Interest Rate Derivatives

Main IR Contracts Theory

Advanced Financial Modeling

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MPS CAPITAL SERVICES

Single Curve Approach

What follows is a detailed exposition of the classic single curve approach for interest derivatives. Today a multicurve approach is used in practical applications. Nevertheless you need to understand deeply this basic approach as a prerequisite for the extension to the multi-curve model.

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Forward Rate Agreement

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- The FRA payout consists of an exchange of interest rate flows calculated for the time period $\tau = S - T$. At the maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T, S)$ whose value is known only in T.

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- The FRA payout consists of an exchange of interest rate flows calculated for the time period $\tau = S - T$. At the maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T, S)$ whose value is known only in T.
- Basically, this contract allows one to lock-in the interest rate between times T and S at a desired value K. (**Note:** interest rate flows are calculated using the simple compounding law).

Forward Rate Agreement

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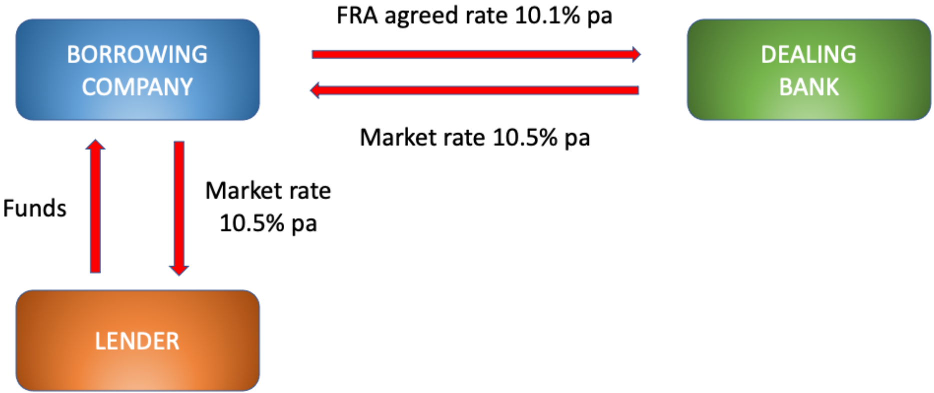
- Consider a 3×6 FRA (3-month into 6-month): the 3 in the 3×6 refers to 3 months' time when settlement (fixing) takes place, and the 6 to the expiry date of the FRA from deal date, i.e. the rate quoted for the FRA is a 3-month rate at the time of settlement

FRA Example



FRA agreed rate 10.1% pa
BORROWING COMPANY
DEALING BANK
Market rate 10.5% pa
Funds
Market rate 10.5% pa
LENDER

FRA: Formalization of the Contract

- Formally, at time S one receives $\tau(T,S)KN$ units of currency and pays the amount $\tau(T,S)L(T,S)N$, where N is the contract nominal value.

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  - estimate the future value of $L(T,S)$;
  - discount the result from S to today (time t).
- There are several ways to arrive at the final result: **no arbitrage is the common denominator.**

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- First interpret A as an amount of money held at time S. At T it's worth 1, indeed

$$A = P(S,S)A \implies P(T,S)A = P(T,S)\frac{1}{P(T,S)} = 1$$

which in turn, at time t, equals to

$$P(t,T)1 = P(t,T)$$

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- On the other hand B in S becomes at time t: $P(t,S)\tau K+P(t,S)$.
- Collecting the terms we get

$$\mathbf{FRA}(t,T,S,\tau,N,K)=N[P(t,S)\tau K–P(t,T)+P(t,S)] \qquad (2)$$

FRA Valuation: a Different Approach

Now we will try to estimate the value of a FRA using a simple replication approach:

- at time t:

$$\begin{cases} \text{lend } P(t,T) \\ \text{borrow } P(t,S)(1+\tau(T,S)K) \end{cases} ;$$

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- at time T: you receive 1 which you will invest at the (in t unknown) EURIBOR rate $L(T,S)$;
- at time S:
$$\begin{cases} \text{receive } (1+L(T,S)\tau(T,S)) \\ \text{pay } 1+\tau(T,S)K \end{cases} .$$

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Adding the cash-flows together and evaluating in t we get:

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According to the *Fundamental Theorem II* its value must equal to the value of the replicating portfolio in t (we assume to be in a complete market). Hence

$$\begin{aligned} \mathbb{E}^Q[D(t,S)(L(T,S) - K)\tau(T,S)|\mathcal{F}_t] = \\ [P(t,S)\tau(T,S)K - P(t,T) + P(t,S)] \end{aligned} \tag{3}$$

FRA Detailed Proof I

- Let's start from the risk neutral pricing formula

$$\mathbf{FRA} = \mathbb{E}_t^{\mathcal{Q}}[D(t,S)\tau(K - L(T,S))] = \mathbb{E}_t^{\mathcal{Q}}[D(t,S)\tau K - D(t,S)\tau L(T,S)]$$

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- From which we can easily derive

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which follows from the definition of **ZCB** value.

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- From the definition of discount factor we know that

$$D(t,S) = D(t,T)D(T,S)$$

FRA Detailed Proof II

- So we get
$$\textbf{FRA} = \tau K P(t,S) - \mathbb{E}_t[\tau D(t,T) D(T,S) L(T,S)]$$

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- Last equation can be transformed as (using another property of expectation)

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- From which again

$$\textbf{FRA} = \tau K P(t,S) - \mathbb{E}_t[\tau D(t,T) L(T,S) P(T,S)]$$

FRA Detailed Proof III

- Then using the definition of LIBOR rate

$$
\begin{aligned}
\textbf{FRA} =& \tau K P(t,S) - \mathbb{E}_t\left[D(t,T)P(T,S)\left(\frac{1}{P(T,S)}-1\right)\right] = \\
&= \tau K P(t,S) - \mathbb{E}_t[D(t,T)] + \mathbb{E}_t[D(t,T)P(T,S)]
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FRA Detailed Proof III

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- The last term can be expressed in terms of D

$$\textbf{FRA} = \tau K P(t,S) - \mathbb{E}_t[D(t,T)] + \mathbb{E}_t\left[D(t,T)\mathbb{E}_T[D(T,S)]\right]$$

- Bringing the discount factor inside the expectation

$$\begin{aligned}\textbf{FRA} =& \tau K P(t,S) - \mathbb{E}_t[D(t,T)] + \mathbb{E}_t\left[\mathbb{E}_T[D(t,T)D(T,S)]\right] = \\ =& \tau K P(t,S) - \mathbb{E}_t[D(t,T)] + \mathbb{E}_t\left[\mathbb{E}_T[D(t,S)]\right]\end{aligned}$$

FRA Detailed Proof IV

- Then applying the law of iterated expectations we get

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- Finally we arrive at the final result

$$\mathbf{FRA} = \tau K P(t, S) - P(t, T) + P(t, S)$$

FRA: Forward Rate as Break-Even Rate

Definition

The value of K which makes the FRA worth zero is called simply-compounded forward rate

$$F(t;T,S) := \frac{1}{\tau(T,S)}\left[\frac{P(t,T)-P(t,S)}{P(t,S)}\right] \tag{4}$$

The forward rate can be interpreted as a rate observed in t and spanning the time period $S-T$.

Forward Rate and No-Arbitrage

- Following Filipovic (2009) we can also directly define the Forward Rate by implementing the following.
- At time t
  - sell one T-ZCB
  - buy $\frac{P(t,T)}{P(t,S)}$ S-ZCB

  This results in a zero net investment.
- At T you must pay 1.
- At time S you receive $\frac{P(t,T)}{P(t,S)}$
- The net effect of the operation is a *forward* investment of one dollar a time T yielding $\frac{P(t,T)}{P(t,S)}$ so

$$1 + \tau(T, S)F(t; T, S) = \frac{P(t, T)}{P(t, S)}$$

Forward Rate

- We can now rewrite the value of the FRA in terms of the simply-compounded forward rate

$$\begin{aligned} \textbf{FRA} &= N[\tau K P(t,S) - P(t,T) + P(t,S)] = \\ &= N\tau P(t,S)\left[K + \frac{1}{\tau}\frac{P(t,S) - P(t,T)}{P(t,S)}\right] = N\tau P(t,S)(K - F(t;T,S)) \end{aligned} \tag{5}$$

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- The forward rate can be interpreted as an *estimate* of the future spot rate, which is unknown at time t (random process based on the market conditions).
- We'll see later that indeed $F(t;T,S)$ is the expectation of $L(T,S)$ under a particular probability measure.

