Matricola: 0900064976

ALMA MATER STUDIORUM - UNIVERSITA' DI BOLOGNA

COLLEGIO SUPERIORE

ELABORATO DI APPROFONDIMENTO

An introduction to American FX basket options hedging: from standard multimarket model to non-gaussian copulas.

An application to a euro-denominated CNH and USD basket in flight-to-quality times.

Supervisore Prof: Giuseppe Pignataro

Co-Supervisore Prof. Sabrina Mulinacci

Presentata da: Matteo Teruggi

Anno accademico
2020-2021

An introduction to American FX basket options hedging:
from standard multimarket model to non-gaussian copulas

An application to a euro-denominated CNH and USD basket in flight-to-quality times.

Abstract: Basket options play a relevant role in corporate finance since they allow a more *bona fide* hedging, allowing to trade the correlation between a couple of assets. Basket options are, in general, index options or currency options: the elephant in the room is the correlation between the assets. In the first pages of this essay, we prove that if the two underlings' assets are correlated, adequate gamma-delta hedging could be reached only using basket options. The standard multimarket model seems to apply not enough. We are interested in the option's price and recovering the main hedging parameters (the single-asset gammas and the cross gamma) and their sensitivity to the interest rate and time. We are highly interested in the cross gamma since the gamma is the only Greek where the European greek is not a good proxy of the American one. The gamma of the early exercise premium plays a significant role in the hedging strategy. That is why, *simili modo,* we cannot rely on previous studies on European contracts, which are unsatisfactory. Results seem relevant enough and suggest further research to be conducted. We refer to American options, though some considerations concerning the pricing scheme interest even the European contracts. The main results concern the inapplicability of a pure first-order hedging strategy.

In contrast, a second-order hedging strategy seems more suitable to be implemented, accounting even for the correlation between the two underlying thanks to the cross gamma. This requires, however, an adequate estimation of the second-order hedging parameters. In addition, even if there is no explicit comparison robustness of the results confirms the actuality of the scientific interest in the Longstaff-Schwarz approximation.

Index

Introduction

In this paper, we would like to consider price and hedge basket american options. Basket american options are contracts written not on a single underlying but a more sophisticated portfolio of currencies, stocks, or bonds. Theoretically speaking, they can be considered a particular case of vanilla options written on a stock index. For example, an index option on MIB could be considered, theoretically, as a basket option written on all the underlying stock indexes. By far, this is only a theoretical equation. Index options are modelled with one random variable, i.e., the random variable of the index, since, in general, there is evidence for modelling the product as a stand-alone product of the index variable.

From a theoretical point of view, there is no restriction on short selling on the basket. By contrast, basket options are modelled as a couple or a tuple of random variables since there is no robust and adequate opportunity to melt the basket into one random variable. The result is even due to the weights of the baskets: while in index options, these weights are constant and predetermined and are the weights of the stock market index, in basket options, they can be arbitrarily defined in the contract. In this case, the buyer of the basket options is the seller of a sub-set of the option: he sells the index part with negative weights. However, this is outside of our most immediate scope. From now on, we will impose short-selling restrictions, so the weights of each instrument in the basket must be in [0,1], and the sum of the weights must be one.

We will now consider the need for a basket option. We are assumed to find risk-prone and risk-averse individuals in all the markets apart from risk-neutral operators (arbitrageurs). Speculators might buy basket options when they believe that the correlation between two or more assets will increase. They buy the basket option and sell the equally weighted portfolio of single options. The contrasting point of view applies when the investors expect a decrease in the correlation. Why a bona fide hedger, instead, should buy a basket option? Going deep in forex options makes it easy to think about a firm with half of the foreign revenues in USD and half in CNH.

As the aim is to decrease the volatility of the revenues due to currency risk, buying two independent options written on a single underling is not an efficient solution. There is no need for position coverage if a relevant drawdown on one currency and a relevant draw upon the other. We depict two different hedging strategies: two vanilla currency american options and one basket option, written on the basket previously defined. Note that the basket has the same structure of revenues. Let us assume that the sales of our model firm are denominated in the following currencies: 50% USD and 50% CNH. However, the cost structure and the income statement are entirely in euros. Therefore, there is no margin for operational currency risk hedging.

Table 1 - The advantages of basket options

	Two vanilla american options	**One basket option**
Sales in USD	100 euros	100 euros
Sales in CNH	100 euros	100 euros
Scenario: +5% on EUR/USD and -5% on EUR/CNH rate or vv.		
P/L on USD currency	+2.5% of the basket	+2.5% of the basket
P/L on CNH currency	-2.5% for the basket	-2.5% for the basket
P/L on all currencies	0	0
Cost of hedging	Call on EUR/USD Call on EUR/CNH	Call on basket EUR/USD and EUR/CNH
P/L	Cost of the two american options	Cost of the single option

The firm will receive a profit, but sales variability is already null. Suppose the firm presents domestic denominated sales but foreign-denominated costs (both operational, like raw material costs or financial, as interests/mortgage repayments). In that case, it will be enough to swing the option from call to put. Unfortunately, suppose part of the currency risk is the revenues structure, and part of currency risk is in the costs structure. Some negative weights should be introduced in the option, selling the currencies on the income statement's negative side. Obviously, the higher the correlation of the two assets, the closest the price to the sum of the two american options. For negative correlations, the price will be significantly lower. A basket option can be the instrument to hedge the risk exposure to or speculate the market movements on the underlying stock basket. Because it involves just one transaction, a basket option often costs less than multiple single american options. The most important feature of a basket option is its ability to efficiently hedge risk on multiple assets simultaneously.

Rather than hedging each asset, the investor can manage risk for the basket or portfolio in one transaction. The benefits of a single transaction can be great, especially when avoiding the costs associated with hedging each component. The increased effectiveness of the hedging is shown even by this graph (the first graph in the following section, the scatterplot). We observe that the number of relevant violations of the barrier of ±0.5% daily movement is significantly lower for the basket than for the single currency pair. In addition, by the symmetry plot of the returns of the two exchange rates, we observe that the EUR/USD is right-skewed, while the EUR/CNH is symmetric. The skewness is detected at as many high distances as possible from the mean observation lying under the symmetry line. This right skewness is due to the flight to the quality of the affected dollar at the early stage of the pandemic.

Figure 1: EUR/CNH vs EUR/USD dailiy scatterplot, bps. From 10/12/2020 to 10/12/2021. Forex returns are evidently negatively correlated. Relevant numbers of outilers.

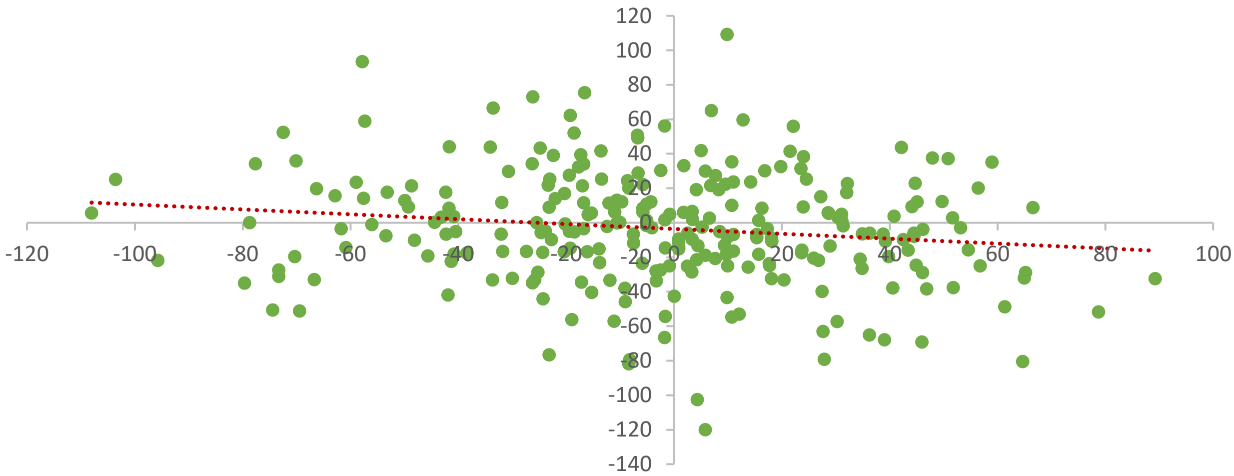


Figure 2 - Quantile regression of EUR/USD rate vs EUR/CNH rate. The interval is still December 2020 to 2021. Note the nonlinearity of the correlation. Even if the correlation is negative for all the quantiles considered, the magnitude doubles for Q>0.9, showing nonlinearity in the pattern. In the most favourable 10% of trading days for CNH, the magnitude of anticorrelation with the dollar was more relevant.

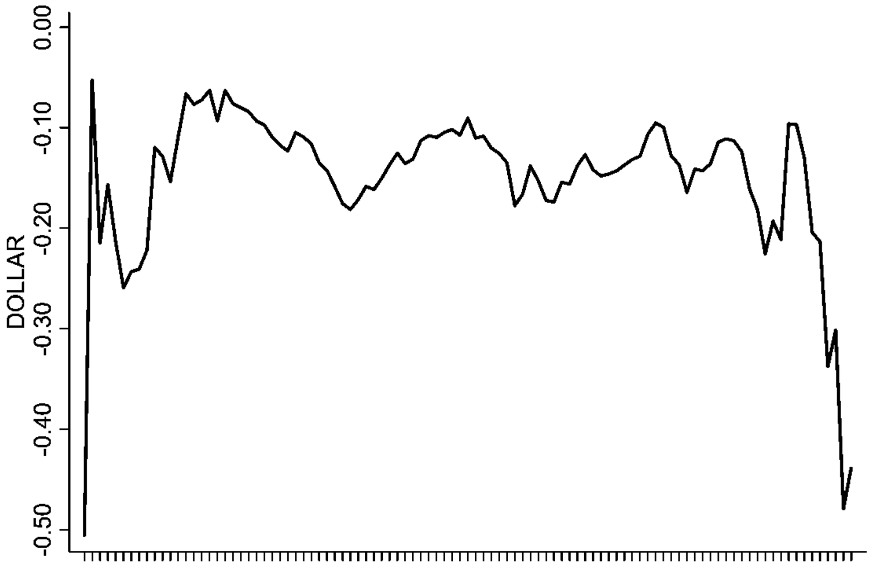
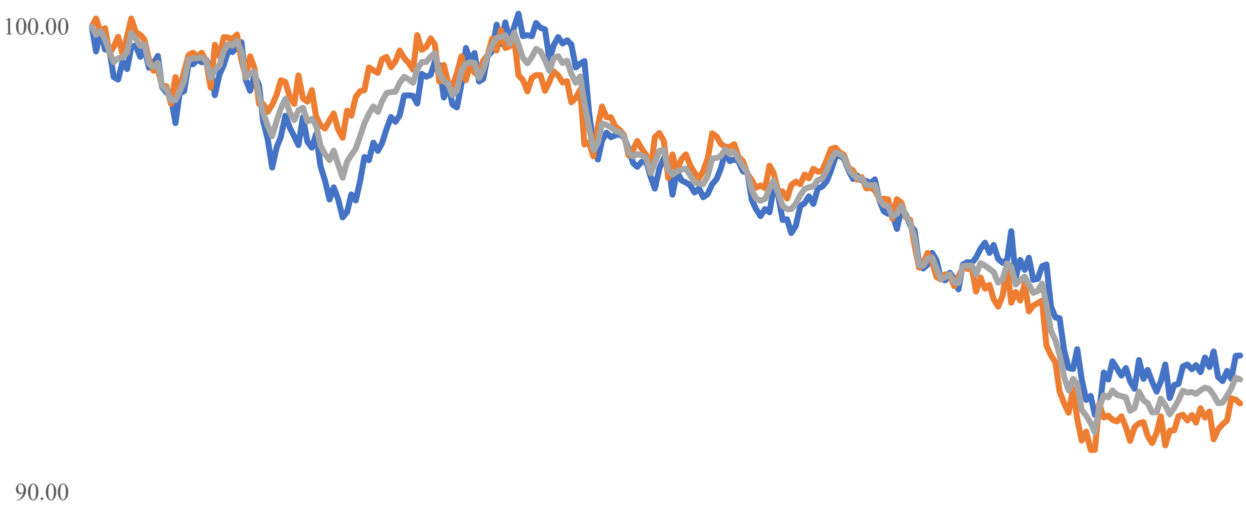


Figure 3: basket pattern of the exchange rate, daily observations from 10/12/2020 to 10/12/2021. Please note the anticorrelation between the two assets and the higher mean reversion tendence.
Blue line: dollar; Orange line: Yuan, Grey line: equally weighted basket



The following inequality summarises the benefit of the basket option:

$$\text{As } \mathrm{E}[1_{\text{Basket}}] = \mathrm{E}\left[1_{\text{Dollar option} \cap \text{Yuan option}}\right]$$

$$\mathrm{E}[1_{\text{Basket}}] < \mathrm{E}\left[1_{\text{Dollar option U Yuan option}}\right] < \mathrm{E}\left[1_{\text{Dollar option}}\right] + \mathrm{E}\left[1_{\text{Yuan option}}\right]$$

Note: in the case of perfect correlation, the midterm converges to the left one, while in the case of perfect anticorrelation, the central term converges to the right one. As the midterm moves to the right, the convenience of the basket option increases.

$$\text{If } \rho=1 \; E[1_{\text{Basket}}] = E[1_{\text{Dollar option U Yuan option}}] = E[1_{\text{Dollar option} \cap \text{Yuan option}}]$$

$$\text{If } \rho = -1 \; E[1_{\text{Dollar option U Yuan option}}] = E[1_{\text{Dollar option}}] + E[1_{\text{Yuan option}}]$$

This benefit, unfortunately, does not come for free, at least in terms of computations and theoretical questions[1]. The main problem in devising an efficient hedging strategy is the cross-gamma risk. Suppose the two sources of risks are covered independently. In that case, vanilla hedging strategies like delta and gamma will apply. The pricing of most products requires at least one stochastic underlying: one discounting rates curve and one credit spread curve. Note that these different curves imply cross-gamma risk between these three underlying. Even if the last two are deterministic, they will be updated from time to time and induce delta hedging on the stochastic underlying. By contrast, in a basket option, the price of one underlying influences, especially when we are close to the money, the hedging parameters of the other one. The effect of the combined movements of two underlying, or cross-gamma, is the most basic cross-risk in the sense that it is based on the first level of observability: the spot prices of the underlying, regardless of the system level of volatility or correlation, in general cross-gamma cannot be hedged using other vanilla instruments, even locally: cross gammas must be accounted for, and their risk is born through the life of the product.[2]

