REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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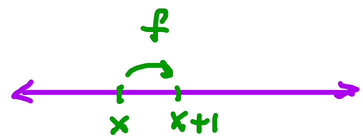
TODAY: THE BROUWER FIXED POINT THEOREM

Consider $f: X \to X$ map between metric spaces.

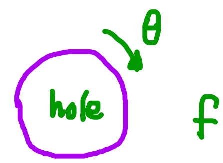
Call $x \in X$ a <u>fixed point</u> if $f(x) = x$.

<u>Ex</u>. $X = \mathbb{R}$ $\quad f(x) = x + 1$.



no fixed pt.

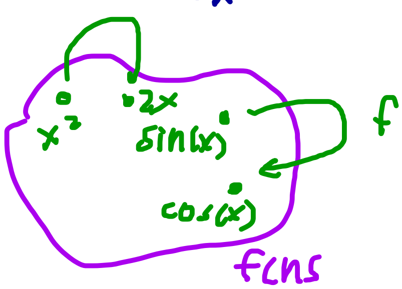
<u>Ex</u>. $X = S^1$ <u>circle</u>, $f(x) =$ rotate x by θ



no fixed pt for $0 < \theta < 2\pi$

<u>Ex</u>. $\frac{d}{dx} f(x) = f(x)$.

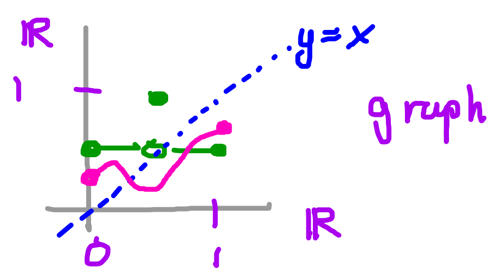
Solving this is a <u>fixed point</u> problem when X = function space and $f: X \to X$ is diff'ation operator.



Q: What spaces have the <u>fixed point property</u>?

any <u>continuous</u> self-map has fixed pt.

<u>Ex</u>. $f: [0,1] \to [0,1]$.



contin f

<u>Idea</u>: consider $g(x) = f(x) - x$.

Then $g(0) \geq 0$, $g(1) \leq 0$.

So Int. Val. Thm $\Rightarrow$

$\exists\, x$ s.t. $g(x) = 0$.

Ex. Newton's method: $x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)}$.

want to solve $x = g(x)$.

A zero for f is a fixed pt for g.

Ex. Fund. Thm. of Algebra

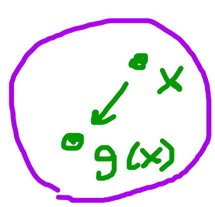
Every polynomial of deg $n \geq 1$ has root in $\mathbb{C}$.

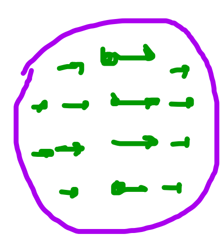
$$\underbrace{x^5 - 3x^2 + \pi x - 2}_{f(x)}$$

proof (via BFPT)

Solve: $\underbrace{-\frac{1}{\pi}x^5 + \frac{3}{\pi}x^2 + \frac{2}{\pi}}_{g(x)} = x$.

BFPT: A contin. $g : \underset{\text{ball}}{B^2} \to B^2$ has fixed pt





map on open ball may not have fixed pt

Ex. map of Claremont

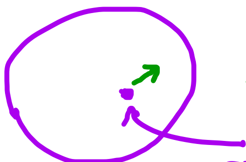
Ex. shaking cup of coffee

Brouwer Fixed Pt. Thm (1910)

If $f: B^n \to B^n$ is contin. then f has a fixed pt.

Ex. Game Thry: Nash equilibrium Thm.

$\exists$ set of strategies: mutual best responses!



f moves in dir of better response.

strategy space

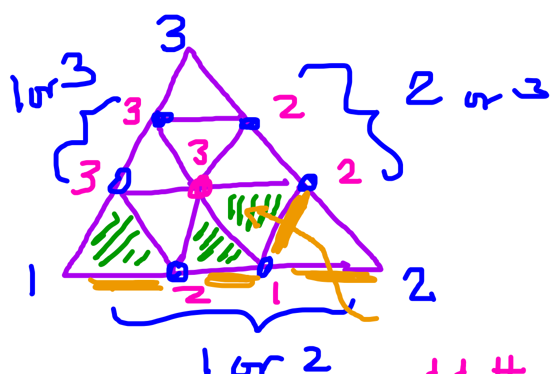
proof. uses "algebraic topology"

another proof: elementary

using Sperner's lemma

proof

odd # bdry doors

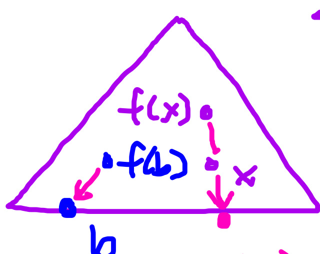


- even # bdry doors matched by paths

"Sperner labelling" $\Rightarrow \exists$ 123 △ . (odd #)

- so at least one path from bdry → 123△.

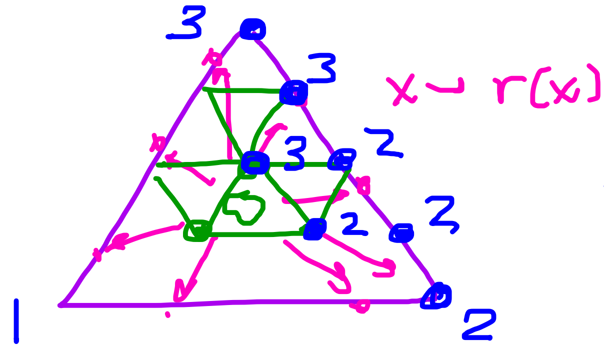
pf of BFPT



Suppose $f: B^n \to B^n$ contin but no fixed pt.

$r(x)$ is contin. in x, if f is.

Note: $r(b)=b$.



- Label by where $r(x)$ goes
- get Sp. labels

$\Rightarrow \exists$ 123 $\triangle$

nearby pts blown apart by $r(x)$.

Notice r is contin on cpt set, so unif contin

But: Let $\varepsilon = \frac{1}{2}$ size $\triangle$.

Know $\boxed{\exists\ \delta}$ s.t.

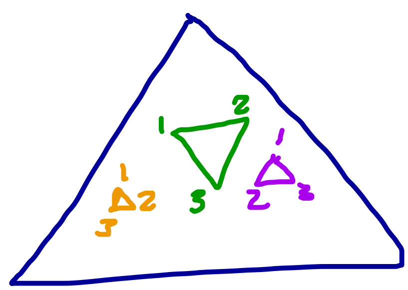
$|x-y|<\delta \Rightarrow$

$|f(x)-f(y)|<\varepsilon$.

Now $\triangle$'ate with $\triangle$'s size $<\delta$.

Then above labelling argument gives contradiction.

Construct'n Idea:



seq. of $\triangle$'s, cptness

$\Rightarrow \exists$ convergent subseq $\rightarrow$ fixed pt!