

Asymptotic Methods

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Part I

Basics of asymptotic approximations

1 Introduction

Some things in maths are easy to understand: x^a , e^x , $\ln x$, and sums of products of these.

Most things in maths are hard to understand: Complicated combinations of the above, integrals, equations, differential equations.

Asymptotic methods allow us to turn hard things into easy things, provided that some quantity is **very small or very large**. For example, Taylor series are an example of an asymptotic expansion, and tell you what the function looks like near the expansion point (when the distance to the point is very small). However, asymptotic expansions are much more general than that, and in this introduction we'll see some brief examples of what is to come in this module. We'll study these examples again later, so don't worry about the details for now and just sit back and enjoy the ride!

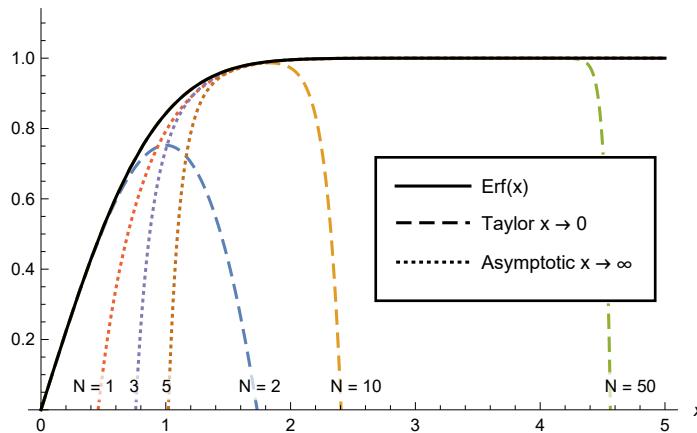
Note that the focus of this module is on methods and ideas for tackling problems rather than rigorous proofs. Some of the problems will be motivated by physical applications, but only the methods themselves will be examinable.

Example(s) 1.1. Consider the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, one of the simplest integrals that can't be evaluated in terms of elementary functions.

Using the Taylor series of e^{-t^2} about $t = 0$, we can integrate each term to obtain a Taylor series

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1) n!}. \quad (1.1)$$

This Taylor series actually converges for all x , but it does so very slowly when x is large. Also, the function is nearly constant for $x > 2$, but this is not at all obvious from the Taylor series.



Let's try to find an approximation of $\text{erf}(x)$ for large x instead. What's the simplest approximation we can do? The function looks nearly constant, so let's just take the limit

$$\text{erf}(x) \approx \text{erf}(+\infty) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt = 1. \quad (1.2)$$

Can we improve on this trivial approximation? Let's write down its error

$$\text{erf}(x) - 1 = \frac{2}{\sqrt{\pi}} \left[\int_0^x - \int_0^{\infty} \right] e^{-t^2} dt = -\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt, \quad (1.3)$$

and try to approximate it for large x . The trick in this case is to integrate by parts by inserting an extra factor t to make te^{-t^2} ,

$$\operatorname{erf}(x) - 1 = -\frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{t} te^{-t^2} dt = \frac{2}{\sqrt{\pi}} \left\{ \left[\frac{1}{2t} e^{-t^2} \right]_x^\infty + \int_x^\infty \frac{1}{2t^2} e^{-t^2} dt \right\} \approx \frac{2}{\sqrt{\pi}} \left\{ -\frac{1}{2x} e^{-x^2} \right\}. \quad (1.4)$$

Was the last step, discarding the next integral, a good approximation? It certainly looks so in the figure, but how do we justify it without plotting it on a computer? Let's look at the discarded integral, and try to make te^{-t^2} again, but this time doing some simplifying approximations using $t \geq x$ before evaluating the integral:

$$\left| \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{2t^2} e^{-t^2} dt \right| = \left| \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{2t^3} te^{-t^2} dt \right| \leq \left| \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{1}{2x^3} te^{-t^2} dt \right| = \frac{2}{\sqrt{\pi}} \frac{1}{4x^3} e^{-x^2}. \quad (1.5)$$

We summarise our result as

$$\operatorname{erf}(x) = 1 + \frac{2}{\sqrt{\pi}} e^{-x^2} \left\{ -\frac{1}{2x} + O(x^{-3}) \right\}, \quad (1.6)$$

where the O stands for “a term that's bounded by some constant times this thing”. When x is large, the $O(x^{-3})$ error is small relative to the term $-1/(2x)$, so we have a good approximation.

We could continue to integrate by parts to obtain arbitrarily many terms, making the remainder smaller each time,

$$\begin{aligned} \operatorname{erf}(x) = 1 + \frac{2}{\sqrt{\pi}} e^{-x^2} & \left\{ -\frac{1}{2x} + \frac{1}{4x^3} - \frac{3}{8x^5} + \frac{3 \times 5}{16x^7} - \frac{3 \times 5 \times 7}{32x^9} + \dots \right. \\ & \left. \dots + (-1)^N \frac{3 \times 5 \times \dots \times (2N-3)}{2^N x^{2N-1}} + O(x^{-(2N+1)}) \right\}. \end{aligned} \quad (1.7)$$

We say that we have a full asymptotic expansion for erf , and write

$$\operatorname{erf}(x) \sim 1 + \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=1}^{\infty} (-1)^n \frac{3 \times 5 \times \dots \times (2n-3)}{2^n x^{2n-1}}, \quad (1.8)$$

but note that the thing on the right-hand side is just a formal series and cannot be evaluated, because it diverges for any fixed value of x . The problem is that although each term is smaller than the previous one for sufficiently large x , the value of x required for this to hold increases with n , so at each fixed value of x the terms eventually start to grow in size instead of shrink.

Example(s) 1.2. Here is an integral example with a very different flavour. Let's estimate the factorial $n!$ for large n .

The trick (or rather, one possible trick) is to use the Gamma function integral expression and make a change variables,

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt \stackrel{t=ns}{=} n^{n+1} \int_0^\infty s^n e^{-ns} ds = n^{n+1} \int_0^\infty e^{n(\ln s - s)} ds. \quad (1.9)$$

According to the Laplace method, which we will learn about in this module, this integral will be dominated by the contribution from near the maximum of $f(s) = \ln s - s$, which occurs at $s = 1$, and the dominant contribution comes from the Taylor expansion of f up to the first non-constant term, so we obtain the estimate

$$n! \approx n^{n+1} \int_{-\infty}^{+\infty} e^{n(-1-(s-1)^2/2)} ds = n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}} = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n. \quad (1.10)$$

This result is called Stirling's approximation, and by keeping more terms in the Taylor expansion of f it's possible to obtain more terms,

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \left[1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \dots \right]. \quad (1.11)$$

Again, this asymptotic expansion is not a convergent series. We note that the first correction, $1/12n$ is surprisingly small even for the not-very-large number $n = 1$, and in fact the leading-order approximation is accurate to within 8% for $n \geq 1$, and within 1% for $n \geq 10$.

Example(s) 1.3. Let's move on to solving equations. Consider the equation $xe^x = y$ for large $y \rightarrow +\infty$. (The exact solution is known as the Lambert W -function, $x = W(y)$.)

It is clear that, in order for y to be large, either x or e^x (or both) must be large. Since e^x is much larger than x , we might expect to obtain a reasonable approximation by approximating

$$e^x \approx y \Rightarrow x \approx \ln y. \quad (1.12)$$

Based on this dominant balance, we can set up an iteration

$$x = \ln y - \ln x, \quad (1.13)$$

in which we repeatedly substitute our estimate for x into the right-hand side to obtain a more accurate estimate. Starting from $x \approx \ln y$, we find

$$x = \ln y - \ln x = \ln y + O(\ln \ln y) \quad (1.14)$$

$$\Rightarrow x = \ln y - \ln(\ln y + O(\ln \ln y)) = \ln y - \ln \ln y + O\left(\frac{\ln \ln y}{\ln y}\right) \quad (1.15)$$

$$\Rightarrow x = \ln y - \ln \left(\ln y - \ln \ln y + O\left(\frac{\ln \ln y}{\ln y}\right) \right) = \ln y - \ln \ln y + \frac{\ln \ln y}{\ln y} + O\left(\frac{(\ln \ln y)^2}{(\ln y)^2}\right). \quad (1.16)$$

This is an example of an asymptotic expansion whose form would be very difficult to guess beforehand!

Example(s) 1.4. We finish with an example of a differential equation, solved using the method of matched asymptotic expansions. Consider $\varepsilon y'' + y' - y = 0$ with boundary conditions $y(0) = 0$, $y(1) = 1$, for small $\varepsilon > 0$.

We first set $\varepsilon = 0$ and try to solve the resulting equation,

$$y' - y \approx 0 \Rightarrow y \approx Ce^x = e^{x-1}, \quad (1.17)$$

where we have chosen C to satisfy the boundary condition $y(1) = 1$. As setting $\varepsilon = 0$ has reduced the order of the differential equation, this solution cannot be made to satisfy the other boundary condition too.

In order to satisfy the boundary condition at $x = 0$, we need to make the second derivative term dominant again, which we achieve by assuming that the solution varies on a much smaller length scale, comparable to ε , near the boundary $x = 0$. We represent this rapid variation using a change of variables $x = \varepsilon X$, noting that $d/dx = \varepsilon^{-1} d/dX$. Writing $y(x) = Y(X)$ for the solution in this region, we obtain

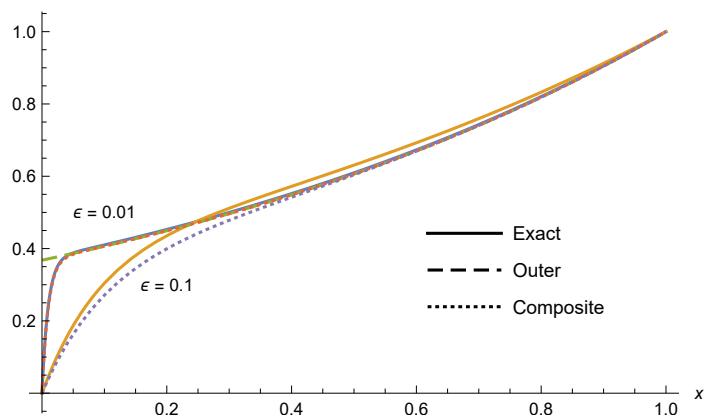
$$\frac{1}{\varepsilon} Y''(X) + \frac{1}{\varepsilon} Y'(X) - Y(X) = 0 \Rightarrow Y'' + Y' \approx 0 \Rightarrow Y = A + Be^{-X} = A(1 - e^{-X}), \quad (1.18)$$

where we have chosen B to satisfy the boundary condition $Y(0) = 0$. The second constant A is clearly somehow determined by the other condition $y(1) = 1$, but how?

The answer is to “match” the “outer” solution $y(x) = e^{x-1}$ and the “inner” solution $Y(X) = A(1 - e^{-X})$ in an intermediate “overlap region”. At leading order, we can simply observe that for small $x = \varepsilon X$, the outer solution tends to the value e^{-1} , while for large $X = x/\varepsilon$, the inner solution tends to the value $Y = A$. Thus, the only consistent choice is $A = e^{-1}$. We can then form a “composite” solution by adding the inner and outer solutions and subtracting the common overlap

$$y(x) + Y(X) - e^{-1} = e^{x-1} + e^{-1} \left(1 - e^{-x/\varepsilon} \right) - e^{-1} = e^{-1} \left(e^x - e^{-x/\varepsilon} \right). \quad (1.19)$$

As seen in the figure below, the asymptotic result agrees well with the exact solution for small ε , and the agreement improves as $\varepsilon \rightarrow 0$. We could in principle calculate the higher-order corrections by expanding $y = y_0 + \varepsilon y_1 + \dots$ and $Y = Y_0 + \varepsilon Y_1 + \dots$ and then solving and matching order by order in ε . However, already at leading order we have obtained a good approximation as well as information about the spatial structure of the solution and insight into which terms form the dominant balance in each region.



2 Definitions

The goal of the various methods taught in this module is to obtain approximate explicit expressions for quantities that it may be difficult or impossible to obtain exact explicit expressions for. Thus, we need to start by defining what it means for something to be a good approximation. (This section contains a lot of mathematical rigour, but don't worry – we'll ignore most of it later!)

2.1 Order notation

Definition 2.1. Let x_0 be a complex number. We say that

- $f(x) = O(\phi(x))$ [“ $f(x)$ is big-oh of $\phi(x)$ ”] as $x \rightarrow x_0$ if

$$\begin{aligned} f/\phi &\text{ is bounded as } x \rightarrow x_0, \\ \text{i.e. there exist } M > 0 \text{ and } \delta > 0 \text{ such that } |f| \leq M|\phi| \text{ for } 0 < |x - x_0| < \delta. \end{aligned} \quad (2.1)$$

- $\begin{cases} f(x) = o(\phi(x)) & [\text{“}f(x)\text{ is little-oh of }\phi(x)\text{”}] \\ \text{or } f(x) \ll \phi(x) & [\text{“}f(x)\text{ is much smaller than }\phi(x)\text{”}] \\ \text{or } \phi(x) \gg f(x) & [\text{“}\phi(x)\text{ is much larger than }f(x)\text{”}] \end{cases}$ as $x \rightarrow x_0$ if

$$\begin{aligned} f/\phi &\rightarrow 0 \text{ as } x \rightarrow x_0, \\ \text{i.e. for all } M > 0 \text{ there exists } \delta > 0 \text{ such that } |f| \leq M|\phi| \text{ for } 0 < |x - x_0| < \delta. \end{aligned} \quad (2.2)$$

- $f(x) = \text{ord}(\phi)$ [“ $f(x)$ is order $\phi(x)$ ”] as $x \rightarrow x_0$ if

$$\begin{aligned} f/\phi &\text{ is bounded and bounded away from zero as } x \rightarrow x_0, \\ \text{i.e. there exist } m > 0, M > 0 \text{ and } \delta > 0 \text{ such that } m|\phi| \leq |f| \leq M|\phi| \text{ for } 0 < |x - x_0| < \delta. \end{aligned} \quad (2.3)$$

Analogous definitions apply for $x_0 = \pm\infty$, replacing the conditions $0 < |x - x_0| < \delta$ with $x > 1/\delta$ or $x < -1/\delta$.

For real x and x_0 , the limit can be specified as being from one side only, using “ $x \searrow x_0$ ” or “ $x \rightarrow x_0^+$ ” for approaching from the right, and “ $x \nearrow x_0$ ” or “ $x \rightarrow x_0^-$ ” for approaching from the left. For complex x and x_0 , the limit is sometimes specified as being in a certain sector, i.e. for a specific range of values of $\arg(x - x_0)$.

If ϕ has infinitely many zeroes as $x \rightarrow x_0$ (e.g. $\sin x$ as $x \nearrow \infty$), then the definitions using f/ϕ don't make sense, but those with $|\phi|$ multiplied out still do. (If ϕ has only finitely many zeroes, then sufficiently close to x_0 there are no zeroes so the ratio f/ϕ does make sense.)

- Remark(s) 2.1.**
- The O , o and ord are like asymptotic analogues of \leq , $<$ and $=$ that ignore $\text{ord}(1)$ prefactors.
 - Typically f is something complicated, and we want to compare it to a simple ϕ like a power, exponential or logarithm.

Example(s) 2.1.

- A simple limit to consider is $x \nearrow \infty$. In this limit (see the problem sheet for proofs):

- The ordering of power terms Ax^α (with $A \neq 0$) is determined by the exponent – a larger exponent yields an asymptotically larger term, e.g. $x^2 \gg x$ and $x^{-1} \gg 10000x^{-2}$.
(We say “**algebraic**” for behaviour that is similar to a power.)
- Exponentials beat all powers, e.g. $\exp(x) \gg x$ and $\exp(0.001x^{0.001}) \gg x^{100}$.
But make sure the exponent is a power and $\nearrow \infty$. For example, $\exp(x^{-1}) \rightarrow 1$.
(We say “**exponential**” for behaviour that is comparable to the exponential of a power.)
- Positive powers beat logarithms to any power, e.g. $x \gg \ln x$ and $x^{1.001} \gg x(\ln x)^{100}$.
(We say “**logarithmic**” for behaviour that is comparable to the power of a logarithm.)

- Similarly, for $x \searrow 0$, the ordering of powers is reversed, while exponentials (now with inverse powers in the exponent!) still beat powers and negative powers beat logarithms.
- The asymptotic behaviour of a sum of terms is determined by the largest one, e.g.

$$x^2 + 3x^3 + 5x^5 = \begin{cases} \text{ord}(x^5) & \text{as } x \nearrow \infty, \\ \text{ord}(x^2) & \text{as } x \rightarrow 0. \end{cases} \quad (2.4)$$

- For finite non-zero limits x_0 , shifting to expanding about zero, e.g. by defining $\varepsilon = x - x_0 \rightarrow 0$, often makes things clearer and saves writing.

For example, what is the behaviour of $f(x) = x^2 - 1$ as $x \rightarrow 1$? Substitute in $x = 1 + \varepsilon$ to find

$$f(x) = (1 + \varepsilon)^2 - 1 = 1 + 2\varepsilon + \varepsilon^2 - 1 = 2\varepsilon + \varepsilon^2 = \text{ord}(\varepsilon) = \text{ord}(x - 1) \quad \text{as } \varepsilon = x - 1 \rightarrow 0. \quad (2.5)$$

2.1.1 Manipulating order notation

Throughout this module we will need to do a lot of manipulation of expressions involving order notation, so we need to know what is allowed and what is not.

Firstly, the analogy with \leq , $<$ and $=$ is helpful in some cases, e.g.

$$f = o(\phi) \Rightarrow f = O(\phi), \quad f = \text{ord}(\phi) \Rightarrow f = O(\phi), \quad f = o(\phi) \Rightarrow f \neq \text{ord}(\phi). \quad (2.6)$$

However, the analogy fails in the following examples, because the ratio f/g can alternate indefinitely between being large and small as $x \rightarrow x_0$:

- There are f and g that satisfy neither $f \ll g$, $f = \text{ord}(g)$ nor $f \gg g$, or neither $f = O(g)$ nor $g = O(f)$. (Not every pair of functions is comparable.)
- There exist $f = O(g)$ that satisfy neither $f = o(g)$ nor $f = \text{ord}(g)$. (You should think of O as meaning “bounded by”, which is a weaker criterion than “either much smaller than or comparable to”.)

