$
- $s = 1$

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



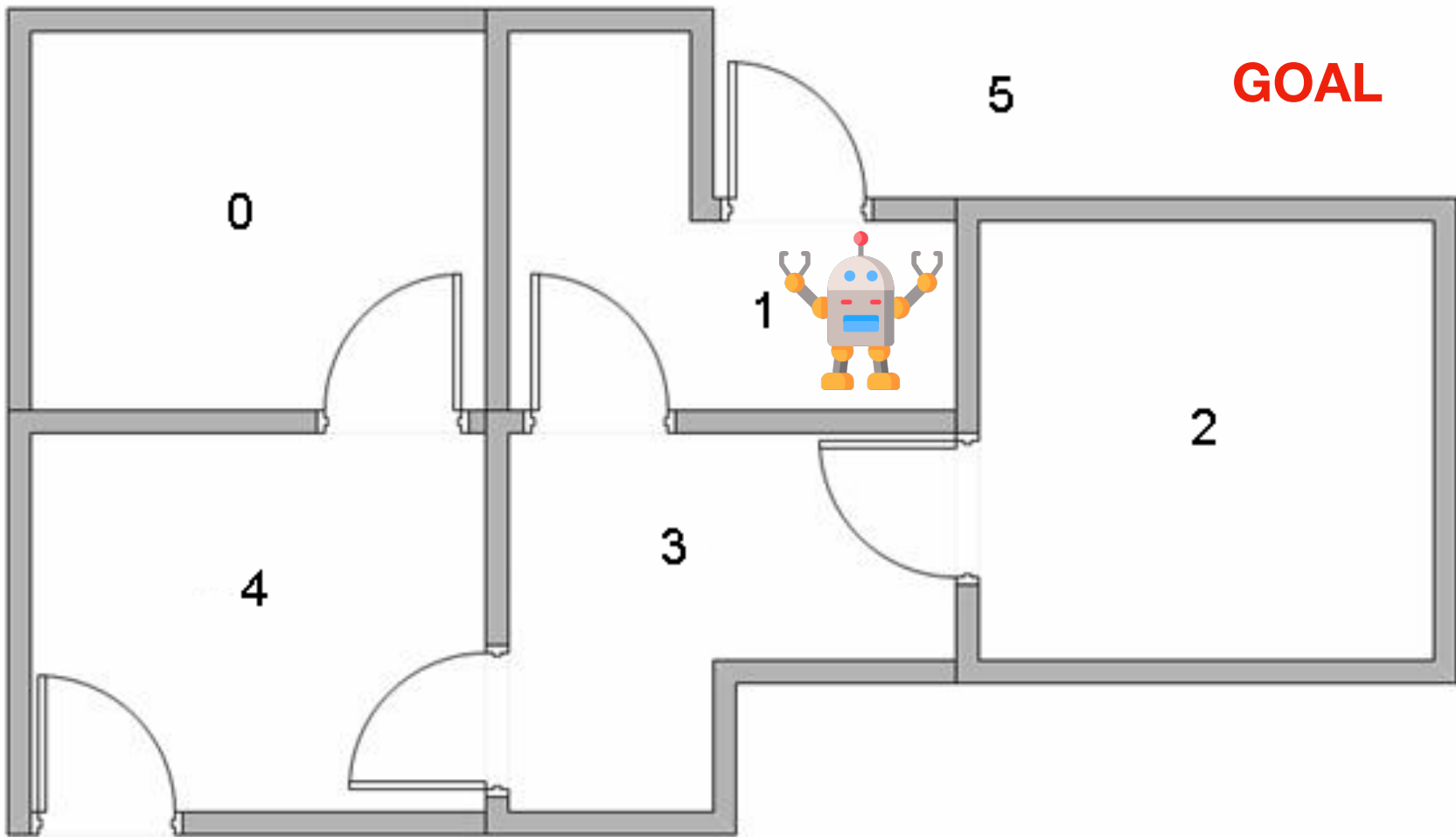
		Action					
		0	1	2	3	4	5
Q =	0	0	0	0	0	0	0
	1	0	0	0	0	0	100
	2	0	0	0	0	0	0
	3	0	80	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	100

