REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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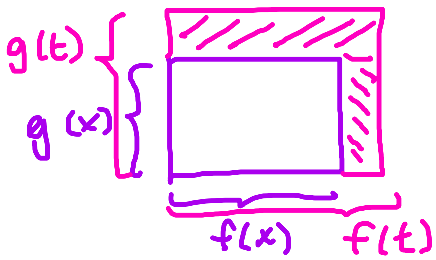
TODAY: THE MEAN VALUE THEOREM

Last time: defined $f'(x)$ $(f+g)' = f' + g'$

Since f' is a <u>limit</u>, then <u>sum</u>, <u>product</u>, <u>quotient</u> rules for derivatives follow...

$(fg)' = f'g + fg'$



why?

$$f(t)g(t) - f(x)g(x)$$

$$-f(t)g(x) + f(t)g(x)$$

$$= f(t)\left[g(t) - g(x)\right] + g(x)\left[f(t) - f(x)\right].$$

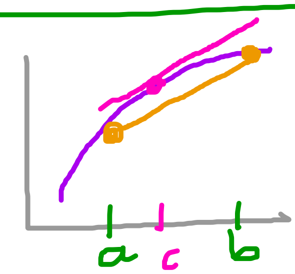
Now divide by $t - x$ and take limits...

THE MEAN VALUE THEOREM

the most important property about derivs.

slope same

MVT: If f is contin on $[a,b]$
and f is diff'ble on (a,b)
then $\exists\ c \in (a,b)$ s.t.



$$f(b)-f(a) = f'(c)\,(b-a)$$

Cauchy 1823 wrong ; Bonnet 1868 correct

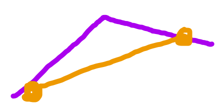
Useful b/c connects value of $f(x)$ to value of derivative without appeal to limits.

- Sample application. If $f'(x) > 0\ \ \forall\, x \in (a,b)$
  show f is strictly increasing in (a,b).

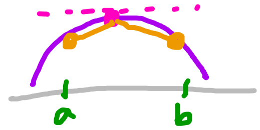
proof. If $s<t$ in (a,b)
then $f(t)-f(s) \overset{\text{MVT}}{=} \overbrace{(t-s)}^{+}\ \overbrace{f'(c)}^{+} > 0$. $\blacksquare$.

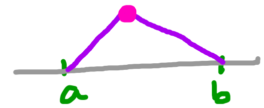
$c \in (s,t)$

- Why diff'bility needed:



First prove special case:



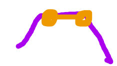


<u>Rolle's Thm.</u> If $h:[a,b]\to\mathbb{R}$ has a <u>local max</u> at $c\in(a,b)$ and $h'(c)$ exists then $h'(c)=0$.

<u>proof</u>: consider <u>sign</u> of slope: $\dfrac{h(t)-h(c)}{t-c}$.

$$\left.\begin{array}{ll} \text{For } t<c & \text{it's } \geq 0 \\ \phantom{\text{For }} t>c & \quad " \quad \leq 0 \end{array}\right\} \text{ so if } \lim_{t\to c}\frac{h(t)-h(c)}{t-c} \ \underline{\text{exists}}, \text{ must be } 0.$$

in an interval on which c is local max



- Version with local min also holds.

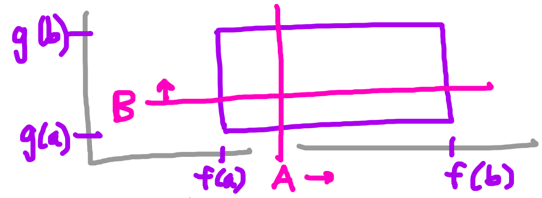
Generalized MVT (Cauchy) If f, g contin on $[a,b]$

and diff'ble on (a,b)

then $\exists\, c \in (a,b)$ s.t.

$$[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c).$$

- If $g(x)=x$, this MVT.



- proof idea: (motivate book's pf)

Say $f(t)$ = pos. of a knife A at time t along x-axis.

$g(t)$ = " " B " " t " y-axis.

Note:

LHS = rate B sweeps area.

RHS = rate A sweep area

Natural to consider: $h(t)$ = difference in area swept by A & B
this of cake

Note: $h(a)=0$ and $h(b)=0$.

So $\exists$ local max or min inside (a,b)

and h is diff'ble (check):

$$h(t) = [f(b)-f(a)]\,\boxed{g(t)-g(a)} - [g(b)-g(a)]\,\boxed{f(t)-f(a)}$$

so $h'(t) = [\quad " \quad]\,g'(t) - [\quad " \quad]\,f'(t)$.

- Rolle's $\Rightarrow$ $\exists\, c \in (a,b)$ where $h'(c)=0$, as desired. ∎

TAYLOR'S THM: is generalization of MVT.
helps us approx f near a
if know several derivatives.

MVT: $f(b) = f(a) + \underbrace{f'(c)(b-a)}_{\text{an "error" term, not precisely known}}$ for some $c \in (a,b)$.

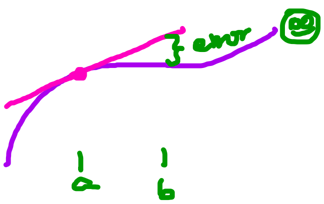
This suggests:

$$f(b) = f(a) + f'(a)(b-a) + \text{error term}$$

eg

In fact: $\dfrac{f''(c)}{2!}(b-a)^2$

for some $c \in (a,b)$
if f'' exists



In gen'l, let $P_{n-1}(x) = f(a) + f'(a)(x-a) + \dots + \dfrac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$

the $(n-1)$th Taylor poly.

- **Taylor's thm.** If $f^{(n-1)}$ contin on $[a,b]$ & $f^{(n)}$ exists on (a,b) then P_{n-1} approxs f: for $x \in (a,b)$

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

for some $c \in (a,x)$

- If $n=1$, this is MVT
- $P_n(x)$ is "best" poly approx of order n
- proof Rudin uses MVT several times

- Different proof

Let $P(a,b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (b-a)^k$.

Define K by $f(b) - P(a,b) = K \frac{(b-a)^n}{n!}$. ☺

<u>Goal</u>: show $K = f^{(n)}(c)$ for some $c \in (a,b)$.

- Let $g(x) = f(b) - P(x,b) - \frac{K(b-x)^n}{n!}$.

<u>Checks</u>: $g(a) = 0$ by ☺.

$g(b) = 0$ b/c $f(b) = P(b,b)$.

MVT $\Rightarrow$ $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$.

But $g'(x) = \cancel{f'(x)} - \cancel{f'(x)} + \cancel{\frac{f''(x)}{1!}(b-x)} - \cancel{\quad}$

$\ldots + \frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1} - K\frac{(b-x)^{n-1}}{(n-1)!}$

If $g'(c) = 0$, then $f'(c) - K = 0$, as desired. $\blacksquare$