Instantaneous Forward Rate

- Now let's introduce the instantaneous forward rate $f(t,T)$.
- It is defined as

$$
\begin{aligned}
f(t,T) &:= \lim_{S \to T^+} F(t;T,S) \\
&= \lim_{\epsilon \to 0} \frac{1}{\tau(T,T+\epsilon)} \frac{P(t,T) - P(t,T+\epsilon)}{P(t,T+\epsilon)} \\
&= \lim_{\epsilon \to 0} -\frac{1}{P(t,T)} \frac{P(t,T+\epsilon) - P(t,T)}{\epsilon} \\
&= -\frac{\partial \log P(t,T)}{\partial T}
\end{aligned}
\tag{6}
$$

Instantaneous Forward Rate

- From the previous equation we can derive

$$P(t, T) = e^{-\int_t^T f(t,s)ds} \tag{7}$$

- The instantaneous forward rate represents the rate for a forward contract with an infinitesimal investment period after the settlement date.
- Notice that

$$r(t) = f(t, t)$$

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Interest Rate Swap

- An Interest Rate Swap (IRS) is a contract that exchanges payments between two differently indexed *legs*, starting from a future time instant. At every pre-specified instant T_i, the fixed leg pays the amount (N is the contract nominal)

$$N\tau(T_{i-1}, T_i)K$$

while the variable leg pays the amount

$$N\tau(T_{i-1}, T_i)L(T_{i-1}, T_i)$$

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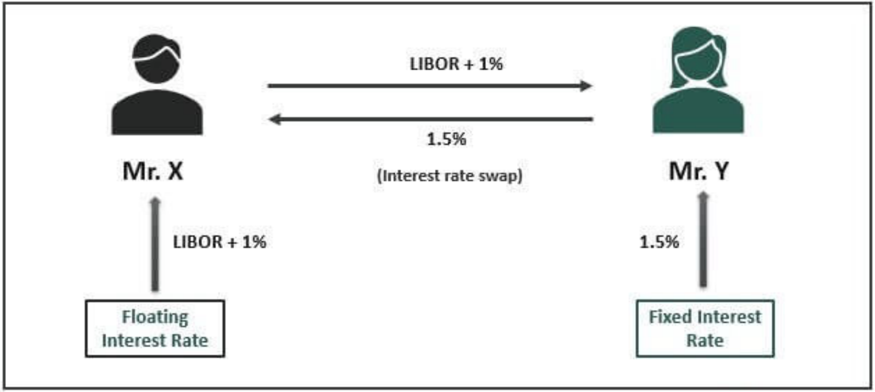
- When the fixed leg is paid, the IRS is termed Payer IRS. If the opposite holds, we have a Receiver IRS.
- The discounted payoff of a Payer IRS can be expressed as:

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)N \underbrace{\tau_i}_{\tau(T_{i-1},T_i)} [L(T_{i-1}, T_i) - K] \tag{8}$$

Interest Rate Swap



IRS and FRA

- We can view the last expression (Eq. (8)) as a portfolio of FRAs though.
- Indeed consider a Receiver IRS and express its payoff as a sum of FRAs with Eq. (5)

$$\begin{aligned}\mathbf{RFS}(t,T,\tau,N,K) &= N\sum_{i=\alpha+1}^{\beta}\mathbb{E}^{Q}[D(t,S)\tau_i(K-L(T_{i-1},T_i))] = \\ &= N\sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i)(K-F(t;T_{i-1},T_i)) \\ &= N\sum_{i=\alpha+1}^{\beta}\tau_i K P(t,T_i) - N\sum_{i=\alpha+1}^{\beta}(P(t,T_{i-1})-P(t,T_i)) \\ &= N\sum_{i=\alpha+1}^{\beta}\tau_i K P(t,T_i) - NP(t,T_\alpha) + NP(t,T_\beta)\end{aligned} \tag{9}$$

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- In a *Payer Swap* with an increasing yield curve the first FRAs will have negative value, the last positive. Still their NPV's sum is zero.
- If the market don't move so much the swap rate payer will have to pay a net interest differential in the first part of the contract life, and to receive net interest in the second.

IRS and FRA

- Clearly not all the FRAs in a swap with zero value have zero value (*except for the degenerate case of flat yield curve*): some of them will have positive NPV, some other negative; the only constraint being their sum to be zero.
- In a *Payer Swap* with an increasing yield curve the first FRAs will have negative value, the last positive. Still their NPV's sum is zero.
- If the market don't move so much the swap rate payer will have to pay a net interest differential in the first part of the contract life, and to receive net interest in the second.
- So the rate payer will have to fund the payments for the first part of the contract and to invest at his best the receipt during the second part. (This is a very hot topic nowadays; this feature is sometimes referred to as the funding profile of the contract).

Swap Rate as Break-Even Rate

Definition

The fixed rate K which makes the above expression null is called forward swap rate:

$$S_{\alpha,\beta}(0) = \frac{P(0, T_\alpha) - P(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)} \tag{10}$$

If $T_\alpha = 0$ we have the spot swap rate (which is published on financial newspapers).

The swap rate makes the contract fair at inception by definition.

Swap Payoff Alternative Expression

- Consider a Payer IRS with $N = 1$ (using Eq. (9))

$$\mathbf{PFS} = P(t, T_\alpha) - P(t, T_\beta) - \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i)$$

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- By multiplying and dividing by (the so called annuity) $A = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$ we get...

$$\mathbf{PFS} = \frac{A}{\sum \tau_i P(t, T_i)} \left[P(t, T_\alpha) - P(t, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i) \right] = \\ = A(S_{\alpha,\beta}(t) - K) \tag{11}$$

(**note:** this expression will be useful when pricing swaptions).

Swap and Bond Switching

- Consider again a Payer Forward Swap (this time with notional N)

$$\mathbf{PFS}(t, T_\alpha, T_\beta, \tau, N, K) = N(P(t, T_\alpha) - P(t, T_\beta)) - NK \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)$$

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- A swap can be considered as an exchange between two kinds of bond with the same notional reimbursed at maturity. In fact, *the fixed leg* of the swap can be viewed as a fixed coupon stream, while the *variable* can be considered a floating rate note coupon stream.

Fixed Rate Side

- More formally, consider a coupon bond that pays the following cash flows $\mathcal{C} = \{c_{\alpha+1}, \dots, c_\beta\}$ on the schedule $\mathcal{T} = \{T_{\alpha+1}, \dots, T_\beta\}$ with

$$\begin{cases} c_i = N\tau K, & i < \beta \\ c_\beta = N\tau K + N, & i = \beta \end{cases}$$

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- Needless to say that in case $K = 0$ the bond reduces to a zero-coupon bond with maturity T_β.

Floating Rate Note (FRN)

- Next consider a floating rate note that pays coupons at dates

$$\mathcal{T}_{\text{payments}} = \{T_{\alpha+1}, \dots, T_{\beta}\}$$

coupons calculated at the LIBOR rate fixed in the previous period

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$$= NP(t, T_\alpha) - \cancel{NP(t, T_\beta)} - \cancel{N \sum_{i=\alpha+1}^{\beta} K \tau_i P(t, T_i)} + \cancel{NP(t, T_\beta)} = NP(t, T_\alpha)$$

Floating Rate Note (FRN)

- The intuition behind this formula is quite straightforward: if we set $T_\alpha = t$. At the date of the first reset, the bond price equals par.
- This result also holds for all the dates equal to the reset of the floating rates (an **FRN** is equivalent to a roll-over strategy of short-term loans).
- This property is sometimes summarized by the sentence "the **FRN** always trade at par".
- The **FRN** is a debt security in which coupon payments adjust according to changes in interest rates. The coupons are closely tied to current short-term spot rates, such that their prices are always at or near par value (no arbitrage).

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Concepts behind the Formulas

Suppose that a generic bank, *MyFavouriteBank*, has the same credit risk of the corresponding average inter-bank entity. So, the spread over the LIBOR rate is 0. Suppose now the bank needs financing and it plans to issue coupon bonds. It has two alternatives:

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1. borrow N and pay floating interests at the rates

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2. borrow N and pay fixed interests given by the coupon

$$K = S_{\alpha,\beta}.$$

   (clearly the overall cost to raise money must be the same at the beginning, i.e. fixed leg NPV equals floating leg NPV).

