REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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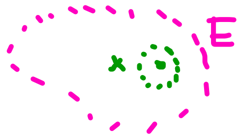
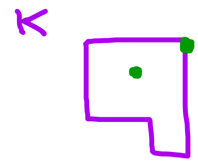
TODAY: TOPOLOGY

Recall: A set E is open if every pt of E is an interior pt:



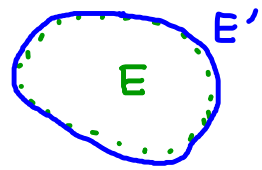
x is interior to E : can perturb x, still stays in E.



A set K is closed if K contains all its limits.

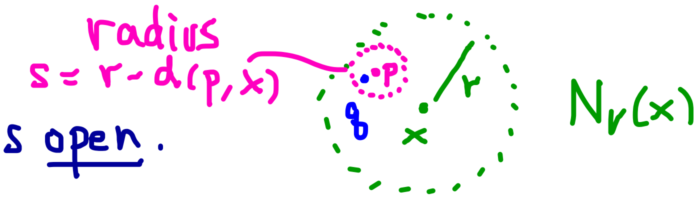
closed: property "preserved" by limits

Def'n Let E' be set of all lim. pts of E.



The closure of E is $\overline{E} = E \cup E'$.

Q. Is $\overline{E}$ closed?



Thm. An open ball (nbhd) is open.

proof. Given ball $N_r(x)$

we'll show any $p \in N_r(x)$ is an interior pt.

Let $s = r - d(p,x)$.

We'll check: $N_s(p) \subset N_r(x)$.

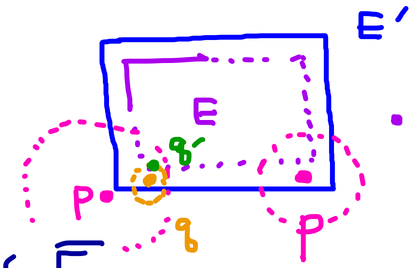
Consider $q \in N_s(p)$. Then $d(q,p) < s$.

But $d(q,x) \leq d(q,p) + d(p,x)$

$\uparrow$ $\triangle$ ineq

$< s + d(p,x) = r$, as desired. $\blacksquare$

Thm. $\overline{E}$ is closed.



proof. Say p is lim pt of $\overline{E}$.

We'll show p is in $\overline{E}$, meaning either $p \in E$ or p is lim pt of E.

If $p \in E$, done; else we'll show p is lim pt of E.

Given nbhd N of p, know N contains some $q \in \overline{E}$.

Since N open, $\exists$ nbhd N' of q s.t. $N' \subset N$

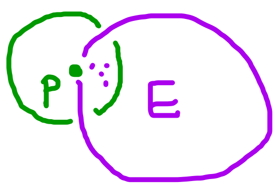
and N' contains some pt. $q' \in E$,

b/c q is lim. pt. of E or is in E.

So q' witnesses that p is lim. pt of E, as desired. ■

Thm. p is lim pt of $E \Rightarrow$ every nbhd of p contains ∞ many pts of E.

proof.



If $\exists$ nbhd w/ finitely many $e_1, \ldots, e_N \in E$

then let $r = \min_i d(p, e_i)$.

Then $N_r(p)$ contains no pts of E,

Contradiction. ■

Def'n. E is dense in X if every pt of X is in E or lim pt of E.

equiv: $\overline{E} = X$.

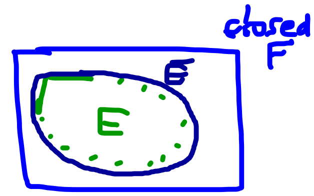
or: every ball in X contains a pt of E.

Ex. $\mathbb{Q}$ is dense in $\mathbb{R}$. Set of polys on $[0,1]$ is dense in contin fcns on $[0,1]$.

• Thm. E closed $\iff E = \bar{E}$.

proof. ($\Rightarrow$) E closed $\Rightarrow E' \subset E$ so $\bar{E} = E \cup E'$.

($\Leftarrow$) $E = \bar{E} \Rightarrow E$ contains its lim pts.



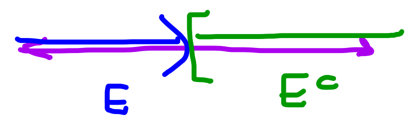
• Thm. If $E \subset$ closed set F, then $\bar{E} \subset F$.

says: $\bar{E}$ is "smallest" closed set containing E.

proof. follows b/c lim pt of E is lim pt of F.
(since pts of E are pts of F).

the complement of E in X: $E^c = \{p : p \in X \setminus E\}$

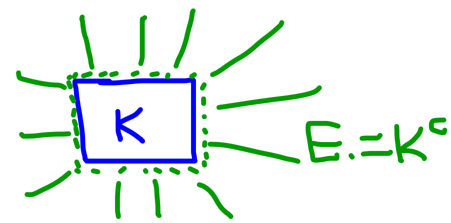
• Thm. E open $\iff E^c$ is closed.



proof.

E open $\iff \forall x \in E$, x is int. pt.

$\iff \forall x \in E$, $\exists$ nbhd N of x containing no pt of E^c



$\iff \forall x \in E$, x is not lim pt of E^c

$\iff E^c$ contains all its lim pts. $\iff E^c$ is closed.

Unions: $\bigcup_{i=1}^{n} A_i$ finite union

$\bigcup_{i=1}^{\infty} A_i$ ctble union

$\bigcup_{\alpha \in \lambda} A_\alpha$ arbitrary, λ is an index set

or $\bigcup_{\alpha} A_\alpha$

Lemma. $\{E_\alpha\}_{\alpha \in \lambda}$ is a collection. Then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$.

Thm. ① Arb union of open sets is open

② " $\cap$ " closed " " closed.

③ Finite $\cup$ of closed sets is closed

④ " $\cap$ " open " " open.