REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

PROF. FRANCIS SU

TODAY: FUNCTION LIMITS

SUMMATION BY PARTS

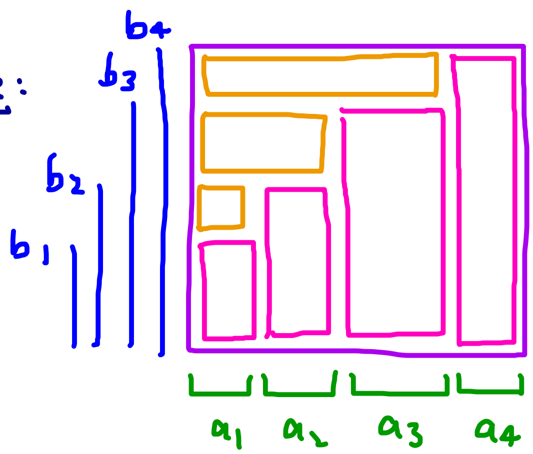
seqs (a_n) (b_n)

$$\sum_{n=p}^{q} a_n b_n = \left[\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) \right] + A_q b_q - A_{p-1} b_p$$

where $A_n = \sum_{k=0}^{n} a_k$ is partial sum, set $A_{-1} = 0$.

LIKE: $\displaystyle\int_a^b u\, dv = -\int_a^b v\, du + uv \Big|_a^b$.

OR:



proof idea:

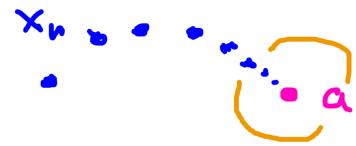
$$\sum a_n b_n = \sum (A_n - A_{n-1}) b_n$$
$$= \dots$$

Thm. A_n bdd, b_n decr $\to 0$ $\Rightarrow$ $\sum a_n b_n$ converges!

EX. $a_n = (\pm 1)^{n+1}$ $b_n = |c_n|$ $\Rightarrow$ $\sum c_n$ converges
(alt series test)

proof idea: use Cauchy criterion
sum by parts allows you to bound

LIMITS FOR FUNCTIONS



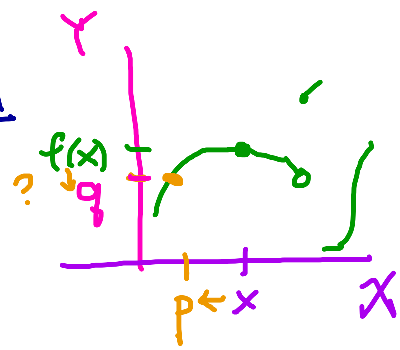
Recall: We defined $\lim_{n\to\infty} x_n = a$.

We want notion like: $\lim_{x\to p} f(x) = q$

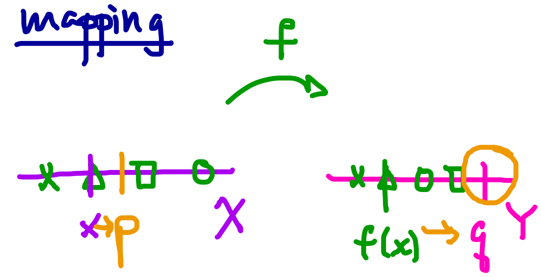
- Let X, Y be metric spaces, $f: X \to Y$ <u>function</u>:

visualize:

as <u>graph</u>



as <u>mapping</u>

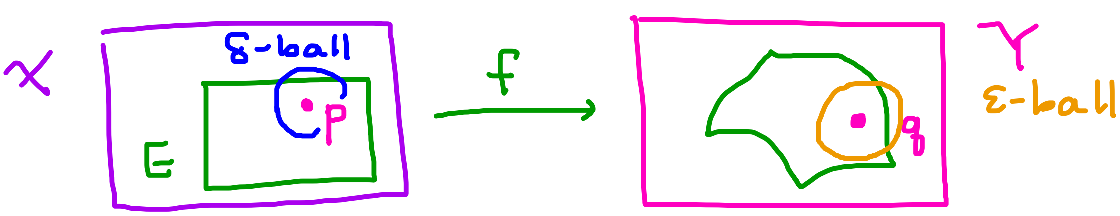


q. (a.) As $x \to p$ ε-ball

Will $f(x)$ get inside ε-ball & stay there?

