REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: THE COMPLEX NUMBER FIELD

- <u>EXTENDED REALS</u> $\mathbb{R} \cup \{-\infty, \infty\} \overset{\text{def}}{=} \overline{\mathbb{R}}$

  <u>put</u> $-\infty < x < +\infty$ for <u>all</u> $x \in \mathbb{R}$

<u>Define</u> $x + \infty = +\infty$

$x - \infty = -\infty$, etc.

$(+\infty) + (-\infty)$ = not def'd

NOT A FIELD

Why?

Now <u>every</u> subset of $\overline{\mathbb{R}}$ has a sup! (possibly $+\infty$).

$\sup \mathbb{R}_+ = +\infty$.

- <u>EUCLIDEAN SPACE</u> $\mathbb{R}^k \overset{\text{def}}{=} \{(x_1, \dots, x_k) : x_i \in \mathbb{R}\}$.

define: $\overbrace{(x_1, \dots, x_k)}^{\vec{x}} \underset{\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k}{+} \overbrace{(y_1, \dots, y_k)}^{\vec{y}} = \overbrace{(x_1+y_1, \dots, x_k+y_k)}^{\vec{x}+\vec{y}}$.

No nice multiplication that's meaningful

but we do have <u>scalar</u> mult:

$a(x_1, \dots, x_k) = (ax_1, \dots, ax_k)$.

Obtain a <u>vector space</u> (check: comm, assoc, dist)

- $\mathbb{R}^k$ has an <u>inner product</u> $\vec{x} \cdot \vec{y} = \overbrace{\sum_{i=1}^{k} x_i y_i}^{\text{real \#}}$.
  - defines what's $\perp$
  - defines <u>length</u> / <u>norm</u> $|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2}$.

COMPLEX NUMBERS

$\mathbb{R}^2$ can be given field structure:

$$(a,b)+(c,d) = (a+c,\ b+d).$$

$$(a,b)\times(c,d) = (ac-bd\ ,\ ad+bc).$$

With this structure $(\mathbb{R}^2, +, \times)$ is called $\mathbb{C}$.

Thm. $\mathbb{C}$ is field.

e.g. check $(0,0)$ is $+$ identity.

$(1,0)$ is $\times$ " .

- Subset $\{(a,0) : a \in \mathbb{R}\}$ "behaves like" $\mathbb{R}$. ← "isomorphic"

  $\mathbb{C}$ contains $\mathbb{R}$ as a subfield

  via $f : \mathbb{R} \to \mathbb{C}$.

  $a \mapsto (a,0)$

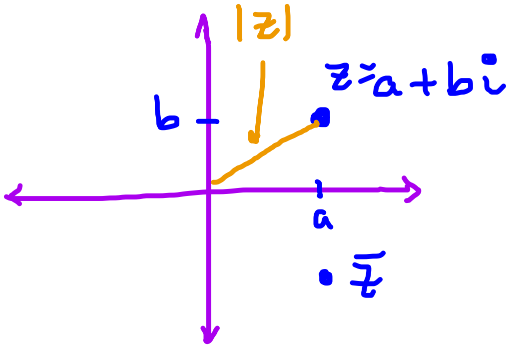
- Observe: $\underset{``\sqrt{-1}"}{(0,1)} \times \underset{``\sqrt{-1}"}{(0,1)} = \underset{``-1"}{(-1,0)}$

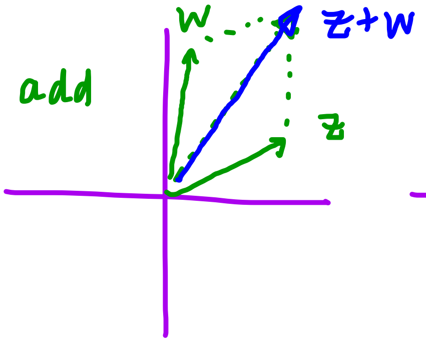
  ↑ call i. So $i^2 = -1$. ☺

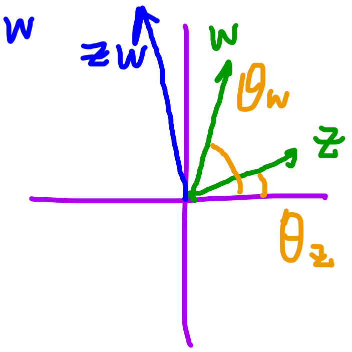
  Thus $x^2+1=0$ has a sol'n in $\mathbb{C}$ but not in $\mathbb{R}$.