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- Given the two strategies are equivalent the bank will then opt for one of two alternatives depending on:
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- Remember that a variable rate mortgage is perceived risky by the average individual as salary is more or less fixed, but for bank this is not the case because it is left unarmed by the rate rise. Indeed if rates go up
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- The same more or less holds for a rate decrease.

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- Banks like floating rates exposure:
  - ✔ the liability value (the bond floating rate) resets with rates;
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  - ✘ higher interest rate risk (risk of rising rates in the future);
  - ✘ with inverted yield curve, the cost of debt may actually be higher than fixed-rate debt (however, this is the exception rather than the norm).

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- So why do not banks issue only floating rate notes ?
- Banks must issue fixed rate coupon bonds to attract customers (think of yourself or insurance companies) but then will hedge the liability with an investment bank. In this way the final risk exposure will be at a variable rate as they like.
- Swaps are both the hedging instrument and the pricing tool.

Basis Swap

- A (Tenor) Basis Swap is defined as a contract in which two floating legs are exchanged.
- The two legs can differ both in the tenor of the underlying rate and in the frequency of payments.
- For instance a typical Basis Swap is one in which a floating leg indexed to EURIBOR-6M is exchanged against one indexed to EURIBOR-3M.

Basis Swap Valuation

- A Tenor Basis Swap can be priced according to the following

$$\mathbf{TBS}(t) = \sum_{i=1}^{n} \tau_i^L F^L(t; T_{i-1}, T_i) P(t, T_i) - \sum_{j=1}^{m} \tau_j^S (F^S(t; T_{j-1}, T_j) + s) P(t, T_j) \tag{13}$$

where L (long index) denotes the index with the longer tenor and S (short index) the other with the shorter tenor.

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- This is made possible by adding a basis spread s to the Ibor index with the lower frequency tenor.

Basis Swap Spread

- But according to the single curve framework there should be no basis as the value of the floating leg is given by (Forward Start case)

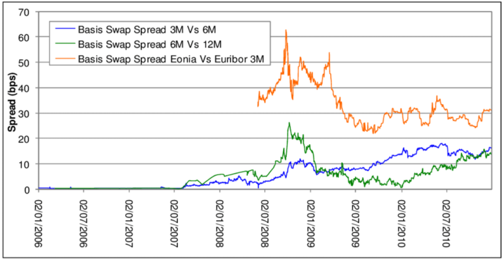
$$P(0, T_0) - P(0, T_n)$$

  or the following (Spot case)

$$1 - P(0, T_n)$$

  (assuming $N = 1$).

- In both cases what matters are the first reset date and the last payment date; hence if they are the same for the two legs (regardless the reset frequency and the index tenor), the basis should be zero.



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Adding Credit Risk

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- This lead us to the next topic, the Asset Swap (and the asset swap spread).

Par Asset Swap

- An Asset Swap (AS) can be defined as a *synthetic floating-rate note*.
- In fact, the Asset Swap transforms a fixed into a floating rate, leaving the credit risk profile unchanged.
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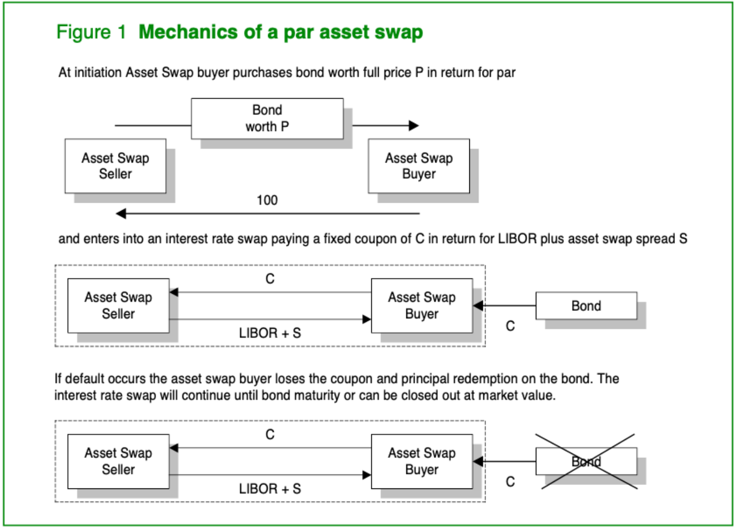
- In case of default of the bond issuer, the Asset Swap buyer must pay the fixed leg and the principal in the swap but does not receive the coupon of the defaulted bond.

Valuation of the Asset Swap



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- the AS buyer buys a generic *risky* coupon bond at the market price $\overline{\textbf{CBP}}(t, T, K, 1)$;
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- A swap is then started between the two counter-parties such that the AS seller receives a fixed leg equal to the coupon stream of the bond and the AS buyer receives the floating leg given by LIBOR rate plus a spread ($ASWS$).

Valuation of the Asset Swap

- From the perspective of the Asset Swap seller the value of the package is given by (we are considering spot trading so $T_\alpha = t = 0$)

$$\begin{aligned} \mathbf{ASW} &= 1 - \overline{\mathbf{CBP}} + K \sum_{i=1}^{\beta} \tau_i P(0, T_i) - \sum_{i=1}^{\beta} \tau_i P(0, T_i)(L(T_{i-1}, T_i) + ASWS) = \\ &= 1 - \overline{\mathbf{CBP}} + K \sum_{i=1}^{\beta} \tau_i P(0, T_i) \\ &\quad - \sum_{i=1}^{\beta} \tau_i P(0, T_i) L(T_{i-1}, T_i) - \sum_{i=1}^{\beta} \tau_i P(0, T_i) ASWS = 0 \end{aligned} \tag{14}$$

Valuation of the Asset Swap

- We can replace the future rates with the forward rates, and by its definition (Eq. (4))

$$\sum_{i=1}^{\beta} \tau_i P(0, T_i) F(t; T_{i-1}, T_i) = \sum_{i=1}^{\beta} \cancel{P(0, T_i)} \frac{P(0, T_{i-1}) - P(0, T_i)}{\cancel{P(0, T_i)}} =$$
$$= P(0,0) \cancel{-P(0, T_1)} + \cancel{P(0, T_1)} \cancel{-P(0, T_2)} + \cancel{P(0, T_2)} +$$
$$\ldots \cancel{-P(0, T_{\beta-1})} \cancel{+P(0, T_{\beta-1})} - P(0, T_\beta) = 1 - P(0, T_\beta)$$

- Substitute into Eq. (14) to get

$$\mathbf{ASW} = 1 - \overline{\mathbf{CBP}} + K \sum_{i=1}^{\beta} \tau_i P(0, T_i) - (1 - P(0, T_\beta)) + \sum_{i=1}^{\beta} \tau_i P(0, T_i) ASWS = 0$$

Valuation of the Asset Swap

- Canceling out the 1s

$$\mathbf{ASW} = -\overline{\mathbf{CBP}} + K\sum_{i=1}^{\beta}\tau_i P(0,T_i) + P(0,T_\beta) + \sum_{i=1}^{\beta}\tau_i P(0,T_i) ASWS = 0$$

- Finally we know that

$$K\sum_{i=1}^{\beta}\tau_i P(0,T_i) + P(0,T_\beta)$$

represents the price of coupons and principal of a risk-free bond which can be denoted by **CBP**.

Asset Swap Spread

Solving for $ASWS$ we arrive at the final expression

$$ASWS = \frac{\mathbf{CBP} - \overline{\mathbf{CBP}}}{\sum_{i=1}^{\beta} \tau_i P(0, T_i)} \tag{15}$$

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- She is still exposed to the loss of the coupons and redemption on the bond, i.e. the difference between the bond price and recovery value.
- In economic terms the purpose of the Asset Swap Spread is to compensate the Asset Swap buyer for taking these risks, which is measured by the spread.

