Matthew Feng November 30, 2018

6.046 Problem Set 9

Collaborators: *Alex Guo*

Problem 1

(a)

First, we want to show that SKIPPING-STONES is in NP. To do this, we must demonstrate that given a *certificate* or *solution*, we can verify that the solution is correct in polynomial time.

Given the moves to make, we can simply construct the board and run through the moves, each move taking at most $O(n)$ time. Finally, verifying that the solution is correct (i.e. checking that each row has a stone) can be done in $O(n^2)$ time. Thus, verification may be completed in polynomial time (given a polynomial sized solution), and thus SKIPPING-STONES is in NP.

(b)

In order to prove SKIPPING-STONES is NP-hard, we will prove the following reduction:

$$\text{3-SAT} \leq_P \text{SKIPPING-STONES}$$

In other words, we will transform an instance of the 3-SAT problem into an instance of SKIPPING-STONES using polynomial time, such that if an efficient algorithm for SKIPPING-STONES existed, then we would likewise have an efficient algorithm for 3-SAT (which, to our knowledge, does not exist).

Suppose our instance of 3-SAT has K clauses and L literals.

If we have more clauses than literals, our SKIPPING-STONES board will be of size $K \times K$. Conversely, if we have more literals than clauses, the board will have shape $L \times L$.

We can imagine the board as follows:

- Each column corresponds a literal x_j.
- Each row corresponds to a clause c_i.

- If we have more literals than clauses, let the additional $L - K$ rows be a duplicate of clause c_K (i.e. duplicate of row K); this is simply to follow the rules that the board must be square, and has no real meaning.

Using this construction, we can place stones on the board according to the following rules:

- Let black stones represent **False** truth values, and white stones represent **True** truth values.
- For each clause $c_i = (a \vee b \vee c)$, place stones in corresponding to literals a, b, and c, placing a white stone where the literal is the same as the column label (e.g. a), or a black stone where the literal is a negation of the column label (e.g. $\bar{a}$).
- Otherwise, the space on the board should be left empty (i.e. literal x_j doesn't appear in clause c_i).

We claim that this construction is an instance of SKIPPING-STONES such that when solved, yields a valid setting of the literals that solves the 3-SAT problem.

We can see this because solutions to SKIPPING-STONES only have a single color in each column, corresponding to a single truth value for each literal. Solutions must also have one stone per row; this is equivalent to requiring that each clause having at least one literal satisfying the clause.

Formally, we can turn any solution to the SKIPPING-STONES problem into a valid solution for the 3-SAT, by simplying assigning literal x_j to **True** if the stones in column x_j are all white, or x_j to **False** if all the stones are black (if no stones are in column x_j, then there does not exist a valid solution). Likewise, given a valid assignment of the literals, we know which stones to remove from each column, such that we can solve the SKIPPING-STONES problem.

Since it only took us $O(n^2), n = \max(K, L)$ time to construct the SKIPPING-STONES board and place the stones, the reduction can be performed in polynomial time.

Thus, if an efficient algorithm exists for SKIPPING-STONES, then we would be able to solve 3-SAT efficiently as well. However, since we do not believe the latter to be the case, we conclude that the former assumption is false; that is, SKIPPING-STONES does not have an efficient, polynomial time algorithm.

Problem 2

(a)

Again, we first want to demonstrate that SCHEDULING is in NP as the first step to proving that SCHEDULING is NP-complete. Our approach again is to show that given a *certificate*, we can verify its validity in polynomial time.

Given a potentially valid schedule, we can simply iterate over each student s, looping through all the pairs of classes that student s is taking, and verify that every pair does not conflict. Since this can be done $O(|S||C|^2)$, where S is the set of all students and C the set of all classes, our verification algorithm runs in polynomial time, which shows that SCHEDULING is indeed in NP.

(b)

In order to prove that SCHEDULING is NP-hard, we first show that it is the equivalent problem as graph 3-colorability (3-COLOR). In other words, we want to transform an instance of SCHEDULING into an instance of 3-COLOR, such that the solution to 3-COLOR directly translates into the solution for SCHEDULING.

Given the list of students S and the classes C_s each student $s \in S$ is taking, and a list of classes $C = \bigcup_{s \in S} C_s$, we can build an instance of the 3-COLOR problem such that the solution to the 3-COLOR yields a solution to SCHEDULING.

The goal is to build a graph G whose coloring represents an exam scheduling. In particular, represent each of the three different exam blocks with a different color (e.g. {red, green, blue}), such that for any student, all the classes they are taking are colored differently.

First, we can construct a node $\hat{c}_i$ for each class $c_i \in C$.

Then, for every student $s_j \in S$ and their list of classes C_{s_j}, build edge $(\hat{c}_a, \hat{c}_b)$, $\forall c_a, c_b \in C_{s_j}, c_a \neq c_b$. These edges encode the fact that no two classes can have exams at the same time, otherwise student s_j would be taking two exams at the same time.

