REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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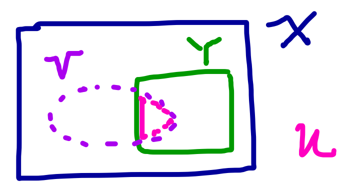
TODAY: THE HEINE-BOREL THEOREM

• If $Y \subset X$ metric then Y also metric space.

• A nbhd $N_r(x)$ in X is $\{p : d(x,p) < r$ for $p \in X.\}$

" in Y ... $p \in Y$

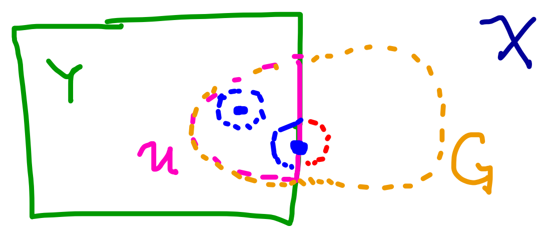


• A set U is open in Y (" open relative to Y")

means: every pt. of U is an interior pt. of U in Y.

using Y-nbhds.

Thm. $E \subset Y \subset X$.



E open in Y

$\Longleftrightarrow$

$E = Y \cap G$

for some G open in X.

U is open in Y

not open in X

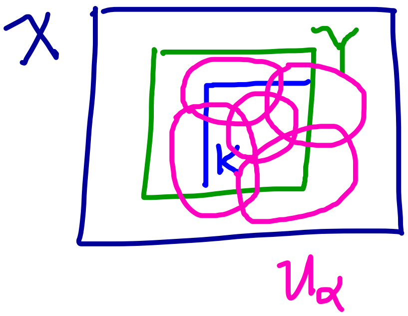
proof idea: ($\Rightarrow$) $\forall e \in E$, let $G = \bigcup_{e \in E} \overbrace{\underbrace{N(e)}_{\text{view in } X}}^{Y\text{-balls showing } e \text{ int. to } E \text{ in } Y}$

($\Leftarrow$) restrict nbhds $N(e)$ in X

to Y. ∎

Compactness is <u>intrinsic property</u>.

Thm. $Y \subset X$. Then K cpt. in $Y \Leftrightarrow K$ cpt. in X.



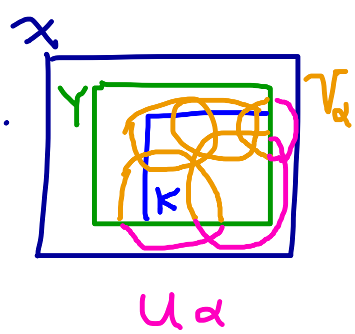
proof ($\Rightarrow$) Suppose K cpt in Y.
to show K cpt in X,
consider $\{U_\alpha\}$ open cover of K in X.

Then let $V_\alpha = Y \cap U_\alpha$ is open in Y.
So: $\{V_\alpha\}$ is open cover of K in Y.
Since K cpt in Y
$\Rightarrow \exists$ fin. subcover:
$\{V_{\alpha_1}, \dots, V_{\alpha_N}\}$.

Then $\{U_{\alpha_1}, \dots, U_{\alpha_N}\}$ are open cover of K in X.

($\Leftarrow$) similar:
START WITH $\{V_\alpha\}$ open cover in Y.

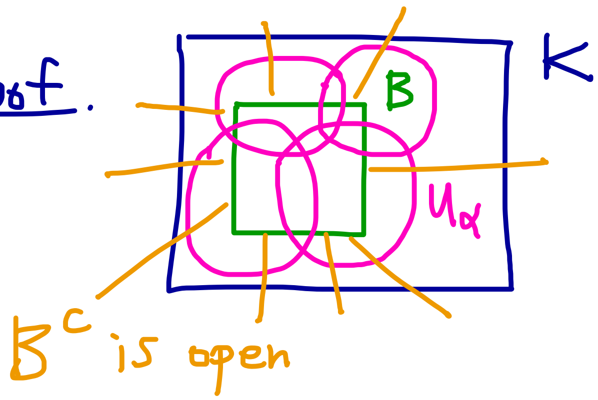
idea $\rightarrow$



• **Thm.** A closed subset B of cpt set K is cpt.

[does not say: "closed ~~sets~~ are cpt"]

proof.



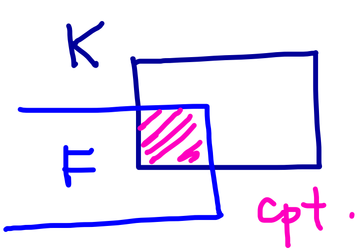
To show B cpt consider $\{U_\alpha\}$ an open cover of B.

Then $\{U_\alpha\} \cup \{B^c\}$ is open cover of K.

