REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: CONTINUOUS FUNCTIONS

Recall: X, Y metric sp's. $p \in E \subset X$, $f: E \to Y$

Say f is <u>continuous</u> at p if

$$\boxed{\begin{array}{l} \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \\ \forall x \in E \qquad d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \epsilon \end{array}}$$

no restriction x can be p

In limit defn this was some q

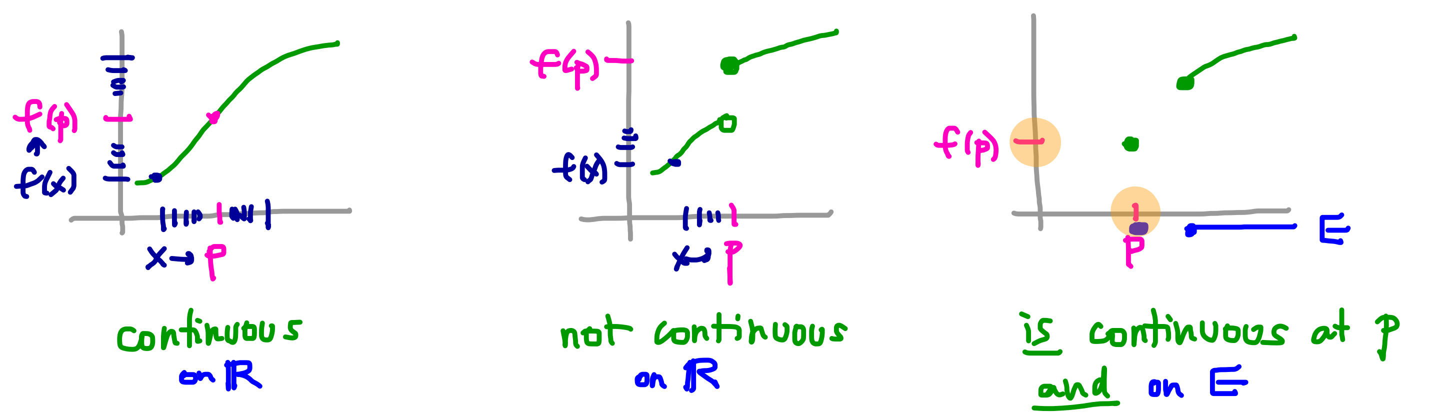
Compare w/ limit defn:

Says: "limit is what you expect"

- Say "f is continuous on E" if f is <u>contin.</u> at <u>all</u> pts of E.
- Continuity: "Close <u>enough</u> pts get mapped to <u>close</u> pts."

δ $\qquad$ ϵ-close

<u>TASK:</u> Find a δ for every ϵ.



- CONTINUOUS FUNCTIONS PRESERVE LIMITS

<u>Thm</u> If p is <u>lim pt</u> of E then

$$f \text{ is } \underline{\text{contin}} \text{ at } p \iff \boxed{\lim_{x\to p} f(x) = f(p)} .$$

$$\text{Hence: } f \text{ is } \underline{\text{contin}} \text{ on } E \iff \boxed{\begin{array}{l}\forall \text{ conv. seq } (x_n) \text{ in } E \\ \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)\end{array}}$$

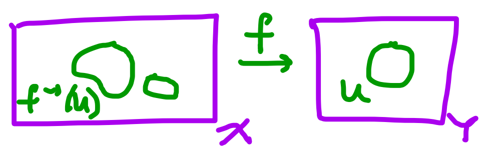
follows from limit defns.

<u>COR</u>. If $Y=\mathbb{R}$, sums/prods of <u>contin</u> fns are <u>contin</u> fns

