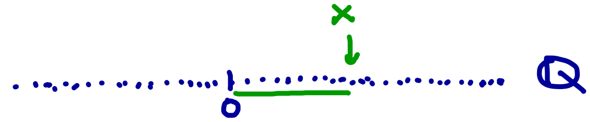
REAL ANALYSIS

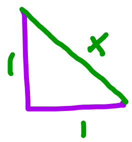
MATH 131, HARVEY MUDD COLLEGE

PROF. FRANCIS SU

TODAY: CONSTRUCTION OF $\mathbb{R}$

Recall: $\mathbb{Q}$ has "holes":





$x^2 = 2$ no sol'n in $\mathbb{Q}$

CONSTRUCTION OF $\mathbb{R}$ → Dedekind cuts 1872
→ Cauchy sequences

• LEAST UPPER BOUND

Def'n Say $E \subset S$ ordered.

there exists ... such that for all

If $\exists\ \beta \in S$ s.t. $\forall\ x \in E$ we have $x \le \beta$

then say E is bounded above, call β an upper bound of E.

[define lower bd similarly.]

EX. $\mathbb{Q}_- =$ neg rat'ls $\subset \mathbb{Q}$. Has upper bd. 17 & many others.

Def'n. If $\exists\ \alpha \in S$ s.t.

1. α is an u.b. of E
2. If $\gamma < \alpha$ then γ is not u.b. of E.

Then α is called a least upper bd (lub) of E

or the supremum of E. Write: $\boxed{\alpha = \sup E}$.

EX. $S = \mathbb{Q}$. $\sup\{\frac{1}{2}, 1, 2\} = 2$. $\sup\{r : r < 2\} = 2$.

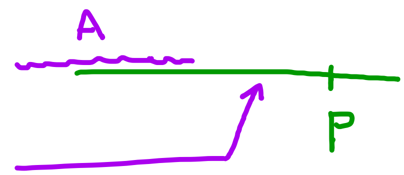
$\sup \mathbb{Q}_- = 0$. $\sup\{r : r^2 < 2\}$ DOES NOT EXIST

Know: $r^2=2$ has no sol'n in $\mathbb{Q}$

Why does $A = \{r : r^2 < 2\}$ have no l.u.b.?

Well, say p is an u.b. for A.

then we exhibit a smaller u.b.



$$q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2}.$$

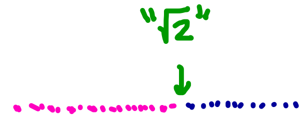
$\frac{p}{p+2} + 1 = \frac{1}{1+\frac{2}{p}} + 1 = \dots$ cont'd frac

(Why not $\frac{p+\sqrt{2}}{2}$?)

- Def'n A set S has the least upper property if every nonempty subset of S that has an u.b. has a least u.b. (in S).
- $\mathbb{Q}$ does not have the l.u.b. property.

But $\mathbb{R}$ will ...

- Construct $\mathbb{R}$ (Dedekind)



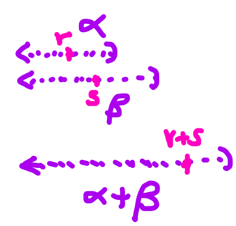
<u>Def'n</u>. A <u>cut</u> α is a <u>subset</u> of $\mathbb{Q}$ s.t.

1. $\alpha \neq \emptyset, \mathbb{Q}$ (non-trivial)
2. If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$ then $q \in \alpha$. (closed downward)
3. If $p \in \alpha$ then $p < r$ for some $r \in \alpha$. (no largest member)

<u>Ex</u>. $\alpha = \mathbb{Q}_-$ is a <u>cut</u>.

$\gamma = \{r : r \leq 2\}$ is <u>not</u> a cut.

- Let $\mathbb{R} = \{\alpha : \alpha \text{ is a } \underline{\text{cut}}\}$. Let's endow it with <u>arithmetic</u> & <u>order</u> & in way extends $\mathbb{Q}$.



- <u>Define</u>: $\alpha < \beta$ to <u>mean</u> $\alpha \subsetneq \beta$.

Check: order.

- <u>Define</u>: $\alpha + \beta = \{ r+s : r \in \alpha \ \& \ s \in \beta \}$.

check satisfies field axioms

A1. $\alpha + \beta$ is a cut

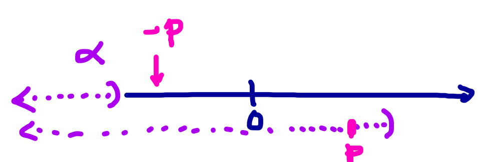
A2. $\alpha + \beta = \beta + \alpha$ follows from comm. of $+$ in $\mathbb{Q}$

A3. $(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$

check $\alpha + 0^* = \alpha$ means checking: $\alpha + 0^* \subset \alpha$ & $\alpha \subset \alpha + 0^*$.

A4. $\exists$ id. for $+$ is $\mathbb{Q}_- = 0^*$

A5. $\exists$ inverse for α :



Use $\beta = \{ p : \exists\ r > 0 \text{ s.t. } -p-r \notin \alpha \}$.

• **Define**: $\alpha \cdot \beta$ first for positive cuts $\{\alpha : \alpha > 0^*\} = \mathbb{R}_+$

b/c $(-)\cdot(-) = (+)$

If $\alpha, \beta \in \mathbb{R}_+$, let $\alpha \cdot \beta = \left\{ p : p < rs \text{ for some } \begin{array}{l} r \in \alpha \\ s \in \beta \\ r, s > 0 \end{array} \right\}$.

Let $1^* = \{q : q < 1\}$.

CHECK: mult. axioms, dist law

extend to products of $\begin{array}{l}(-)(+) = (-) \\ (-)(-) = (+)\end{array}$ in natural way.

check axioms again!

Thm. $\mathbb{R}$ is ordered field with l.u.b. property.

$\mathbb{R}$ contains $\mathbb{Q}$ as subfield!

FACT: $\mathbb{R}$ is only such.