REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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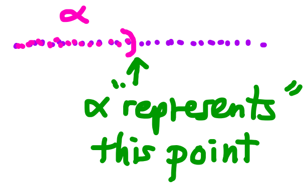
TODAY: THE LEAST UPPER BOUND PROPERTY

• Recall: constructed $\mathbb{R} = \{\alpha : \alpha \text{ is a } \underline{\text{cut}}\}$.

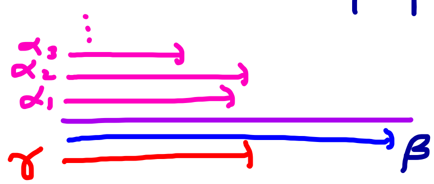


$\mathbb{R}$ has the <u>least upper bd</u> property:

every <u>non-empty</u> subset <u>that has u.b.</u> has a <u>l.u.b.</u>

($\mathbb{Q}$ doesn't have this prop.)

See:



If $\mathcal{A}$ is a <u>collection</u> of cuts with u.b. β

let $\gamma = \bigcup \{\alpha : \alpha \in \mathcal{A}\}$.

<u>Claim</u>: γ is a <u>cut</u> & $\gamma = \sup \mathcal{A}$.

<u>Sketch</u>: γ non-trivial: not empty (since $\mathcal{A}$ not empty)

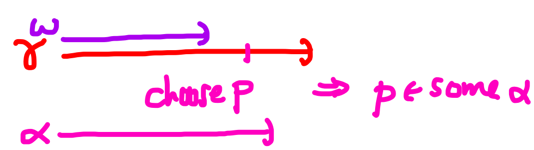
not $\mathbb{Q}$ (since β is u.b.)

γ closed down (b/c for $x \in \gamma \Rightarrow x \in$ some α, closed down...)

γ has no largest member (b/c $x \in \gamma \Rightarrow x \in$ some α, but α has no largest).

γ is <u>u.b.</u> (since all $\alpha \in \mathcal{A}$ are contained in γ)

and any smaller $\omega < \gamma$, see:



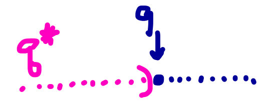
so ω not <u>u.b.</u> $\blacksquare$

The l.u.b. prop of $\mathbb{R}$: "the completeness axiom" of $\mathbb{R}$.

<u>EX</u>. Know: $\sup\{1, 1.4, 1.41, 1.414, 1.4142, 1.41421, \dots\}$ exists!

"$\sqrt{2}$"

- $\mathbb{R}$ contains $\mathbb{Q}$ as <u>subfield</u>:



How? Associate $q \in \mathbb{Q} \longrightarrow$ cut $q^* = \{ r \in \mathbb{Q} : r < q \}$.

- Exists: cut $\{ r \in \mathbb{Q} : r^2 < 2$ <u>OR</u> $r < 0 \}$ $\leftarrow$ call $\sqrt{2}$

  Check $\sqrt{2} \cdot \sqrt{2} = 2^*$ as cuts.

- Gen'l <u>roots</u> exist:

  <u>define</u>: $a^{1/n} = \sup \underbrace{\{ r : r^n < a \}}_{\text{a set of real \#'s}}$.

- <u>Define</u>: infimum $\inf E =$ <u>greatest lower bd</u> of E.

  [HW] $\inf E = -\sup(-E)$.

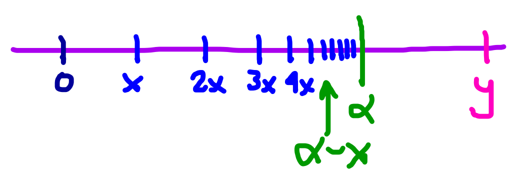
CONSEQS OF LUB PROP:

- The archimedean property of $\mathbb{R}$: interesting: $x=1$ or $y=1$

$$\boxed{x, y \in \mathbb{R},\ x > 0 \Rightarrow \exists \text{ pos. integer } n \text{ s.t. } nx > y.}$$

<u>proof</u>: let $A = \{ nx : n \in \mathbb{N} \}$. If cond. false, then y is u.b. for A.

So A has a <u>lub</u> α.

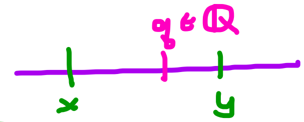


So $\alpha - x$ is <u>not</u> u.b.

Therefore $\alpha - x < mx$ for some m.

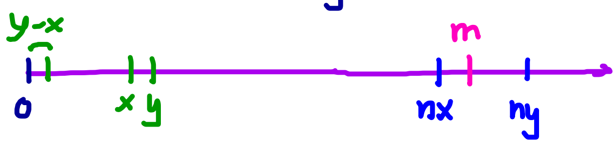
Thus $\alpha < (m+1)x$, contradicts α being u.b. ▨

- $\mathbb{Q}$ is **dense** in $\mathbb{R}$:



$$\boxed{\text{for any } x, y \in \mathbb{R},\ x<y,\ \exists\, q\in\mathbb{Q} \text{ s.t. } x<q<y.}$$

Proof. Note $y-x>0$, so archim. prop $\Rightarrow$ $n(y-x)>1$ for some $n\in\mathbb{N}$.



Choose first m s.t. $m-1 \le nx < m$. Then $x<\frac{m}{n}$ by ☺.

And $\frac{m}{n}<y$ by ☺ since: $ny > nx+1 \ge m$. $\blacksquare$

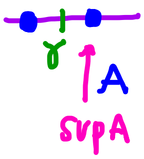
- PROPERTIES OF SUP

(a) γ is u.b. for $A \iff \sup A \leq \gamma$

(b) If $\forall a \in A$, we have $a \leq \gamma \Rightarrow \sup A \leq \gamma$.

THINK neg. rat'ls → (c) " " $a < \gamma \Rightarrow \sup A \leq \gamma$.

(d) If $\gamma < \sup A \Rightarrow \exists\, a \in A$ s.t. $\gamma < a \leq \sup A$.



(e) To show $\sup A = \gamma$, show γ is u.b. & any smaller $x < \gamma$ is not u.b.

or show γ is u.b. & any u.b. b satisfies $\gamma \leq b$.

(f) If $A \subset B$ then $\sup A \leq \sup B$.

why? by (a), show $\sup B$ is u.b. for A.

This follows from noting: $a \in A \Rightarrow a \in B$ (b/c $A \subset B$)

so $a \leq \sup B$ (by def'n of $\sup B$).

(g) To show $\sup A = \sup B$:

show $\leq$ & $\geq$ use (a).

(h) let $A + B = \{a + b : a \in A, b \in B\} \Rightarrow \sup(A+B) = \sup A + \sup B$.

(i) $A \cdot B = \{ab : a \in A, b \in B\} \Rightarrow A, B \subset \mathbb{R}_+$

$\sup A \cdot B = \sup A \cdot \sup B$.