REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: SETS & RELATIONS

IN MATH OR LIFE: THE STRUGGLE IS WORTHY & WORTHWHILE

"Every time that a human being succeeds in making
an effort of attention with the sole idea
of increasing his grasp of truth,
he acquires a greater aptitude for grasping it
even if his effort produces no visible fruit."

— SIMONE WEIL (1951)

WHY DO MATH?

"It is impossible to be a mathematician
without being a poet in soul."

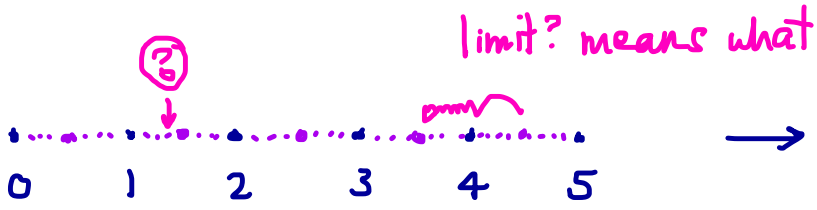
—SONIA KOVALEVSKY (1895)

WHAT ARE THE REAL NUMBERS?

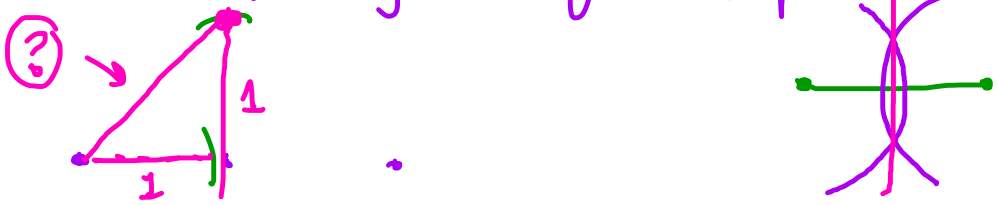
"God created the integers.
All else is the work of man."

— LEOPOLD KRONECKER (1886)

WHAT IS THE REAL LINE?



Greeks: constructible #'s by straightedge & compass



- Constructible #'s are <u>algebraic</u>: root of polynomial w/ coeffs in $\mathbb{Z}$.

  ex. $x^2 - 2 = 0$ has root $\sqrt{2}$.

- Not all #'s are <u>algebraic</u>: those are <u>transcendental</u>.

- Can you <u>construct</u> π? ("squaring the circle")
- Lindemann (1882): π is <u>transcendental</u>.

- limits: for calculus in late 1600's - Newton, Leibniz <u>vague</u>

  1820's - Cauchy made precise

  WE WANT TO CONSTRUCT REAL #'s CAREFULLY

  1858 - Dedekind constructs real #'s w/o idea of <u>limit</u>

SET AND RELATIONS

- A set is "collection" of objects, such as:

$B = \{ 1, \bigstar, \ddot{\smile}, \{\Delta, \pi\} \}$

or $\{ x : P(x) \text{ is true} \}$.

$P(x)$ ↑ a statement about x
e.g., "x is less than 1"

B has 4 elements.

Shorthand:

$x \in S$ means "x is in S".
$x \notin S$ " " " not in "
$\emptyset$ " empty set $\{\}$.

- If $x \in A \Rightarrow x \in B$ then write: $A \subset B$.
  ($\Rightarrow$ ↑ implies; $\subset$ ↑ "is a subset")
- If $A \subset B$ but $B \not\subset A$ then A is proper
  ($\not\subset$ ↑ not a subset)
- If $A \subset B$ and $B \subset A$ then write $A = B$, else $A \neq B$.

- More sets:

union $A \cup B = \{ x : x \in A \text{ or } x \in B \}$

intersection $A \cap B = \{$ " AND " $\}$

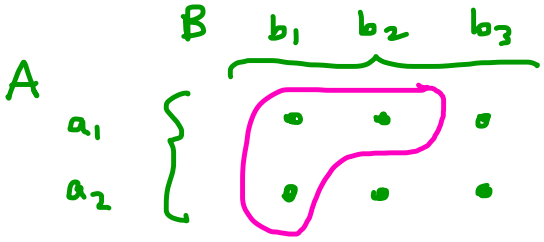
complement of A in B: A^c (if B is whole universe)

$B \setminus A = \{ x : x \in B \text{ and } x \notin A \}$.

↑ setminus

product $A \times B = \{ (a,b) : a \in A \text{ & } b \in B \}$.

↑ ordered pair



- A (binary) relation R is a subset of $A \times B$

If $(a,b) \in R$ write: $a \, R \, b$

Ex. L "loves" is a rel'n on $P \times P$. ← people

A "ancestor of" " $P \times P$.

S "sibling of" " $P \times P$.

K "likes" " $P \times F$.

$<$ "less than" $\mathbb{Z} \times \mathbb{Z}$. ↑ food.

Some important relations:

- **Defn.** An **equivalence rel'n** on a **set** S is a **rel'n** R on $S \times S$

  ① aRa — reflexive

  ② $aRb \Rightarrow bRa$ — symmetric

  ③ $aRb \;\&\; bRc \Rightarrow aRc$ — transitive

  for all $a, b, c \in S$.

  Often write $\sim$ or $\approx$ or $\cong$.

Ex. "is related to", "in same class mod 5"

- A **function** F from A to B is a **relation** $\vdash$ on A s.t. (such that)

  if aFb & aFb' then $b = b'$.

  Write: $F(a) = b$ for aFb.

  ↑ rule assigns to each a unique b.