REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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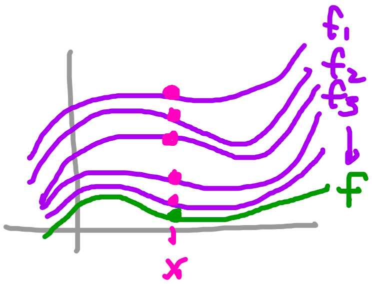
TODAY: SEQUENCES OF FUNCTIONS

SEQUENCES OF FUNCTIONS (preview of Analysis II in Ch. 7)

Q. What does it mean for $f_1(x), f_2(x), f_3(x), \ldots$ to **converge**?



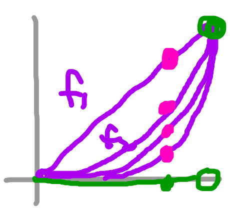
pointwise: fix x. Ask: does $\{f_n(x)\}$ converge?

If so, **ptwise limit** is $\boxed{f(x) = \lim_{n\to\infty} f_n(x)}$

EX. ① $f_n(x) = \dfrac{x}{n}$

$\xrightarrow{\text{ptwise}} f(x) = 0.$

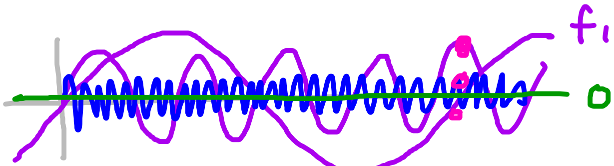
EX ② $f_n(x) = x^n$ on $[0,1]$



$\xrightarrow[\text{ptwise}]{} f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}.$

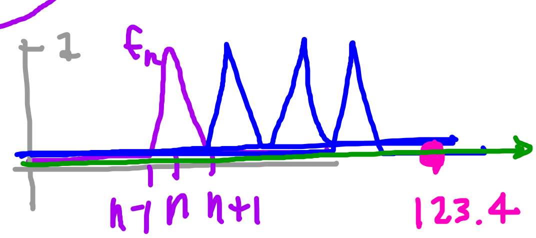
CONTINUITY NOT PRESERVED

EX ③ $f_n(x) = \frac{1}{n}\sin(n^2 x)$ $\xrightarrow{\text{ptwise}} f(x) = 0$



DERIVATIVES NOT PRESERVED

EX ④ $f_n(x) =$



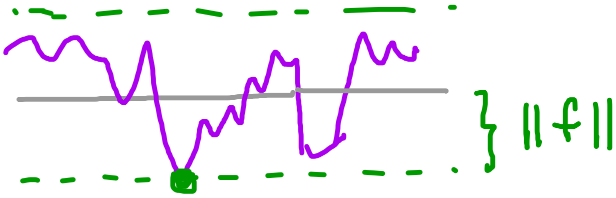
$\xrightarrow{\text{ptwise}} f(x) = 0$

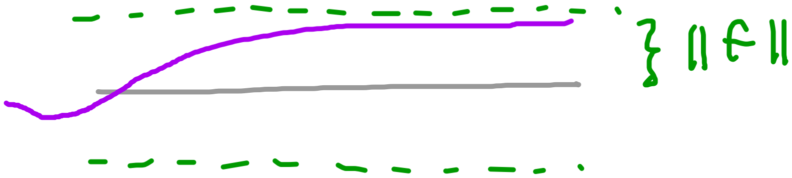
"INTEGRALS" NOT PRESERVED

Need stronger notion.

For bounded $f: E \to \mathbb{R}$, define $\boxed{\|f\| = \sup_{x \in E} |f(x)|}$.

← also works





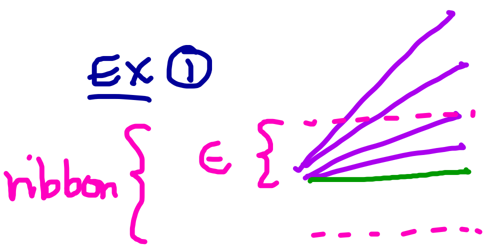
Def'n. Say $f_n \xrightarrow{u} f$ ("f_n converges uniformly to f")

if $\boxed{\forall \varepsilon > 0 \;\; \exists N \text{ s.t. } n \geq N \Rightarrow \|f_n - f\| < \varepsilon}$.

same N works for every x

"ribbon convergence"

