Interpolation - Practical Lesson 3

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1 Interpolation

1.1 Recap

Last lesson we looked at:

- `print` statements and variables
- mathematical expressions, also importing functions from a module (specifically the log and exp functions from the math module)
- boolean expressions
- string expressions
- indentation, `if/elif/else` blocks and `for` loops
- lists
- dictionaries
- tuples
- dates
- functions and modules

1.2 Exercise Interpolation / Extrapolation

1.2.1 Linear interpolation

Interpolation is a method of constructing new data points within the range of a discrete set of known data points.

It may happen to have few data points, obtained by sampling or experimenting, which represent the values of a function for a limited number of values of an independent variable (e.g. in recording a trip: distances at certain times). It is often required to interpolate, i.e. estimate the value of that function for an intermediate value of the independent variable (e.g. in our previous example what is the distance at new times ?).

Let's exercise on linear interpolation (and extrapolation because yes in same cases you can also get new points outside the measured range) with a couple of examples.

Example 1 Assume you are going on holidays by car and that luckily there isn't much traffic so that you can drive at constant speed (which gives a linear relation between travelled space and time i.e. $s = v \cdot t$, which means that if you'd plot the distances s as a function of the time t you get a line with slope v). Given two samples of the car travelled distance s_1 and s_2 taken at two different times t_1 and t_2 you can linearly interpolate to find your position at different times using the following relations:

$$w = \frac{t - t_1}{t_2 - t_1} \tag{1}$$

t generic time at which we want to know the distance s,

$$s = (1 - w) \cdot s\_1 + w \cdot s\_2 \tag{2}$$

Derivation The equation of a line for two points (t_1, s_1) and (t_2, s_2) can be written as:

$$\frac{t - t_1}{t_2 - t_1} = \frac{s - s_1}{s_2 - s_1} \tag{3}$$

Setting $w = \frac{t - t_1}{t_2 - t_1}$ and solving for s we find the desired solution:

$$w = \frac{s - s_1}{s_2 - s_1} \Rightarrow (s_2 - s_1) \cdot w = s - s_1 \Rightarrow ... \tag{4}$$

Back to our example, if $s_1 = 25.75$ km ($@t_1 = 15$ min) and $s_2 = 171.7$ km ($@t_2 = 100$ min) let's compute:

```
In [1]: # let's find distance travelled in 1 hour (interpolation)

        s_1 = 25.75 # distance in km
        t_1 = 15    # elapsed time in minutes
        s_2 = 171.7
        t_2 = 100

        t = 60

        w = (t - t_1)/(t_2 - t_1)
        s = (1 - w)*s_1 + w*s_2

        print ("{:.1f} km".format(s))
```

```
103.0 km
```

```
In [2]: # distance travelled in first 10 minutes
        # (extrapolation because we are outside of the measured points)

        s_1 = 25.75 # distance in km
        t_1 = 15    # elapsed time in minutes
        s_2 = 171.7
        t_2 = 100

        t = 10

        w = (t - t_1)/(t_2 - t_1)
        s = (1 - w)*s_1 + w*s_2

        print ("{:.1f} km".format(s))
```

```
17.2 km
```

```
In [3]: # distance travelled in a 3 hour trip (extrapolation)

        s_1 = 25.75 # distance in km
        t_1 = 15    # elapsed time in minutes
        s_2 = 171.7
        t_2 = 100

        t = 180

        w = (t - t_1)/(t_2 - t_1)
        s = (1 - w)*s_1 + w*s_2

        print ("{:.1f} km".format(s))
```

```
309.1 km
```

1.2.2 Log-linear interpolation

When the variable we would like to interpolate has an exponential relation with the unknown we can fall back to the previous case by applying the logarithm. In this case the previous formulas apply again except that at the end we have to exponentiate to get back the original variable:

$$p = \exp(c \cdot h) \tag{5}$$

$$s = \log(p) = \log(\exp(c \cdot h)) = c \cdot h \tag{6}$$

$$w = \frac{h - h_1}{h_2 - h_1} \tag{7}$$

$$s = (1 - w) \cdot s\_1 + w \cdot s\_2;;(\text{remember now}; s = \log(p)) \tag{8}$$

$$p = \exp(s) \tag{9}$$

Let's see a practical example.