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Asset Swap: Credit Considerations

- If the credit worthiness of the issuer reduces ($\overline{\textbf{CBP}}$ decreases), rates remain constant, so $ASWS$ increases.
- To avoid arbitrage opportunities, $ASWS$ of bond with maturity T_N must be very close to corresponding maturity CDS spread X_N
  - if $ASWS - X_N > 0$ the investor can buy the bond, financing the purchase, enters in AS and buying protection, making an (almost) risk free profit (*negative basis trading*);
  - if $ASWS - X_N < 0$ can do the opposite.
- Since the $ASWS$ is quoted as a spread to LIBOR, for assets of better credit quality than AA-rated banks it may be negative.

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- However there are a number of complications...
- Interest rate derivatives depend on a variety of instruments, used in the determination of the interest rate curve, rather than a single asset.
- Because there are many ways of shifting the interest rate curve, many different deltas can be computed.

Basis Point Value

- The Basis Point Value (BPV) is computed by changing the value of the fixed coupon by 1 bp and evaluating the impact on the IRS NPV. It is equal to the discounted value of the cash-flows for a rate of 0.01% for all periods of the fixed leg

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- This is a useful measure for dealers calculating the exact P&L generated by applying a spread (or margin) to a fixed rate away from the mid-market rate.
- Nevertheless BPV can also be used as an approximation for the interest rate risk.

Basis Point Value

- Indeed in the single-curve framework the "fair" swap rate can be expressed as

$$K_{\text{fair}} = \frac{1 - P(0, T_\beta)}{\sum_i P(0, T_i)}$$

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- When perturbing the term structure, it turns out that K_{fair} changes a lot more than the individual discount factors do.
- Therefore, NPV_{float} variation can be neglected and the total NPV variation can be approximated by

$$\Delta\mathbf{PS} = K'_{\text{fair}} \sum_i P(0, T_i) - K_{\text{fair}} \sum_i P(0, T_i) = \sum_i \underbrace{(K'_{\text{fair}} - K_{\text{fair}})}_{\text{typically few bps}} P(0, T_i)$$

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- In the *real* scenario the fixed rate is, well, fixed and floating rates move, so you can consider what happens to the NPV if every forecast rate r_i are changed in parallel by the same amount

$$\mathbf{DV01} = \sum_j \frac{\partial \mathbf{PFS}}{\partial r_j} = -\sum_j \tau_j P_j + \sum_j \left(K \sum_{i=\alpha+1}^{\beta} \tau_i \frac{\partial P_i}{\partial r_j} - \sum_{k=\alpha+1}^{\beta} L_k \tau_k \frac{\partial P_k}{\partial r_j} \right)$$

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- In the multi-curve framework DV01 would be calculated for the forecast curve, and for the discounting curve, resulting in two actual DV01 measurements.

DV01 Numerical Calculation

- Assume the P&L on a swap could be approximated by its linear P&L plus its convexity

$$\Delta \mathbf{PFS}(\delta r) \approx \frac{\partial \mathbf{PFS}}{\partial r}\delta r + \frac{1}{2}\frac{\partial^2 \mathbf{PFS}}{\partial r^2}\delta r^2$$

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- Then bumping δr by ± 1 bp and dividing by 2 eliminates the convexity element and very accurately approximates the real DV01

$$\mathbf{DV01} = \frac{\Delta\mathbf{PFS}(+1\ bp) - \Delta\mathbf{PFS}(-1\ bp)}{2} = \frac{\partial\mathbf{PFS}}{\partial r}$$

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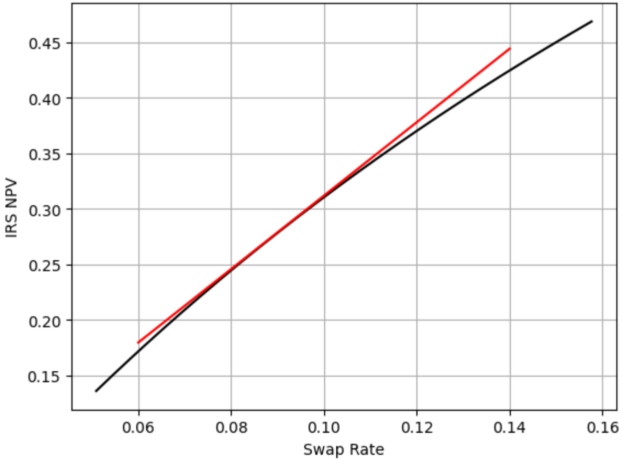
- Another common method of calculation is to use a single bumped curve by, say, $\frac{1}{100}$th of a bp, and scale the result by 100 (less accurate, since the convexity is marginalised and not eliminated)

$$100\Delta \mathbf{PFS}\left(\frac{1}{100}\ bp\right) = \frac{\partial \mathbf{PFS}}{\partial r} + \frac{1}{200}\frac{\partial^2 \mathbf{PFS}}{\partial r^2}$$

DV01 Numerical Calculation



Market Rate Sensitivity

- Another popular, though more complicated, method is to shock each input instrument used in the *bootstrapping* by 1 bp one by one (i.e., change the futures rate by 1 bp, the swap rates by 1bp, OIS rates by 1 bp, etc. . .), rebuild the curves, and then reprice the instrument of interest to obtain its curve sensitivity.

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- This of course is not quite a "parallel" shift of any curve (e.g., a 1 bp change in futures rate won't correspond to a 1bp change in the others).
- In a plain vanilla swap is very close but not equal to BPV. The mark to market value of the swap is not linear (as in a future contract), but rather it is a convex function of the rates (just like a bond is a convex function of the yield).

Trading and Hedging Swaps

- With a swap you can have a clear picture of the market. For example if you own the $5y$, you make money if rates go higher, loose money if they go lower.
- With two swaps instead you can bet on the slope of the interest rate curve.
- If you bet on steepening of the $30y - 10y$ slope, you pay the $10y - 30y$; if you bet on flattening on the same portion of the yield curve you receive the $10y - 30y$.
- With swaps you can bet on the *bund basis*: this quantity is linked to the evolution of credit and liquidity in the inter-bank and Government bond markets.

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Caps and Floors

- A Cap is a Payer IRS in which the payment is done only if the payoff is positive. Its value is the expectation of

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i \max\left[L(T_{i-1}, T_i) - K, 0\right] \tag{17}$$

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- A Floor is the same kind of object but analogous to a Receiver IRS:

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i \max\left[K - L(T_{i-1}, T_i), 0\right] \tag{18}$$

Caplet and Floorlet

- Considering each element of the sum in Eq. (17) or Eq. (18) we see that Cap/Floor can be split into forward starting options over a floating rate, called Caplet/Floorlet.

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- A Caplet/Floorlet payoff is defined as

$$D(t, T_i) N \tau_i \max\left[L(T_{i-1}, T_i) - K, 0\right]$$

and its value is given by

$$\mathbf{Cpl}(t, T_{i-1}, T_i, \tau, N, K) = \mathbb{E}^{\mathcal{Q}}\left(e^{-\int_t^{T_i} r_s ds} N\tau(L(T_{i-1}, T_i) - K)^+ | \mathcal{F}_t\right) \tag{19}$$

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- This can also be written

$$\mathbf{Cpl} = N \mathbb{E}^{\mathcal{Q}} \left(e^{-\int_t^{T_{i-1}} r_s ds} \tau P(T_{i-1}, T_i)(L(T_{i-1}, T_i) - K)^+ | \mathcal{F}_t \right)$$

Caplets as ZCB Put Options

- Using the LIBOR rate definition we get

$$\mathbf{Cpl} = N\mathbb{E}^{\mathcal{Q}}\left(e^{-\int_t^{T_{i-1}} r_s ds} P(T_{i-1}, T_i)\left[\frac{1}{P(T_{i-1}, T_i)} - 1 - K\tau\right]^+ |\mathcal{F}_t\right)$$
$$= N\mathbb{E}^{\mathcal{Q}}\left(e^{-\int_t^{T_{i-1}} r_s ds}\left[1 - (1 + K\tau)P(T_{i-1}, T_i)\right]^+ |\mathcal{F}_t\right)$$

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- Multiplying by $\frac{1}{1+K\tau}$ we finally get

$$\mathbf{Cpl} = N(1 + K\tau)\mathbb{E}^{\mathcal{Q}}\left(e^{-\int_t^{T_{i-1}} r_s ds}\left[\frac{1}{1 + K\tau} - P(T_{i-1}, T_i)\right]^{+} |\mathcal{F}_t\right) \tag{20}$$

Caplets as ZCB Put Options

Intepretation

Caplets can then be seen as put options on ZCBs. In the same way, floorlets can be seen as call options on ZCBs.