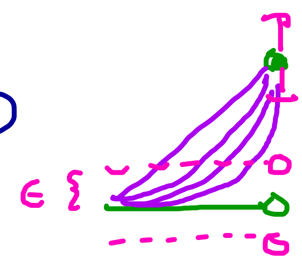
Ex ①



f_n NOT CONV. UNIFORMLY on $\mathbb{R}$

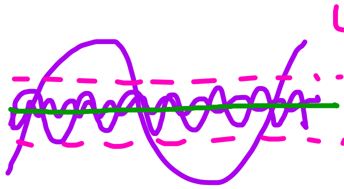
(DOES if domain is bdd)

Ex ②



NOT UNIFORMLY CONV.

Ex ③



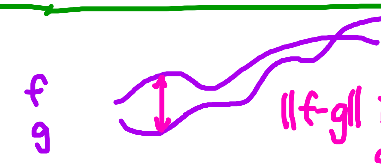
UNIF. CONV.

Ex ④

NOT

- Uniform convergence of functions is convergence in metric space

$$\boxed{\mathcal{C}_b(E) = \underline{\text{contin. bdd. fcns}}\quad E \to \mathbb{R} \quad \text{with metric} \quad d(f,g) = \|f-g\|.}$$



$\|f-g\|$ is "max" difference

Fact (Analysis II). $\mathcal{C}_b(E)$ is complete.

So we have Cauchy criterion!

Thm. f_n converges unif'ly to some f on E

$$\Leftrightarrow \boxed{\forall\, \varepsilon > 0 \ \exists\, N \text{ s.t. } n, m \geq N \text{ and } \forall\, x \in E \Rightarrow |f_n(x) - f_m(x)| < \varepsilon.}$$

same: $\|f_n - f_m\| < \varepsilon$

Completeness follows from:

Thm. $f_n \xrightarrow{u} f$, f_n contin $\Rightarrow$ f contin.

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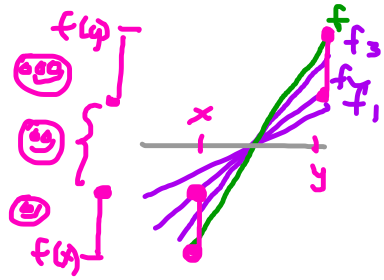


proof. ($\epsilon/3$ - argument)

use bound:

$$|f(x)-f(y)| \le |f(x)-f_n(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)| .$$

Fix x. We'll show contin. at x.

Fix $\epsilon > 0$.

① Choose f_n s.t. $\|f_n - f\| < \epsilon/3$ by unif. convergence so , $< \epsilon/3$.

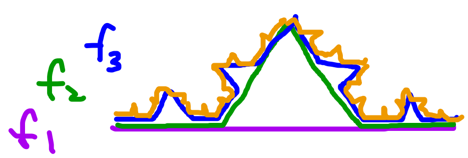
② f_n contin, so $\exists\ \delta > 0$ s.t.

$|x-y| < \delta \Rightarrow$ $< \epsilon/3$.

This is desired δ so $|f(x)-f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

So f is contin. $\square$

- Ex. Koch curve:

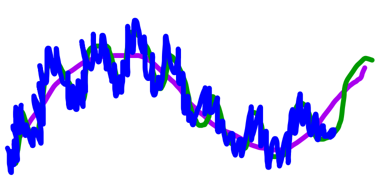


$f_n : [0,1] \to \mathbb{R}^2$ conv. uniformly (show) & continuous, so limit fcn is continuous.

- Ex. (Weierstrass) $\boxed{f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)}$ $0 < b < 1$, a odd integer, $ab > 1 + 3\pi/2$

converges uniformly

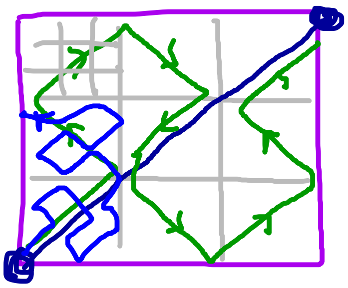
So limit function exists, is continuous everywhere but can show it's diff'ble nowhere!



Ex. Q:  Is there curve whose image covers the entire square?

A: YES!

"Space-filling curve"



$f_1 \quad f_2 \quad f_3 \longrightarrow$

converges to a continuous "space-filling curve"!