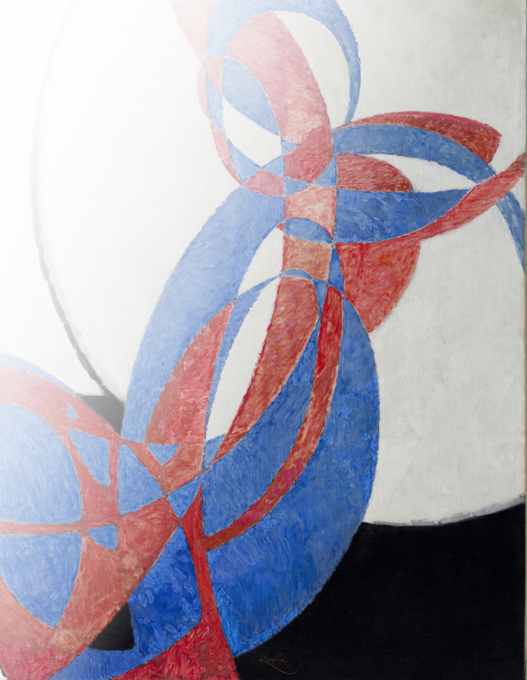
Change of Measure and Its Applications

Introduction to Libor Market Models



Advanced Financial Modeling

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Few Definitions

It may be helpful to explain (and recall) some of the more technical terms we are going to use.

Sample space: all possible future states or outcomes (Ω) of a random process.

(Probability) Measure ($\mathcal{P}, \mathcal{Q} \ldots$): is a mapping which associates a probability to each element in the sample space. Two measures are **equivalent** if they agree "on what is possible". Note the word *possible*: the two measures can have different probabilities for the same event, but must have the same *null-set* $\{x \in \mathcal{P} \mid p(x) = 0\}$.

Few Definitions

Contingent claim: is a derivative whose future payoff depends on the value of another "underlying" asset, or more generally, that is dependent on the realization of some uncertain future event $(S, X \ldots)$.

Filtrations: are totally ordered collections of subsets that are used to model the information that is available at a given point in time $(\mathcal{F}_t)$.

Martingale: is a stochastic process for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values. It can be imagined as a drift-less process.

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- And these states plus the corresponding probabilities are supposed to reflect the subjective beliefs of traders or investors about what will happen in the future.
- Unfortunately, under $\mathcal{P}$ it is usually quite complicated to price derivatives, and the probabilities themselves cannot easily be derived.
- This makes it hard to work out the price processes and it is necessary to use simulation techniques.

Changing Measure

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- At the beginning of the course we have seen how by simply assuming the absence of arbitrage it is possible to define a new measure under which the price of a derivative is given by the discounted expectation of its payoff.
- This result has been formalized by *Harrison* and *Pliska* in 1981.

Equivalent Martingale Measure

Definition

An equivalent martingale measure $\mathcal{Q}$ is a probability measure on the space Ω such that

1. $\mathcal{Q}$ is equivalent to $\mathcal{P}$;
2. for any asset A and for each time t, $0 \leq t \leq T$ there exists a price π_t

$$\pi_t = \mathbb{E}^{\mathcal{Q}^0}[D(t,T)V_A|\mathcal{F}_t]$$

3. the "discounted asset price" is a $\mathcal{Q}$-martingale

$$\pi_u = \mathbb{E}^{\mathcal{Q}^0}[D(0,t)V_A(t)|\mathcal{F}_u], \quad \text{with } (t > u)$$

Fundamental Results Summary

Harrison and Pliska proved and formalized also the following results:

- the market is free of arbitrage if and only if there exists an equivalent martingale measure;
- the market is complete if and only if the martingale measure is unique;
- in an arbitrage-free market the price of any claim is uniquely given, either by the value of an associated replicating strategy, or by the expectation of the discounted payoff under any of the equivalent martingale measures.

Numeraires

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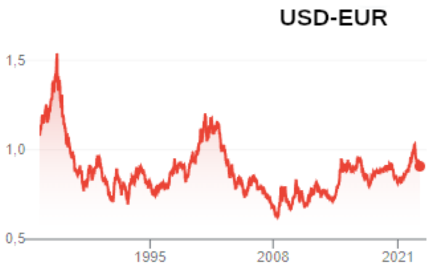
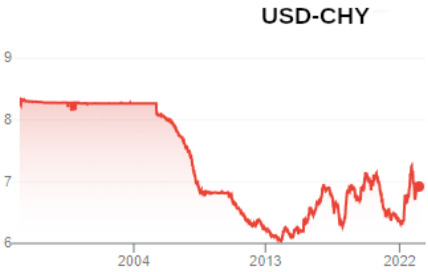
- The question that arises is: how can we determine such measure $\mathcal{Q}^X$?
- The answer is through change of numeraire.
- A numeraire is any strictly positive stochastic process N_t that is taken as a unit of reference when pricing an asset S_t

$$\tilde{S}_t := \frac{S_t}{N_t}, \quad t \geq 0$$

Numeraires

- We may compute asset values w.r.t. USD, EUR or JPY.
- Others might prefer use commodities: 1 oz of gold could be a numeraire.
- In any case, once we choose a numerarie e.g. 1 USD, we determine the value of other assets:





Numeraires

- Of course, in practice there are reasons to prefer gold to other commodities e.g. corn, live cattle, or one currency with respect to another (i.e. political reasons)...
- But intuitively there should be no theoretical reason (at least at the scale of investors) to measure value in gold or USD (there used to be also the "gold standard")
- Basically we want to exploit this fact in our financial models.

