Given:

- $\mathbb{P}(O_n|O) = p_n$, the probability the bathroom is occupied by a "do nothing" user, given that its occupied, i.e. the percentage of bathroom users who do nothing

- $\mathbb{P}(O_s|O) = p_s$, the probability the bathroom is occupied by a "switch to occupied" user, given that its occupied, i.e. the percentage of bathroom users who switch the sign to "Occupied"

- $\mathbb{P}(O_c|O) = p_c$, the probability the bathroom is occupied by a "correct" user, given that its occupied, i.e. the percentage of bathroom users who use the sign correctly

- $\mathbb{P}(O) = p_O$, the probability the bathroom is occupied, i.e. the percentage of time the bathroom is occupied

We want to know two things:

1. $\mathbb{P}(O|\text{``}O\text{''})$, i.e. if you go to the bathroom and see that the sign on the door reads "Occupied," what is the probability that the bathroom is actually occupied?

2. $\mathbb{P}(V|\text{``}V\text{''})$, i.e. if the sign reads "Vacant," what is the probability that the bathroom actually is vacant?

Tackling item one first, we start by invoking Bayes Rule:

$$\mathbb{P}(O|\text{``}O\text{''}) = \frac{\mathbb{P}(\text{``}O\text{''}|O) \cdot \mathbb{P}(O)}{\mathbb{P}(\text{``}O\text{''})} = \frac{\mathbb{P}(\text{``}O\text{''}|O) \cdot p_O}{\mathbb{P}(\text{``}O\text{''})}$$

Since $\mathbb{P}(O) = p_O$ from our given, we can focus on $\mathbb{P}(\text{"}O\text{"}|O)$, which can be decomposed into the three separate bathroom users:

$$\mathbb{P}(\text{"}O\text{"}|O) = \mathbb{P}(\text{"}O\text{"}|O_n) \cdot \mathbb{P}(O_n|O) + \mathbb{P}(\text{"}O\text{"}|O_s) \cdot \mathbb{P}(O_s|O) + \mathbb{P}(\text{"}O\text{"}|O_c) \cdot \mathbb{P}(O_c|O)$$
$$= \mathbb{P}(\text{"}O\text{"}|O_n) \cdot p_n + \mathbb{P}(\text{"}O\text{"}|O_s) \cdot p_s + \mathbb{P}(\text{"}O\text{"}|O_c) \cdot p_c$$

Again from our given, we can fill-in the probabilities for each group of users. As well, $\mathbb{P}(\text{"}O\text{"}|O_s) = \mathbb{P}(\text{"}O\text{"}|O_c) = 1$, but $\mathbb{P}(\text{"}O\text{"}|O_n)$ depends on who used the bathroom before the "do nothing" user:

$$\mathbb{P}(\text{"}O\text{"}|O_n) = \frac{p_s}{p_s + p_c}$$

With that result, we can now solve for $\mathbb{P}(\text{"}O\text{"}|O)$:

$$\mathbb{P}(\text{"}O\text{"}|O) = \frac{p_s}{p_s + p_c} \cdot p_n + 1 \cdot p_s + 1 \cdot p_c$$

Now, we just have to solve for $\mathbb{P}(\text{"}O\text{"})$. This time, we'll decompose this into the bathroom being vacant or occupied:

$$\mathbb{P}(\text{"}O\text{"}) = \mathbb{P}(\text{"}O\text{"}|V) \cdot \mathbb{P}(V) + \mathbb{P}(\text{"}O\text{"}|O) \cdot \mathbb{P}(O)$$
$$= \mathbb{P}(\text{"}O\text{"}|V) \cdot (1 - p_O) + \left(\frac{p_s}{p_s + p_c} \cdot p_n + 1 \cdot p_s + 1 \cdot p_c \right) \cdot p_O$$

That leaves us with just needing to find $\mathbb{P}(\text{"}O\text{"}|V)$, i.e. the odds the bathroom sign says "Occupied" when its actually vacant. Since this depends on who used the bathroom previously, it turns out this is the same as $\mathbb{P}(\text{"}O\text{"}|O_n)$:

$$\mathbb{P}(\text{``}O\text{''}|V) = \frac{p_s}{p_s + p_c}$$

$$\Rightarrow \mathbb{P}(\text{``}O\text{''}) = \frac{p_s}{p_s + p_c} \cdot (1 - p_O) + \left(\frac{p_s}{p_s + p_c} \cdot p_n + 1 \cdot p_s + 1 \cdot p_c \right) \cdot p_O$$

$$= \frac{p_s}{p_s + p_c} + p_c \cdot p_O$$

Finally, we're able to calculate the original question, the probability the sign reads "Occupied" while the bathroom is actually occupied:

$$\mathbb{P}(O|\text{``}O\text{''}) = \frac{\left(\frac{p_s \cdot p_n}{p_s + p_c} + p_s + p_c \right) \cdot p_O}{\frac{p_s}{p_s + p_c} + p_c \cdot p_O}$$

Moving onto the second question, the probability the sign reads "Vacant" while the bathroom is actually vacant, we again start with Bayes Rule:

$$\mathbb{P}(V|\text{``}V\text{''}) = \frac{\mathbb{P}(\text{``}V\text{''}|V) \cdot \mathbb{P}(V)}{\mathbb{P}(\text{``}V\text{''})}$$

All our work answering part one makes this a breeze:

$$\mathbb{P}(V) = 1 - \mathbb{P}(O) = 1 - p_O$$

$$\mathbb{P}(\text{``}V\text{''}) = 1 - \mathbb{P}(\text{``}O\text{''}) = 1 - \left(\frac{p_s}{p_s + p_c} + p_c \cdot p_O \right) = \frac{p_c}{p_s + p_c} - p_c \cdot p_O$$

$$\mathbb{P}(\text{``}V\text{''}|V) = 1 - \mathbb{P}(\text{``}O\text{''}|V) = 1 - \frac{p_s}{p_s + p_c} = \frac{p_c}{p_s + p_c}$$

$$\Rightarrow \mathbb{P}(V|\text{``}V\text{''}) = \frac{\frac{p_c}{p_s + p_c} \cdot (1 - p_O)}{\frac{p_c}{p_s + p_c} - p_c \cdot p_O}$$