Arbitrage-Free Pricing Theory

Basic Definitions and a Bit of Stochastic Calculus

Advanced Financial Modeling

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MPS CAPITAL SERVICES

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Random Variables

Definition

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Random variables are very different from usual *algebraic variables*:

$$x^2 - 3 = 0 \implies x = \pm\sqrt{3}$$

no matter how many times I solve this equation.

Random Variables

A random variable instead is kind of a mapping between

$$X(\omega) : \Omega \rightarrow \mathbb{R} \quad \forall \omega \in \Omega$$

such that $X(\omega)$ represents the occurrence probability of the "outcome" ω. Ω is called sample space and is the set of all possible future states (or outcomes) of a process.

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Hence the random variable X will take values distributed according to the probability distribution of the random process, e.g. if X represents the outcomes of rolling a *fair* die $\Omega = [1, 2, 3, 4, 5, 6]$ and each value has equal probability of $1/6$.

A random variable is always associated to a probability distribution.

Discrete and Continuous Random Variables

Definition

If a random variable takes only a countable number (finite) of values, it is called discrete.

Example: when 3 coins are tossed, the number of heads obtained is the random variable, which assumes the values $\Omega = \{0, 1, 2, 3\}$ (Ω ia a countable set).

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Definition

A random variable X which can take any value between a certain interval is called continuous.

Example: the height of students in a particular class lies between 160 and 190 cm ($X = \{x | 160 \leq x \leq 190\}$).

Probability Distribution

Let X be a random variable defined on a domain Ω of possible outcomes.

Discrete	Continuous
Probability Mass	Probability Density
$P(X = x_i)\ \forall x_i \in \Omega$	$P(X = x) = \int_x^{x+dx} f(x)dx$
$P(x_i) \geq 0;\ \forall i$	$f(x) \geq 0;\ -\infty < x < \infty$
$\sum_{i=0}^{n} P_i = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
Cumulative Distribution	
$F(x_i) = P(X < x_i) = \sum_{x<x_i} P(x)$	$F(x) = P(X < a) = \int_{-\infty}^{a} f(x)dx$

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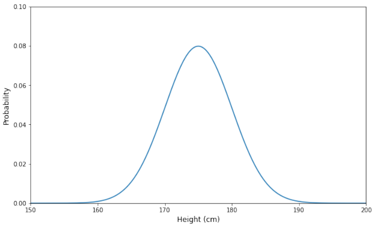
Characterizing a Random Variable

If we know the distribution of a random variable, we pretty much know all is needed about it.



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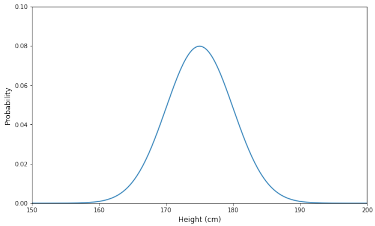

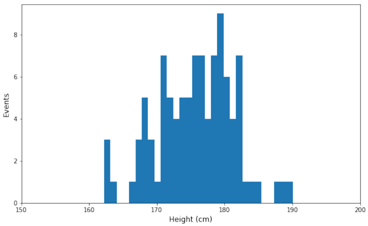

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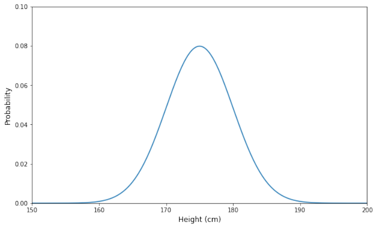

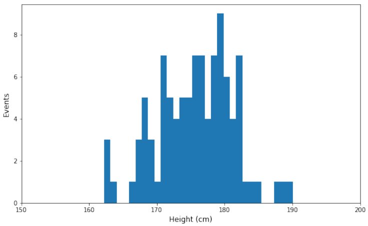

But with *real data*, we don't know the full distribution. Fortunately we can characterize it by a couple of numbers (*statistics*).