Numeraires

- Deterministic numeraires are easy to handle as they imply just an algebraic transformation (i.e. do not involve any risk),
  - the exchange rate of the Italian Lira to the Euro was locked at EUR $1 =$ ITL 1936.27 on 31 December 1998.

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- When the numeraire is a stochastic process, and we want to move to it, the pricing of a claim has to be changed in order to take into account the new risks (the intrinsic randomness of the new numeraire).
- In particular a change of numeraire implies also a change in the measure (probability distribution). Indeed (starting with the bank account numeraire)

$$\begin{cases} \dfrac{S_1}{B_t} = \dfrac{\sum_i S_1^i p_i}{e^{rt}} \\ \dfrac{S_2}{B_t} = \dfrac{\sum_i S_2^i p_i}{e^{rt}} \end{cases} \implies \frac{S_1}{S_2} = \frac{\sum_i S_1^i p_i}{\sum_j S_2^j p_j} = \sum_i S_1^i \frac{p_i}{\sum_j S_2^j p_j} = \sum_i S_1^1 \pi_i$$

Numeraire Examples (I)

- **Money market account.** Given r_t, a possibly random and time dependent risk-free interest rate process, let

$$N_t := \exp\left(\int_0^t r_s ds\right)$$

In this case

$$\tilde{S}_t := \frac{S_t}{N_t} = e^{-\int_0^t r_s ds} S_t, \quad t \geq 0$$

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- **Currency exchange rate.** In this case $N_t := R_t$ denotes e.g. the EUR/SGD exchange rate. Let

$$\tilde{S}_t := \frac{S_t}{R_t}, \quad t \geq 0$$

denotes the price of a local (SG) asset quoted in units of the foreign currency (EUR) (notice the difference with previous ITL/EUR example above).

Numeraire Examples (II)

- **Forward numeraire.** The price $P(t,T)$ of a bond paying $P(T,T)=1$ at maturity T. In this case

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- **Annuity numeraire.** Processes of the form

$$N_t = P(t,T_0,T_n) := \sum_{k=1}^{n}(T_k - T_{k-1})P(t,T_k), \quad 0 \le t \le T$$

where $P(t,T_1), P(t,T_2), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \ldots < T_n$.

Radon-Nikodym Derivative

- Now we need to understand how to pass from a numeraire to another, and hence by a measure to another, in an arbitrage free setting.
- Notice that until now we have implicitly assumed the bank account B as numeraire.

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Definition

When two measures are equivalent it is possible to express the first in terms of the second through the Radon-Nikodym derivative. Indeed there exists a martingale process ζ_t such that

$$\mathcal{Q}^* = \int_A \zeta_t(\omega) d\mathcal{Q}(\omega)$$

which can be written in a more concise form as

$$\frac{d\mathcal{Q}^*}{d\mathcal{Q}} = \zeta_t \tag{1}$$

Intuition from Expected Value

- To get a sense of what a Radon-Nikodym derivative is we can write the expected value of a generic function $\Pi(x)$ under a measure $\mathcal{F}$, with associated density function $f(x)$ as

$$\mathbb{E}^{\mathcal{F}} = \int \Pi(x) f(x) dx$$

- Suppose there exists a function $g(x)$, which satisfies the mathematical conditions required to be a density function. Then we can write

$$\mathbb{E}^{\mathcal{F}} = \int \Pi(x) f(x) \frac{g(x)}{g(x)} dx$$

- If we define $\psi(x) = \Pi(x)\frac{f(x)}{g(x)}$ the expected value can be written as

$$\mathbb{E}^{\mathcal{F}}[\Pi(x)] = \int \psi(x) g(x) dx = \mathbb{E}^{\mathcal{G}}[\psi(x)] = \mathbb{E}^{\mathcal{G}}\left[\Pi(x)\frac{f(x)}{g(x)}\right]$$

Radon-Nikodym Derivative

- The expectations corresponding to the two measures are related by

$$\mathbb{E}^{\mathcal{Q}^*}[X] = \mathbb{E}^{\mathcal{Q}}\left[X \frac{d\mathcal{Q}^*}{d\mathcal{Q}}\right]$$

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$$\mathbb{E}^{\mathcal{Q}^*}[X] = \mathbb{E}^{\mathcal{Q}}\left[X\frac{d\mathcal{Q}^*}{d\mathcal{Q}}\right]$$

- In case of a conditioned expectation

$$\mathbb{E}^*[X|\mathcal{F}_t] = \frac{\mathbb{E}\left[X\frac{d\mathcal{Q}^*}{d\mathcal{Q}}\middle|\mathcal{F}_t\right]}{\mathbb{E}[\zeta_t|\mathcal{F}_t]} \tag{2}$$

which is an "equivalent" formulation of the famous *Bayes theorem*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Change of Numeraire

Theorem

Assume exists a numeraire N_t and the associated measure $\mathcal{Q}^N$, equivalent to $\mathcal{P}$, such that the price of any traded asset S_t relative to N is a martingale under $\mathcal{Q}^N$

$$\frac{S_t}{N_t} = \mathbb{E}^N\left[\frac{S_T}{N_T}\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T$$

Let U be another arbitrary numeraire. Then there exists a measure $\mathcal{Q}^U$, also equivalent to $\mathcal{P}$, such that the price of any traded asset S_t, normalized to U, is a martingale under $\mathcal{Q}^U$

$$\frac{S_t}{U_t} = \mathbb{E}^U\left[\frac{S_T}{U_T}\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T$$

