REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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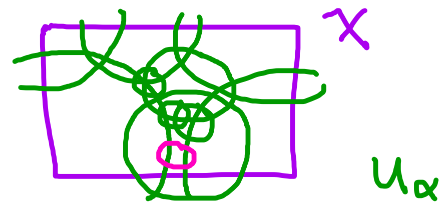
TODAY: DISCONTINUOUS FUNCTIONS

- Lebesgue covering lemma.

  If $\{U_\alpha\}$ is open cover of cpt metric space X
  then $\exists\ \delta > 0$ s.t.
  $\forall x \in X$, $B_\delta(x)$ is contained in some U_α.

- δ is called a Lebesgue number of the cover.

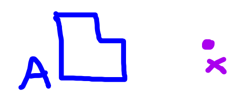


EXTRA CREDIT: FIND YOUR OWN PROOF.

proof: Since X cpt, $\exists$ finite subcover of $\{U_\alpha\}$, say $\{U_{\alpha_i}\}_{i=1}^n$.

For any closed set A,

define $\boxed{d(x, A) = \inf_{a \in A} d(x, a)}$.



Claim: $d(x, A)$ is a continuous function of x. [show this.]

Then let

$$\boxed{f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, U_{\alpha_i}^c)}.$$

closed

So f is continuous fcn on cpt set, so it achieves min value, call it δ.

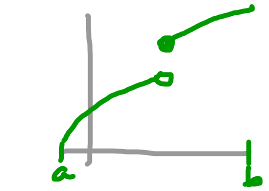
See that $\delta > 0$ b/c $\{U_{\alpha_i}\}$ is an open cover.

$(\text{so } x \in U_{\alpha_i} \Rightarrow d(x, U_{\alpha_i}^c) > 0)$.

For each x, $f(x) \geq \delta \Rightarrow$ at least one $d(x, U_{\alpha_i}^c) \geq \delta$.

So $B_\delta(x) \subset U_{\alpha_i}$, as desired. ∎

DISCONTINUITIES



$f:(a,b)\to\mathbb{R}$ not contin.

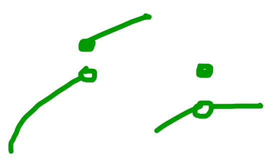
Here, can define $\boxed{f(x^-)=\lim_{t\to x^-} f(t)}$ left-hand limit

and $\boxed{f(x^+)=\lim_{t\to x^+} f(t)}$ right-hand limit

restrict attention domain (a,x) or (x,b)

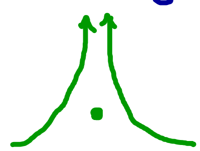
Say $\lim_{t\to x} f(t)$ exists when $f(x^+)=f(x^-)$ exist & equal.

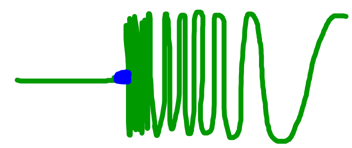
- If f is not contin at x but $f(x^-)$ and $f(x^+)$ exist,



say f has a simple discontinuity ("a discont. of 1^{st} kind")

Else: "of 2^{nd} kind"

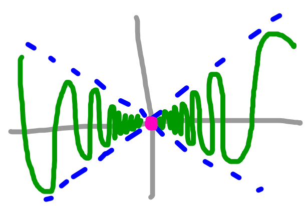




topologist's sine curve

$$f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & x>0\\ 0 & x\le 0\end{cases}.$$

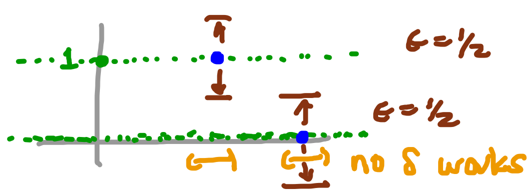
Ex. $f(x)=\begin{cases}x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & x=0\end{cases}.$



contin (incl at 0).

Ex. (Dirichlet function)

$$f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}.$$

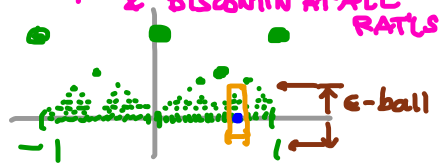


NOT CONTINUOUS AT ANY PT!

