Libor Market Models

Advanced Financial Modeling

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MPS CAPITAL SERVICES

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  3. finally the expectation could be viewed as the price of a call option in a market with zero risk-free rate, therefore obtained through the Black's formula.
- Unfortunately this is logically inconsistent.

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  - chooses a different numeraire than the risk-free bank account such that these rates can be modeled log-normally;
  - produces pricing formulas of the Black-76 type for caps/floors and swaptions;
  - is relatively easy to calibrate to market data and can be used to price more exotic products.

Setting the Model

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We have already seen that F_i is a martingale under the corresponding T_i-forward measure, on the interval $[0, T_{i-1}]$, hence each F_i has a drift-less dynamics under $\mathcal{Q}^i$.

Forward Rate Dynamics

Proposition

A discrete tenor Libor Market Model assumes that the forward rates have the following dynamics under their associated forward measures:

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t), t \leq T_{i-1}, \qquad \text{for } i = 1, \dots, n \tag{1}$$

where σ_i is assumed to be deterministic and scalar, whereas dZ_i is the i-th component of the Brownian motion.

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We know that, if σ_i is bounded, the solution for F_i is

$$F_i(T) = F_i(t) \exp\left(\int_t^T \sigma_i(s)dZ_i(s)ds - \frac{1}{2}\int_t^T \sigma_i^2(s)ds\right), \quad 0 \leq t \leq T \leq T_{i-1}$$

(for a non-rigorous proof see next slide.)

Exponential Martingale

- We have already seen that the solution of the SDE $dF_t = \mu_t F_t dt + \sigma_t F_t dW_t$ is

$$\frac{F_T}{F_t} = e^{\int_t^T (\mu_s - \frac{1}{2}\sigma_s^2) ds + \int_t^T \sigma_s dW}$$

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- If we now assume the drift coefficient $\mu_t = 0$

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- This is a pretty straightforward result, but what is the F_i dynamics under the T^j-measure $(i \neq j)$?

Forward Rate Dynamics in LMM

Proposition

Under the assumptions of the Libor Market Model, the dynamics of each F_k, for $k = 1, \ldots, n$, under the forward measure $\mathcal{Q}^i$ with $i \in \{1, \ldots, n\}$, is:

$$
\begin{cases}
k < i : dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i} \dfrac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k^i(t) \\
k = i : dF_k(t) = \sigma_k(t)F_k(t)dZ_k^i(t) \\
k > i : dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=k+1}^{i} \dfrac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k^i(t)
\end{cases}
\tag{2}
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for $t \leq \min\{T_{k-1}, T_i\}$.

Forward Rate Dynamics in LMM (Proof)

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- We would like to determine the deterministic functions $\mu_k^i(t, \bar{F}(t))$, with ($\bar{F}(t)$ is the vector $(F_1(t), \ldots, F_n(t))$), that satisfies

$$dF_k(t) = \mu_k^i(t, \bar{F}(t))F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t), \quad k \neq i \tag{3}$$

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- Let's apply the change of measure from $\mathcal{Q}^i$ to $\mathcal{Q}^k$, then impose that the $\mathcal{Q}^k$ resulting drift is null.
- The Radon-Nikodym derivative of $\mathcal{Q}^{i-1}$ w.r.t. $\mathcal{Q}^i$ at time t is

$$\frac{dQ^{i-1}}{dQ^i}\bigg|\mathcal{F}_t = \frac{P(t, T_{i-1})P(0, T_i)}{P(0, T_{i-1})P(t, T_i)} = \zeta_t^i$$

Forward Rate Dynamics in LMM (Proof)

- From the forward rate definition $\dfrac{P(t,T_{i-1})}{P(t,T_i)} = 1 + \tau_i F_i$ so

$$\zeta_t^i = \frac{P(0,T_i)}{P(0,T_{i-1})}(1 + F_i(t)\tau_i)$$

therefore, assuming Eq. (1), the dynamics of ζ_t^i under $\mathcal{Q}^i$ is

$$d\zeta_t^i = \frac{P(0,T_i)}{P(0,T_{i-1})} dF_i(t)\tau_i = \frac{P(0,T_i)}{P(0,T_{i-1})}\tau_i\sigma_i(t)F_i(t)dZ_i^i(t) =$$

$$= \frac{\tau_i\sigma_i(t)F_i(t)}{1 + F_i(t)\tau_i}\zeta_t^i dZ_i^i(t) = \lambda\zeta_t^i dZ_i^i(t)$$