		Action					
		0	1	2	3	4	5
R =	0	-1	-1	-1	-1	0	-1
	1	-1	-1	-1	0	-1	100
	2	-1	-1	-1	0	-1	-1
	3	-1	0	0	-1	0	-1
	4	0	-1	-1	0	-1	100
	5	-1	0	-1	-1	0	100

# Q-Learning Example

**While** state room 1  
**iter 2** - choose action randomly from go to room {5, 3} -> 3  
**(epoch)** - received reward 0  
-  $Q[1,3] = R[1,3] + 0.8 \cdot \max\{Q[3,1], Q[3,2], Q[3,4]\} = 0 + 0.8 \cdot 80 = 64$   
-  $s = 3$

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



		Action					
		0	1	2	3	4	5
$Q =$	0	0	0	0	0	0	0
	1	0	0	0	64	0	100
	2	0	0	0	0	0	0
	3	0	80	0	0	0	0
	4	0	0	0	0	0	0
	5	0	0	0	0	0	100

		Action					
		0	1	2	3	4	5
$R =$	0	-1	-1	-1	-1	0	-1
	1	-1	-1	-1	0	-1	100
	2	-1	-1	-1	0	-1	-1
	3	-1	0	0	-1	0	-1
	4	0	-1	-1	0	-1	100
	5	-1	0	-1	-1	0	100