Example 2 Atmospheric pressure decreases with the altitude (i.e. the highest I flight the lower is the pressure) following an exponential law:

$$p = p_0 \cdot e^{-\alpha h} \tag{10}$$

where

- h is the altitude
- p_0 is the pressure at sea level
- α is a constant

Taking the logarithm of each side of the equation I get a linear relation which can be interpolated as before:

$$\tilde{s} = \log(p) = \log(p_0 \cdot e^{-\alpha h}) \propto -\alpha \cdot h \tag{11}$$

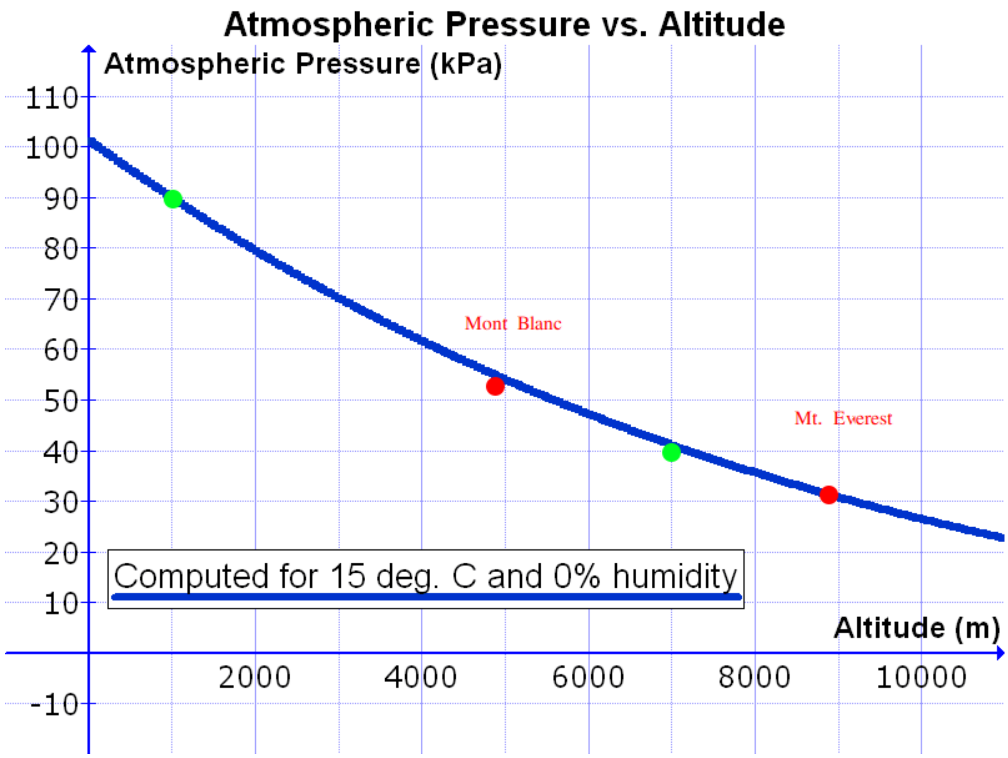
Now assume that we have measured $p_1 = 90$ kPa $(h_1 = 1000$ m$)$ and $p_2 = 40$ kPa $(h_1 = 7000$ m$)$ what will be the atmospheric pressure on top of the Mont Blanc (4812 m) ? and on top of Mount Everest (8848 m) ?