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- Thus, the Radon-Nikodym derivative ζ^i is an exponential martingale with associated process λ that is the n-dimensional vector $\lambda = \left(0, \ldots, \frac{\tau_i \sigma_i F_i}{1+F_i\tau_i}, \ldots, 0\right)$

Forward Rate Dynamics in LMM (Proof)

- So from the Girsanov theorem:

$$dZ^i(t) = dZ^{i-1}(t) + \rho\lambda dt = dZ^{i-1}(t) + \rho^{ji}\frac{\tau_i\sigma_i(t)F_i(t)}{1+F_i(t)\tau_i}dt$$

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- Applying this inductively we obtain

$$\begin{cases} k < i : dZ_j^i = dZ_j^k + \sum_{h=k+1}^{i} \rho^{jh}\frac{\tau_h\sigma_h(t)F_h(t)}{1+F_h(t)\tau_h}dt; \\ k > i : dZ_j^i = dZ_j^k - \sum_{h=k+1}^{i} \rho^{jh}\frac{\tau_h\sigma_h(t)F_h(t)}{1+F_h(t)\tau_h}dt; \end{cases}$$

Forward Rate Dynamics in LMM (Proof)

- Then, inserting these into Eq. (3) and requiring the drift to be zero, we have:

$$k < i : F_k(t)\left(\mu_k^i(t, F(t)) + \sigma_i(t)\sum_{h=k+1}^{i}\rho^{jh}\frac{\tau_h\sigma_h(t)F_h(t)}{1+F_h(t)\tau_h}\right)dt = 0$$

$$\Longrightarrow \mu_k^i(t, F(t)) = -\sigma_i(t)\sum_{h=k+1}^{i}\rho^{jh}\frac{\tau_h\sigma_h(t)F_h(t)}{1+F_h(t)\tau_h}$$

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  and similarly for $k > i$.

- At this point, we can turn around the argument to have the following existence result.

Log-normal Forward Libor Model

Proposition

Consider a given volatility structure $\sigma_1, \dots, \sigma_n$, where each σ_i is bounded, and the terminal measure $\mathcal{Q}^n$ with associated n-dimensional correlated Brownian motions Z^n. If we define the processes $F_1, \dots, F_n$ by

$$dF_i(t) = -\sigma_i(t) F_i(t) \sum_{j=i+1}^{n} \rho^{ij} \frac{\tau_j \sigma_j(t) F_j(t)}{1 + F_j(t)\tau_j} dt + \sigma_i(t) F_i(t) dZ_i^n(t) \tag{4}$$

for $i = 1, \dots, n$, then the $\mathcal{Q}^i$-dynamics of F_i is given by Eq. (1), i.e. there exists a Libor Model with the given volatility structure. This model is often called log-normal forward Libor model from the log-normal distribution of each forward rate under the related forward measure.

LFM and Black Price Equivalence

Proposition

The price of the *i*-th caplet implied by the Libor market model coincides with that given by the corresponding Black caplet formula:

$$\textbf{Caplet}^{LFM}(0, T_{i-1}, T_i, K, v_i) = \tau_i P(0, T_i) Bl(K, F(0; T_{i-1}, T_i), v_i \sqrt{T_{i-1}}) \tag{5}$$

where

$$v_i^2 = \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i^2(t) dt \tag{6}$$

LFM and Black Price Equivalence (Proof)

- According to the risk-neutral pricing formula the price of a cap is

$$\mathbf{Cap} = \mathbb{E}^B\left[\sum_{i=\alpha+1}^{\beta} \tau_i D(0, T_i)(F(T_{i-1}; T_{i-1}, T_i) - K)^+\right]$$

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- Given the linearity of the expectation operator, we can apply the change of measure technique to move to $\mathcal{Q}^i$, i.e. the T-forward measure associated to bond price $P(\cdot, T_i)$

$$\textbf{Cap} = \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)\mathbb{E}^i[(F(T_{i-1}; T_{i-1}, T_i) - K)^+] = \sum_{i=\alpha+1}^{\beta} P(0, T_i)\textbf{Caplet}$$

LFM and Black Price Equivalence (Proof)

- In order to compute
$$\textbf{Caplet} = \mathbb{E}^i[(F_i(T_{i-1}) - K)^+]$$
just remember that under $\mathcal{Q}^i$ the process F_i is a martingale.
- Given its lognormal distribution, the above expectation is computed as a Black and Scholes price for a stock call option whose underlying "stock" is F_i, struck at K, with maturity T_{i-1}, with zero constant "risk-free rate" and instantaneous percentage volatility $\sigma_i(t)$.

