CME 241: Assignment 6

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Question 1.

Given

$$U(x) = x - \frac{\alpha x^2}{2} \mid x \sim \mathcal{N}(\mu, \sigma^2)$$

we can compute

$$\mathbb{E}\left[U(x)\right] = \mathbb{E}\left[x - \frac{\alpha x^2}{2}\right] = \mathbb{E}\left[x\right] - \frac{\alpha}{2} \cdot \mathbb{E}\left[x^2\right]$$

Using the fact that x follows a normal distribution and the formula $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, we can derive an expression for the expected utility:

$$\mathbb{E}\left[U(x)\right] = \mu - \frac{\alpha}{2} \cdot (\sigma^2 + \mu^2)$$

To find x_{CE}, we determine what value of x would generate $\mathbb{E}\left[U(x)\right]$. Making the proper substitutions, we see this value is the subtractive root of the quadratic equation

$$\alpha x^2 - 2x + (2\mu - \alpha\sigma^2 - \alpha\mu^2) = 0$$

$$x_{CE} = \frac{2 - \sqrt{4 - 4\alpha(2\mu - \alpha\sigma^2 - \alpha\mu^2)}}{2\alpha} = \frac{1 - \sqrt{1 - 2\alpha\mu + \alpha^2\sigma^2 + \alpha^2\mu^2)}}{\alpha}$$

From here, we can compute π_A:

$$\pi_A = \mathbb{E}[x] - x_{CE} = \mu - \frac{1 - \sqrt{1 - 2\alpha\mu + \alpha^2\sigma^2 + \alpha^2\mu^2)}}{\alpha}$$

If we define $y(\delta)$ as our future wealth in one year for a given ratio δ invested in the risky asset, we can express:

$$y(\delta) = \delta(1 + x) + (1 - \delta)(1 + r)$$

Notice that $y(\delta) \sim \mathcal{N}(1 + r + \delta(\mu - r), \delta^2\sigma^2)$. For convenience, let $k = \mu - r$.

$$\mathbb{E}\left[U\left(y\left(\delta\right)\right)\right] = 1 + r + \delta k - \frac{\alpha}{2} \cdot \left\{\delta^2\sigma^2 + (1 + r + \delta k)^2\right\}$$

To maximize with respect to δ, we take the partial derivative

$$\frac{\partial \mathbb{E}\left[U(y(\delta))\right]}{\partial \delta} = k - \alpha\sigma^2\delta - \alpha(1 + r + \delta k)k$$

Setting equal to zero and solving for δ, we see the value that maximizes $\mathbb{E}\left[U(y(\delta))\right]$ is

$$\delta = \frac{k(1-\alpha-\alpha r)}{\alpha(\sigma^2+k^2)}$$

To see how δ varies with α, see the associated Jupyter notebook.

2. Kelly Criterion

After a single bet of $f \cdot W_0$ your wealth can either be

$$f \cdot W_0 \cdot (1+\alpha) + W_0 \cdot (1-f) = W_0 \cdot (1+f\alpha)$$

or

$$f \cdot W_0 \cdot (1-\beta) + W_0 \cdot (1-f) = W_0 \cdot (1-f\beta)$$

The two utility outcomes are therefore

$$\log\left(W_0 \cdot (1+f\alpha)\right)$$

and

$$\log\left(W_0 \cdot (1-f\beta)\right)$$

Taking it a step further, we can write the expected utility of wealth

$$\begin{aligned}
\mathbb{E}\left[\log(W)\right] &= p \cdot \log\left(W_0 \cdot (1+f\alpha)\right) + q \cdot \log\left(W_0 \cdot (1-f\beta)\right) \\
\frac{\partial \mathbb{E}\left[\log(W)\right]}{\partial f} &= \frac{p \cdot W_0 \cdot \alpha}{W_0 \cdot (1+f\alpha)} + \frac{-q \cdot W_0 \cdot \beta}{W_0 \cdot (1-f\beta)} = 0 \\
&\implies \frac{p \cdot \alpha}{1+f\alpha} = \frac{q \cdot \beta}{1-f\beta} \\
&\implies \alpha p - \alpha\beta f = \beta q + \alpha\beta f \\
&\implies f^* = \frac{\alpha p - \beta q}{2\alpha\beta}
\end{aligned}$$

Intuitively, this formula makes a lot of sense. For a fixed payout structure α, β the equation encourages larger bets for higher win percentages (greater values of p). Similarly, when the win percentage is lower (low p, high q), it recommends smaller bets. The Kelly Criterion also takes into account the relative payout difference of a loss versus a win, which should indeed inform your decision.