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The Black Model - Overview

- The Black Model extends the Black-Scholes formula to *caplets*, *swaptions* and *bond options*.
- The main difference with respect to the Black-Scholes set up is that forward rates (or swap rates...) are log-normally distributed, rather than the spot price of the underlying.
- So $F(t; T_{i-1}, T_i)$ (or $S_\alpha(t)$) are modeled as log-normal random variables. But not at the same time ! If $F(t; T_{i-1}, T_i)$ is log-normal, then $S_\alpha(t)$ cannot be.

The Black Model: Overview

- It is widely used in practice.
- Black formula was indeed the metric by which traders translated volatilities into prices until rates became too low and the model collapsed under the assumption of positive rates.
- ...but for the moment we cannot consider it as a model ! Just a bunch of formulas. Their formal justification will be provided later in the context of the Libor Market Model.

Pricing Caps with the Black-76 Formula

Definition

$$
\begin{aligned}
\mathbf{Cap}_{Bl}(0,\tau,N,K,\sigma_{\alpha,\beta}) &= N\sum_{i=\alpha+1}^{\beta}\mathbf{Caplet}_{Bl}(T_i,\tau,K,\sigma_{\alpha,\beta}) = \\
&= N\sum_{i=\alpha+1}^{\beta}\tau P(0,T_i)\mathbf{Bl}(K,F(0;T_{i-1},T_i),v_i)
\end{aligned} \tag{21}
$$

where

$$\mathbf{Bl}(K,F_i,v_i) = F\Phi(d_1(K,F_i,v_i)) - K\Phi(d_2(K,F_i,v_i))$$

$$d_{1,2} = \frac{\log\frac{F_i}{K} \pm \frac{v_i^2}{2}}{2}$$

$$v_i = \sigma_{\alpha,\beta}\sqrt{T_{i-1}}$$

Problems with the Black Model

- In the Black model negative rates are not allowed. Hence a zero strike floor cannot be priced

$$d_{1,2} = \frac{\boxed{\log \frac{F}{K}} \pm \frac{v^2}{2}}{2}$$

  but in the last years in the inter-bank market it was not so unusual to find prices for -1% strike floors.
- Moreover in the Black model the empirical evidence of the "smile" (volatility vs K) is not accounted for, i.e. σ is a constant. Two caps identical but for the strike need a different volatility to recover two different market prices if one uses Black formula.

The Practitioner Solutions

- To face the "smile" issue, the model is used with different input volatilities for different strikes. In practice the model is a mapping of implied volatilities into prices and viceversa
- To face the non-positive rates, Black model has been shifted. The technique was already known but in the last years has become crucial to shift the lower bound of prices admitted by the model.

Shifted Lognormal Model for Caplets

- It can be shown that Black formula provides valid solutions if strike and forward rate are shifted. For a (T, S) caplet with strike K we get

$$\mathbf{Caplet}_{Bl}(t, T, S, \tau, K, v_T, \alpha) = P(t, S)Bl(K + \alpha, F(t; T, S) + \alpha, v_T) \qquad (22)$$

where d_1 and d_2 read as before and instead v_i is now given by

$$v_i = \sigma^{\text{shifted}} \sqrt{T_{i-1}}$$

- The market quotes of σ^{shifted} refer to shifts α of the order of [2%,3%].

Flat Volatilities

- When comparing to other vanilla derivatives, Cap/Floor pricing offers an additional complexity, as it does not involve a single volatility number.
- As seen Cap/Floor can be stripped into Caplet/Floorlet which should be priced with a different volatility each.
- However, in the market, **Cap** quotes are computed according to the following

$$\mathbf{Cap}(0, \mathcal{T}_j, K) = N \sum_{i=1}^{j} \tau P(0, T_i) \mathbf{Bl}(K, F_i(0), v_{\mathcal{T}_j}^{cap})$$

  where the same "average" volatility value $v_{\mathcal{T}_j}^{cap}$ is used in each caplet.
- $v_{\mathcal{T}_j}^{cap}$ is called flat volatility, and is typically quoted for a range of strikes and expiries over liquid floating rates (e.g. 3M and 6M Euribor).

Spot Volatilities

- Notice that the same average volatility $v_{T_j}^{cap}$ is assumed for all caplets in the T_j-maturity cap.
- This appears to be somehow inconsistent. In the cap volatility system, the same caplet is linked to different volatilities when concurring to different caps.
- The correct volatilities can be recovered with a numerical procedure (i.e. *bootstrapping*) from

$$\sum_{i=1}^{j} \tau P(0, T_i)\mathbf{Bl}(K, F_i(0), v_{T_j}^{cap}) = \sum_{i=1}^{j} \tau P(0, T_i)\mathbf{Bl}(K, F_i(0), v_{T_{i-1}}^{caplet})$$

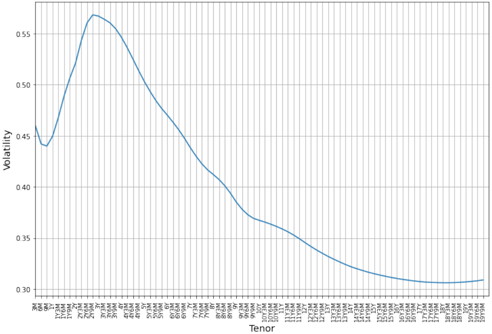
- $v_{T_{i-1}}^{caplet}$ are called spot volatilities (different $v_{T_{i-1}}^{caplet}$ are assumed for different caplets concurring to the Tj-maturity cap).
- Although being quoted, flat volatilities have little financial meaning. Conversely spot volatilities cannot be observed directly but are the quantities naturally tied to forward rates as a measure of their uncertainty.

Main Challenges

1. Produce Caplet/Floorlet prices consistent with current levels of Cap/Floor volatilities and therefore be able to re-price the market;
2. being able to "rebase" volatilities when pricing Cap/Floor over not quoted floating rates according to multiple curves framework (e.g. Cap/Floor over 1M or 12M Euribor);
3. to make things more complicated, some Caps/Floors are not always quoted over the same LIBOR, for example EUR Cap/Floor are quoted over 3M EURIBOR up to $2y$ maturity and then over 6M EURIBOR.

The Volatility Hump

- Empirical studies have pointed out two very important facts:
  - the first one is that interest rates volatility can depend on the level of the interest rates themselves;
  - moreover the volatility function is increasing in the short end of the curve, and decreasing in the long end, with an humped type movement.



- Uncertainty is bigger in the intermediate region and lower in the front of the maturity spectrum. For long maturities volatility tends to decay.
- When the hump does not appear it is regarded as *stressed market*.

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- There are two types of swaptions (as the underlying swaps): a *payer* in which at maturity the buyer could become the fixed-rate payer and a *receiver* where she could become the fixed-rate receiver.
- A swaption provides protection for a borrower as it ensures a maximum fixed interest rate payable in the future. Furthermore, it gives her the flexibility, if the rate does not rise above the swaption strike rate at expiry, to choose not to exercise it and take advantage of the lower market rates.

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  - The primary risk with a Swaption occurs after you have exercised your right and proceeded with the Swap. Should interest rate movements be different to your expectations, the Swap may have the opposite effect to what you were trying to achieve with the transaction.
  - If interest rates do not rise above the strike on the exercise date, you have not obtained any benefit from the premium paid for the purchase of the Swaption. The premium is the cost of obtaining protection against a rise in interest rates.

Swaption Payoff

- The discounted payoff of a payer Swaption (with maturity T_α) is given, recalling the value of a payer IRS (Eq. (9)) by

$$\mathbf{PSw} = D(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^{+} \tag{23}$$

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- Nevertheless it can be simplified by writing it in a different way...

Swaption Payoff

- Recall that we have expressed the swap payoff also as (Eq. (11))

$$\mathbf{PFS} = \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(S_{\alpha,\beta} - K) = A(S_{\alpha,\beta} - K)$$

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- If we look at the swaption payoff through this expression modeling as stochastic variable directly $S_{\alpha,\beta}(t)$, instead of each forward rate $F(t; T_{i-1}, T_i)$, we can write the swaption price as the expectation of

$$\mathbf{PSw} = \mathbb{E}^{\mathcal{Q}}\left[D(t, T_\alpha) A \max(S_{\alpha,\beta}(T_\alpha) - K, 0)\right] \tag{24}$$

which looks like easier and more intuitive than previous Eq. (23).

