REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

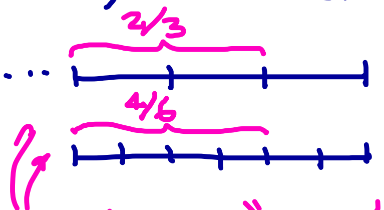
PROF. FRANCIS SU

TODAY: CONSTRUCTION OF $\mathbb{Q}$

We assume $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \}$

arithmetic order.

Hmmn...



one whole unit

2 parts of 3 $(2,3)$

4 " " 6 $(4,6)$

in $\mathbb{Z} \times \mathbb{Z}$

want say: "same"

put in same class, call $\frac{2}{3}$ name.

Goal of "constructing" $\mathbb{Q}$:

- define classes, extend $\mathbb{Z}$, so that arithmetic on $\mathbb{Z}$ works as before.

- **Define:** $\boxed{\mathbb{Q} = \left\{ \frac{p}{q} : q \neq 0 \ \& \ p, q \in \mathbb{Z} \right\}}$.

where $\frac{p}{q}$ is an equiv. class defined by equiv. rel'n on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$

let $(p,q) \sim (m,n)$ if $pn = qm$ and $q, n \neq 0$.

- **Check:** $\sim$ is an equiv. rel'n: reflexive $(p,q) \sim (p,q)$ ✓

symmetric...

transitive: if $(p,q) \sim (m,n)$ & $(m,n) \sim (a,b)$ then is $(p,q) \sim (a,b)$? } HW

Sol'n: ... have $pn = qm$ & $bm = an$ ... want $pb = qa$

YOU'LL NEED: [CANCELLATION LAW IN $\mathbb{Z}$: if $ab = ac$ & $a \neq 0$ then $b = c$.]

Arithmetic on $\mathbb{Q}$?

- View $\mathbb{Z}$ is <u>subset</u> of $\mathbb{Q}$:

$$n \longmapsto \frac{n}{1} = \{ (n,1), (2n,2), \dots \}$$

<u>Define</u>: $\frac{a}{b} + \frac{c}{d} \overset{?}{=} \frac{a+c}{b+d}$ <u>BAD</u>

PROBLEM: this addition depends on choice of representative:

$$\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$$
$$\frac{2}{4} + \frac{1}{3} = \frac{3}{7}$$

equiv. — not equiv!

<u>BETTER</u> $\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$.

this addition is "well-defined".

but arithmetic on $\mathbb{Z}$ is not same as before.

<u>Define</u>: $\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$ ← check: well-def'd (HW)

<u>Define</u>: $\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$ ← check: well-def'd

See: $\left.\frac{n}{1} + \frac{m}{1} = \frac{n+m}{1}, \quad \frac{n}{1} \cdot \frac{m}{1} = \frac{nm}{1}.\right\}$ $\mathbb{Z}$ as before.

<u>well-def'd</u>: assume $(a,b) \sim (a',b')$.
& $(c,d) \sim (c',d')$.

must check: $(ad+bc, bd) \sim (a'd'+b'c', b'd')$.

- $\mathbb{Q}$ has <u>order</u>.

<u>Def'n.</u> An <u>order</u> on set S is a <u>rel'n</u> $<$ satisfying

trichotomy ① <u>exactly</u> one true: $x<y$, $x=y$, $y<x$.

transitive ② $x<y,\ y<z \Rightarrow x<z$.

S is <u>ordered set</u>: $(S,<)$.

<u>EX.</u> in $\mathbb{Z}$, say $m<n$ if $n-m \in \{1,2,3,4,\dots\}$. ("positive")

in $\mathbb{Q}$, say $\frac{m}{n}$ is <u>positive</u>: both m,n <u>positive</u>

or both m,n non-pos & non-0.

So $\frac{a}{b} < \frac{c}{d}$ if $\frac{c}{d} + \frac{-a}{b}$ is <u>positive</u>.

- Write $y>x$ for $x<y$.

$x \leq y$ for $x<y$ or $x=y$.

- $\mathbb{Q}$ is a field. ($\mathbb{Z}$ is ring).

  A field has properties:

  it is $(F, +, \times)$

  set — operations $F \times F \to F$

  satisfying some "axioms": closure

  commut.

- $\mathbb{Z}$ not field (no $\times$ inv's) — assoc.

- $\mathbb{Q}$ is. For $+$ identity, use $(0,1)$ — identities: 1 for $\times$

  $\times$ " " $(1,1)$. — 0 for $+$

  inverse, except 0 has no mult. inv.

  distributive law: $a(b+c) = ab+ac$.

Axioms: important props

- $\mathbb{Q}$ is an ordered field:

  field with order and order "preserved" by field ops:

  (i) $y < z \Rightarrow x+y < x+z$.

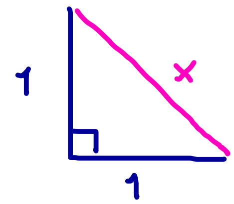
  book $\rightarrow$ (ii) $y < z$ & $x > 0 \Rightarrow xy < xz$.

  eq: $x>0, y>0 \Rightarrow xy > 0$.

- $\mathbb{Q}$ helps solve eqns $\mathbb{Z}$ could not!

  <u>Ex</u>. $5x = 3$ can't be solved in $\mathbb{Z}$.

  But $\mathbb{Q}$ not "<u>big enough</u>"
  to solve



$x^2 = 1^2 + 1^2 = 2$.

- <u>Thm</u>. $x^2 = 2$ has no <u>sol'n</u> in $\mathbb{Q}$.

<u>proof</u>. (by contradiction)

Suppose $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q \neq 0$

<u>and</u> p, q have no <u>common factors</u>.

Assume $\left(\frac{p}{q}\right)^2 = 2$. Then $p^2 = 2q^2$ in $\mathbb{Z}$.

So p^2 is <u>even</u> (div. by 2). Then p is <u>even</u>. (why?)

So $p = 2k$, for some $k \in \mathbb{Z}$.

So $4k^2 = 2q^2$, or $2k^2 = q^2$. So q^2 is <u>even</u>.

So q is <u>even</u>., contradicting

p, q no common factor

Thus there is no sol'n to $x^2 = 2$ in $\mathbb{Q}$. ∎

NEXT TIME: we enlarge $\mathbb{Q}$ by constructing $\mathbb{R}$.