Matthew Feng, R06 September 14, 2018

6.046 Problem Set 1

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Problem 1-1

(a)

$$T(n) = 5T(\frac{n}{4}) + n$$

Answer $T(n) = \Theta(n^{\log_4 5})$

Solution This falls under case 1 of the Master Theorem, which states that if $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$. In this case, $a = 5$, $b = 4$, and $f(n) = n$. Since $n = O(n^{\log_b a - \epsilon})$ for $\epsilon = n^{\log_b a} - 1$, we have $T(n) = \Theta(n^{\log_4 5})$.

(b)

$$T(n) = 9T(\frac{n}{3}) + n^2$$

Answer $T(n) = \Theta(n^2 \log n)$

Solution This falls under case 2 of the Master Theorem, which states that if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$. In particular, $a = 9$, $b = 3$, and $f(n) = n^2$. Since $n^2 = \Theta(n^{\log_3 9}) = \Theta(n^2)$, $T(n) = \Theta(n^2 \log n)$.

(c)

$$T(n) = T(n-2) + n$$

Answer $T(n) = \Theta(n^2)$.

Solution For this problem, we do not use the Master Theorem, but instead rely on the tree method. In particular, we create a tree that has $O(\frac{n}{2})$ layers, since at every layer we decrease n by 2, and the cost per layer is the size of that layer. The total cost then is

$$T(n) = n + (n-2) + (n-4) + \ldots + 4 + 2 = \frac{n}{2}(\frac{n}{2} + 1)$$

and so we find that $T(n) = \Theta(n^2)$.

(d)

$$T(n) = 3T(\sqrt{n}) + \log n$$

Answer $T(n) = \Theta((\log n)^{\log_2 3})$

Solution To solve this problem, we use the Master Theorem after involving a change of variable, since we know that we can turn powers into multiplication via logarithms. Concretely, let $m = \log(n)$. Then we have $T(2^m) = 3T(2^{m/2}) + m$. Let $S(m) = T(2^m)$. Then $S(m) = 3S(\frac{m}{2}) + m$, which by case 1 of the Master Theorem, dictates that $S(m) = T(2^m) = \Theta(m^{\log_2 3})$. Switching our variable back to n, we get $S(\log n) = T(n) = \Theta((\log n)^{\log_2 3})$.

(e)

$$T(n) = 2T(\frac{n}{4}) + T(\frac{n}{3}) + \Theta(n)$$

Answer $T(n) = \Theta(n)$

Solution We will use the Akra-Bazzi method to solve the problem. First, we notice that $0 < p < 1$, since $Q(p) = 2(\frac{1}{4})^p + (\frac{1}{3})^p$ is a decreasing function, and $Q(0) = 3 > 1 > \frac{5}{6} = Q(1)$. Akra-Bazzi tells us that

$$T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{f(u)}{u^{p+1}} du\right)\right)$$

which, when applied to this recurrence, yields

$$T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{cu}{u^{p+1}}du\right)\right)$$

for some positive real number constant c. Completing the calculation of the integral,

$$\begin{aligned} T(n) &= \Theta\left(n^p\left(1 + \int_1^n \frac{cu}{u^{p+1}}du\right)\right) && (1) \\ &= \Theta\left(n^p\left(1 + c\int_1^n u^{-p}du\right)\right) && (2) \\ &= \Theta\left(n^p\left(1 + \frac{cn^{1-p}}{1-p}\right)\right) && (3) \\ &= \Theta(n) && (4) \end{aligned}$$

(f-1)

$\log\log n = o(\log n)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{\log n}{\log\log n}$ is unbounded.
$\log n^2 = \Theta(\log_5 n) = \Theta(\log n)$, because of rules of logarithms (change of base, exponents inside logarithms).
$\log n = o(\log^2 n)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{\log^2 n}{\log n}$ is unbounded.
$\log^2 n = o(\sqrt{n})$, because the limit as $n \longrightarrow \infty$ of $\dfrac{\sqrt{n}}{\log^2 n}$ is unbounded, as determined by L'Hopital's rule.

(f-2)

$n\log n = \Theta(\log n!)$, by both Stirling's Approximation and the rules of logarithms ($\log ab = \log a + \log b$).
$n\log n = o(n^{4/3})$, because the limit as $n \longrightarrow \infty$ of $\dfrac{n^{4/3}}{n\log n}$ is unbounded, as determined by L'Hopital's rule.
$n^{4/3} = o(n^2)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{n^{4/3}}{n^2}$ is 0.
$n^2 = o(2^n)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{n^2}{2^n}$ is 0.

(f-3)

$2^{\log n^{\log\log n}} = o(e^n)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{2^{\log n^{\log\log n}}}{e^n}$ is 0, which we can see by first taking the logarithm of the limit and noticing that $n > \log n^{\log\log n}$ for large n.
$e^n = o(n^{n-1})$, because the limit as $n \longrightarrow \infty$ of $\dfrac{ne^n}{n^n}$ is 0
$n^{n-1} = o(n^n)$, because the limit as $n \longrightarrow \infty$ of $\dfrac{1}{n}$ is 0.