Swaption Characterization

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  1. The swaption is said to be at-the-money (ATM) if

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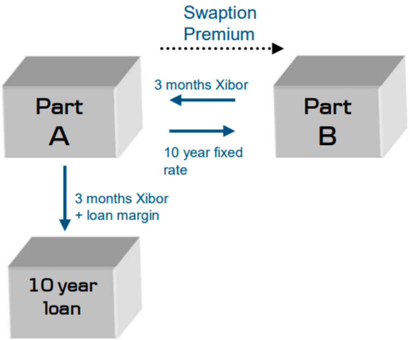
- ATM swaptions are quoted for maturities ranging between $1m$ and $30y$, and for tenors between $1y$ and $30y$.

An Example

- **A** has raised a 10y loan with floating interest rates fixed every three months (IBOR + margin).
- **A** wants to *hedge the loan against rising interest rates but also to benefit from the floating rate*, i.e. should interest rates not rise above a certain level (the swaptions strike-rate K).
- The purchase of a payer swaption could hedge this risk:

  — **interest rates increase**: **A** may exercise the swaption and be a party of a swap as a payer of a fixed interest rate;

  — **swap-rate below** K: it will not be exercised and **A** will continue to have floating-rate funding.



Differences between Caps and Swaptions

- As we have seen caps can be decomposed into more elementary products: caplets. You can simply value each caplet one by one and then add their prices
  - you can value them modeling each forward rate as a random variable;
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- If you take as "fundamental" entity the LIBOR rates you have to deal with the joint action of the simple forward LIBOR rates and so with the **terminal correlation** between rates of different portions of the yield curve.

An Option to Exchange Fixed with Float

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- To fully characterized this point we need an expression for a Coupon Bond Option. Most interest rate models (e.g. short rate models) have closed formulas for the Zero Coupon Bond price. So it would be even better (simpler) if we could express a Coupon Bond Option as a portfolio of Zero Coupon Bond Options.

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- Luckily we can do that thanks to a recipe known as Jamshidian's decomposition.

Jamshidian's Trick

Theorem

Consider a sequence of functions f_i, a random variable W and a constant $K \geq 0$. If each f_i is monotone (decreasing), that is $\dfrac{\partial f_i}{\partial W} < 0;\ \forall i$, then

$$\left(K - \sum_i f_i(W)\right)^+ = \sum_i (K_i - f_i(W))^+$$

In financial terms it means that the payoff of an option on a portfolio of assets can be expressed in terms of a portfolio of options on each asset.

Jamshidian's Trick Proof

- Since each f_i is monotone also $\sum_i f_i$ is decreasing. Hence there is a unique solution $\hat{w}$ to
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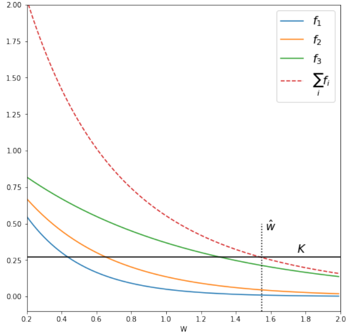

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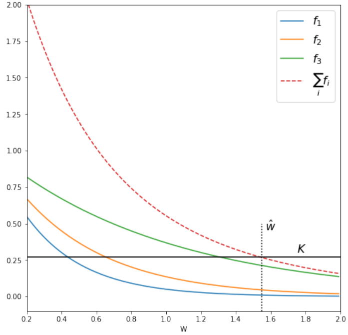

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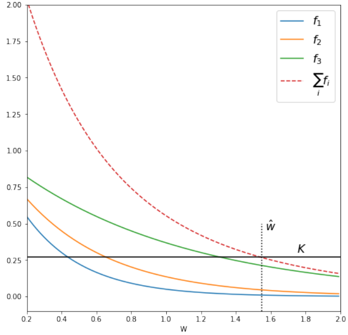

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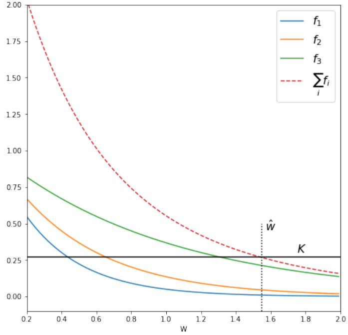

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- Suppose we would like to calculate the price of a put option with strike K on this coupon bond. The payoff reads

$$\mathbf{CBP} = [K - \mathbf{CB}(t, \mathcal{C}, \mathcal{T})]^+ = \left[K - \sum_{i=1}^{n} c_i \Pi(t, T_i, r(t))\right]^+$$

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Coupon Bond Option

- Now apply the Jamshidian's decomposition to previous payoff.
- First need to find the interest rate value r^* such that $\sum_{i=1}^{n} c_i \Pi(t, T_i, r^*) = K$.
- Assuming the interest rate model satisfies the required condition

$$\frac{\partial \Pi(t, T_i, r(t))}{\partial r} < 0, \ \forall 0 < t < s$$

we can rewrite the payoff as

$$\mathbf{CBP}(t, T_i, \Pi, r^*) = \sum_{i=1}^{n} c_i [\Pi(t, T_i, r^*) - \Pi(t, T_i, r(t))]^+ \tag{25}$$

Coupon Bond Option

- Eq. (25) tells us that we can price a coupon bond option as a portfolio of options on ZCBs.
- The strike of these option is calculated as the value of a ZCB given a *particular* value of the short rate, determined with a root finding procedure.
- In formulas the CBO with maturity T and strike K reads

$$\mathbf{CBP}(t, \mathcal{T}, \mathcal{C}, K) = \sum_{i=1}^{n} c_i \mathbf{ZBP}(t, T_i, \Pi, r^*) \tag{26}$$

Adapting to Swaptions

- When interest rates are modeled using Affine Short Rate Models it is rather simple to arrive to the swaption pricing formula.
- *Affine Models* indeed relates ZCB price to a spot rate model according to

$$P(t,T) = A(t,T)e^{-B(t,T)r}$$

- Hence the value r^* can be determined as a solution of

$$\sum_{i=1}^{n} A(t,t_i)e^{-B(t,t_i)r^*}$$

Swaption Pricing via Affine Models

- Consider an option on a swap which pays a fixed rate X and receives LIBOR.
- Modeling the short rate with an affine model we can define r^* at time T, such that

$$\sum_{i=1}^{n} c_i A(t, T_i) e^{-B(t,T_i)r^*} = 1$$

where c_i is the coupon value.
- Setting $X_i = A(t, T_i)e^{-B(t,T_i)r^*}$ the payer swaption price is thus given by

$$\mathbf{PSw}(t, T, N) = N \sum_{i=1}^{n} c_i \mathbf{ZBP}(t, T_i, X_i) \tag{27}$$

while the receiver swaption price reads

$$\mathbf{RSw}(t, T, N) = N \sum_{i=1}^{n} c_i \mathbf{ZBC}(t, T_i, X_i) \tag{28}$$

Black Formula for Swaptions

- Replacing the forward rate $F(0; t_{i-1}, t_i)$ with the swap rate $S_{\alpha,\beta}(0)$ and plugging in the quoted swaption volatility you get Black's formula for swaptions

$$\begin{aligned}&\mathbf{PS}_{Bl}(0, T, N, K, S_{\alpha,\beta}) = \\ &\qquad N\left[S_{\alpha,\beta}(0)\Phi(d_1) - K\Phi(d_2)\right]\sum_{i=\alpha+1}^{\beta} P(0, T_i)\tau_i\end{aligned} \tag{29}$$

where

$$d_{1,2} = \frac{\log \frac{S_{\alpha,\beta}}{K} \pm \frac{v^2}{2}}{2}$$

and

$$v = \sigma_{\alpha,\beta}\sqrt{T_\alpha}$$

Swaptions Volatility Calibration

- Swaption volatilities are quoted for different maturities and tenors (length of the underlying swap).
- Both for ATM and away from ATM on both sides ("swaption smile").
- So swaptions have an additional dimension with respect to caps: the quotes are parametrized according to
  - maturites;
  - tenors;
  - strikes.