Note also that if $f = o(\phi)$ and $g = O(\phi)$ then $f + g = O(\phi)$, not $f + g = o(\phi)$.

Here is an example of a longer algebraic manipulation with order notation, as $x \rightarrow 0$:

$$(x + O(x^3))^2 = x^2 + 2xO(x^3) + O(x^3)^2 = x^2 + O(x^4) + O(x^6) = x^2 + O(x^4) = x^2 + o(x^3). \quad (2.7)$$

The use of equals signs is actually an abuse of notation, because the symbols are treated differently on the left- and right-hand sides: At each equals sign, we mean that for every possible choice of functions satisfying the order symbols on the left, the equality holds for some specific choice of functions satisfying the order symbols on the right. As a consequence, the equality sign is not symmetric, e.g. $o(\phi) = O(\phi)$ holds but $O(\phi) = o(\phi)$ does not.

Various algebraic manipulations you can do with O , o :

- Replace a strict bound with a weaker bound, e.g.

$$o(\phi) = O(\phi), \quad \text{if } f \ll g \text{ then } o(f) = o(g) \text{ and } O(f) = O(g). \quad (2.8)$$

- Add together (keeping only the one with the largest argument and the largest symbol), e.g.

$$O(x^2) + o(x^2) + o(x) = \begin{cases} O(x^2) & \text{as } x \rightarrow \infty, \\ o(x) & \text{as } x \rightarrow 0. \end{cases} \quad (2.9)$$

- Multiply by a non-zero constant or by an $\text{ord}(1)$ function without any effect, e.g.

$$2O(x) = O(x), \quad (2 + \sin(x))o(x^2) = o(x^2) \text{ (provided } x \text{ is real).} \quad (2.10)$$

- Multiply together or multiply by a function, e.g.

$$o(f)o(g) = o(fg) \text{ and } fO(g) = O(fg). \quad (2.11)$$

- Take a positive power, e.g. $O(\phi)^\alpha = O(\phi^\alpha)$ for $\alpha > 0$
- Integrate from the limiting point, provided ϕ doesn't change sign and the integral converges, e.g.

$$\int_{x_0}^x O(\phi(x)) dx = O\left(\int_{x_0}^x \phi(x) dx\right). \quad (2.12)$$

(But if ϕ changes sign then $\int_{x_0}^x \phi(x) dx$ might become “surprisingly small” due to cancellation.)

However, you cannot do the following:

- Exponentiate, e.g. $2x = O(x)$ but $\exp(2x) \gg \exp(x)$ as $x \nearrow \infty$.
- Differentiate, e.g. $x \sin x = O(x)$ but $x \cos x + \sin x \neq O(1)$ as $x \nearrow \infty$.

Although handling O and o is relatively intuitive, with ord things are more tricky due to the risk of cancellation, so that even something as simple as $\text{ord}(\phi) + \text{ord}(\phi) = \text{ord}(\phi)$ is false (because e.g. $x + (-x) = 0 \neq \text{ord}(x)$). Since the main point is to bound errors from above using O and o , we avoid manipulating ord and just convert to O before manipulation. When bounding errors I will often accidentally say “order” when I mean “big-oh”…

We do still use ord , typically when measuring overall quantities as opposed to approximation error bounds.

2.2 Asymptotic expansions

Definition 2.2. We say that $f(x) \sim g(x)$ [“ $f(x)$ is asymptotic to $g(x)$ ” or “ $f(x)$ twiddles $g(x)$ ”] as $x \rightarrow x_0$ if

$$f/g \rightarrow 1 \text{ as } x \rightarrow x_0, \quad \text{i.e. } f - g = o(g) \quad (2.13)$$

Remark(s) 2.2. • The \sim is stricter than ord as it does distinguish $\text{ord}(1)$ prefactors.

- If $f \sim \phi$ then $f = \text{ord}(\phi)$.

Definition 2.3. We say that $f(x)$ is an asymptotic approximation of $g(x)$ to order $\delta(x)$ as $x \rightarrow x_0$ if

$$f - g = o(\delta). \quad (2.14)$$

Definition 2.4. Given a sequence $\delta_0(x) \gg \delta_1(x) \gg \delta_2(x) \gg \dots$, we say that $f(x)$ has the (finite) asymptotic expansion

$$f(x) \sim \sum_{n=0}^N a_n \delta_n(x) \quad \text{if} \quad f(x) = \sum_{n=0}^N a_n \delta_n(x) + o(\delta_N(x)), \quad (2.15)$$

i.e. the error is smaller than the last term in the expansion. We say that $f(x)$ has the (infinite/full) asymptotic expansion

$$f(x) \sim \sum_{n=0}^{\infty} a_n \delta_n(x) \quad \text{if} \quad f(x) = \sum_{n=0}^N a_n \delta_n(x) + o(\delta_N(x)) \text{ for all } N, \quad (2.16)$$

i.e. for any partial sum the error is smaller than the last term kept.

Remark(s) 2.3. • We have now redefined the “ \sim ” symbol to have a stricter meaning than before (the difference must be small relative to the last term kept in the sum on the right-hand side, rather than just small relative to the whole left- or right-hand side).

- A common mistake is to treat the infinite asymptotic expansion as a function itself, or try to sum up the tail of the series, but this is not correct, as it can be divergent.
- Asymptotic expansions can be added and multiplied (provided the terms are appropriately reordered and/or deleted in the result).

- Like o and O , asymptotic expansions can be integrated with respect to the parameter (provided the δ_n do not change sign), but not differentiated (although we'll try anyway).

Definition 2.5. The sequence $\delta_n(x)$ is called the asymptotic scale for the expansion.

Remark(s) 2.4. • A Taylor series is an asymptotic expansion, whose scale functions are natural powers. (Indeed, Taylor's theorem with the Peano form of the remainder is exactly the definition of an asymptotic expansion.)

- Often asymptotic expansions are of the form $f(x) \sim e^{Ax^\alpha} x^\beta \sum_{n=0}^{\infty} a_n (x^\gamma)^n$, i.e. have terms that reduce by a factor x^γ each step, with possibly a pre-multiplying power and/or exponential.
- Sometimes there are logarithms inserted, e.g.

$$f(x) \sim (\dots) [a_0 + a_{1,1}x \ln x + a_{1,0}x + a_{2,2}x^2(\ln x)^2 + a_{2,1}x^2(\ln x) + a_{2,0}x^2 + \dots] \text{ as } x \rightarrow 0. \quad (2.17)$$

- For a given function f and asymptotic scale $\{\delta_n(x)\}$, the coefficients a_N can be determined by

$$a_N = \lim_{x \rightarrow x_0} \frac{f(x) - \sum_{n=0}^{N-1} a_n \delta_n(x)}{\delta_N(x)}, \quad (2.18)$$

so the expansion is unique. But a given function f can have different expansions for different scales.

- There is usually only one reasonable/simplest choice of asymptotic scale for a function, and it comes out naturally from the calculation of the expansion.
- Different functions can have the same asymptotic expansion, if they differ by a quantity smaller than all the scale functions:

$$\text{If } f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n} \text{ as } x \nearrow \infty \text{ then } f(x) + e^{-x} \sim \sum_{n=0}^{\infty} a_n x^{-n} \text{ too.} \quad (2.19)$$

The simplest approximation task we might think of is converting an exact (but complicated) explicit expression into an approximate (but simple) explicit expression, like determining the Taylor expansion of a function, but more general.

Example(s) 2.2. Well-known Taylor expansions for small x :

$$e^x = 1 + x + \frac{x^2}{2} + \dots, \quad \sin(x) = x - \frac{x^3}{6} + \dots, \quad \cos(x) = 1 - \frac{x^2}{2} + \dots \quad (2.20)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots, \quad (1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \dots, \quad (2.21)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots. \quad (2.22)$$

We can use these to obtain other expansions such as:

$$\text{As } x \rightarrow 0 : \quad \sin(2x) = 2x - \frac{4x^3}{3} + O(x^5), \quad (2.23)$$

$$\text{As } x \nearrow \infty : \quad \ln(1+e^{-x}) = e^{-x} - \frac{e^{-2x}}{2} + O(e^{-3x}). \quad (2.24)$$

Remark(s) 2.5. In more complicated cases, we have to substitute expansions into each other, and it is important to pay attention to what point is being expanded about. If x is the leading-order term and $y \ll x$ represents all the corrections, then we need to expand a function of $(x+y)$ about x . For powers and logarithms, we can use

$$(x+y)^p = x^p(1+(y/x))^p = x^p(1+p(y/x) + p(p-1)(y/x)^2/2 + \dots), \quad (2.25)$$

$$\ln(x+y) = \ln\{x[1+(y/x)]\} = \ln x + \ln(1+(y/x)) = \ln x + (y/x) - \frac{(y/x)^2}{2} + \dots, \quad (2.26)$$

and in particular at leading order we obtain just x^p and $\ln(x)$.

However, for exponentials we have $e^{x+y} = e^x e^y$, which does not reduce to e^x in general. (The same applies to $\cos(x+y)$ and $\sin(x+y)$, but with the trigonometric addition formulae.) We need to additionally have $y \ll 1$ in order to deduce

$$e^{x+y} = e^x (1 + y + y^2/2 + \dots) \sim e^x. \quad (2.27)$$

Example(s) 2.3. As $x \rightarrow 0$,

$$(\sin x)^{-1} = \left[x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7) \right]^{-1} = x^{-1} \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + O(x^6) \right) \right]^{-1} = \quad (2.28)$$

$$= x^{-1} \left[1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + O(x^6) \right) + \left(\frac{x^2}{6} + O(x^4) \right)^2 + O(x^2)^3 \right] = \quad (2.29)$$

$$= x^{-1} \left[1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + O(x^6) \right) + \left(\frac{x^4}{36} + O(x^6) \right) + O(x^6) \right] = \quad (2.30)$$

$$= x^{-1} \left[1 + \frac{x^2}{6} + \frac{7x^4}{360} + O(x^6) \right] = x^{-1} + \frac{x}{6} + \frac{7x^3}{360} + O(x^5). \quad (2.31)$$

Hence,

$$\exp[(\sin x)^{-1}] = \exp \left[x^{-1} + \frac{x}{6} + \frac{7x^3}{360} + O(x^5) \right] = \exp(x^{-1}) \exp \left[\frac{x}{6} + O(x^3) \right] = \quad (2.32)$$

$$= \exp(x^{-1}) \left[1 + \frac{x}{6} + \frac{x^2}{72} + O(x^3) \right]. \quad (2.33)$$

Note how it is useful here to have kept the error estimate $O(x^3)$, so that in the end we know that the $x^2/72$ term is accurate. Without that estimate, we would only be able to conclude

$$(\sin x)^{-1} \sim x^{-1} + \frac{x}{6} \Rightarrow \exp[(\sin x)^{-1}] \sim x^{-1} \left[1 + \frac{x}{6} \right]. \quad (2.34)$$

If we only had the leading-order estimate $(\sin x)^{-1} \sim x^{-1}$, then we would not be able to calculate the leading-order behaviour of $\exp[(\sin x)^{-1}]$ at all.

3 Equations with a small/large parameter

In this section, we seek to obtain explicit approximations to the solutions of algebraic (i.e. non-differential) equations.

3.1 Polynomial equations

We start with polynomial equations, i.e. equations of the form

$$a_n(\varepsilon)x^n + \cdots + a_1(\varepsilon)x + a_0(\varepsilon) = 0, \quad (3.1)$$

where the goal is to obtain an expansion for the solution $x = x(\varepsilon)$ as $\varepsilon \searrow 0$.

Example(s) 3.1. Let's solve the equation $\varepsilon x^2 + (\cos \varepsilon)x - e^\varepsilon = 0$ as $\varepsilon \searrow 0$.

Let's forget for a moment that we can solve general quadratic equations exactly, and instead attempt to solve this equation using asymptotic methods. We note that $\cos \varepsilon$ and e^ε are easily expanded as $\varepsilon \searrow 0$ so won't pose a big problem.

What is a good leading-order approximation x_0 to the solution when ε is very small? Let's just set $\varepsilon = 0$:

$$x_0 - 1 = 0 \Rightarrow x_0 = 1. \quad (3.2)$$

How do we improve on this estimate? The most naive **power-series method** is to assume that the solution takes the form of a power series $x \sim \sum_{n=0}^{\infty} x_n \varepsilon^n$. In order to find the first four terms, we then substitute

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + O(\varepsilon^4) \quad (3.3)$$

into the equation and collect terms of the same order:

$$0 = \varepsilon [x_0^2 + \varepsilon(2x_0 x_1) + \varepsilon^2(2x_0 x_2 + x_1^2) + O(\varepsilon^3)] \quad (3.4)$$

$$+ [1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)] [x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + O(\varepsilon^4)] \quad (3.5)$$

$$- [1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{6}\varepsilon^3 + O(\varepsilon^4)] = \quad (3.6)$$

$$= x_0 - 1 + \varepsilon [x_0^2 + x_1 - 1] + \varepsilon^2 [2x_0 x_1 - \frac{1}{2}x_0 + x_2 - \frac{1}{2}] \quad (3.7)$$

$$+ \varepsilon^3 [2x_0 x_2 + x_1^2 - \frac{1}{2}x_1 + x_3 - \frac{1}{6}] + O(\varepsilon^4). \quad (3.8)$$

(By thinking ahead we knew we could stop the expansion in the first term at the $O(\varepsilon^3)$ remainder.) Now it's simply a question of solving the equation order by order to determine the coefficients x_n :

$$\text{ord}(\varepsilon^0) : 0 = x_0 - 1 \Rightarrow x_0 = 1 \quad (3.9)$$

$$\text{ord}(\varepsilon^1) : 0 = x_0^2 + x_1 - 1 = x_1 \Rightarrow x_1 = 0 \quad (3.10)$$

$$\text{ord}(\varepsilon^2) : 0 = 2x_0 x_1 - \frac{1}{2}x_0 + x_2 - \frac{1}{2} = x_2 - 1 \Rightarrow x_2 = 1 \quad (3.11)$$

$$\text{ord}(\varepsilon^3) : 0 = 2x_0 x_2 + x_1^2 - \frac{1}{2}x_1 + x_3 - \frac{1}{6} = x_3 + \frac{11}{6} \Rightarrow x_3 = -\frac{11}{6}. \quad (3.12)$$

We conclude that one solution is

$$x = 1 + \varepsilon^2 - \frac{11}{6}\varepsilon^3 + O(\varepsilon^4), \quad (3.13)$$

and we note the expansion can be continued to arbitrary order, as the $O(\varepsilon^n)$ equation will be of the form $0 = \cdots + x_n + \cdots$, which always has a single solution x_n in terms of the previous coefficients. This is an example of a **regular** solution. Obviously the power-series method only works if the solution is actually of power-series form.

A more general method of finding the corrections to the leading-order result is the **iteration method**. The idea is to emulate the solution steps for the leading-order result while treating all other occurrences of x as constant, in order to rewrite the equation to the form $x = f_\varepsilon(x)$, where f_ε does not depend on x

at leading order. We can then iteratively apply f_ε to our approximation and obtain successively better approximations to the solution. In our example, we can rewrite the equation as follows:

$$\varepsilon x^2 + (\cos \varepsilon)x - e^\varepsilon = 0 \quad \Rightarrow \quad x = \frac{e^\varepsilon - \varepsilon x^2}{\cos \varepsilon}. \quad (3.14)$$

We can then iteratively produce, starting from the leading-order scaling $x = O(1)$,

$$x = \frac{e^\varepsilon - \varepsilon O(1)}{\cos \varepsilon} = 1 + O(\varepsilon) \quad (3.15)$$

$$\Rightarrow x = \frac{e^\varepsilon - \varepsilon(1 + O(\varepsilon))}{\cos \varepsilon} = \frac{1 + \varepsilon - \varepsilon + O(\varepsilon^2)}{1 + O(\varepsilon^2)} = 1 + O(\varepsilon^2) \quad (3.16)$$

$$\Rightarrow x = \frac{e^\varepsilon - \varepsilon(1 + O(\varepsilon^2))}{\cos \varepsilon} = \frac{1 + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)}{1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)} = 1 + \varepsilon^2 + O(\varepsilon^3) \quad \text{etc.} \quad (3.17)$$

Why did we gain one more power of ε of accuracy in each iteration? If x is the exact solution and x_n is an approximation with error $E_n = x_n - x$, applying the function once results in a new error

$$E_{n+1} = f_\varepsilon(x + E_n) - x \approx f_\varepsilon(x) + f'_\varepsilon(x)E_n - x = f'_\varepsilon(x)E_n. \quad (3.18)$$

Since we ensured that f_ε only depends on x in the small corrections, in this case $O(\varepsilon)$ terms, we expect to have $f'_\varepsilon(x) = O(\varepsilon)$, and hence the error would shrink by a factor ε in each iteration step. (This method doesn't always work, because $f'_\varepsilon(x)$ can turn out to be not small – see the problem sheet!)

A third, more robust, method of finding the correction is the **method of dominant balance**, which we will discuss below.

Having found one solution, we realise that we are far from done: The number of solutions to the equation changed when we set $\varepsilon = 0$, because the highest-degree term in the equation is multiplied by a quantity that vanishes in the limit $\varepsilon \searrow 0$. Where is the second solution? Clearly, we need the quadratic term to become important, so we might expect x to be larger than $O(1)$ and diverge as $\varepsilon \searrow 0$. This is called a **singular** solution.

Let's use the **method of dominant balance** to look for a rescaling of x that yields the singular solution. We define

$$x = \delta(\varepsilon)X, \quad (3.19)$$

where δ (which will be $\gg 1$) is meant to capture the approximate size of x (e.g. a power of ε) while $X = \text{ord}(1)$ captures its specific value. This yields the equation

$$\begin{array}{rcl} \varepsilon \delta^2 X^2 & + & (\cos \varepsilon)\delta X & - & e^\varepsilon & = 0. \\ \text{ord}(\varepsilon \delta^2) & & \text{ord}(\delta) & & \text{ord}(1) & \end{array} \quad (3.20)$$

If δ is such that a single of these terms is the largest while the other two are smaller, then the resulting leading-order equation is that the single term must be equal to zero, which is impossible (partly because we require that $X = \text{ord}(1)$). Hence, we need at least two of these terms to be the largest at the same time, while the remaining term may be smaller. This is a key principle in the asymptotic solution of both algebraic equations and differential equations – the scales for the solution are determined by requiring that two (or more) terms form part of a **dominant balance**, i.e. have the same asymptotic order that is larger than all the other terms.

