Assuming u is an exact result and v is an approximation

Types of Error

Absolute error (A): $|u - v|$

Relative error (R): $\frac{|u-v|}{|u|}, u \neq 0$

Cancellation Error (R): when adding 2 numbers of similar absolute value, but opposite sign and there's a loss of precision

Overflow: when a number is too large (fatal)

Underflow: when a number is too small (usually rounded down to 0)

$\min_x f$: minimum value of f, with changing x

Other sources:

- **(A)** when y >> x, x + y
- **(A, R)** when |y| >> 1, xy
- **(A, R)** when |y| << 1, x/y

Problem 2 (4 points) Let x and y be real numbers that are stored as floating-point numbers $\widetilde{x}$ and $\widetilde{y}$ respectively. Derive a bound for the relative error

$$\frac{(x-y)-(\widetilde{x}-\widetilde{y})}{x-y}$$

Problem 1 (8 points) For what argument(s) cancellations can occur when evaluating each of the following expressions. Show how to rewrite them to avoid cancellations.

- $\sqrt{x+1}-1$ $\quad \sqrt{x+1}-1=(\sqrt{x+1}-1)\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\frac{x}{\sqrt{x+1}+1}$
- $\sin x - \sin y$ $\quad \sin x-\sin y=2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$ $\quad \frac{1-\cos x}{\sin x}=\frac{1-\cos x}{\sin x}\frac{1+\cos x}{1+\cos x}=\frac{\sin x}{1+\cos x}$
- x^2-y^2 $\quad x^2-y^2=(x-y)(x+y)$
- $(1-\cos x)/\sin x$

$$\text{Condition number} = \frac{\left|\frac{f(\hat{x})-f(x)}{f(x)}\right|}{\left|\frac{\hat{x}-x}{x}\right|} = \frac{\left|\frac{\hat{y}-y}{y}\right|}{\left|\frac{\Delta x}{x}\right|} = \|A\|\cdot\|A^{-1}\|$$

$$\text{Condition number} = \frac{\frac{|\Delta y|}{|y|}}{\frac{|\Delta x|}{|x|}} \quad \begin{matrix}\leftarrow \text{relative forward error}\\ \leftarrow \text{relative backward error}\end{matrix}$$

Summary: condition number = change in *solution*/change in *input*

Well-conditioned: low condition #

Ill-conditioned: high condition #

Horner's Method (or rule or scheme): nested polynomial form, i.e.

$$p_n(x)=c_0+c_1x+\cdots+c_nx^n \Rightarrow p_n(x)=(\cdots((c_nx+c_{n-1})x+c_{n-1})x\cdots)x+c_0$$

Problem 9 (5 points) Set up the linear least squares system $Az \approx b$ for fitting $y = ax+be^x$, where the data is

x	1	2	3
y	2	3	5

That is, show the matrix A and the right-hand side b. How would you find A and b in Matlab?

$$Ax=\begin{bmatrix}1 & e\\ 2 & e^2\\ 3 & e^3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}\approx\begin{bmatrix}2\\ 3\\ 5\end{bmatrix}=c.$$

In Matlab, solve using $A\backslash c$.

Condition Number

Forward error: difference between result and solution

Backward error: the difference between the value of x used to find the result and the value of x that would give the solution

Condition number: how sensitive a function is to changes or errors in the input

Chapter 6 – Linear Least Squares

Data fitting: forming a curve that fits between the points

Interpolation: drawing a curve along the points, guessing the shape

This method is a method of data-fitting

Linear least square: find a function that best fits the given data points. The function that you want is the one that minimizes the sum of the square of the distances from the point to the curve. Why squared?

Problem 5 (5 points)

t_i	1	2	4
y_i	0.8	2.1	3.8

$$A=\begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 4\end{bmatrix},\quad A^TA=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}3 & 7\\ 7 & 21\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}0.8\\ 2.1\\ 3.8\end{bmatrix}=\begin{bmatrix}5.7\\ 20.2\end{bmatrix}$$

derive the normal equations for the linear least-squares fit $p(t) = a + bt$. Show matrix and right hand side. You do not have to solve the linear system.

