REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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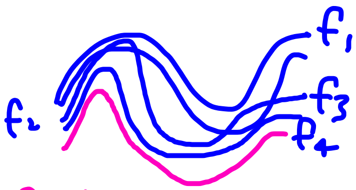
TODAY: SEQUENCES

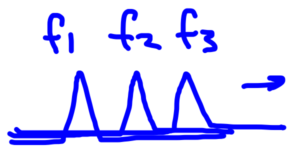
SEQUENCES in metric spaces

__Ex.__ Let $X = C_b(\mathbb{R})$ contin, bdd functions $\mathbb{R} \to \mathbb{R}$.

one metric: $d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$.



Converge? to some f?

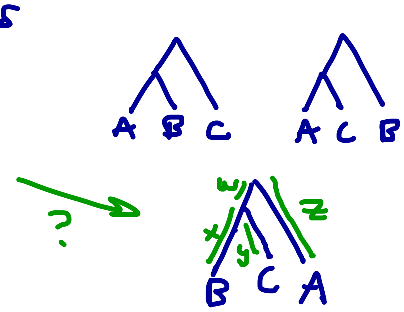


does not converge under this metric

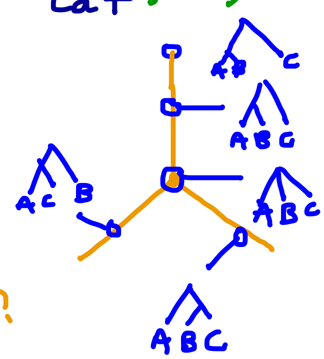
__EX.__ math bio: reconstruct phylogenetic trees

given ACGATAG⋯ ape
GATTACA⋯ bug
TTTTTTT⋯ cat

7, 5, 6



"__Space__" of phylo trees?



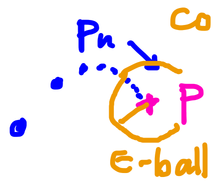
"average" pts in this space?

"converge"?

__EX.__ in $\mathbb{R}^2$

__converge?__

①



converge: get close & __stay there__

②

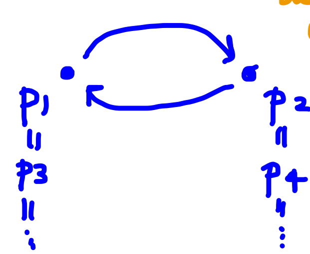
no convergence

③

$p_n = p$

converges

④



does not converge

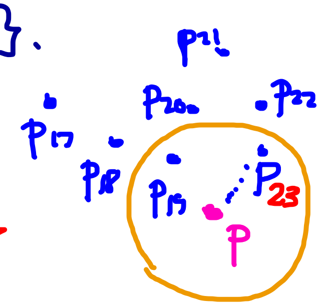
• **Def'n** A <u>sequence</u> $\{p_n\}$ in metric space X is a <u>function</u> $f: \mathbb{N} \to X$ mapping $n \mapsto p_n$, a <u>point</u> in X.

The <u>range</u> of $\{p_n\}$ is <u>set</u> $\{x : x = p_n \text{ for some } n\}$.

A <u>seq.</u> is <u>bounded</u> its range is <u>bdd</u>.

Ex. The seq: $\overset{p_1}{\overset{\shortparallel}{p}}, \overset{p_2}{\overset{\shortparallel}{p}}, \overset{p_3}{\overset{\shortparallel}{p}}, p, \ldots$ has range $\{p\}$.

Def'n. $\{p_n\}$ <u>converges</u> if $\exists\, p \in X$ s.t. $\forall \varepsilon > 0$ $\boxed{\exists \text{ integer } N}$ * s.t. $n \geq N$ <u>implies</u> $d(p, p_n) < \epsilon$.



<u>Write:</u> $\boxed{p_n \to p}$ or $\boxed{\lim\limits_{n\to\infty} p_n = p}$.

"p_n converges to p" "p is the <u>limit</u> of $\{p_n\}$"

* To <u>show</u> $p_n \to p$, you must, for each ϵ, <u>find</u> an N that works.