# Q-Learning Example

**While iter 2 (epoch)**

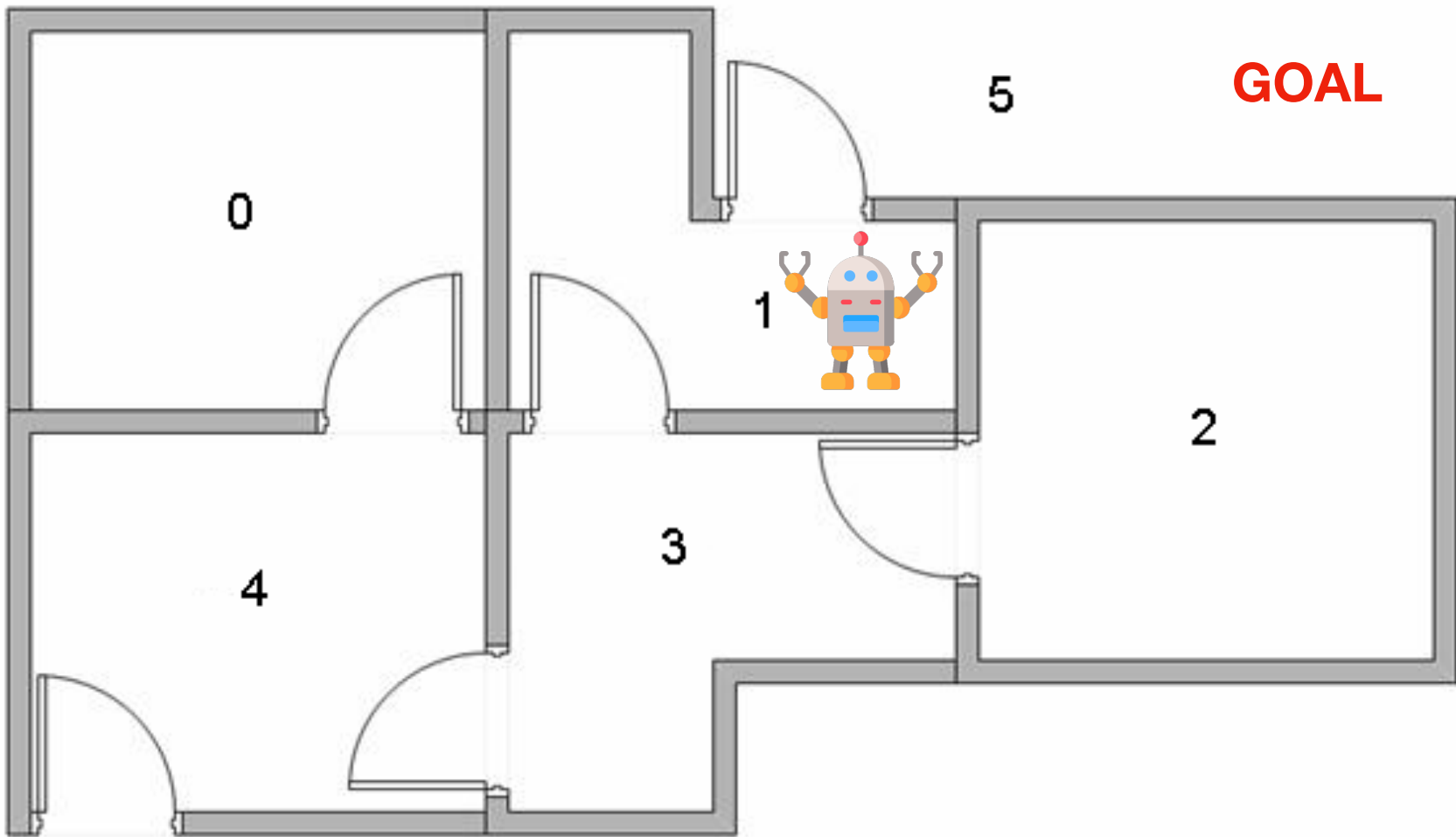
**state room 1**

- choose action randomly from go to room {5, 3} -> 3
- received reward 0
- $Q[1,3] = R[1,3] + 0.8 \cdot \max\{Q[3,1], Q[3,2], Q[3,4]\} = 0 + 0.8 \cdot 80 = 64$
- $s = 3$

**state room 3**

- choose action randomly from go to room {1, 2, 4} -> 2
- received reward 0
- $Q[3,2] = R[3,2] + 0.8 \cdot \max\{Q[2,3]\} = 0 + 0.8 \cdot 0 = 0$
- $s = 2$

- set  $Q[s, a] = 0 \forall s \in S, a \in A$
- while true:
  - ▶ from state  $s$  (random at the start of the epoch) let the process evolve until  $s$  is end
    - \* choose action **initially random** with decaying probability, then  $a = \arg \max_{a' \in A} Q(s, a')$
    - \* take action and observe  $s', r$
    - \*  $Q[s, a] = (1 - \alpha)Q[s, a] + \alpha(r + \gamma \max_{a' \in A} Q[s', a'])$
    - \* set  $s = s'$
  - ▶ exit when converged



	Action					
	0	1	2	3	4	5
State 0	-1	-1	-1	-1	0	-1
1	-1	-1	-1	0	-1	100
2	-1	-1	-1	0	-1	-1
3	-1	0	0	-1	0	-1
4	0	-1	-1	0	-1	100
5	-1	0	-1	-1	0	100

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	64	0	100
2	0	0	0	0	0	0
3	0	80	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	100