(2 points) $1-\cos x$

When $x \approx 0$, $\cos(x) \approx 1$ and cancellations may occur. Using

$$\sin^2\frac{x}{2}=\frac{1-\cos x}{2}$$

we can evaluate $2\sin^2\frac{x}{2}$ since

$$1-\cos x = 2\sin^2\frac{x}{2}.$$

(2 points) $\sqrt{x+2}-\sqrt{x}$

Cancellations may occur for large x. It is better to evaluate $2/(\sqrt{x+2}+\sqrt{x})$:

$$\sqrt{x+2}-\sqrt{x}=\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}(\sqrt{x+2}-\sqrt{x})=\frac{2}{\sqrt{x+2}+\sqrt{x}}$$

(2 points) $e^x - e$ Cancellations can arise for $x \approx 1$. Let $x = 1+\epsilon$, where $\epsilon \approx 0$. Then

$$e^x-e=e^{1+\epsilon}-e=e(e^\epsilon-1)$$

$$=e\left(\epsilon+\frac{\epsilon^2}{2!}+\frac{\epsilon^3}{3!}+\cdots\right)$$

We can evaluate a few terms in

$$\left(\epsilon+\frac{\epsilon^2}{2!}+\frac{\epsilon^3}{3!}+\cdots\right)$$

and then multiply by e.

Another approach suggested by a student in class is to evaluate

$$\log(e^{e^x}/e^e)=e^x-e$$

Problem 1 (6 points) For each of the functions below, explain how accuracy can be lost and suggest how to evaluate the corresponding function (in floating-point arithmetic) without loss of precision:

a) $\exp(x) - \sin x - \cos x$

b) $\log x - \log(1/x)$

c) $x^{-2}(\sin x - e^x + 1)$

Floating Point

Floating Point (FP) System (β, t, L, u)

β – base

t – number of digits

L – lower bound for exponent

u – upper bound for exponent

Round-off Errors

$f(x)=x(1+\delta_x)$

$f(y)=y(1+\delta_y)$

$f(xy)=xy(1+\delta_{xy})$

Rounding unit (a.k.a. machine epsilon): $\eta=\frac{1}{2}\beta^{1-t}$

Rounding

Types:

a) To nearest
b) Towards $+\infty$ (i.e. upper bound)
c) Towards $-\infty$ (i.e. lower bound)
d) Towards 0 / chopping

Ceiling function: $\lceil x \rceil$

Floor: $\lfloor x \rfloor$

$f(x)=\ln x$

$f'(x)=x^{-1}$

$f''(x)=-1\cdot x^{-2}$

$f'''(x)=2\cdot x^{-3}$

$f^{(4)}(x)=-3\cdot 2x^{-4}$

$f(2)=\ln(2)$

$f'(2)=\frac{1}{2}$

$f''(2)=-\frac{1}{4}$

$f'''(2)=2\left(\frac{1}{8}\right)$

Convergence

where $0<\rho<1$, RoC $= -\log_{10}\rho$

x*: root

More accurate as k $\to \infty$

Rate of Convergence: given $\rho = |g'(x^*)|$

$$|x_k-x^*|\le\rho|x_{k-1}-x^*|\le\cdots\le\rho^k|x_0-x^*|$$

$$|x_{k+1}-x^*|=\frac{m-1}{m}|x_k-x^*|, x^*=0$$

Problem 4 (5 points) Consider the Taylor series

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots=\sum_{i=0}^{\infty}(-1)^i\frac{x^{2i+1}}{2i+1}.$$

- (4 points) Derive how many terms in this series are needed to approximate $\arctan(x)$ for any $x \in [-0.1, 0.1]$ with (absolute) error of at most 10^{-10}.

  Denoting this approximation by $f(x) = x - \frac{x^3}{3} + \cdots$, you need to determine how many terms are needed in $f(x)$.