Swaption Volatility Calibration

AK	AL	AM	AN	AO	AP	AQ	AR	AS	AT	AU	AV	AW	AX	AY	AZ	BA
		ATM														
		1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	25y	30y	
	1m	0.973%	1.214%	1.188%	1.157%	1.133%	1.117%	1.096%	1.072%	1.053%	1.020%	0.971%	0.910%	0.871%	0.835%	
	2m	1.031%	1.226%	1.194%	1.167%	1.139%	1.122%	1.106%	1.084%	1.069%	1.046%	0.986%	0.934%	0.902%	0.870%	
	3m	1.029%	1.208%	1.188%	1.162%	1.136%	1.109%	1.096%	1.076%	1.064%	1.038%	0.985%	0.930%	0.898%	0.869%	
	6m	1.103%	1.212%	1.197%	1.173%	1.149%	1.131%	1.111%	1.089%	1.073%	1.048%	0.997%	0.948%	0.927%	0.895%	
	9m	1.140%	1.212%	1.190%	1.171%	1.141%	1.125%	1.105%	1.086%	1.067%	1.047%	1.000%	0.953%	0.934%	0.904%	
	1y	1.168%	1.207%	1.179%	1.156%	1.133%	1.116%	1.098%	1.082%	1.067%	1.045%	1.000%	0.952%	0.932%	0.909%	
	18m	1.165%	1.180%	1.154%	1.129%	1.105%	1.089%	1.073%	1.059%	1.045%	1.029%	0.983%	0.930%	0.913%	0.894%	
	2y	1.141%	1.151%	1.127%	1.108%	1.082%	1.069%	1.056%	1.047%	1.037%	1.020%	0.962%	0.920%	0.897%	0.884%	
	3y	1.103%	1.107%	1.085%	1.069%	1.047%	1.035%	1.022%	1.010%	0.998%	0.983%	0.927%	0.885%	0.864%	0.844%	
	4y	1.062%	1.064%	1.039%	1.024%	1.006%	0.996%	0.984%	0.970%	0.956%	0.941%	0.887%	0.845%	0.826%	0.800%	
	5y	1.017%	1.016%	0.995%	0.977%	0.963%	0.947%	0.936%	0.924%	0.912%	0.898%	0.843%	0.804%	0.783%	0.759%	
	6y	0.989%	0.984%	0.964%	0.942%	0.926%	0.912%	0.898%	0.889%	0.877%	0.862%	0.811%	0.773%	0.751%	0.727%	
	7y	0.954%	0.952%	0.926%	0.906%	0.894%	0.876%	0.865%	0.852%	0.843%	0.832%	0.782%	0.745%	0.722%	0.700%	
	8y	0.925%	0.924%	0.899%	0.879%	0.864%	0.850%	0.838%	0.825%	0.817%	0.805%	0.752%	0.718%	0.698%	0.675%	
	9y	0.901%	0.900%	0.877%	0.860%	0.842%	0.827%	0.815%	0.803%	0.794%	0.780%	0.725%	0.692%	0.673%	0.652%	
	10y	0.881%	0.881%	0.857%	0.839%	0.820%	0.810%	0.793%	0.783%	0.771%	0.759%	0.703%	0.672%	0.650%	0.631%	
	12y	0.841%	0.840%	0.822%	0.804%	0.785%	0.773%	0.762%	0.746%	0.734%	0.721%	0.665%	0.632%	0.614%	0.594%	
	15y	0.800%	0.799%	0.780%	0.763%	0.738%	0.725%	0.709%	0.696%	0.684%	0.676%	0.618%	0.595%	0.575%	0.552%	
	20y	0.746%	0.746%	0.726%	0.704%	0.682%	0.668%	0.654%	0.642%	0.631%	0.618%	0.558%	0.539%	0.521%	0.499%	
	25y	0.710%	0.706%	0.689%	0.664%	0.641%	0.627%	0.610%	0.597%	0.584%	0.575%	0.518%	0.494%	0.476%	0.456%	
	30y	0.680%	0.679%	0.659%	0.636%	0.608%	0.595%	0.578%	0.562%	0.545%	0.538%	0.478%	0.454%	0.439%	0.421%	

Swaption Volatility Calibration

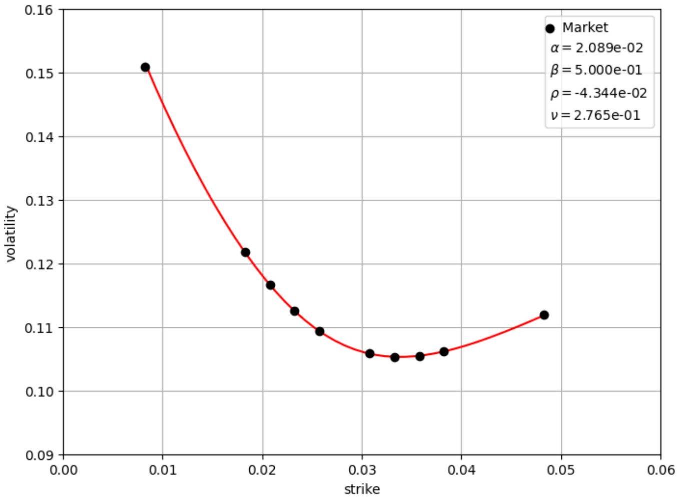
AL	AM	AN	AO	AP	AQ	AR	AS	AT	AU	AV	AW	AX	AY	AZ		
	SKEW SPREADS															
expiry	tenor	-2.00%	-1.50%	-1.00%	-0.75%	-0.50%	-0.25%	0.00%	0.25%	0.50%	0.75%	1.00%	1.50%	2.00%		
1m	1y	1.09%	0.82%	0.55%	0.40%	0.26%	0.12%	0.00%	-0.06%	-0.04%	0.03%	0.13%	0.36%	0.59%		
3m	1y	0.69%	0.508%	0.32%	0.23%	0.14%	0.06%	0.00%	-0.03%	-0.03%	0.00%	0.06%	0.20%	0.37%		
6m	1y	0.53%	0.38%	0.23%	0.16%	0.09%	0.04%	0.00%	-0.01%	0.00%	0.03%	0.08%	0.20%	0.35%		
9m	1y	0.40%	0.28%	0.16%	0.11%	0.06%	0.03%	0.00%	-0.01%	0.00%	0.02%	0.05%	0.15%	0.26%		
1y	1y	0.30%	0.20%	0.11%	0.08%	0.04%	0.02%	0.00%	-0.01%	0.00%	0.02%	0.04%	0.11%	0.20%		
2y	1y	0.11%	0.06%	0.02%	0.01%	0.00%	0.00%	0.00%	0.01%	0.02%	0.04%	0.07%	0.13%	0.20%		
3y	1y	0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.01%	0.00%	0.01%	0.03%	0.04%	0.06%	0.11%	0.16%		
5y	1y	-0.05%	-0.04%	-0.03%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.05%	0.08%	0.11%		
10y	1y	-0.05%	-0.04%	-0.03%	-0.02%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.04%	0.06%	0.09%		
15y	1y	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.03%	0.05%	0.07%		
20y	1y	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.01%	0.01%	0.02%	0.03%	0.05%		
1m	2y	1.02%	0.76%	0.49%	0.36%	0.22%	0.10%	0.00%	-0.05%	-0.04%	0.02%	0.12%	0.34%	0.57%		
3m	2y	0.61%	0.44%	0.27%	0.19%	0.11%	0.05%	0.00%	-0.02%	-0.02%	0.01%	0.06%	0.20%	0.36%		
6m	2y	0.47%	0.33%	0.19%	0.13%	0.08%	0.03%	0.00%	-0.01%	0.00%	0.03%	0.07%	0.18%	0.32%		
9m	2y	0.34%	0.23%	0.14%	0.09%	0.06%	0.02%	0.00%	-0.01%	-0.01%	0.00%	0.02%	0.09%	0.19%		
1y	2y	0.22%	0.15%	0.08%	0.06%	0.03%	0.01%	0.00%	0.00%	0.00%	0.01%	0.03%	0.08%	0.15%		
2y	2y	0.10%	0.05%	0.02%	0.01%	0.00%	0.00%	0.00%	0.01%	0.02%	0.04%	0.06%	0.11%	0.18%		
3y	2y	0.02%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.01%	0.02%	0.04%	0.05%	0.10%	0.14%		
5y	2y	-0.04%	-0.04%	-0.03%	-0.02%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.04%	0.07%	0.10%		
10y	2y	-0.05%	-0.04%	-0.03%	-0.02%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.04%	0.06%	0.09%		
15y	2y	-0.03%	-0.03%	-0.02%	-0.02%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.03%	0.05%	0.07%		
20y	2y	-0.03%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.04%	0.06%		
1m	5y	0.94%	0.70%	0.44%	0.32%	0.19%	0.08%	0.00%	-0.02%	0.01%	0.09%	0.20%	0.43%	0.67%		
3m	5y	0.50%	0.35%	0.20%	0.13%	0.07%	0.03%	0.00%	0.00%	0.02%	0.07%	0.13%	0.28%	0.44%		
6m	5y	0.32%	0.21%	0.11%	0.07%	0.03%	0.01%	0.00%	0.01%	0.03%	0.07%	0.12%	0.24%	0.37%		
9m	5y	0.21%	0.13%	0.06%	0.04%	0.02%	0.00%	0.00%	0.01%	0.03%	0.05%	0.09%	0.17%	0.28%		
1y	5y	0.16%	0.10%	0.05%	0.02%	0.01%	0.00%	0.00%	0.01%	0.02%	0.05%	0.08%	0.15%	0.24%		