Deeper, looking for a taxonomy of the cross-gammas effects, we can find the following ones:

- on the expected P&L[3] through the difference between realised and implied covariance, as we will consider deeply in the next paragraph.
- On the expected P&L through hedging costs (intersecting bid-offers).
- on the P&L variance since the lack of a local cross-gamma hedge implies potentially significant

daily delta rebalancing until maturity

[1] There are, in truth, even some relevant illiquidity issues that we will not consider in this stage.
2 Chicot, Measuring cross gamma risk (December 5, 2019). Available at SSRN: https://ssrn.com/abstract=3502251 or http://dx.doi.org/10.2139/ssrn.3502251, pag.8
3 Linders and Shoutens, Basket Option Pricing and Implied Correlation in a One-Factor Lévy Model in K. Glau et al. (eds.), Innovations in Derivatives Markets, Springer Proceedings (2016), page 335

The pricing and hedging problem

Our primary research interest is to find an arbitrage-free price for currency basket american options and recover some relevant hedging parameters. We are interested in the two main *greeks* that allow hedging against the price movements in the underlying. The delta measures the change in the price of the option wrt to the change in the price of the underlings. In our case

$$\Delta_i = \frac{\mathrm{dPrice}}{\mathrm{d}S_i}$$

The overall delta could be summarised for all the underlings in:

$$\Delta = \|\Delta_1, .., \Delta_n\|$$

The gamma measures the change in the option's delta wrt to the change in the price of the underlings. In our case

$$\Gamma_i = \frac{\mathrm{d}\Delta_i}{\mathrm{d}S_i} = \frac{\mathrm{d}Price}{\mathrm{d}S_i{}^2}$$

In the basket of american options, we have also cross gamma:

$$\Gamma_{i,j} = \frac{\mathrm{d}\Delta_i}{\mathrm{d}S_j} = \frac{\mathrm{d}Price}{\mathrm{d}S_i\,\mathrm{d}S_j}$$

Going to more exotic Greeks and increasing the number of underlings concerning the derivative is possible. However, this is not useful. The total number of cross gammas is equal to:

$$\text{Number of cross gammas} = \sum_{\mathrm{i=1}}^{\mathrm{n-1}} \mathrm{i} = 0.5\mathrm{n}(\mathrm{n}-1)$$

$$\text{Number of overall gammas} = \sum_{\mathrm{i=1}}^{\mathrm{n-1}} \mathrm{i} + \mathrm{n} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{i} = 0.5\mathrm{n}(\mathrm{n}+1)$$

We are interested in American options, as there are already some results in the literature for European contracts[4]. One way to think of American-exercise options is to break their value V_A down into a value due to European exercise, V_E, and a "premium" due to the possibility of early exercise, V_P

$$V_A=V_E+V_P$$

We have similar deltas, gammas, and cross-gammas since the differentiation operator is linear.

$$\Delta_A=\Delta_E+\Delta_P$$

$$\Gamma_A=\Gamma_E+\Gamma_P$$

The interest in this hedging parameter is explained as it is necessary to design a hedging strategy meticulously. The more common hedging strategy is the delta-neutral first-order strategy. Defining our portfolio as π, $S_1$and S_2our assets, we are interested in immunising the portfolio value to shifts in the value of $S_1$and S_2. It means looking for the combination $N_1, N_2$of underlying quantities to be purchased[5].

$$\pi(\mathrm{t}) = \mathrm{N}_1(\mathrm{t})\mathrm{S}_1(\mathrm{t}) + \mathrm{N}_2(\mathrm{t})\mathrm{S}_2(\mathrm{t}) + \text{Basket Put on } [\mathrm{S}_1 + \mathrm{S}_2](\mathrm{t})$$

4 Choi, J. Sum of all Black–Scholes–Merton models: An efficient pricing method for spread, basket, and Asian options. J Futures Markets. 2018; 38: 627– 644. https://doi.org/10.1002/fut.21909

5 We obtain the same result studying the writer case, thought here gammas and cross-gamma will play a bigger role.

$$\pi(\mathrm{t}) = \mathrm{N}_1(\mathrm{t})\mathrm{S}_1(\mathrm{t}) + \mathrm{N}_2(\mathrm{t})\mathrm{S}_2(\mathrm{t}) + \text{Basket Call on } [\mathrm{S}_1 + \mathrm{S}_2](\mathrm{t})$$

$$\begin{cases} \frac{d\pi(t)}{dS_1} = N_1(t) + \Delta_{S_1}(t) = 0 \text{ i.e. } N_1(t) = -\Delta_{S_1}(t) \\ \frac{d\pi(t)}{dS_2} = N_2(t) + \Delta_{S_2}(t) = 0 \text{ i.e. } N_2(t) = -\Delta_{S_2}(t) \end{cases}$$

The strategy works pretty if $\frac{d\pi(t)}{dS_2 dS_1} = 0$, and consequently, the two equations show independent behaviour. Unfortunately, in general $\frac{d\pi(t)}{dS_2 dS_1} \neq 0$ and $\frac{d\Delta_1(t)}{dS_2} = \frac{d\Delta_{S_2}(t)}{dS_1} \geq 0$. Cross-Gamma risk is the instability risk of a first-order hedging strategy due to the elasticity of hedging parameters to the other underlying price. Our main goal is to estimate the pattern and the behaviour of this delta of the delta, technically speaking, the Cross-Gamma.

$$\frac{d\pi(t)}{dS_2 S_1} = \frac{d\Delta_{S_2}(t)}{dS_1} = \Gamma_{S1,S2}$$

$$\frac{d\pi(t)}{dS_2 S_1} = \frac{d\Delta_1(t)}{dS_2} = \Gamma_{S1,S2}$$

Knowing this parameter, we can put in place are more complex and responding Gamma Hedging strategy, so second-order hedging that covers even relevant shocks in the price of S_1, S_2. We assume a delta-hedge portfolio.

$$\pi(t) = -\Delta_{S_1}(t)S_1(t) + -\Delta_2(t)S_2(t) + M\,(t)\text{Basket Put on }[S_1 + S_2](t)$$

$$\begin{cases} \frac{d\pi(t)}{dS_1 dS_1} = -\frac{d\left[\Delta_{S_1}(t) + \Gamma_{S1}(t)S_1(t)\right]}{dS_1} + M\,(t)\Gamma_{S1} = [-2 + M\,(t)]\Gamma_{S1}(t) \\ \frac{d\pi(t)}{dS_2 dS_2} = -\frac{d\left[\Delta_{S_2}(t) + \Gamma_{S2}(t)S_2(t)\right]}{dS_2} + M\,(t)\Gamma_{S2} = [-2 + M\,(t)]\Gamma_{S2}(t) \\ \frac{d\pi(t)}{dS_2 dS_1} = -\frac{d\left[\Delta_{S_1}(t) + \Gamma_{S1}(t)S_1(t)\right]}{dS_2} - \frac{d\left[\Delta_{S_2}(t) + \Gamma_{S2}(t)S_2(t)\right]}{dS_1} + M\,(t)\Gamma_{S1,S2}(t) = \\ \qquad = [-2 + M\,(t)]\Gamma_{S1,S2}(t) \end{cases}$$

Imposing the null determinant for the relevant hessian matrix, we have:

$$[-2 + M\,(t)]\Gamma_{S1}[-2 + M\,(t)]\Gamma_{S2} = [-2 + M\,(t)]^2\Gamma_{S1,S2}^2$$

Assuming $M\,(t) \neq 2$, the necessary and sufficient condition for our portfolio to be even gamma-hedged is

$$\Gamma_{S1}(t)\Gamma_{S2}(t) = \Gamma_{S1,S2}^2$$

$$\Gamma_{S1,S2}(t) = \mu_g(\Gamma_{S1}, \Gamma_{S2})$$

That is a reliable and empirically consistent assumption.

Moreover, consequently, our Delta-Gamma hedged portfolio becomes:

$$\pi(t) = -\Delta_{S_1}(t)S_1(t) + -\Delta_2(t)S_2(t) + \text{Basket Put on }[S_1 + S_2](t)$$

This portfolio, which is precisely the previous shown portfolio, is neutral even to more relevant shifts in the underlying assets. A short pricing introduction is necessary since hedging parameters are derived from price sensitivities. Consequently, to derive the hedging parameters instructions, we need to know how to manage price. In addition, to responsibly buy this portfolio, we need to estimate the gamma matrix and the cross gamma to verify the suitability of our assumption.

We can furthermore derive the following proposition: let the portfolio be delta-hedged. There is consequently no need for gamma rebalancing. Let us assume that we opted for two independent vanilla put options instead of a basket option.

$$\pi(t) = -\Delta_{S_1}(t)S_1(t) + -\Delta_2(t)S_2(t) + M\,(t)\text{Put on }[S_1](t) + M\,(t)\text{Put on }[S_1](t)$$

The second-order condition becomes:

$$\begin{cases} \dfrac{d\pi(t)}{dS_1 dS_1} = -\dfrac{d\left[\Delta_{S_1}(t) + \Gamma_{S1}(t)S_1(t)\right]}{dS_1} + \Gamma_{S1} = [-2 + M\,(t)]\Gamma_{S1}(t) \\ \dfrac{d\pi(t)}{dS_2 dS_2} = -\dfrac{d\left[\Delta_{S_2}(t) + \Gamma_{S2}(t)S_2(t)\right]}{dS_2} + \Gamma_{S2} = [-2 + M\,(t)]\Gamma_{S2}(t) \\ \dfrac{d\pi(t)}{dS_2 dS_1} = -\dfrac{d\left[\Delta_{S_1}(t) + \Gamma_{S1}(t)S_1(t)\right]}{dS_2} - \dfrac{d\left[\Delta_{S_2}(t) + \Gamma_{S2}(t)S_2(t)\right]}{dS_1} = \\ \qquad = -2\Gamma_{S1,S2}(t) \end{cases}$$

Imposing the null determinant for the relevant hessian matrix, we have:

$$[-2 + M\,(t)]\Gamma_{S1}[-2 + M\,(t)]\Gamma_{S2} = 4\Gamma^2_{S1,S2}(t)$$

$$[-2 + M\,(t)]^2\Gamma_{S1}(t)\Gamma_{S2}(t) = 4\Gamma^2_{S1,S2}(t)$$

Calling $\alpha(t) = \frac{\Gamma^2_{S1,S2}(t)}{\Gamma_{S1}(t)\Gamma_{S2}(t)}$ we have $[-1 + 0.5M\,(t)]^2 = \alpha(t)$

Which is not verified ∀ x in **R**, since $\frac{\Gamma_{S1,S2}(t)}{\Gamma_{S1}(t)\Gamma_{S2}(t)} > 0$ unless the two assets are completely independent in the dynamic. There is only in this case of independence perfect correspondence in the delta-gamma hedging strategy between a basket option and a couple of vanilla options.

The hedging solution if this case is $M = 2 \pm 2\sqrt{\alpha(t)}$ which is not easily applicable and costly, as it increases the number of options that the hedger must buy.

We have the following result consequently: Let S_2 and S_1be two correlated assets of our portfolio. A gamma-delta hedging strategy is harder to implement without a suitable basket option. A portfolio composed only of the two underlying and vanilla options can be gamma hedged only with the high cost and extremely frequent rebalancing due to the instability of the parameters. Unfortunately, the rationale is that while we consider small movements in delta hedging and the assumption of independence between the two assets is dependable, the same assumption does not hold when considering more significant shifts.

The hedging parameters we are interested in are the ones we mentioned till now. Let us start this journey.

Let us start by introducing the pricing mechanism. The Black-Scholes currency Model consists of three processes: the domestic money market and the foreign (more than one, in our case) money market and the exchange rate processes. Holding a foreign currency is a risk for the investor. The more exotic the currency, the higher the risk. We start considering a market with only one exchange rate. We derive the pricing formula for a derivative written on it (we follow the fourth chapter of Baxter and Rennie, 1996, to which we refer the reader for more details).

- Domestic bond: $B_t = e^{rt}$
- Foreign bond: $D_t = e^{it}$
- Exchange rate: $C_t = C_0 e^{\sigma W + \mu t}$

W is a P-Brownian Motion and constants r, i, σ and μ. We denote by X the original claim, denominated in foreign currency. In our case, X is simply an amount of foreign currency due at a specific date in the future time T, quoted in foreign currency. It could be even more complicated, e.g., a bond or a stock denominated in the foreign currency of interest.

The followings are the steps towards the replication that will allow us to recover a domestic price for the contract:

- Find a measure Q under which the foreign bond discounted by the domestic bond is a martingale measure.

$$Z(t) = \frac{S(t)}{B(t)} = \frac{C(t)D(t)}{B(t)}$$

- Form the process E(t) such that

$$E(t) = E_Q\left(\frac{X}{B(T)} | \boldsymbol{F}\,(t)\right)$$

- Find the reversible process Φ(t), given by the ratio between the two differentials

$$\frac{dE(t)}{dZ(t)}$$

The domestic currency discounted worth of the foreign zero-coupon bond

$$Z_t = C_0 e^{\sigma W + (\mu + i - r)t}$$

Then under Q measure, applying the Cameron-Martin-Girsanov theorem[6]:

$$Z_t = C_0 e^{\sigma \widetilde{W}_t - 0.5\sigma^2 t}$$

$$C_t = C_0 e^{\sigma \widetilde{W}_t + (r - i - 0.5\sigma^2)t}$$

Given this Q, we define the conditional process E(t) to be the conditional expectation process, which, as noted before, is a Q-Martingale. After this, thanks to the martingale representation theorem produces an ***F***-previsible process that connects E(t) and Z(t), such that

$$E(t) = E\,(0) + \int_0^t \Phi(s)dZ(s)$$

After this, a replication strategy (Φ(t), k(t)) is needed to clearly state the amounts of the two investments: foreign and domestic currency. So, we hold Φ(t) units of the foreign currency (bond) and k(t)=E(t)- Φ(t)Z(t) units of the domestic currency. The resulting portfolio value is, at time T,

$$V(t) = \Phi(t)S(t) + k(t)B(t)$$

This portfolio V(t) is self-financing under two situations, first if the change in the portfolio is only due to changes in the asset prices

---

[6] We state the theorem as in Meyer.
Let F = (Ω, F, {Ft}t≥0, P) be a stochastic basis with the usual assumptions, let Z' be a non-negative random variable with unitary expectation, and let the martingale Z = Z(t) be the right-continuous version of E(Z'|F_t). Define the probability measure P' by dP' = ZdP. If M is a local martingale on F = (Ω, F, {*Ft*}t≥0, P) and the measures P' and P are equivalent, then the process Mf = Mf(t) defined by Mf(t) = M(t) − Zt₀d [Z, M] (s) Z(s), is a local martingale on (Ω, F, {*Ft*} t≥0, P').

$$\text{i.e., } dV(t)= \Phi(t)dS(t)+ k(t)dB(t)$$

$$\text{or if } dE(t)= \Phi(t)dZ(t),$$

that is what is precisely stated in the martingale representation theorem.

