REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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TODAY: COMPACT SETS

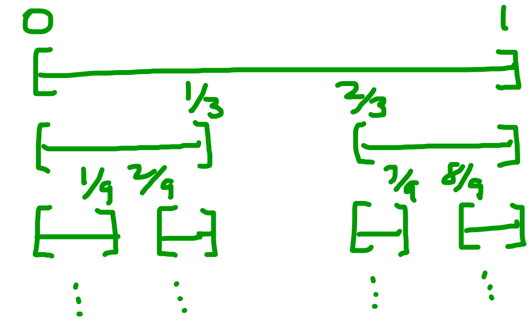
Ex. Cantor set.

A_0

A_1

A_2

$\vdots$



Let $C = \bigcap_{i=1}^{\infty} A_i$.

called the standard Cantor set.

C is closed, perfect, uncountable, "totally disconnected"

closed and every pt is lim. pt.

It's "compact."

- COMPACT SETS

"next best thing to being finite."

Finite sets nice: contain their sup

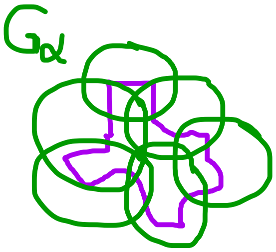
bdd

closed

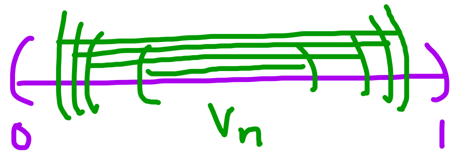
processes on such sets "end"

CPT. SETS too!

• **Def'n.** A (open) cover of E in X is a collection $\{G_\alpha\}$ of open sets whose union contains E.



EX.



boring: $\{(-1,2)\}$ covers $(0,1)$.

interesting: $V_n = (\frac{1}{n}, 1-\frac{1}{n})$, $n \geq 3$.

$\{V_n\}_{n=3}^{\infty}$ covers $(0,1)$, too.

Did I need them all to cover $(0,1)$?

NO. but need ∞ many.

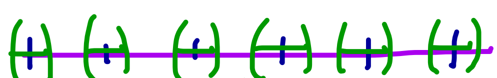
• A subcover of cover is a subcollection that still covers E.

• **Def'n.** A set K in X is compact (in X) if every open cover of K has a finite subcover.

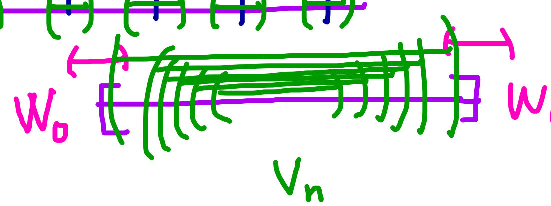
EX. $(0,1)$ is not cpt, b/c $\{V_n\}$ is cover w/no fin. subcover.

EX. $\mathbb{Z}$ in $\mathbb{R}$.

not cpt: see

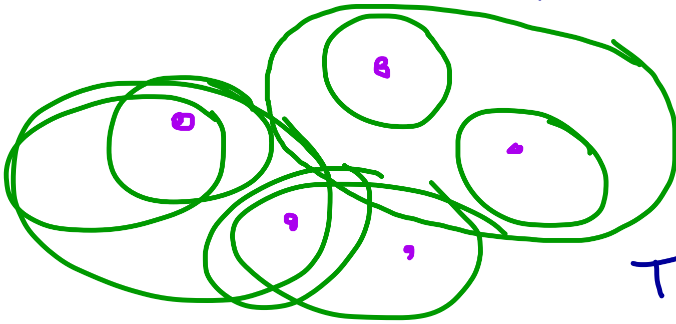


EX. $[0,1]$ is compact.



one example won't suffice, to show it!

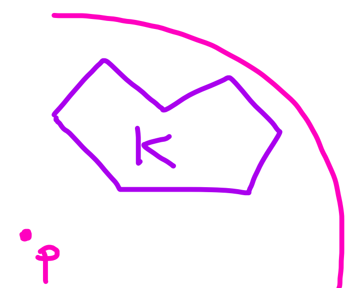
EX. Finite sets cpt? YES! Given $\{G_\alpha\}$ covering $x_1, \ldots, x_N$, we'll choose one G_{α_i} that contains x_i.



Then $\{G_{\alpha_i}\}_{i=1}^{N}$ is a finite subcover of $\{G_\alpha\}$.

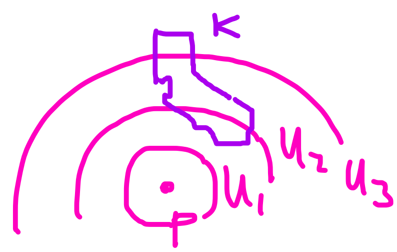
WARNING: Showing set is compact $\neq$ showing it has finite cover. (every set does!)

Def'n. Set K is bounded if $\exists M \in \mathbb{R}, p \in X$ s.t. $K \subset N_M(p)$, some ball.



Thm. K cpt in $X \Rightarrow K$ bdd.

proof. Let $U_n = N_n(p)$, fix p.



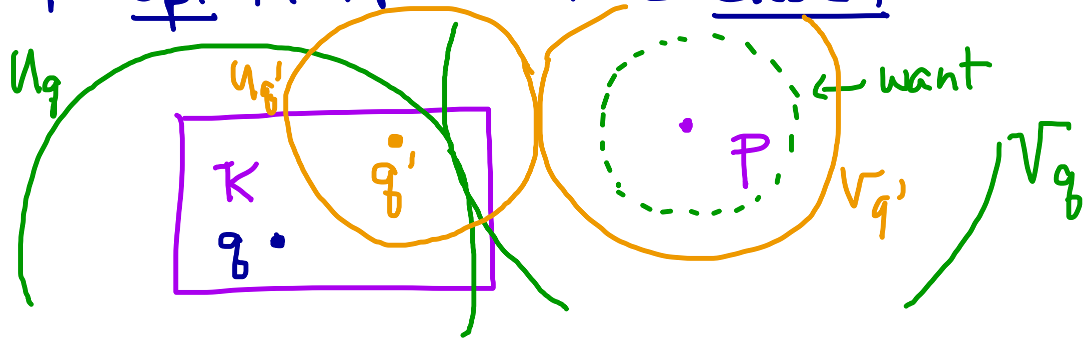
Now $\{U_n\}_{n=1}^{\infty}$ is a cover.

B/c K cpt. $\Rightarrow \exists$ fin. subcover $U_{n_1} \subset \ldots \subset U_{n_M}$

so U_{n_M} contains $K \Rightarrow K$ bdd. ▨

Thm. K cpt in $X \Rightarrow K$ is closed.

proof.



If $p \notin K$, we'll show p is not a lim pt of K.

$\forall q \in K$, let $U_q = B_{r/2}(q)$, $V_q = B_{r/2}(p)$ for $r = d(p,q)$.

Then $\{U_q\}_{q \in K}$ is open cover of K.

K cpt $\Rightarrow \exists$ fin. subcover $\{U_{q_i}\}_{i=1}^{N}$.

Let $V = \bigcap_{i=1}^{N} \{V_{q_i}\}$. Claim: $V \cap K = \emptyset$.

Why? If $s \in V \Rightarrow s \in V_{q_i} \Rightarrow s \notin U_{q_i} \ \forall q_i$.
So $s \notin K \subset \bigcup_{i=1}^{N} U_{q_i}$, as desired. $\blacksquare$