Swaption Volatility Calibration

After we have constructed the volatility matrix we can fit the "smile" at each (expiry, tenor) pair.
This is done assuming the volatilities evolves according to the SABR model. An approximated solution has been given by *Hagan et al.*

$$
\begin{aligned}
\sigma_B(K,f) &= \frac{\alpha\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T\right\}}{(fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}\right]}\times\frac{z}{\chi(z)}\\
z &= \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln\frac{f}{K}\\
\chi(z) &= \ln\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right].
\end{aligned}
$$

Swaption Volatility Calibration



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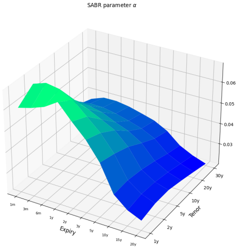

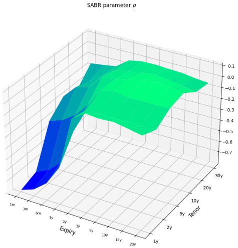

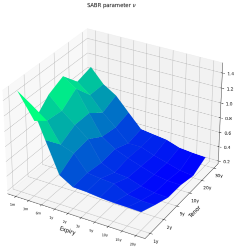

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- A Bermudan Swaption gives the holder the right but not the obligation to enter in an interest rate swap contract at different dates (usually the swap reset dates) with some days of notification to the counter-party.
- The interest rate swap the holder can enter into, is the same existing contract, so if the holder does not exercise at the first date in the call schedule, the option for the following periods is written on shorter swaps.

Bermudan Swaption Example

- As an example consider the following: receiver Bermudan Swaption written on a 3 years swap with the first call date $2y$ from now (we suppose semi-annual payments).

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- If at the end of the second year she will not exercise, six months later she will have to decide if entering or not in the same remaining swap which now has become a $2y6m$ swap.
- If again she will not exercise the last possibility will involve the decision of whether or not to enter on the $2y6m - 3y$ FRA.

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- Given the complexity of Bermudan swaption valuation, there is no closed form solution. Therefore, we need to select an interest rate term structure model and a numeric solution to price this contract numerically.
- Typically tree techniques or the Longstaff-Schwartz method are used.

Bermudan Swaption Pricing: Outline

- Consider a tenor structure $\mathcal{T} = \{T_i\}_{i=\alpha}^{\beta}$ and a Bermudan receiver swaption with time t value $\mathbf{RBS}(t, K)$.

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- Assuming no prior exercise, at any time point T_i the swaption holder has the right to receive the exercise value V_e of the swaption, i.e., present value of the underlying swap:

$$V_e(T_i) = (K - S_{i,\beta}(T_i)) + \sum_{k=i+1}^{\beta} P(T_i, T_k)\tau_k \tag{30}$$

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- The exercise value has to be compared to the so-called continuation value, V_c, of holding the option beyond T_i:

$$V_c(T_i) = \mathbb{E}[\mathbf{RBS}(T_{i+1}, K)|S_{i,\beta}(T_i)] \tag{31}$$

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- This allows to update the continuation value at $T_{\beta-2}$ with Eq. (31) and compare it to the exercise value

$$\mathbf{RBS}(T_j, K) = \max(V_e(T_j), V_c(T_j)), \quad \text{for } j = \beta - 2, \beta - 3, \ldots, n$$

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- This procedure of comparing "backwardly-cumulated" continuation value with exercise value and deciding upon a swaption exercise is repeated until the initial valuation date is reached, at which point the algorithm yields a price estimate for the Bermudan swaption.

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- This procedure of comparing “backwardly-cumulated” continuation value with exercise value and deciding upon a swaption exercise is repeated until the initial valuation date is reached, at which point the algorithm yields a price estimate for the Bermudan swaption.
- The calculation of the continuation value is clearly model-dependent and the choice of modeling framework itself often determines the scope of available numerical techniques.

Callable Coupon Bond

- A callable bond is a bond in which, on the call date(s) (there can be more than one), the issuer has the right, but not the obligation, to buy back (redeem) the bonds from the bond holders at a defined call price.

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- If there are multiple callability dates is clear that the swaption we need is a Bermudan one.
- With a receiver bermudan swaption with the same contractual conventions of the Swap (i.e. the strike of the swaption is equal to the coupon of the bond) we can offset the swap.

Callable vs Non Callable Coupon Bonds

- *Ceteris paribus* a non callable coupon bond has an higher price than a callable one because the callability option adds value to the issuer

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- As a result, the company has refinanced its debt by paying off the higher-yielding callable bonds with the newly-issued debt at a lower interest rate.

Risk Analysis of Callable Bonds

- The investor might not make out as well as the company when the bond is called. Not only she loses the remaining interest payments but unlikely she will be able to match the original coupon. This situation is known as reinvestment risk.

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Few Useful Math Tricks

rule 1	$\max(J,K) = K + \max(J-K,0)$
rule 2	$\max(J-K,0) = J-K+\max(K-J,0)$
rule 3	$\max(\alpha J,K) = \alpha\max(J,\frac{K}{\alpha})$
rule 4	$\max(\alpha J,K) = K + \alpha\max(J-\frac{K}{\alpha},0)$
rule 5	$\max(J,0) = -\min(-J,0)$
rule 6	$\min(\max(J-K_{\max},0),K_{\min}) =$ $\max[J-K_{\max},0] - \max[J-K_{\max}-K_{\min},0]$

Reverse Floater Bond

- Denote with $F(T)$ the short-term rate observed in T. We can write the Reverse Floater coupon in general form as

$$\mathbf{RF} = \max[0, K - \alpha F(T)] = \underbrace{K - \alpha F(T) + \max[\alpha F(T) - K, 0]}_{rule2}$$

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- Previous equation gives the payoff as the sum of a fixed leg of an IRS and a Cap with strike K

$$\mathbf{RF} = \underbrace{K - \alpha F(T)}_{\text{IRS fixed leg}} + \underbrace{\max[\alpha F(T) - K, 0]}_{\text{Cap}}$$

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- Hedging against the risk of falling interest rates. If an investor has invested in regular bonds, and if the interest rate falls, then she will receive lower returns than expected. In this scenario, it is extremely helpful to have inverse floaters in the portfolio as it gives higher returns when interest rates fall.

Reverse Floater Bond

- As with all investments that employ leverage, inverse floaters introduce a significant amount of interest rate risk.

- When short-term interest rates fall, both MV and yield of the reverse floater increases, magnifying the fluctuation in the bond's price.

- When short-term interest rates rise, bond value can drop significantly, and holders may end up with a security that pays little interest (i.e. magnified interest rate risk).

- Other typical investors are long vega, if K is close to the forward rates, vega is much higher.
- A long Vega portfolio means there is positive exposure to increases in implied volatility, while a short Vega portfolio is indicative of volatility vulnerability.

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