- (1 points) Evaluate the error $|\arctan(0.1) - f(0.1)|$. If you have the correct $f(x)$, this error should be much smaller than 10^{-10}.

Solution.

- We have

$$\arctan(x)=\sum_{i=0}^{n-1}(-1)^i\frac{x^{2i+1}}{2i+1}+(-1)^n\frac{\xi^{2n+1}}{2n+1}$$

for some ξ between 0 and x. Since $x \in [-0.1, 0.1]$, $|\xi| \le 0.1$. Then we require that

$$\left|(-1)^n\frac{\xi^{2n+1}}{2n+1}\right|\le\frac{0.1^{2n+1}}{2n+1}\le 10^{-10}.$$

If $n = 4$, $0.1^{2n+1}/(2n+1) \approx 1.1111 \times 10^{-10}$ and if $n = 5$, $0.1^{2n+1}/(n+1) \approx 9.0909 \times 10^{-13}$

Hence, we can approximate by

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}.$$

- Then for any $x \in [-0.1, 0.1]$ Then

$$|\arctan(x)-f(x)|\le 10^{-10}.$$

- The actual error is

$$|\arctan(0.1)-f(0.1)|\approx 9.0145\times 10^{-13}.$$

Taylor Series

Rewrite a function, f(x) as an infinite sum of other functions.

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

a is a constant that either you choose or it is given to you

It is easiest to do if you start off by finding the first couple derivatives and then plugging in a, so you can see how the function changes with each iteration.

Taylor's Theorem

If you know the value of f(x$_0$), but want the value of f(x$_0$ + h):

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2}f''(x_0)+\ldots+\frac{h^k}{k!}f^{(k)}(x_0)+\frac{h^{k+1}}{(k+1)!}f^{(k+1)}(\xi)$$

Where ξ is some point between x$_0$ and x$_0$ + h.

(2 points) Given a nonsingular $n \times n$ matrix A, describe (in words or/and pseudo code) an efficient algorithm for computing A^{-1}.

We search for matrix X such that $AX = I$, the identity matrix. Denote the ith column of X by $x^{(i)}$ and the ith column of I by e_i.

We can compute the LU factorization of A, $A = LU$. Then, from $L(Ux^{(i)}) = e_i$, we can solve $2n$ triangular systems

b) (2 points) What is the complexity of your algorithm in big-O notation?

Computing the LU factorization takes $O(n^3)$. Since solving each of these systems takes $O(n^2)$, and we have n of them, we have in total $O(n^3)$ for computing A^{-1}.

c) (2 points) How would you tell a user of your algorithm how accurate the computed A^{-1} is?

A simple way is to compute the condition number of A. If it is $\approx 10^k$, about k decimal digits are lost when solving for each column of X.

ts) Consider the bisection method on the interval [1,10]. What is the smallest number of iterations of this method that are needed to determine a root with an error of at most 10^{-12}.

Solution. We have

$$n>\frac{\log(b-a)-\log(2\epsilon)}{\log 2}=\frac{\log 9-\log(2\times 10^{-12})}{\log 2}\approx 42.033.$$

We can take $n = 43$.

Chapter 3 – Solving Non-linear Equations

Find the roots.

Tolerance: a value that tells the computer what precision to stop computation

Absolute Tolerance: $|x_n - x_{n-1}|$ < atol

Relative Tolerance: $|x_n - x_{n-1}|$ < rtol$\cdot|x_n|$

$|f(x_n)|$ < ftol ← tolerance based on function

Error

Secant Method

$$f'(x_k)=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$$

$$x_{k+1}=x_k-\frac{f(x_k)(x_k-x_{k-1})}{f(x_k)-f(x_{k-1})}$$

Bisection Method

When given an upper and lower estimate for a number, find the midpoint between the two points. If the midpoint is negative, it becomes the new lower limit. If it is positive, it becomes the new upper limit. Keep doing this until the end of time…

This is used on graphing calculators.

It is linearly convergent.