We've already seen that balancing the second and third terms yields the scaling for the regular solution,

$$\text{ord}(\delta) = \text{ord}(1) \quad \Rightarrow \quad \delta = 1. \quad (3.21)$$

(Strictly speaking, we could have δ be any multiple of 1, and have smaller corrections such as $\delta = 1 + \varepsilon$, but we take the simplest possible expression.) The first term, $\text{ord}(\varepsilon)$, is then smaller than the two terms we balanced, which are $\text{ord}(1)$, so we have a consistent dominant balance. This recovers the regular solution with $x = X$.

Let's try balancing the first and the third term,

$$\text{ord}(\varepsilon \delta^2) = \text{ord}(1) \quad \Rightarrow \quad \delta = \varepsilon^{-1/2}. \quad (3.22)$$

This yields the neglected second term as $\text{ord}(\varepsilon^{-1/2})$ which is larger than the supposedly dominant terms which are $\text{ord}(1)$, so this is not a consistent balance and does not yield a solution.

Finally, we balance the first and the second term,

$$\text{ord}(\varepsilon\delta^2) = \text{ord}(\delta) \Rightarrow \delta = \varepsilon^{-1}. \quad (3.23)$$

This yields the two balanced terms as $\text{ord}(\varepsilon^{-1})$, while the neglected third term is $\text{ord}(1)$, so we again have a consistent dominant balance. We obtain the rescaled equation

$$\frac{1}{\varepsilon}X^2 + \frac{\cos \varepsilon}{\varepsilon}X - e^\varepsilon = 0 \Rightarrow X^2 + (\cos \varepsilon)X - \varepsilon e^\varepsilon = 0. \quad (3.24)$$

Taking the limit $\varepsilon \searrow 0$ now yields the dominant balance

$$X^2 + X \approx 0 \Rightarrow X \approx -1, \quad (3.25)$$

(The discarded solution $X \approx 0$ represents the regular solution with $x = \text{ord}(1)$ and $X = \text{ord}(\varepsilon)$.) We can proceed to find the corrections to this result using the power-series ansatz $X \sim \sum_{n=0}^{\infty} X_n \varepsilon^n$ or using an iteration with

$$X = -\cos \varepsilon + \varepsilon \frac{e^\varepsilon}{X} = -1 + O(\varepsilon) \quad (3.26)$$

$$\Rightarrow X = -1 + O(\varepsilon^2) + \varepsilon \frac{1 + O(\varepsilon)}{-1 + O(\varepsilon)} = -1 - \varepsilon + O(\varepsilon^2) \quad (3.27)$$

$$\Rightarrow X = -1 + \frac{1}{2}\varepsilon^2 + \varepsilon \frac{1 + \varepsilon + O(\varepsilon^2)}{-1 - \varepsilon + O(\varepsilon^2)} = -1 + \frac{1}{2}\varepsilon^2 - \varepsilon(1 + O(\varepsilon^2)) = -1 - \varepsilon + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3), \quad (3.28)$$

and hence

$$x = \varepsilon^{-1}X = -\varepsilon^{-1} - 1 + \frac{1}{2}\varepsilon + O(\varepsilon^2). \quad (3.29)$$

We can also find the corrections to a leading-order result using a version of the **method of dominant balance**. Let's consider the regular solution $x \approx 1$ again, and simply make a change of variables to the correction $\tilde{x}_1 = x - 1$. Substituting $x = 1 + \tilde{x}_1$ into the equation yields

$$\varepsilon(1 + 2\tilde{x}_1 + \tilde{x}_1^2) + (\cos \varepsilon)(1 + \tilde{x}_1) - e^\varepsilon = 0. \quad (3.30)$$

We can treat this as a fresh equation for the unknown $\tilde{x}_1(\varepsilon)$, but with the additional constraint that $\tilde{x}_1 \ll 1$. Thanks to this constraint, we often don't need to seek the scaling for \tilde{x}_1 , but can just expand everything and drop terms that are unlikely to feature in the dominant balance,

$$0 = \varepsilon + O(\varepsilon \tilde{x}_1) + (1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^4))(1 + \tilde{x}_1) - (1 + \varepsilon + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)) \quad (3.31)$$

$$= -\frac{1}{2}\varepsilon^2 + \tilde{x}_1 - \frac{1}{2}\varepsilon^2 + O(\varepsilon \tilde{x}_1, \varepsilon^3) = \tilde{x}_1 - \varepsilon^2 + O(\varepsilon \tilde{x}_1, \varepsilon^3). \quad (3.32)$$

(Here, I use $O(f, g) = O(f) + O(g)$ to denote an error that is bounded by one of the two bounds.) In this case, the $\text{ord}(\varepsilon)$ terms surprisingly cancelled, so we had to make sure to expand to $\text{ord}(\varepsilon^2)$. We conclude that $\tilde{x}_1 = \varepsilon^2 + O(\varepsilon^3)$, and could then in principle continue to find the next correction with $x = 1 + \varepsilon^2 + \tilde{x}_2$ and $\tilde{x}_2 \ll \varepsilon^2 \dots$

Remark(s) 3.1. • The difficult bit is often finding each leading-order balance, which can involve rescaling and nonlinearity (and the leading-order problem might require numerical solution using a computer). After that, the corrections are usually easier to find, as their equations are linear, or they just involve expanding the iterated function. This principle is often true for more general asymptotics problems too.

- There are two competing definitions for **regular** and **singular**. Some people say that a solution is regular if it has a power-series expansion, and singular if not. Some people say that a solution is regular if it converges (to a finite value in the limit) and singular if not. Thus, everyone agrees that a power-series expansion is regular, and that a divergent expansion is singular, but a convergent non-power-series expansion (e.g. $x \sim \varepsilon^{1/2}$) is disputed. An equation, or a more general asymptotic problem, is classified as regular if all its solutions/results are regular, and singular otherwise. Singular problems are the most exciting, as the basic characteristics of the problem (e.g. number of solutions, convergence of an integral, order of a differential equation) are different at small $\varepsilon > 0$ compared with at exactly $\varepsilon = 0$.

- Consider a general n th degree polynomial equation where the coefficients have been rescaled so that all the coefficients converge as $\varepsilon \rightarrow 0$ and at least one is $\text{ord}(1)$.
 - If some of the coefficients don't have power-series expansions as $\varepsilon \rightarrow 0$, but contain more exotic terms (logarithms, fractional powers, or exponentials), then we can't expect the solution to have a power-series expansion either.
 - If any of the leading-order solutions is a multiple root (e.g. $x^2 - \varepsilon = 0$), then there is a risk that a fractional power of ε appears in the expansion
 - If the highest-degree term is multiplied by a small factor (e.g. $\varepsilon x - 1 = 0$), then at least one solution is divergent so the problem is singular.

If none of these exceptions hold, then we expect all solutions to have power-series expansions.

- In the method of dominant balance, it is a matter of taste whether to split up the unknown x as a scale $\delta(\varepsilon)$ times an $\text{ord}(1)$ quantity X , or to just work directly with x in the scaling argument as well as when subsequently solving for the value. I would recommend rescaling to $X = \text{ord}(1)$ at least when solving the equation in detail, because it is easier to keep track of the different sizes of terms when the scalings are explicit. Later in this module, with differential equations, using explicit scalings is a necessity.
- Sometimes, more than two terms can form part of the dominant balance. (You will find that when trying to balance two terms, another term becomes the same order at the same time.)
- When there are many terms to balance, it is better to work through the possible balances systematically rather than trying all pairs, as shown in the next example.

Example(s) 3.2. Let's find the correct scalings for the solutions of the equation

$$\tan(\varepsilon^2)x^4 + \varepsilon x^3 + (\cos \varepsilon)x^2 + \varepsilon^2 x - \varepsilon = 0 \quad (3.33)$$

as $\varepsilon \searrow 0$.

This time, let's work with x directly and estimate the terms as

$$\text{ord}(\varepsilon^2 x^4), \quad \text{ord}(\varepsilon x^3), \quad \text{ord}(x^2), \quad \text{ord}(\varepsilon^2 x), \quad \text{ord}(\varepsilon). \quad (3.34)$$

For very very small x , the constant term $\text{ord}(\varepsilon)$ alone is dominant. As x increases, the other terms increase in magnitude until one of them becomes equal to $\text{ord}(\varepsilon)$. Which one? Let's try the lowest degree, x^1 , first. If $\text{ord}(\varepsilon^2 x) = \text{ord}(\varepsilon)$ then $x = \text{ord}(\varepsilon^{-1})$. This is not consistent since the x^2 term becomes $\text{ord}(x^2) = \text{ord}(\varepsilon^{-2}) \gg \text{ord}(\varepsilon)$. Since the x^2 term posed a problem by being too large, we try to use it next. If $\text{ord}(x^2) = \text{ord}(\varepsilon)$ then $x = \text{ord}(\varepsilon^{1/2})$. This is a consistent balance since the other terms become $\text{ord}(\varepsilon^4)$, $\text{ord}(\varepsilon^{5/2})$ and $\text{ord}(\varepsilon^{5/2})$ which are all smaller. We conclude that the x^2 and ε terms have a dominant balance at $x = \text{ord}(\varepsilon^{1/2})$.

As x increases past $\text{ord}(\varepsilon^{1/2})$, the x^2 term becomes dominant alone. The lower-degree terms that are negligible will not become dominant again, as they grow slower with x than the x^2 term. Which is the next term to become part of a dominant balance together with x^2 ? Let's try the next term up, $\text{ord}(\varepsilon x^3) = \text{ord}(x^2)$. This yields $x = \text{ord}(\varepsilon^{-1})$, so these two terms become $\text{ord}(\varepsilon^{-2})$. We find that the last remaining term, $\text{ord}(\varepsilon^2 x^4)$ is also $\text{ord}(\varepsilon^{-2})$, so this is a balance between three terms.

As x increases past $\text{ord}(\varepsilon^{-1})$, the highest-degree term becomes dominant alone, and all other terms become small compared with it.

The dominant balances can be represented graphically (figure 3.1).

3.2 Transcendental equations

Moving on to non-polynomial equations, the principle for how to find a dominant balance between terms is the same, but one difficulty is how to obtain an initial guess for the leading-order value of the root. For this, we'll need to use some intuition, or maybe draw a sketch.

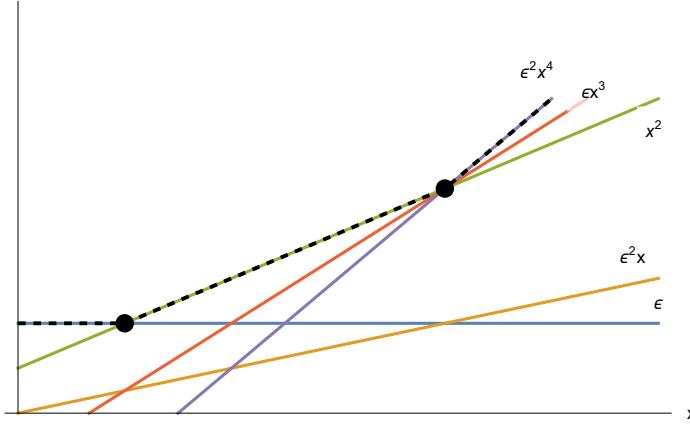


Figure 3.1: Schematic log-log plot of the magnitude of the terms in the equation (3.33). The black circles indicate a dominant balance, while dotted lines indicate a lone dominant term.

Example(s) 3.3. We first consider a simple but important example, namely inverting a function using its expansion near a known point. For example, let's find an expansion for the (real) inverse $x = f^{-1}(\varepsilon)$ of the function $f(x) = x^3 e^{-x}$ near the point $f(0) = 0$.

Near $x = 0$, we simply expand the function and use the first term to obtain a leading-order approximation,

$$\varepsilon = f(x) = x^3 e^{-x} = x^3 - x^4 + \frac{x^5}{2} + O(x^6) \approx x^3 \quad \Rightarrow \quad x \approx \varepsilon^{1/3}. \quad (3.35)$$

The corrections are easily obtained using the iteration method, by moving all remaining terms to the other side and taking the cube root,

$$x = \left[\varepsilon + x^4 - \frac{x^5}{2} + O(x^6) \right]^{1/3} = \varepsilon^{1/3} \left[1 + \frac{x^4}{\varepsilon} - \frac{x^5}{2\varepsilon} + O\left(\frac{x^6}{\varepsilon}\right) \right]^{1/3} = \varepsilon^{1/3} \left[1 + O(\varepsilon^{1/3}) \right]. \quad (3.36)$$

It is good practice to at least obtain the estimate for the leading-order approximation, even if we don't want to calculate more terms.

Are there any other real solutions to $x^3 e^{-x} = \varepsilon$ for small ε ? A sketch reveals that there should exist a solution $x \rightarrow \infty$ for positive $\varepsilon \searrow 0$. For this solution, the exponential is the dominant factor, and hence we solve for the x in the exponent to set up the iteration,

$$3 \ln x - x = \ln \varepsilon \quad \Rightarrow \quad x = \ln \frac{1}{\varepsilon} - 3 \ln x = \ln \frac{1}{\varepsilon} + O\left(\ln \ln \frac{1}{\varepsilon}\right). \quad (3.37)$$

Example(s) 3.4. Let's find the positive solutions x to the equation $\ln x + e^{1/x} = y$, as $y \nearrow \infty$. A simple sketch shows that either of the two terms on the left-hand side can be dominant, resulting in a small and a large solution.

For the small solution, $e^{1/x}$ is dominant, so we rewrite the equation to solve for the x in the exponent:

$$e^{1/x} = y - \ln x \quad \Rightarrow \quad \frac{1}{x} = \ln(y - \ln x) \quad (3.38)$$

$$\Rightarrow \quad x = \frac{1}{\ln(y - \ln x)} = \frac{1}{\ln[y + O(\ln \ln y)]} = \frac{1}{\ln y + O[(\ln \ln y)/y]} = \frac{1}{\ln y} + O\left(\frac{\ln \ln y}{y(\ln y)^2}\right). \quad (3.39)$$

For the large solution, $\ln x$ is dominant. Here, we have to be careful as we are exponentiating, so a naive approach

$$\ln x \sim y \quad \Rightarrow \quad x \sim e^y \quad (3.40)$$

would be invalid. We instead exponentiate the whole equation and obtain (using $x \nearrow \infty$)

$$x = \exp[y - e^{1/x}] = \exp[y - 1 + O(1/x)] = \exp(y - 1)[1 + O(e^{-y})] = \exp(y - 1) + O(1). \quad (3.41)$$

Example(s) 3.5. Let's find approximations to the real solutions of $(\sin x)^2 = \varepsilon x$, as $\varepsilon \searrow 0$.

Sketching the functions $(\sin x)^2$ and εx reveals that the solutions are near the solutions $\sin x = 0$, i.e. $n\pi$ for integer $n \geq 0$. Additionally, there are two such solutions for each n . We write

$$x = n\pi + x_1, \quad x_1 \ll 1 \quad (3.42)$$

and expand

$$(\sin x)^2 = (\sin x_1)^2 = \left(x_1 - \frac{x_1^3}{6} + O(x_1^5) \right)^2 = x_1^2 - \frac{x_1^4}{3} + O(x_1^6), \quad (3.43)$$

resulting in the equation

$$\varepsilon n\pi + \varepsilon x_1 = x_1^2 - \frac{x_1^4}{3} + O(x_1^6). \quad (3.44)$$

Let's first consider the case $n > 0$. As $\varepsilon \searrow 0$ with $x_1 \ll 1$, we find that the main balance must be

$$\varepsilon n\pi \approx x_1^2 \Rightarrow x_1 \approx \pm\sqrt{\varepsilon n\pi}. \quad (3.45)$$

We can improve on this estimate using the iteration

$$x_1 = \pm\sqrt{\varepsilon n\pi + \varepsilon x_1 + \frac{x_1^4}{3} + O(x_1^6)}. \quad (3.46)$$

For the case $n = 0$, one root is exactly $x = 0$ and the other satisfies

$$\varepsilon = x - \frac{x^3}{3} + O(x^5) \Rightarrow x \approx \varepsilon, \quad x = \varepsilon + \frac{x^3}{3} + O(x^5) = \varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5). \quad (3.47)$$

Part II

Integrals

4 Term-by-term integration

In simple cases, an integral might be calculated by asymptotically expanding the integrand and then integrating the result term by term. Note that the integration can be either with respect to the small/large parameter of the integrand expansion or with respect to another quantity.

When integrating with respect to the parameter of the expansion, we use the fact that

$$\int_{x_0}^x O(\delta(t)) dt = O\left(\int_{x_0}^x \delta(t) dt\right), \quad (4.1)$$

where the O are with respect to the limits $t \rightarrow x_0$ and $x \rightarrow x_0$, respectively. This is provided that the integrals converge and $\delta(x)$ does not change sign (infinitely many times as $x \rightarrow x_0$). The same relation holds for o , and also for infinite x_0 . It immediately follows from the definitions that

$$\text{if } f(x) \sim \sum a_n \delta_n(x) \text{ as } x \rightarrow x_0 \quad \text{then} \quad \int_{x_0}^x f(t) dt \sim \sum a_n \left(\int_{x_0}^x \delta_n(t) dt \right) \text{ as } x \rightarrow x_0. \quad (4.2)$$

But what if the fixed limit of the integral is not the limiting point x_0 ?

Example(s) 4.1. Let's obtain an expansion for $I(x) = \int_x^1 \frac{\ln(1+t)}{t^3} dt$ as $x \searrow x_0 = 0$.

