of the present book), the problem still stands and reduces one's enthusiasm for the theoretical setup of market models, if not for the practical one.

In this chapter, we will derive the LFM dynamics under different measures by resorting to the change-of-numeraire technique. We will show how caps are priced in agreement with Black's cap formula, and explain how swaptions can be priced through a Monte Carlo method in general. Analytical approximations leading to swaption-pricing formulas are also presented, as well as closed-form formulas for terminal correlations based on similar approximations.

We will suggest parametric forms for the instantaneous covariance structure (volatilities and correlations) in the LFM. Part of the parameters in this structure can be obtained directly from market-quoted cap volatilities, whereas other parameters can be obtained by calibrating the model to swaption prices. The calibration to swaption prices can be made computationally efficient through the analytical approximations mentioned above.

We will derive results and approximations connecting semi-annual caplet volatilities to volatilities of swaptions whose underlying swap is one-year long. We will also show how one can obtain forward rates over non-standard periods from standard forward rates either through drift interpolation or via a bridging technique.

We will then introduce the LSM and show how swaptions are priced in agreement with Black's swaptions formula, although Black's formula for caps does not hold under this model.

We will finally consider the "smile problem" for the cap market, and introduce some possible extensions of the basic LFM that are analytically tractable and allow for a volatility smile.

6.2 Market Models: a Guided Tour

> *"Fischer had many remarkable qualities that became apparent almost as soon as you met him. If you knew about the Black-Scholes breakthrough that allowed the somehow miraculous determination of the fair price of an option independent of what you thought about the stock, and appreciated what a giant leap forward that was in the world of financial economics, then you expected him to be deep and brilliant. But what struck you even more forcefully was how meticulous was his devotion to clarity and simplicity in presentation and speaking. [...]".* Emanuel Derman, December 3, 1996.
>
> Induction Speech for Fischer Black (1938-1995).

Before market models were introduced, short-rate models used to be the main choice for pricing and hedging interest-rate derivatives. Short-rate models are still chosen for many applications and are based on modeling the instantaneous spot interest rate ("short rate") via a (possibly multi-dimensional) diffusion process. This diffusion process characterizes the evolution of the complete yield curve in time. We have seen examples of such models in Chapters 3 and 4.

To fix ideas, let us consider the time-0 price of a T_2-maturity caplet resetting at time T_1 $(0 < T_1 < T_2)$ with strike X and a notional amount of 1. Caplets and caps have been defined in Chapter 1 and will be described more generally in Section 6.4. Let τ denote the year fraction between T_1 and T_2. Such a contract pays out the amount

$$\tau(L(T_1, T_2) - X)^+$$

at time T_2, where in general $L(u, s)$ is the LIBOR rate at time u for maturity s.

Again to fix ideas, let us choose a specific short-rate model and assume we are using the shifted two-factor Vasicek model G2++ given in (4.4). The parameters of this two-factor Gaussian additive short-rate model are here denoted by $\theta = (a, \sigma, b, \eta, \rho)$. Then the short rate r_t is obtained as the sum of two linear diffusion processes x_t and y_t, plus a deterministic shift φ that is used for fitting the initially observed yield curve at time 0:

$$r_t = x_t + y_t + \varphi(t; \theta)\,.$$

Such model allows for an analytical formula for forward LIBOR rates F,

$$\begin{aligned} F(t; T_1, T_2) &= F(t; T_1, T_2; x_t, y_t, \theta), \\ L(T_1, T_2) &= F(T_1; T_1, T_2; x_{T_1}, y_{T_1}, \theta)\ . \end{aligned}$$

At this point one can try and price a caplet. To this end, one can compute the risk-neutral expectation of the payoff discounted with respect to the bank account numeraire $\exp\left(\int_0^{T_2} r_s ds\right)$ so that one has

$$E\left[\exp\left(-\int_0^{T_2} r_s ds\right)\tau(F(T_1; T_1, T_2, x_{T_1}, y_{T_1}, \theta) - X)^+\right]\,.$$

This too turns out to be feasible, and leads to a function

$$U_C(0, T_1, T_2, X, \theta).$$

On the other hand, the market has been pricing caplets (actually caps) with Black's formula for years. One possible derivation of Black's formula for caplets is based on the following approximation. When pricing the discounted payoff

$$E\left[\exp\left(-\int_0^{T_2} r_s ds\right)\tau(L(T_1, T_2) - X)^+\ \right] = \cdots$$

one first assumes the discount factor $\exp\left(-\int_0^{T_2} r_s ds\right)$ to be deterministic and identifies it with the corresponding bond price $P(0,T_2)$. Then one factors out the discount factor to obtain:

$$\cdots \approx P(0,T_2)\tau E\left[(L(T_1,T_2)-X)^+\right] = P(0,T_2)\tau E\left[(F(T_1;T_1,T_2)-X)^+\right] \; .$$