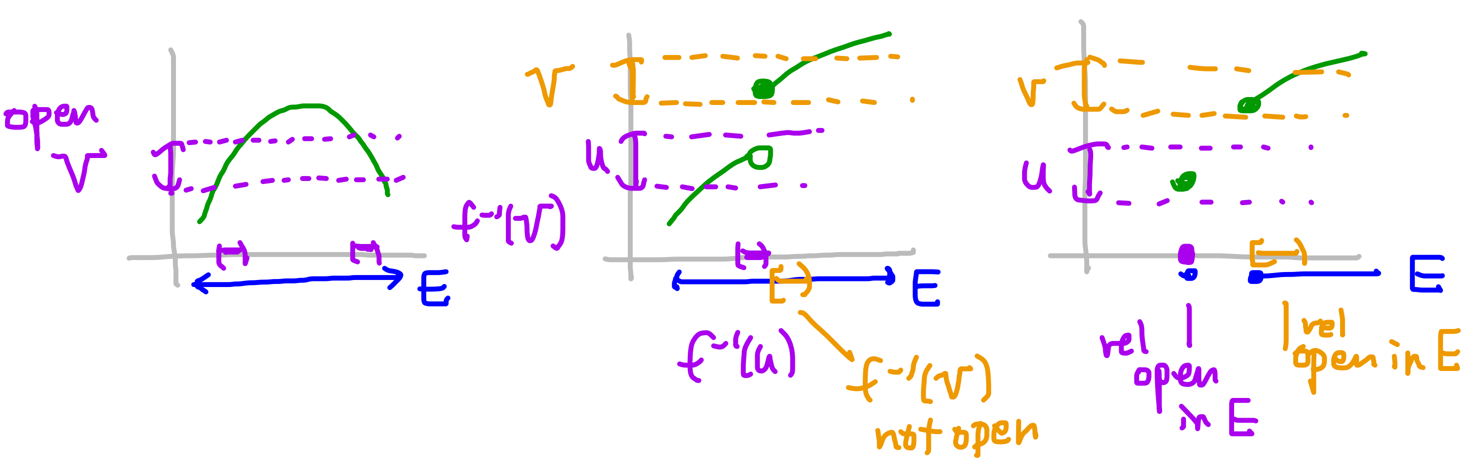
BOOK: If $Y=\mathbb{R}^k$ (vector version)

f contin $\iff$ all components f_i contin.

- Another characterization:



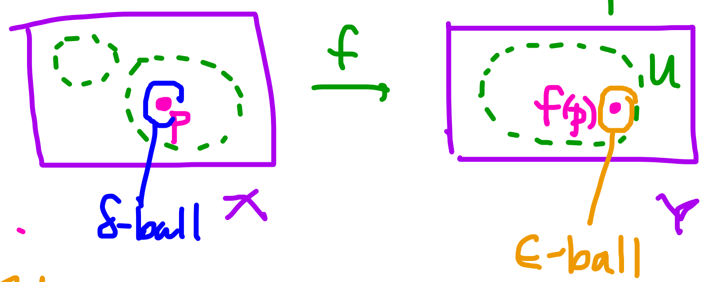
Recall $f^{-1}(U) = \{x : f(x) \in U\}$ the <u>inverse image</u> of U



Thm. $f: X \to Y$ contin $\iff$ $\boxed{\begin{array}{l}\forall \text{ open } \mathcal{U} \text{ in } Y \\ f^{-1}(\mathcal{U}) \text{ is open in } X\end{array}}$

"the inverse image of open sets is open"

proof idea: $(\Rightarrow)$



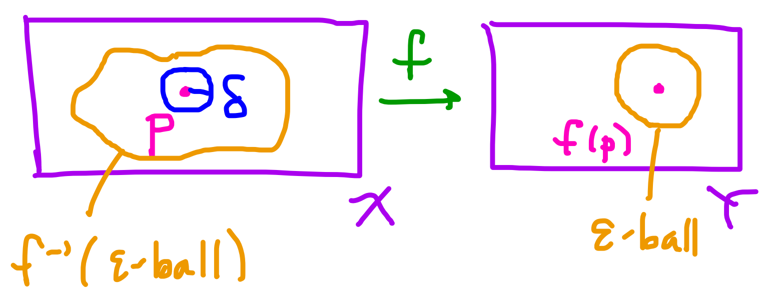
Pick $p \in f^{-1}(\mathcal{U})$,
we'll show it's interior pt.

Note $f(p)$ is interior to $\mathcal{U}$,
witnessed by some ε-ball.

By contin. of f, $\exists$ δ-ball whose image lies inside ε-ball,
which is inside $\mathcal{U}$.