We start by expanding the integrand about the limiting point $x_0 = 0$. In this case we have a full convergent Taylor series, but the approach would be the same if we just had the first few terms of an asymptotic expansion:

$$\frac{\ln(1+t)}{t^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{n-3}}{n} = \frac{1}{t^2} - \frac{1}{2t} + \frac{1}{3} + O(t) \quad \text{as } t \searrow 0. \quad (4.3)$$

Because the expansion only gives us local information near x_0 , we want to rewrite the integral as an integral from x_0 to x . Before we can do that, we need to check if the expansion has any term that are (non-integrably) singular at x_0 , and separate those out. We identify the first two terms as being singular, so calculate

$$I(x) = \int_x^1 \underbrace{\frac{\ln(1+t)}{t^3} - \frac{1}{t^2} + \frac{1}{2t}}_{g(t)} dt + \int_x^1 \frac{1}{t^2} - \frac{1}{2t} dt = \int_x^1 g(t) dt + (x^{-1} - 1) + \frac{1}{2} \ln x. \quad (4.4)$$

The remaining integrand $g(t)$ is now integrable at 0, but the integral still has the “wrong” upper limit $x = 1$. We thus “shift” the limits of the integral by rewriting

$$\int_x^1 g(t) dt = \int_0^1 g(t) dt - \int_0^x g(t) dt. \quad (4.5)$$

The first integral looks very difficult to evaluate, but it's equal to some constant A independent of x , so from an asymptotic standpoint we don't need to do anything further with it. Meanwhile, we can finally evaluate the second integral term by term,

$$g(t) \sim \sum_{n=3}^{\infty} \frac{(-1)^{n-1} t^{n-3}}{n} \Rightarrow \int_0^x g(t) dt \sim \sum_{n=3}^{\infty} \frac{(-1)^{n-1} x^{n-2}}{n(n-2)}. \quad (4.6)$$

We can now collect everything together, being careful to put the terms in the right order,

$$I(x) \sim \frac{1}{x} + \frac{1}{2} \ln x + (A - 1) + \sum_{n=3}^{\infty} \frac{(-1)^n x^{n-2}}{n(n-2)}, \quad A = \int_0^1 \frac{\ln(1+t)}{t^3} dt - \frac{1}{t^2} + \frac{1}{2t}. \quad (4.7)$$

(According to Mathematica, $A = 1/4$.)

Remark(s) 4.1. • When solving problems numerically, it can still be useful (or indeed necessary) to subtract a singularity analytically in order to be left with a convergent integral that the computer can handle. In fact, even if the computer can handle the integral you might find that subtracting some not very well-behaved terms improves the accuracy of the numerical result.

- Sometimes the integrand expansion has terms that are not just a power of x but also have a multiplying exponential and/or a logarithm. In these cases, we might not be able to integrate the term directly, but we can instead obtain an asymptotic expansion of it, using **integration by parts** as in the next example.

Example(s) 4.2. Let's find an expansion for the exponential integral $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$, as $x \nearrow \infty$.

We seek to integrate by parts, and since x and t are large we want to have higher and higher powers of t in the denominator. So let's repeatedly integrate the exponential and differentiate the power:

$$E_1(x) = \left[e^{-t} \left(-\frac{1}{t} \right) \right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} dt = \left[e^{-t} \left(-\frac{1}{t} + \frac{1}{t^2} \right) \right]_x^\infty + 2 \int_x^\infty \frac{e^{-t}}{t^3} dt = \dots = \quad (4.8)$$

$$= \left[e^{-t} \left(-\frac{1}{t} + \frac{1}{t^2} - \frac{2}{t^3} + \dots + \frac{(-1)^N (N-1)!}{t^N} \right) \right]_x^\infty + (-1)^N N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt = \quad (4.9)$$

$$= e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + \dots + \frac{(-1)^{N-1} (N-1)!}{x^N} \right) + R_N(x), \quad (4.10)$$

We bound the remainder by replacing the power with a constant,

$$|R_N(x)| = N! \int_x^\infty \frac{e^{-t}}{t^{N+1}} dt \stackrel{t \geq x}{\leq} N! \int_x^\infty \frac{e^{-t}}{x^{N+1}} dt = \frac{N! e^{-x}}{x^{N+1}}. \quad (4.11)$$

Since this is smaller than the last term kept (in fact it is equal to the next term!), we conclude that

$$E_1(x) \sim e^{-x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n-1)!}{x^n}. \quad (4.12)$$

Note here that it is crucial to provide a proper bound for the remainder R_N – it is not sufficient to simply observe that the terms coming out from the integration by parts are decreasing in size.

In the bounding process, there was a choice of whether to make the power or the exponential constant. If we tried to make the exponential constant instead, then we would get $R_N(x) = O(e^{-x}/x^N)$, i.e. the same as the last term kept. This isn't too big of a problem, since it means that the previous error R_{N-1} is the sum of two $O(e^{-x}/x^N)$ terms and hence small enough.

When integrating an asymptotic expansion term by term with respect to another quantity, i.e.

$$f(t; \varepsilon) \sim \sum a_n(t) \delta_n(\varepsilon) \Rightarrow \int_{t_1}^{t_2} f(t; \varepsilon) dt \sim \sum \left(\int_{t_1}^{t_2} a_n(t) dt \right) \delta_n(\varepsilon), \quad (4.13)$$

the expansion of the integrand must be **uniformly valid** throughout the range of integration. We will revisit this concept later, but for now we just assume that if an expansion is valid throughout the range $t_1 \leq t \leq t_2$ (which is not always the case - see the next section) then it is also uniformly valid.

For completeness, uniform validity means that the error $R_N(t; \varepsilon)$ of the asymptotic expansion satisfies the definition $R_N(t; \varepsilon)/\delta_N(\varepsilon) \rightarrow 0$ **uniformly in t** as e.g. $\varepsilon \rightarrow 0$. That is, for every $M > 0$ there exists $\varepsilon_* > 0$ **independent of t** such that $|R_N/\delta_N| < M$ for all $0 < |\varepsilon| < \varepsilon_*$ (and $t_1 \leq t \leq t_2$).

5 Splitting the range of integration

Sometimes the integrand cannot be expanded in way that is valid throughout the entire range of integration, but does have different expansions that are valid in different regions. We can then split the range of integration into separate parts, which can be calculated using different expansions of the integrand, and the final result is the sum of the separate results. (This section is a bit algebra-heavy, as any non-trivial application of the method requires an integrand that has two different expansions.)

Example(s) 5.1. Consider the integral

$$I(\varepsilon) = \int_0^1 \frac{dx}{(1+x)(\varepsilon+x)^{1/2}}, \quad \text{as } \varepsilon \searrow 0. \quad (5.1)$$

If we expand the integral for small ε we obtain

$$I(\varepsilon) = \int_0^1 \frac{1}{(1+x)^{1/2}} \left(1 + \frac{\varepsilon}{x}\right)^{-1/2} dx = \int_0^1 \frac{1}{(1+x)x^{1/2}} \left[1 - \varepsilon \frac{1}{2x} + O\left(\frac{\varepsilon^2}{x^2}\right)\right] dx. \quad (5.2)$$

We can evaluate the integral of the first term using a change of variables,

$$\int_0^1 \frac{dx}{(1+x)x^{1/2}} \stackrel{x=y^2}{=} \int_0^1 \frac{2dy}{1+y^2} dy = [2 \arctan y]_0^1 = \frac{\pi}{2}. \quad (5.3)$$

However, the next term in the integrand is

$$\frac{1}{(1+x)x^{1/2}} \left(-\varepsilon \frac{1}{2x}\right) = \text{ord}(x^{-3/2}) \text{ as } x \searrow 0, \quad (5.4)$$

so its integral diverges. The problem is that the expansion is not uniformly valid, as it requires ε/x to be small and hence breaks down for $x = \text{ord}(\varepsilon)$. We can estimate the **local** contribution to the integral from $x = \text{ord}(\varepsilon)$ as being the integrand scale, $\text{ord}(\varepsilon^{-1/2})$, times the interval length scale, $\text{ord}(\varepsilon)$, which yields $\text{ord}(\varepsilon^{1/2})$ – larger than the next term in the naive expansion. For comparison, the **global** contribution is an $\text{ord}(1)$ integrand over a $\text{ord}(1)$ interval, resulting in an $\text{ord}(1)$ contribution (namely $I \approx \pi/2$).

To proceed, we use the **method of splitting the range of integration**. We have identified that the integrand above has a global scale $x = \text{ord}(1)$ and a local scale $x = \text{ord}(\varepsilon)$, in which the integrand can be expanded in different ways. The idea is to split the integral into two parts at an intermediate scale δ satisfying $\varepsilon \ll \delta \ll 1$:

$$I = I_1 + I_2, \quad I_1 = \int_0^\delta \frac{dx}{(1+x)(\varepsilon+x)^{1/2}}, \quad I_2 = \int_\delta^1 \frac{dx}{(1+x)(\varepsilon+x)^{1/2}}. \quad (5.5)$$

In the second integral, which captures the global contribution, we expand as before, which is now allowed since we have removed the troublesome part $x = \text{ord}(\varepsilon)$, and the expansion is now uniformly valid for $\delta \leq x \leq 1$. With the local region removed, we have

$$I_2 = \int_\delta^1 \frac{1}{(1+x)x^{1/2}} \left[1 - \varepsilon \frac{1}{2x} + O(\varepsilon^2/x^2)\right] dx \stackrel{x=y^2}{=} 2 \int_{\delta^{1/2}}^1 \frac{1}{1+y^2} \left[1 - \varepsilon \frac{1}{2y^2} + O(\varepsilon^2/y^4)\right] dy = \quad (5.6)$$

$$= \left[2 \arctan(y) + \varepsilon \left(\frac{1}{y} + \arctan(y)\right) + O(\varepsilon^2/y^3)\right]_{\delta^{1/2}}^1 = \quad (5.7)$$

$$= \left[\frac{\pi}{2} - 2 \arctan(\delta^{1/2})\right] + \varepsilon \left[1 - \delta^{-1/2} + \frac{\pi}{4} - \arctan \delta^{1/2}\right] + O(\varepsilon^2/\delta^{3/2}). \quad (5.8)$$

Note that although we say that the global contributions are from $x = \text{ord}(1)$, the range of integration actually reaches down to small $x = \delta \ll 1$, but not so small that (ε/x) becomes large and the expansion becomes invalid. That is why the error bounds need to include x (or y), and we have then assumed that the error can simply be integrated directly. We see that the first square bracket recovers the leading-order result in the limit $\delta = 0$, while the second square bracket contains the $\delta^{-1/2}$ divergence due to the $x^{-3/2}$ integrand.

Moving on to the first integral, which captures the local contribution, we change to the local scale $x = \varepsilon X$, and thus obtain an expandable $1/(1+x) = 1/(1+\varepsilon X)$,

$$I_1 = \int_0^{\delta/\varepsilon} \frac{\varepsilon^{1/2} dX}{(1+\varepsilon X)(1+X)^{1/2}} = \varepsilon^{1/2} \int_0^{\delta/\varepsilon} \frac{1}{(1+X)^{1/2}} [1 - \varepsilon X + O(\varepsilon^2 X^2)] dX. \quad (5.9)$$

$$= \left[\varepsilon^{1/2} 2(1+X)^{1/2} + \varepsilon^{3/2} \left(-\frac{2}{3}(1+X)^{3/2} + 2(1+X)^{1/2} \right) + O(\varepsilon^{5/2} X^{5/2}) \right]_0^{\delta/\varepsilon}. \quad (5.10)$$

Adding the two results will give us the answer, with errors $O(\delta^{5/2})$ and $O(\varepsilon^2/\delta^{3/2})$. Since δ is still unspecified, we can choose it to give us the smallest possible error, so we equate $\delta^{5/2} = \varepsilon^2/\delta^{3/2}$ and find $\delta = \varepsilon^{1/2}$ with optimal error $O(\varepsilon^{5/4})$. We could now substitute this value of δ in, add the results together and simplify the result.

However, it is safer to keep δ around for a little bit longer, and first expand both integrals in large δ/ε and small δ , respectively, keeping sufficiently many terms that the resulting error is at most $O(\varepsilon^{5/4})$ when $\delta = \varepsilon^{1/2}$:

$$I_1 = \varepsilon^{1/2} \left[2 \left(1 + \frac{\delta}{\varepsilon} \right)^{1/2} - 2 \right] + \varepsilon^{3/2} \left[-\frac{2}{3} \left(1 + \frac{\delta}{\varepsilon} \right)^{3/2} + 2 \left(1 + \frac{\delta}{\varepsilon} \right)^{1/2} + \frac{4}{3} \right] + O(\delta^{5/2}) \quad (5.11)$$

$$= \left[2\delta^{1/2} + \frac{\varepsilon}{\delta^{1/2}} - 2\varepsilon^{1/2} + O(\varepsilon^2/\delta^{3/2}) \right] + \left[-\frac{2}{3}\delta^{3/2} + O(\varepsilon\delta^{1/2}) \right] + O(\delta^{5/2}), \quad (5.12)$$

$$I_2 = \left[\frac{\pi}{2} - 2 \arctan \delta^{1/2} \right] + \varepsilon \left[1 - \frac{1}{\delta^{1/2}} + \frac{\pi}{4} - \arctan \delta^{1/2} \right] + O(\varepsilon^2/\delta^{3/2}) \quad (5.13)$$

$$= \left[\frac{\pi}{2} - 2\delta^{1/2} + 2\frac{\delta^{3/2}}{3} + O(\delta^{5/2}) \right] + \varepsilon \left[1 - \frac{1}{\delta^{1/2}} + \frac{\pi}{4} + O(\delta^{1/2}) \right] + O(\varepsilon^2/\delta^{3/2}). \quad (5.14)$$

$$\Rightarrow I = \frac{\pi}{2} - 2\varepsilon^{1/2} + \varepsilon \left(1 + \frac{\pi}{4} \right) + \underbrace{O(\delta^{5/2}, \varepsilon\delta^{1/2}, \varepsilon^2/\delta^{3/2})}_{=O(\varepsilon^{5/4}) \text{ for } \delta=\varepsilon^{1/2}}. \quad (5.15)$$

We note that all terms depending on δ have cancelled out, as the result should not depend on where exactly we split the integrals. If a δ -dependent term did not cancel, as is not smaller than the error terms, then we know that we've made a mistake somewhere – that's why it's good to keep δ until the very end.

The fact that the δ -dependent terms are the same (but with opposite sign) between the two integrals also reflects the fact that the integrand can be doubly expanded on the intermediate scale $\varepsilon \ll x \ll 1$:

$$\int x^{-1/2} (1+x)^{-1} (1+\varepsilon/x)^{-1/2} dx = \int x^{-1/2} (1-x+\dots) \left(1 - \frac{\varepsilon}{2x} + \dots \right) dx = \quad (5.16)$$

$$= \int x^{-1/2} \begin{pmatrix} 1 & -x & \dots \\ -\frac{1}{2}\varepsilon x^{-1} & +\frac{1}{2}\varepsilon & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} dx = \begin{pmatrix} 2x^{1/2} & -\frac{2}{3}x^{3/2} & \dots \\ \varepsilon x^{-1/2} & +\varepsilon x^{1/2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} + C(\varepsilon). \quad (5.17)$$

These terms appear, with $x = \delta$, with a positive sign in the I_1 expansion (as δ is the upper limit of that integral) and a negative sign in the I_2 expansion (as δ is the lower limit of that integral).

Example(s) 5.2. Now consider $I(\varepsilon, \alpha) = \int_0^1 \frac{dx}{(1+x)(\varepsilon+x)^\alpha}$ for general fixed $\alpha > 0$ as $\varepsilon \searrow 0$.

We start by studying the integrand to identify the key scales for x , and estimate the order of magnitude of the contributions from each scale:

$$x = \text{ord}(1) : \text{Integrand: } \sim \frac{1}{(1+x)x^\alpha} = \text{ord}(1), \quad \text{Contribution to integral: } \text{ord}(1), \quad (5.18)$$

$$x = \text{ord}(\varepsilon) : \text{Integrand: } \sim \frac{1}{(\varepsilon+x)^\alpha} = \text{ord}(\varepsilon^{-\alpha}), \quad \text{Contribution to integral: } \text{ord}(\varepsilon^{1-\alpha}). \quad (5.19)$$

Thus, if $0 < \alpha < 1$ then we expect that the **global** contribution dominates and the integral is $\text{ord}(1)$, while for $\alpha > 1$ the **local** contribution dominates and the integral is $\text{ord}(\varepsilon^{-(\alpha-1)}) \gg 1$. This immediately

tells us that

$$0 < \alpha < 1 : \quad I \sim \int_0^1 \frac{dx}{(1+x)x^\alpha}, \quad 1 < \alpha : \quad I \xrightarrow{x=\varepsilon X} \varepsilon^{-(\alpha-1)} \int_0^\infty \frac{dX}{(1+X)^\alpha}. \quad (5.20)$$

The corrections to these results would be obtained by splitting the range of integration as in the previous example.

What about if $\alpha = 1$? It seems the local and global contributions would both be $\text{ord}(1)$, but that turns out to be not quite right. We find that

$$I_1 = \int_0^{\delta/\varepsilon} \frac{1}{1+X} [1 + O(\varepsilon X)] dX = [\ln(1+X) + O(\varepsilon X)]_0^{\delta/\varepsilon} = \ln \left(1 + \frac{\delta}{\varepsilon} \right) + O(\delta). \quad (5.21)$$

$$I_2 = \int_\delta^1 \frac{1}{(1+x)x} [1 - O(\varepsilon/x)] dx = \left[\ln \frac{x}{1+x} + O(\varepsilon/x) \right]_\delta^1 = \ln \frac{1}{2} - \ln \frac{\delta}{1+\delta} + O(\varepsilon/\delta). \quad (5.22)$$

We equate the errors $O(\delta) = O(\varepsilon/\delta)$ to find $\delta = \varepsilon^{1/2}$ and the overall error $O(\varepsilon^{1/2})$. Then, expanding the results to this order,

$$I_1 = \left[\ln \frac{\delta}{\varepsilon} + O(\varepsilon/\delta) \right] + O(\delta), \quad I_2 = \left[\ln \frac{1}{2\delta} + O(\delta) \right] + O(\varepsilon/\delta) \quad (5.23)$$

$$\Rightarrow \quad I = \ln \frac{1}{2\varepsilon} + O(\delta, \varepsilon/\delta) = \ln \frac{1}{\varepsilon} - \ln 2 + O(\varepsilon^{1/2}). \quad (5.24)$$

Note that our naive estimates of the integral contribution as being the integrand scale times the size of the interval fails, as the logarithm is special. The dominant contribution comes from the intermediate region $\varepsilon \ll x \ll 1$ and is larger, $\text{ord}(\ln \varepsilon)$, than the estimates $\text{ord}(1)$ for either side, but not by very much:

$$\int_{\text{ord}(\varepsilon)}^{\text{ord}(1)} \frac{A}{x} dx = [A \ln x]_{\text{ord}(\varepsilon)}^{\text{ord}(1)} = A \ln \frac{1}{\varepsilon} + O(1). \quad (5.25)$$

Although strictly speaking the leading-order $\text{ord}(\ln \varepsilon)$ result is independent of the local and global details, in practice the $\text{ord}(1)$ contributions should be calculated if possible.