Now, inconsistently with the previous approximation, one goes back to assuming rates to be stochastic, and models the forward LIBOR rate $F(t;T_1,T_2)$ as in the classical Black and Scholes option pricing setup, i.e as a (driftless) geometric Brownian motion:

$$dF(t;T_1,T_2) = vF(t;T_1,T_2)dW_t \; , \tag{6.1}$$

where v is the instantaneous volatility, assumed here to be constant for simplicity, and W is a standard Brownian motion under the risk-neutral measure Q.

Then the expectation

$$E\left[(F(T_1;T_1,T_2)-X)^+\right]$$

can be viewed simply as a T_1-maturity call-option price with strike X and whose underlying asset has volatility v, in a market with zero risk-free rate.

We therefore obtain:

$$\begin{aligned}\mathbf{Cpl}(0,T_1,T_2,X) &:= P(0,T_2)\tau E(F(T_1;T_1,T_2)-X)^+ \\ &= P(0,T_2)\tau[F(0;T_1,T_2)\Phi(d_1(X,F(0;T_1,T_2),v\sqrt{T_1})) \\ &\qquad -X\Phi(d_2(X,F(0;T_1,T_2),v\sqrt{T_1}))],\end{aligned}$$

$$d_1(X,F,u) = \frac{\ln(F/X)+u^2/2}{u}, \quad d_2(X,F,u) = \frac{\ln(F/X)-u^2/2}{u},$$

where Φ is the standard Gaussian cumulative distribution function.

From the way we just introduced it, this formula seems to be partly based on inconsistencies. However, within the change-of-numeraire setup, the formula can be given full mathematical rigor as follows. Denote by Q^2 the measure associated with the T_2-bond-price numeraire $P(\cdot,T_2)$ (T_2-forward measure) and by E^2 the corresponding expectation. Then, by the change-of-numeraire approach, we can switch from the bank-account numeraire $B(t) = B_0 \exp\left(\int_0^t r_s ds\right)$ associated with the risk-neutral measure $Q = Q^B$ to the bond-price numeraire $P(t,T_2)$ and obtain (set $F_2 = F(\cdot,T_1,T_2)$ for brevity, so that $L(T_1,T_2) = F(T_1;T_1,T_2) = F_2(T_1)$)

$$E\left[\exp\left(-\int_0^{T_2} r_s ds\right)\tau(L(T_1,T_2)-X)^+\right] =$$

$$= E\left[\exp\left(-\int_0^{T_2} r_s ds\right)\tau(F_2(T_1)-X)^+\right] =$$

$$= E^{\boxed{B}}\left[\frac{\boxed{B(0)}}{\boxed{B(T_2)}}\tau(F_2(T_1)-X)^+\right] = E^{\boxed{2}}\left[\frac{\boxed{P(0,T_2)}}{\boxed{P(T_2,T_2)}}\tau(F_2(T_1)-X)^+\right] = \dots$$

where we have used Fact Two on the change of numeraire technique (Section 2.3), obtaining an invariant quantity by changing numeraire in the three boxes. Take out $P(0,T_2)$ and recall that $P(T_2,T_2)=1$. We have

$$E\left[\exp\left(-\int_0^{T_2} r_s ds\right)\tau(L(T_1,T_2)-X)^+\right] = P(0,T_2)\tau E^2\left[(L(T_1,T_2)-X)^+\right].$$

What has just been done, rather than assuming deterministic discount factors, is a change of measure. We have "factored out" the stochastic discount factor and replaced it with the related bond price, but in order to do so we had to change the probability measure under which the expectation is taken. Now the last expectation is no longer taken under the risk-neutral measure but rather under the T_2-forward measure. Since by definition $F(t;T_1,T_2)$ can be written as the price of a tradable asset divided by $P(t,T_2)$, it needs follow a martingale under the measure associated with the numeraire $P(t,T_2)$, i.e. under Q^2 (Fact One on the change of numeraire, again Section 2.3). As we have hinted at in Appendix C, martingale means "driftless" when dealing with diffusion processes. Therefore, the dynamics of $F(t;T_1,T_2)$ under Q^2 is driftless, so that the dynamics