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- The implied flat volatilities are the solutions $v_{T_1-cap}, \ldots, v_{T_n-cap}$ of the equations

$$\mathbf{Cap}(t, \mathcal{T}_j, K) = \sum_{i=1}^{j} \mathbf{Caplet}^{Black}(t, T_{i-1}, T_i, K, v_{T_j-cap}), \quad j = 1, \ldots, n$$

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- While the implied spot volatilities are the solutions $v_{T_1-caplet}, \ldots, v_{T_n-caplet}$ of

$$\mathbf{Caplet}(t, T_{i-1}, T_i, K) = \mathbf{Caplet}^{Black}(t, T_{i-1}, T_i, K, v_{T_i-caplet}), \quad i = 1, \ldots, n$$

where $\mathbf{Caplet}(t, T_{i-1}, T_i, K) = \mathbf{Cap}(t, \mathcal{T}_i, K) - \mathbf{Cap}(t, \mathcal{T}_{i-1}, K)$

Flat and Spot Volatilities

- Notice that flat volatility v_{T_1-cap} is that implied by the Black formula by putting the same average volatility in all caplets up to T_j, whereas the spot volatility $v_{T_i-caplet}$ is just the implied average volatility from caplet over $[T_{i-1}, T_i]$.

Flat and Spot Volatilities

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- To recover correctly cap prices according to the LFM dynamics, we need to have

$$\mathbf{Cap} = \sum_{i=1}^{j} \tau_i P(t, T_i)\mathbf{Caplet}^{Black}(K, F(t; T_{i-1}, T_i), \sqrt{T_{i-1} v_{T_j-cap}}) =$$

$$\sum_{i=1}^{j} \tau_i P(t, T_i)\mathbf{Caplet}^{Black}(K, F(t; T_{i-1}, T_i), \sqrt{T_{i-1} v_{T_i-caplet}}),$$

$$\forall j = 1, \ldots, n$$

Remarks on LFM and Cap Pricing

- The joint dynamics of forward rates is not involved in the pricing of a cap, its payoff is just a sum of caplet payoffs, i.e. marginal distributions of forward rates are enough to compute the expectation, correlation does not matter.

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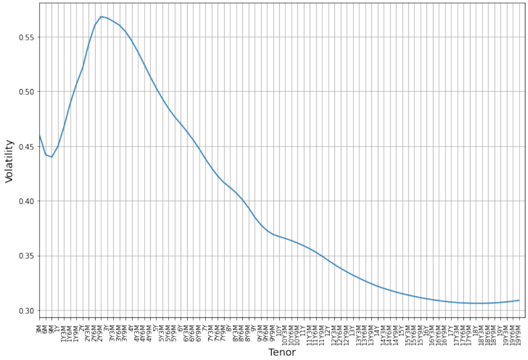
- Indeed, there are no expectations involving two or more forward rates at the same time, so that correlations are not relevant.
- From Eq. (6) it is clear how it is impossible to uniquely determine the instantaneous volatility $\sigma_i(t)$ as there exist plenty of functions that would integrate to v_i.
- Since usually one looks at particular interest rate fixings at a finite amount of times, it is customary to specify $\sigma_i(t)$ as a piecewise-constant function.

Remarks on LFM and Cap Pricing

- In general, term structures of forward rate volatilities shape does not change significantly over time.
- Further, forward rate volatilites are low close to expiry, peak around 1-2 years and then fall off again (hump-shaped).