Change of Numeraire

...continued

The Radon-Nikodym derivative defining the measure $\mathcal{Q}^U$ is given by

$$\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N} = \frac{U_T N_0}{U_0 N_T} \tag{3}$$

Change of Numeraire (Proof p.2)

Let's prove first this second part. By definition of $\mathcal{Q}^N$, for any asset price S_t holds

$$\begin{cases} \dfrac{S_0}{N_0} = \mathbb{E}^N\left[\dfrac{S_T}{N_T}\right] \\ \dfrac{U_0}{N_0}\mathbb{E}^U\left[\dfrac{S_T}{U_T}\right] = \dfrac{\cancel{U_0}S_0}{N_0\cancel{U_0}} = \dfrac{S_0}{N_0} \end{cases} \implies \mathbb{E}^N\left[\dfrac{S_T}{N_T}\right] = \mathbb{E}^U\left[\dfrac{U_0 S_T}{N_0 U_T}\right]$$

since both equal S_0/N_0.

Change of Numeraire (Proof p.2)

Also, by definition of Radon-Nikodym derivative

$$\mathbb{E}^N\left[\frac{S_T}{N_T}\right] = \mathbb{E}^U\left[\frac{S_T}{N_T}\frac{d\mathcal{Q}^N}{d\mathcal{Q}^U}\right]$$

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But from the previous result

$$\mathbb{E}^N\left[\frac{S_T}{N_T}\right] = \mathbb{E}^U\left[\frac{U_0 S_T}{N_0 U_T}\right]$$

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This last equation shows that the risk-neutral price is invariant under change of numeraire.

$$Price_0 = \mathbb{E}^N\left[\frac{N_0 S_T}{N_T}\right] = \mathbb{E}^U\left[\frac{U_0 S_T}{U_T}\right]$$

Change of Numeraire (Proof p.1)

- Now we can prove the first part of the change of numeraire theorem. The conditional expectation formula Eq. (2) gives

$$\mathbb{E}^U\left[\frac{S_T}{U_T}\middle|\mathcal{F}_t\right] = \frac{\mathbb{E}^N\left[\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N}\frac{S_T}{U_T}\middle|\mathcal{F}_t\right]}{\mathbb{E}^N\left[\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N}\middle|\mathcal{F}_t\right]}$$

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- But

$$\begin{cases}\mathbb{E}^N\left[\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N}\frac{S_T}{U_T}\middle|\mathcal{F}_t\right]=\mathbb{E}^N\left[\frac{\cancel{U_T}N_0}{N_T U_0}\frac{S_T}{\cancel{U_T}}\middle|\mathcal{F}_t\right]=\frac{N_0 S_t}{U_0 N_t}\\ \mathbb{E}^N\left[\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N}\middle|\mathcal{F}_t\right]=\mathbb{E}^N\left[\frac{U_T N_0}{N_T U_0}\middle|\mathcal{F}_t\right]=\frac{N_0 U_t}{U_0 N_t}\end{cases}\implies \frac{S_t}{\cancel{N_t}}=\mathbb{E}^U\left[\frac{S_T}{U_T}\middle|\mathcal{F}_t\right]\frac{U_t}{\cancel{N_t}}$$

Change of Numeraire Remarks

- The powerful theorem we have just proved allows to
  - find a characterization of our process by means of which we can work-out more easily the fundamental pricing formula. In particular allows to find a measure associated to the new numeraire such that the price of any asset divided by that numeraire is a martingale;
  - give a simple rule to write (the otherwise difficult to derive) Radon-Nikodym derivative;
  - show that the risk-neutral price is invariant under change of numeraire.

Asset Price divided by Numeraire

- Let B be the money bank numeraire and $\mathcal{Q}^B$ the corresponding risk-neutral measure. Also let N be another numeraire (note that N/B is a $\mathcal{Q}^B$-martingale).

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- Then, for any asset S such that S/B is a $\mathcal{Q}^B$-martingale

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- So S/N is a $\mathcal{Q}^N$-martingale.

Examples

Any asset divided by the bank account B_t (recall $dB_t = r_t B_t dt$) $\boxed{\frac{S_t}{B_t} = e^{-\int_0^t r_s ds} S_t}$	It is a martingale under the measure Q^B associated to the bank account numeraire, i.e. the risk neutral measure.
The forward rate $\boxed{F(t; T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)}\right)}$ can be interpreted as a portfolio of two ZCBs divided by another ZCB.	Under the measure $\mathcal{Q}^2$ associated to the numeraire $P(\cdot, T_2)$ it is a martingale.
The swap rate $\boxed{S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)}}$ can be interpreted as a portfolio of two ZCBs divided by a portfolio of ZCBs.	It is a martingale under the measure associated to the annuity numeraire.

A Useful Separation

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- Given a contingent claim whose payoff at time T is χ, we have the following formula for its price Π

$$\Pi_\chi(t, T) = \mathbb{E}^B \left[e^{-\int_t^T r_s ds} \chi \middle| \mathcal{F}_t \right] = \mathbb{E}^B \left[e^{\int_0^t r_s ds} e^{-\int_0^T r_s ds} \chi \middle| \mathcal{F}_t \right] = B_t \mathbb{E}^B \left[B_T^{-1} \chi | \mathcal{F}_t \right]$$

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- If χ and the short rate process were independent under $\mathcal{Q}^B$ (recall $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$) then we could write

$$\Pi_\chi(t,T) = \mathbb{E}^B\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right]\mathbb{E}^B\left[\chi|\mathcal{F}_t\right] = P(t,T)\mathbb{E}^B\left[\chi|\mathcal{F}_t\right]$$

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- The forward measure $\mathcal{Q}^T$ (also called the T-measure) is defined as the martingale measure for the numeraire process $P(\cdot, T)$, the ZCB maturing in T indeed, i.e. what we called $\mathcal{Q}^2$ in the example above.