In [4]:
```
# pressure on top of the Mont Blanc (interpolation)
from math import log, exp

# first we take the logarithm of our measurements to use the linear
# relation to interpolate
h_1 = 1000 # height in meters
s_1 = log(90) # logarithm of the pressure at heigth h1
h_2 = 7000 # height in meters
s_2 = log(40) # logarithm of the pressure at heigth h2

h = 4812

w = (h - h_1)/(h_2 - h_1)
s = (1 - w)*s_1 + w*s_2

print ("{:.1f} kPa".format(exp(s)))
```

```
53.8 kPa
```

In [5]:
```
# pressure on top of the Mount Everest (extrapolation)
from math import log, exp

# first we take the logarithm of our measurements to use the linear
# relation to interpolate
h_1 = 1000 # height in meters
s_1 = log(90) # logarithm of the pressure at heigth h1
h_2 = 7000 # height in meters
s_2 = log(40) # logarithm of the pressure at heigth h2

h = 8848

w = (h - h_1)/(h_2 - h_1)
s = (1 - w)*s_1 + w*s_2

print ("{:.1f} kPa".format(exp(s)))
```

```
31.2 kPa
```



Atmospheric pressure versus altitude (wikipedia). Green points represent our measurements, red points represent interpolation/extrapolation.

1.3 Discount curve interpolation

Now we can come back to finance and using what we have just learnt try to write a function which interpolates some given discount factors.

Needed data: * a list of pillars dates specifying the value dates of the given discount factors, $t_0, ..., t_{n-1}$ * a list of given discount factors, $D(t_0), ..., D(t_{n-1})$ * a pricing date ('today' date) which corresponds to $t = 0$

The input argument to the function will be the value date at which we want to interpolate the discount factor.

Since the discount factor can be expresses as $D = e^{-r(T-t)}$ the function will use a log-linear interpolation to return the value we are looking for.

$$D(t) = \exp\Big((1-w)\cdot\ln(D(t_i)) + w\cdot\ln(D(t_{i+1}))\Big); \quad w = \frac{t-t_i}{t_{i+1}-t_i}$$

where i is such that $t_i \le t \le t_{i+1}$. More technically we can say that we are doing a linear interpolation over time in the log space:

$$d(t_i) := \ln(D(t_i))$$

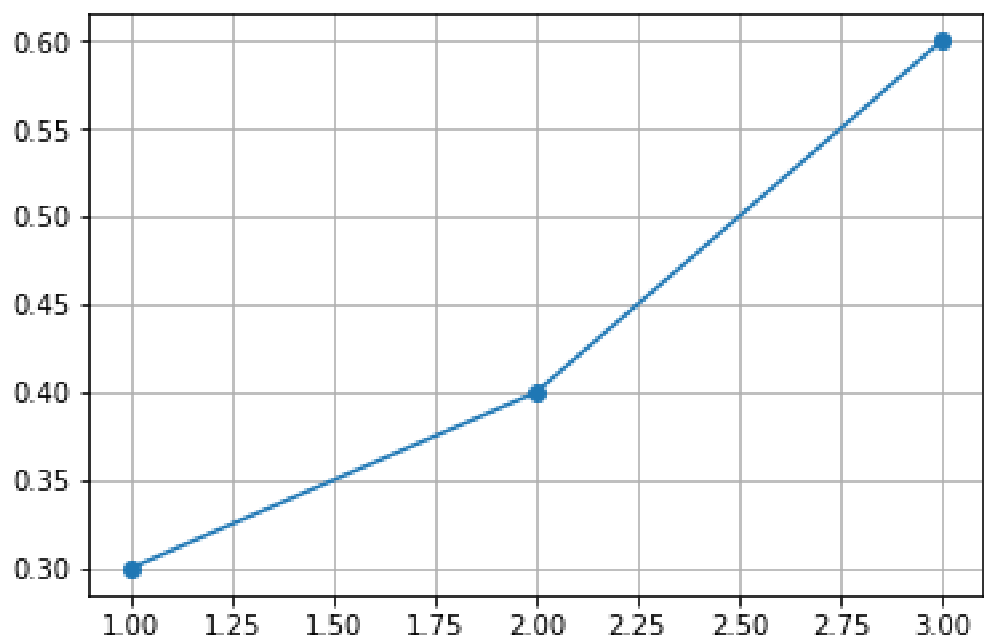
$$d(t) = (1-w)d(t_i) + wd(t_{i+1}); \quad w = \frac{t-t_i}{t_{i+1}-t_i}$$

