REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: UNCOUNTABLE SETS, METRIC SPACES

Last time: A ctble set: can be put into some bijection with $\mathbb{N}$.
An unctble set cannot.
Saw: $\mathbb{Q}$ is ctble.

Thm. $\mathbb{R}$ is uncountable. (Cantor 1874)
"Cantor's diagonal argument" ← this proof, 1891

proof. Enough to show $[0,1)$ unctble.
Suppose, by way of contradiction, that $\mathbb{R}$ is ctble.
Then it can be "listed" (1-1 corr. with $\mathbb{N}$)

$1 \to 0.14159\ldots$
$2 \to 0.26535\ldots$
$3 \to 0.89793\ldots$
$4 \to 0.23846\ldots$

I'll show you some # not on your list!
By: construct # that differs from
i^{th} # in the i^{th} decimal place!

e.g. $x = 0.7717\ldots$ ←

See: x cannot be in the image of this proposed bijection.
Since this construction works for any proposed bijection,
then no such bijection exists! ▨

• Given set A, let 2^A denote the power set of A,
the set of all subsets of A.

Ex. $A = \{ ☺, \square, \triangle \}$ then 2^A has 2^3 elements.

subset $D = \{ ☺, \triangle \} \longleftrightarrow$ binary 3-tuple: 1 0 1

$E = \{ \triangle \} \longleftrightarrow$ 0 0 1

Thm (Cantor, 1891) For any set A, we have $A \not\sim 2^A$.

proof. ← THINK ABOUT IT! (next time)

Def'n. The set A and B have same cardinality (size) if $A \sim B$.

So These sets have different cardinalities:

$$\mathbb{N}, \quad 2^{\mathbb{N}}, \quad 2^{2^{\mathbb{N}}}, \quad \ldots$$

$$2^{\mathbb{N}} \sim \mathbb{R} \sim \{f : \mathbb{N} \to \{0,1\}\}$$

$$2^{2^{\mathbb{N}}} \sim 2^{\mathbb{R}} \sim \{f : \mathbb{R} \to \{0,1\}\} \sim \{f : \mathbb{R} \to \mathbb{R}\}$$ not necess. continuous

Schröder-Bernstein Thm (extra credit!)

If A, B are sets s.t. $\exists\, f : A \hookrightarrow B$ & $\exists\, g : B \hookrightarrow A$ then $A \sim B$.

(To prove, use f, g to construct a bijection $h : A \hookrightarrow\!\!\!\!\!\rightarrow B$.)

Cardinal numbers:

aleph-null

$0, 1, 2, 3, \ldots \aleph_0, \aleph_1, \aleph_2, \ldots \aleph_\alpha$

card of $\mathbb{N}$

which is card of $\mathbb{R}$ ("the continuum")?

indexed by "ordinal" #'s

$\boxed{\text{The Continuum Hypothesis: } \aleph_1 = c, \text{ the card of } \mathbb{R}.}$

indep of axioms of set theory!

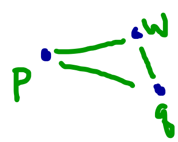
METRIC SPACES

(Q.) How to define "distance"? – between pts, functions, genome sequences

Def'n. A set X is a <u>metric space</u> if $\exists$ a <u>metric</u> $d: X \times X \to \mathbb{R}$

s.t. $\forall\, p, q \in X$ (a) $d(p,q) \geq 0$ ($=$ <u>iff</u> $p=q$)

symmetric → (b) $d(p,q) = d(q,p)$



triangle ineq. → (c) $d(p,q) \leq d(p,w) + d(w,q)$ $\forall w \in X$.

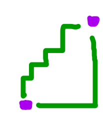
Say: "(X, d) is a <u>metric space</u>"

<u>EX</u>. $\mathbb{R}$ with $d(x,y) = |x-y|$.

<u>EX</u>. $\mathbb{R}^n$ with $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$ with $\|\vec{w}\| = \sqrt{w_1^2 + \dots + w_n^2}$.