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- The forward measure $\mathcal{Q}^T$ (also called the T-measure) is defined as the martingale measure for the numeraire process $P(\cdot, T)$, the ZCB maturing in T indeed, i.e. what we called $\mathcal{Q}^2$ in the example above.
- It is easy to see that using Eq. (3), the Radon-Nykodim derivative is given in this case by

$$\zeta_t = \frac{d\mathcal{Q}^T}{d\mathcal{Q}^B} = \frac{P(t,T)\overbrace{B(0)}^{=1}}{B_t P(0,T)}, \quad \left(\zeta_T = \frac{\overbrace{P(T,T)}^{=1} B(0)}{B(T)P(0,T)} = \frac{1}{B(T)P(0,T)} \right) \tag{4}$$

A Useful Separation

- Applying the change of numeraire to the pricing formula, we get

$$
\begin{aligned}
\Pi_\chi(t,T) &= B_t \mathbb{E}^B\left[B_T^{-1}\chi|\mathcal{F}_t\right] \\
&= B_t \mathbb{E}^B\left[P(0,T)\zeta_T\chi|\mathcal{F}_t\right] \quad (\text{using } B_T^{-1} = \zeta_T P(0,T)) \\
&= B_t P(0,T)\mathbb{E}^B\left[\zeta_T|\mathcal{F}_t\right]\mathbb{E}^T\left[\chi|\mathcal{F}_t\right] \quad (\text{by Eq. (2)}) \\
&= \cancel{B_t P(0,T)}\frac{P(t,T)}{\cancel{B_t P(0,T)}}\mathbb{E}^T\left[\chi|\mathcal{F}_t\right] \\
&= P(t,T)\mathbb{E}^T\left[\chi|\mathcal{F}_t\right]
\end{aligned}
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  which achieves the desired separation (although under a new measure).

- Clearly this kind of transformation is useful when χ has known dynamics under the forward measure.

Identity between $\mathcal{Q}^B$ and $\mathcal{Q}^T$

By construction of the martingale measure $\mathcal{Q}^B$, the following relationship holds

$$\frac{P(t,T)}{B_t} = \mathbb{E}^B\left[\frac{P(T,T)}{B_T}\right]$$

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Plugging the result into the Radon-Nikodym derivative gives

$$\frac{d\mathcal{Q}^T}{d\mathcal{Q}^B} = \frac{B_t}{B_T}\frac{1}{P(t,T)} = \frac{B_t/B_T}{\mathbb{E}^B[B_t/B_T]}$$

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Proposition

If interest rates are deterministic (i.e. the Radon-Nikodym derivative is 1), then the measures $\mathcal{Q}^B$ and $\mathcal{Q}^T$ are identical.

Clarification on Time

- Clearly as the Radon-Nikodym derivative is a martingale for valuation time t, we have
$$\frac{d\mathcal{Q}^U}{d\mathcal{Q}^N} = \frac{U_t N_0}{U_0 N_t}$$
- Do not confuse the maturity of the numeraire bond T with the times at which you have to take the values of the numeraire, in this case t and 0.
- If you want to switch from the T measure to the S measure, i.e. the one induced by the bond $P(., S)$, for the valuation time t we get

$$\frac{d\mathcal{Q}^S}{d\mathcal{Q}^T} = \frac{P(t,S)P(0,T)}{P(t,T)P(0,S)}$$

The Forward Rate Under $\mathcal{Q}^T$

Proposition

Consider the forward numeraire $P(t, T)$ and denote with $\mathcal{Q}^T$ its associated measure. The forward rate spanning the interval $[S, T]$ is the $\mathcal{Q}^T$-expectation of the future spot rate at time S for the maturity T

$$F(t; S, T) = \mathbb{E}^T[L(S, T)|\mathcal{F}_t] \tag{5}$$

The Forward Rate Under $\mathcal{Q}^T$ (Proof)

$$F(t; S, T) = \frac{1}{\tau}\left[\frac{P(t,S) - P(t,T)}{P(t,T)}\right]$$

$$F(t; S, T)P(t,T) = \frac{P(t,S) - P(t,T)}{\tau}$$

This is the price at time t of an asset (difference of two bonds). Therefore by the change of numeraire theorem and by definition of forward measure

$$\frac{F(t; S, T)P(t,T)}{P(t,T)} = F(t, S, T)$$

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is a martingale under $\mathcal{Q}^T$-measure. Hence

$$F(t;S,T) = \mathbb{E}^T[F(S;S,T)|\mathcal{F}_t] = \mathbb{E}^T\left[\frac{1}{\tau}\left(\frac{1-P(S,T)}{P(S,T)}\right)\Big|\mathcal{F}_t\right] = \mathbb{E}^T[L(S,T)|\mathcal{F}_t]$$

The Forward Rate Under Q^T

A similar result can be derived for the corresponding instantaneous quantities

$$f(t, T) = \mathbb{E}^T[r_t|\mathcal{F}_t] \tag{6}$$

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Indeed from the definition of $\mathcal{Q}^B$

$$\frac{P(t, T)}{B_t} = \mathbb{E}^B \left[\frac{P(T, T)}{B_T} \middle| \mathcal{F}_t \right]$$

but $P(T, T) = 1$ so

$$P(t, T) = \mathbb{E}^B \left[\frac{B_t P(T, T)}{B_T} \middle| \mathcal{F}_t \right] = \mathbb{E}^B \left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right]$$