$$D(t) = \exp(d(t))$$

where i is such that $t_i \leq t \leq t_{i+1}$

Instead of reinventing the wheel and perform the interpolation with our own code, we'll use the function `interp` provided by the Python module `numpy`. So first let's try it with some simple examples:

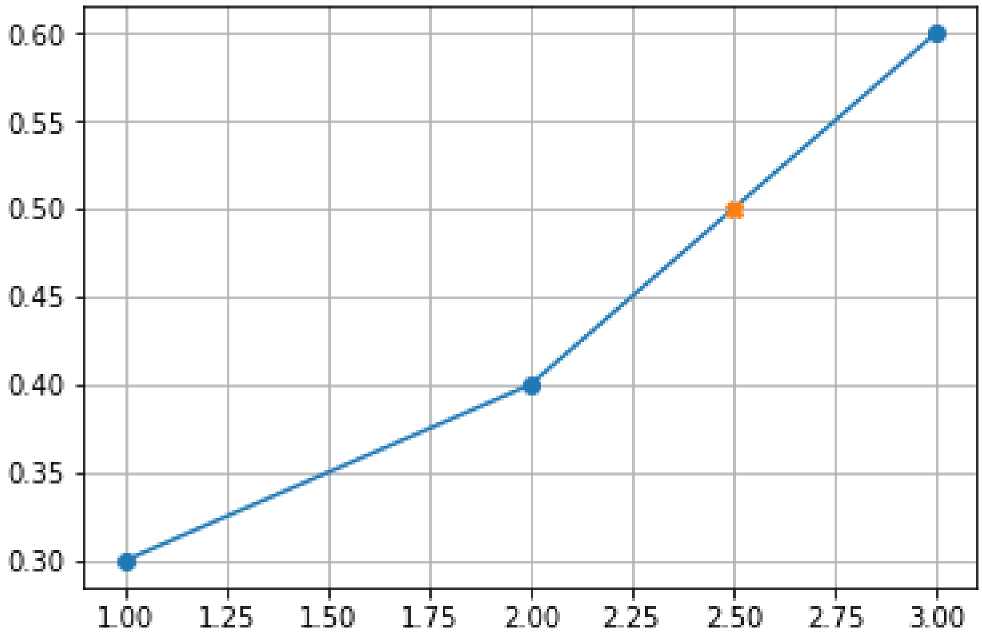
In [8]:
```
# let's assume we have a list 'xp' of x-axis coordinates,
# and the corresponding values 'fp' of a function evaluated
# at those x coordinates.
xp = [1, 2, 3]
fp = [0.3, 0.4, 0.6]

# let's see what this looks like when plotted on a graph
# we briefly introduce here the matplotlib module
# the keyword as allows to give shorter names to modules
from matplotlib import pyplot as plt
plt.plot(xp, fp, marker='o')
plt.grid(True)
plt.show()
```



In [9]:
```
# the numpy.interp function linearly interpolates these points to
# estimate the value of f at other x coordinates.
# For example, say we want to interpolate the points at x = 2.5::
import numpy as np
np.interp(2.5, xp, fp)
```

Out[9]: 0.5

In [10]:
```
# let's see what this looks like when plotted on a graph

from matplotlib import pyplot as plt
plt.plot(xp, fp, marker='o')
plt.grid(True)
plt.plot(2.5, np.interp(2.5, xp, fp), marker='X')
plt.show()
```



1.3.1 Interlude

How to learn about a given function ? Use the `help` keyword !