Problem 1-2

(a)

Let every person in the group G representing the MIT Class of 2050 be denoting p_i, for $i \in [1, n]$. Let G_1 represent the group of students that belong to the same universe as p_1, and G_x represent those who do not belong the same universe as p_1. If we ask queries "Are person p_1 and p_i from the same universe?" for all $i \in [2, n]$, then we can determine the members of G_1 and G_x, which are mutually exclusive sets by construction. The larger group must be those from Earth-X because there are more X-ers than Earthers, and everyone in that group must be from the same universe, due to transitivity of equivalence.

Since we make $O(n)$ queries each taking $O(1)$ time, the total running time this algorithm is $O(n)$.

(b)

We will show that it is impossible to determine the group of Earthers if X-ers can lie and there are more X-ers than Earthers. We will do this by construction.

First, we notice that if we ask the same two people (A, B) for their response, they will always respond in the same way because switching answers gives away whether or not one is an X-er.

After this observation, we realize that the maximum number of meaningful queries we can make is $n(n-1)/2$, one query for every possible pairing. We aim to demonstrate that the X-ers can always respond in a way that makes it impossible to determine, because of symmetry, whether or not a subgroup is from Earth or from Earth-X.

Essentially, if the X-ers work to maintain that the number of (E, E) responses they give is the same as the number of (E, E) responses that real Earthers give, then Ben is unable to distinguish between a fake (E, E) response and a real (E, E) response, meaning that Ben won't be able to confidently create a group of all Earthers. In particular, the X-ers can always accomplish this, because the number of real (E, E) responses that can possibly be given is fewer than the number of fake (E, E) responses that can be given (because there are more X-ers than Earthers).

As such, the X-ers can always respond in a way that makes then indistinguishable from Earthers because of symmetry, and thus Ben cannot determine who is from Earth.

(c)

Let (a, b) be the response to query (A, B), where $a, b \in E, X$, and a, b correspoonding to the response that A, B gave about the other person, respectively. Then, there are three unique responses:

1. (E, E), where A says B is from Earth, and B says A is from Earth.
2. (E, X), where A says B is from Earth, and B says A is from Earth-X.
3. (X, X), where A says B is from Earth-X, and B says A is from Earth-X.

For response 1, there are either 0 X-ers or 2 X-ers. If both A and B are from Earth, then they would respond truthfully with (E, E). If A and B are both from Earth-X, then they can collaborate and lie to give the response (E, E). However, if one is from Earth and the other is from Earth-X, then the person from Earth would be truthful and respond with X, which is impossible because we are only considering the case of the response (E, E).

For response 2, there are either 1 X-ers or 2 X-ers. There cannot be 0 X-ers, because then the response would be (E, E), not (E, X). One X-er is possible because the one from Earth will respond with X truthfully, while the X-er can be truthful as well as respond with E. Two X-ers is also possible because one may tell the truth and the other can lie and give the response (E, X).

For response 3, again there are either 1 or 2 X-ers. It is impossible to have 0 X-ers, because then the response would be (E, E), not (X, X). 1 X-er is possible because the Earther will respond truthfully with X, while the X-er may lie and respond with X as well. Likewise, 2 X-ers is possible, because they can both decide to be truthful and both say that the other is from Earth-X.

(d)

One of the key insights to solving the problem is to realize that once we have identified a single person is from Earth, we are done, because we may use that person as an "oracle" who will tell us the true identity of every other person in the group.

In order to accomplish this, we will maintain the invariant that in our group G of students we are focusing on, **there will always be more Earthers than X-ers**. Why? Because if we can whittle down the size of the group we are considering while still maintaining this invariant, then if we reduce the size of the group to a single person, we are done: the last one standing must be from Earth.

We will thus proceed as follows: pair every student with another student as (A_i, B_i), $i \in [1, \frac{n}{2}]$, leaving one student by himself/herself if n is odd. Then, ask each pair (A_i, B_i) for their response to the question as described in part (b).

The second key insight is as follows: if the response to (A_i, B_i) is either (E, X) or (X, X), you may immediately remove them from the group G of students we are focusing on while still maintaining the invariant. This is because there is at least 1 X-er in the pair, and that we begin with more Earther than X-ers, so removing one of each still maintains that there are more Earthers and X-ers in group G. Of course, if both students in the pair turn out to be X-ers, then we still will have maintained the invariant, because only the number of X-ers will have decreased.

The more difficult case is when the response to (A_i, B_i) is (E, E). Then, either there are 0 X-ers, or there are 2. The final insight is to realize that this forces the identity of A_i and B_i to be the same: either they are both X-ers, or they are both Earthers. Thus, we may immediately remove either A_i or B_i from group G (or, in other view, merging A_i and B_i into a single person) while still maintaining the invariant, because if there were more Earthers beforehand, then there must be more Earther/Earther pairs than X-er/X-er pairs, so removing one person from every pair will still leave more Earthers than X-ers. In this way, we are able to consider every possible response, and strictly decrease the size of the group at every iteration, which means that we will always terminate with a single Earther remaining.

Finally, once we have identified a single Earther, we interrogate every other member of the class of 2050 using this Earther and determine whether or not the other member is from Earth.

The running time of this algorithm is given by the recurrence $T(n) = T(n/2)+O(n)$, because at each iteration, we need to make $O(n)$ queries each of $O(1)$ time, and in the worst case reduce the size of the problem by half (if every pair answers (E, E)). The solution to this recurrence is $\Theta(n)$ by case 3 of the Master Theorem. Adding the additional $\Theta(n)$ time needed to interrogate every other student using the Earther, the final runtime complexity of our algorithm is $\Theta(n)$.