mean: $\boxed{\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx}$ variance: $\boxed{\sigma^2 = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx}$

Properties of Expectation

Scalar multiplication	$\mathbb{E}[aX] = a\mathbb{E}[X]$
Sums	$\mathbb{E}[X_1 + \ldots + X_K] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
Linear combinations	$\mathbb{E}[a_1X_1 + \ldots + a_KX_K] = a_1\mathbb{E}[X_1] + \ldots + a_K\mathbb{E}[X_K]$
Expected value of a constant	$\mathbb{E}[a] = a$
Products (independent variables)	$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Essentially all the expectation properties come from integration properties, e.g.

$$\mathbb{E}[aX] = \int_{-\infty}^{\infty} axf(x)dx = a\int_{-\infty}^{\infty} xf(x)dx = a\mathbb{E}[X]$$

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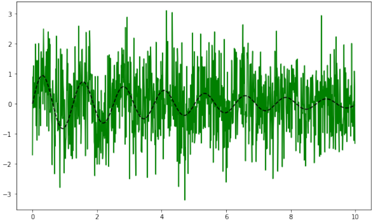

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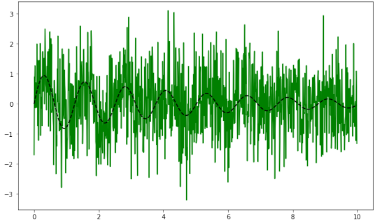


Need to switch to stochastic processes in order to model the uncertainty of data.

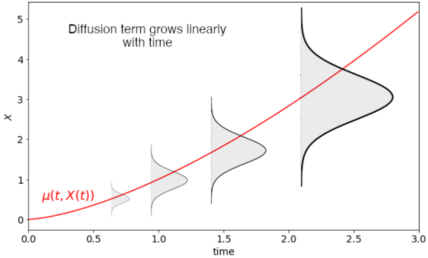
Definition

A collection of random variables that is indexed by some mathematical set (usually time) is called a stochastic processes.

Stochastic Differential Equation

Stochastic processes are described by *stochastic differential equations* (SDE):

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) =$$
$$= \underbrace{\mu(t, X(t))dt}_{\text{deterministic}} + \underbrace{\sigma(t, X(t))\mathcal{N}(0,1)\sqrt{dt}}_{\text{stochastic}}$$



- The mean of dW is zero and its variance is dt
- the standard deviation grows with the square root of time: $W(t) \simeq \mathcal{N}(0, t)$ because each dW is distributed like independent standard Gaussian.

Martingale

Definition

A martingale is a (integrable and adapted) stochastic process which models a fair game with the following remarkable feature

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \tag{1}$$

so the best prediction for the future value X_t, given the knowledge $\mathcal{F}_s$ at time s is the value at time s itself, X_s.

Martingale

Properties

If X_t is a stochastic process with diffusion coefficient (i.e. volatility) σ_t, which satisfies $\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{\frac{1}{2}}\right] < \infty$, and SDE $dX_t = \mu_t dt + \sigma_t dW_t$

$$X \text{ is a martingale} \iff X \text{ is drift-less } (\mu_t = 0)$$

A martingale corresponds to the common notion that "an efficient price, changes randomly" so we cannot know if it will go up or down. That is why this mathematical concept is brought into finance.

Geometric Brownian Motion

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- Putting all together, the resulting SDE is

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ \frac{dS_t}{S_t} &= d\log(S_t) = \mu dt + \sigma dW_t \end{aligned} \tag{2}$$

Itô's Formula

Proposition

For any given continuous and differentiable function $G(S,t)$ where S satisfies $dS = adt + bdW_t$, holds

$$dG = \left(a\frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \underbrace{\frac{1}{2}b^2\frac{\partial^2 G}{\partial S^2}}_{\text{additional term}} \right) dt + b\frac{\partial G}{\partial S}dW \tag{3}$$

This is essentially an extension of the *Taylor series* for stochastic functions, in the expansion an extra term appears.