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Differentiating with respect to T ($\frac{d}{dx}\int_c^x f(t)dt = f(x)$)

$$\frac{\partial P(t,T)}{\partial T} = \mathbb{E}^B\left[-r(T)e^{-\int_t^T r_u du}\middle|\mathcal{F}_t\right]$$

The Forward Rate Under $\mathcal{Q}^T$

Now we can change numeraire to $P(\cdot, T)$ so that, using reciprocal of Eq. (4) $\left(\zeta^{-1} = \frac{B_t/B_T}{P(t,T)/P(T,T)}\right)$

$$\frac{\partial P(t,T)}{\partial T} = \mathbb{E}^T\left[-r(T)\cancel{e^{-\int_t^T r_u du}}\frac{P(t,T)}{\cancel{e^{-\int_t^T r_u du}}}\middle|\mathcal{F}_t\right] = P(t,T)\mathbb{E}^T\left[-r(T)|\mathcal{F}_t\right]$$

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Hence

$$f(t,T) = \frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = -\frac{\partial \ln P(t,T)}{\partial T} = \mathbb{E}^T\left[r(T)|\mathcal{F}_t\right] = \mathbb{E}^T\left[f(T,T)|\mathcal{F}_t\right]$$

Which demonstrates the initial statement and also shows that the instantaneous forward rate is a martingale under the T-forward measure.

The Expectation Hypothesis

- Previous result
$$f(t,T) = \mathbb{E}^T[r(T)|\mathcal{F}_t]$$
has a nice connection with the expectation hypothesis of the term structure of interest rates.
- Its basic idea is that the long-term rate is determined purely by current and future expected short-term rates.
- We cannot dive into it, but there are tons of papers on the subject, among which I suggest
  - *The Expectation Hypothesis, A. Sangvinatsos*
  - *A Re-Examination of Traditional Hypotheses about the Term Structure of Interest Rates*, J.C. Cox, J.E. Ingersoll, and S.A. Ross

Which Dynamics ?

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- Will see that it evolves as the sum of a Brownian motion under $\mathcal{Q}$ and a drift related to the Radon-Nikodym derivative characterizing $\mathcal{Q}$.
- We therefore want to choose the Radon-Nikodym derivative so that the drift of W_t w.r.t. $\mathcal{Q}$ exactly cancels out the drift of S_t, leaving us with a pure diffusion process (martingale).

Girsanov Theorem

Theorem

Consider the SDE

$$dX_t = f_t dt + \sigma_t dW_t$$

under $\mathcal{P}$.

Let be given a new drift f_t^* and assume $\gamma_t = \frac{f_t^* - f_t}{\sigma_t}$ such that $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_t^2 dt\right)\right] < \infty$. Define the measure

$$\frac{d\mathcal{P}^*}{d\mathcal{P}} = \exp\left(-\frac{1}{2}\int_0^t \gamma_s^2 ds + \int_0^t \gamma_s dW_s\right) \tag{7}$$

Then $\mathcal{P}^*$ is equivalent to $\mathcal{P}$. The Radon-Nikodym derivative process is an exponential martingale.

Girsanov Theorem

Theorem continued

Also the process

$$dW_t^* = -\gamma_s dt + dW_t \tag{8}$$

is a Brownian motion under $\mathcal{P}^*$, and

$$dX_t = f_t^* dt + \sigma_t dW_t^*$$

The condition $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_t^2 dt\right)\right] < \infty$ is a sufficient but non-necessary, and it is know as the Novikov condition.

An Example

- Consider the stochastic differential equation

$$dX_t = b(X_t, t)dt + a(X_t, t)dW_t$$

- Let's assume that the drift and diffusion coefficients are such that there exists a unique solution to the equation which is X.
- We want to find a probability measure $\mathcal{P}^*$ such that the drift of X is $\tilde{b}(X_t, t)$ instead of $b(X_t, t)$.

An Example

$$
\begin{aligned}
dX_t &= \tilde{b}(X_t,t)dt + b(X_t,t)dt - \tilde{b}(X_t,t)dt + a(X_t,t)dW_t = \\
&= \tilde{b}_t dt + (b_t - \tilde{b}_t)dt + a(X_t,t)dW_t = \\
&= \tilde{b}_t dt + a_t \overbrace{\left(\frac{b_t - \tilde{b}_t}{a_t}\right)}^{-\gamma_t} dt + a_t dW_t = \\
&= \tilde{b}_t dt + a_t dW_t - a_t \gamma_t dt \\
&= \tilde{b}_t dt + a_t d\tilde{W}_t
\end{aligned}
$$

where $d\tilde{W}_t = dW_t - \gamma_t dt$.

An Example

- If the Novikov condition is satisfied then we can apply the Girsanov theorem and we have that

$$\mathcal{P}^* = \mathbb{E}^{\mathcal{P}} \left[\exp \left(-\frac{1}{2} \int_0^t \gamma_s^2 ds + \int_0^t \gamma_s dW_s \right) \right] \tag{9}$$

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and that $\tilde{W}$ is a Brownian motion on $\mathcal{P}^*$.
- In practice, don't need to determine the new measure $\mathcal{P}^*$.
- It is enough to know it exists, such that we can work with the resulting SDE of the process of interest under the new measure.

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Moving Away from $\mathcal{P}$ Measure

- Let's go back to the real-world probabilities and assume that a stock price has the following dynamics (Geometric Brownian Motion) under the real-world measure $\mathcal{P}$

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  - risk-neutral measure (bank account numeraire);
  - stock measure (stock numeraire).