Newton's Method

It runs faster than bisection method. When given a single guess (x$_0$),

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

The order of convergence is the golden ratio, $\alpha=\frac{1+\sqrt{5}}{2}\approx 1.618$. It is super-linearly convergent

Problem 5 (4 points) You have to interpolate e^x by a polynomial of degree five using equally spaced points in [0, 1].

(a) What error would you expect if you use this polynomial?

To have degree $n = 5$, we need $(n + 1)$ points. Let $h = 1/5 = 0.2$. Since e^x is bounded by e on [0, 1], the error is

$$\frac{M}{4(n+1)}h^{n+1}\le\frac{e}{4(5+1)}0.2^{5+1}\approx 7.2488\times 10^{-6}.$$

(b) Using equally spaced points, what degree polynomial would you use to achieve a maximum error of 10^{-8}?

Let $h = 1/n$. Then we want

$$\frac{M}{4(n+1)}h^{n+1}\le\frac{e}{4(n+1)}(1/n)^{n+1}\le 10^{-8}.$$

By trial and error, one can easily find that $n \ge 8$, so we need at least 9 equally spaced points.

Problem 3 (2 points) How accurately can we determine $\sin(x)$ by linear interpolation, given a table of $\sin(x)$ to 10 decimal places for $x \in [0, 2]$ with $h = 0.01$? That is, we have values for $\sin(x)$ at $0, 0.01, 0.02, \ldots, 2$.

On each subinterval $[x_i, x_{i+1}]$, we approximate $\sin(x)$ the error

$$\frac{M}{4(n+1)}h^{n+1},$$

using $M = 1$, as $|\sin(x)| \le 1$ and $n = 1$, the bound is

$$\frac{1}{4(1+1)}0.01^2=1.25\times 10^{-5}.$$

Gaussian Elimination

A method of solving matrices that is useful for solving for x.

Cost: $2\sum_{k=1}^{n-1}(n-k)=2\left((n-1)^2+(n-2)^2+\ldots+1^1\right)=O(n^3)$

LU Decomposition

It is useful if you aren't given m number of **b** vectors.

Do not switch rows! | Solving a Matrix

L: lower triangular
U: upper triangular
If L is a lower triangular, non-singular matrix, its inverse is also lower triangular.

1. Use Naïve Gaussian to find the upper and lower triangular matrices.
2. $A = LU \Rightarrow L(U\mathbf{x}) = \mathbf{b}$, which splits up into $\mathbf{y} = U\mathbf{x}$ & $L\mathbf{y} = \mathbf{b}$
3. Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. $\leftarrow O(n^2)$
4. Use $\mathbf{y}$ to solve the equation $\mathbf{y} = U\mathbf{x}$ for $\mathbf{x} \leftarrow O(n^2)$

$\det(A)=\det(L)\det(U)$

Cost: $O(n^3)$ for the Gauss, but $O(n^2)$ for solving for **b**.

Problem 3 (5 points) Solve

$$\begin{bmatrix}0&1&0&0\\1&0&3&0\\-0.5&0&-0.2&1\\-0.5&-0.3&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

using partial pivoting. Show all your calculations.

Solution

$$\left[\begin{array}{cccc|c}0&1&0&0&1\\1&0&3&0&1\\-0.5&0&-0.2&1&1\\-0.5&-0.3&1&0&1\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&3&0&1\\0&1&0&0&1\\-0.5&0&-0.2&1&1\\-0.5&-0.3&1&0&1\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&3&0&1\\0&1&0&0&1\\0&0&1.3&1&1.5\\0&-0.3&2.5&0&1.5\end{array}\right]$$

$$\to\left[\begin{array}{cccc|c}1&0&3&0&1\\0&1&0&0&1\\0&0&1.3&1&1.5\\0&0&2.5&0&1.8\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&3&0&1\\0&1&0&0&1\\0&0&2.5&0&1.8\\0&0&1.3&1&1.5\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&3&0&1\\0&1&0&0&1\\0&0&2.5&0&1.8\\0&0&0&1&0.5645\end{array}\right]$$