Itô's Formula "Proof"

- Suppose X_t is an stochastic process that satisfies the SDE

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t$$

.

- If $f(t,x)$ is a twice-differentiable scalar function of X, its expansion in a Taylor series is

$$df = \frac{\partial f}{\partial t} \, dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2} \, dt^2 + \cdots + \frac{\partial f}{\partial x} \, dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \, dx^2 + \cdots$$

- Substituting X_t for x and dX_t with the SDE gives

$$df = \frac{\partial f}{\partial t} \, dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2} \, dt^2 + \cdots + \frac{\partial f}{\partial x}(\mu_t \, dt + \sigma_t \, dW_t) + \\ + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \left(\mu_t^2 \, dt^2 + 2\mu_t\sigma_t \, dt \, dW_t + \sigma_t^2 \, dW_t^2\right) + \cdots$$

Itô's Formula "Proof"

- Stopping the expansion up to the first order (i.e. negletting dt^2 and $dtdW_t$ terms), and collecting dt and dW terms, we obtain

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

as required.

Geometric Brownian Motion

- Let's apply this expansion to $G = \log(S_t)$

$$\frac{\partial G}{\partial S} = \frac{1}{S_t}, \ \frac{\partial G}{\partial t} = 0, \ \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S_t^2}$$

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$$d(\log S_t) = \left[\mu S_t \frac{1}{S_t} + \frac{1}{2}\sigma^2 S_t^2 \left(-\frac{1}{S_t^2} \right) \right] dt + \sigma dW$$

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$$S_t = S_{t-1} \exp\left[\left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma \mathcal{N}(0,1)\sqrt{dt} \right] \tag{4}$$

Log-normality

- The variation in $\log(S_t)$ equals a constant (the *drift* $\mu - \frac{1}{2}\sigma^2$) plus a Gaussian distributed random variable. Therefore at some time t

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Definition

A random variable whose logarithm is normally distributed is said to be log-normal. One of the most important properties of a log-normal distribution is to be positive definite (a good characteristic for stock prices).

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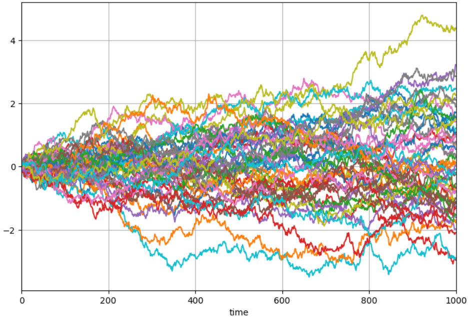
Final Remark on Stochastic Processes



When you have to get something out of a stochastic process you cannot rely on a single realization. Instead you need to take into account "all" the possible paths the process can go through in a *statistical* way with an expectation ($\mathbb{E}$). Hence it is mandatory to know (or assume) the proper probability distribution.

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When it comes to making things sound and look much more complicated than they are, financial maths really does reign supreme. Fortunately it's not just me who thinks it's all overly complicated but even Paul Wilmott thinks the profession takes it just a bit too far and he's a bit better at maths than me.

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As an example, this is number one recommended easier way to to remember what the risk-neutral measure is on Wikipedia:
"The probability measure of a transformed random variable. Typically this transformation is the utility function of the payoff. The risk-neutral measure would be the measure corresponding to an expectation of the payoff with a linear utility."

Portfolio

Definition

A portfolio is a vector $\theta \in \mathbb{R}^K$ whose j components represent the number of shares of asset A_j (asset A_0 is risk-free). It's value is

$$V_t(\theta, \omega) = \sum_{j=1}^{K} \theta_j S_t^j(\omega) \tag{5}$$

where S_t^j is the value of j-th asset, and ω a market situation. A portfolio is self-financing if its value changes only due to variations of the asset prices.

