REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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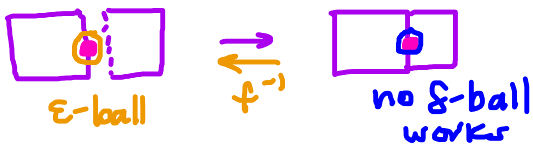
TODAY: UNIFORM CONTINUITY

Topologically the "same": "homeomorphism"

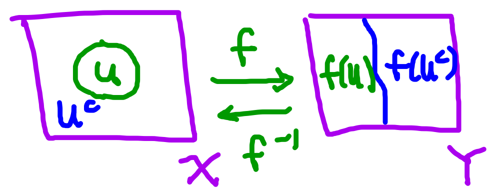
- If $f: X \to Y$ continuous bijection, notice f^{-1} not necessarily continuous.



But

- Thm If $f: X \to Y$ is contin. bijection and X cpt then f^{-1} is continuous.

- proof. Suppose $\mathcal{U}$ open in X.
  [GOAL: show $f(\mathcal{U})$ open.]



So $\mathcal{U}^c$ is closed, hence cpt b/c $\mathcal{U}^c$ is closed subset of cpt space.

Since f contin, $f(\mathcal{U}^c)$ is cpt.

Cpt sets are closed, so $f(\mathcal{U}^c)$ is closed.

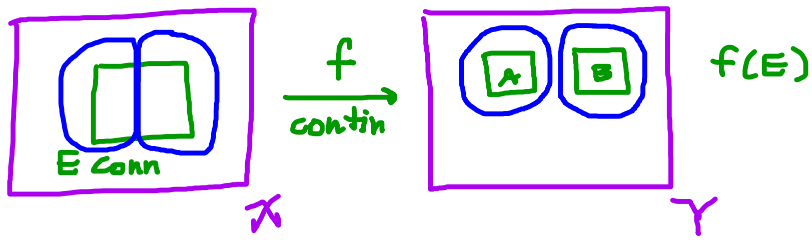
Thus $f(\mathcal{U}) = f(\mathcal{U}^c)^c$ is open.

∎

• CONTINUOUS FUNCTIONS PRESERVE CONNECTEDNESS

Thm. $f: X \to Y$ contin, E conn $\subset X \Rightarrow f(E)$ conn.

proof idea:



If $f(E)$ not conn, then ...

$\exists\ A, B$ s.t. $\bar{A} \cap B = \bar{B} \cap A = \emptyset$ and $A \cup B = f(E)$.

This means A is closed in $A \cup B$.

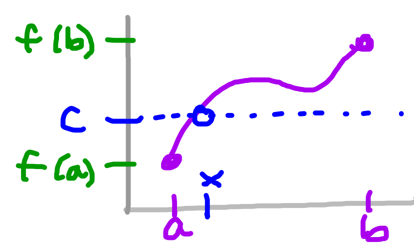
B " " " .

So A & B are open in $A \cup B$.

So $f^{-1}(A)$, $f^{-1}(B)$ are open in E, a separation. ■

check:

• COR. (Intermediate Value Thm)



$f: [a,b] \longrightarrow \mathbb{R}$ contin.

and $f(a) < c < f(b) \Rightarrow \exists\ x \in (a,b)$ s.t. $f(x) = c$.

proof idea: if not then $[f(a), c)$ and $(c, f(b)]$ would separate $f([a,b])$.

• UNIFORM CONTINUITY

Recall: X, Y metric spaces. $f: X \to Y$ continuous if

$$\boxed{\forall p \in X \text{ and } \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x \in X, \ d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon.}$$

δ may depend on p

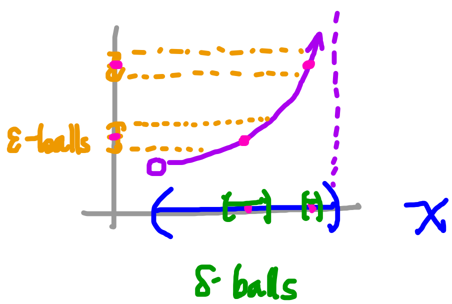
Defn. Call $f: X \to Y$ uniformly continuous if

$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall x \text{ and } p \text{ in } X, \ d(x,p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon.}$$

same δ works for every p

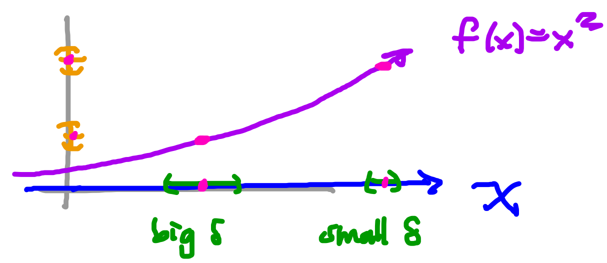
Unif. continuity $\Rightarrow$ continuity

Q. When is a contin. fcn not uniformly continuous?



no δ works for all p

f not unif contin.



f not unif contin.

but it IS if X were restricted to a bounded set!

• **Thm.** $f: X \to Y$ contin. and X cpt.

Then f is unif. contin. on X

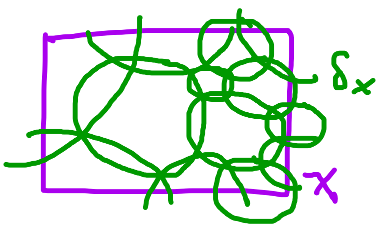
proof. #1 book: open covers by δ-balls have fin. subcover...

#2 Let $\varepsilon > 0$. [Goal: find δ for all p.]

Each x has δ_x-ball that shows f contin at x:

$$d(x,p) < \delta_x \Rightarrow d(f(x), f(p)) < \varepsilon/2$$

these balls cover X.



Q. Can I find δ s.t. every δ-ball in X must lie completely in a δ_x-ball?

If so, then:

$$d(f(p), f(q)) < d(f(p), f(x)) + d(f(x), f(q))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which would show f is unif. contin. ∎

- <u>Lebesgue covering lemma</u>.

  If $\{U_\alpha\}$ is open cover of <u>cpt</u> metric space X

  then $\exists\ \delta > 0$ s.t.

  $\forall\ x \in X$, $B_\delta(x)$ is <u>contained</u> in <u>some</u> U_α.

  - δ is called a <u>Lebesgue number</u> of the <u>cover</u>.