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Remarks on LFM and Cap Pricing

- A possible explanation of the humped shape is obtained by segmenting the caplet market across three maturities, (see Rebonato, 2002)
  1. **Very short end of the curve**: central banks nowadays clearly communicate their strategy so that their actions are largely anticipated, leading to low volatilities in this region;
  2. **6M to 12-18M**: market participants continuously assess their expectations of future monetary policy, this is the region where subject beliefs have largest impact;

Remarks on LFM and Cap Pricing

- A possible explanation of the humped shape is obtained by segmenting the caplet market across three maturities, (see Rebonato, 2002)
  1. **Very short end of the curve**: central banks nowadays clearly communicate their strategy so that their actions are largely anticipated, leading to low volatilities in this region;
  2. **6M to 12-18M**: market participants continuously assess their expectations of future monetary policy, this is the region where subject beliefs have largest impact;
  3. **Longer maturities**: lastly, the third segment is much more affected by structural, long-term changes in expectations related to inflation and real rates/real returns. Thus, these long-term concerns are more or less independent of short-term monetary policies and the forward rate volatility is relatively low at the long end of the curve.

Cap Price Characterization

- Recall that the T_{i-1}-caplet is said to be *at-the-money* when its strike price $K = K_i$ equals the current value F_i of the underlying forward rate.
- The caplet is said to be *in-the-money* when $F_i > K_i$ and *out-of-the-money* when $F_i < K_i$.
- What about caps ? they are collections of caplets with a common strike K. Each caplet, however, would be ATM for a different strike K_i.
- Since it is not possible to define the ATM cap strike K in terms of the single ATM caplet K_i, we can however select a single rate that takes into account all the forward rates of the underlying caplets: that is the forward swap rate $K = S_{\alpha,\beta}$.

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- It describes the evolution of the forward swap rates instead of the forward LIBOR rates, these two kind of rates being the bases of the two main markets in the interest rate derivatives world.
- The settings of this model are similar to the LFM, the relevant exception being the choice of a more convenient numeraire.

Choice of the Numeraire

- From the pricing formula of a payer swaption

$$\mathbf{PSw} = \mathbb{E}^{\mathcal{Q}}\left[D(t,T_\alpha)A_{\alpha,\beta}\max(S_{\alpha,\beta}(T_\alpha)-K,0)\right]$$

  it comes clearly that the natural choice of numeraire to model the dynamics of the forward swap rate is

$$A_{\alpha,\beta} := \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$$

  which is referred to as the annuity or the present value of a basis point, given $\alpha,\beta \in \{0,\dots,n\}, \alpha < \beta$.

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  which is referred to as the annuity or the present value of a basis point, given $\alpha,\beta \in \{0,\ldots,n\}, \alpha < \beta$.

- Moreover it has the representation of the value at a time t of a traded asset that is a buy-and-hold portfolio consisting, for each i, of τ_i units of the zero coupon bond maturing at T_i, thus it is a plausible numeraire.

Choice of the Numeraire

- Denoted by $\mathcal{Q}^{\alpha,\beta}$ the EMM associated with the numeraire $A_{\alpha,\beta}$, the forward swap rate process $S_{\alpha,\beta}$ is a martingale under $\mathcal{Q}^{\alpha,\beta}$, on the interval $[t, T_\alpha]$.

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- The probability measure $\mathcal{Q}^{\alpha,\beta}$ is called the (forward) swap measure related to α, β.
- We may note that the annuity plays for the swap rate the same role as the zero coupon bond prices did for the forward rates in the LFM.

Log-normal Forward Swap Model

Definition

Consider a fixed subset $\mathcal{T}$ of all the pairs of integer indices (α, β) of the resettlement dates in the tenor structure $\{T_0, T_1, \ldots, T_n\}$ such that $0 \leq \alpha < \beta \leq n$ and consider for each pair a deterministic function of time $t \to \sigma_{\alpha,\beta}(t)$. A swap market model (LSM) with volatilities $\sigma_{\alpha,\beta}$ assumes that the forward swap rates have the following dynamics under their associated swap measures:

$$dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta}(t) S_{\alpha,\beta}(t) dW^{\alpha,\beta}(t), \quad t \leq T_\alpha \tag{7}$$

for $(\alpha, \beta) \in T$ pairs, where $W^{\alpha,\beta}$ is a scalar standard $\mathcal{Q}^{\alpha,\beta}$-Brownian motion.

