REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: SERIES

Recall: Given $\{a_n\}$, the series $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \underbrace{\left(\sum_{k=1}^{n} a_k\right)}_{\text{partial sum } s_n}$

Q. When does a series converge?

A. When its partial sums do! But when is that?

EX. $a_n = \frac{1}{n}$ the harmonic series $\underbrace{\underbrace{\underbrace{\underbrace{1}_{s_1} + \frac{1}{2}}_{s_2} + \frac{1}{3}}_{s_3} + \frac{1}{4}}_{s_4} + \dots$

In $\mathbb{R}$, seq s_n converges iff its Cauchy.

Saw: if $m>n$ $\quad s_m - s_n = \frac{1}{n+1} + \dots + \frac{1}{m} \geq \frac{m-n}{m}$

so $s_{2n} - s_n \geq \frac{1}{2}$ so $\{s_n\}$ can't be Cauchy!

The harmonic series "diverges".

• In gen'l there is a Cauchy criterion for series in $\mathbb{R}$:

Thm. $\sum a_n$ converges $\iff$ $\boxed{\begin{array}{l}\forall \varepsilon > 0 \quad \exists N \text{ s.t.} \\ m,n \geq N \Rightarrow \left| \sum_{k=n}^{m} a_k \right| < \varepsilon.\end{array}}$

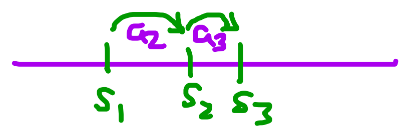
proof idea: the Cauchy difference $s_n - s_m = \sum_{k=m+1}^{n} a_k$.

• Let $m=n$, get:

Cor. $\sum a_n$ converges $\Rightarrow$ $\boxed{\lim_{n\to\infty} a_n = 0}$ "term test": terms $\to 0$

The converse is false: see harmonic series

• Thm. (non-neg series)



If $a_n \geq 0$ then

$\boxed{\sum a_n \text{ converges} \iff \text{p. sums are bounded.}}$

proof: follows from bdd. mon. seqs converge!

• Thm. (comparison test)

(a) If $|a_n| \leq c_n$ for n large enough

and $\sum c_n$ converges then $\sum a_n$ converges

(b) If $\boxed{a_n \geq d_n \geq 0}$ for n large enough

non-neg. and $\sum d_n$ diverges then $\sum a_n$ diverges.

proof. (a) Since $\sum c_n$ converges $\overset{\text{by Cauchy criterion}}{\Rightarrow}$ $\forall \varepsilon > 0\ \exists N$ s.t.

$m \geq n \geq N \Rightarrow \sum_{k=n}^{m} c_k < \varepsilon$.

So, given $\varepsilon > 0$, use this N (& large enough so $|a_n| \leq c_n$).

Then $m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$.

So by Cauchy criterion, $\sum a_n$ converges.

(b) contrapositive: show if $\sum a_n$ converges $\Rightarrow$ $\sum d_n$ converges.

Now use (a).

- <u>Geometric Series</u> if $|x|<1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

  if $|x| \geq 1$ then series <u>diverges</u>.

<u>proof</u>. If $|x|<1$,

let $s_n = 1+x+x^2+\ldots+x^n \underset{\text{by poly. arithmetic}}{=} \frac{1-x^{n+1}}{1-x}$

Then $\lim_{n\to\infty} s_n = \frac{1}{1-x} \lim_{n\to\infty} (1-x^{n+1}) = \frac{1}{1-x}\left(1 - \underbrace{\lim_{n\to\infty} x^{n+1}}_{0 \text{ if } |x|<1}\right) = \frac{1}{1-x}$.

If $|x|\geq 1$, terms $\not\to 0$, so series <u>diverges</u>. $\boxtimes$

- $\sum_{n=0}^{\infty} \frac{1}{n!}$ <u>converges</u> to limit called $\boxed{e}$.

  <u>Why?</u> Its p. sums $s_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$

  <u>idea:</u> compare to geo. series, $x = \frac{1}{2}$.

  $< 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}}_{\text{geom}}$.

- Converges <u>rapidly</u>.

$$e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots$$

$$< \frac{1}{(n+1)!} \underbrace{\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots \right]}_{\text{geom.}}$$

yes $\frac{1}{n+2} < \frac{1}{n+1}$.

$$< \frac{1}{n!\,n}.$$

- See: e is <u>irrational</u>! b/c if $e = \frac{m}{n}$ some m, n, <u>then</u>:

$$0 < \underbrace{n!\,(e - s_n)}_{\text{integer}} < \frac{1}{n}.$$

(ALL DENOMS CANCEL)

Also: $e \approx 2.718281828459045\ldots$

ALSO $\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$