Remark(s) 5.1. It may be the case that we are unable to integrate the terms found after splitting. But luckily we don't need the exact answer, only an expansion of the integral as its limit (δ or δ/ε in the examples above) tends to 0 or ∞ . Hence, we can use the methods from the last chapter to obtain the required expansion, albeit including a constant integral that we would have to evaluate numerically.

6 Integrals with a large exponent

Integrals of the form

$$I(x) = \int_a^b e^{xh(t)} f(t) dt \quad (6.1)$$

often occur as the result of Fourier transforms, Laplace transforms, or other methods for solving differential equations. We can exploit the singular behaviour of the exponential function for large arguments to obtain an asymptotic expansion for $I(x)$ as $x \nearrow \infty$.

Remark(s) 6.1. We will make frequent use of the following facts about the Gamma function:

- The integral formula $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.
- The recurrence relation $\Gamma(x+1) = x\Gamma(x)$ and relation to the factorial $n! = \Gamma(n+1)$.
- The half-integer values are given by

$$\Gamma(\frac{1}{2}) = 2 \int_0^\infty e^{-u^2} du = \sqrt{\pi}, \quad \Gamma(n + \frac{1}{2}) = (n - \frac{1}{2})(n - \frac{3}{2}) \cdots (\frac{1}{2})\Gamma(\frac{1}{2}) = \frac{(2n-1)!!\sqrt{\pi}}{2^n}, \quad (6.2)$$

where the double factorial is defined by $(2n-1)!! = (2n-1) \times (2n-3) \times \cdots \times 3 \times 1$, and $(-1)!! = 1$.

- The related integral

$$\int_0^\infty e^{-u^2} u^n du = \frac{1}{2} \int_0^\infty e^{-t} t^{(n-1)/2} dt = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right), \quad (6.3)$$

where the Γ function can be further simplified depending on whether n is odd or even.

6.1 Exponential integrals

In this section we consider

$$I(x) = \int_a^b e^{xh(t)} f(t) dt, \quad x \nearrow \infty, \quad (6.4)$$

where the parameter x , the integration variable t and the function $h(t)$ in the exponent are all real. (The limits a and b can also be infinite, provided that the integral converges.)

Example(s) 6.1. Consider the integral

$$I(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt, \quad x \nearrow \infty. \quad (6.5)$$

The integrand is $\text{ord}(1)$ for $t = O(1/x)$, and exponentially small for $t \gg 1/x$, so we would intuitively expect to obtain all algebraic contributions from the local region $t = \text{ord}(1/x)$. Hence, we might intuitively expect to be able to use the small- t expansion of $1/(1+t)$ to integrate term-by-term and obtain

$$I(x) \sim \sum_{n=0}^{\infty} \int_0^\infty e^{-xt} (-t)^n dt \stackrel{t=s/x}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{x^{n+1}} \int_0^\infty e^{-s} s^n ds = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}}. \quad (6.6)$$

Note that the result is not a convergent series (for any fixed x , the terms start to increase in magnitude after $n \approx 1/x$), so there is no way this could be a valid manipulation. However, according to Watson's lemma below, our intuition and this result are correct. (By the way, the substitution $t = (s/x) - 1$ can be used to relate this integral to the exponential integral $E_1(x)$ which we have expanded before, and the expansions agree.)

6.1.1 Watson's lemma

Theorem 6.1 (Watson's lemma). If $f(t)$ is continuous and has the asymptotic power-law expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha_n} \text{ as } t \searrow 0, \quad \text{where } -1 < \alpha_0 < \alpha_1 < \dots, \quad (6.7)$$

then the following integral, where $0 < T \leq \infty$, has an asymptotic expansion given by

$$\int_0^T e^{-xt} f(t) dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha_n + 1)}{x^{\alpha_n + 1}} \text{ as } x \nearrow \infty, \quad (6.8)$$

provided that $\int_0^T |f(t)| dt$ exists, or more generally that $\int_0^T e^{-Xt} |f(t)| dt$ exists for some X .

Remark(s) 6.2. • The result should not be a surprise – the theorem simply states that, although not allowed, substituting the expansion and integrating term by term yields the correct answer:

$$\int_0^T e^{-xt} f(t) dt \sim \sum_{n=0}^{\infty} \int_0^T e^{-xt} a_n t^{\alpha_n} dt \stackrel{s=xt}{\sim} \sum_{n=0}^{\infty} \frac{a_n}{x^{\alpha_n + 1}} \int_0^{\infty} e^{-s} s^{\alpha_n} ds \quad (6.9)$$

- The result is valid also for a finite number of terms in the expansion of f .
- The various conditions ensure that the integral exists: Continuity of $f(t)$ ensures that the integrand is continuous and hence (locally) integrable. The limitation $-1 < \alpha_0$ ensures that the integral doesn't diverge at $t = 0$. The condition that $\int_0^T |f(t)| dt$ exists ensures that the integral doesn't diverge at $t = T$. For $T = \infty$, the weaker condition that $e^{-Xt} |f(t)|$ is integrable for some X allows f to grow like some $e^{\alpha t}$, while still ensuring that $I(x)$ exists for sufficiently large $x > X$.
- The key point is that any contribution to the integral from $t \geq \delta$ has an $O(e^{-\delta x})$ suppression factor, which makes it exponentially small compared with the terms $O(1/x^{\alpha_n + 1})$ even if δ is very small (as long as it remains fixed, independent of x). Thus, we were able to use the asymptotic expansion for $f(t)$, which is only known to be valid for small t , to obtain a full asymptotic expansion for the whole integral.
- The fact that the contribution comes from $t = \text{ord}(1/x)$ explains why each t^α term becomes an $x^{-\alpha-1}$ term: The integrand is $\text{ord}(x^{-\alpha})$ and the length of the interval is $\text{ord}(x^{-1})$, resulting in an $\text{ord}(x^{-\alpha-1})$ contribution.

Proof. Here is a sketch of a proof of Watson's lemma, the details of which are not important.

We denote the integral by $I(x)$ and fix an integer $N \geq 0$, a constant $M > 0$ and a small value of $\delta > 0$ such that

$$f(t) = \sum_{n=0}^N a_n t^{\alpha_n} + \underbrace{R_N(t)}_{O(t^{\alpha_{N+1}})}, \quad \text{where } |R_N(t)| < Mt^{\alpha_{N+1}} \text{ for all } 0 < t < \delta. \quad (6.10)$$

When we split the integral at δ ,

$$I(x) = \underbrace{\int_0^{\delta} e^{-xt} f(t) dt}_{I_1} + \underbrace{\int_{\delta}^T e^{-xt} f(t) dt}_{I_2}, \quad (6.11)$$

we can then neglect the contribution I_2 from $t \geq \delta$ as being exponentially smaller as follows. If $\int_0^T |f(t)| dt$ exists, then it's easy to see that

$$|I_2| = \int_{\delta}^T e^{-xt} |f(t)| dt \leq e^{-x\delta} \int_{\delta}^T |f(t)| dt \leq e^{-x\delta} \underbrace{\int_0^T |f(t)| dt}_{\text{a constant}} = O(e^{-x\delta}). \quad (6.12)$$

However, for the more general condition that $\int_0^T e^{-Xt} |f(t)| dt$ exists for some X , we have for sufficiently large x that

$$|I_2| = \int_{\delta}^T e^{-(x-X)t} e^{-Xt} |f(t)| dt \stackrel{x-X \geq 0}{\leq} e^{-(x-X)\delta} \int_{\delta}^T e^{-Xt} |f(t)| dt \leq \quad (6.13)$$

$$\leq e^{-x\delta} e^{X\delta} \underbrace{\int_0^T e^{-Xt} |f(t)| dt}_{\text{a constant}} = O(e^{-x\delta}). \quad (6.14)$$

We then expand the lower part to obtain

$$I_1(x) = \left(\sum_{n=0}^N a_n \int_0^\delta e^{-xt} t^{\alpha_n} dt \right) + \int_0^\delta e^{-xt} R_N(t) dt. \quad (6.15)$$

In each of the main terms, we use

$$\int_0^\delta e^{-xt} t^{\alpha_n} dt = \int_0^\infty e^{-xt} t^{\alpha_n} dt - \int_\delta^\infty e^{-xt} t^{\alpha_n} dt = \frac{\Gamma(\alpha_n + 1)}{x^{\alpha_n + 1}} + O(e^{-x\delta}), \quad (6.16)$$

where the $t > \delta$ integral is exponentially small for the same reason as I_2 . Similarly, the integral of the remainder term is bounded by

$$\left| \int_0^\delta e^{-xt} R_N(t) dt \right| \leq \int_0^\delta e^{-xt} M t^{\alpha_{N+1}} dt \leq M \int_0^\infty e^{-xt} t^{\alpha_{N+1}} dt = O(x^{-(\alpha_{N+1} + 1)}). \quad (6.17)$$

Collecting these together, we find that $I(x)$ is the required partial sum plus a sufficiently small error. (An analogous proof works with little- o of the last term instead of big- O of the next, but is more complicated to phrase.) \square

Remark(s) 6.3. Watson's lemma can be applied to any integral of the form $\int_a^b e^{\alpha xt} f(t) dt$ with $\alpha \neq 0$, although some work may be needed to transform the integral into the specific form $\int_0^T e^{-xt} f(t) dt$.

Example(s) 6.2. The confluent hypergeometric function ${}_1F_1(a; b; x)$ is proportional to the integral

$$I(x) = \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt \quad \text{for} \quad \operatorname{Re} b > \operatorname{Re} a > 0. \quad (6.18)$$

(You can check that this satisfies Kummer's equation $xy''(x) + (b-x)y'(x) - ay = 0$.)

What is the behaviour of $I(x)$ as $x \searrow -\infty$? We note that the integral is of the required form for Watson's lemma, with the large parameter being $(-x) \nearrow +\infty$ and

$$f(t) = t^{a-1} (1-t)^{b-a-1} \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(b-a)}{\Gamma(b-a-n)n!} t^{a-1+n} \quad \text{as} \quad t \searrow 0. \quad (6.19)$$

We thus obtain

$$I(x) = \int_0^1 e^{(-x)t} f(t) dt \sim \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(b-a)}{\Gamma(b-a-n)n!} \frac{1}{(-x)^{a+n}}. \quad (6.20)$$

What is the behaviour of $I(x)$ as $x \nearrow \infty$? Now the integrand is exponentially largest near $t = 1$ instead of $t = 0$, so we use the transformation $t = 1-s$ to rewrite the integral as

$$I(x) = \int_0^1 e^{x(1-s)} (1-s)^{a-1} s^{b-a-1} ds = e^x \int_0^1 e^{-xs} (1-s)^{a-1} s^{b-a-1} ds, \quad (6.21)$$

and can then expand $f(s)$ and apply Watson's lemma as before.

Remark(s) 6.4. Watson's lemma can also be used when the exponent has more complicated form, provided that it is x times a function that decreases with t , as we can then rewrite the integral using a change of variables $h(t) = h(a) - s$ to

$$\int_a^b e^{xh(t)} f(t) dt = e^{xh(a)} \int_0^{h(a)-h(b)} e^{-xs} \underbrace{\frac{f(h^{-1}(h(a)-s))}{-h'(h^{-1}(h(a)-s))}}_{\text{new } f(s)} ds, \quad (6.22)$$

but this can be very tedious due to the need to expand the new $f(s)$.

6.1.2 Laplace's method

When the function $h(t)$ in the exponent is complicated, we can instead use Laplace's method, in which we expand $h(t)$ in place in the exponent. Note that when using Laplace's method we typically do not justify all the approximations made in detail, but the principle is the same as for Watson's lemma: The dominant contribution to the integral comes from a small region near the maximum of h , and any contribution from outside this region is exponentially smaller, compared with at the maximum. This allows us to expand the integrand near the maximum and evaluate the integral. Often, a function h attains its maximum on an interval of integration either at one of its endpoints, or at a stationary point with $h' = 0$ and $h'' < 0$, so let's consider those cases first.

Example(s) 6.3. (Linear endpoint maximum.) Let's find a few terms in an asymptotic expansion for

$$I(x) = \int_0^1 e^{-xe^t} \cos \sqrt{t} dt \quad \text{as } x \nearrow \infty. \quad (6.23)$$

We identify $h(t) = -e^t$ and $f(t) = \cos \sqrt{t}$. For $0 \leq t \leq 1$, h attains its maximum value $h = -1$ at $t = 0$, so the (exponentially) dominant contribution comes from near $t = 0$, and we only need to consider this local contribution, in which we can expand h (and later f):

$$I(x) = \int_0^\delta e^{x(-1-t-t^2/2+O(t^3))} \cos \sqrt{t} dt + \text{EST} = e^{-x} \int_0^\delta e^{-xt} e^{-xt^2/2+O(xt^3)} \cos \sqrt{t} dt + \text{EST}, \quad (6.24)$$

where "EST" stands for "exponentially smaller terms". In the expansion for $h(t)$ in the exponent, the constant term simply yields an exponential prefactor. The first varying term, which in this case is linear, is responsible for the exponential suppression of the integrand away from the maximum so that only the region $t = O(1/x)$ contributes. We thus rescale $t = u/x$ to make that term $\text{ord}(1)$ and clarify the order of the contributions from the other terms. The higher-order terms, which have higher powers of t than the $\text{ord}(1)$ term, become small corrections in the exponent, which allows their exponential to be expanded:

$$\exp \left[-\frac{xt^2}{2} + O(xt^3) \right] \stackrel{t=u/x}{=} \exp \left[-\frac{u^2}{2x} + O(x^{-2}) \right] = 1 - \frac{u^2}{2x} + O(x^{-2}). \quad (6.25)$$

Similarly, the higher-order terms in the expansion of f also become small corrections:

$$\cos \sqrt{t} = 1 - \frac{t}{2} + O(t^2) = 1 - \frac{u}{2x} + O(x^{-2}). \quad (6.26)$$

Hence,

$$I(x) = e^{-x} \int_0^{x\delta} e^{-u} \left[1 + \frac{-u^2}{2x} + O(x^{-2}) \right] \left[1 - \frac{u}{2x} + O(x^{-2}) \right] \frac{du}{x} + \text{EST} \quad (6.27)$$

Since $x\delta$ is large (δ fixed and $x \nearrow \infty$), we can replace it by ∞ and only pick up another "EST" error. Then

$$I(x) = \frac{e^{-x}}{x} \int_0^\infty e^{-u} \left[1 + \frac{-u^2}{2x} - \frac{u}{2x} + O(x^{-2}) \right] du + \text{EST} = \frac{e^{-x}}{x} \left[1 - \frac{3}{2x} + O(x^{-2}) \right]. \quad (6.28)$$

(Strictly speaking, the big- O bounds in the intermediate steps are wrong because u can be as large as $\delta x \gg 1$, so we should be keeping track of the powers of u , but since the exponential at the end suppresses any contribution from $u \gg 1$, this is not a problem and we're too lazy to do it.)

Example(s) 6.4. (Quadratic interior maximum.) We shall calculate the first few terms in an asymptotic expansion of the integral

$$I(x) = \int_0^\infty e^{x(\ln t - t)} dt, \quad x \nearrow \infty. \quad (6.29)$$

We first check where $h(t) = \ln t - t$ attains its maximal value on the integration interval $0 \leq t \leq \infty$: The function decays to $-\infty$ at both endpoints, and it has a single stationary point, $0 = h'(t) = 1/t - 1 \Rightarrow t = 1$. We expand $h(t)$ near this point

$$h(1+s) = \ln(1+s) - (1+s) = -1 - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + O(s^5), \quad (6.30)$$

and verify that it's a maximum since the first varying term is negative.

The integral is exponentially dominated by the contribution from near $t = 1$, so

$$I(x) = \int_{-\delta}^{\delta} \exp \left[x \left(-1 - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + O(s^5) \right) \right] ds + \text{EST}. \quad (6.31)$$

In order to make the first varying term be $\text{ord}(1)$, we set $s = (2/x)^{1/2}u$. (The choice of whether to include the numerical factor 2 in the rescaling is up to you.) Thus,

$$I(x) = \sqrt{\frac{2}{x}} e^{-x} \int_{-\sqrt{x/2}\delta}^{\sqrt{x/2}\delta} e^{-u^2} \exp \left(\frac{2^{3/2}u^3}{3x^{1/2}} - \frac{u^4}{x} + O(x^{-3/2}) \right) du + \text{EST} \quad (6.32)$$

$$= \sqrt{\frac{2}{x}} e^{-x} \int_{-\infty}^{\infty} e^{-u^2} \left[1 + \frac{2^{3/2}u^3}{3x^{1/2}} + \left(\frac{4u^6}{9x} - \frac{u^4}{x} \right) + O(x^{-3/2}) \right] du + \text{EST} \quad (6.33)$$

$$= \sqrt{\frac{2\pi}{x}} e^{-x} \underbrace{\left[1 + \frac{0}{x^{1/2}} + \frac{1}{x} \left(\frac{4}{9} \times \frac{15}{8} - \frac{3}{4} \right) + O(x^{-3/2}) \right]}_{1/12}. \quad (6.34)$$

The first correction term in the integrand was an odd function, so its integral from $-\infty$ to $+\infty$ cancelled out. In fact, this occurs at every other order, resulting in the corrections being integer powers of x (relative to the leading order).