Arbitrage

Definition

An arbitrage is a self financing portfolio θ such that

$$P(V_t \geq 0) = 1 \text{ and } P(V_t \neq 0) > 0,\ 0 < t \leq T \tag{6}$$

where $V_0 = 0$, V_t denotes the portfolio value at time t and T is the time the portfolio ceases to be available on the market. This means that the value of the portfolio is never negative, and guaranteed to be positive at least once over its lifetime.

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Informally, arbitrage is a way to make a guaranteed profit from nothing, by short-selling certain assets at time $t = 0$, using the proceeds to buy other assets, and then settling accounts at time t. Arbitrage may take place when: the same asset does not trade at the same price on all markets, or two assets with identical cash flows do not trade at the same price.

Fundamental Theorems of Arbitrage Pricing

Theorem I

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But what does it mean "risk-neutral measure" ?

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- Consequently to price assets, the calculated expected values need to be adjusted for an investor's risk preferences
- Unfortunately, these adjustments would vary between investors and an individual's risk preference is very difficult to quantify.
- It turns out that, under few assumptions, there is an alternative way to do this calculation.

Risk-Neutral Measure

Denote the risk-free rate with r and assume today's stock price to be S_0. In one period of time from now, the price could be

$$\begin{cases} S_0 \cdot u = S_u \\ S_0 \cdot d = S_d \end{cases}, \quad \text{with } (u > d)$$

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To *avoid arbitrage opportunities*, conditions must be imposed on u and d.
(Example: in case $e^r > u$, I could short the stock in t_0 and invest the proceeds S_0 into the risk-free account: in both future states in t_1, I could buy the stock back for less than my proceeds $S_0 e^r$ because S_u and S_d would both be lower. Similarly for $e^r < d \ldots$)

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So imposing $d \leq e^r \leq u$, will ensure no arbitrage.

Risk-Neutral Measure

$$
\begin{aligned}
S_0 &= \frac{S_0(u-d)e^r}{(u-d)e^r} = \frac{S_0(u-d)e^r + (S_0ud - S_0ud)}{(u-d)e^r} = \\
&= \frac{1}{e^r}\left(\frac{S_0ue^r - S_0ud}{u-d} + \frac{-S_0de^r + S_0ud}{u-d}\right) = \\
&= \frac{1}{e^r}\left(S_0u\frac{e^r-d}{u-d} + S_0d\frac{u-e^r}{u-d}\right)
\end{aligned}
$$

Risk-Neutral Measure

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\begin{aligned}
S_0 &= \frac{S_0(u-d)e^r}{(u-d)e^r} = \frac{S_0(u-d)e^r + (S_0 ud - S_0 ud)}{(u-d)e^r} = \\
&= \frac{1}{e^r}\left(\frac{S_0 ue^r - S_0 ud}{u-d} + \frac{-S_0 de^r + S_0 ud}{u-d}\right) = \\
&= \frac{1}{e^r}\left(S_0 u \frac{e^r - d}{u-d} + S_0 d \frac{u - e^r}{u-d}\right)
\end{aligned}
$$

The no arbitrage condition implies the following bounds

$$
0 \leq \frac{e^r - d}{u-d} \leq 1, \quad 0 \leq \frac{u - e^r}{u-d} \leq 1
$$

Risk-Neutral Measure

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also

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Risk-Neutral Measure

So we can interpret $p_u = \dfrac{e^r - d}{u - d}$ and $p_d = \dfrac{u - e^r}{u - d}$ as (risk-neutral) probability measure ($\mathcal{Q}$).

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A probability measure is a real-valued function that assigns probabilities to a set of events in a sample space that satisfies measure properties such as countable additivity, and assigning value 1 to the entire space.

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Rewriting previous expression of S_0 in terms of the newly defined probabilities

$$S_0 = \frac{S_u p_u + S_d p_d}{e^r} = e^{-r}\mathbb{E}^{\mathcal{Q}}[S_1] \tag{7}$$

So the stock price, "under the chosen probability measure" is the discounted stock expectation at t_1.