↑ Euclidean metric

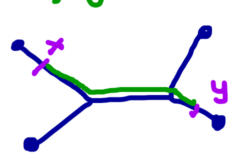
<u>EX</u>. $\mathbb{R}^n$, $d(\vec{x},\vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ ← the staircase metric or "L^1 metric"



<u>EX</u>. $\mathbb{R}^n$ with L^p-metric: $d(x,y) = \left(\sum |x_i - y_i|^p\right)^{1/p}$.

(If $p=1$, get staircase; if $p=2$, get Euclidean.)

<u>EX</u>. X = tree



$d(x,y)$ = shortest path x to y

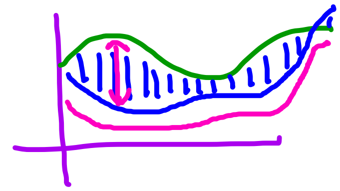
<u>EX</u>. $X = \{$genome seqs.$\}$ $d(x,y) = \#$ letters differ

GATTACA
AGATCGA
$d=5$

<u>Ex</u>. any set X, $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.

the discrete metric

<u>Ex</u>. $X = \{\text{contin functions}: \mathbb{R} \to \mathbb{R}\}$



L^1-metric: $d(f,g) = \int_{\mathbb{R}} |f(x) - g(x)|\, dx$ ← restrict space further so

sup metric: $d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$.

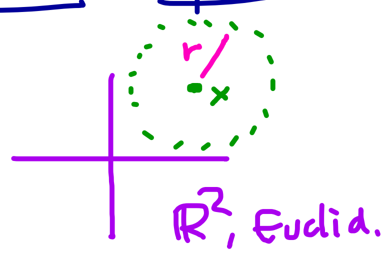
these aren't ∞

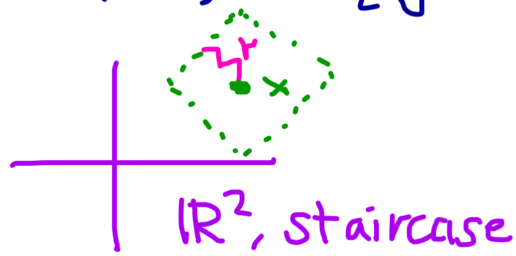
<u>Ex</u>. $\mathbb{R}^n$ with <u>entropy</u> distance $d(\vec{p}, \vec{q}) = \sum p_i \log \frac{p_i}{q_i}$

not metric

• The topology of $\mathbb{R}^n$: what's close?

Def'n. Open "nbhd" ball $N_r(x) = B(x,r) = \{ y : d(x,y) < r \}$.



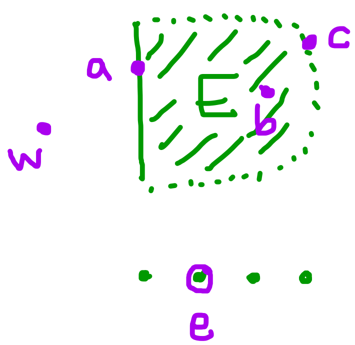


Closed ball $\overline{B(x,r)} = \{ y : d(x,y) \le r \}$.

MORAL: Balls tell us what's "close".

Def'n A point $p \in X$ is a limit pt of E if every nbhd of p contains a point $q \in E$ with $q \neq p$.

any $B(p,r)$



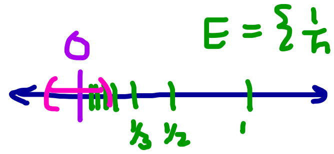
Ex. a, b, c are l.p. of E.

w, e are not l.p. of E.

even though $e \in E$.

Ex. $\mathbb{R}$

$E = \{ \frac{1}{n} : n \in \mathbb{Z} \}$



nbhd of 0 is $(-r, r)$

so 0 is l.p. of E.

Ex. ($\mathbb{R}$, discrete) ↑ Is 0 a l.p of E? $\boxed{\text{NO}}$. b/c $B(0, \frac{1}{2})$ contains no other pts.