Risk-Neutral Measure Dynamics

- We have seen that under the bank account induced measure the process defined as an asset divided by the numeraire is a martingale

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- So if we define $Z_t = \frac{S_t}{B_t}$, since Z_t is a martingale (no-drift process) it's evolution could be described by

$$dZ_t = \sigma Z_t dW_t^B \tag{10}$$

where dW_t^B is a Brownian motion under the $\mathcal{Q}^B$ measure.

Risk-Neutral Measure Dynamics

- Computing directly the Z_t differential (by Itô's rule at first order)

$$
\begin{aligned}
d\left(\frac{S_t}{B_t}\right) &= \frac{dS_t}{B_t} + S_t d\left(\frac{1}{B_t}\right) = \\
&= \frac{dS_t}{B_t} + S_t d\left(e^{-rt}\right) = \frac{dS_t}{B_t} - S_t r e^{-rt} dt \\
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\end{aligned}
$$

- Now substitute for dS_t

$$
d\left(\frac{S_t}{B_t}\right) = \frac{\mu S_t dt + \sigma S_t dW_t}{B_t} - r\frac{S_t}{B_t} dt = \sigma \frac{S_t}{B_t}\left(\frac{\mu - r}{\sigma} dt + dW_t\right)
$$

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- Replacing the Brownian Motion into the real-world dynamics

$$dS_t = \mu S_t dt + \sigma S_t \left(dW_t^B - \frac{\mu - r}{\sigma} dt \right) =$$

$$= \cancel{\mu S_t dt} \cancel{- \mu S_t dt} + r S_t dt + \sigma S_t dW_t^B = \boxed{r S_t dt + \sigma S_t dW_t^B}$$

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- So under the risk-neutral measure the drift equals the risk-free rate.

Stock Numeraire Measure Dynamics

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- By the Change of Numeraire Theorem under the measure $\mathcal{Q}^A$ induced by asset numeraire A

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$$S_0 = \mathbb{E}^A\left[A_0\frac{S_t}{A_t}\middle|\mathcal{F}_0\right]$$

- Since both expressions represent a price of an asset they must be the same and we can equal the terms inside the expectations. Note that the expectations are computed according two different measures so we keep the factors $d\mathcal{Q}^X$.

Stock Numeraire Measure Dynamics

$$\frac{B_0}{B_t} d\mathcal{Q}^B = \frac{A_0}{A_t} d\mathcal{Q}^A \implies \frac{d\mathcal{Q}^A}{d\mathcal{Q}^B} = \frac{B_0 A_t}{B_t A_0}$$

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- We have already derived the analytical GBM solution in the risk-neutral measure (see pg. 14 to 18 of "no_arbitrage" slides)

$$A_t = A_0 \exp\left(rt - \frac{1}{2}\sigma^2 t + \sigma W_t^B\right)$$

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$$\frac{B_0}{B_t} d\mathcal{Q}^B = \frac{A_0}{A_t} d\mathcal{Q}^A \implies \frac{d\mathcal{Q}^A}{d\mathcal{Q}^B} = \frac{B_0 A_t}{B_t A_0}$$

- We have already derived the analytical GBM solution in the risk-neutral measure (see pg. 14 to 18 of "no_arbitrage" slides)

$$A_t = A_0 \exp\left(rt - \frac{1}{2}\sigma^2 t + \sigma W_t^B\right)$$

- So we can replace the numeraire definition into the Radon-Nikodym derivative

$$\frac{d\mathcal{Q}^A}{d\mathcal{Q}^B} = \frac{\cancel{A_0} e^{\cancel{rt} - \frac{1}{2}\sigma^2 t + \sigma W_t^B}}{\cancel{e^{rt}} \cancel{A_0}} = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t^B\right)$$

Stock Numeraire Measure Dynamics

- From the Girsanov theorem, setting the function $y_t = \sigma$, we can get the transformed diffusion process

$$dW_t^A = dW_t^B - \sigma dt$$

Stock Numeraire Measure Dynamics

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- Substituting back into the risk-neutral dynamics we get

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^B = rS_t dt + \sigma S_t (dW_t^A + \sigma dt) \\ &= (r + \sigma^2) S_t dt + \sigma S_t dW_t^A \end{aligned}$$

Summarizing the Results

- To summarize all the results

$$
\begin{array}{llr}
(i) & dS_t = \mu S_t dt + \sigma S_t dW_t & \text{Real-world measure} \\
(ii) & dS_t = r S_t dt + \sigma S_t dW_t^B & \text{Risk-neutral measure} \\
(iii) & dS_t = (r + \sigma^2) S_t dt + \sigma S_t dW_t^A & \text{Stock measure}
\end{array}
$$

Summarizing the Results

- To summarize all the results

$$\begin{array}{lll} (i) & dS_t = \color{red}{\mu} S_t dt + \sigma S_t dW_t & \text{Real-world measure} \\ (ii) & dS_t = \color{red}{r} S_t dt + \sigma S_t dW_t^B & \text{Risk-neutral measure} \\ (iii) & dS_t = \color{red}{(r + \sigma^2)} S_t dt + \sigma S_t dW_t^A & \text{Stock measure} \end{array}$$

- In practical terms, this means that it possible to use equation (ii), instead of equation (i), to simulate future payoffs, and hence that it is possible to get rid of the big problem of the equity premium estimation. Equation (ii) just needs estimates of the risk-free rate r and of the volatility σ, which can be derived from real market quotes.