Since V(t)=B(t)E(t) and Z(t) is the discounted claim $\frac{S(t)}{B(t)}$ we have a self-financing strategy ($\Phi(t)$, k(t)) that replicates our arbitrary claim S. For a call option, the payoff X is:

$$X = MAX\ (\ C(t) - K, 0)$$

The value of the payoff at the time t is:

$$V(t) = B(t)E_Q\left(\frac{X}{B(T)}|\boldsymbol{F}\ (t)\right)$$

The option's price is unique, so it is a no-arbitrage price. Even if the currencies in which the instrument can be computed are two, the domestic and the foreign account, the valuation is unique and consistent. The reverse of the Cameron-Martin-Girsanov theorem allows us to migrate from the foreign to the domestic price through a reverse martingale measure. The payoff is easily valuable, assuming lognormal distribution for C(t) and adapting the Black and Scholes formula.

$$C_t^{-1} \ = \ C_0^{-1}e^{-\sigma W - \mu t}$$

$$Y_t = \ C_0^{-1}e^{-\sigma W-(\mu+i-r)t}$$

This discounted price process Y_t will be a martingale under the new measure Qforeing if Qforeing is a Brownian motion. Consequently, the previously mentioned option values:

$$U(t) = D(t)E_{Qforeing}\left(\frac{X}{D(T)}|\boldsymbol{F}\ (t)\right)$$

Where Qforeing is the measure under which the foreign discounted claim Y_t is a martingale[7].

We now want to extend this measure change to the multivariate model suitable for the basket option. We need to introduce our basket. Since the option is denominated in € (the domestic currency) and is written in two currencies (the foreign currencies) \$ and ¥ (USD and CNH), defined by the following formula and being quoted in €:

$$\text{Basket (T)} = 100\left(\frac{S_1(T)}{S_1(0)} + \frac{S_2(T)}{S_1(0)}\right) €$$

Where $S_1$is the exchange rate \$/€ (USD/EUR), while S_2 is the ¥ /€ (CNH/USD) exchange rate. Note that we must invert the basket to the definition given in the previous chapter, as we are now working in the domestic currency. T denotes the maturity of the specific financial instruments. We will now define our option.

A Call option on the basket pays:

---

[7] Thus, the payoff of X dollars at time T is worth the same to either investor at any time beforehand. Similar calculations show that the dollar and sterling investors' replicating strategies for X are identical. So, they agree not only on the prices but also on the hedging strategy. The difference in martingale measures only reflected the different numberaires of the two investors rather than any fundamental disagreement over prices. All investors, whatever their currency of account, will agree on the current value of a derivative or other security (BR, ch.4)

$$\text{MAX}\left(100\left(\frac{S_1(T)}{S_1(0)}+\frac{S_2(T)}{S_1(0)}\right)-K,0\right)$$

K is the strike and is set equal to 200 for ATM options. A call option protects from an appreciation of the Euro in front of the basket. The option will not be exercised as the basket will appreciate versus the Euro, the domestic currency.

A Put option on the basket pays

$$\text{MAX}\left(K-100\left(\frac{S_1(T)}{S_1(0)}+\frac{S_2(T)}{S_1(0)}\right),0\right)$$

K is the strike and is set equal to 200 for ATM american options. A put option protects from an appreciation of the basket in front of the Euro. The domestic currency will appreciate in front of the basket. A put option will not be exercised as the Euro appreciates versus the selected basket.

Suppose we assume the absence of correlation between the assets for a moment. In that case, the problem is reduced to N-dimensional integration of the payoff. This method is accurate but slow for more than two basket american options. We obtain the following multimarket model by applying the considerations already made in the one-variable model to the basket currency option. We define C(t) as the vector of the underlying assets (so the foreign exchanges, in our specific case), r as the interest rates matrix and σ as the variance-covariance matrix (historical) and W(t) as an n-dimension Brownian motion. The interest rate matrix is assumed constant over the option's whole life. The interest rate matrix is a diagonal matrix where each diagonal element is the difference between the risk-free rate and the net cost of carry of each specific asset. This net cost of carry is the foreign annualised interest rate. As a result, our pricing problem becomes defined under the martingale measure Q_m

$$r=\text{diag}\,(r_{\text{domestic}}-q_1,\ r_{\text{domestic}}-q_2\ldots r_{\text{domestic}}-q_n)=\text{diag}\,(r_{\text{domestic}}-r_{\text{dollar}},\ r_{\text{domestic}}-r_{\text{yuan}})$$

$$dC(t)=rC(t)dt+\sigma C(t)dW(t)$$

The crucial point is how to model the correlation between the assets. First, we assume that the basket is homogeneous in the asset class, i.e., the assets composing the baskets are all currencies or all stock indexes or stocks/bonds. How do we model the correlation between two currencies? How do we model the correlation between two stocks? It is known that the sum of a series of lognormal random variables is not a lognormal random variable unless they are perfectly correlated. A shifted lognormal random variable (SLN) approximates the weighted summation of the distributions. A good proxy, adopted by different practitioners, is to match central moments to the shifted lognormal variable or use approximations[8]. We have introduced the concept of implied covariance. The real *punctum dolens* is the implied correlation: calling ρ' (x, y) the implied correlation between the two assets, σ' (x) and σ' (y) as the implied volatility of the two assets, we have - simply applying the correlation definition:

$$\sigma'(x,y)=\rho'(x,y)\,\sigma'(x)\,\sigma'(y)$$

The implied correlation ρ' (x, y) is the rescaled difference between the implied volatility of the basket option σ' (Basket (x, y)) and the product of the implied volatilities of the single underlying asset american options, σ' (x) σ' (y). Going to a more general basket option, we have:

$$\text{Implied correlation}=\frac{\sigma'\,(\text{Basket option})-\sum w_i^2\sigma'\,(X_i)}{2\sum_{1\le i\le j\le n} w_i w_j \sigma'\,(X_i)\sigma'\,(X_j)}$$

8 On this point a concise, bust complete introduction is given by Dufresne, Sum of Lognormal, on http://ozdaniel.com/A/DufresneLognormalsARC2008.pdf. The working paper is deeply instructive, as it proposes the history of the problem, dating back to years 30s, up to the Laplace transform approximation by Wo et al. (2005)

Where in the previous formula, the product components are the product of the implied volatility of all the single-underling asset american options and w_i is the product weight in case the basket is not equally weighted. As a result, we can explain dispersion trading: i.e., the betting on the similarity or dissimilarity of the paths of the different components of the portfolio. Dispersion definition is:

$$\text{Dispersion} = \frac{1}{N}\sum w_i \sigma(x)_i^2 \; - \sigma(x)_{\text{index}}^2$$

For american options, correlation can break down the components of implied volatility and compare the relative value of an index option with a basket portfolio of its component american options. A more robust understanding of the drivers affecting correlation in equity or currency markets will benefit even volatility market participants. It will provide them with a more effective way to hedge underlying risk and construct speculative strategies by isolating volatility components.

After considering option pricing, we believe it is fruitful to switch to hedging.

Let us consider an exotic option with price P (t, S), where S is a tradable stock. At the time t, we hedge this option with a quantity Δ of stock. Over a short timeframe δt, the P&L of the hedged position is given, for a European basket option, by:

$$\text{P\&L} = -\,[P\,(t + \delta t, S + \delta S) - P\,(t, S)] + r_{\text{domestic}} P\,(t, S)\,\delta t + \Delta\,(\delta S - r_{\text{domestic}} S\delta t + qS\delta t)$$

Where r_{domestic} is the deterministic interest rate, and q is the deterministic repo rate, including the dividend yield: i.e., in our case, q is a vector of all the foreign interest rates

$$q = [r_{\text{dollar}}, r_{\text{yuan}}]$$

Setting $\Delta = \partial P/\partial S$ to cancel the first-order term in δS and expanding the P&L to second order

powers of δS and δt, we get the following standard expression:

$$\text{P\&L} = -\left(\frac{\delta p}{\delta t} - r_{\text{domestic}} P + (r_{\text{domestic}} - q) S \frac{\delta p}{\delta S}\right)\delta t - \frac{1}{2} S^2 \frac{dP}{dS^2} \left(\frac{dS}{S}\right)^2$$

Assuming that the historical volatility is fitted by the volatility of the lognormal distribution corresponding to the asset's returns, we obtain the following simplification, putting in evidence the difference between the historical volatility and the volatility implied by the price: the difference is the term is round brackets. We also need to use the risk-management argument stating that the portfolio should not make any profit or loss (fully hedged)[9].

$$\text{P\&L} = -\,\frac{1}{2} S^2 \frac{dP}{dS^2} \left(\frac{dS^2}{S^2} - \sigma^2 \delta t\right)$$

However, since underlings do not follow lognormal processes, and the hedging process occurs (at maximum) daily, P&Ls are not negligible. They do not vanish even in minimal considered time intervals. Let us switch to a multi-asset (Basket) option. We assume that all the assets are delta-hedged in the same way, and, by defining C_{ij} as the specific element of the covariance matrix.

We define

$$\Phi(i, j) = S(i)S(j)\frac{dP}{dS(i)dS(j)}$$

As dollar gamma (the exposure for one dollar) for i=j and dollar cross-gamma (the exposure for one dollar) for i≠j and C is a positive definite matrix linked to the covariance matrix and represents the

9 Even if we are not put in a Black-Scholes world, for the previously stated problem of summing the lognormals distributions, we must remember that the sum of the above P&Ls vanishes as $\delta t \to 0$.

implied covariance. We do obtain by consequence the following expression, which puts in evidence the implied covariance spread[10]:

$$\text{P\&L}= -\frac{1}{2}\sum_{i,j}\Phi(i,j)\left(\frac{\delta S(i)}{S(j)}\frac{\delta S(j)}{S(j)} - C_{ij}\delta t\right)$$

As a result, we can split the case where i=j (gamma risk) and the case where i≠j (cross gamma risk). So, we can separate

P&L = Gamma P&L+ Cross-Gamma P&L =

= P&L attributable to a single currency + P&L coming from the correlation effect

These concerned the analytical part, but the correlation could be modelled slightly differently in simulation and computation finance. Let us start with the analytical results. The first noise process is a traditional Brownian motion model whose index correlated Gaussian random draws drive the portfolio price process. The Brownian motion benchmark is compared to noise processes driven by Gaussian and Student's t copulas, collectively called Brownian Copula.

A copula is a multivariate cumulative distribution function (CDF) with uniformly distributed margins[11]. These Copula draws produce dependent random variables, subsequently transformed into individual variables (margins). Although the theoretical foundations were established decades ago, copulas have experienced a tremendous surge in popularity over the last few years. They are primarily used as a technique for modelling non-Gaussian portfolio risks. Even if numerous families exist, all copulas represent a statistical device for modelling the dependence structure between random variables. In addition, essential statistics, such as rank correlation and tail dependence, are properties of a given copula and are unchanged by monotonic transforms of their margins.

The risk-neutral market model to simulate is, with C(t) being the vector of exchange rates at time t, dt the increment simulated in the model (not necessarily coinciding with the early exercise of american options of an American contract or the value revelation for an Asian contract), while σ is the variance/covariance matrix of the market model

$$dC(t)=rCtdt+\sigma C(t)dWt \quad \text{for t being in } 1,2\ldots N=252.$$

Where n is the number of time intervals of the discrete simulations

where the interest rate, r (the above-mentioned diagonal matrix, given by the risk-free rates spread) is assumed constant over the option's life.

In contrast, the specification of the exposure matrix, sigma, depends on how the driving source of uncertainty is modelled. We can model it directly as a Brownian motion, correlated Gaussian random

---

10 By the implied volatility spread

11 Copulas can be defined informally as follows: Let X and Y be continuous random variables with distribution functions

$$F(x) = P(X \leq x) \text{ and } G(y) = P(Y \leq y),$$

And joint distribution function

$$H(x, y) = P(X \leq x, Y \leq y).$$

For every (x, y) in [-∞, ∞] consider the point in I3 (I = [0, 1]) with coordinates (F(x), G(y), H (x, y)). This mapping from I2 to I is a copula. Copulas are also known as dependence functions or uniform representations.
Important is the Sklar's theorem: Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then there exists a copula C such that H (x, y) = C(F(x), G(y)). Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with marginals F and G. Furthermore, if F and G are continuous, C is unique. It is easy to show that, because of the 2-increasing property, for any copula C we have that C is nondecreasing in each variable, and that C satisfies the following Lipschitz condition: for every a, b, c, d in I,

$$|C(b, d) - C(a, c)| \leq |b - a| + |d - c|.$$

Consequently, copulas are uniformly continuous.

numbers implicitly mapped to Gaussian margins, or as a Brownian copula, correlated Gaussian or t random number explicitly mapped to semi-parametric margins.

As the CDF and inverse CDF of univariate distributions are monotonic transforms, a copula conveniently simulates dependent random variables. The previous statement holds even when margins are dissimilar and arbitrarily distributed. In addition, since a copula defines a given dependence structure regardless of its margins, copula parameter calibration is typically easier than estimating the joint distribution function.

Once we have simulated sample paths, American options are priced by the least square's regression method of Longstaff & Schwartz[12]since we are interested in American options. This approach uses least squares to estimate the expected payoff of an option if it is not immediately exercised. It does so by regressing the discounted option cash flows received on the current price of the underlying associated with all in-the-money sample paths. All cash flows and prices in the regression are normalised by the option strike price, improving numerical stability. A simple third-order polynomial estimates the continuation value function in function of the spot measured price S.

$$Continuation\ value \cong a + bS + cS^2 + dS^3$$

Let us consider the study's outcomes of the correlation between the two variables of interest, the EUR-CHN and EUR-USD exchange rate and their correlations. The "Spearman's Rho" is the traditional correlation, giving a negative (anticorrelation) value of -0.15. However, this result is biased toward zero: switching to a gaussian copula, so computing correlations independently from the marginal distributions gives an outcome of -0.17, so the bias not using copulas is of a relevant 33%. However, once we move to the T-copula correlation, results change sharply in terms of the distribution of the extreme scenario. When a T-copula estimation provides a low (<20) DoF multivariate distribution (4.8 in this case), it means that a significant undervaluation of the correlation happens if the Gaussian Copula or the correlation matrix are used. By contrast, when the degree of freedom is high enough, the "Option" of the Gaussian assumptions expires in the money. Although Gaussian [13]copulas have convenient computational properties, the market-implied joint distribution is often skewed or fat-tailed at another.