Remark(s) 6.5. • The main steps of the Laplace method are:

- Identify the maximum of $h(t)$ in the integration interval. Claim that the integral is exponentially dominated by the sum of the contributions from the neighbourhoods of these t_* , to obtain integrals $\int_{-\delta}^{\delta} ds$ where $s = t - t_*$. (Use \int_0^δ or $\int_{-\delta}^0$ if on the boundary.)
- Expand h and f for small s .
- Rescale s to make the first varying term in h be $\text{ord}(1)$. Keep this term in the exponent.
- Expand the exponential of any higher-order terms in h and multiply with the expansion for f .
- Replace the integration limit(s) with ∞ and evaluate the integral (using the Γ function).
- The general leading-order contribution for a linear endpoint maximum at $t = a$ or $t = b$ is given by

$$\int_a^b e^{xh(t)} f(t) dt \sim \begin{cases} \int_0^\infty e^{xh(a)-x(-h'(a))s} f(a) ds = \frac{e^{xh(a)} f(a)}{x(-h'(a))} & \text{or} \\ \int_0^\infty e^{xh(b)-xh'(b)s} f(b) ds = \frac{e^{xh(b)} f(b)}{xh'(b)}, \end{cases} \quad (6.35)$$

and for a quadratic interior maximum at $t = c$,

$$\int_a^b e^{xh(t)} f(t) dt \sim \int_{-\infty}^\infty e^{xh(c)-x(-h''(c)/2)s^2} f(c) ds = \frac{\sqrt{2\pi} e^{xh(c)} f(c)}{\sqrt{(-h''(c))x}}. \quad (6.36)$$

If the quadratic maximum is at either endpoint, then one of the integral limits is replaced with 0, resulting in the contribution being halved.

- What terms to expect:

- In general, for an n th order maximum, i.e. $h(t_* + s) = h(t_*) + As^n + o(s^n)$ with $A \neq 0$, the dominant contribution comes from a region of size $s = \text{ord}(x^{-1/n})$, and hence the integral is $\text{ord}(e^{xh(t_*)}/x^{1/n})$.
- If additionally the multiplying function f vanishes or diverges like s^α (with $\alpha > -1$ so that the integral still exists), then Laplace's method can proceed as usual with f expanded in (possibly non-integer) powers of s starting from s^α , and the integral will be $\text{ord}(e^{xh(t_*)}/x^{(1+\alpha)/n})$.
- If the corrections come from integer powers of s , then they become powers of $\text{ord}(x^{-1/n})$.
- At an interior maximum, for integer powers of s , every other term vanishes due to being an integral of an odd function.

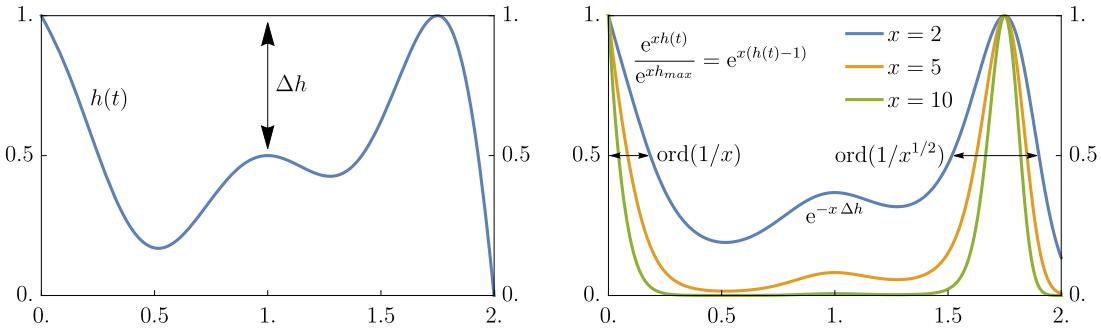


Figure 6.1: Illustration of how the maxima of $h(t)$ become exponentially dominant when exponentiated, in a region whose width depends on the local behaviour of f near the maximum.

- If instead the pre-multiplying function $f(t)$ vanishes exponentially at $t = t_*$, or h diverges to $+\infty$ in the range of integration, then that's an indication that we need to have a **moving maximum** t_* that depends on x . In order to find the true maximum of the integrand properly, we should be maximising

$$|e^{xh(t)} f(t)| = e^{xh(t)+\ln|f(t)|}, \quad (6.37)$$

i.e. find the maximum of $xh(t) + \ln|f(t)|$. We can then carefully expand around that maximum (sometimes just a sufficiently close approximation to it), or in some cases we can change variables via a rescaling such as $s = t/t_*$ to place the maximum in a fixed location and recover a classical Laplace problem.

Example(s) 6.5. Let's calculate the large- n behaviour of $n!$, using the integral expression for the Gamma function,

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n dt. \quad (6.38)$$

Here, the large parameter n is in the exponent once we rewrite $t^n = e^{n \ln t}$. However, $h(t) = \ln t$ grows indefinitely as $t \rightarrow \infty$, instead of attaining a maximum. The integral doesn't diverge, though, because of the exponential decay of the e^{-t} term. Combining the two terms and seeking a maximum yields

$$e^{-t} t^n = e^{-t+n \ln t}, \quad \frac{d}{dt}(-t+n \ln t) = -1 + \frac{n}{t} \Rightarrow t_* = n. \quad (6.39)$$

This is a **moving maximum**, and hints that we should rescale the integration variable as $t = ns$, to obtain an exponent which attains its maximum at a fixed location $s = 1$,

$$n! = \Gamma(n+1) = n \int_0^\infty e^{-ns+n \ln n+n \ln s} ds = n^{n+1} \int_0^\infty e^{n(-s+\ln s)} ds. \quad (6.40)$$

We calculated this integral in the previous example, so using that result we obtain

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \left[1 + \frac{1}{12n} + O(n^{-2}) \right]. \quad (6.41)$$

The moving maximum could also be handled without rescaling: When we expand in “small” $s = t - n$ the exponent becomes

$$-n - s + n \ln(n+s) = -n + n \ln n - s + n \ln \left(1 + \frac{s}{n} \right) = -n + n \ln n - \frac{s^2}{2n} + \frac{s^3}{3n^2} + O\left(\frac{s^4}{n^3}\right). \quad (6.42)$$

Although this is not of the standard form $nh(t)+\ln|f(t)|$ (plus some n -dependent “constant”) for Laplace’s method, we can still proceed in the same way. Looking at the first varying term in the expansion, $-s^2/2n$, and insisting that it becomes $\text{ord}(1)$ yields the rescaling $s^2/2n = u^2$, after which the calculation becomes identical to the previous one.

6.2 Oscillatory integrals

In this section we consider the case of a purely imaginary exponent, i.e.

$$I(x) = \int_a^b e^{ix\psi(t)} f(t) dt, \quad x \nearrow \infty, \quad (6.43)$$

where the parameter x , the variable t and the phase function $\psi(t)$ are all real. Note that by taking the real or imaginary part we can obtain results for integrals with $\cos(x\psi(t))$ or $\sin(x\psi(t))$. (Again, we allow a and/or b to be infinite provided that the integral converges.) Unlike in the previous section, here the exponential does not change the size of the integrand, as $|e^{ix\psi}| = 1$, but only causes it to oscillate rapidly.

We start with a crude estimate for the very simplest case of $\psi(t) = t$.

Theorem 6.2 (Riemann–Lebesgue lemma). For $-\infty \leq a < b \leq \infty$, if $\int_a^b |f(t)| dt$ exists, then

$$\int_a^b e^{ixt} f(t) dt \rightarrow 0 \quad \text{as } x \nearrow \infty. \quad (6.44)$$

Why should we expect this to hold? The idea is that the integrand oscillates rapidly, on the small scale $\text{ord}(1/x)$, while the amplitude $f(t)$ of the oscillation changes on a much longer scale ($\text{ord}(1)$), so each oscillation approximately cancels itself out (figure 6.2(a)).

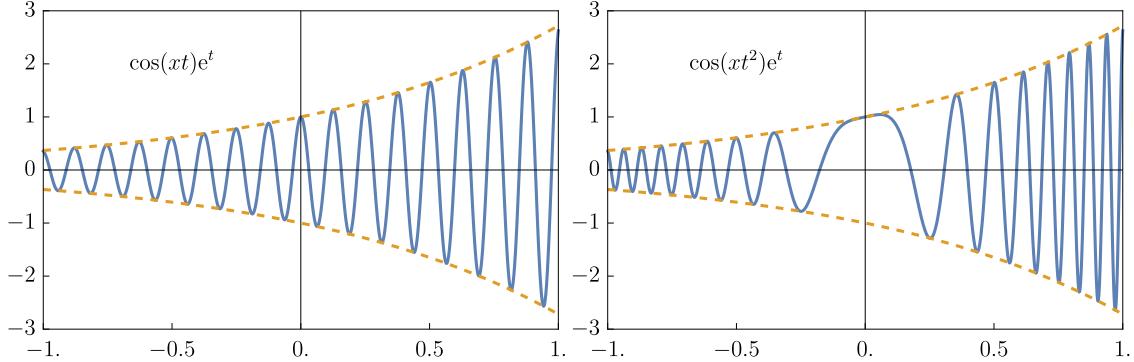


Figure 6.2: Oscillations with amplitude $f(t) = e^t$ and phase function (a) $\psi = t$, (b) $\psi = t^2$ for $x = 50$.

The proof of this theorem is outside the scope of this module, and proceeds by suitably approximating f with a piecewise constant function and evaluating the integral on each interval. For sufficiently well-behaved f , an analogous calculation is to integrate by parts, which yields

$$\int_a^b e^{ixt} f(t) dt = -\frac{i}{x} \underbrace{[e^{ixt} f(t)]_a^b}_{=O(1)} + \frac{i}{x} \underbrace{\int_a^b e^{ixt} f'(t) dt}_{\text{bounded by } \int_a^b |f'(t)| dt} = O(1/x) \rightarrow 0. \quad (6.45)$$

Example(s) 6.6. (Endpoint/boundary contributions.) If f is (continuously) differentiable, then looking more carefully at the previous integration-by-parts result yields

$$I(x) = \int_a^b e^{ixt} f(t) dt = -\frac{i}{x} [e^{ixt} f(t)]_a^b + \frac{i}{x} \underbrace{\int_a^b e^{ixt} f'(t) dt}_{\rightarrow 0 \text{ by R-L lemma}} = \frac{i}{x} [e^{ixa} f(a) - e^{ibx} f(b)] + o(1/x). \quad (6.46)$$

We thus have an asymptotic approximation to I , given by the $O(1/x)$ boundary contributions. (If a or b is infinite, then we require f and its derivative(s) to decay there, and there is no contribution.)

If f is N times (continuously) differentiable, then repeating yields a finite expansion with all terms coming from the boundary,

$$I(x) = - \left[e^{ixt} \left(\frac{i}{x} f(t) + \frac{i^2}{x^2} f'(t) + \frac{i^3}{x^3} f''(t) + \cdots + \frac{i^N}{x^N} f^{(N-1)}(t) \right) \right]_a^b + \frac{i^N}{x^N} \underbrace{\int_a^b e^{ixt} f^{(N)}(t) dt}_{o(1) \text{ by R-L lemma}}, \quad (6.47)$$

and for smooth f we obtain an infinite asymptotic expansion.

Remark(s) 6.6. What is the implication for more general oscillatory integrals, with a general function $\psi(t)$ in the exponent? If ψ is monotonic and ψ' bounded away from 0, then we could just use the change of variables $s = \psi(t)$ to change the integral into the previous form

$$I(x) = \int_{\psi(a)}^{\psi(b)} e^{ixs} \frac{f(\psi^{-1}(s))}{\psi'(\psi^{-1}(s))} ds \quad (6.48)$$

and obtain $O(1/x)$ terms from the endpoints. (This is like transforming $s = h(t)$ to apply Watson's lemma in the previous section.)

However, if $\psi' = 0$ somewhere then the transformation cannot be used, as the integrand would become singular. This indicates that the dominant contribution will come from near the point of stationary phase. Intuitively, the rapid oscillations (due to the rapid change of the phase $x\psi(t)$ with t) which cause cancellation slow down near the point of stationary phase, resulting in less cancellation there (figure 6.2(b)). We'll need to use the **method of stationary phase** to obtain the leading-order contribution.

Example(s) 6.7. Let's find the leading-order behaviour as $x \nearrow \infty$ of the Bessel function of the first kind, which for integer order n can be written as

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - nt) dt. \quad (6.49)$$

(You can check that this satisfies Bessel's equation $x^2 y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0$.)

To obtain an exponential with imaginary exponent, we write

$$J_n(x) = \frac{1}{\pi} \operatorname{Re} I(x), \quad \text{where} \quad I(x) = \int_0^\pi e^{i(x \sin t - nt)} dt = \int_0^\pi e^{ix \sin t} \underbrace{e^{-int}}_{f(t)} dt. \quad (6.50)$$

The phase function $\psi(t) = \sin t$ is stationary at $t = \pi/2$, so that's where we expect the dominant contribution to come from, while the contributions away from the stationary point are $O(1/x)$ as they are integrals over an interval where ψ' is bounded away from zero.

Since, unlike in Laplace's method, there is no exponential decay away from the stationary point, we will only work at leading order and not keep track of any errors. We thus expand ψ near the stationary point to only the first varying term, $\sin(\pi/2 + s) = \cos(s) \approx 1 - s^2/2$, and take the leading-order term for $f(t) \approx e^{-in\pi/2}$. Thus, we obtain

$$I(x) \sim \int_{-\delta}^{\delta} e^{ix(1-s^2/2)} e^{-in\pi/2} ds \stackrel{s=(2/x)^{1/2}u}{\sim} e^{i(x-n\pi/2)} \sqrt{\frac{2}{x}} \int_{-\infty}^{\infty} e^{-iu^2} du = e^{i(x-n\pi/2)} \sqrt{\frac{2\pi}{x}} e^{-i\pi/4}. \quad (6.51)$$

In the last step, we had to evaluate the Fresnel integral $\int_0^\infty e^{iu^2} du$, which we explain how to do below. It's possible to cheat by using the formula $\int_{-\infty}^\infty e^{-au^2} du = \sqrt{\frac{\pi}{a}}$ with $a = i$, but you have to make sure you choose the correct branch of the square root, which is that the argument of a should be the principal one, $\pi/2$.

We conclude that

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + o(1/x^{1/2}), \quad \text{or} \quad J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad (6.52)$$

but the latter expression is strictly speaking not correct, as the right-hand side has zeroes in not quite the same locations as the left-hand side. In general, with oscillatory terms on the right-hand side of \sim , we mean that the error is small compared with the algebraic factor of the last term, so in this case $o(x^{-1/2})$, ignoring the imaginary-exponential or trigonometric factor. Also, in general there can be multiple terms of the same algebraic order but different oscillation frequency (including none), in which case there is no way to sort them in a strictly decreasing way so we just write them in any order.

Example(s) 6.8. In general, the method of stationary phase ends with needing to evaluate an integral of the form

$$\int_0^\infty e^{iu^p} du \stackrel{u=v^{1/p}}{=} \frac{1}{p} \int_0^\infty e^{iv} v^{(1/p)-1} dv, \quad \text{where } p > 1. \quad (6.53)$$

This requires a contour deformation, which we will discuss briefly now and cover in more detail in the next section.

We would like to convert the exponential from e^{iv} to e^{-w} to make a Gamma function, so we rotate the contour from the positive real axis to the positive imaginary axis, and obtain, with $v = iw = e^{i\pi/2}w$,

$$\frac{1}{p}e^{i\pi/2p} \int_0^\infty e^{-w} w^{(1/p)-1} dw = \frac{e^{i\pi/2p}\Gamma(1/p)}{p}. \quad (6.54)$$

(If you want to be extra careful, which is not needed in this module, you check that the contour rotation is allowed since, firstly, the contribution from a quarter circle at infinity vanishes, which follows from Jordan's lemma since $|v^{(1/p)-1}| \searrow 0$ as $|v| \nearrow \infty$ for $p > 1$, and, secondly, the contribution from a quarter circle avoiding the branch point at the origin vanishes, which follows from estimating the modulus of the integrand as $|v|^{(1/p)-1}$ and multiplying by the arclength $|v|\pi/2$.)

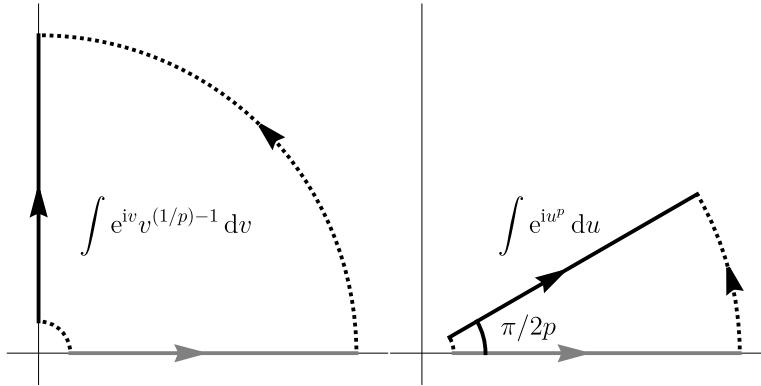


Figure 6.3: Contour deformation for calculating the integral $\int_0^\infty e^{iu^p} du$.

It is also possible to do the contour rotation first, by an angle $\pi/2p$, to obtain

$$\int_0^\infty e^{iu^p} du \stackrel{u=e^{i\pi/2p}v}{=} e^{i\pi/2p} \int_0^\infty e^{-v^p} dv \stackrel{v=w^{1/p}}{=} \frac{e^{i\pi/2p}}{p} \underbrace{\int_0^\infty e^{-w} w^{(1/p)-1} dw}_{\Gamma(1/p)}. \quad (6.55)$$

(But if you wanted to show that the contribution from the arc at infinity vanishes, you'd still have to do the other variable change and apply Jordan's lemma.)

If the exponent has the opposite sign (i.e. if the p th derivative of ψ is negative), then we get the complex conjugate of the result. If the integral is double-ended, then we split it into two parts, and flip the sign of s in the lower half, so that for even p we get twice this result, while for odd p we get twice the real part of this result.

Remark(s) 6.7. The method of stationary phase has several drawbacks due to the lack of exponential decay away from the stationary point.

- We can only obtain the leading-order result – trying to put in corrections like in Laplace's method results in a divergent integral of the form $\int e^{iu^p} u^q du$ with $q \geq 0$.
- It is important that the multiplying function $f(t)$ is bounded over the whole range of integration including the endpoints, as the estimate $O(1/x)$ for the global/endpoint contribution relies on an integration by parts.
- If $f(t)$ vanishes at the point of stationary phase then the local contribution from there is smaller, which is problematic if the result becomes no larger than the $O(1/x)$ global/endpoint contributions.

We can avoid these issues by using the method of steepest descent instead, which is the next topic.

