REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

PROF. FRANCIS SU

TODAY: INDUCTION

INDUCTION

Recall $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural #'s.

- The well-ordering property (WOP) of $\mathbb{N}$:

"$\mathbb{N}$ is well-ordered" $\Rightarrow$ any non-empty subset of $\mathbb{N}$ has a least element.

- can take as an axiom of $\mathbb{N}$.

- Principle of Induction (POI)

Let S = a subset of $\mathbb{N}$ such that

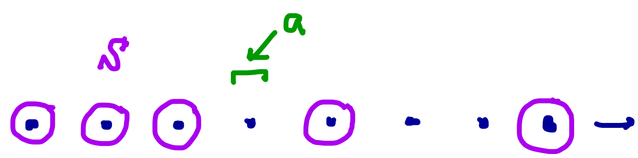
① $1 \in S$

② if $k \in S$ then $k+1 \in S$

$\Big\}$ say S is an "inductive set"

Then $S = \mathbb{N}$.

- Thm. WOP $\Rightarrow$ POI.



proof. (by contradiction)

Suppose $S \neq \mathbb{N}$. Then $A = \mathbb{N} \setminus S$ non-empty, so has least element a by WOP.

If $a = 1$, contradicts ①. Else:

consider $a-1$. Note $a-1 \in S$ b/c a was smallest not in S.

But ② would imply $a \in S$, contradiction. $\blacksquare$

- In fact POI $\Rightarrow$ WOP. (HW)

- Proofs by induction:

  Let $P(n)$ = statement indexed by $n \in \mathbb{N}$

  To show $P(n)$ true for all n:

  base case → ① show $P(1)$ is true

  inductive step → ② show $\underbrace{\text{if } P(k) \text{ true}}_{\text{inductive hypothesis}}$ then $P(k+1)$ is true.

  Then by POI, $P(n)$ holds for all $n \in \mathbb{N}$.

Strong induction uses $\boxed{2'}$: show if $P(1) \ldots P(k)$ true then $P(k+1)$ true.

WRITING! proofs by induction should:

- at start, indicate you'll use induction:

  "proof (by induction on n)."

- show base case*   * = assume reader knows term
- show ind. step

  & indicate where ind. hyp* is used.

- state conclusion & cite POI.

ERRORS:

- forget to check base case.
- bad logic in ind. step.
- not proving the most gen'l $P(n+1)$.

pos. comb.

__Ex.__ Prove $\forall n \geq 14$, n is __sum__ of 3's & 8's.

__proof.__ (by induction).

__base case:__ Note $14 = 8+3+3$, as __desired__.

__inductive step:__ Assume result holds for $n=k$; we'll show holds for $n=k+1$.

To build a __sum__ for $k+1$, consider a __sum__ for k.

By ind. hyp., k is sum of 3's & 8's.

If it has an 8, replace with $3+3+3$.

If no 8, then since $k \geq 15$ (b/c only uses 3's) replace five 3's with $8+8$.

This gives __sum__ for $n=k+1$.

By POI, statement holds for all $n \geq 14$. ∎

__Alternative:__ prove w/ several base case, jump by 3.

BAD

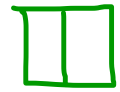
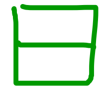
__Thm.__ All natural #'s are __even__.

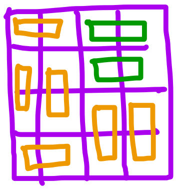
BAD __proof.__ Assume all #'s $\leq n$ are __even__ (strong ind. hyp.)

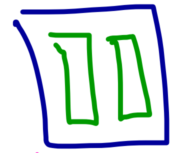
Notice: $n+1 = \underbrace{(n-1)}_{\text{even by ind. hyp.}} + \underbrace{2}_{\text{even}}$, so $n+1$ is even. ∎

no base case!

BAD

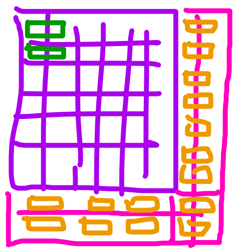
- **Thm.** All tilings of $2n \times 2n$ square using dominoes must have  or  in one corner.



proof. base case: 2×2 .

MUST START with $(2n+2)\times(2n+2)$ board

ind step: $\boxed{\text{Given } 2n \times 2n \text{ square}}$, by ind. hyp it has tiling w/ desired prop.



Add 2 rows & 2 cols: tile this part & get new tiling for $(2n+2)\times(2n+2)$ board. ∎

START with $P(k+1)$ AND FIND a way to use $P(k)$.

- **Thm.** A non-empty finite set A in $\mathbb{R}$ contains its sup.

proof. (by induction on size of A).

- For base case: note if $A = \{a\}$, then $\sup A = a$.

(justify ...)

- Now assume claim holds for n-element sets. ← ind. hyp.

NO! { Say A is n-element set. Then $A \cup \{a\}$ is $(n+1)$-element set...

better: Let A be set with $(n+1)$ elements.

Choose $a \in A$.

Let $B = A \setminus \{a\}$, it has n elements...

BAD **Thm.** All horses have same color.

BAD **pf.** (by ind. on # horses).

For base case: in set w/ one horse, all have same color!

Assume any set of n horses has same color (ind. hyp.)

Consider $n+1$ horse set: $\{\underbrace{h_1, \dots,}_{\text{these } n \text{ have same color}} h_{n+1}\}$ — last n same color

bad ind. step.

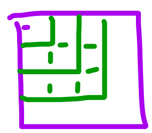
So all have same color!

Thm. $S_n = 1+3+5+\dots+(2n-1)$ is perfect square. $= n^2$

Strengthen the induction to prove the theorem.

INSIGHT

1
3
5
7



but if you don't have insight,

INDUCT: $S_n = 1$ is square ✓

ind. step: $S_{n+1} = \underbrace{1+\dots+(2n-1)}_{\text{some } k^2 \to n^2} + (2n+1)$

now this proof is easier!