REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

PROF. FRANCIS SU

TODAY: SERIES TESTS

EX. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

Thm (Cauchy) If $a_1 \geq a_2 \geq \dots \geq 0$ (non-decr., non-neg)

then
$$\sum_n a_n \text{ converges} \iff \sum_k 2^k a_{2^k} \text{ converges}$$
$$= a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

proof idea: Compare

$s_n = a_1 + \dots + a_n \quad = a_1 + (a_2 + a_3) + (a_4 + \dots + a_7) + \dots + a_n$

$t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k} = a_1 + (a_2 + a_2) + (a_4 + \dots + a_4) + \dots + a_{2^k}$

If $n < 2^k$, then $s_n \leq t_k$. Shows if t_k conv, then s_n does.

If $n > 2^k$, then $t_k \leq 2s_n$. Show if s_n conv, then t_k does.

compare $2a_1 + 2a_2 + 2(a_3 + a_4) + \dots$

$a_1 + 2a_2 + 4a_4 + \dots$

Application. Thm. $\sum \frac{1}{n^p}$ conv if $p > 1$, div if $p \leq 1$.

proof. If $p \leq 0$, terms $\not\to 0$, so series div's.

If $p > 0$, use $\sum_k 2^k \frac{1}{2^{kp}} = \sum_k 2^{(1-p)k}$ geometric

conv iff $2^{(1-p)} < 1$

iff $1 - p < 0 \iff p > 1$.

as desired. $\square$

BOOK: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ etc...

H.S. use lim if exists

Thm. (Root test).

Given $\sum a_n$, let $\alpha = \limsup \sqrt[n]{|a_n|}$

then $\alpha < 1 \Rightarrow$ series conv.

$\alpha > 1 \Rightarrow$ " div.

$\alpha = 1 \Rightarrow$ test inconclusive.

proof. by comparison w/ geom series!

- If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$.

By def'n of lim sup $\exists N$ s.t. $n \geq N \Rightarrow \sqrt[n]{|a_n|} < \beta$.

$\Rightarrow |a_n| < \beta^n$.

But $\sum \beta^n$ converges, so $\sum a_n$ does. ↑ condition for comparison test

- If $\alpha > 1$, $\exists$ subseq $\sqrt[n_k]{|a_{n_k}|} \longrightarrow \alpha > 1$.

So $|a_{n_k}| > 1$ for ∞ many terms, so terms $\not\to 0$.

So $\sum a_n$ div's.

- If $\alpha = 1$, notice: $\sum 1$ (div's) $\sum \frac{1}{n}$ (div's) $\sum \frac{1}{n^2}$ (conv's) ← all $\alpha = 1$

Thm. (Ratio Test)

(a) $\sum a_n$ conv's if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(b) $\sum a_n$ div's if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for $n \geq$ some N_0.

proof. (compare w/ geom series)

(a) $\exists \beta$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$ for $n \geq$ some N.

So $a_{N+k} < \beta a_{N+k-1} < \cdots < \beta^k a_N$.

Compare: $\sum_{k=0}^{\infty} a_{N+k} < a_N \sum_{k=0}^{\infty} \beta^k$. ← converges so tail of $\sum a_n$ conv's $\Rightarrow \sum a_n$ conv's.

(b) terms $\not\to 0$, so series div's. $\boxtimes$

BOOK: root test more powerful, ratio test easier.

POWER SERIES

c_n cplx, then $\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \ldots$

is a power series in z, a cplx variable.

EX $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

Q When does a power series converge?

Thm. Let $\alpha = \limsup \sqrt[n]{|c_n|}$. Let $R = \frac{1}{\alpha}$ $\begin{pmatrix} = \infty \text{ if } \alpha = 0 \\ = 0 \text{ if } \alpha = \infty \end{pmatrix}$.

Then if $|z| < R$ then $\sum c_n z^n$ converges!
if $|z| > R$ " " diverges!

surprise: there's a radius R of convergence!

pf idea. use root test:

$$\limsup \sqrt[n]{|a_n|} = |z| \cdot \limsup \sqrt[n]{|c_n|} \overset{?}{\underset{\text{when}}{<}} 1$$

Ex. For $\sin(x)$ power series,

check $R = \infty$. series always conv's.

- ABSOLUTE CONVERGENCE

Def'n. $\sum a_n$ conv's absolutely means: $\sum |a_n|$ conv's.

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ does not converge absolutely,
but it does converge (to $\ln 2$).

But:

Thm. If $\sum a_n$ converges absolutely, then it converges.

pf idea: $\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon$ CAN MAKE for $n, m \geq$ some N

Cauchy sum for $\sum a_k$ — Cauchy sum for $\sum |a_k|$

- PRODUCTS OF POWER SERIES

$$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots)$$
$$= \sum c_n z^n \quad \text{where}$$
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$
a finite sum.

$\sum c_n$ may not converge,

but

Thm. If $\underset{A}{\sum a_n}$, $\underset{B}{\sum b_n}$ converge absolutely, then $\sum c_n$ converges and to AB

REARRANGEMENTS:

If I rearrange terms $\sum a_n = A$, must it converge to A?

$\boxed{\text{ANS.}}$ NO!

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2.$$

BUT:

$$\underbrace{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2k+1}}_{\text{just pass } \pi} \underbrace{- \frac{1}{2}}_{\text{just under } \pi} + \underbrace{\qquad\qquad}_{\text{pass } \pi} \quad \text{etc.}$$

will converge to π.

Riemann: If $\sum a_n$ conv's but not absolutely,
we can form rearrangement
that has any lim sup, lim inf you like!

If $\sum a_n$ conv. absolutely, every rearrangement same sum!