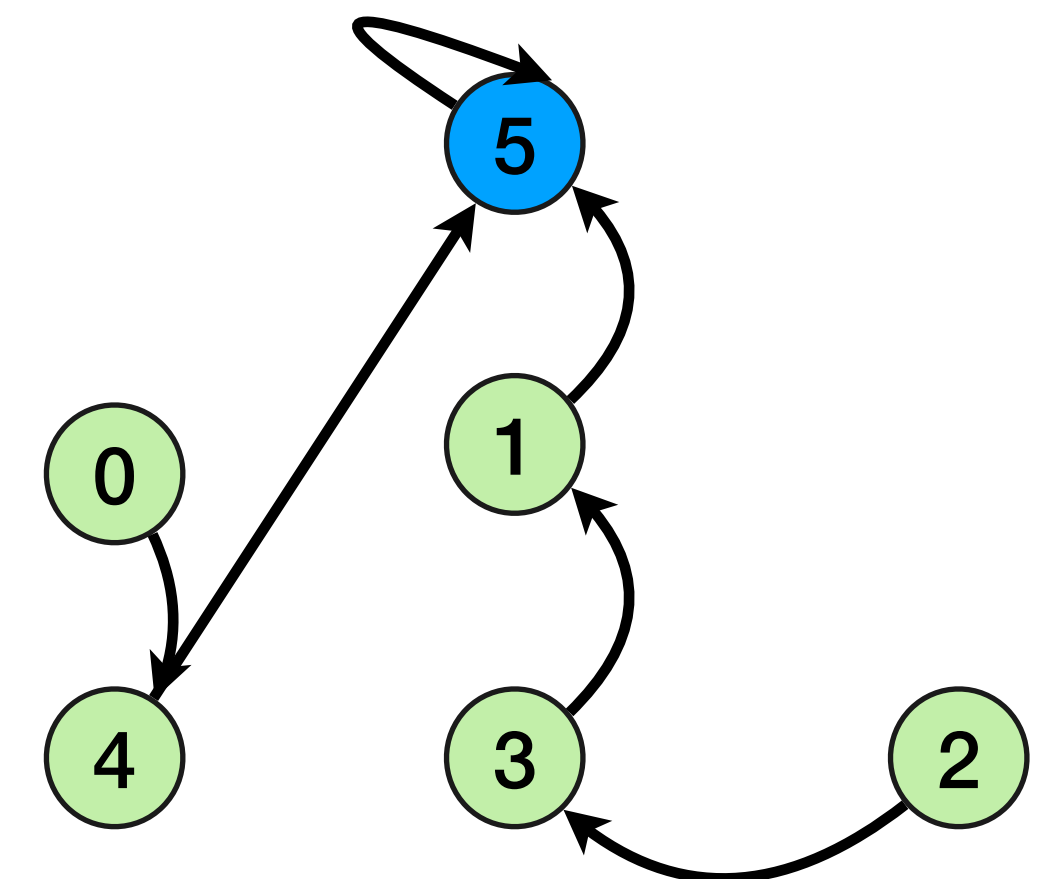