$$dF(t;T_1,T_2) = vF(t;T_1,T_2)dW_t \tag{6.2}$$

is correct under the measure Q^2, where W is a standard Brownian motion under Q^2. Notice that the driftless (lognormal) dynamics above is precisely the dynamics we need in order to recover *exactly* Black's formula, without approximation. *We can say that the choice of the numeraire $P(\cdot,T_2)$ is based on this fact: It makes the dynamics (6.2) of F driftless under the related Q^2 measure, thus replacing rigorously the earlier arbitrary assumption on the F dynamics (6.1) under the risk-neutral measure Q.* Following this rigorous approach we indeed obtain Black's formula, since the process F has the same distribution as in the approximated case above, and hence the expected value has the same value as before. Nonetheless, it can be instructive to derive Black's formula in detail at least once in the book. We do so now. The reader that is not interested in the derivation may skip the derivation and continue with the guided tour.

Begin(detailed derivation of Black's formula for caplets)

I wear black on the outside / 'cause black is how I feel on the inside
The Smiths, 1987

"Blackness coming... this Earth next..."
Superboy from Hyper-time, "Hyper-tension" , 1999, DC Comics

We change slightly the notation to derive a formula for time-varying volatilities. As seen above, by Fact One on the change of numeraire (again Section 2.3) F_2 is a martingale (no drift) under Q^2. Take a geometric Brownian motion

$$dF(t;T_1,T_2) = \sigma_2(t)\ F(t;T_1,T_2)dW(t),$$

where σ_2 is the instantaneous volatility, and W is a standard Brownian motion under the measure Q^2.

Let us solve this equation and compute the caplet price term $E^{Q^2}\left[(F_2(T_1)-X)^+\right]$. By Ito's formula:

$$\begin{aligned}
d\ln(F_2(t)) &= \ln'(F_2)dF_2 + \tfrac{1}{2}\ln''(F_2)\ dF_2\ dF_2\\
&= \frac{1}{F_2}dF_2 + \tfrac{1}{2}\left(-\frac{1}{(F_2)^2}\right)dF_2\ dF_2 =\\
= \frac{1}{F_2}\sigma_2 F_2 dW - \tfrac{1}{2}&\frac{1}{(F_2)^2}(\sigma_2 F_2 dW)(\sigma_2 F_2 dW) =\\
&= \sigma_2 dW - \tfrac{1}{2}\frac{1}{(F_2)^2}\sigma_2^2 F_2^2 dW dW =\\
&= \sigma_2(t)dW(t) - \tfrac{1}{2}\sigma_2^2(t)dt
\end{aligned}$$

(where $'$ and $''$ denote here the first and second derivative and where we used $dW\ dW = dt$). So we have

$$d\ln(F_2(t)) = \sigma_2(t)dW(t) - \tfrac{1}{2}\sigma_2^2(t)dt.$$

Integrate both sides:

$$\begin{aligned}
\int_0^T d\ln(F_2(t)) &= \int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\\
\ln(F_2(T)) - \ln(F_2(0)) &= \int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\\
\ln\frac{F_2(T)}{F_2(0)} &= \int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\\
\frac{F_2(T)}{F_2(0)} &= \exp\left(\int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\right)\\
F_2(T) = F_2(0)&\exp\left(\int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\right).
\end{aligned}$$

Compute the distribution of the random variable in the exponent.

It is Gaussian, since it is a stochastic integral of a deterministic function times a Brownian motion (roughly, sum of independent Gaussians is Gaussian).

Compute the expectation:

$$E\left[\int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\right] = 0 - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt$$

and the variance

$$\begin{aligned}\mathrm{Var}\left[\int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt\right] &= \mathrm{Var}\left[\int_0^T \sigma_2(t)dW(t)\right] \\ = E\left[\left(\int_0^T \sigma_2(t)dW(t)\right)^2\right] - 0^2 &= \int_0^T \sigma_2(t)^2 dt\end{aligned}$$

where we have used Ito's isometry in the last step. We thus have

$$\begin{aligned}I(T) := \int_0^T \sigma_2(t)dW(t) - \tfrac{1}{2}\int_0^T \sigma_2^2(t)dt \sim \\ \sim m + V\mathcal{N}(0,1), \quad m = -\tfrac{1}{2}\int_0^T \sigma_2(t)^2 dt, \\ V^2 = \int_0^T \sigma_2(t)^2 dt.\end{aligned}$$