Table 2 - Copula fitting parameters

Kind of copula	Gaussian	T
Correlation	-0.17	-0.198
Degrees of freedom	/	4.8

12 see Valuing American Options by Simulation: A Simple Least-Squares Approach, The Review of Financial Studies, Spring 2001. We use a third-degree polynomial approximation

13 Recall that when the marginal CDFs are continuous, we have that

C(u) = F (F1 ← (u1), . . ., F1 ← (ud)).

Now let X ~ MultivariateNormal (0, P), where P is the correlation matrix of X. Then the corresponding Gaussian copula is defined as

C Gauss P (u): = Φ (P (Φ −1 (u1), . . ., Φ −1 (ud)))

where Φ (·) is the standard univariate normal CDF and ΦP (·) denotes the joint CDF of X. We can therefore conclude that a Gaussian copula is fully specified by a correlation matrix, P. For d = 2, we obtain the counter monotonic, independence and comonotonic copulas in when ρ = −1, 0, and 1, respectively.

Results of the different simulations

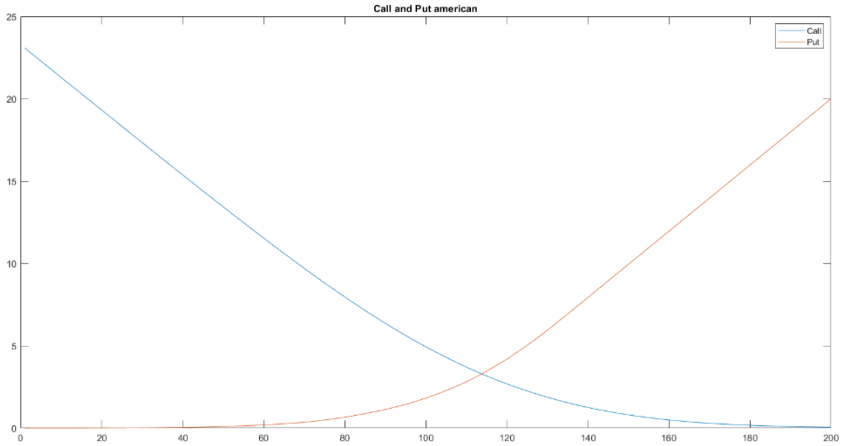
We have computationally solved the pricing and hedging problems through a Montecarlo simulation. The data sources for the simulation are investing.com, both the exchange rates and the money market rates. We used MATLAB with proprietary pre-installed default packages for copula fitting and Bermudan Longstaff-Schwarz approximation[14].

- Domestic money market rate: 1Y EURIBOR
- Foreign money market rate, USD: 1Y LIBOR
- Foreign money market rate, CNH: 1Y SHIBOR

The first analysis of our basket american options shows that the intersection point between the call and the Put American option is skewed to the right concerning the ATM option. The ATM forward option is significantly higher than the actual basket level (200). The result is consistent with the IR differential shown and implies a revaluation of the basket concerning the Euro. Please consider that in all the graphs that will be reported, the x-axis shows the strike price or, better said, a proportional indicator.

[14] The Least Squares Monte Carlo method for pricing American-style options that Longstaff & Schwartz proposed in 2001 is an algorithm that uses backward induction for the estimation of the option price, while the optimal exercise strategy is obtained via the combination of the Monte Carlo simulation with the least square's regression. Its main purpose is to provide a pathwise approximation of the optimal stopping rule that maximizes the value of the American option. The desired number of sample paths is simulated in the beginning. Every option is considered as a Bermudan-style one in this approach. In other words, the time steps are assumed to be discrete allowing early exercise. In practice, however, there are a lot of options that are continuously exercisable, but that is a case that can be covered by the LSM algorithm taking the number of exercise dates to be sufficiently large. At each time step, a least squares regression takes place according to the no-arbitrage valuation theory. That is, the discounted optimal payoffs from continuation of the next time step are regressed on the desired set of the basis functions chosen towards underlying asset prices. The key here is that Longstaff and Schwartz allow in the regression only the paths of the underlying asset price that are in-the-money and that the values obtained are used only for comparison. In other words, paths that do not generate any cash flows are neglected from the regression. At the final date, the holder of the option exercises it if it is in-the-money or allows it to expire if it's out-of-the-money. At every other time step, a comparison is done between the fitted value from the regression and the expected payoff from immediate exercise. If the fitted value from the regression is larger than the payoff from immediate exercise, then the optimal strategy is to hold the option alive from at least one more time step. In the opposite case, even if the two values are equal, the holder of the option should immediately exercise it to benefit from the instant payoff. After completing all the steps described above, the lattice of the optimal exercise strategy and that of the optimal cash flows at each time step will have been constructed. Each optimal payoff is, then, discounted back to time zero and, finally, the average of all payoffs produces the value of the option.

Figure 4- Call and Put american option. Prices refer to different strikes that are increasing



Concerning the early exercise premium, we observe that the early exercise premium for the call is not particularly relevant. The premium is 0,2 at the top ad is valuable only for deeply the money american options. By contrast, the early exercise premium increases sharply with moneyness and reaches 20% of the option's value.

The baseline scenario is estimated with a T-copula. We want to consider what happens when we decide to use a Gaussian copula: the other said, which is the error associated with underestimating the fat tails' distributions? Results are significant and must be scrutinised. We define the "Gaussian Bias" as a T-copula[15] option vs a Gaussian copula option. Not surprisingly, using a Gaussian copula leads to a crucial relative devaluation of the option. Using the gaussian Copula, we risk "underestimating" the value and the cost of this insurance. However, this bias becomes relevant only sufficiently close to the moneyness for a put option.

15 Recall that X = (X1, . . ., Xd) has a multivariate t distribution with ν degrees of freedom (d.o.f.) if

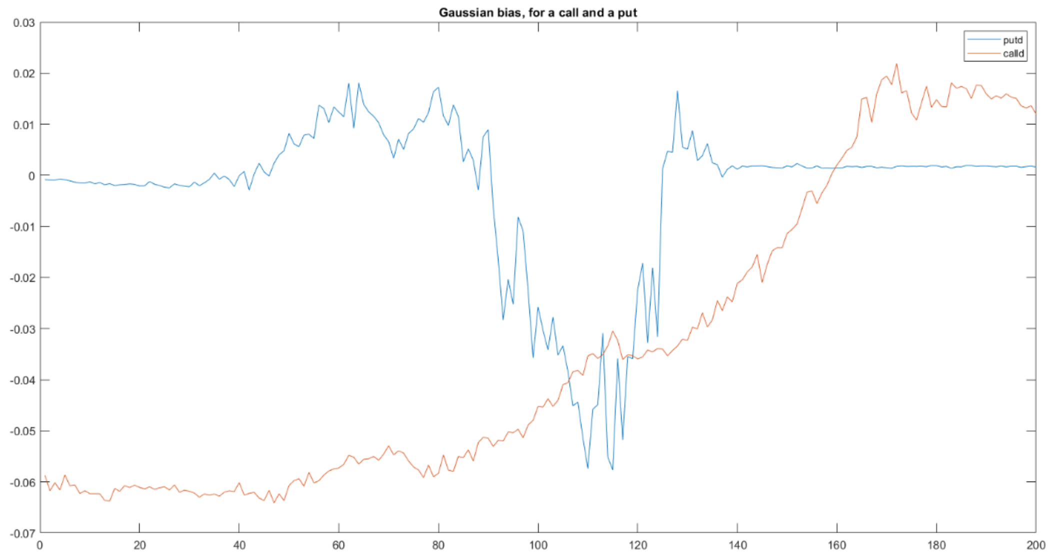
$$X = \frac{Z}{\sqrt{\xi/\nu}} \text{ where } Z \sim \text{Multivariate normal } (0, \Sigma) \text{ and } \xi \sim \chi 2\ \nu \text{ independently of Z.}$$

The d-dimensional t-copula is then defined as C t ν, P (u): = tν, P (t −1 ν (u1), . . ., t−1 ν (ud))
where again P is a correlation matrix, tν, P is the joint CDF of X ∼ td (ν, 0, P) and tν is the standard univariate CDF of a t-distribution with ν d.o.f.
According to Embrechts et al (2001), fitting the t copula is like fitting the Gaussian copula since both the multivariate t and multinormal distributions are elliptical distributions. (As noted in Lindskog (2000), the t copula converges to the Gaussian copula as ν increases to infinity.) Accordingly, τ (− 1≤τ ≤1) is estimated for each pair of random variables, and r (− 1≤ r ≤1) is computed as

$$r = \sin (\pi \sin \frac{t}{2}),$$

which is used to construct the dispersion matrix of the t copula. Again, the dispersion matrix must be positive definite and symmetric.

Figure 5 - Gaussian Bias wrt to a T-student copula. It is the difference between Gaussian and non-gaussian copula pricing. Prices refer to different strikes which are increasing.



By contrast, the bias is more relevant in the case of a Call option: the lower the moneyness, the higher the bias. For deeply in the money call american options, the bias is negative. At the same time, it becomes even positive for sincerely OTM call american options. After, we are interested in comparing the baseline scenario with the T-copula and the GBM modelling, the standard multimarket model. In this case, the approximation cost is significantly higher. It is the difference between the T-copula pricing and the geometric Brownian motion one. In the case of a call option, there is a significant under-pricing in the BM case, which is 5% on average, with a constant bias concerning the moneyness, measuring the bias in relative terms. By contrast, the BM model has a slight overpricing for american options close to the money, which is still at 5%. In terms of implied volatility consideration, it is interesting to remark that this overpricing of BM happens only near the money, for strikes going from 90 to 120.

Figure 6 - the difference in price prices for a call option under the standard multimarket model and the more reliable T-copula pricing scheme.

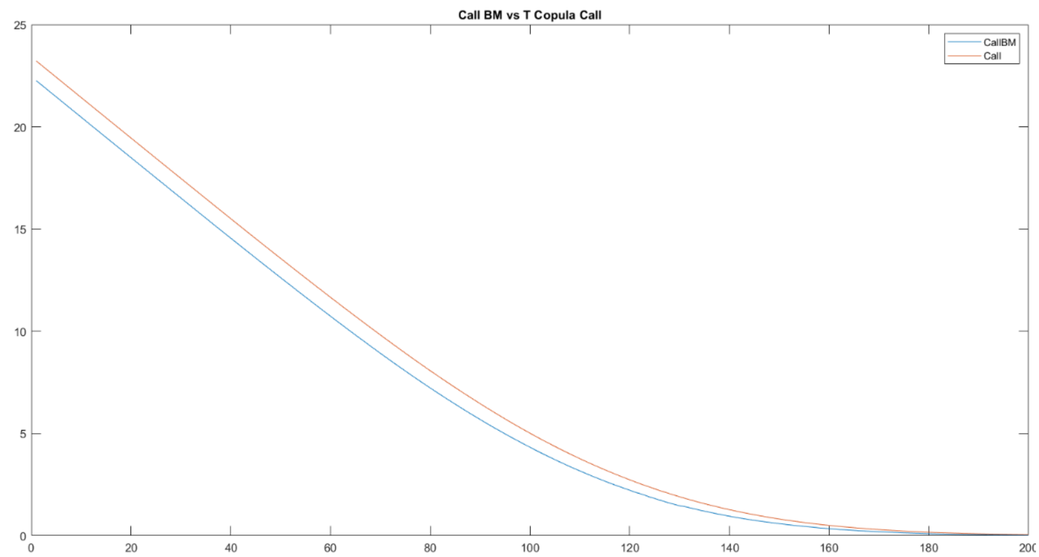


Figure 7 - shows the price difference for a put option under the standard multimarket model and the more reliable T-copula pricing scheme.

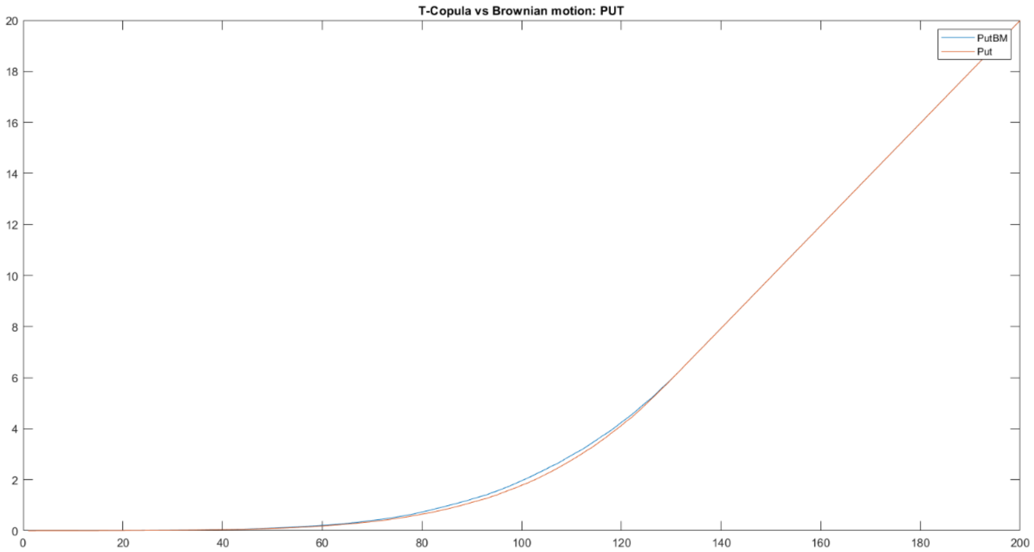
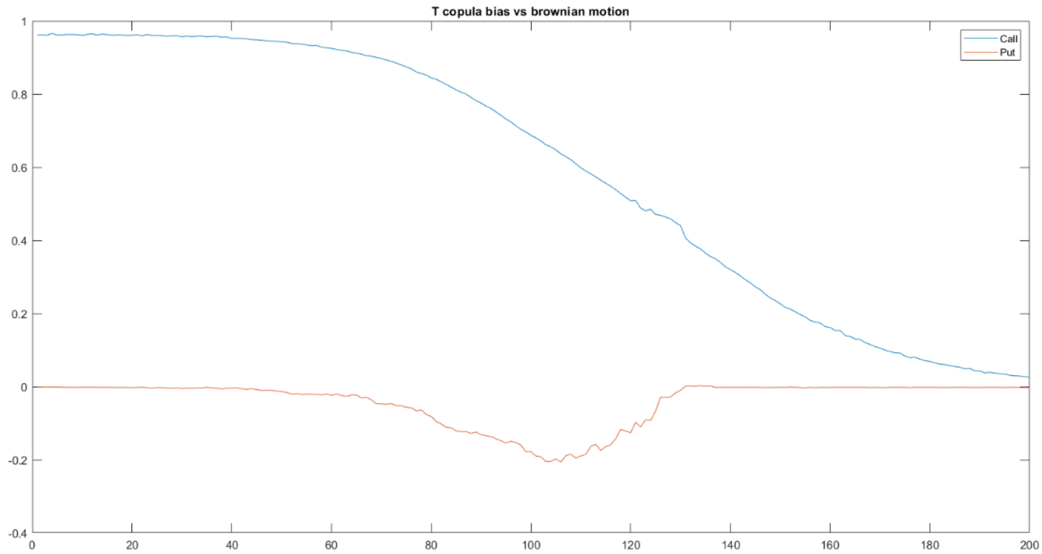


Figure 8 - The quantification of the bias experienced by the standard multimarket model, note the decreasing pattern for the call option and the moneyness pattern for the put option.