Def'n. X, Y metric space, $E \subset X$

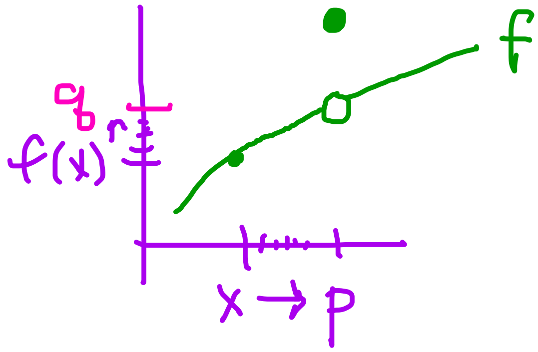
p is a <u>lim pt</u> of E, and $f: E \to Y$



Say "$f(x) \to q$ as $x \to p$" or "$\lim_{x\to p} f(x) = q$"

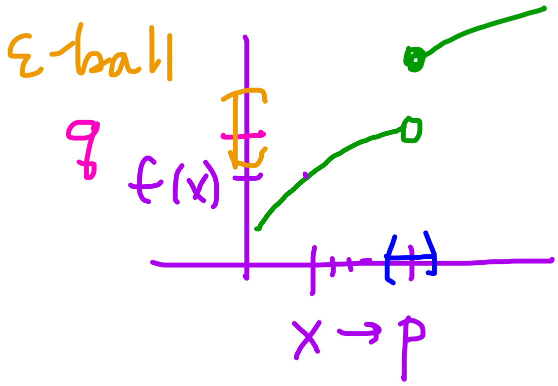
if $\exists\, q \in Y$ s.t.

$$\boxed{\forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } \forall x \in E \quad \text{if } \underset{\text{(exclude } x=p)}{0 <} d_X(x,p) < \delta \quad \text{then } d_Y(f(x), q) < \varepsilon.}$$



Here $f(p) \neq q$

but $\lim_{x \to p} f(x) = q$.



← This ε-ball shows q is not $\lim_{x \to p} f(x)$

No δ-ball works for this ε.

- Similarly you can show there's no q that could be $\lim_{x \to p} f(x)$.

- Note: for $f: \mathbb{R} \to \mathbb{R}$, we do have notion of left-hand, right-hand limits where domain E is restricted.

Say "$f(x) \to q$ as $x \to p$" or "$\lim_{x \to p} f(x) = q$"

if $\exists\, q \in Y$ s.t.

$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\forall x \in E$

if $0 < d_X(x,p) < \delta$ then $d_Y(f(x),q) < \varepsilon$.

(exclude $x=p$)

<u>Thm</u>. $\boxed{\lim_{x \to p} f(x) = q} \iff \boxed{\begin{array}{l} \forall \text{ seqs } \{p_n\} \text{ in } E \\ \quad \text{s.t. } p_n \neq p,\ p_n \to p \\ \text{we have } f(p_n) \to q. \end{array}}$

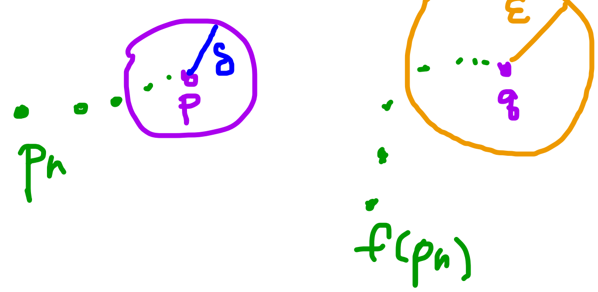
ε-δ def'n

<u>proof</u>. ($\Rightarrow$) $\forall \varepsilon > 0$ [must find N...]

Know $\exists \delta > 0$

s.t. $0 < d(x,p) < \delta$

$\overset{☺}{\Longrightarrow}$ $d(f(x),q) < \varepsilon$.



Given $p_n \to p$, $p_n \neq p$, by def'n:

we have N s.t. $n \geq N \Rightarrow d(p_n,p) < \delta$.

By ☺ for this N, (noting $p_n \neq p$)

then $n \geq N \Rightarrow d(f(p_n),q) < \varepsilon$, as <u>desired</u>.

($\Leftarrow$) Suppose $\lim_{x \to p} f(x) \neq q$, then [we'll show a bad seq...]

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0\ \exists x \in E$ for which

$0 < d(x,p) < \delta$ <u>but</u> $d(f(x),q) \geq \varepsilon$.

Find bad seq: Using ϵ, Let $\delta = 1/n$, get p_n

st. $0 < d(p_n, p) < \delta$ but

$d(f(p_n), q) \geq \epsilon$.

Do for all n, get $p_n \to p$ but

$f(p_n) \not\to q$ (b/c all images $\geq \epsilon$ away!)

$\blacksquare$

COR. From thms about seqs, see:

- $\lim_{x \to p} f(x)$ is unique (if exists)
- for real-valued fcns, sum of limits = limit of sum
  prods " = " prods
  etc.

CONTINUITY

Def'n X, Y metric space, $p \in E \subset X$, $f: E \to Y$

Say f is continuous at p if

$$\boxed{\begin{array}{l} \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t. } \forall x \in E \\ \text{if} \quad d(x,p) < \delta \quad \text{then} \quad d(f(x), f(p)) < \varepsilon \end{array}}$$