REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

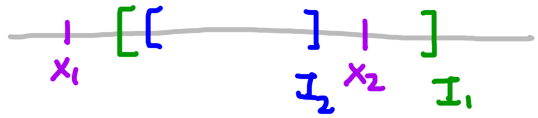
PROF. FRANCIS SU

TODAY: COMPACTNESS & CONNECTEDNESS

FUN: Another proof $\mathbb{R}$ is unctble.

If $\mathbb{R}$ were ctble, then $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$ listed!

Then



choose $I_1 \subset \mathbb{R} \setminus \{x_1\}$.

Choose $I_2 \subset I_1 \setminus \{x_2\}$.

Notice: $x_1, x_2 \notin I_2$.

Choose $I_{n+1} \subset I_n \setminus \{x_{n+1}\}$ so that $x_1, \dots, x_{n+1} \notin I_{n+1}$.

Then $\cap I_n \neq \emptyset$, so $z \in \cap I_n$ is real # not on my list,

contradiction. $\blacksquare$

Recall: Heine-Borel says in $\mathbb{R}^k$

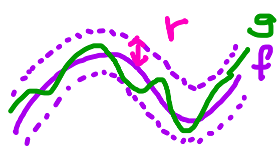
K cpt $\Leftrightarrow$ K is closed & bdd

EX. Discrete metric on ∞ set. Subsets are closed & bdd,

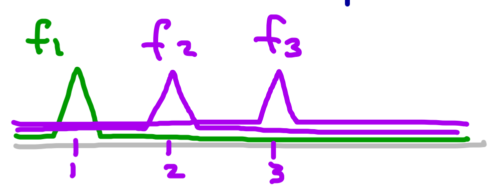
but Inf. subsets are not cpt.

EX. In spaces of fcns, (e.g., contin. bdd fcns on $\mathbb{R}$, w/ sup metric: $d(f,g)=\sup_x |f(x)-g(x)|$

there are closed, bdd sets that are not cpt.



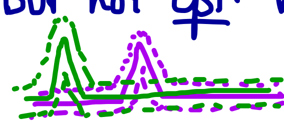
$g \in N_r(f)$



$\{f_n\}$ has no limit pts.

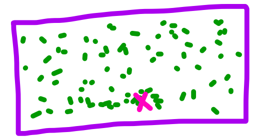
So it is closed. Also bdd.

But not cpt because this open cover:

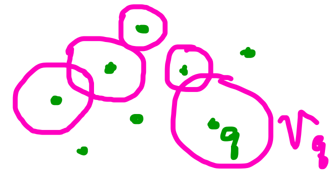


has no fin. subcover.

Thm. K cpt. $\iff$ Every ∞ subset E of K has a lim pt. in K.



proof. ($\Rightarrow$) If no pt of K is a l.p. of E
then each pt $q \in K$
has V_q nbhd containing no pts of E
(other than possibly q itself).

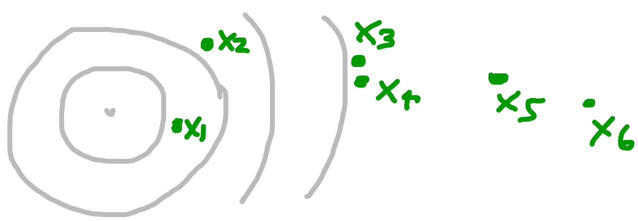


Then $\{V_q\}_{q \in K}$ is a cover with no fin subcover
b/c V_e for $e \in E$ is the only elt of cover
that covers e. And E is infinite.
So any subcover of $\{V_q\}$ must include $\{V_e\}_{e \in E}$.

($\Leftarrow$) [proof here for $\mathbb{R}^k$, but it's true for gen'l metric spaces HW 2.26.]

We'll show K is closed & bdd.

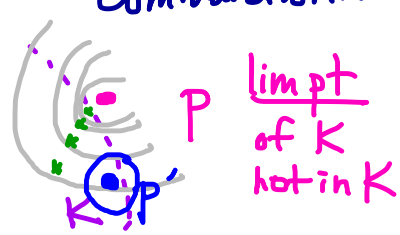
- Suppose K not bdd.



Choose x_n s.t. $|x_n| > n$.

Idea: If p is l.p. of $\{x_n\}$
it would be somewhere & have ∞ pts in any nbhd...
contradiction...

- Suppose K not closed.

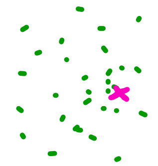


$\exists\, p \notin K$ but p a lim pt of K.
Choose $\{x_n\}$ s.t. $|x_n - p| < \frac{1}{n}$.

Idea: This ∞ set, with lim pt p and no other lim pt.
use: $|x_n - p'| \geq |p - p'| - |x_n - p|$.

Thm. (Bolzano-Weierstrass Thm)

Every bdd ∞ subset of $\mathbb{R}^k$ has a lim pt in $\mathbb{R}^k$.

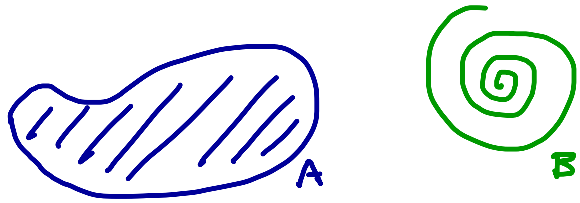


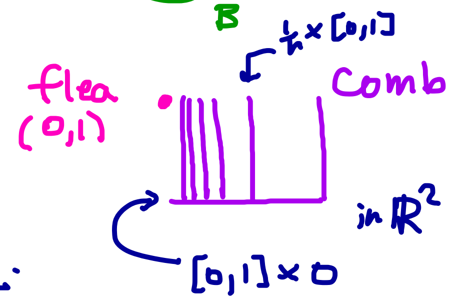
pf. E bdd $\Rightarrow$ $E \subset$ k-cell, which is cpt.

• **FACT:** K cpt. $\Leftrightarrow$ Any collection of closed subsets K_α
that has the Finite Intersection Prop
[any finite subcollection has non-$\emptyset$ intersection]
will have non-empty intersection.

This generalizes Nested Intervals Thm!

CONNECTED SETS





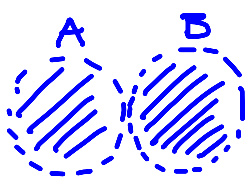
Def'n $A, B \subset X$ metric.

Call A, B **separated** if **both** $A \cap \bar{B}$ and $\bar{A} \cap B$ are **empty**.

EX. flea & comb are **not** separated.

Def'n. Say E is **connected** if E is **not** union of 2 non-$\emptyset$ **separated** sets.

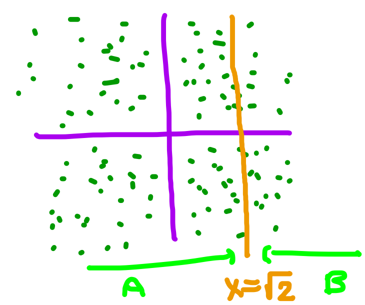
EX. flea & comb is **connected**, but not "**path-connected**".

EX. 

not connected (use A, B as separation)

EX. $E = \{(x,y) : x, y \in \mathbb{Q}\}$ in $\mathbb{R}^2$

not connected, use A, B as **separation**.



EX. $[a,b]$ is **connected**. (why?)