REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

PROF. FRANCIS SU

TODAY: CAUCHY SEQUENCES

"small enough"

Thm. In a compact metric space X

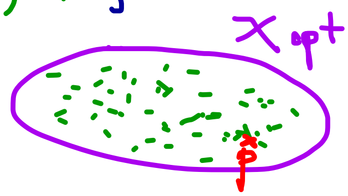
[every seq. has a convergent subseq. (converging to a point of X).] say "X is <u>sequentially compact</u>."

proof. Let $R = \text{range}\{p_n\}$.

If R <u>finite</u>, then some $p \in R$ is p_i for ∞ many i (pigeonhole) say $\{p_{n_1}, p_{n_2}, \ldots\}$.

If R <u>infinite</u>, then X cpt

$\Rightarrow$ R has <u>lim pt</u> p.



Use (E) to get seq. in R <u>conv.</u> to p (take smaller & smaller nbhds, for subseq. of <u>increasing</u> indices).

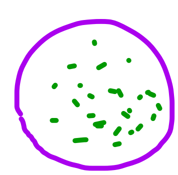
FACT. X <u>cpt.</u> $\Leftrightarrow$ X is <u>seq. cpt.</u>

($\Rightarrow$) just did

($\Leftarrow$) extra credit

COR. (Bolzano-Weierstrass) Every <u>bounded</u> seq in $\mathbb{R}^k$ has a <u>conv. subsequence</u>.

pf. View <u>seq</u> as subset of large, closed <u>disk</u>.



• In $\mathbb{R}$, call seq $\{s_n\}$ monotonically increasing if $s_n \le s_{n+1}$ $\forall n$.

" " mon. decreasing if $s_n \ge s_{n+1}$ $\forall n$.

Thm. Bounded monotonic seqs. converge. [to its sup or inf]

pf. Let $s = \sup(\text{range}\{s_n\})$ ← exists b/c s_n bdd.

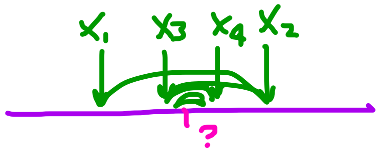
Then $\forall \varepsilon > 0$, $\exists N$ s.t. $s - \varepsilon < s_N \le s$.

But then for this N, $n \ge N \Rightarrow s - \varepsilon < s_N \le s_n \le s$,

as desired. $\blacksquare$

Cauchy:

Q: How to tell if $\{p_n\}$ converges if you don't know its limit?



EX. Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

Defn. $\{p_n\}$ is Cauchy if:

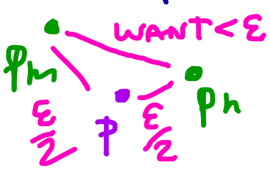
$$\boxed{\begin{array}{l} \forall\, \varepsilon > 0 \ \exists N \text{ such that} \\ \qquad n, m \geq N \Rightarrow d(p_n, p_m) < \varepsilon. \end{array}}$$

"past index N all p_i are ε-close";

Thm. If $\{p_n\}$ converges, then $\{p_n\}$ is Cauchy.

pf. Say $p_n \to p$. Then $\forall\, \varepsilon > 0$, $\exists N$ s.t.

$$n \geq N \Rightarrow d(p, p_n) < \frac{\varepsilon}{2}.$$



Now, for this N, we see $n, m \geq N$ implies:

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. ∎

- But not every Cauchy seq converges.

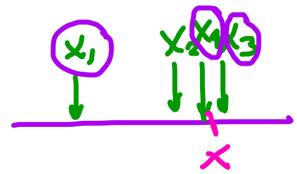
EX. Let $X = \mathbb{Q}$. Seq: $3, 3.1, 3.14, 3.141, 3.1415, \ldots$

is Cauchy, but does not converge in $\mathbb{Q}$.

Defn. A metric space X is complete means every Cauchy seq in X converges (to a point of X).

Q. Which spaces are complete?

Thm. Compact metric spaces are complete.



proof (diff from Rudin)

Let $\{x_i\}$ be Cauchy seq in X.

Since X cpt $\Rightarrow$ X is seq. cpt:

every seq has conv. subseq:

so $\exists$ subseq of $\{x_i\}$, call it $\{x_{i_k}\}$ conv to $x \in X$.

Fix $\varepsilon > 0$. Conv. of subseq $\Rightarrow \exists N_1$ s.t. $i_k \geq N_1 \Rightarrow d(x, x_{i_k}) < \frac{\varepsilon}{2}$.

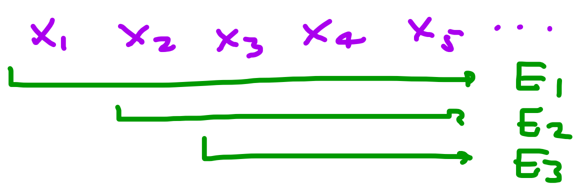
$\{x_i\}$ Cauchy $\Rightarrow \exists N_2$ s.t. $i, j \geq N_2 \Rightarrow d(x_i, x_j) < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. For this N, choose $i_k \geq N$.

Then $n \geq N \Rightarrow$ $d(x, x_n) \leq d(x, x_{i_k}) + d(x_{i_k}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

as desired. ☒

Rudin: uses nested seq of non-empty cpt sets not empty.



$\overline{E_1} \supset \overline{E_2} \supset \dots$

$\exists x$ in all . . .

E

Thm. Closed subsets of complete spaces are complete.

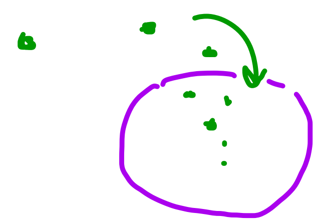
Why: Cauchy seq $\{x_i\} \in E$ conv's in big space to some $x \in X$.

But then $x \in E$ since E closed.

Thm. $\mathbb{R}^k$ is complete.

idea: show Cauchy seq is bdd.

so in some cpt subset of $\mathbb{R}^k$.



Ex. in $\mathbb{R}$, does $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ converge?

No: check it's not Cauchy:

If $n > m$, $\quad x_n - x_m = \frac{1}{m+1} + \ldots + \frac{1}{n} \geq \frac{n-m}{n} = 1 - \frac{m}{n}$.

let $n = 2m$ see $x_{2m} - x_m > \frac{1}{2}$. NOT CAUCHY.