TRUE/FALSE?

T (A) $p_n \to p$ and $p_n \to p'$ $\Rightarrow$ $p = p'$

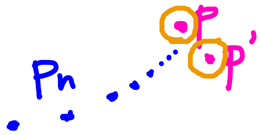
F (B) $\{p_n\}$ bdd $\Rightarrow$ p_n converges?

T (C) p_n converges $\Rightarrow$ p_n is bdd

F (D) $\lim\limits_{n\to\infty} p_n = p$ $\Rightarrow$ p is <u>lim pt</u> of range of $\{p_n\}$

T (E) p is <u>lim pt</u> of $E \subset X$ $\Rightarrow$ $\exists$ <u>seq</u> $\{p_n\}$ in E s.t. $p_n \to p$

T (F) $p_n \to p$ $\overset{iff}{\Leftrightarrow}$ every nbhd of p contains all but finitely many p_n.

(A) 

Idea: use $\epsilon = d(p, p')$

Suppose $p_n \to p$ and $p_n \to p'$, where $p \neq p'$.

Then b/c $p_n \to p$, $\exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \frac{\epsilon}{2}$.

And b/c $p_n \to p'$, $\exists N'$ s.t. $n \geq N' \Rightarrow d(p_n, p') < \frac{\epsilon}{2}$.

So: if $n \geq \max\{N, N'\}$

then $d(p, p') \leq d(p, p_n) + d(p_n, p') < \frac{\epsilon}{2} + \frac{\epsilon}{2} = d(p, p')$

Δ ineq

a contradiction. ∎

Remove purple: get a direct proof
showing $d(p, p') < \epsilon$ forall $\epsilon > 0$
hence $d(p, p') = 0$, so $p = p'$.

(B) see ex (4.)

(C.) 

Idea: use $\epsilon = 1$.

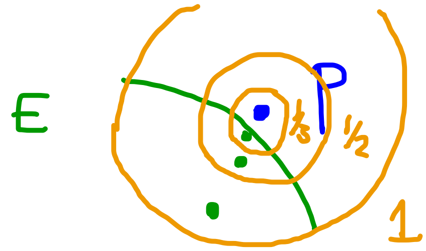
Since $p_n \to p$ $\exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < 1$.

Let $R = \max\{1, d(p_1, p), \ldots, d(p_{N-1}, p)\}$.

So all pts are in $B_{R+1}(p)$.

(D.) see EX (3)

E: Idea



$\forall n \in \mathbb{N}$, choose $p_n \in N_{1/n}(p)$. (such pt exist by def of lim pt.)

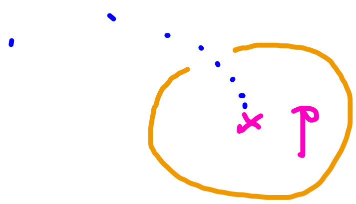
We'll show: $p_n \to p$.

So given $\varepsilon > 0$, let $N = \lceil \frac{1}{\varepsilon} \rceil$.

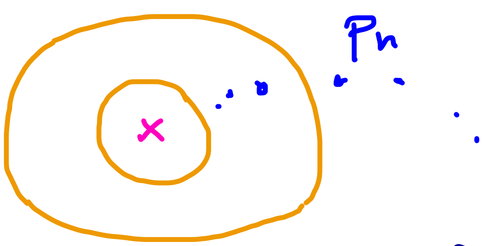
<u>Check:</u> $n \geq \lceil \frac{1}{\epsilon} \rceil \Rightarrow \frac{1}{n} \leq \epsilon$ <u>so</u> $d(p_n, p) < \frac{1}{n} \leq \epsilon$,

as <u>desired</u>. ∎

F: $(\Rightarrow)$



$(\Leftarrow)$



$\forall \varepsilon > 0$ & ball $N_\epsilon(p)$ contains all but

$p_{i_1}, p_{i_2}, \dots p_{i_N}$

for $i_1 < i_2 < \dots < i_N$

then $n \geq i_N + 1 \Rightarrow d(p_n, p) < \epsilon$.