In [11]:
```
help(np.interp)
```

```
Help on function interp in module numpy:

interp(x, xp, fp, left=None, right=None, period=None)
    One-dimensional linear interpolation.

    Returns the one-dimensional piecewise linear interpolant to a function
    with given discrete data points (`xp`, `fp`), evaluated at `x`.

    Parameters
    ----------
    x : array_like
        The x-coordinates at which to evaluate the interpolated values.
```

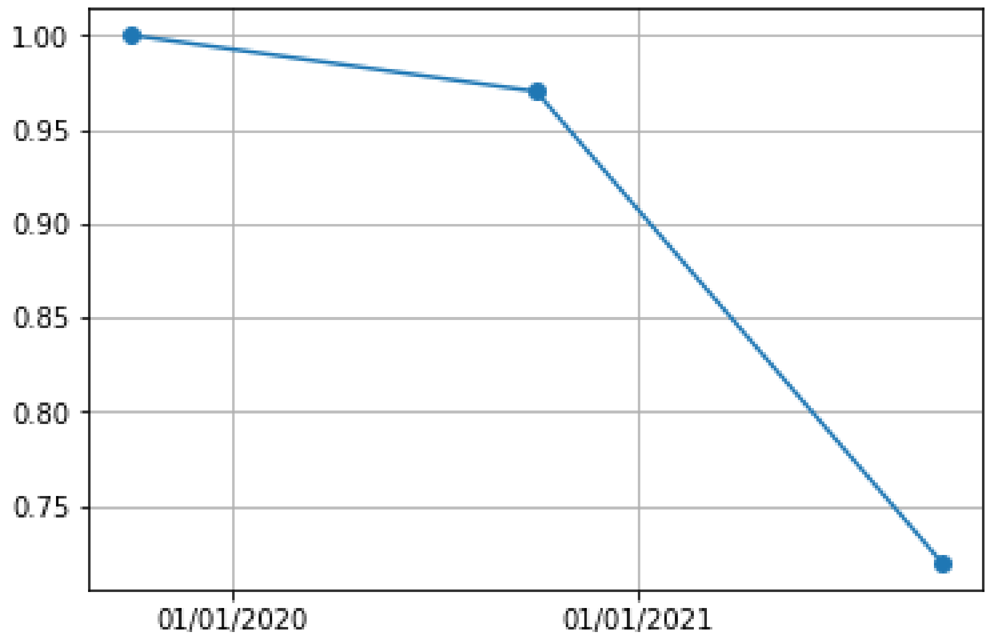
```
xp : 1-D sequence of floats
    The x-coordinates of the data points, must be increasing if argument
    `period` is not specified. Otherwise, `xp` is internally sorted after
    normalizing the periodic boundaries with ``xp = xp % period``.

fp : 1-D sequence of float or complex
    The y-coordinates of the data points, same length as `xp`.

left : optional float or complex corresponding to fp
    Value to return for `x < xp[0]`, default is `fp[0]`.

right : optional float or complex corresponding to fp
    Value to return for `x > xp[-1]`, default is `fp[-1]`.

period : None or float, optional
    A period for the x-coordinates. This parameter allows the proper
    interpolation of angular x-coordinates. Parameters `left` and `right`
    are ignored if `period` is specified.

    .. versionadded:: 1.10.0

Returns
-------
y : float or complex (corresponding to fp) or ndarray
    The interpolated values, same shape as `x`.

Raises
------
ValueError
    If `xp` and `fp` have different length
    If `xp` or `fp` are not 1-D sequences
    If `period == 0`

Notes
-----
Does not check that the x-coordinate sequence `xp` is increasing.
If `xp` is not increasing, the results are nonsense.
A simple check for increasing is::

    np.all(np.diff(xp) > 0)

Examples
--------
>>> xp = [1, 2, 3]
>>> fp = [3, 2, 0]
>>> np.interp(2.5, xp, fp)
1.0
```

```
>>> np.interp([0, 1, 1.5, 2.72, 3.14], xp, fp)
array([ 3. ,  3. ,  2.5 ,  0.56,  0. ])
>>> UNDEF = -99.0
>>> np.interp(3.14, xp, fp, right=UNDEF)
-99.0

Plot an interpolant to the sine function:

>>> x = np.linspace(0, 2*np.pi, 10)
>>> y = np.sin(x)
>>> xvals = np.linspace(0, 2*np.pi, 50)
>>> yinterp = np.interp(xvals, x, y)
>>> import matplotlib.pyplot as plt
>>> plt.plot(x, y, 'o')
[<matplotlib.lines.Line2D object at 0x...>]
>>> plt.plot(xvals, yinterp, '-x')
[<matplotlib.lines.Line2D object at 0x...>]
>>> plt.show()

Interpolation with periodic x-coordinates:

>>> x = [-180, -170, -185, 185, -10, -5, 0, 365]
>>> xp = [190, -190, 350, -350]
>>> fp = [5, 10, 3, 4]
>>> np.interp(x, xp, fp, period=360)
array([7.5, 5., 8.75, 6.25, 3., 3.25, 3.5, 3.75])

Complex interpolation:

>>> x = [1.5, 4.0]
>>> xp = [2,3,5]
>>> fp = [1.0j, 0, 2+3j]
>>> np.interp(x, xp, fp)
array([ 0.+1.j ,  1.+1.5j])
```