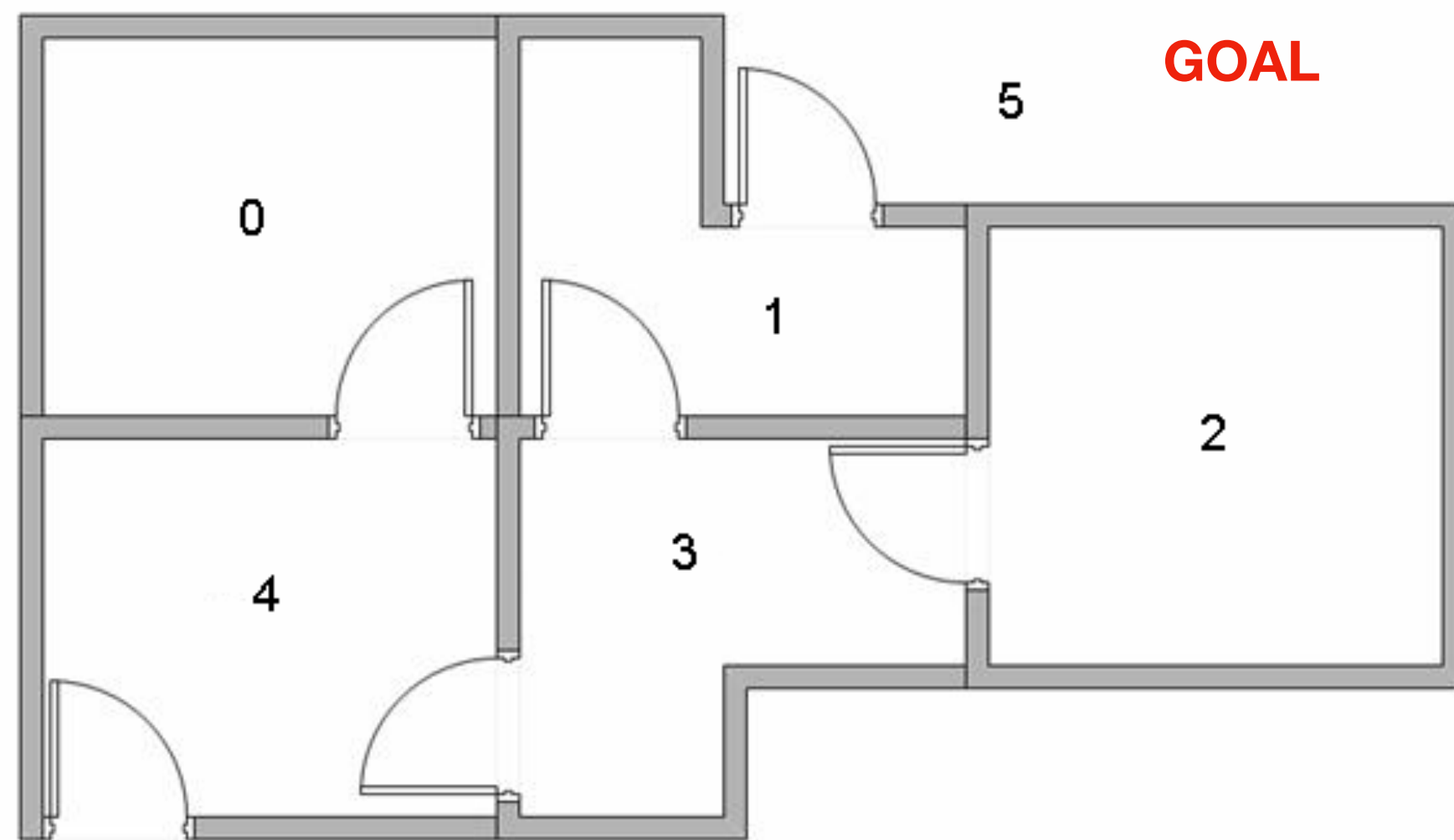
# Q-Learning Example

