CME 241: Assignment 6

Matteo Santamaria (msantama@stanford.edu)

February 20, 2021

1. Merton Portfolio Problem with log Utility

Define our utility function as

$$U(x) = \log(x)$$

We will define State as two-tuples (t, W_t) and Actions as two-tuples (π_t, c_t). Reward is given by

$$\mathcal{R}(t, W_t, \pi_t, c_t) = \begin{cases} U(c_t) = \log(c_t) & t < T \\ B(T) \cdot U(W_T) = \epsilon \cdot \log(W_T) & t = T \end{cases}$$

The HJB equation tells us that

$$\max_{\pi_t, c_t} \left\{ \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t} \cdot ((\pi_t(\mu - r) + r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \frac{\pi_t^2 \cdot \sigma^2 \cdot W_t^2}{2} + \log(c_t) \right\} = \rho \cdot V^*(t, W_t)$$

Or more concisely

$$\max_{\pi_t, c_t} \{\Phi(t, W_t; \pi_t, c_t)\} = \rho \cdot V^*(t, W_t)$$

Subject to a boundary constraint

$$V^*(T, W_T) = \epsilon \cdot \log(W_T)$$

To find optimal π_t^*, c_t^* we solve the first order conditions

$$\frac{\partial \Phi}{\partial \pi_t} = (\mu - r)W_t \cdot \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \pi_t \cdot \sigma^2 \cdot W_t^2 = 0$$

$$\implies \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}$$

$$\frac{\partial \Phi}{\partial c_t} = -\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t} = 0$$

$$\implies c_t^* = \left(\frac{\partial V^*}{\partial W_t}\right)^{-1}$$

Substituting these optimizing values into Φ, we see that

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t} \cdot ((\pi_t^*(\mu - r) + r)W_t - c_t^*) + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \frac{(\pi_t^*)^2 \cdot \sigma^2 \cdot W_t^2}{2} + \log(c_t^*) = \rho \cdot V^*(t, W_t)$$

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} \cdot r \cdot W_t - \log\left(\frac{\partial V^*}{\partial W_t}\right) - 1 = \rho \cdot V^*(t, W_t)$$

To solve the PDE, we will guess a solution of the form

$$V^*(t, W_t) = c \cdot \log W_t + f(t)$$

for some constant $c \in \mathbf{R}$. This has partial derivatives

$$\frac{\partial V^*}{\partial t} = f'(t)$$
$$\frac{\partial V^*}{\partial W_t} = \frac{c}{W_t}$$
$$\frac{\partial^2 V^*}{\partial W_t^2} = \frac{-c}{W_t^2}$$

Substituting these equations into our PDE, we get

$$f'(t) = \rho \cdot f(t) + \rho c \cdot \log(W_t) - \frac{c(\mu - r)^2}{2\sigma^2} - cr + \log\left(\frac{c}{W_t}\right) + 1$$
$$f'(t) = \rho \cdot f(t) + \nu$$

where

$$\nu = \rho c \cdot \log(W_t) - \frac{c(\mu - r)^2}{2\sigma^2} - cr + \log\left(\frac{c}{W_t}\right) + 1$$

which is solvable.