6.3 Method of steepest descent

We finally consider the case of

$$I(x) = \int_C e^{xh(t)} f(t) dt, \quad x \nearrow \infty, \quad (6.56)$$

where $h(t)$ can be complex, and the contour of integration C is allowed to be in the complex plane with finite or infinite endpoints, while x is real. (If x is complex but has fixed argument θ then we can write $x = re^{i\theta}$ with $r \nearrow \infty$ and redefine $e^{i\theta}h(t)$ to be a new $h(t)$.)

Let's first quickly review the relevant facts about contour integrals.

- A contour is a directed (piecewise) smooth curve in the complex plane, with finite or infinite endpoints. We use the endpoint “ $e^{i\alpha}\infty$ ” for the contour going to infinity in the direction $\arg t = \alpha$.
- A contour integral can be evaluated by parametrisation (just like a change of variables). If the contour C is parametrised by $t = \gamma(s)$ for real s ranging from a to b then $dt = \gamma'(s) ds$ so

$$\int_C f(t) dt = \int_{s=a}^b f(\gamma(s)) \gamma'(s) ds.. \quad (6.57)$$

- Contours can be chained together (provided one ends where the other begins), which results in the integrals being added:

$$\int_{C_1} f(t) dt + \int_{C_2} f(t) dt = \int_{C_1+C_2} f(t) dt. \quad (6.58)$$

(Similarly, a single integral can be split into two parts at an intermediate point on the contour.)

- The integral around a closed contour (loop) is zero, provided the integrand is **complex analytic / holomorphic** inside the loop. As a consequence, integrals along two different contours with the same end points have the same value (so we can write $\int_{t_1}^{t_2} f(t) dt$ without specifying the path taken from t_1 to t_2). For a given integral, we can thus “deform” the contour while keeping the endpoints fixed, without changing the value of the integral. (If the integrand has singularities, then we need to specify which side of the singularities the contour passes, and deforming the contour across a singularity results in picking up its residue.)

The idea in the **method of steepest descent** is to deform the contour to obtain a new integral that is exactly equal to the original one, but can be approximated using Laplace's method. The tricky part is mainly the first step, choosing a suitable contour to deform onto. Let's start with a relatively simple example.

6.3.1 Steepest descent between endpoints

Example(s) 6.9. Let's consider the simple stationary-phase problem $I(x) = \int_0^1 e^{ixt^2} dt$, as $x \nearrow \infty$.

We know that the dominant contribution will come from the point of stationary phase, $t = 0$, which happens to also be an endpoint. But how do we calculate the higher-order corrections to this?

We seek to deform the contour in the complex t -plane onto so-called “steepest-descent” (SD) contours where the exponent has **constant imaginary part**, leaving just its real part varying with t . We first write $t = p + iq$ and identify the real and imaginary parts of h ,

$$h(t) = it^2 = i(p^2 + 2ipq - q^2) = \underbrace{-2pq}_{\text{Re } h} + i\underbrace{(p^2 - q^2)}_{\text{Im } h}. \quad (6.59)$$

The steepest-descent contours emanating from the first endpoint $(p, q) = (0, 0)$ and the second endpoint $(p, q) = (1, 0)$ then satisfy

$$p^2 - q^2 = 0^2 - 0^2 = 0 \Rightarrow q = \pm p, \quad \text{and} \quad p^2 - q^2 = 1^2 - 0^2 = 1 \Rightarrow q = \pm \sqrt{1 + p^2}. \quad (6.60)$$

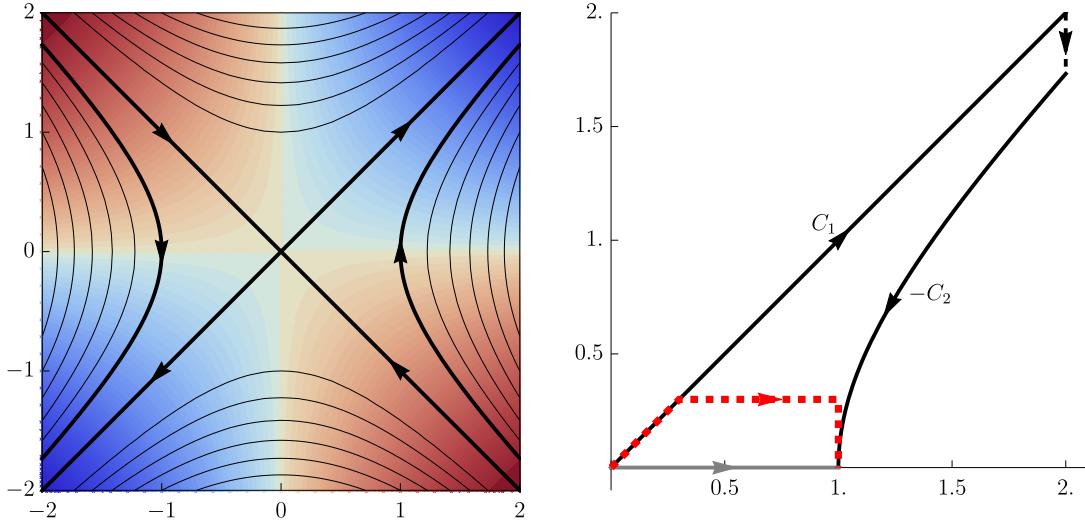


Figure 6.4: Steepest-descent contours $\text{Im } h = \text{const}$ for $h(t) = it^2$. (a) Arrows pointing downhill. Colours indicate value of $\text{Re } h$ from positive (red) to negative (blue). (b) Appropriate contours running from $t = 0$ to $t = 1$, both the full SD contour (black) and a shortcut one that just has the correct directions at the dominant points (red).

We sketch these contours in the complex plane (figure 6.4a), and also put arrows on to indicate the **descending** directions, meaning that $\text{Re } h = -2pq$ decreases in the direction of the arrow. Since the size of the exponential is $e^{x \text{Re } h}$, it becomes exponentially smaller when descending.

We wish to select contours that descend from the endpoints, and join together. Hence, we select the contours with increasing q from $t = 0$ and $t = 1$, that “meet” at $e^{i\pi/4}\infty$. To keep things positive, we parametrise the second contour in the opposite direction (figure 6.4b), and hence have

$$I(x) = I_1(x) - I_2(x), \quad I_i(x) = \int_{C_i} e^{ixt^2} dt. \quad (6.61)$$

Strictly speaking, to justify this we need consider joining the contours together at a large but finite value of $|t|$ using a “bypass” contour, and then taking the limit $|t| \nearrow \infty$ with the contribution from the bypass contour vanishing as its integrand decays exponentially with $|t|$. Note that this connection at a location with $\text{Re } h \searrow -\infty$ allows us to **jump between contours with different constant value of $\text{Im } h$** .

We are now ready to estimate the integral contribution from each contour, using the parametrisations we obtained when solving for the contours above.

On the first contour, from $t = 0$, we have $q = p$ so

$$t = p + ip \Rightarrow dt = (1 + i) dp, \quad h(t) = -2p^2 + 0i \Rightarrow I_1(x) = \int_{p=0}^{\infty} e^{-2xp^2} (1 + i) dp. \quad (6.62)$$

Ignoring the fact that we can evaluate this integral exactly, we observe that it’s a Laplace-type integral with dominant contribution from near the maximum at $p = 0$, so

$$I_1(x) = (1 + i) \int_0^{\delta} e^{-2xp^2} dp + \text{EST} \stackrel{p=u/\sqrt{2x}}{=} \frac{1+i}{\sqrt{2x}} \int_0^{\infty} e^{-u^2} du + \text{EST} = \frac{1+i}{2} \sqrt{\frac{\pi}{2x}} + \text{EST}. \quad (6.63)$$

On the second contour, from $t = 1$, we have $q = \sqrt{p^2 - 1}$ so

$$t = p + i\sqrt{p^2 - 1} \Rightarrow dt = \left(1 + \frac{ip}{\sqrt{p^2 - 1}}\right) dt, \quad h(t) = -2p\sqrt{p^2 - 1} + i \quad (6.64)$$

$$\Rightarrow I_2(x) = e^{ix} \int_{p=1}^{\infty} e^{-2xp\sqrt{p^2 - 1}} \left(1 + \frac{ip}{\sqrt{p^2 - 1}}\right) dp. \quad (6.65)$$

This is again a Laplace-type integral that we can estimate, but it's a bit messy. Another approach that can work out nicer is to use h directly in the parametrisation,

$$h(t) = h(1) - s \Rightarrow it^2 = i - s \Rightarrow t = (1 + is)^{1/2}, \quad dt = \frac{i ds}{2(1 + is)^{1/2}} \quad (6.66)$$

$$\Rightarrow I_2(x) = e^{ix} \int_{s=0}^{\infty} e^{-sx} \frac{i}{2(1 + is)^{1/2}} ds. \quad (6.67)$$

Note that h is enforced to have constant imaginary part and decreasing real part due to s being real and increasing, and that we end up with an integral whose form is suitable for Watson's lemma. However, this relies on us being able to invert h to solve for t , which is not always possible, although we could also get away with just finding an expansion for the inverse near $s = 0$.

Let's instead use an even lazier approach! Since the contribution away from the endpoint is exponentially small, it is irrelevant and we don't need to follow the steepest-descent contour exactly. Instead, let's just use a short line segment that is tangent to the steepest-descent direction at the dominant points, and join the contours together arbitrarily in the exponentially smaller region (see figure 6.4b). Let $t = 1 + is$ with $0 \leq s \leq \delta$, and we obtain

$$t = 1 + is \Rightarrow dt = i ds, \quad h(t) = i(1 + is)^2 = i - 2s - is^2 \quad (6.68)$$

$$\Rightarrow I_2(x) = e^{ix} \int_0^{\delta} e^{-2xs - ixs^2} i ds + \text{EST}. \quad (6.69)$$

We observe that the first varying term is real and negative, which indicates that we chose the line segment in the correct direction. The higher-order terms from h are complex (whereas they would be real when following the exact SD contour), but this is not a problem. We can then continue like in Laplace's method, with the rescaling $-2xs = -u$ and hence

$$I_2(x) = \frac{ie^{ix}}{2x} \int_0^{x\delta} e^{-u} e^{-iu^2/(4x)} ds + \text{EST} = \frac{ie^{ix}}{2x} \int_0^{\infty} e^{-u} \left[1 - i \frac{u^2}{4x} + O(x^{-2}) \right] ds + \text{EST} = \quad (6.70)$$

$$= \frac{ie^{ix}}{2x} \left[1 - \frac{i}{2x} + O(x^{-2}) \right]. \quad (6.71)$$

As the full SD contour typically takes effort to calculate and use, we prefer to use this lazy method, but the full SD contour is useful for example when evaluating the integral numerically without asymptotic approximation, as it avoids any issues with oscillation and cancellations.

Let's revisit the first integral using this lazy method. We use $t = e^{i\pi/4}s$ and obtain

$$I_1 = \int_0^{\delta} e^{ix(e^{i\pi/4}s)^2} e^{i\pi/4} ds + \text{EST} = e^{i\pi/4} \int_0^{\infty} e^{-xs^2} ds + \text{EST} = e^{i\pi/4} \frac{1}{2} \sqrt{\frac{\pi}{x}} + \text{EST}, \quad (6.72)$$

which is the same result as before.

Collecting the contributions together, we find that

$$I = I_1 - I_2 = \frac{e^{i\pi/4}}{2} \sqrt{\frac{\pi}{x}} - \frac{ie^{ix}}{2x} \left[1 - \frac{i}{2x} + O(x^{-2}) \right]. \quad (6.73)$$

How does the method of steepest descent compare with the method of stationary phase? Note that the contour deformation near $t = 0$ is exactly the deformation we would do to calculate the final integral in the method of stationary phase. However, by doing the deformation before doing the expansion of the integral, we have made the contribution from the rest of the contour exponentially smaller (relative to the stationary point and endpoint) and can thus obtain a full expansion by just considering the neighbourhoods of the dominant points.

6.3.2 Steepest descent with saddle points

Remark(s) 6.8. • In the previous example, following the SD contours “downhill” from the endpoints resulted in the contours converging towards the same infinite direction, so the full contour would just

be given by going out on one and back in on the other. More generally, following the SD contours downhill we may end up in different locations, meaning that we “reach” $\text{Re } h = -\infty$ in different directions at infinity (or at a singularity of h , which we won’t discuss further but is treated similarly). Hence, we need to understand how to connect two infinite end points using SD contours. This is also needed if our original contour is specified with infinite endpoints.

- First, we need to understand how the contours of constant $\text{Im } h$ behave. We estimate how h changes near a point t_* with non-zero $h'(t_*) = Ae^{i\alpha}$ (and $A > 0$), by Taylor expanding to first order:

$$h(t_* + s) - h(t_*) \approx h'(t_*)s = A|s|e^{i(\alpha+\arg s)} = A|s|[\cos(\alpha + \arg s) + i \sin(\alpha + \arg s)]. \quad (6.74)$$

The fact that $\text{Re } h \propto \cos$ while $\text{Im } h \propto \sin$ means that the contours of $\text{Re } h$ and $\text{Im } h$ are perpendicular to each other, and that the direction of no variation in $\text{Im } h$, i.e. $\arg s = -\alpha$ or $\pi - \alpha$, is also the direction of steepest variation in $\text{Re } h$, i.e. **steepest ascent** $\arg s = -\alpha$ or **steepest descent** $\arg s = \pi - \alpha$. This explains why we refer to the constant- $\text{Im } h$ contours as steepest-descent contours (saying “descent” because we descend away from the dominant point, regardless of which direction the integral contour travels in).

- So following a contour of constant $\text{Im } h$, we see that $\text{Re } h$ is ascending or descending when $h'(t_*) \neq 0$. This means that its maximum, and hence the dominant contribution, must be either at an **endpoint** or at a **saddle point** where $h'(t_*) = 0$. To connect two endpoints at infinity, we thus need a steepest-descent contour that passes over a saddle point, with the saddle point having the maximum $\text{Re } h$.
- What do the contours look like near a saddle point? The most common type of saddle point is quadratic, with $h''(t_*) \neq 0$, but we can just consider the general n th order saddle (meaning that the derivatives up to order $n - 1$ vanish and the first non-zero derivative is the n th one). Thus, with $h^{(n)}(t_*) = Ae^{i\alpha}$ (and $A > 0$), the change in h is

$$h(t_* + s) - h(t_*) \approx \frac{A}{n!}|s|^n e^{i(\alpha+n\arg s)} = \frac{A}{n!}|s|^n [\cos(\alpha + n\arg s) + i \sin(\alpha + n\arg s)]. \quad (6.75)$$

Here, we see that the directions of constant $\text{Im } h$ are evenly distributed and alternating between ascending and descending (figure 6.5). In particular, the steepest-descent directions are found by seeking

$$\pi + 2\pi k = \arg(h^{(n)}(t_*)s^n) = \alpha + n\arg s \Rightarrow \arg s = \frac{\pi - \alpha + 2\pi k}{n}, \quad k = 0, 1, \dots, n-1. \quad (6.76)$$

Note that if $h \sim Ct^n$ as $|t| \nearrow \infty$ then the large- $|t|$ behaviour of h is also described by this.

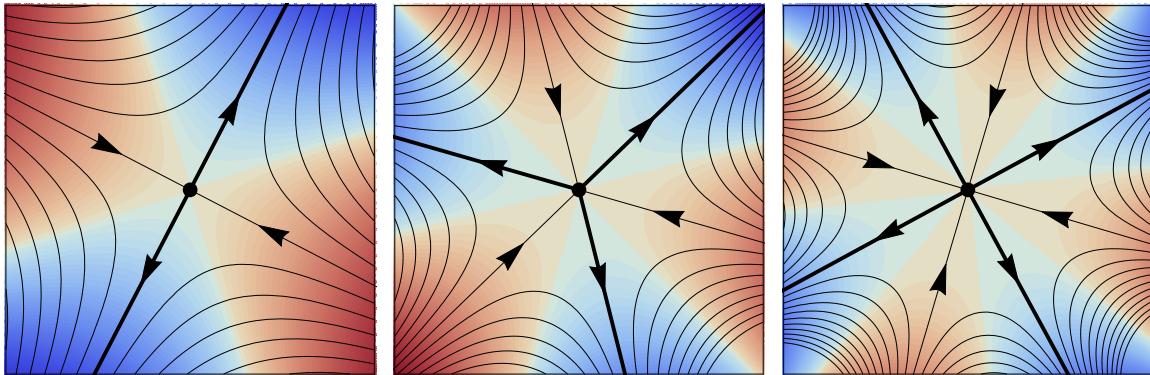


Figure 6.5: Shape of the contours of $\text{Im}(h)$ near a saddle point of order 2, 3 or 4. The arrows indicate directions of descent, and the thick lines are the directions of steepest descent from the saddle.

- The behaviour of h for large $|t|$ is also given by the schematics in figure 6.5 if $h \sim Bt^n$.

We are now ready to tackle a saddle example, whose results will be important later in this module.

Example(s) 6.10. The Airy function $\text{Ai}(x)$ is a solution of the equation $\text{Ai}''(x) - x\text{Ai}(x) = 0$ that decays as $x \nearrow \infty$, and has the integral expression

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xs+s^3/3)} ds. \quad (6.77)$$

How exactly does it decay as $x \nearrow \infty$?

We note that the original form of the integral is not very suitable since the supposedly large term xs becomes dominated by $s^3/3$ for large s , so we rewrite it using a change of variables that balances xs with $s^3/3$, i.e. $s = x^{1/2}t$,

$$\text{Ai}(x) = \frac{x^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{x^{3/2}i(t+t^3/3)} dt, \quad h(t) = i\left(t + \frac{1}{3}t^3\right), \quad x^{3/2} \nearrow \infty. \quad (6.78)$$

Where will the dominant contribution come from? We note that the contour has no finite endpoints, so it must be one or more of the saddle points. We calculate

$$h'(t) = i(1+t^2) \Rightarrow t_* = \pm i, \quad h(t_*) = \mp \frac{2}{3}, \quad h''(t_*) = 2it_* = \mp 2. \quad (6.79)$$

The expansions tell us which the steepest-descent directions are:

$$t_* = +i : h(t_* + s) \approx -\frac{2}{3} - s^2 \Rightarrow \text{SD: } \arg s = 0, \pi \quad (s \propto \pm 1), \quad (6.80)$$

$$t_* = -i : h(t_* + s) \approx +\frac{2}{3} + s^2 \Rightarrow \text{SD: } \arg s = \pi/2, -\pi/2 \quad (s \propto \pm i). \quad (6.81)$$

To complete our rough picture of the contour landscape, we also look at the large- $|t|$ behaviour,

$$h(t) \sim i \frac{t^3}{3} = \frac{|t|^3}{3} e^{i(\pi/2 + 3\arg t)} \Rightarrow \text{SD: } \arg t = \pi/6, 5\pi/6, -\pi/2. \quad (6.82)$$

We use this information to first sketch the local SD contours around $\pm i$ and at large $|t|$, and then join up the contours in a sensibly-looking way (figure 6.6(a)).