LSM and Black Price Equivalence

Proposition

The price of a payer swaption implied by the swap market model coincides with that given by the corresponding Black swaptions formula:

$$\mathbf{PSw}^{LSM}(T_\alpha, T_\beta, K, v_{\alpha,\beta}) = A_{\alpha,\beta}(0)\mathbf{Sw}^{Black}(K, S_{\alpha,\beta}, v_{\alpha,\beta}) \tag{8}$$

where

$$v_{\alpha,\beta}^2(T) = \frac{1}{T_\alpha}\int_0^T \sigma_{\alpha,\beta}(t)^2 dt$$

LSM and Black Price Equivalence (Proof)

- From the risk-neutral pricing formula after a change of numerarie (to the annuity)

$$\begin{aligned}\mathbf{Sw} &= \mathbb{E}^{B}\left[D(0, T_\alpha)A_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+\right] = \\ &= \mathbb{E}^{\alpha,\beta}\left[\frac{A_{\alpha,\beta}(0)}{A_{\alpha,\beta}(T_\alpha)}A_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)^+\right] = \\ &= A_{\alpha,\beta}(0)\mathbb{E}^{\alpha,\beta}\left[(S_{\alpha,\beta}(T_\alpha) - K)^+\right]\end{aligned}$$

- Given the swap rate lognormal distribution of Eq. (7), computing the last expectation leads to Black's formula for swaptions.
- Indeed, the above expectation is the classical Black and Scholes price for a call option whose underlying "asset" is $S_{\alpha,\beta}$, struck at K, with maturity T_α, with 0 constant "risk-free rate" and instantaneous percentage volatility $\sigma_{\alpha,\beta}(t)$.

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Unfortunately, the answer is negative.

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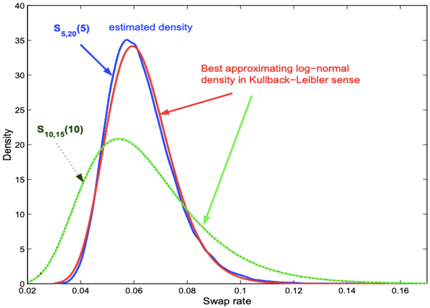
- However, from a practical point of view, this incompatibility seems to weaken considerably.
- Indeed, simulating a large number of realizations of $S_{\alpha,\beta}(T_\alpha)$ with the dynamics induced by the LFM one can compute its numerical density and compare it with the log-normal density.



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- Once ascertained the mathematical inconsistency of these two models, we must admit that the LSM is particularly convenient when pricing a swaption, because it yields the practice Black's formula for swaptions. However, for different products, even those involving the swap rate, there is no analytical formula in general.

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- Once ascertained the mathematical inconsistency of these two models, we must admit that the LSM is particularly convenient when pricing a swaption, because it yields the practice Black's formula for swaptions. However, for different products, even those involving the swap rate, there is no analytical formula in general.
- The problem left is choosing either of the two models for the whole market. After that choice, the half market consistent with the model is calibrated almost automatically, thanks to Black's formulas, but the remaining half is more intricate to calibrate.

Proposed Solution

- Since the LIBOR forward rates, rather than swap rates, are more natural and representative coordinates of the yield curve usually considered, besides being mathematically more manageable, the better choice of modeling may be to assume as framework the Log-normal Forward Libor Model.

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- If we choose one of the discount bonds $P(\cdot, T_i)$ as a numeraire, one forward rate will be a martingale, however, the swap rate being a combination of several forward rates, will not.
- We thus conclude that swaption pricing via Black's formula is not possible in the LFM.

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- But it is not much manageable. There exist, however, very good approximate formulas to the swaption volatility which can be directly used to calibrate to a matrix of quoted swaption volatilities.
- Also, performing a Monte Carlo simulation to obtain the swaption price is anyway feasible.

Table of Contents

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- In general, MC intends to estimate an expectation value through an arithmetic mean of realizations of i.i.d. random variables and it proceed as follows:
  1. let X be the r.v., with known distribution, on which the expectation we need to estimate depends;
  2. a pseudo-random number generator provides a sequence of realizations $X(k)$ of theoretical independent r.v. $X_1, X_2, \dots$ distributed as X;
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- Indeed, by the Law of large numbers, the sample average converges to the expected value, under the hypothesis that X_i is an infinite sequence of i.i.d. random variables with finite expected value.