If the resulting graph G has a valid 3-coloring, then there does exist a feasible exam schedule. We can see this in two steps:

- Given the 3-coloring, simply assign each class to the exam block determined by its coloring. Since exam conflicts were defined by the edges, and graph coloring ensures that the endpoints of edges always have different colors, there cannot exist any exam conflicts.

- Given a valid exam scheduling, we can construct a valid 3-coloring on the graph G by assigning all nodes representing classes in the same exam block the same color. Again, because the edges represent exam conflicts, and the exam schedule is valid, the colors of the endpoints of every edge should be different, satisfying the requirement for a valid coloring. Since only three exam blocks exist, only three colors will be used, making it a valid 3-coloring.

Thus, we can see that SCHEDULING and 3-COLOR are equivalent problems.

(c)

Now that we know SCHEDULING and 3-COLOR are equivalent, we want to show the following reduction can be peformed in polynomial time:

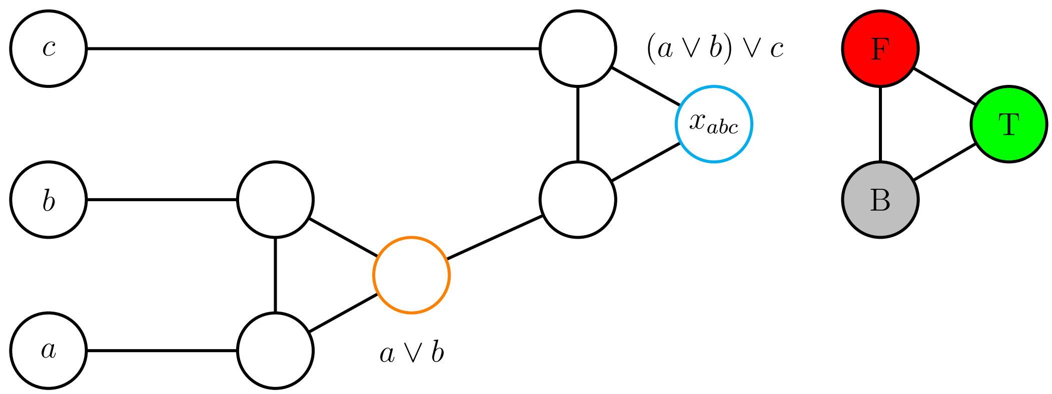
$$\text{3-SAT} \leq_P \text{3-COLOR}$$

Given an instance of 3-SAT with K clauses and L literals, we need to build an instance of 3-COLOR, such that the existence of a valid solution to 3-COLOR implies the existence valid setting of the L literals that satisfies all K clauses in 3-SAT.

First, we need some way of mapping colorings to truth values. We can do this as follows: for every literal x, construct a triangle of nodes x_T, x_F, B, where B is a common node shared among all triangles. We can then imagine that three "colors" in our graph will be {T, F, B}. Any node colored "T" takes on the truth value **True**, while any node colored "F" takes on the truth value **False**, and nodes colored "B" should not map to any literal.

Next, we show that given a clause $a \vee b \vee c$, we can add edges to the node representation of literals given by the previous paragraph to constrain the coloring of a special node x_{abc} that will hold the truth value of $a \vee b \vee c$.

Consider the following construction:



The construction has the two following properties that allow it to guarantee that x_{abc} is colored accordingly:

- When a, b, c are all colored "F" (noting that they are also connected to their respective triangles as previously described but not shown), the only color that x_{abc} may have is "F".
- If any of a, b, or c is colored "T", then there exists some coloring of the intermediate nodes such that x_{abc} can be colored "T". We can force x_{abc} to be colored as such by adding edges (x_{abc}, F) and (x_{abc}, B), which also has the side effect of invalidating all colorings if a, b, c are all colored "F", since x_{abc} will be forcibly connected to a node pre-colored "F".

Thus, this small subgraph allows us to map the existence of a valid coloring to the existence of a valid setting of a, b, c such that that clause is satisfied.

Now we build k of these "subgraphs", one for each clause, where all the nodes highlighted in blue in the figure connect to common nodes F, B, and that B connects to the literals in a triangular fashion described earlier (i.e. (B, x_T, x_F)).

If a valid coloring exists in this final graph, then every subgraph must have a valid coloring, which means all k clauses must be satisfied.

This reduction is correct because, by taking the nodes colored "T" in the (B, x_T, x_F) triangles, we get a assignment of variables that solves our 3-SAT problem, since we had previously demonstrated the one-to-one mapping between clause and coloring. Conversely, given a valid assignment of variables, we can construct a valid coloring of the graph by assigning all the nodes corresponding to true literals with the color "T".

The reduction can be constructed in polynomial time since we do constant amount of work constructing nodes and edges for each of the k clauses.

Thus, if we had an efficient algorithm to 3-COLOR, we could efficiently solve 3-SAT. However, to our knowledge, no such algorithm exists for 3-SAT, and thus no such algorithm exists for 3-COLOR.