Stochastic Processes

Very Brief Introduction to Stochastic Calculus

Advanced Financial Modeling

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Outline

Random Variables

A variable whose value is a number determined by the outcome of a random experiment is called a random variable.

There are two kinds of random variable:

- Discrete Random Variable
- Continuous Random Variable

Discrete and Continuous Random Variables

Discrete Random Variable

If a random variable takes only a countable number (finite) of values, it is called discrete.

Example: when 3 coins are tossed, the number of heads obtained is the random variable X, which assumes the values $\Omega = \{0, 1, 2, 3\}$ (Ω ia a countable set).

Continuous Random Variable

A random variable X which can take any value between certain interval is called continuous.

Example: the height of students in a particular class lies between 160 and 190 cm ($X = \{x|160 \leq x \leq 190\}$).

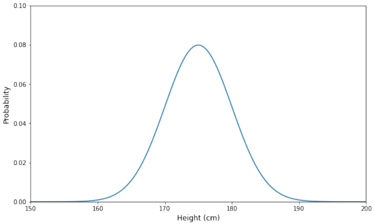
Probability Distribution

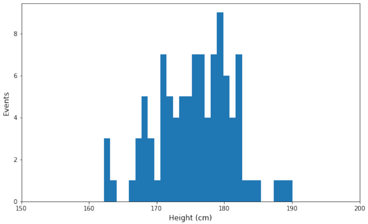
Let X be a random variable defined on a domain Ω of possible outcomes.

Discrete	Continuous
Probability Mass	Probability Density
$f(x) = P(X = x)\ \forall x \in \Omega$	
$P_i \geq 0;\ \forall i$	$f(x) \geq 0;\ -\infty < x < \infty$
$\sum_{i=0}^{n} P_i = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
Cumulative Distribution	
$F(x) = P(X < x_i) = \sum_{x_i < x} P_i$	$F(x) = P(X < a) = \int_{-\infty}^{a} f(x)dx$

Characterizing a Random Variable

If we know the distribution of a random variable, we pretty much know all there is to know about it.





But with real data, we don't know the full distribution. So we want to characterize it by a couple of numbers (statistics).

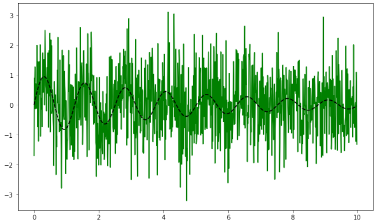
- mean, $\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$
- variance, $\sigma^2 = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$,

Properties of Expectation

Scalar multiplication	$\mathbb{E}[aX] = a\mathbb{E}[X]$
Sums	$\mathbb{E}[X_1 + \ldots X_K] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$
Linear combinations	$\mathbb{E}[a_1 X_1 + \ldots a_K X_K] = a_1\mathbb{E}[X_1] + \ldots + a_K\mathbb{E}[X_K]$
Expected value of a constant	$\mathbb{E}[a] = a$
Products (independent variables)	$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Stochastic Process

Real world data is noisy (i.e. distorted), and exhibits behaviours that cannot be described by a deterministic model (always produce same result from same inputs).



Need to switch to stochastic processes in order to model the uncertainty of data.

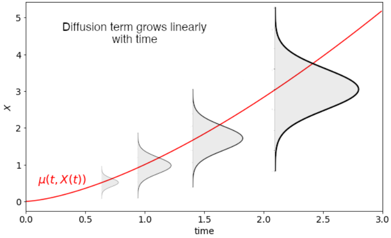
Stochastic process

A collection of random variables that is indexed by some mathematical set (usually time).

Stochastic Process

Stochastic processes are described by stochastic differential equation (SDE):

$$\begin{aligned} dX(t) &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t) \\ &= \underbrace{\mu(t, X(t))dt}_{\text{deterministic}} + \underbrace{\sigma(t, X(t))\mathcal{N}(0,1)\sqrt{dt}}_{\text{stochastic}} \end{aligned}$$



- The mean of dW is zero and its variance is dt
- the standard deviation grows with the square root of time: $W(t) \simeq \mathcal{N}(0, t)$ because each dW is distributed like independent standard Gaussian.

Martingale

Definition

A martingale is a (integrable and adapted) stochastic process which models a fair game with the following remarkable feature

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s \tag{1}$$

so the best prediction for the future value X_t, given the knowledge $\mathcal{F}_s$ at time s is the value at time s itself, X_s.

Martingale

Properties

If X_t is a stochastic process with volatility σ_t, which satisfies $\mathbb{E}[(\int_0^T \sigma_s^2 ds)^{\frac{1}{2}}] < \infty$, and SDE $dX_t = \sigma_t dW_t + \mu_t dt$

$$X \text{ is a martingale} \iff X \text{ is drift-less } (\mu_t = 0)$$

A martingale corresponds to the common notion that "an efficient price, changes randomly" so we cannot know if it will go up or down. That is why this mathematical concept is brought into finance.

Geometric Brownian Motion

- Trade random fluctuations deviate a stock price S_t from a steady state.
- The price relative change in dt can be decomposed into two parts
  - deterministic: the expected return from holding the stock during dt. It can be expressed as $\mu S_t dt$ (with μ being the expected rate of return);
  - stochastic: model the random changes of the market. A reasonable assumption is to equal this term to $\sigma S_t dW_t$.
- Putting all together, the resulting SDE is

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ \frac{dS_t}{S_t} &= d\log(S_t) = \mu dt + \sigma dW_t \end{aligned} \tag{2}$$

Ito's Lemma

Ito's Lemma

For any given continuous and differentiable function $G(S, t)$ where S satisfies $dS = adt + bdW_t$, holds

$$dG = \left(a\frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + \underbrace{\frac{1}{2}b^2\frac{\partial^2 G}{\partial S^2}}_{\text{additional term}} \right) dt + b\frac{\partial G}{\partial S}dW \tag{3}$$

This is essentially an extension of the Taylor series for stochastic functions, in the expansion an extra (2^{nd} order) term appears.

Geometric Brownian Motion

- So setting $G = \log(S_t)$

$$\frac{\partial G}{\partial S} = \frac{1}{S_t}, \ \frac{\partial G}{\partial t} = 0, \ \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S_t^2}$$

- Substituting back into Ito's lemma Eq. 3 and taking a and b values from Eq. 2

$$\begin{aligned} d(\log S_t) &= \left(\mu S_t \frac{1}{S_t} + \frac{1}{2}\sigma^2 S_t^2 \left(-\frac{1}{S_t^2}\right)\right) dt + \sigma dW \\ \log(S_t) - \log(S_{t-1}) = \log \frac{S_t}{S_{t-1}} &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW \qquad (4) \\ S_t &= S_{t-1} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \mathcal{N}(0,1)\sqrt{dt}\right] \end{aligned}$$

Log-normality

- The variation in $\log(S_t)$ equals a constant (the drift $\mu - \frac{1}{2}\sigma^2$) plus a Gaussian distributed random variable. Therefore at some time t

$$\log S_t = \mathcal{N}\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

Log-normal Variable

A random variable whose logarithm is normally distributed is said to be log-normal.
One of the most important properties of a log-normal distribution is to be positive definite (a good characteristic for stock prices).