REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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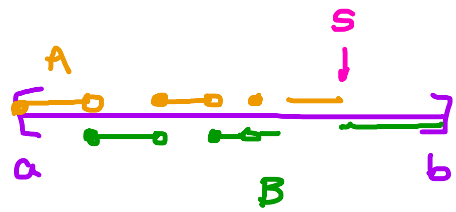
TODAY: CONNECTED SETS

__Recall:__ E is __connected__ if E is __not__ union of 2 non-$\emptyset$ __separated__ sets.

Call A, B __separated__ if __both__ $A \cap \bar{B} = \bar{A} \cap B = \emptyset$.

__Thm.__ $[a,b]$ is __connected__.



__proof.__ If not, $\exists$ __separation__ A and B, without loss of generality, with $b \in B$.

Let $s = \sup A$.

Then either $s \in A$ or __lim.pt__ of A (b/c we know $s-\epsilon$ is not an upper bd., so $\exists\, a \in A$ s.t. $s - \epsilon < a \le s$.)

Then $s \in \bar{A}$.

By def'n of separated, $s \notin B$. Then $s \in A$ b/c $A \cup B = [a,b]$. note now $s \neq b$.

" $s \notin \bar{B}$.

So $\exists$ interval $(s-\epsilon, s+\epsilon)$ around s __disjoint__ from B.

This contradicts s as sup A. $\blacksquare$

Rudin: shows $E \subset \mathbb{R}^1$ is connected $\Leftrightarrow$ E has "interval-like" property:

$[x, y \in E,\ x < z < y \Rightarrow z \in E]$

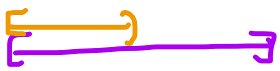
- Another characterization:

A set E is connected $\Leftrightarrow$ E is not the disjoint union of A and B where A & B are open relative to E.

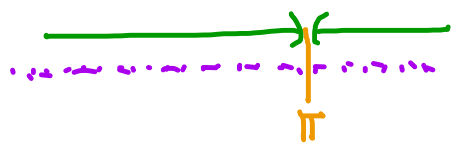
(& therefore closed in E)

"clopen"

E



$\mathbb{Q}$



DENSE SETS

Rudin: set E is dense in X if

$\forall x \in X$, x is a lim pt of E or in E.

Ex. $\mathbb{Q}$ is dense in $\mathbb{R}$

EX. Set of polys over $[0,1]$ is dense in set of contin fcns on $[0,1]$ (using sup metric on fcns).

Alternatively: Thm. The following are equivalent:

(a) E dense in X.

(b) Every open set in X contains a pt of E.

(c) $\overline{E} = X$.

proof. (a) $\Rightarrow$ (c) by def'n of closure.

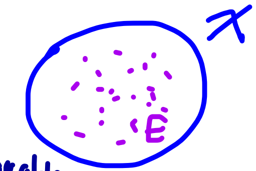
(c) $\Rightarrow$ (b) " " " lim pt.

(b) $\Rightarrow$ (a) " " " lim pt.

HW: X separable means: X has ctble dense subset.

do not confuse with separation

size of X is "small"



HW: cpt. metric space $\Rightarrow$ ctble base $\Rightarrow$ X is separable.