$x_4=0.564,\ x_3=0.72,\ x_2=1,\ x_1=1.16$

$$\begin{bmatrix}1&1&1&3\\0&-1&0&-4\\0&0&-5&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&-1&0\\0&0&-5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}3\\-4\\4\end{bmatrix}$$

$-5x_3=4$

$\boxed{x_3=-\frac{4}{5}}$

$-x_2=-4$

$\boxed{x_2=4}$

$x_1+x_2+x_3=3$

$x_1+4+\frac{-4}{5}=3$

$x_1=\frac{4}{5}-1$

$x_1=\frac{4-5}{5}$ $\quad\left(-\frac{1}{5},4,-\frac{4}{5}\right)$

$\boxed{x_1=-\frac{1}{5}}$

Problem 3 (4 points) Find the LU factorization of

$$\begin{bmatrix}2&2&1\\4&7&2\\2&11&5\end{bmatrix}$$

(No pivoting is required.)

Problem 6 (5 points) Using the naive (without pivoting) Gauss elimination, calculate the L and U factors in the LU factorization of

$$A=\begin{bmatrix}1&0&0&0\\0&1&3&-1\\3&-3&0&6\\0&2&4&-6\end{bmatrix}\quad L=\begin{bmatrix}1&0&0&0\\0&1&0&0\\3&-3&1&0\\0&2&-1/4&1\end{bmatrix}\quad U=\begin{bmatrix}1&0&1/3&0\\0&1&3&-1\\0&0&8&3\\0&0&0&-13/4\end{bmatrix}$$

Chapter 10–Interpolation

Interpolation is determining the value of a point based on the values of the points around it. You do not need to use all given points.

Multiple methods:

1. Monomial basis functions: $\varphi_j(x)=X^j$

Newton Basis Functions

x_i	-1	0	1	2
y_i	1	1	2	0

e the interpolating polynomials using

Linear

Given you have to find the value on the curve at a certain point, find 2 points, x_0 and x_1, such that x_0 is below the value and x_1 is above the value.

Polynomial

The order of your polynomial will be the 1 – the number of points you are given. The order of your polynomial will be the subscript of your function. For example, a quadratic equation would have the form:

$f_2(x)=b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)$

$$p(x)=1+c_1(x-x_0)+c_2(x-x_1)(x-x_0)+c_3(x-x_2)(x-x_1)(x-x_0)$$
$$=1+0(x+1)+\frac{1}{2}(x-0)(x+1)-\frac{2}{3}(x-1)(x-0)(x+1)$$
$$=1+\frac{1}{2}x(x+1)-\frac{2}{3}(x-1)x(x+1)$$

$$L_0(x)=\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}=\frac{x(x-1)(x-2)}{-6}$$
$$L_1(x)=\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}=\frac{(x+1)(x-1)(x-2)}{2}$$
$$L_2(x)=\frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)}=\frac{(x+1)x(x-2)}{-2}$$
$$L_3(x)=\frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)}=\frac{(x+1)x(x-1)}{6}$$

$p(x)=1\cdot L_0(x)+1\cdot L_1(x)+2\cdot L_2(x)$.

Problem 5 (6 points) Suppose that you are given values for e^{-x} as

x	e^{-x}
0	1
0.5	0.60653
1	0.36788

a) Approximate $e^{0.25}$ using the interpolating polynomial through these points.

b) Derive a bound for the error for any $x\in[0,1]$.

Problem 2 (5 points) Given an $a>0$, you want to compute $a^{1/3}$, that is, the cubic root of a. Describe how you can compute it. Then compute $3^{1/3}$ up to 3 accurate digits.

Solution Write $x^3=a$ and $f(x)=x^3-a$. Then we can apply Newton's method:

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}=x_k-\frac{x_k^3-a}{3x_k^2}$$

```
>> x = x-(x.3-3)\(3*x.3)
>> x = 3
```

Problem 11 (5 points) The explicit trapezoid method for integrating the ODE $y'=f(t,y)$ is

$$y_{i+1}=y_i+\frac{h}{2}[f(t_i,y_i)+f(t_{i+1},y_i+hf(t_i,y_i))].$$

Determine for what h this method is stable when applied to $y'=\lambda y$ with $\lambda<0$.