Monte Carlo Pricing of Swaptions

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- In order to simulate all the processes involved in the payoff, their joint dynamics is discretized with a numerical scheme for SDEs, e.g. the Euler scheme.
- Recall the price of a payer swaption:

$$\begin{aligned}\mathbf{PSw} &= \mathbb{E}^{\mathcal{Q}}\left[D(0,T_\alpha)(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right]\\ &= P(0,T_\alpha)\mathbb{E}^{\alpha}\left[(S_{\alpha,\beta}(T_\alpha)-K)^+\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i)\right]\end{aligned}$$

by considering this time $P(\cdot,T_\alpha)$ as numeraire.

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- The dynamics of the k-th forward rate, for each $k = \alpha + 1, \dots, \beta$, under $\mathcal{Q}^\alpha$ is (Eq. (2))

$$dF_k(t) = \sigma_k F_k \sum_{j=\alpha+1}^{k} \frac{\rho_{kj}\tau_j \sigma_j F_j}{1 + \tau_j F_j} dt + \sigma_k F_k dW_k^\alpha, \quad t < T_\alpha \tag{9}$$

and, in order to evaluate the swaption payoff we have to generate a number of realization of $F_{\alpha+1}(T_\alpha), \dots, F_\beta(T_\alpha)$.

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  and, in order to evaluate the swaption payoff we have to generate a number of realization of $F_{\alpha+1}(T_{\alpha}), \dots, F_{\beta}(T_{\alpha})$.
- Finally the Monte Carlo price of the swaption is given by the mean of all the payoff evaluations.

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- Choosing a time grid with step $\Delta t = \dfrac{T_\alpha}{N}$ we get

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- Which provides us with approximated realizations of the true process $F_k(T_\alpha)$.

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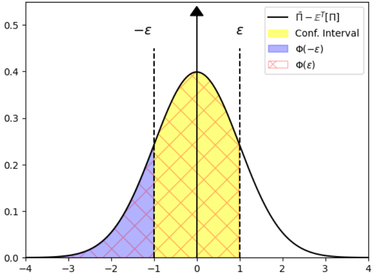
- Since $\Pi^1(T), \Pi^2(T), \ldots$ is a sequence of realizations of i.i.d. random variables distributed as $\Pi(T)$, the Central Limit Theorem tell us that for $m \to \infty$

$$\frac{\sum_{j=1}^{m}\Pi^{j}(T) - \mathbb{E}^{T}[\Pi(T)]}{\sqrt{m}Std(\Pi(T))} \to \mathcal{N}(0,1)$$

Confidence Interval

- Thus, for large m, the following approximation holds

$$\frac{\sum_{j=1}^{m}\Pi^j}{m} - \mathbb{E}^T[\Pi] \approx \frac{Std(\Pi)}{\sqrt{m}}\mathcal{N}(0,1)$$



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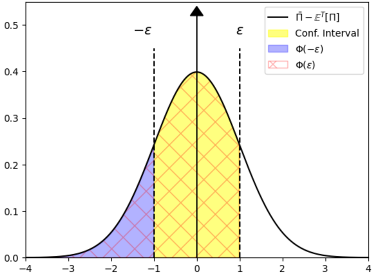


- The probability that the MC estimate is closer than ϵ to the true price is

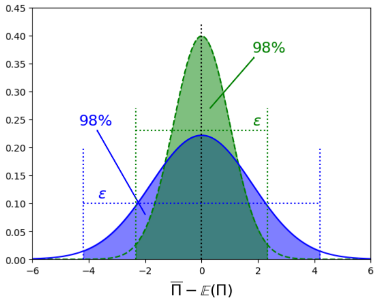
$$P\left(\left|\frac{\sum_{j=1}^{m}\Pi^j}{m}-\mathbb{E}^T[\Pi]\right|<\epsilon\right)=P\left(|\mathcal{N}(0,1)|<\epsilon\frac{\sqrt{m}}{Std(\Pi)}\right)=2\Phi\left(\epsilon\frac{\sqrt{m}}{Std(\Pi)}\right)-1$$

where Φ denotes the c.d.f. of the standard Gaussian distribution.

$$(P(|x|<\epsilon)=\Phi(\epsilon)-\Phi(-\epsilon)=\Phi(\epsilon)-(1-\Phi(\epsilon))=2\Phi(\epsilon)-1)$$

Confidence Interval

- Once we have chosen the desired value for such a probability, we can find the corresponding value for ϵ. For a typical choice of accuracy of 98%: $\epsilon \approx 2.33 \dfrac{Std(\Pi)}{\sqrt{m}}$
- Notice that as m increases, the window shrinks as $1/\sqrt{m}$.



