REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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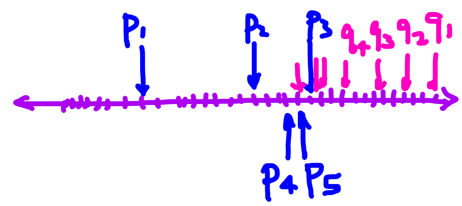
TODAY: COMPLETE SPACES, LIM SUP

Recall: Cauchy seqs: pts get close to each other

In metric space, convergent seqs are Cauchy.

In complete space, Cauchy seqs converge.

Not complete: $\mathbb{Q}$



← Cauchy seq — make equivalence class of — a point in X^*

Amazing thm. Every metric space (X, d) has a completion (X^*, Δ).

So: X^* "extends" X and Δ "extends" d.

How? Given X,

- Let $X^* = \left\{ \begin{array}{c} \text{set of all Cauchy seqs in } X \\ \text{under an equiv. rel'n } \sim \end{array} \right\}$

  where seqs $\{p_n\}$ and $\{q_n\}$ are equivalent if $\underbrace{\lim_{n\to\infty} d(p_n, q_n) = 0}_{\text{HW: exists if seqs Cauchy}}$.

- For $P, Q \in X^*$

  let $\Delta(P,Q) = \lim_{n\to\infty} d(p_n, q_n)$. (rep. of P, Q)

- Then X is isometrically embedded in X^* and X^* is complete.

  $p \in X \longrightarrow \{p, p, p, p, \dots\}$ ← extra credit.

EX. If $X = \mathbb{Q}$ then X^* is isometrically same as $\mathbb{R}$.

Remark. Cauchy seqs are the other way to construct $\mathbb{R}$ from $\mathbb{Q}$.

↑ show ordered field with lub property

• LIM SUP:

For seq $\{s_n\}$, let $E = \{x : x \text{ is a } \underline{\text{subseq.}} \text{ limit}\}$ $\in \mathbb{R}$ may include $\pm\infty$

Let $s^* = \sup E \stackrel{\text{def}}{=} \limsup s_n$ ← "the upper limit"

$s_* = \inf E \stackrel{\text{def}}{=} \liminf s_n$ ← "the lower limit"

always exist!

EX. $s_n \to s$ then $\limsup s_n = \liminf s_n = s$.

EX. $s_n = \{.1, \frac{3}{2}, .11, \frac{4}{3}, .111, \frac{5}{4}, \dots\}$

$\limsup s_n = 1$

$\liminf s_n = \frac{1}{9}$

Thm. (a) $s^* \in E$

(b) If $x > s^*$, $\exists N$ s.t. $n \geq N \Rightarrow s_n < x$.

s^* is the only # with these properties.

• Alt defn of lim sup: $\limsup s_n = \lim\limits_{n\to\infty} \left(\sup\limits_{k \geq n} s_k \right)$

limit of sup of tails.

- LIMITS

<u>Thm.</u> If $s_n \le t_n$ for all $n \ge N$ $\Rightarrow$ $\limsup s_n \le \limsup t_n$

For conv. seqs, $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$

<u>Special limits:</u> If $p > 0$

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) $\lim_{n\to\infty} p^{1/n} = 1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$ $\leftarrow$ let $x_n = n^{1/n} - 1$.

$\Rightarrow \frac{n(n-1)}{2} x_n^2 \le \underset{\text{binom}}{(x_n+1)^n} = n$

(d) $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ $\leftarrow$ exponents "win"

(e) $|x| < 1 \Rightarrow \lim_{n\to\infty} x^n = 0$.

SERIES

- What does this mean? $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots = \frac{\pi^2}{6}$.

Or this: $1-1+1-1+1-\ldots = ?$

$$\left.\begin{array}{l}(1-1)+(1-1)+(1-1)+\ldots = 0\,? \\ 1+(-1+1)+(-1+1)+\ldots = 1\,?\end{array}\right\}\text{ which?}$$

Some have learned:

$$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots = \frac{1}{1-\frac{1}{3}}$$

as special case of:

$$1+x+x^2+x^3+\ldots = \frac{1}{1-x} \longleftarrow \text{Why?}$$

$$(1-x)(1+x+x^2+\ldots)$$

So then:

$$1+2+4+8+\ldots = \frac{1}{1-2} = -1.$$

$$1-2+4-8+\ldots = \frac{1}{1-(-2)} = \frac{1}{3}.$$

Euler accepted this:

$$1-1+1-1+\ldots = \frac{1}{1-(-1)} = \frac{1}{2}.$$

- We'll define a <u>series</u>:

given $\{a_n\}$ let $\sum_{n=p}^{q} a_n = a_p + \dots + a_q$.

let $\boxed{s_n = \sum_{k=1}^{n} a_k}$ the n-th <u>partial sum</u>: sum of 1st n terms

- Then $\boxed{\{s_n\}}$ is a <u>seq</u>., sometimes written $\boxed{\sum_{n=1}^{\infty} a_n}$ called an <u>infinite series</u>.

It may or may not <u>converge</u>, but if it does to <u>some</u> s, $\boxed{\text{write: } s = \sum_{n=1}^{\infty} a_n}$.