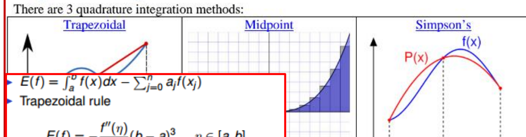
Solution.

$$y_{i+1}=y_i+\frac{h}{2}[\lambda y_i+\lambda(y_i+h\lambda y_i)]$$
$$=y_i+h\lambda y_i+\lambda\frac{h^2}{2}y_i$$
$$=\left(1+h\lambda+\frac{h^2}{2}\lambda\right)y_i$$

The method is stable for h such that $|1+h\lambda+\frac{h^2}{2}\lambda|\le 1$.

Chapter 15 – Integration Methods

There are 3 quadrature integration methods:



- $E(f)=\int_a^b f(x)dx-\sum_{j=0}^n a_j f(x_j)$
- Trapezoidal rule

$$E(f)=-\frac{f''(\eta)}{12}(b-a)^3,\quad \eta\in[a,b]$$

- Midpoint rule

$$E(f)=-\frac{f''(\zeta)}{24}(b-a)^3,\quad \zeta\in[a,b]$$

- Simpson's rule

$$E(f)=-\frac{f^{(iv)}(\xi)}{90}\left(\frac{b-a}{2}\right)^5,\quad \xi\in[a,b]$$

Problem 4 (10 points)

(a) (5 points) Let A be an $n\times n$ lower-triangular matrix. Write an algorithm in pseudo-code (or Matlab if you wish) for computing x in $Ax=b$, where b is an n column vector.

(b) (5 points) Derive a formula for the number of arithmetic operations to find x.

The inner for loop executes $k-1$ times and has in total $2(k-1)$ operations. The total number of operations is

$$\sum_{k=1}^n(2+2(k-1))=2n+2\sum_{k=1}^n(k-1)=2n+2\sum_{k=1}^{n-1}k=2n+2\frac{n(n-1)}{2}=2n+n^2-n$$
$$=n^2+n.$$

for $k=1:n$
  $s=b_k$
  for $j=1:k-1$
    $s=s-a_{kj}x_j$
  $x_k=s/a_{kk}$

Power Method

Finds largest eigenvalue. Works well on large, sparse matrices.

Chapter 8–Eigenvalue

$A\in R^{n\times n}$
$Ax=\lambda x$
λ = eigenvalue
x = eigenvector

s

Deflation: once you have computed the largest eigenvalue, you can subtract it and repeat the method to find the other eigenvalues from highest to lowest

Chapter 16–ODE's

To solve for $\frac{dy}{dx}=f(x,y),\ y(0)=y_0$, you could use Euler's formula: $y_{i+1}=y_i+f(x_i,y_i)h$, where $h=x_{i+1}-x_i$, but this is inaccurate, so use Runge

Runge-Kutta

Uses Taylor Series

$$y_{i+1}=y_i+\left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i)+\frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2+\frac{1}{3!}\left.\frac{d^3y}{d^3x}\right|_{x_i,y_i}(x_{i+1}-x_i)^3+\ldots$$
$$=y_i+f(x_i,y_i)h+\frac{1}{2!}f'(x_i,y_i)h^2+\frac{1}{3!}f''(x_i,y_i)h^3+\ldots$$

The second order Runge-Kutta only includes up to the part where you need to find the second derivative: $y_{i+1}=y_i+f(x_i,y_i)h+\frac{1}{2!}f'(x_i,y_i)h^2$

Problem 10 (10 points) Consider the ODE $y'=-5y$ with $y(0)=1$. Suppose you solve this ODE with constant stepsize $h=0.5$. Provide sufficient detail when answering the following questions.

Are solutions to this ODE stable? 2 points) Compute the numerical value for the approximate solution at $t=0.5$ by the orward Euler method.

Is the forward Euler method stable for this ODE using this stepsize? backward