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1. Consider another payoff Π_{an} which we can be evaluated analytically, whose expectation is denoted by $\mathbb{E}[\Pi_{an}] = \pi_{an}$, and simulate it together with Π under the same scenarios.
2. Define an unbiased estimator for $\mathbb{E}[\Pi]$ as the sample mean of the random variable

$$\Pi_c(\gamma) = \Pi + \gamma(\Pi_{an} - \pi_{an})$$

Hence $\Pi_c(\gamma)$ has expectation $\mathbb{E}[\Pi]$ and variance

$$Var(\Pi_c(\gamma)) = Var(\Pi) + \gamma^2 Var(\Pi_{an}) + 2\gamma Cov(\Pi, \Pi_{an})$$

Monte Carlo Variance Reduction

3. This can be minimized by choosing the appropriate γ $\left(\text{from } \frac{\partial Var(\Pi_c)}{\partial \gamma} = 0\right)$

$$\gamma^* = -\frac{Cov(\Pi, \Pi_{an})}{Var(\Pi_{an})} = -\frac{Corr(\Pi, \Pi_{an}) Std(\Pi)}{Std(\Pi_{an})}, \quad \left(Corr_{XY} = \frac{Cov_{XY}}{Std_X Std_Y}\right)$$

and the minimum variance of Π_c is computed as

$$Var(\Pi_c(\gamma^*)) = Var(\Pi)(1 - Corr(\Pi, \Pi_{an})^2)$$

that is smaller than the variance of Π. Moreover, the larger the correlation between Π and Π_{an} the larger the difference between the two variances.

Monte Carlo Variance Reduction

4. Moving to the standard deviation

$$Std(\Pi_c(\gamma^*)) = Std(\Pi)\sqrt{(1 - Corr(\Pi, \Pi_{an})^2)}$$

the variance reduction will increase with the correlation between Π and Π_{an}.

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 the variance reduction will increase with the correlation between Π and Π_{an}.

5. Now if we consider the confidence interval for Π_c

$$98\% C.L. = \left[\Pi_c(\gamma, m) - 2.33 \cdot \frac{Std(\Pi_c)}{\sqrt{m}}; \Pi_c(\gamma, m) + 2.33 \cdot \frac{Std(\Pi_c)}{\sqrt{m}}\right]$$

 we get a narrower width by a factor of

$$\sqrt{1 - Corr(\Pi, \Pi_{an}; m)^2}$$

Monte Carlo Variance Reduction

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- In the case of the pricing of swaptions in the LFM, we select as Π_{an} one of the simplest payoff with underlying rates $F_{\alpha+1}, \dots, F_{\beta}$, as may be a portfolio of FRA contracts at time T_α on each single time interval $(T_{i-1}, T_i]$.
- Consider the payoff of such portfolio where the $K_i = F_i(0)$ and rewrite it by a change of measure as follows:

$$
\begin{aligned}
\textbf{FRAs} = & \mathbb{E}^{\mathcal{Q}}\left[D(0,T_\alpha)\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(F_i(T_\alpha)-F_i(0))\right] = \\
& = \mathbb{E}^{j}\left[\frac{P(0,T_j)}{P(T_\alpha,T_j)}\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(F_i(T_\alpha)-F_i(0))\right] = \\
& = P(0,T_j)\mathbb{E}^{j}\left[\frac{\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(F_i(T_\alpha)-F_i(0))}{P(T_\alpha,T_j)}\right]
\end{aligned}
$$

Monte Carlo Variance Reduction

- Thus we can set

$$\Pi_{an}(T_\alpha) = \frac{\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F_i(T_\alpha) - F_i(0))}{P(T_\alpha, T_j)}$$

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- Indeed, the payoff $\Pi_{an}(\cdot)$ is a sum of traded assets divided by $P(\cdot, T_j)$, hence it is a martingale under the T^j-forward measure $\mathcal{Q}^j$, which implies that

$$\mathbb{E}^j[\Pi_{an}(T_\alpha)] = \mathbb{E}^j[\Pi_{an}(0)] = 0$$

Table of Contents

Correlated Brownian Motions

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- This model feature is included because the value of a swaption at maturity is influenced by the joint distribution of forward rates and thus by the correlation amongst them.
- For example, since the underlying in a swaption is a swap rate which in turn is a weighted average of forward rates, we expect the price of a swaption to increase if the forward rates become more correlated.