Concerning the interest rate sensitivity, we depicted some upward and downward shocks of ±100 bps and ±50 bps concerning the baseline scenario. In the case of a put, decreasing the interest rate increases the option value, with a constant relative change concerning the moneyes. In the case of a call option, the opposite relationship holds, enhanced in the intensity.

Figure 9 - Sensitivity of call option to different shocks in domestic interest rate. Different increasing strike levels.

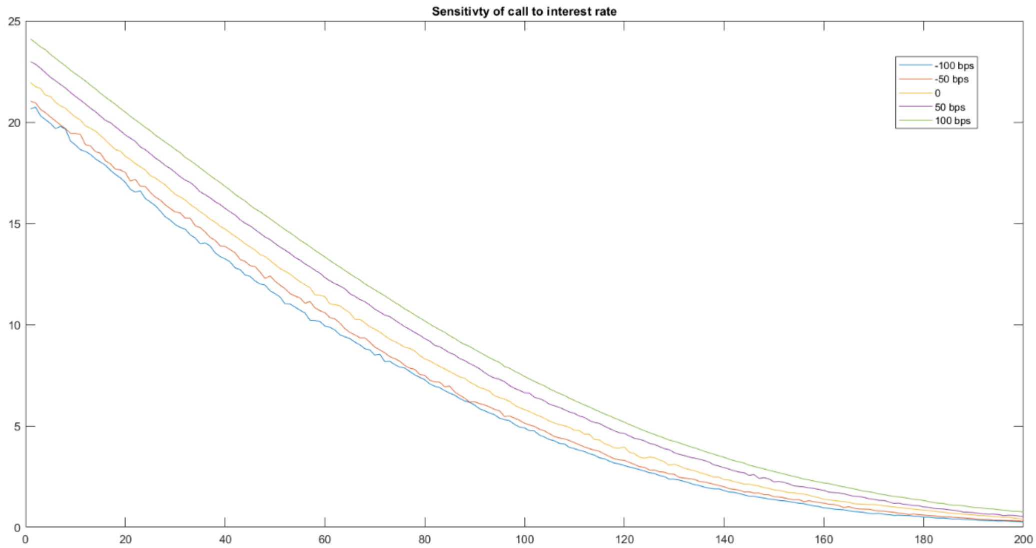
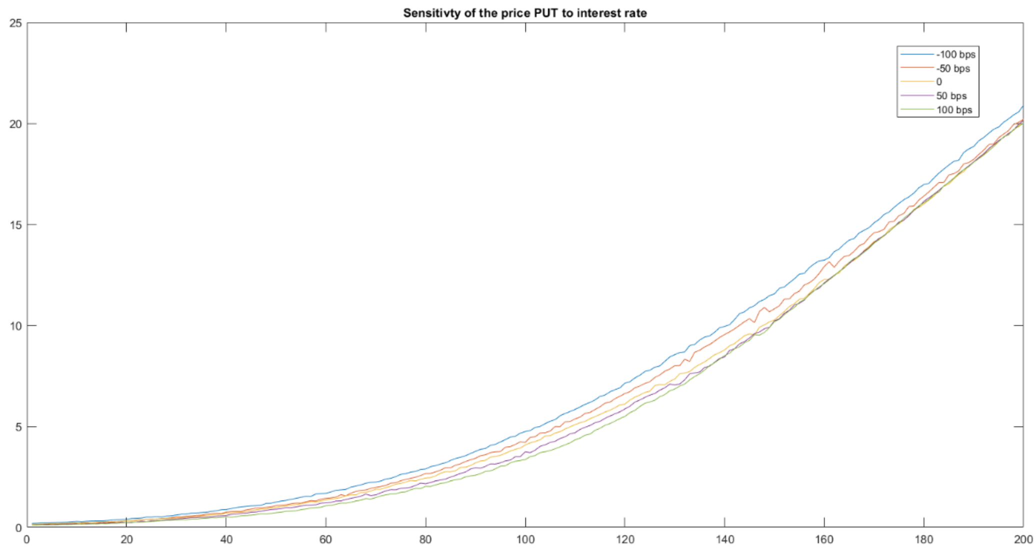


Figure 10 - Sensitivity of call option to different shocks in domestic interest rate. Different increasing strike levels.



We then consider some hedging parameters recovered through the finite difference method. In a basket option, there are two deltas, the partial derivatives concerning each of the underlying. Delta might be in an additive (Interest rate products) or multiplicative (Equity, currency) way:

$$\text{Moltiplicative delta up: } \frac{P(S \times (1+\varepsilon)) - P(S)}{\varepsilon S}$$

$$\text{Additive delta up: } \frac{P(S+\varepsilon) - P(S)}{\varepsilon}$$

$$\text{Moltiplicative delta down: } \frac{P(S) - P(S \times (1-\varepsilon))}{\varepsilon S}$$

$$\text{Additive delta down: } \frac{P(S) - P(S-\varepsilon)}{\varepsilon}$$

The resulting delta can be different for products with large gamma depending on this choice. It will therefore imply different P&Ls and P&L variances. The Up Delta will induce over hedging; the Down Delta will induce under hedging. The Average Delta is the closest to the theoretical value.

Moreover, the average delta:

$$\text{Additive average: } \frac{P(S+\varepsilon)-P(S-\varepsilon)}{2\varepsilon}$$

$$\text{Multiplicative average: } \frac{P(S\times(1+\varepsilon))-P(S\times(1-\varepsilon))}{2\varepsilon S}$$

We believe, in our case, multiplicative expression of the delta is more suitable.

In this case, we have ΔDollar and ΔYuan, as,

$$\Delta\text{Dollar} = \frac{\Delta\text{Option(Dollar,Yuan)}}{\Delta S_{Dollar}}$$

$$\Delta\text{Yuan} = \frac{\Delta\text{Option(Dollar,Yuan)}}{\Delta S_{Yuan}}.$$

However, the results of the two deltas are particularly close, even considering that the two assets in the baskets are equally weighted. The delta of the call option is close to the traditional results, with a higher level for the American option. The same results apply to the delt of the put. However, the dyscrasia due to the early exercise premium is significantly more relevant. From slightly in the money option, the delta starts to be profoundly higher than the delta of the European contract with the steepest decrease. When we plot the absolute value of the delta of a call and the corresponding put, we obtain a χ that is steepest and skewed for the american options, differently from the European one. We present the delta of the first currency S_1 in our case, the dollar.

Figure 11 - Delta of the call option, recovered by finite difference method. The american delta dominates the European delta, with a relevant difference for ITM contracts.

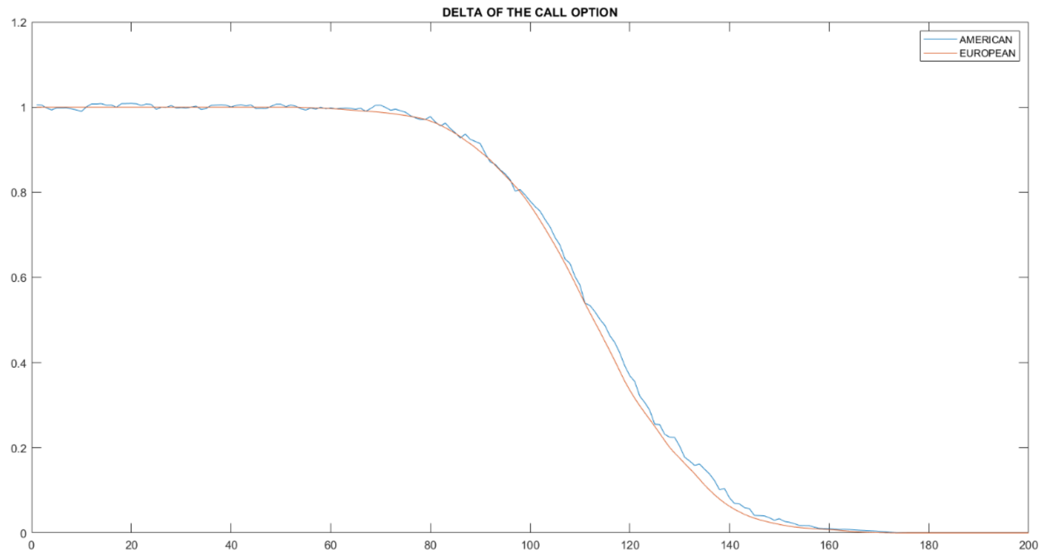


Figure 12 - Delta of the put option, recovered by finite difference method. The american delta dominates the European delta, with a relevant difference for ITM contracts.

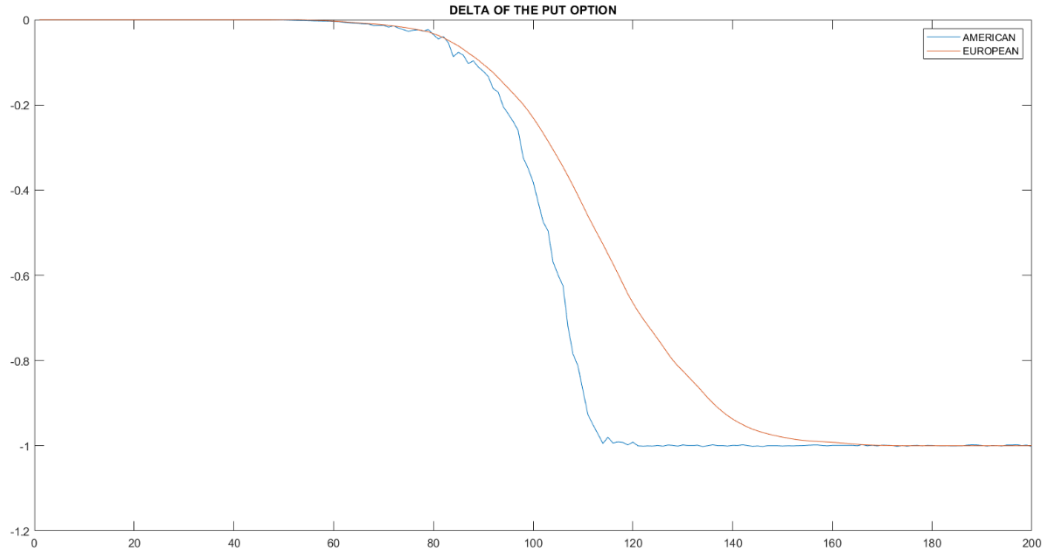


Figure 13 - Delta of the call and put american options, recovered by finite difference method. The intersection is the ATM forward contract. Note that, on the ATM forward option, the delta is not 0.5. We should also consider the delta of the early exercise premium that explains the difference.

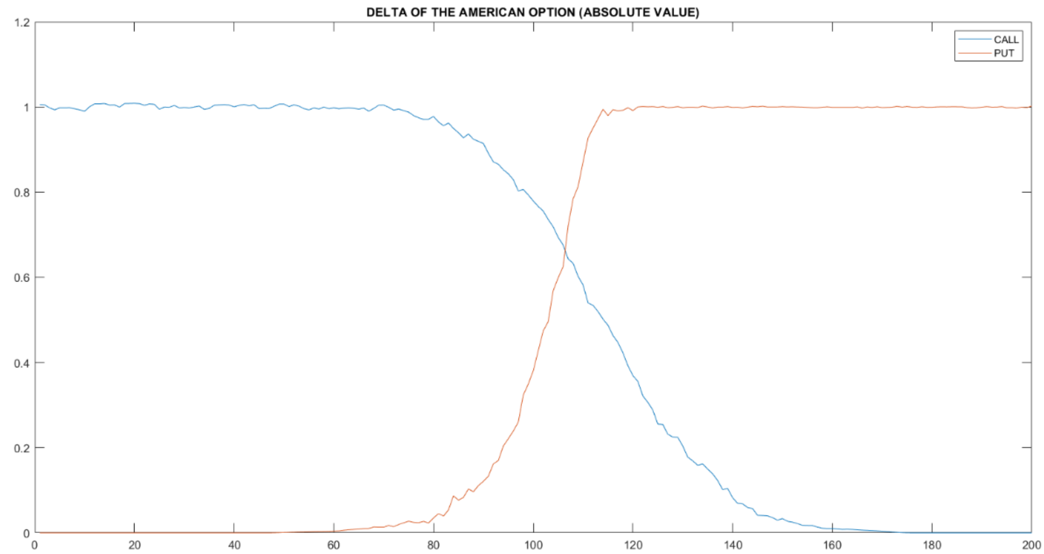
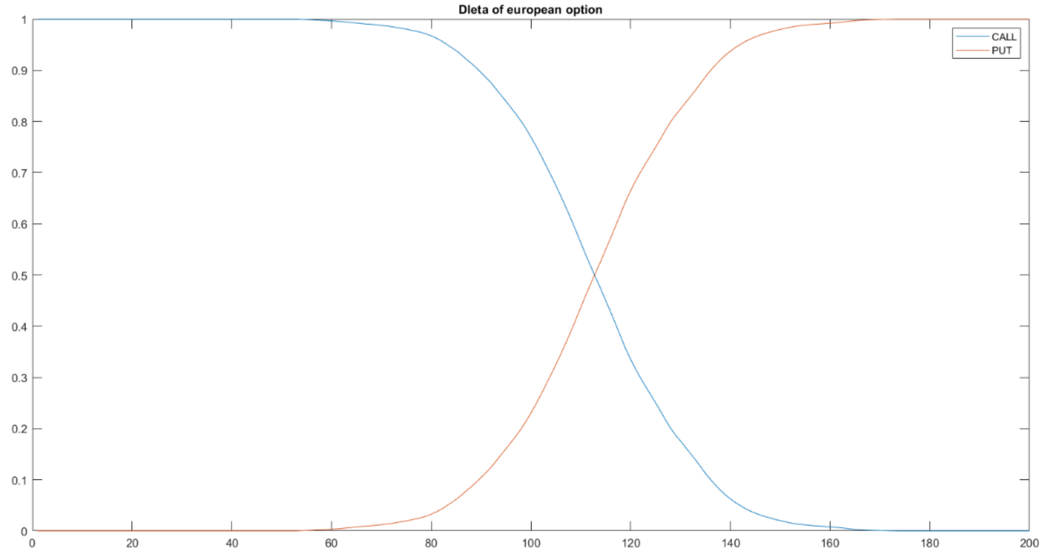
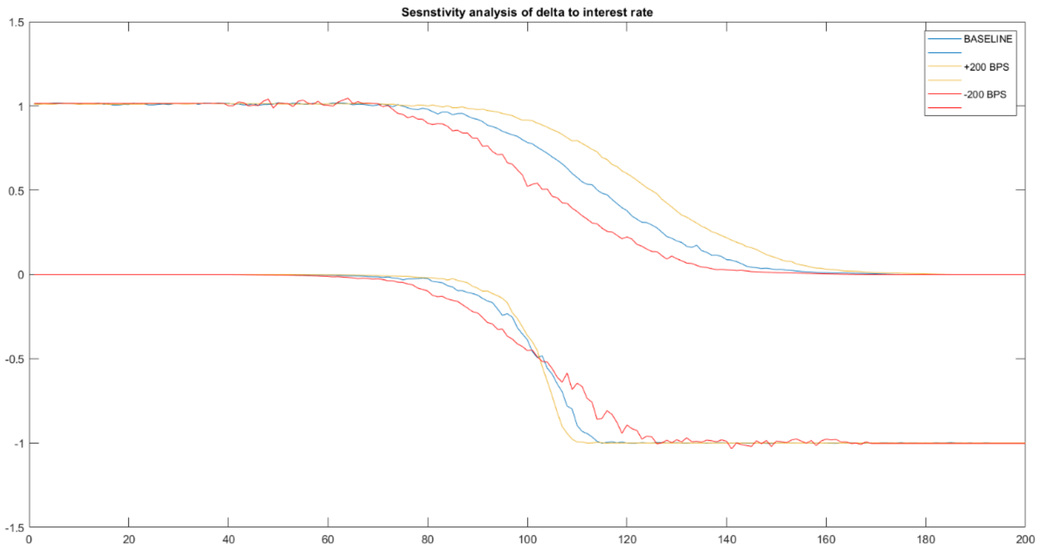