This δ ball witnesses that p is interior to $f^{-1}(\mathcal{U})$.

$(\Leftarrow)$ Fix $p \in X$, $\varepsilon > 0$.



Let $\mathcal{U}$ be ε-ball around
$f(p)$; it's open.

By assumption, $f^{-1}(\mathcal{U})$ is open, so $\exists$ δ-ball around p
showing p is interior to $f^{-1}(\mathcal{U})$.

This is desired δ-ball that maps inside ε-ball,
shows f is contin at p. And p arbitrary. $\blacksquare$

Thm. $X \xrightarrow{f} Y \xrightarrow{g} \overset{\text{open } \mathcal{U}}{Z}$, f & g contin

$\Rightarrow g \circ f$ is contin.

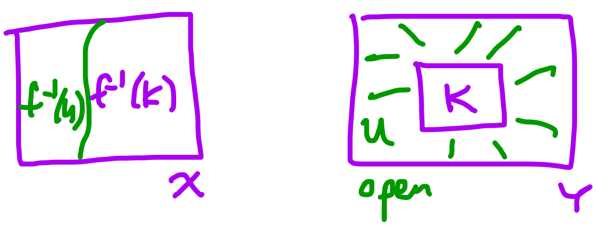
(book proof: uses ε-δ defn)

proof. Given $\overset{\text{open}}{\mathcal{U}}$ in Z, want to show $(g\circ f)^{-1}(\mathcal{U})$ is open in X.

But $(g\circ f)^{-1}(\mathcal{U}) = \underbrace{f^{-1}(\underbrace{g^{-1}(\mathcal{U})}_{\text{open by contin of } g})}_{\text{open by '' of } f}$. ▨

- **Thm.** $f: X \to Y$ contin $\Leftrightarrow$ $\boxed{\forall \text{ closed } K \text{ in } Y,\ f^{-1}(K) \text{ is closed in } X.}$

inv. images of closed sets is closed.

proof idea. 

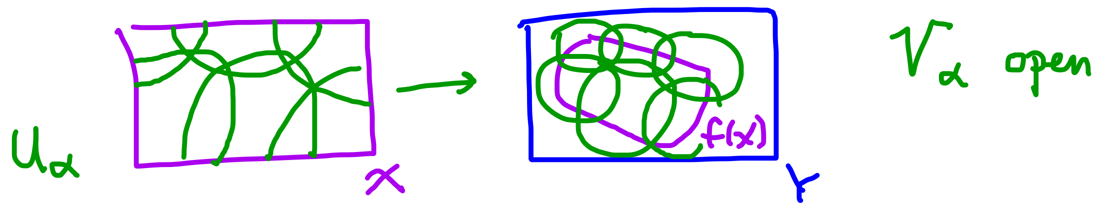
use:

$$f^{-1}(K) = [f^{-1}(\underbrace{K^c}_{\text{open}})]^c$$

- **CONTINUOUS FCNS PRESERVE COMPACTNESS**

<u>Thm</u>. $f: X \to Y$ <u>contin</u>, X cpt. $\Rightarrow$ $f(X)$ is <u>cpt</u>.

pf.



Say $\{V_\alpha\}$ is open cover of $f(X)$ in Y.

Let $U_\alpha = f^{-1}(V_\alpha)$ in X. Note it's <u>open cover</u> of X.

by <u>contin of</u> f ← (why?)

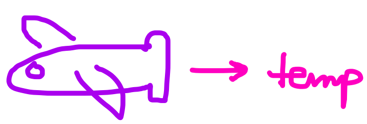
X <u>cpt</u>, so $\exists$ fin. subcover

$\{U_{\alpha_i}\}_{i=1}^n$ covering X.

Then $\{V_{\alpha_i}\}_{i=1}^n$ <u>covers</u> $f(X)$. ▨

(why?)

- <u>COR</u>. $f: X_{cpt} \to \mathbb{R}$ contin. Then $f(X)$ is <u>closed</u> & <u>bdd</u>.



<u>COR</u>. A <u>contin. fcn</u> $f: X \to \mathbb{R}$ on <u>cpt set</u>.

must achieve its <u>max</u> and <u>min</u>.