REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: SUBSEQUENCES

Recall: Say $p_n \to p$ or $\lim_{n\to\infty} p_n = p$ if

$$\exists\, p \in X \text{ s.t. } \forall \varepsilon > 0\ \exists N \text{ s.t. } n \geq N \Rightarrow d(p_n, p) < \varepsilon.$$

SEQS in $\mathbb{R}$ or $\mathbb{C}$:

Thm. $\{s_n\}, \{t_n\} \in \mathbb{C}$, $s_n \to s$, $t_n \to t$.

Then (a) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

(b) $\lim_{n\to\infty} c s_n = cs$, $\lim_{n\to\infty} (s_n + c) = s + c$.

(c) $\lim_{n\to\infty} s_n t_n = st$.

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ (as long as $s_n \neq 0, s \neq 0$).

IMPORTANT: to show convergence, given ε, you must find an N.

proof (a). IDEA: bound $|(s_n+t_n) - (s+t)| = |(s_n - s) + (t_n - t)|$

scratchwork (ineq) $\to \leq |s_n - s| + |t_n - t|$

Given $\varepsilon > 0$, $\exists N_1, N_2$ s.t.

$n \geq N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$, (by conv of $s_n \to s$)

$n \geq N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$. (" " $t_n \to t$)

Then let $N = \max\{N_1, N_2\}$. For this N,

$n \geq N \Rightarrow |(s_n + t_n) - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

as desired. ∎

(b) IDEA: bound $|cs_n - cs| = |c|\,|s_n - s|$. Given $\varepsilon > 0$, $\exists N_1$ s.t. $|s_n - s| < \frac{\varepsilon}{|c|}$ ...

... let $N = N_1$ ...

(c) IDEA: bound $|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$

(d) use $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| \dots$

SUBSEQUENCES

$\{p_n\}$ a sequence. Let $n_1 < n_2 < \cdots$ increasing set of indices

then

$\{p_{n_1}, p_{n_2}, p_{n_3}, \dots\}$ is a subsequence $\{p_{n_i}\}_{i=1}^{\infty}$.

EX. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$

subseq: $p_2 \quad p_3 \quad p_5 \dots$ (here $n_1 = 2$, $n_2 = 3$, etc.)

Q. If a seq conv's must every subseq converge?

YES. If $p_n \to p$ by (F) every nbhd of p contains all but fin many p_n

EX. $\{1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \dots\}$ does not conv.

has conv. subseqs.

say "0 is a subsequential limit".

Q. Must every seq. have conv. subseq?

NO. $1, 2, 3, 4, 5, \dots$

Thm. In a compact metric space X any sequence has a conv. subseq. [converging to a point of X].

Cor. (Bolzano-Weierstrass) Every bdd seq in $\mathbb{R}^k$ has a conv. subsequence.