1.3.2 Now back to our discount factor function df.

In [12]:
```
# import modules and objects that we need
from datetime import date
import numpy, math
from matplotlib import pyplot as plt
import matplotlib.dates as mdates
# with this notation we tell python to use mdates as an alias
# for matplotlib.dates, I told you I'm lazy...

# define the input data
```

```python
today_date = date(2019, 10, 1)

pillar_dates = [date(2019, 10, 1), date(2020, 10, 1), date(2021, 10, 1)]
discount_factors = [1.0, 0.97, 0.72]

# let's see what this looks like when plotted on a graph
# here a more complicated usage of matplotlib to
# get a nicer plot
plt.plot(pillar_dates, discount_factors, marker='o')
plt.gca().xaxis.set_major_formatter(mdates.DateFormatter('%m/%d/%Y'))
plt.gca().xaxis.set_major_locator(mdates.YearLocator())
plt.grid(True)
plt.show()

# define the df function
def df(d):
    # first thing we need to do is to apply the logarithm function
    # to the discount factors since we are doing log-linear and
    # not just linear interpolation
    log_discount_factors = []
    for discount_factor in discount_factors:
        log_discount_factors.append(math.log(discount_factor))

    # perform the linear interpolation of the log discount factors
    interpolated_log_discount_factor = \
        numpy.interp(d, pillar_dates, log_discount_factors)

    # return the interpolated discount factor
    return math.exp(interpolated_log_discount_factor)
```



This is almost OK, **but it won't work** because `numpy.interp` only accepts numbers/lists of numbers as arguments i.e. it doesn't automatically convert or interpret dates as numbers in any way, so it doesn't know how to interpolate them. So we need to do the conversion ourselves before passing the data into the `numpy.interp` function.

```
In [13]: def df(d):
             # first thing we need to do is to apply the logarithm function
             # to the discount factors since we are doing log-linear and
             # not just linear interpolation
             log_discount_factors = []
             for discount_factor in discount_factors:
                 log_discount_factors.append(math.log(discount_factor))

             # convert the pillar dates to pillar 'days'
             # i.e. number of days from today
             # to write shorter code we can use this NEW notation
             # which condenses for and list creation in one line
             pillar_days = \
                 [(pillar_date - today_date).days for pillar_date in pillar_dates]

             # obviously we need to do the same to the value date
             # argument of the df function
             d_days = (d - today_date).days

             # perform the linear interpolation of the log discount factors
             interpolated_log_discount_factor = \
```

```
                numpy.interp(d_days, pillar_days, log_discount_factors)

            # return the interpolated discount factor
            return math.exp(interpolated_log_discount_factor)
```