Figure 14 - Delta of call and put European option. Note how the patterns are more smoothed.



Concerning the sensitivity of the delta of american options to the domestic interest rate r_d.We produced a scenario of ±200 bps, and the results are significantly interesting. The following graph first reports the call option and then the put option. More interesting is the put option, where an increase in the interest rate decreases, in absolute value, the delta before the moneyness and the increases after the moneyness. The increase in interest rate increases the option delta value in the call option. In contrast, a decrease in interest rate increases the value of the delta. By contrast, a decrease in the interest rate decreases the value of the hedging parameter before the moneyness and increases it after the moneyness. There is, practically speaking, reflexivity of the pattern concerning moneyness that is missing in the call option.

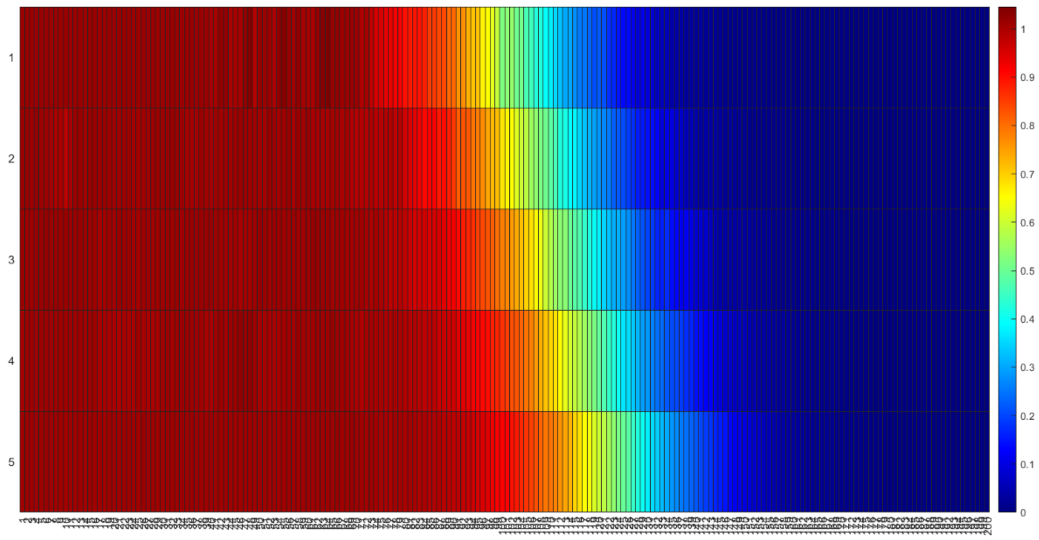
Figure 15 - Sensitivity to the interest rate of the first order hedging parameters, both call and put options. The IR shift is 200 bps upward and downward. Different increasing strikes.



The colourmap can also measure the sensitivity of the delta to the interest rate, which shows from top to bottom increasing interest rates (-100 bps, -50 bps, baseline, 50 bps, 100 bps): for the call option, we see the rainbow moving to the right, meaning that there is an increase on the average delta, while for the put american options the rainbow significantly decreases the space needed to complete the excursion of the whole colourmap: it means that the decrease of the delta is becoming steepest as interest rates increase.

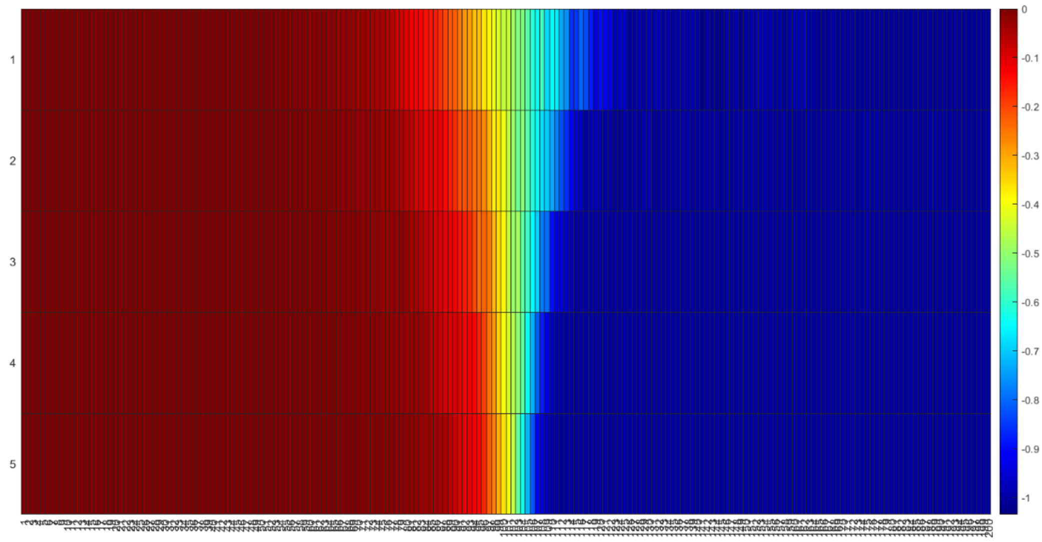
The yellow-green part, moving, identifies the ATM forward contract for different domestic interest rates, where the delta is 0.5. We note that the delta pattern moves to the right as the domestic interest rate increase. The higher the interest rate, the higher the sensitivity to the spot price of the call option. This change in sensitivity concerns the close to the money contracts.

Figure 16 - Colourmap of the call option, descending we increase the interest rate (First row: -100 bps, second row: -50 bps, third row: baseline, fourth row: + 50 bps, fifth row: + 100 bps, increasing strike on the x axis).



We note that the delta pattern becomes more concentrated and steeped as the domestic interest rate increase. The higher the interest rate, the shorter the evolution of the sensitivity to the spot price of the put option. The lower the interest rate, the more diluted the pattern depicted by the delta. Note that we are not considering the absolute value of the delta, but precisely the delta, which is negative or null in this contract.

Figure 17 - Colourmap of the put option, descending we increase the interest rate (First row: -100 bps, second row: -50 bps, third row: baseline, fourth row: + 50 bps, fifth row: + 100 bps, increasing strike on the x axis).



Concerning the cross gamma, as the discrete version of the Schwartz Theorem holds, we have

$$\Gamma\ (\text{Yuan, Dollar}) = \Gamma\ (\text{Dollar, Yuan}) =$$

$$\frac{\Delta\text{Option(Dollar,Yuan)}}{\Delta S_{\text{Dollar}}\Delta S_{\text{Yuan}}} = \frac{\Delta\text{Dollar}}{\Delta S_{\text{Yuan}}} = \frac{\Delta\text{Yuan}}{\Delta S_{\text{Dollar}}}$$

$$\Gamma\ (\text{Yuan, Dollar}) = \Gamma\ (\text{Dollar, Yuan}) = \frac{\Delta(S\times(1+\varepsilon),T)-\Delta(S\times(1-\varepsilon),T)}{2\varepsilon S}$$

$$= \frac{\frac{P(S\times(1+\varepsilon),T\times(1+\varepsilon))-P(S\times(1-\varepsilon),T\times(1+\varepsilon))}{2\varepsilon T} - \frac{P(S\times(1+\varepsilon),T\times(1-\varepsilon))-P(S\times(1-\varepsilon),T\times(1-\varepsilon))}{2\varepsilon T}}{2\varepsilon S}$$

That leads, after some relevant simplifications, since in our case $\varepsilon T = \varepsilon S = 1$

$$\Gamma\ (\text{Yuan, Dollar}) = \Gamma\ (\text{Dollar, Yuan}) =$$

$$\frac{P(S\times(1+\varepsilon),T\times(1+\varepsilon))+P(S\times(1-\varepsilon),T\times(1-\varepsilon))-P(S\times(1+\varepsilon),T\times(1-\varepsilon))-P(S\times(1-\varepsilon),T\times(1+\varepsilon)}{4} =$$

$$\frac{P(S\times(1+\varepsilon),T\times(1+\varepsilon))+P(S\times(1-\varepsilon),T\times(1-\varepsilon))-[P(S\times(1+\varepsilon),T\times(1-\varepsilon))+P(S\times(1-\varepsilon),T\times(1+\varepsilon)]}{4}$$

And, moving to a more immediate notation

$$\Gamma\ (\text{Yuan, Dollar}) = \frac{P(\text{up,up})+P(\text{down,down})-[P(\text{down,up})+P(\text{up,down})]}{4} =$$

$$\frac{\text{the total effect of the variation}-\text{effect attributable to specific shifts}}{4}$$

The results are relevant enough, as using the multiplicative average delta,[16] we obtain a finite difference cross gamma quoted as the difference between two sums: the first sum is the sum of the two variations in the same direction (up, up) and (down, down), while the second sum is the sum of the two crossed components (up, down) and (down, up). The interpretation is immediate enough, as the first sum is the total effect. In contrast, the second is the sum of the effects specifically attributable to one price movement: their difference will represent the interaction effect between the two price movements. The interpretation and analytical computations are not so immediate when considering more underlings: we opted for a couple of baskets of american options. In truth, when passing to n-assets basket american options, we have n-1 cross Gamma, each with a specific meaning. The interpretation of the Cross-Gamma risk might be the norm of this vector of cross gammas. This measures how, in front of a parallel shift of all the other basket assets, the delta of the i-th asset changes.

$$\Gamma(S_j, S_{i\ldots n}) = \|\Gamma(S_j, S_1),\ldots,\Gamma(S_j, S_n)\|$$

---

16 The result fully holds, in the view of the author, even in the case of the additive delta and gamma, if consistency is kept (in the sense that multiplicative gammas are derived by multiplicative delta and additive gammas are derived by additive deltas). In addition, results hold even when instead of the average delta the "positive" or "negative" deltas are considered:

$$\Gamma\text{ADDITIVE (Yuan, Dollar)} = \Gamma\text{ADDITIVE (Dollar, Yuan)} =$$

$$\frac{P(S+\varepsilon,T+\varepsilon)+P(S-\varepsilon,T-\varepsilon)-P(S+\varepsilon,T-\varepsilon)-P(S-\varepsilon,T+\varepsilon)}{4ST} = \frac{P(S+\varepsilon,T+\varepsilon)+P(S-\varepsilon,T-\varepsilon)-[P(S+\varepsilon,T-\varepsilon)+P(S-\varepsilon,T+\varepsilon)]}{4ST} =$$

$$\frac{P(\text{up,up})+P(\text{down,down})-[P(\text{down,up})+P(\text{up,down})]}{4} = \frac{\text{total effect of the variation}-\text{effect attribuable to specific prices}}{4}$$