Ex. (HW) ANOTHER DIRICHLET FCN

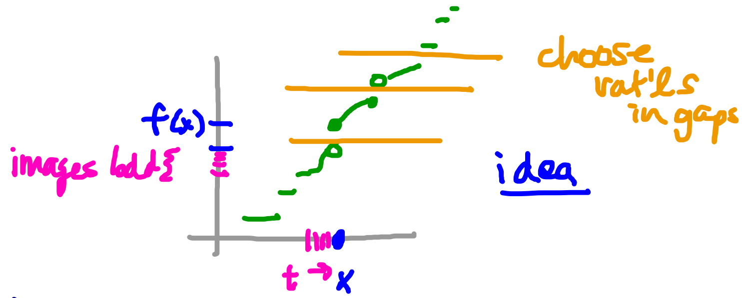
$$f(x)=\begin{cases}1/q & \text{if } x=p/q \text{ (lowest terms)}\\ 0 & \text{else}\end{cases}$$

CONTIN AT ALL IRRAT'LS & DISCONTIN AT ALL RAT'LS!



idea: only finitely many dots $\geq\varepsilon$.

MONOTONIC FUNCTIONS



Thm. f mon incr. on (a,b)

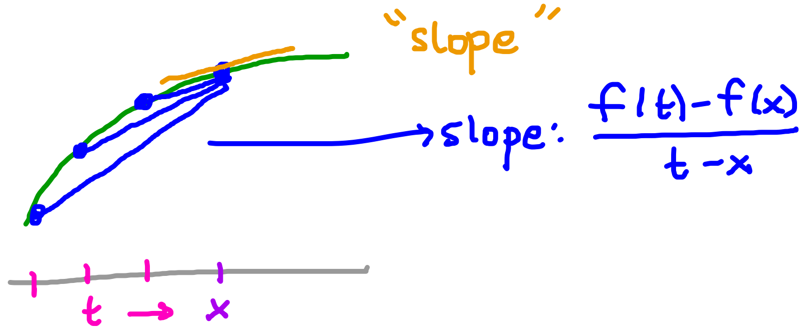
$\Rightarrow$ $f(x^+)$ & $f(x^-)$ exist for all $x \in (a,b)$

COR. Mon fcns can only have simple discontins!

COR. Set of discontinuities of mon fcn must be at most ctble!

HW 4.17: Any f: # simple disconts is at most ctble.

DERIVATIVES



Defn. A function $f:[a,b]\to\mathbb{R}$ is differentiable at $x\in[a,b]$ if this limit exists:

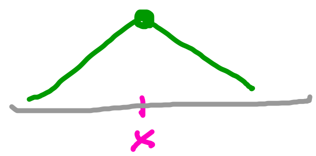
$$\boxed{f'(x) \overset{\text{def}}{=} \lim_{t\to x} \frac{f(t)-f(x)}{t-x}}.$$

the "derivative" of f at x

$t\neq x,\ t\in(a,b)$

think: inst. rate of change.

Q. Contin $\Rightarrow$ diff'bility? $\boxed{\text{NO}}$



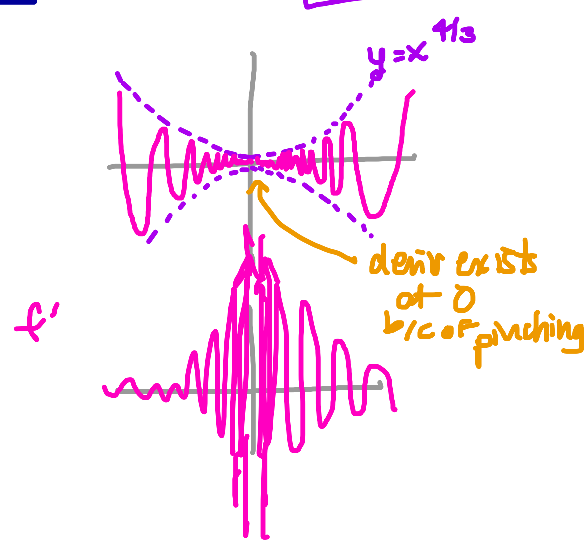
Q. Diff'ble $\Rightarrow$ contin.? $\boxed{\text{YES}}$

Why? If $t\to x$

$$\lim_{t\to x}\left[f(t)-f(x)\right] = \underbrace{\lim_{t\to x}\frac{f(t)-f(x)}{t-x}}_{f'(x)}\cdot\underbrace{\lim_{t\to x}(t-x)}_{0} = 0,$$ as desired. ∎

Q. If f diff'ble on $[a,b]$, is f' contin on $[a,b]$? $\boxed{\text{NO}}$

$$f(x)=\begin{cases} x^{4/3}\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}.$$



so $f'(0)$ does not exist!

ORDERS OF DIFF'BILITY

A function is C^0 if its contin.

is C^1 " " diff'ble & deriv is contin.

C^k " " k^{th} deriv exists & is contin!

C^∞ all deriv. exists "smooth"

C^ω ← has power series expansion "analytic"

For cplx-diff'bility: no distinction $C^1 \to C^\omega$.

[MATH 136].