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Drift Changes Generalization

Proposition

Assume that, under a generic N-measure, we have the following dynamics for a n-vector diffusion process X

$$dX_t = \mu_t^N(X_t)dt + \sigma_t(X_t)CdW_t^N$$

where dW_t^N is a n-dimensional standard Brownian motion whose correlation is modeled by the $n \times n$ matrix C, μ is an $n \times 1$ vector and σ_t a $n \times n$ diagonal matrix. Under the U-measure, we have

$$\mu_t^U(X_t) = \mu_t^N(X_t) - \rho\sigma(X_t)\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \tag{12}$$

or . . .

Drift Changes Generalization

Proposition contd.

$$CdW_t^U = CdW_t^N + \rho \left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t} \right)' dt \qquad (13)$$

$\rho = CC'$ is the correlation matrix of $< dW_i^N, dW_j^N >$ and σ_t^N and σ_t^U are the (vector) volatilities of numeraires N and U.

Drift Changes (Proof)

Indicate by $\mathcal{Q}^N$ and $\mathcal{Q}^U$ the N-measure and U-measure. By Girsanov theorem we have the following expression for the Radon-Nikodym derivative

$$\zeta_t = \frac{d\mathcal{Q}^N}{d\mathcal{Q}^U} = e^{-\frac{1}{2}\int_0^t \gamma_s^2 ds + \int_0^t \gamma_s dW_s^U}$$

with

$$\gamma_t = \frac{[\mu_t^N(X_t) - \mu_t^U(X_t)]'}{(\sigma_t(X_t)C)'} \tag{14}$$

Drift Changes (Proof)

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$$\gamma_t = \frac{[\mu_t^N(X_t) - \mu_t^U(X_t)]'}{(\sigma_t(X_t)C)'} \tag{14}$$

We also know that ζ_t is an exponential martingale hence its dynamics is such that

$$d\zeta_t = \gamma_t \zeta_t dW_t^U \tag{15}$$

Drift Changes (Proof)

By the main theorem on numeraire change Eq. (3), and using the fact that ζ_t is a $\mathcal{Q}^U$-martingale,

$$\zeta_t = \frac{d\mathcal{Q}^N}{d\mathcal{Q}^U} = \frac{U_0 N_t}{N_0 U_t} \tag{16}$$

thus

$$d\zeta_t = \frac{U_0}{N_0} d\left(\frac{N_t}{U_t}\right) = \frac{U_0}{N_0}\sigma_t^{N/U} C dW_t^U \tag{17}$$

where $\sigma_t^{N/U}$ is the volatility of the process N_t/U_t, which is also a martingale under $\mathcal{Q}^U$.

Drift Changes (Proof)

Comparing the two results for $d\zeta_t$ (Eq. (15), Eq. (17) and using Eq. (16)) we get

$$\gamma_t \zeta_t dW_t^U = \gamma_t \frac{\cancel{U_0} N_t}{\cancel{N_0} U_t} \cancel{dW_t^U} = \frac{\cancel{U_0}}{\cancel{N_0}} \sigma_t^{N/U} C \cancel{dW_t^U} \implies \gamma_t = \frac{U_t}{N_t} \sigma_t^{N/U} C$$

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Using the definition of γ_t (Eq. (14)) and remembering that given two matrices A and B it holds $((AB)' = B'A')$

$$\begin{aligned} (\mu_t^N(X_t) - \mu_t^U(X_t))' &= \gamma_t (\sigma_t(X_t) C)' = \frac{U_t}{N_t} \sigma_t^{N/U} CC' (\sigma_t(X_t))' \\ \mu_t^U(X_t) &= \mu_t^N(X_t) - \frac{U_t}{N_t} \sigma_t(X_t) \rho (\sigma_t^{N/U})' \end{aligned} \tag{18}$$

Intermezzo

- One of the classical formulas of differential calculus is th Leibniz rule $d(xy) = x(dy) + y(dx)$
- For stochastic processes this becomes, applying Itô's formula to the function $F(X, Y) = XY$

$$dF(x_i) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i + \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial x_i \partial x_j} dx_i dx_j$$

For $n = 2$:

$$\frac{\partial F}{\partial X} = Y, \frac{\partial F}{\partial Y} = X$$

$$\frac{\partial^2 F}{\partial X^2} = 0, \frac{\partial^2 F}{\partial X \partial Y} = \frac{\partial^2 F}{\partial Y \partial X} = 1, \frac{\partial^2 F}{\partial Y^2} = 0$$

$$d(XY) = XdY + YdX + dXdY$$

Drift Changes (Proof)

Now let N_t and U_t have dynamics under $\mathcal{Q}^U$ given by

$$dN_t = (\ldots)dt + \sigma_t^N C dW_t^U$$
$$dU_t = (\ldots)dt + \sigma_t^U C dW_t^U$$

Drift Changes (Proof)

Now let N_t and U_t have dynamics under $\mathcal{Q}^U$ given by

$$dN_t = (\ldots)dt + \sigma_t^N C dW_t^U$$
$$dU_t = (\ldots)dt + \sigma_t^U C dW_t^U$$

From what we have just seen about the product rule in stochastic calculus

$$d\left(\frac{N_t}{U_t}\right) = \frac{1}{U_t}dN_t + N_t d\frac{1}{U_t} + dN_t d\frac{1}{U_t} \qquad \left(d\frac{1}{U_t} = -\frac{1}{U_t^2}dU_t + \cancel{\frac{1}{U_t^3}dU_t dU_t}\right)$$