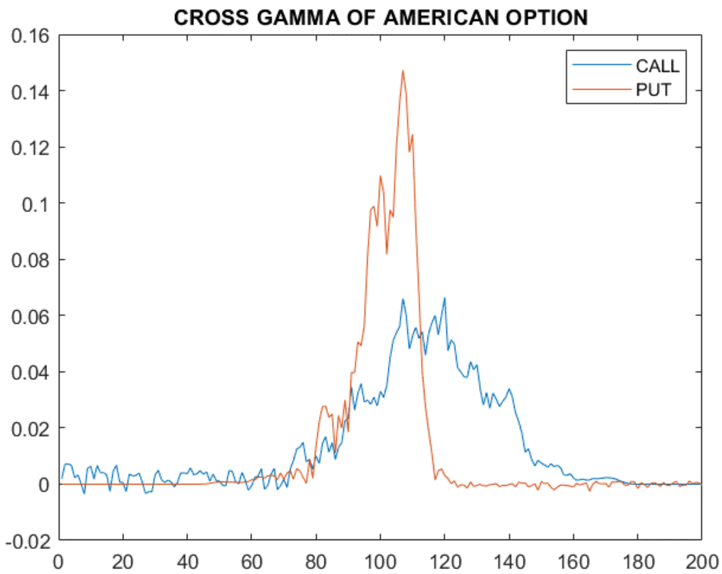
$$\Gamma\text{ increasing (Yuan, Dollar)} = \frac{P(\text{up,up}) + P(0,0) - [P(0,\text{up}) + P(\text{up},0)]}{2}$$

$$\Gamma\text{ decreasing (Yuan, Dollar)} = \frac{P(\text{down,down}) + P(0,0) - [P(0,\text{down}) + P(\text{down},0)]}{2}$$

$$\Delta\left(\Delta_j\right) = \sum_{i=1}^{n-1} \Delta S_i\, \Gamma(S_j, S_i)$$

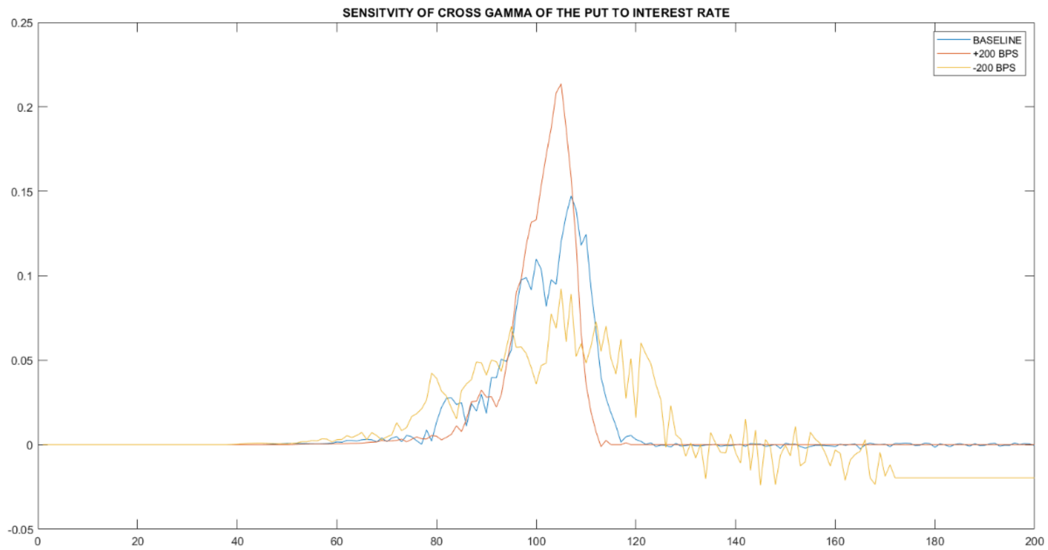
Results show a potential problem in implementing the first order traditional hedging strategy, the delta-hedging, as the relevant parameters are highly variable and close to the money. So, the partial derivatives of the Call Option price concerning the two asset prices, i.e., the derivative of the delta concerning the other asset's price, we first consider the pattern for the different moneyness. With a basket option, a change in the price of the first assets influences the delta of the latter and vice versa. The pattern is analogous between the call and the put for the European option, consistent with other results. While considering the American option, which is our main interest of study, the pattern for the call option is close to one of the European american options. In contrast, the American put is significantly more skewed and higher (3 times more). However, it collapses to zero for a +20% ITM option.

Figure 18 - Cross Gamma of call and put options



Concerning the domestic interest rate sensitivity, we do not find such exciting results for the call option.

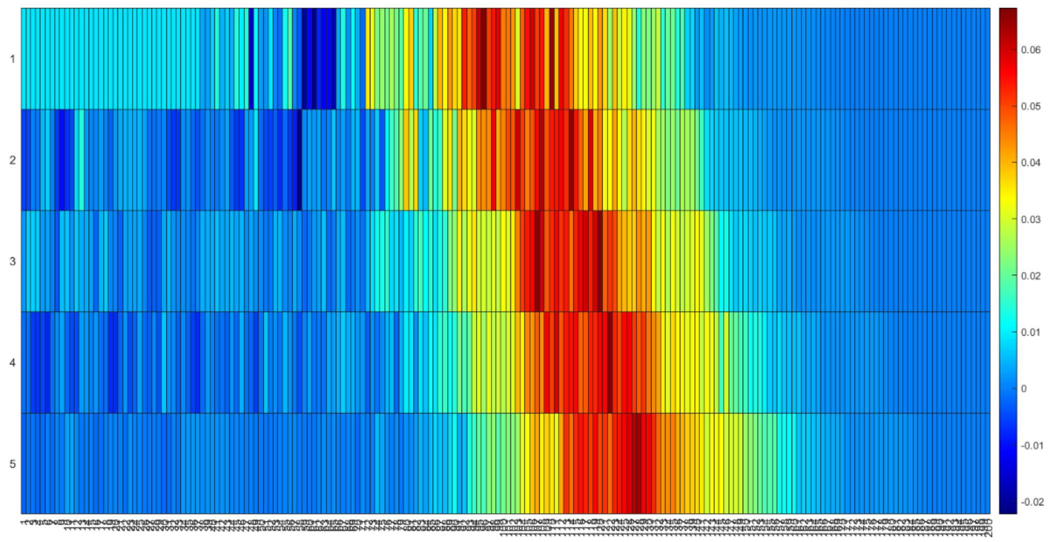
Figure 19 - Sensitivity of cross gamma to different interest rate shocks, 200 bps upward and downward



Considering the colourmap, by contrast, we see that for the call option, the distribution of the cross gamma shifts to the right as the interest rate increase, as there is a switch into the ATM forward.

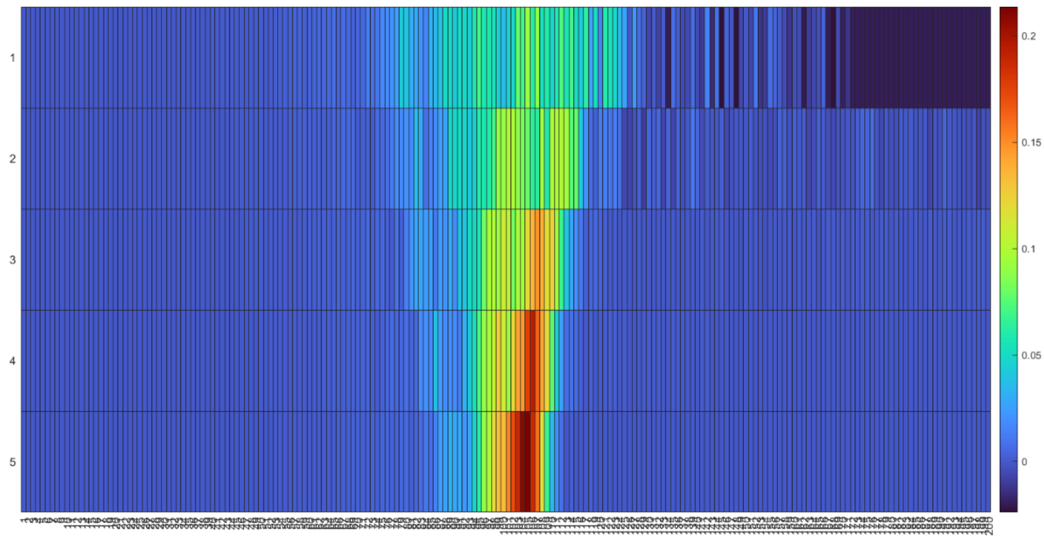
We note that the cross-gamma pattern moves to the right as the domestic interest rate increase. The greater the interest rate, the higher the sensitivity to the spot price of the call option. This change in sensitivity concerns the close to the money contracts. The red part identifies the ATM forward contracts, where the cross-gamma pattern presents a peak.

Figure 20 - Colourmap of the call option, descending we increase the domestic interest rate
(First row: -100 bps, second row: -50 bps, third row: baseline,
fourth row: + 50 bps, fifth row: + 100 bps, increasing strike on the x axis).



We note that the cross-gamma pattern becomes more concentrated and steeped as the domestic interest rate increase. The higher the interest rate, the shorter the evolution of the sensitivity to the spot price of the put option. The lower the interest rate, the more diluted the pattern depicted by the cross-gamma. As the interest rate increases, the value of the cross-gamma increases, doubling from -100 bps to +100 bps wrt to the baseline scenario.

Figure 21 - Colourmap of the put option, descending we increase the domestic interest rate. (First row: -100 bps, second rod: -50 bps, third row: baseline, fourth row: + 50 bps, fifth row: + 100 bps, increasing strike on the x axis).



We are now interested in considering the diverse ways of modelling our correlation problem: Copula (T or Gaussian) vs Geometric Brownian Motion. In the previous point, we have already explored what happens when we return to the standard multimarket model for pricing our instrument(s) since we refer to European contracts. Regarding the hedging parameters, the differences between the T-copula and the gaussian Copula are so light that they cannot be in evidence with a rollover graph for the different moneyness. Robust regression shows how returning to a gaussian word affects the two vanillas delta hedging parameters.

Concerning the vanilla delta, using a gaussian copula slightly decreases the delta of the call option. At the same time, there is no immediate and suitable implication for the delta of the put option[17]. The bias seems more relevant for call american options for lower moneyness (higher strikes). In contrast, for put american options, the lower bias is for higher strikes that, in this case, however, mean higher moneyness. No relevant result concerning cross gamma comes from comparing the T-copula and the gaussian one. The differences are too small, and sign problems disturb the interpretation of regression outcomes. Adequate regression analysis should be implemented for scrutinising this outcome in further studies. We now want to consider the more robust differences between the multimarket model outcomes and the T-copula one. Unlike the precedent situation where we studied the two copulas, the differences are visible even in the traditional strike-rollover graph. Concerning the delta, the delta of the call option is significantly underestimated (10%, for all the moneyness) when pricing with a GBM. At the same time, no significant difference appears in the Put option. Concerning the cross gamma, no significant difference appears, even considering the numerical error, between the two proposed approaches. The only remarks are the lower stability of the hedging parameters (higher standard error), and the lower values for close-to-the money put american options.

Let the author conclude this section by apologising for having omitted some relevant cases and variables in the sensitivity study. In particular, the same problem with gamma should be presented for vega (as we should have the vega of the first asset, the vega of the second asset, and the vega of

17 Regression parameters (Gaussian delta vs original T delta), are, in fact, the following:
Call: α: 0.002 β:0.9926, decrease
Put: α: -0.002 β:0.9948, no immediate interpretation as deltas is negative.

the covariance/correlation). In addition, some sensitivity analysis on theta (the time decay factor) should be conducted while bearing in mind that basket options present a maturity T ≤ 1 year. This is since the correlation pattern is, by definition, volatile and challenging to forecast for longer time horizons.

Figure 22 - Delta of the call option, the difference between standard multimarket model and nongaussian copula price

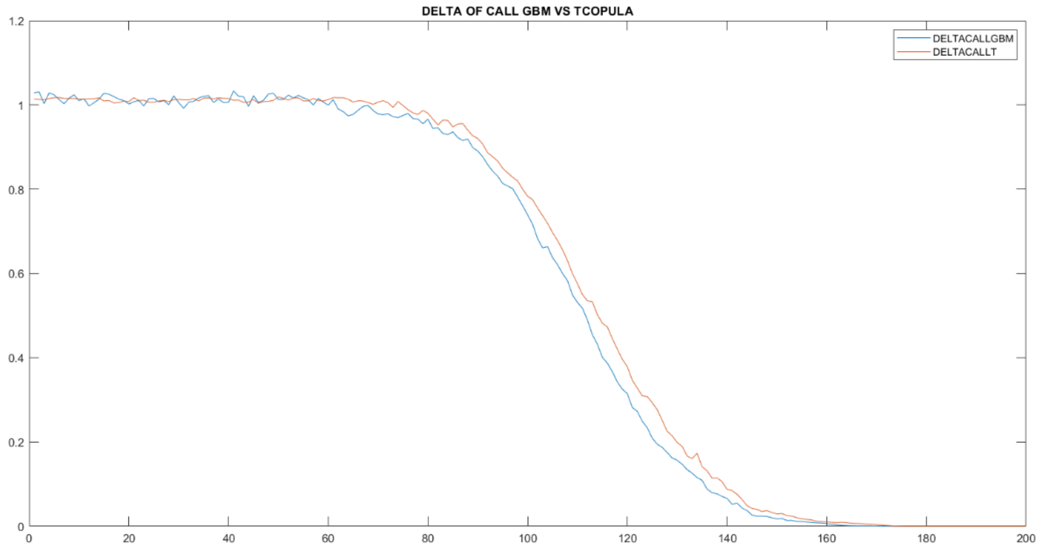


Figure 23 - Cross Gamma of the call option, the difference between standard multimarket model and non-gaussian copula price

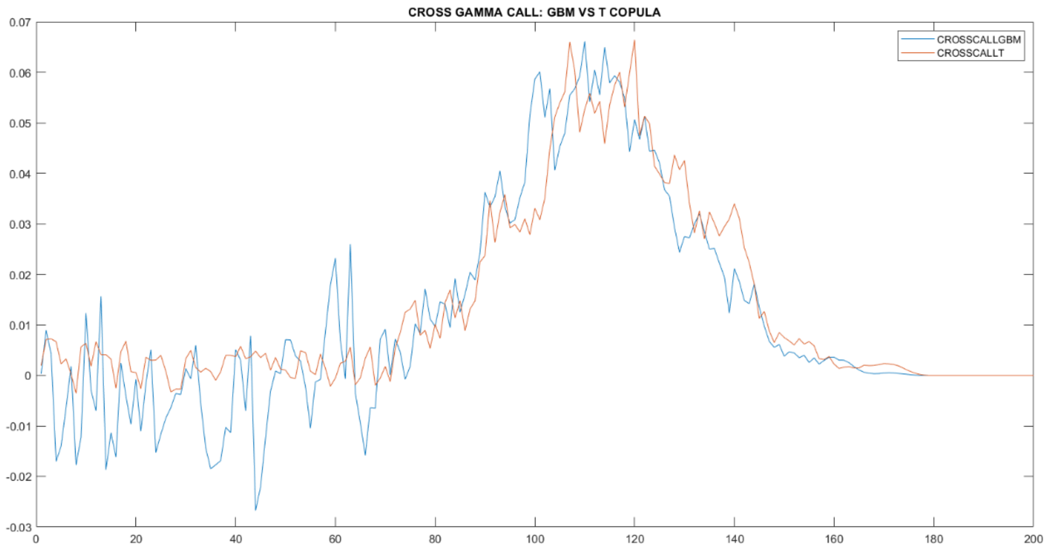


Figure 24 - Delta of the put option, the difference between standard multimarket model and non-gaussian copula price