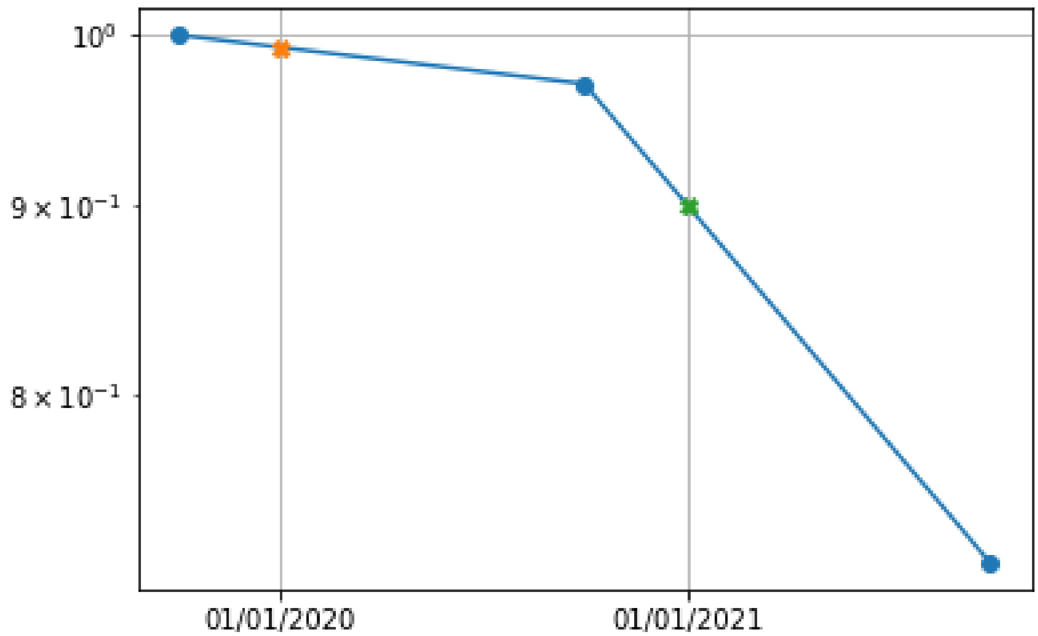
In [18]:
```
# now we can use the df function to get discount factors
# on value dates between the given pillar dates
d0 = date(2020, 1, 1)
df0 = df(d0)
print (df0)
```

```
0.9923728228571693
```

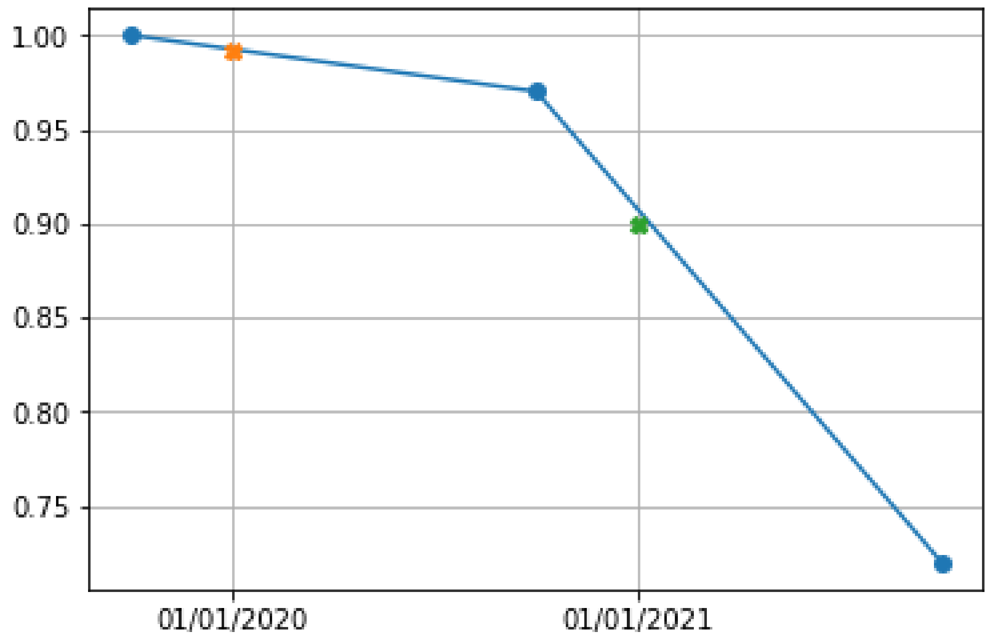
In [15]:
```
d1 = date(2021, 1, 1)
df1 = df(d1)
print (df1)
```

```
0.8997999273630835
```