Drift Changes (Proof)

Now let N_t and U_t have dynamics under $\mathcal{Q}^U$ given by

$$dN_t = (\ldots)dt + \sigma_t^N C dW_t^U$$
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Replacing the dynamics for N_t and U_t (ignoring the terms in dt since we know that $d\dfrac{N_t}{U_t}$ is a martingale)

$$d\left(\frac{N_t}{U_t}\right) = \frac{dN}{U_t} - \frac{N_t dU}{U_t^2} - \cancel{\frac{dNdU}{U_t^2}} = \frac{\sigma_t^N C dW_t^U}{U_t} - \frac{N_t \sigma_t^U C dW_t^U}{U_t^2} \qquad (19)$$

Drift Changes (Proof)

Taking $d(N_t/U_t)$ definition from Eq. (17) and comparing it with Eq. (19)

$$d\left(\frac{N_t}{U_t}\right) = \sigma_t^{N/U}\cancel{CdW_t^U} = \frac{\sigma_t^N \cancel{CdW_t^U}}{U_t} - \frac{N_t\sigma_t^U \cancel{CdW_t^U}}{U_t^2} \implies \sigma_t^{N/U} = \frac{\sigma_t^N}{U_t} - \frac{N_t\sigma_t^U}{U_t^2} \quad (20)$$

Replacing above expression for $\sigma_t^{N/U}$ into Eq. (18)

$$\begin{aligned} \mu_t^U(X_t) &= \mu_t^N(X_t) - \frac{\cancel{U_t}}{N_t}\sigma_t(X_t)\rho\left(\frac{\sigma_t^N}{\cancel{U_t}} - \frac{N_t}{U_t^{\cancel{2}}}\sigma_t^U\right)' \\ &= \mu_t^N(X_t) - \sigma_t(X_t)\rho\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \end{aligned} \quad (21)$$

which proves the first part of the statement.

Drift Changes (Proof)

Expressing γ_t coefficient in terms of the numeraires volatilities

$$\begin{cases} \gamma_t = \frac{[\mu_t^N(X_t) - \mu_t^U(X_t)]'}{(\sigma_t(X_t)C)'} \\ \mu_t^N(X_t) - \mu_t^U(X_t) = \sigma_t(X_t)\rho\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \end{cases} \implies \gamma_t = \frac{\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right) CC'(\sigma_t(X_t))}{C'(\sigma_t(X_t))'}$$

$$\gamma_t = \left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right) C = C'\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \quad (C \text{ is a symmetric matrix}) \tag{22}$$

Drift Changes (Proof)

Expressing γ_t coefficient in terms of the numeraires volatilities

$$\begin{cases} \gamma_t = \frac{[\mu_t^N(X_t) - \mu_t^U(X_t)]'}{(\sigma_t(X_t)C)'} \\ \mu_t^N(X_t) - \mu_t^U(X_t) = \sigma_t(X_t)\rho\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \end{cases} \implies \gamma_t = \frac{\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right) CC'(\sigma_t(X_t))}{C'(\sigma_t(X_t))'}$$

$$\gamma_t = \left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right) C = C'\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' \quad (C \text{ is a symmetric matrix}) \tag{22}$$

Finally from the Girsanov theorem we get the diffusion process under the new numeraire

$$\begin{aligned} CdW_t^N &= CdW_t^U - C\gamma_t dt \\ CdW_t^N &= CdW_t^U - \rho\left(\frac{\sigma_t^N}{N_t} - \frac{\sigma_t^U}{U_t}\right)' dt \end{aligned} \tag{23}$$

which proves also the second part of the proposition.

Asset/Numeraraire by Girsanov

Assuming an asset S and a numeraire N with the following evolutions under the risk-neutral measure

$$\begin{cases} \dfrac{dS_t}{S_t} = r_t dt + \sigma_t^S dW_t^B \quad \text{(asset)} \\ \dfrac{dN_t}{N_t} = r_t dt + \sigma_t^N dW_t^B \quad \text{(numeraire)} \end{cases} \tag{24}$$

by Girsanov Theorem (Eq. (23)), under $\mathcal{Q}^N$, we get

$$dW_t^N = dW_t^B - \sigma_t^N dt \tag{25}$$

which is a Brownian motion.

Asset/Numeraraire by Girsanov

Now let's apply Itô's lemma to S_t/N_t

$$d\left(\frac{S}{N}\right) = \frac{1}{S}dS - \frac{S}{N^2}dN + \frac{S}{N^3}dN^2 - \frac{1}{N^2}dSdN = \frac{S}{N}\left(\frac{dS}{S} - \frac{dN}{N} + \frac{dN^2}{N^2} - \frac{dSdN}{SN}\right) =$$
$$= \frac{S}{N}\left(rdt + \sigma^S dW^B - rdt - \sigma^N dW^B + (\sigma^N)^2 dt - \sigma^S\sigma^N dt\right) = \quad (with\ Eq.\ (24))$$
$$= \frac{S}{N}((\sigma^N)^2 - \sigma^S\sigma^N)dt + \sigma^S dW^B - \sigma^N dW^B = \quad (with\ Eq.\ (25))$$
$$= \frac{S}{N}((\sigma^N)^2 - \sigma^S\sigma^N)dt + \sigma^S(dW^N + \sigma^N dt) - \sigma^N(dW^N + \sigma^N dt) =$$
$$= \frac{S}{N}(\sigma^S - \sigma^N)dW^N$$

which shows that $\frac{S}{N}$ is a $\mathcal{Q}^N$-martingale (no drift in dynamics).