- Usually write (a,b) as $\underset{\text{real part}}{a} + \underset{\text{imag. part}}{b}\,i$, a "complex" #.

- $\mathbb{C}$ is algebraically closed: every non-const poly has root!







zw has angle $\theta_w + \theta_z$ & length $|z||w|$.

- If $z = \underset{\mathrm{Re}(z)}{a} + \underset{\mathrm{Im}(z)}{b}\, i$, the <u>conjugate</u> is $\bar z = a - bi$.

<u>Check:</u>

$$z + \bar z = 2\,\mathrm{Re}(z)$$

$$\left.\begin{aligned} \overline{z+w} &= \bar z + \bar w . \\ \overline{zw} &= \bar z\, \bar w \end{aligned}\right]$$ preserves +, ×

$$z \bar z = a^2 + b^2$$ ← always ≥ 0.

- <u>Define:</u> $|z| = (z\bar z)^{1/2}$ the <u>absolute value</u> of z.

↑ same: length of <u>vector</u> (a,b).

Check: $|z| \geq 0$.

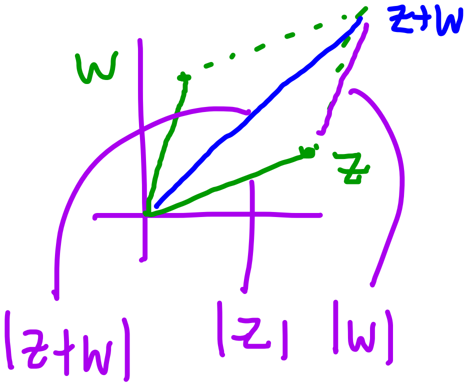
$$|\bar z| = |z|$$

$$|zw| = |z||w|$$ ← uses: $(ac-bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2)$.

Also $\mathrm{Re}(z) \leq |z|$.

- The <u>triangle inequality</u> holds:

$$|z+w| \le |z| + |w|.$$



<u>proof</u>.
$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2 \\
&\le |z|^2 + 2\,|z\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z|+|w|)^2 .
\end{aligned}$$

Taking sq. roots both sides yields desired ineq. ▨

- <u>Define</u> $\mathbb{C}^n = \{ (z_1, \dots, z_n) : z_i \in \mathbb{C} \}$.

Has <u>inner product</u>: for $\vec{a}, \vec{b} \in \mathbb{C}^n$

$$\langle \vec{a}, \vec{b} \rangle = \sum_{j=1}^{n} a_j \overline{b_j}$$

$\uparrow$

ensures $\langle \vec{a}, \vec{a} \rangle$ is real ≥ 0.

So can define $|\vec{a}| = \langle \vec{a}, \vec{a} \rangle^{1/2}$.

- The Cauchy-Schwarz ineq says:

For $\vec{a}, \vec{b} \in \mathbb{C}^n$, $\quad |\langle \vec{a}, \vec{b} \rangle|^2 \le \langle \vec{a}, \vec{a} \rangle \langle \vec{b}, \vec{b} \rangle$.

OR:

$$\boxed{\left| \sum a_j \bar{b}_j \right|^2 \le \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2}$$

basis of Heisenberg uncertainty

proof (#1) for $\mathbb{R}^n$, consider $p(x) = \sum (a_i - x b_i)^2 \ge 0$.

so discriminant $D \le 0$.

but $D = \underbrace{4\left(\sum a_i b_i\right)^2}_{B^2} \underset{-}{-} \underbrace{4 \sum a_i^2 \sum b_i^2}_{4AC}$

≤ 0 gives desired ineq.

This motivates:

proof (#2) for $\mathbb{C}^n$, if $\vec{b} = \vec{0}$ done.

Else for any $x \in \mathbb{C}$:

$$0 \le |\vec{a} - x\vec{b}|^2 = \langle \vec{a} - x\vec{b}, \vec{a} - x\vec{b} \rangle$$

$$= \langle a, a \rangle - \bar{x} \langle a, b \rangle - x \langle b, a \rangle + x\bar{x} \langle b, b \rangle.$$

Now put $x = \dfrac{\langle a, b \rangle}{\langle b, b \rangle}$. Get $0 \le \langle a, a \rangle - \dfrac{|\langle a, b \rangle|^2}{\langle b, b \rangle}$ as desired. ☒

Equality holds only when $\vec{a} = x\vec{b}$ for some x.