In [19]:
```
# let's see what these look like when plotted on a semi-log graph

from matplotlib import pyplot as plt
import matplotlib.dates as mdates
plt.semilogy(pillar_dates, discount_factors, marker='o')
plt.semilogy(d0,df0 , marker='X')
plt.semilogy(d1,df1 , marker='X')
plt.gca().xaxis.set_major_formatter(mdates.DateFormatter('%m/%d/%Y'))
plt.gca().xaxis.set_major_locator(mdates.YearLocator())
plt.grid(True)
plt.show()
```



In [20]:
```
# let's see what these look like when plotted on a linear graph

from matplotlib import pyplot as plt
import matplotlib.dates as mdates
plt.plot(pillar_dates, discount_factors, marker='o')
plt.plot(d0,df0 , marker='X')
plt.plot(d1,df1 , marker='X')
plt.gca().xaxis.set_major_formatter(mdates.DateFormatter('%m/%d/%Y'))
plt.gca().xaxis.set_major_locator(mdates.YearLocator())
plt.grid(True)
plt.show()
```



1.4 Exercises

1.4.1 Exercise 3.1

Take the code for the Black-Scholes formula from Exercise 2.3 and wrap it in a function. Then, use this function to calculate the prices of calls with various strikes, using the following data.

```
S_t = 800
# strikes expressed as % of spot price
moneyness = [ 0.5, 0.75, 0.825, 1.0, 1.125, 1.25, 1.5 ]
vol = 0.3
ttm = 0.75
r = 0.005
```

The output should be a dictionary mapping strikes to call prices.

1.4.2 Exercise 3.2

Python has a useful command called `assert` which can be used for checking that a given condition is satisfied, and raising an error if the condition is not satisfied.

The following line does not cause an error, in fact it does nothing

```
assert 1 < 2
```

This causes an error

```
assert 1 > 2
```

`assert` can take a second parameter with a message to display in case of failure:

```
assert 1 > 2, ``Two is bigger than one''
```

Take the `df` function from this lesson and modify it by adding some assertions to check that:

- the pillar date list contains at least 2 elements
- the pillar date list is the same length as the discount factor list
- the first pillar date is equal to the today date
- the value date argument 'd' is greater or equal to the first pillar date and also less than or equal to the last pillar date

Then try using the function with some invalid data to make sure that your assertions are correctly checking the desired conditions.

1.4.3 Exercise 3.3

Python has a module called `matplotlib` which can be used for plotting graphs and charts. In particular, we can use a sub-module called `pyplot` which provides slightly easier-to-use interface for plotting interactively.

```
from matplotlib import pyplot

# plot some data
pyplot.plot(
    [1, 2, 3],    # x-axis coordinates
    [5, 3, 10],   # y-axis coordinates
    marker='o'    # we want the points to be marked with circles
)
```

Use this function to plot the call prices from exercise 3.1. Remember to use `help` and `dir` to have some help (or to look in Google ;-)).

1.5 Advanced hint

Interpolation using `scipy.interpolate`: https://docs.scipy.org/doc/scipy-0.15.1/reference/interpolate.html#module-scipy.interpolate