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By contrast, if you tried to estimate the anticipated value of a stock based on how likely it is to go up or down, considering unique factors or market conditions that influence that specific asset, you would be including risk into the equation and, thus, would be looking at real or physical probability.

Risk-Neutral Measure and Pricing

Proposition

Assume there exists a risk-neutral measure $\mathcal{Q}^0$ and let A be an asset. Then, for each time t, $0 \leq t \leq T$ there exists a unique price π_t associated with A

$$\pi_t = \mathbb{E}^{\mathcal{Q}^0}[D(t,T)V_A|\mathcal{F}_t] \tag{8}$$

Such a price is given by the expectation of the discounted payoff under the measure $\mathcal{Q}^0$. Note that $\mathcal{F}_t$ is called filtration and represents our knowledge of the system up to time t, i.e. the expectation is indeed *conditioned* to what happened until time t.

Risk-Neutral Measure and Pricing

- Later in this course, in the context of the change of measure, we are going to formalize the previous slide statement.
- In summary, we will show a generalization of the original ideas of Black and Scholes, showing that, under complete markets with no arbitrage, it is possible to use for pricing purposes (only) stochastic models that do not factor in the risk premium.
- **Example:** imagine an asset such that $S_0 = 100$ and that $S_u = 120$ and $S_d = 80$. If the risk-free rate is 5% the risk-neutral probability is

$$q = \frac{e^r - d}{u - d} \approx 63\%$$

Hedging

- A portfolio θ in the assets A is a replicating portfolio for the asset B if

$$S_t^B(\omega_i) = \sum_{j=1}^{K} \theta_j S_t^j(\omega_i) \quad \forall i = 1, 2, \ldots, N \tag{9}$$

- In particular it can be demonstrated if the market is *arbitrage-free* then the relation holds for all t.
- The importance of replicating portfolios is that they enable financial institutions that sell asset B (e.g. a call options) to hedge: for each sold share of asset B, buy θ_j shares of asset A_j and hold them to time $t+1$. Then at time $t+1$,

$$\text{net gain} \;=\; \text{net loss} \;= 0$$

Fundamental Theorems of Arbitrage Pricing

- In some circumstances, an arbitrage-free market may admit more than one risk-neutral measure, i.e. incomplete markets.
- By contrast, a complete market is one that has a unique risk-neutral measure.

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Theorem II

Let $\mathcal{M}$ be an arbitrage-free market with a risk-less asset. If for every derivative security there is a replicating portfolio in the assets A_j then the market $\mathcal{M}$ is complete. Conversely, if the market $\mathcal{M}$ is complete, and if the unique risk-neutral measure $\mathcal{Q}$ gives positive probability to every market scenario ω, then for every derivative security there is a replicating portfolio in the assets A_j.

Summary of Basic Definitions

- The market is free of arbitrage if (and only if) there exists an equivalent martingale measure (EMM) (i.e. a risk-neutral measure).
- The market is complete if and only if the martingale measure is unique.
- In a complete and arbitrage-free market the price of any derivative is uniquely given, either by the value of the associated replicating strategy, or by the expectation of the discounted payoff under the risk-neutral measure.

Money Market Account

- The money market account represents a risk-less investment, where profit is accrued continuously at the risk-free rate, and its value is denoted by $B(t)$.
- We assume $B(0) = 1$ and by definition

$$dB(t) = r(t)B(t)dt \tag{10}$$

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- This evolution can be solved through variable separation

$$\begin{aligned} \frac{dB_t}{B_t} = r_t dt \implies \int_0^t \frac{dB_t}{B_t} = \int_0^t r_s ds \\ \implies \log \frac{B_t}{B_0} = \int_0^t r_s ds \implies \boxed{B(t) = \exp\left(\int_0^t r_s ds\right)} \end{aligned} \tag{11}$$

where r_t is referred to as instantaneous spot rate or short rate.