Correlated Brownian Motions

- In models for short rate it is assumed full correlation, i.e. $\rho_{ij} = 1$, which is a tight constraint on the dynamics of the forward rates.
- When trying to improve these models, one of the objectives is to lower the correlation of the forward rates implied by the model. Some authors refer to this objective as to achieving decorrelation.
- In the LFM, we can allow it by fine-tuning the corresponding parameters into the dynamics

$$dF_k(t) = \sigma_k F_k \sum_{j=\alpha+1}^{k} \frac{\boxed{\rho_{kj}}\,\tau_j \sigma_j F_j}{1 + \tau_j F_j} dt + \sigma_k F_k dZ_k^\alpha$$

Instantaneous and Terminal Correlation

Definition

The instantaneous correlation is a quantity summarizing the degree of "dependence" between changes of different forward rates.

$$\rho_{ij} = \frac{dF_i(t)dF_j(t)}{Std(dF_i(t))Std(dF_j(t))}$$

The terminal correlation is a quantity summarizing the degree of "dependence" between two different forward rates at a given *terminal* time-instant. Typically, the T_1 terminal correlation between F_i and F_j is the correlation between $F_i(T_1)$ and $F_j(T_1)$.

It is important to understand how instantaneous correlation in the forward-rate dynamics translates into a terminal correlation of simple rates.

Instantaneous Correlation

- Insantaneous correlation can be estimated from historical market data.
  1. first derive yield curves for some period of time in the past;
  2. then, forward curves may be calculated off the yield curves;
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- This has several advantages: it is computationally convenient to work with an analytical formula. Noise (e.g. non-synchronous data or illiquidity) is removed by focusing on general properties of correlation. Furthermore, the correlation matrix rank can be controlled through the functional form.
- Anyway it would be nice to imply these correlations out of liquidly traded swaption prices.

Decorrelation

- Generally, inferring instantaneous correlations from actively traded swaptions is desirable as they reflect current market conditions, thus not suffering from the backward-looking nature of historically estimated correlations.

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- There is no liquidly traded fixed income derivative that solely depends on correlation, as opposed to caplets, which solely depend on volatility.
- Another problem concerns the relationship between instantaneous and terminal correlations.

Decorrelation

- For terminal correlations, (Rebonato, 1998) shows that an appropriate quantity summarizing the amount of decorrelation between two stochastic variables from time 0 to time t is

$$\tilde{\rho}_{ij}(t) = \frac{\int_0^t \sigma_i(u)\sigma_j(u)\rho_{ij}(u)du}{\sqrt{\int_0^t \sigma_i(u)^2 du \int_0^t \sigma_j(u)^2 du}} \tag{10}$$

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- From this equation, we see that the terminal correlation not only depends on the instantaneous correlation ρ_{ij} but also on the instantaneous volatilities. Hence, even for perfectly instantaneously correlated random variables, $\rho_{ij} = 1$, terminal decorrelation could be achieved by time-dependent instantaneous volatilities.

An Example

- Considering Rebonato's analytical formula for terminal correlation Eq. (10) it can be shown that in the case of piecewise-constant volatilities, terminal correlation is just the average correlation over the period.

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$$\tilde{\rho}_{ij}(t) = \frac{\cancel{\sigma_i}\cancel{\sigma_j}\int_0^t \rho_{ij}(u)du}{\sqrt{\cancel{\sigma_i^2}\cancel{\sigma_j^2}t^2}} = \frac{\int_0^t \rho_{ij}(u)du}{t}$$

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  which indeed leads to lower terminal correlation.
- Swaption payoffs depend on the terminal correlation between several different forward rates which leads (Brigo and Mercurio, 2006) to the conclusion that swaption volatilities are more directly linked with terminal correlations rather than with instantaneous ones.