Recall that we have

$$F_2(T) = F_2(0)\exp(I(T)) = F_2(0)e^{m+V\mathcal{N}(0,1)}.$$

Compute now the option price term

$$\begin{aligned}E^{Q^2}[(F_2(T_1) - X)^+] &= E^{Q^2}[(F_2(0)e^{m+V\mathcal{N}(0,1)} - X)^+] \\ &= \int_{-\infty}^{+\infty} (F_2(0)e^{m+Vy} - X)^+ p_{\mathcal{N}(0,1)}(y)dy = \ldots\end{aligned}$$

Note that $F_2(0)\exp(m+Vy) - X > 0$ if and only if

$$y > \frac{-\ln\left(\frac{F_2(0)}{X}\right) - m}{V} =: \bar{y}$$

so that

$$\begin{aligned}\ldots = \int_{\bar{y}}^{+\infty} (F_2(0)\exp(m+Vy) - X)p_{\mathcal{N}(0,1)}(y)dy = \\ = F_2(0)\int_{\bar{y}}^{+\infty} e^{m+Vy}p_{\mathcal{N}(0,1)}(y)dy - X\int_{\bar{y}}^{+\infty} p_{\mathcal{N}(0,1)}(y)dy\end{aligned}$$

$$\begin{aligned}= F_2(0)\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}y^2+Vy+m}dy - X(1-\Phi(\bar{y})) \\ = F_2(0)\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2+m-\frac{1}{2}V^2}dy - X(1-\Phi(\bar{y})) =\end{aligned}$$

$$= F_2(0)e^{m-\frac{1}{2}V^2}\frac{1}{\sqrt{2\pi}}\int_{\bar{y}}^{+\infty} e^{-\frac{1}{2}(y-V)^2}dy - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m-\frac{1}{2}V^2}\frac{1}{\sqrt{2\pi}}\int_{\bar{y}-V}^{+\infty} e^{-\frac{1}{2}z^2}dz - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m-\frac{1}{2}V^2}(1-\Phi(\bar{y}-V)) - X(1-\Phi(\bar{y})) =$$

$$= F_2(0)e^{m-\frac{1}{2}V^2}\Phi(-\bar{y}+V) - X\Phi(-\bar{y}) =$$

$$= F_2(0)\Phi(d_1) - X\Phi(d_2), \quad d_{1,2} = \frac{\ln\frac{F_2(0)}{X} \pm \frac{1}{2}\int_0^{T_1}\sigma_2^2(t)dt}{\sqrt{\int_0^{T_1}\sigma_2^2(t)dt}}.$$

When including the initial discount factor $P(0,T_2)$ and the year fraction τ this is exactly the classic market Black's formula for the $T_1 - T_2$ caplet.

End(detailed derivation of Black's formula for caplets)

The example just introduced is a simple case of what is known as "lognormal forward-LIBOR model". It is known also as Brace-Gatarek-Musiela (1997) model, from the name of the authors of one of the first papers where it was introduced rigorously. This model was also introduced by Miltersen, Sandmann and Sondermann (1997). Jamshidian (1997) also contributed significantly to its development. At times in the literature and in conversations, especially in Europe, the LFM is referred to as "BGM" model, from the initials of the three above authors. In other cases, colleagues in the U.S. called it simply an "HJM model", related perhaps to the fact that the BGM derivation was based on the HJM framework rather than on the change-of-numeraire technique. However, a common terminology is now emerging and the model is generally known as "LIBOR Market Model". We will stick to the "Lognormal Forward-LIBOR Model", since this is more informative on the properties of the model: Modeling forward LIBOR rates through a lognormal distribution (under the relevant measures).

Let us now go back to our short-rate model formula U_C and ask ourselves whether this formula can be compatible with the above reported Black's market formula. It is well known that the two formulas are not compatible. Indeed, by the two-dimensional version of Ito's formula we may derive the Q^2-dynamics of the forward LIBOR rate between T_1 and T_2 under the short-rate model,

$$dF(t;T_1,T_2;x_t,y_t,\theta) = \frac{\partial F}{\partial(t,x,y)}\, d[t\ x_t\ y_t]' + \frac{1}{2}\, d[x_t\ y_t]\, \frac{\partial^2 F}{\partial^2(x,y)}\, d[x_t\ y_t]',\ Q^2, \tag{6.3}$$

where the Jacobian vector and the Hessian matrix have been denoted by their partial derivative notation. This dynamics clearly depends on the linear-Gaussian dynamics of x and y under the T_2-forward measure. The thus obtained dynamics is easily seen to be incompatible with the lognormal dynamics leading to Black's formula. More specifically, for no choice of the