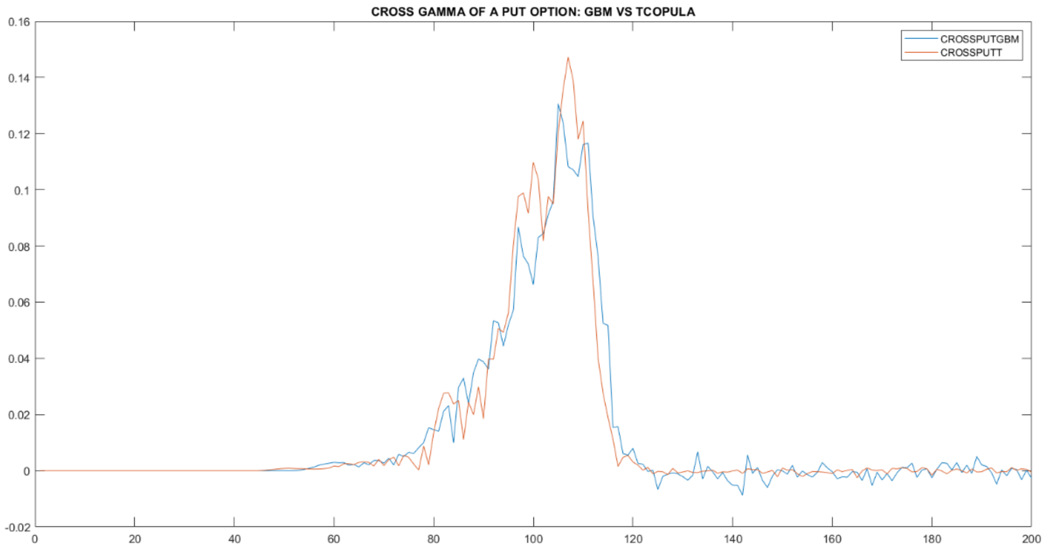
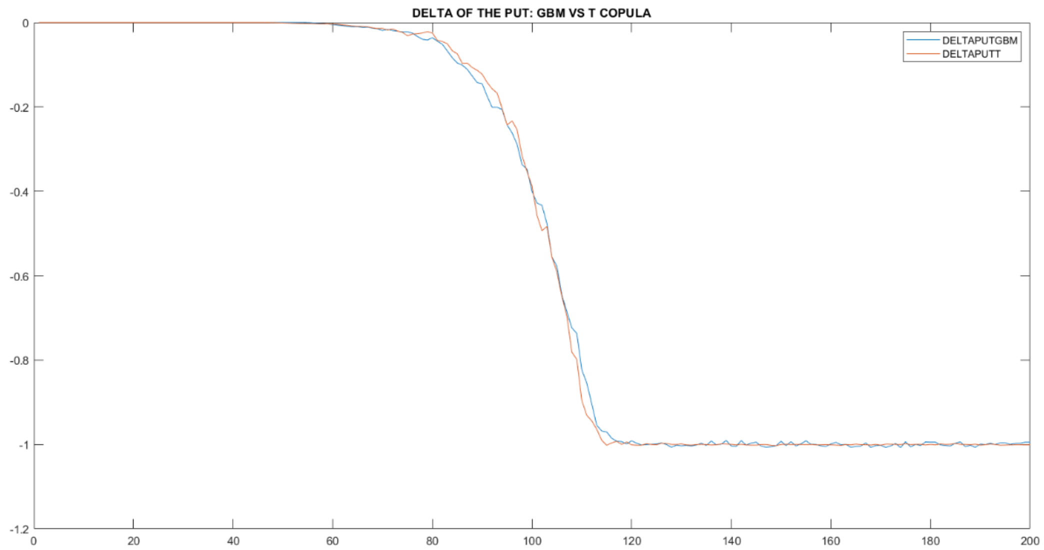


Figure 25 - Delta of the put option, the difference between standard multimarket model and non-gaussian copula price



Conclusion

In this short work, we have shown the critical features of Basket (currency) american options. Furthermore, we have developed enough robust analytical computations for the case of two assets. Thanks to Montecarlo simulations, we have found relevant results in terms of pricing, both in the standard multimarket model and for the different copulas. Concerning hedging parameters, the main result is the finite difference method for recovering the cross gamma, in conjunction with deltas estimations and sensitivities to domestic interest rate under different market models. Further research will concern:

- Extension to three or more currencies
- studies specific currency patterns that show relevant macroeconomic meaning.
- On different stock market indexes
- Basket american options (oil, natural gas, and energy) or interest rate basket american options (different currency denominations and interest rate maturity).

More exotic basket american options are under development: Asian and rainbow american options. Only the best k assets determine the index and basket binary american options. It is effectively possible to bet on the direction of the correlation shift. However, the modus operandi will still be the same: it is relevant to understand the economic intuition associated with the basket entirely.

Basket american options are genuinely going to play a role in hedging the currency risk. However, particular attention should concern delta hedging of the position. Our results show that, close to the money, the methods become unstable as the spot prices mutually affect the first order hedging parameter. Consequently, the i-th delta is elastic to the j-th spot price. A gamma-hedging strategy might be more suitable: this is more complicated to implement. Basket options are, in general, cheaper than the corresponding single currency options couple. However, the absence of free meals, which gains efficiency in pricing, reverses a more complex (but even more complete and concise) hedging strategy for basket options. Basis risks will decrease in currency hedging thanks to these instruments once these instruments are studied and implemented.

We may cut the Gordian knot by an augmented delta-hedging strategy, including the two deltas and the cross gamma: it is a first-order hedging strategy accounting for the mixed term.

Bibliography

- Alexander, C. and Venkatramanan, A. (2012), Analytic approximations for multi-asset option pricing. Mathematical Finance, 22: 667-689. https://doi.org/10.1111/j.1467-9965.2011.00481.x
- Baxter M, Rennie A. (1996): Financial calculus- An introduction to derivative pricing, Cambridge University press
- Chicot, Harold, Measuring Cross-Gamma Risk (Dec 5, 2019). Available at SSRN: https://ssrn.com/abstract=3502251 or http://dx.doi.org/10.2139/ssrn.3502251
- Choi, J. Sum of all Black–Scholes–Merton models: An efficient pricing method for spread, basket, and Asian options. J Futures Markets. 2018; 38: 627– 644. https://doi.org/10.1002/fut.21909
- D'Aspremont Alexandre, Pricing Basket american options with an eye to swaption, ORF557, stochastic analysis seminar, BCF, Sep. 2004. https://www.di.ens.fr/~aspremon/PDF/BCF04a.pdf
- Hamza Hanbali & Daniel Linders (2019) American-type basket option pricing: a simple two-dimensional partial differential equation, Quantitative Finance, 19:10, 1689-1704, DOI:10.1080/14697688.2019.1588987
- Iancu, Aniela Karina. "Numerical methods for pricing basket american options." (2004).
- Letourneau, Pascal and Stentoft, Lars, Simulated Greeks for American American options (Dec 14, 2019).
- Pricing American Basket American options by Monte Carlo Simulation - MATLAB & Simulink
- Rosin J.P. Hedging Basket American options with Deep Learning - Jan 29, 2021 – conference by FIN-ML
- Su, Xia: Essays on Basket Options Hedging and Irreversible Investment Valuation. - Bonn, 2008. - Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn.

```
% With this code we want to price the american basket option trought
% different montecarlo methods, according to the evolution of the
% dependence structure. First of all, we load the data table, that
% consists of a vector of dates, to allow the time-series dailiy
% representation and a maxtrix of foreing currency prices. In the first
% colum vector we have the dollar price, while in the second colum vector
% we have the yuhan price of the euro.
load("dati.mat");
dt       = 1 / 252;  % time increment = 1 day = 1/252 years
% continuously-compounded, annualized yields
% In the previous step we have set the bermudan daily approximation.
% Now we define the vector of foreing rates, i.e. the dollar 1year
% overnight rate and the shibor chinese money market rate. The domestic
% rate is the european one EURIBOR for 1 year.
rforeing        = [0.03 0.05];             % TASSO
rdomestic= 0.01
r= rdomestic*ones(1,2)-rforeing;
% We recover the risk-neutral measure that we still represente, for
% computational efficency troubles, as a vector. We will transform it in a
% a marrix when necessary. The following vector describes the initial
% state vector, where each component of the basket start at pair.
X         = [100*1,100*1]'; % initial state vector
% The at the money strike is 100*number of component of the basket, if the
% assets are all equally weighted. This method is not really scalable for
% not equally wieghter basket, where it is suitable to define the weight
% vectors and to re-normalize the basket to 100.
strike    = 200;                     % initialize an at-the-money basket
% Number of trials of the montecarlo experiment. We prefer to perform a
% significant number of trials, but without any variance reduction
% tecnique.
nTrials   = 10;                        % # of independent trials
% The option maturity is one year
nPeriods = 252;
% Since we do not want to recover only the price of an ATM option, but we
% are interest in a bidirectional shift of 20% in the moneyess we define
% the following vectore to adequate the strike. We use continuos
% capitalization for recovering returns.
moneyess=linspace(0.8,1.2,200);
prices = Data(:,1:end);
returns = tick2ret(prices);        % convert prices to returns

% technical section for graphical representation
dollar = returns(:,1);
yuhan = returns(:,2);
u = ksdensity(dollar,dollar,'function','cdf');
v = ksdensity(yuhan,yuhan,'function','cdf');
U=[u v]
% Now we define the three correlation patterns.
% We start by the standard multimarket model, so we compute the
% correlation coefficient, the we shift to the gaussian and to the
% T-Copula, the only one that presents 2 parameters. ML for gaussian
% copula is more precise, but less consistent with the non-gaussian
% parameters estimation.
C=corrcoef(returns); % this is for standard multimarket model
options     = statset('Display', 'off', 'TolX', 1e-4);
[rhoG] = copulafit('Gaussian', U,'Options', options);
[rhoT,dof] = copulafit('T', U);

% We then reset the pseudorandom generator parameters and we compute the yearly volatility
% matrix.
```

```matlab
s = RandStream.getGlobalStream();
reset(s)
sigma= std(returns) * sqrt(252);
% We simulate the two brownian motions, one with the correlation matrix
% and the second one without any correlation assumption for the couplas
% simulations.
GBM1=gbm(diag(r(ones(1,2))), diag(sigma), 'StartState', X, 'Correlation', C);
%alternatively the GBM suitable for the copulas
GBM2 = gbm(diag(r(ones(1,2))), eye(2), 'StartState', X);
% here we set the LongstaffSchwartz bermudan aproximation consistently
% with the discretization we had choosen (daily).
f = Example_LongstaffSchwartz(nPeriods, nTrials)
reset(s)
% We recover the basket path for standard multimarket model and the price.
% This prices are derived without any copula, but with the standard
% multimarket.
% The prices are quotes as a vector because we thake into account
% different moneyess (different strike prices). Please consider the vector
% is odered wrt to the strike, the same for call and put options. Since
% from now on we are working with domestic currency, when discounting the
% payofss we will consider only domestic rate
simByEuler(GBM1, nPeriods, 'nTrials'  , nTrials, 'DeltaTime', dt, ...
                         'Processes', f.LongstaffSchwartz);
for i=1:200
 bCall(i) = f.CallPrice(strike*moneyess(i), rdomestic);
 bPut(i)  = f.PutPrice (strike*moneyess(i), rdomestic);
end
BOUTCOME=[bCall;bPut]

% the gaussian copula price
reset(s)

z = Example_CopulaRNG(returns * sqrt(252), nPeriods, 'Gaussian')
f = Example_LongstaffSchwartz(nPeriods, nTrials)

simByEuler(GBM2, nPeriods, 'nTrials'  , nTrials, 'DeltaTime', dt, ...
                         'Processes', f.LongstaffSchwartz, 'Z', z);

for i=1:200
 gCall(i) = f.CallPrice(strike*moneyess(i), rdomestic);
 gPut(i)  = f.PutPrice (strike*moneyess(i), rdomestic);
end

GOUTCOME=[gCall;gPut]

% the T copula price
reset(s)
z = Example_CopulaRNG(returns * sqrt(252), nPeriods, 't');
f = Example_LongstaffSchwartz(nPeriods, nTrials);

simByEuler(GBM2, nPeriods, 'nTrials'  , nTrials, 'DeltaTime', dt, ...
                         'Processes', f.LongstaffSchwartz, 'Z', z);

for i=1:200
 tCall(i) = f.CallPrice(strike*moneyess(i), rdomestic);
 tPut(i)  = f.PutPrice (strike*moneyess(i), rdomestic);
end
TOUTCOME=[tCall;tPut]
moneyess=linspace(0.8,1.2,200);

% this is to join the prices of call and put options
CALL=[BOUTCOME(1,:); GOUTCOME(1,:); TOUTCOME(1,:)]
PUT=[BOUTCOME(2,:); GOUTCOME(2,:); TOUTCOME(2,:)]
```

```matlab
% Recovering the first two deltas, trought finite difference methods
% The first row of the vector is the output for the call, the second for the put.
% these convection applies to all the greeks.
% We use here the percentage increment, not the additional increment
% (the so called, multiplicative finite different, not
% the additive one). We make excursion in both upside and downside
% direction, to increase the accuray and avoid systematic bias
% We call the function CODICE (the previously three pages of code)
% The first argument is the increment of dollar price,
% the second argument is the increment of yuhan price
% in this first step we work out the two deltas, wrt to
% dollar and yuhan currencies, as the weighted average of an upward
% and a downward shock.
DELTAFIRSTDOLLAR=0.5*(CODICE(1.01,1)-CODICE(0.99,1))
DELTASECONYUHAN=0.5*(CODICE(1,1.01)-CODICE(1,0.99))
```

```matlab
%Here we compute the cross gamma. We profit from the equality of second
% derivatives: the equality of mixed partials in discrete approximation
% is consistent with Schwarz's theorem in continuos pricing. Still, we
% use the upward and downward shifts to avoid under and over estimations
% of the hedging parameter
%%CROSS GAMMA
CROSSDELTA=0.25*(CODICE(1.01,1.01)+CODICE(0.99,0.99)-CODICE(0.99,1.01)-CODICE(1.01,0.99));
% The road to multilevel montecarlo estimates of delta and gammas.
% First of all, we generate random values in the interval 0.98 to 1.02 as
% multipliers for the two underlying prices. we set the number of
% simulations of the montecarlo simulations. On this stage we do not need
% a really huge number of sensitvity simulations. 100 to 1000 should be
% fairly sufficient. Since the intervall is small enoguh uniform
% distribution could be used, normal distribution is however, welocome.
nmultilevel=100
sensitvities=randi([9800000 10200000],2,nmultilevel)./10000000;
prezzi=[]
for i=1:nmultilevel
    prezzi=[prezzi CODICE(sensitvities(1,i),sensitvities(2,i))];
end
% If you prefer, this is the moment suitable for switching to STATA
% that allows an higher variety of regressions, specifically designed for
% finance (as quantile). It is even the moment to open the curve fitting
% toolbox. We stay in vanilla MATLAB and we create the cross term for estimating
% the cross gamma
sensitvities=[sensitvities; sensitvities(1,:).*sensitivities(2,:)];
% First we estimate augmented delta hedging paramters, i.e. deltas and
% cross-gamma
regress(prezzi,sensitvities)
% Now we go for full and complete gamma hedging
sensitvities=[sensitvities; sensitivities(1,:).^2; sensitivities(2,:).^2];
regress(prezzi,sensitvities)
```