- Once Q converged, normalise it for convenience and find the optimal policy

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 320 & 0 & 500 \\ 0 & 0 & 0 & 320 & 0 & 0 \\ 0 & 400 & 256 & 0 & 400 & 0 \\ 320 & 0 & 0 & 320 & 0 & 500 \\ 0 & 400 & 0 & 0 & 400 & 500 \end{bmatrix} \end{matrix} \xrightarrow{\text{Normalize}} Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 64 & 0 & 100 \\ 0 & 0 & 0 & 64 & 0 & 0 \\ 0 & 80 & 51 & 0 & 80 & 0 \\ 64 & 0 & 0 & 64 & 0 & 100 \\ 0 & 80 & 0 & 0 & 80 & 100 \end{bmatrix} \end{matrix} \xrightarrow{\text{Optimal policy}}$$

$$\pi^*(s) = \arg \max_{a \in A} Q(s, a)$$

$$\pi^* = [4, 5, 3, 1, 5, 5]$$



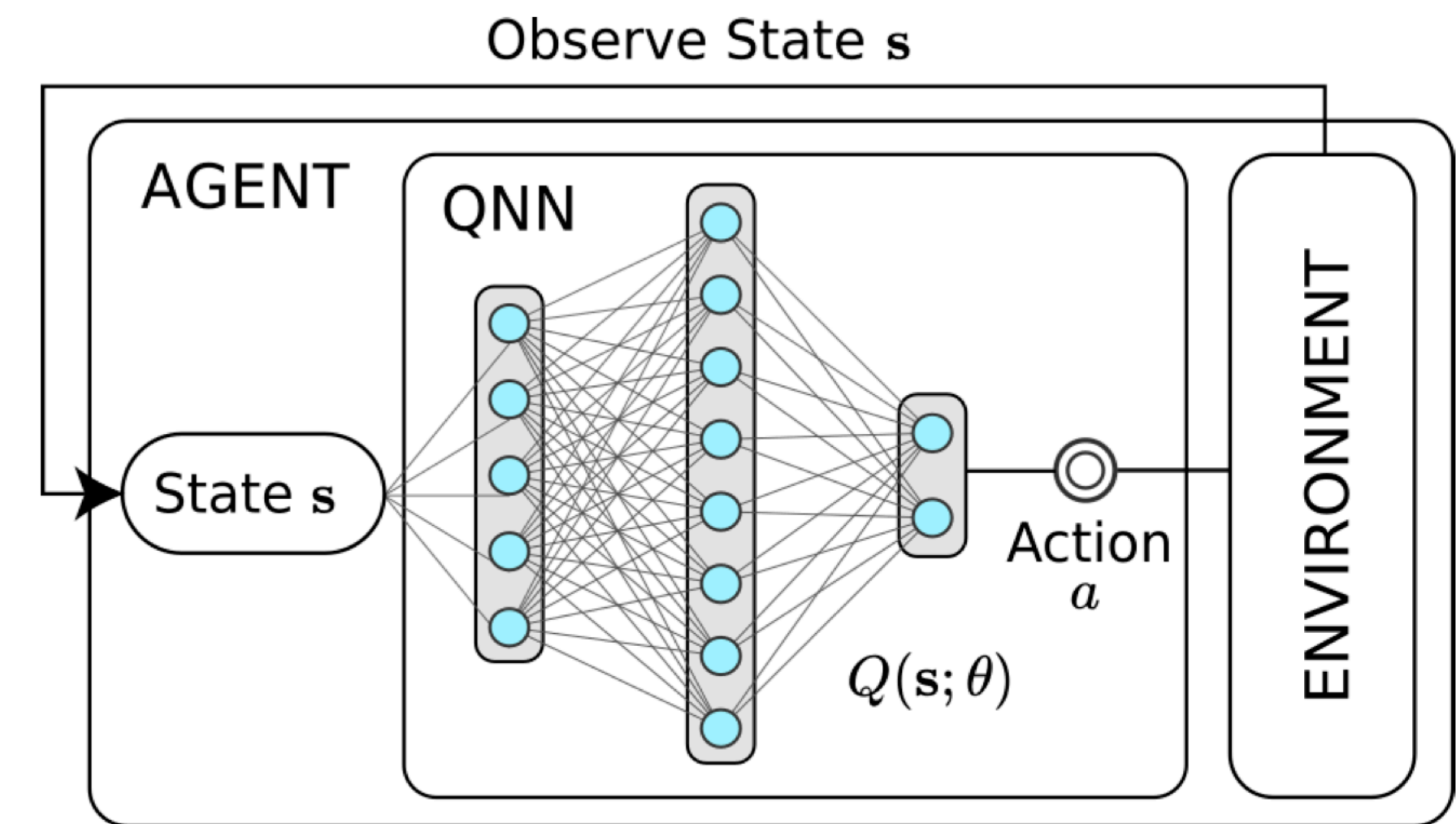


# Deep Reinforcement Learning

- What's the problem with Q-learning?
- We need to keep in memory a table of  $|S| \cdot |A|$  entries. State space is generally **HUGE!**
- If you need a dense state space, exploit **Deep Reinforcement Learning (DRL)**
- A neural network (a function approximator) can predict Q values (one for each action) given a state as input. Just learn to minimise this loss function

$$L(\theta) = \left( \underbrace{r + \gamma \max_{a'} Q(\hat{s}; \theta)[a']}_{\text{target}} - \underbrace{Q(s; \theta)[a]}_{\text{prediction}} \right)^2$$

- Read more about it, or ask: Mnih, Volodymyr, et al. "Playing atari with deep reinforcement learning."



# Go Code Some RL Algorithm

- Go to <https://github.com/matteoprata/markov-processes-class>
- To get these slides and a Jupyter Notebook with **Value Iteration**, **Policy Iteration** and **Q-Learning** algorithms implemented on the two examples showed in class