REAL ANALYSIS

MATH 131, HARVEY MUDD COLLEGE

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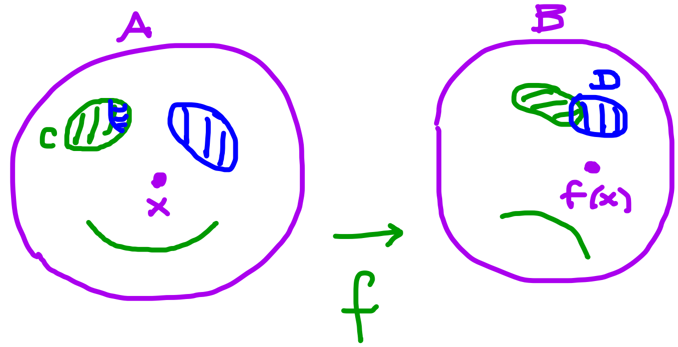
TODAY: COUNTABLE SETS

Q: How do we count?

A: We construct a function.

Recall: $f: A \to B$

associates $x \mapsto f(x)$
"maps"



Write: $f(C) = \{ f(x) : x \in C \}$ the image of C

$f^{-1}(D) = \{ x : f(x) \in D \}$ the pre-image of D

- When $f(A) =$ all B, say f is onto (surjective) $\twoheadrightarrow$
- When $f(x) = f(y)$ implies $x = y$, say f is 1-1 (injective) $\hookrightarrow$
- When f is 1-1 and onto, say f is bijective. $\hookrightarrow\!\!\!\!\!\twoheadrightarrow$

A bijection puts A, B into "1-1 correspondence"

Write: $A \sim B$.

Ex. $\underbrace{\{1, ☺, \propto\}}_{A} \sim \{1, 2, 3\}$. Say A has 3 elts.

Def'n. Call A finite if $A \sim \underbrace{\{1, 2, \dots, n\}}_{[n]}$ for some $n \in \mathbb{N}$.

or if A is empty.

Else A is infinite.

Def'n. Call an infinite A **countable** if $A \sim \mathbb{N}$.

Else an infinite A is **uncountable**.

Ex. $\mathbb{N}$ is ctble: use bijection $f(n) = n$.

Ex. A sequence $x_1, x_2, x_3, \ldots$ of distinct terms is ctble: use $f(n) = x_n$.

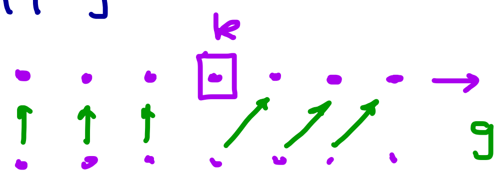
$$\begin{matrix} x_1, & x_2, & x_3, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & \end{matrix}$$

NOTICE: Any ctble set can be "listed" in a seq.

Ex. $\{2, 3, 4, \ldots\}$ is ctble: use $f: \mathbb{N} \to \mathbb{N} - \{1\}$ mapping $n \mapsto n+1$.

$$\begin{matrix} 2, & 3, & 4, & \ldots & n+1 \\ \uparrow & \uparrow & \uparrow & & \uparrow \\ 1 & 2 & 3 & & n \end{matrix}$$

Ex. $\mathbb{N} \setminus \{k\}$ is ctble:



use $g: \mathbb{N} \to \mathbb{N} - \{k\}$ def'd by
$$\begin{cases} g(n) = n & \text{if } n < k \\ g(n) = n+1 & \text{if } n \geq k. \end{cases}$$

Thm. $\mathbb{N}$ is infinite.

proof. We'll show $\nexists$ bijection $[n] \leftrightarrow \mathbb{N}$ by induction on n.

base case: If $\exists f_1 : [1] \leftrightarrow \mathbb{N}$, consider $\mathbb{N} \setminus f_1(1)$. It's not empty. So f_1 not bijective.

ind. step: we'll show $[k] \not\leftrightarrow \mathbb{N} \Rightarrow [k+1] \not\leftrightarrow \mathbb{N}$.

equiv: show if $\exists f_{k+1} : [k+1] \leftrightarrow \mathbb{N} \Rightarrow \exists f_k : [k] \leftrightarrow \mathbb{N}$.

idea: $[k+1] \xleftrightarrow{f_{k+1}} \mathbb{N}$

restricts to $[k] \xleftrightarrow{f_k} \mathbb{N} - f_{k+1}(k+1) \xleftrightarrow{g^{-1}} \mathbb{N}$ is desired bijection.

Ex. $2\mathbb{N}$ (even #'s) is ctble: use $f(n)=2n$.

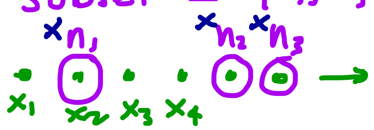
Ex. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is ctble.

$$\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \\ 5 & 3 & 1 & 2 & 4 \end{array}$$

Thm. Every infinite subset of ctble set is ctble.

pf idea:

subset $E = \{n_1, \ldots\}$



set $A = \{x_1, x_2, \ldots\}$.

Let $n_1 = \inf\{i : x_i \in E\}$ ← $\exists$ least elt by WOP

$\ldots \quad n_k = \inf\{i : x_i \in E \;\&\; i > n_{k-1}\}$.

Then $E = \{x_{n_1}, x_{n_2}, \ldots\}$. Or use $f(k) = x_{n_k}$.

Thm. $\mathbb{Q}$ is ctble.

Idea: $\mathbb{Q}_+$ is ctble:

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & 1/4 & \cdots \\ 2/1 & 2/2 & 2/3 & 2/4 & \cdots \\ 3/1 & 3/2 & 3/3 & 3/4 & \cdots \\ \vdots & & & & \end{array}$$

See elts in array are ctble.
Subset (w/ one rep of each class of m/n) is ctble!

Thm. A (at most) ctble union of (at most) ctble sets is (at most) ctble.

pf. same idea: if $A_1, A_2, \ldots$ ctble sets

$$\begin{array}{lccccc} A_1 \to & a_{11} & a_{12} & a_{13} & \cdots \\ A_2 \to & a_{21} & a_{22} & a_{23} & \cdots \\ & \vdots & & & \end{array}$$

Appl. # computer progs is ctble.

Q. Are there uncountble sets?
Q. Is $\mathbb{R}$ unctble?
If so there are #'s whose decimal exp is not the output of comp. prog!
"computable" #'s