Since K cpt, $\exists$ finite subcover: $\{U_{\alpha_1}, \ldots, U_{\alpha_N}, \text{possibly } B^c\}$ covering K.

Then $\{U_{\alpha_1}, \ldots, U_{\alpha_N}\}$ covers B (since $B^c \cap B = \emptyset$). $\blacksquare$

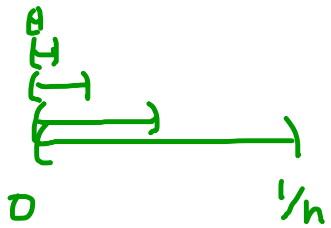
Cor. F closed, K cpt $\Rightarrow F \cap K$ cpt.



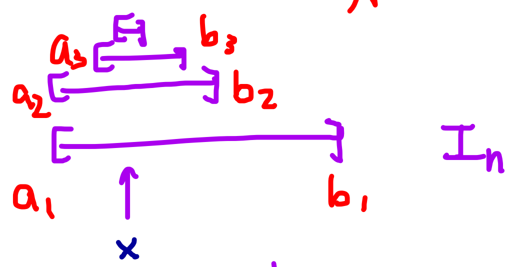
pf. K cpt $\Rightarrow K$ closed.

So F closed $\Rightarrow F \cap K$ closed subset of cpt set, so $F \cap K$ cpt. $\blacksquare$

• <u>Thm.</u> Nested closed intervals $[a_n, b_n]$ in $\mathbb{R}$ have nonempty intersection.



empty intersection



non-empty

$$\boxed{m > n \Rightarrow \quad ☺ \quad a_n \le a_m \le b_m \le b_n}$$

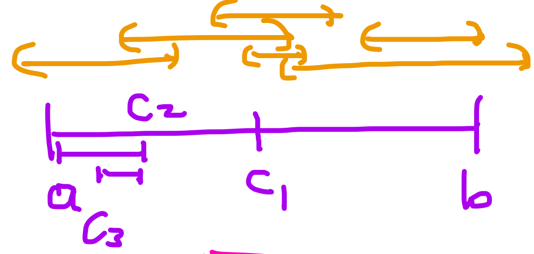
<u>proof</u>. Let $x = \sup\{a_i : i \in \mathbb{N}\}$, exists b/c set bdd by b_1.

Clear: $a_n \le x \quad \forall n$ by def'n of sup.

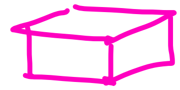
And $x \le b_n \quad \forall n$ b/c b_n is u.b. for <u>all</u> a_i's. by ☺ & cases...

So $x \in$ all I_n. ∎.

Thm. $[a,b]$ is cpt.



(**Thm.** k-cell in $\mathbb{R}^k$ is cpt.)



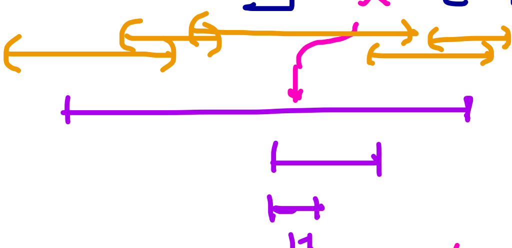
proof. Suppose not. Then $\exists$ cover $\{G_\alpha\}$ with no fin. subcover.

Then this $\{G_\alpha\}$ covers $[a, c_1]$ & $[c_1, b]$ but at least one of them with no fin. subcover..

WLOG: say $[a, c_1]$ has no FS.

Subdivide, repeat: get $I_1 \supset I_2 \supset I_3 \supset \cdots$ closed, nested, $\boxed{\text{size} \to 0}$ ⓔ $\boxed{\text{with no FS}}$ of this $\{G_\alpha\}$.

$\exists\ x^* \in$ all I_n.



Notice $x^* \in$ some G_{α_0}, which is open, so this

$x^* \in N_r(x^*) \subset G_{\alpha_0}$.

So by ⓔ some $I_n \subset N_r(x^*)$,

contradicting that for I_n, the $\{G_\alpha\}$ has no FS of I_n. $\blacksquare$

Heine-Borel Thm.

In $\mathbb{R}$ or $\mathbb{R}^k$, K cpt $\Leftrightarrow$ K is closed & bdd.

proof. ($\Rightarrow$) already. (true in all metric sp.)

($\Leftarrow$) not true in arbitrary metric sp.

K bdd $\Rightarrow$ $K \subset [-r, r]$ or closed k-cell in $\mathbb{R}^k$.

but K is closed by assumption,

& subset of cpt set $[-r, r]$ or (k-cell in $\mathbb{R}^k$)

$\Rightarrow$ K is cpt. ▨.