Money Market Account

- Clearly the *short rate* r_t can be modeled either by a deterministic or a stochastic process.
- In case r_t is *deterministic*, from the definition of money market account it follows that
$$V(0) = A \implies V(t) = A \cdot B(t)$$

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- In case r_t is *deterministic*, from the definition of money market account it follows that
$$V(0) = A \implies V(t) = A \cdot B(t)$$
- If we want to have at time T exactly 1 unit of currency
$$AB(T) = 1 \implies AB(t) = \frac{B(t)}{B(T)}$$

hence $\frac{B(t)}{B(T)}$ is the value of one unit of currency payable at time T seen from t.

Stochastic Discount Factor

Defintion

The (stochastic) discount factor $D(t, T)$ is the amount at time t that is *equivalent* to one unit of currency payable at time T and is given by

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  - this is motivated by the small influence interest rate variations have on equity prices.
- When dealing with interest rate products, r becomes the main actor, so the deterministic assumption must be dropped.

Money Market Account

- Looking at Eq. (7), it is apparent that we are computing prices w.r.t. the bank account (the *numeraire*). The corresponding risk-neutral measure is often denoted by $\mathcal{Q}^0$ or $\mathcal{Q}^B$, and similarly for the expectations.

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- Recalling the risk-neutral pricing formula (Eq. (8))

$$\pi_t = \mathbb{E}^{\mathcal{Q}^B}[D(t,T)A|\mathcal{F}_t]$$

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$$\pi_t = e^{-\int_t^T r_s ds}\mathbb{E}^B\left[A|\mathcal{F}_t\right]$$

- Instead when rates are constant (e.g. Black-Scholes or Heston models)

$$\pi_t = e^{-r(T-t)}\mathbb{E}^B\left[A|\mathcal{F}_t\right]$$

Zero Coupon Bond

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- What is the relation between $P(t,T)$ and $D(t,T)$?

$D(t,T)$	$P(t,T)$
equivalent amount of money	value of a contract
deterministic rates	
D(t,T)=P(t,T)	
stochastic rates	
random quantity at time t	being the value of a contract with a payoff at time T, must be known in t. Indeed $P(t,T)=\mathbb{E}^Q[D(t,T)\vert\mathcal{F}_t]$

Year Fraction, Day-Count Convention

- Time to maturity: $T - t$
- Denote by $\tau(t, T)$ the chosen time measure between t and T. In theory, you can think of $\tau(t, T) = T - t$. In practice there are many possible ways to compute that time difference.
- Among the plethora of conventions, the followings are worth mentioning:
  - Actual/365
  - Actual/360
  - 30/360
- Go and find their definitions: they are also embedded in Excel as financial functions.

shorturl.at/chzT2

Compounding

- Simply-compounded spot interest rate

$$L(t,T) = \frac{(1 - P(t,T))}{\tau(t,T)P(t,T)} \tag{13}$$

- The yield curve at time t is the graph of:

$$T \rightarrowtail L(t,T)$$

- In the market these are the so called LIBOR (or EURIBOR) rates and are typically compounded with the actual/360 convention. They are the main rates underlying interest rate derivatives.

Compounding

In the following we recall some useful definition **you should be always familiar with**.

- Annually-compounded spot interest rate

$$Y(t,T) = \frac{1}{P(t,T)^{\frac{1}{\tau(t,T)}}} - 1$$

- k-times-per-year compounded spot interest rate

$$Y^k(t,T) = \frac{k}{P(t,T)^{\frac{1}{k\tau(t,T)}}} - 1$$

- When $k \to \infty$, we get the continuously compounded rate

$$R(t,T) = -\frac{\log P(t,T)}{\tau(t,T)} \implies P(t,T) = e^{-R(t,T)\tau(t,T)}$$