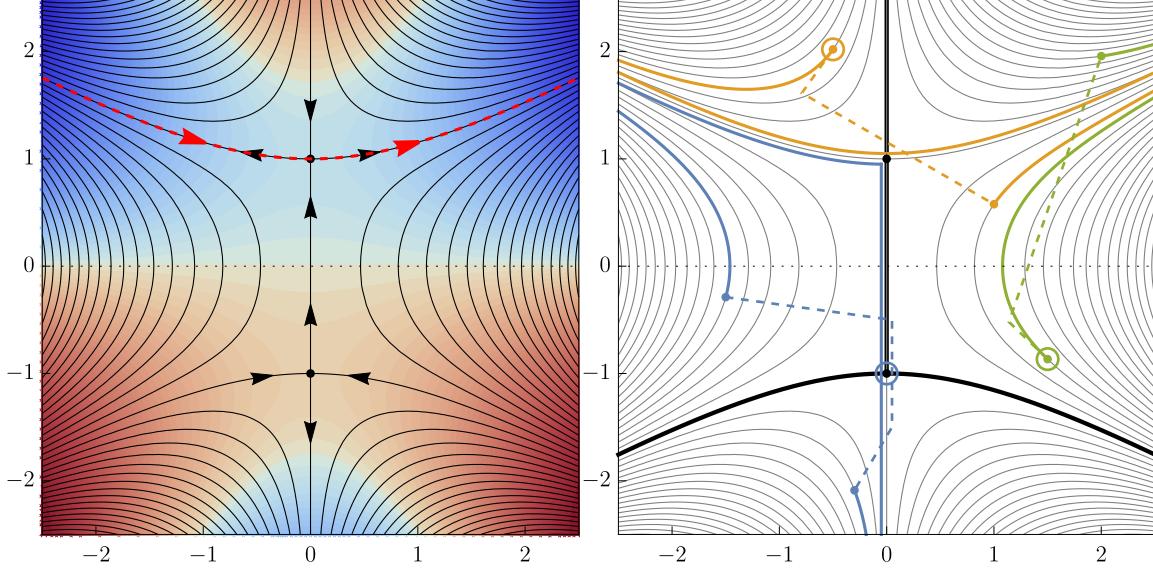


Figure 6.6: Steepest-descent contours for $h(t) = i(t + t^3/3)$. (a) Colours indicate the value of $\text{Re } h$ (positive red, negative blue), and black arrows show descending directions. The red dashed contour is the full SD contour for $\text{Ai}(x)$. (b) The ridges connecting the growing infinity directions, and three examples of full SD contours (solid) and shortcut contours (dashed) connecting given endpoints, with the dominant point circled.

Given the local sketches, we suspect that there is a contour joining the two saddle points, and that the contours coming out from the saddle points in the other directions join up with the corresponding far-field descending/ascending directions. (We could verify this by solving $\text{Im } h(p + iq) = 0$.)

We are now ready to find an appropriate contour of integration. The original contour has infinite endpoints on the real axis, where the integrand is purely oscillatory. We first deform the endpoints onto the adjacent decaying directions $\arg t = \pi/6$ and $\arg t = 5\pi/6$. (Strictly speaking, this requires a

change of variables and an application of Jordan's lemma, like when deforming $\int e^{iu^p} du$ in the method of stationary phase, but we don't worry about this detail.) Note that it would not be possible to deform either onto the decaying direction $\arg t = -\pi/2$, as that would involve crossing a growing region where the integrand blows up exponentially as $|t| \rightarrow \infty$.

We then expect to be able to use a SD contour that passes horizontally through the saddle at $+i$, as shown in figure 6.6(a). The exponentially dominant contribution comes from near the saddle point, so using a short line segment that is locally tangent to the SD direction, $t = i + s$ with s real, we obtain

$$\text{Ai}(x) \sim \frac{x^{1/2}}{2\pi} \int_{-\delta}^{\delta} e^{x^{3/2}(-\frac{2}{3}-s^2)} ds \sim \frac{x^{1/2}}{2\pi} e^{-\frac{2}{3}x^{3/2}} \sqrt{\frac{\pi}{x^{3/2}}} = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, \quad \text{as } x \nearrow \infty. \quad (6.83)$$

6.3.3 Choice of steepest-descent contour

Remark(s) 6.9. • There should be only one way to connect two given endpoints using full SD contours (while avoiding locations where $\text{Re } h \nearrow +\infty$). This is because the functions $\text{Re } h$ and $\text{Im } h$ cannot have any local maxima, only saddle points. If you could find two different SD contours between the same endpoints, then you would have a closed curve on which $\text{Re } h$ is bounded from above, and hence it would attain a maximum somewhere within that curve, which is a contradiction.

- In the example above, why couldn't we choose a contour passing through the saddle at $-i$, which has larger $\text{Re } h$ and hence would be exponentially dominant? Looking at the full contour picture we can see that there is a “ridge” (direction of steepest ascent) passing through that saddle horizontally, which no SD contours cross (except at the saddle point). Hence, any full SD contour that crosses the $-i$ saddle ends up stuck in that region unless it returns the same way it came.
- In fact, the three growing directions at infinity connect via ridges that divide the complex plane up into three separate regions, see figure 6.6b. In each region, all descending contours go to the same infinite direction. Hence, any full SD contour between two points in the same region will not cross any saddle points and hence be endpoint dominated. Any full SD contour connecting the two upper regions will pass over the saddle at $+i$, so can be dominated by that saddle or the endpoints. Any full SD contour connecting the lower region with either upper region will pass over both saddles, so will be dominated by either the higher saddle at $-i$ or by the endpoints. In a general problem, if you can determine where the ridges from the saddle points go, then you'll know which saddles need to be crossed.
- As before, we don't need to use the full contour – what we need is that the **maximum of $\text{Re } h$ on the contour must be attained at an endpoint or saddle point, while moving in the SD direction**. Hence, if you are unsure where the contour should go, you can just go through all endpoints and saddle points one by one and try to make it the dominant point, i.e. draw a contour along the SD direction through the point and then try to connect to the endpoints together without passing any region with larger $\text{Re } h$ than at that point.
- Why can't we just choose any contour, find where $\text{Re } h$ is maximal on the contour, and then the contribution from everywhere else will be exponentially smaller? Because if the maximum is not attained at a saddle or endpoint, then it will have oscillations and cancellation. Even though the integrand is $\text{ord}(e^{xh(t_*)})$, after integration the result can be exponentially smaller than that. The original contour for the Airy integral illustrates this: For real t the exponent is purely imaginary and the integrand is $\text{ord}(1)$ everywhere, but as we have seen after integration the result is exponentially smaller, $\text{ord}(e^{-\frac{2}{3}x^{3/2}})/x^{1/4}$.

Example(s) 6.11. For later in the module we will need some more Airy results, which you will calculate in detail on the problem sheet.

To investigate the behaviour of $\text{Ai}(x)$ as $x \searrow -\infty$, we write $x = -r$ with $r \nearrow \infty$ and instead use the change of variables $s = r^{1/2}t$, which yields a different exponent function $h(t) = i(-t + t^3/3)$ and hence a different contour landscape (but the same behaviour at infinity, $h \sim it^3/3$), see figure 6.7. This time, the saddles are at ± 1 with diagonal SD directions. The deformed contour for $\text{Ai}(x)$ needs to pass through both saddles, using the parametrisation $t = -1 + se^{-i\pi/4}$ and $t = +1 + se^{+i\pi/4}$ with $-\delta \leq s \leq \delta$.

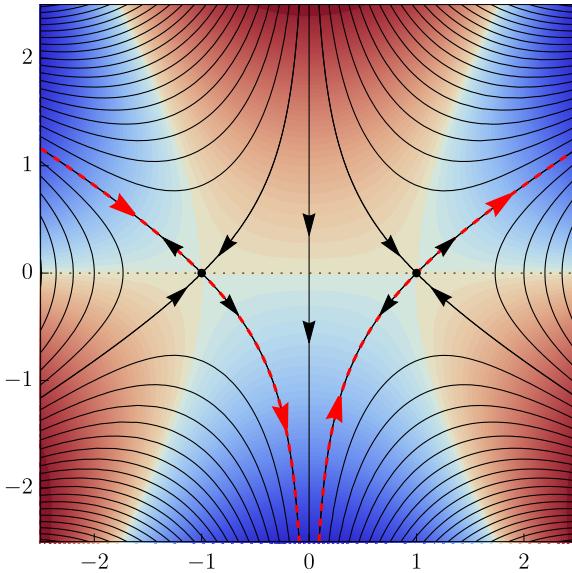


Figure 6.7: Steepest-descent contours for $h(t) = i(-t + t^3/3)$. Colours indicate the value of $\operatorname{Re} h$ (positive red, negative blue), and black arrows show descending directions. The red dashed contour is the full SD contour for $\operatorname{Ai}(x)$.

A second independent solution of the Airy equation is $\operatorname{Bi}(z)$, sometimes called the “Bairy” function, which can be defined as the integral of the same integrand but over two different contours, added together,

$$\operatorname{Bi}(x) = \frac{1}{2\pi i} \left[\int_{-i\infty}^{-\infty} + \int_{i\infty}^{\infty} \right] e^{i(xs + \frac{1}{3}s^3)} ds. \quad (6.84)$$

The asymptotic behaviour of $\operatorname{Bi}(x)$ as $x \rightarrow \pm\infty$ can be analysed in the same way as for Ai , and the contour landscape looks the same, but the choice of deformed contour is different as it the infinite endpoints to connect are different.

The end result is the asymptotic behaviours

$$\operatorname{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, \quad \operatorname{Bi}(x) \sim \frac{e^{\frac{2}{3}x^{3/2}}}{\sqrt{\pi}x^{1/4}} \quad \text{as } x \nearrow \infty, \quad (6.85)$$

$$\operatorname{Ai}(x) \sim \frac{\sin(\frac{2}{3}(-x)^{3/2} + \pi/4)}{\sqrt{\pi}(-x)^{1/4}}, \quad \operatorname{Bi}(x) \sim \frac{\cos(\frac{2}{3}(-x)^{3/2} + \pi/4)}{\sqrt{\pi}(-x)^{1/4}} \quad \text{as } x \searrow -\infty. \quad (6.86)$$

These are plotted together with the exact Ai and Bi in figure 6.8.

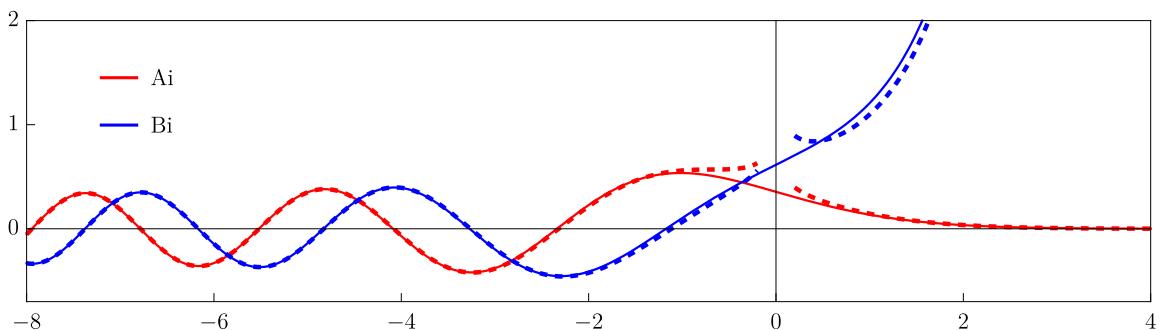


Figure 6.8: Plots of $\operatorname{Ai}(x)$ and $\operatorname{Bi}(x)$ (solid curves) together with their asymptotic approximations as $x \rightarrow \pm\infty$ (dashed curves).

Part III

Ordinary differential equations

7 Local analysis of ordinary differential equations

Moving on to differential equations, we begin by solving them locally, i.e. determining the asymptotic behaviour of their solutions $y(x)$ near a finite value of x , which we can shift to be $x = 0$, or as $x \rightarrow \pm\infty$. Although rigorous methods exist in some cases (e.g. the Frobenius method), we will use more general ideas without proper justification – in particular, we are going to be differentiating asymptotic relationships without worrying about the fact that it is not actually allowed. You will recognise some of the ideas from solving algebraic equations at the beginning of the module, with the added complications of derivatives being involved.

7.1 Linear equations

7.1.1 Algebraic behaviour

We first consider solutions with algebraic behaviour, i.e. behaving like a power of x . A key estimate we will use is that differentiating a power yields

$$y = Cx^\alpha \Rightarrow \frac{d^n y}{dx^n} = C\alpha(\alpha - 1) \cdots (\alpha - n + 1)x^{\alpha-n} = \alpha(\alpha - 1) \cdots (\alpha - n + 1) \frac{y}{x^n}, \quad (7.1)$$

so apart from for specific values of the exponent α where the derivative vanishes, we can think of the derivative as being of size $d/dx = \text{ord}(1/x)$, for both $x \rightarrow \pm\infty$ and $x \rightarrow 0$. This estimate can also be used when differentiating powers times logarithms, because

$$y = Cx^\alpha(\ln x)^\beta \Rightarrow \frac{dy}{dx} = C\alpha x^{\alpha-1}(\ln x)^\beta + Cx^\alpha \frac{\beta(\ln x)^{\beta-1}}{x} = \underbrace{Cx^{\alpha-1}(\ln x)^\beta}_{y/x} \left[\alpha + \underbrace{\frac{\beta}{\ln x}}_{\ll 1} \right]. \quad (7.2)$$

However, it is not valid for differentiating exponential/trigonometric functions of large quantities, e.g. $d(e^x)/dx \gg \text{ord}(e^x/x)$ for both $x \rightarrow \pm\infty$.

Example(s) 7.1. Consider the Airy equation but with a forcing term on the right-hand side,

$$y''(x) - xy(x) = e^{-x}. \quad (7.3)$$

As is customary for forced linear equations, we can first seek any particular solution of the equations and then add any solution of the homogeneous equation $y'' - xy = 0$ (of which there are two independent ones since the equation is of second order) to get the overall general solution.

Can we find the asymptotic behaviour of a particular solution as $x \rightarrow 0$? Let's look for a **dominant balance**, like in algebraic equations. Using $d/dx = \text{ord}(1/x)$, we expect xy to be negligible compared with $y'' = \text{ord}(y/x^2)$ as $x \rightarrow 0$. On the right-hand side, we expand the exponential and take the first term. This leaves us with

$$y'' \approx 1 \Rightarrow y \approx \frac{1}{2}x^2. \quad (7.4)$$

The general solution of the equation $y'' = 1$ actually contains two larger terms $A + Bx$, which we could keep if we wanted to find the general solution in one go, but let's just find one particular solution and keep the number of terms small. Note also the use of the vague “ \approx ” throughout the calculations, meaning that the error is small in some suitable sense, instead of the more strictly defined “ \sim ”.

We then solve for the next-order correction by writing

$$y = \frac{1}{2}x^2 + \tilde{y}_1, \quad \tilde{y}_1 \ll x^2 \Rightarrow 1 + \tilde{y}_1'' - x \left(\frac{1}{2}x^2 + \tilde{y}_1 \right) = e^{-x} \Rightarrow \tilde{y}_1'' - x\tilde{y}_1 = e^{-x} - 1 + \frac{1}{2}x^3. \quad (7.5)$$

Like at leading order, we expect $\tilde{y}_1'' \gg x\tilde{y}_1$, and hence (after expanding on the right-hand side) obtain

$$\tilde{y}_1'' \approx -x \Rightarrow \tilde{y}_1 \approx -\frac{1}{6}x^3. \quad (7.6)$$

We could then continue with $\tilde{y}_1 = -\frac{1}{6}x^3 + \tilde{y}_2$ etc. In each step, after neglecting the $-x\tilde{y}_n$ term we get an approximation for \tilde{y}_n'' which we can integrate twice to obtain an approximation for \tilde{y}_n . The result we get should be the expansion for a particular solution,

$$y \sim \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots \quad (7.7)$$

For the homogeneous solutions, we start with the equation $y'' - xy = 0$ and again estimate $xy \ll y''$, so obtain the dominant balance

$$y'' \approx 0 \Rightarrow y \approx A. \quad (7.8)$$

Note here the difference with algebraic equations, where a dominant balance involving a single term only would yield the zero solution and hence be disallowed. Also, strictly speaking this is not a dominant balance, because the values of the term y'' are not larger than the values of the term xy , since according to the equation they are exactly equal.

Once we have found a leading-order term in the homogeneous solution, the equation for the correction is a forced version of the equation again: We write $y = A + \tilde{y}_1$ with $\tilde{y}_1 \ll 1$ and obtain, as above,

$$0 + \tilde{y}_1'' - x(A + \tilde{y}_1) = 0 \Rightarrow \tilde{y}_1'' - x\tilde{y}_1 = Ax \Rightarrow \tilde{y}_1'' \approx Ax \Rightarrow \tilde{y}_1 \approx \frac{1}{6}Ax^3. \quad (7.9)$$

Here in the last step we have ignored the possibility of a Bx term. To calculate the second independent homogeneous solution, we start over from the beginning but considering that term only,

$$y'' \approx 0 \Rightarrow y \approx Bx, \quad (7.10)$$

$$y = Bx + \tilde{y}_1 \Rightarrow \tilde{y}_1'' - x\tilde{y}_1 = Bx^2 \Rightarrow \tilde{y}_1'' \approx Bx^2 \Rightarrow \tilde{y}_1 \approx \frac{1}{12}Bx^4. \quad (7.11)$$

We conclude that the general solution is of the form

$$y \approx [\frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots] + A[1 + \frac{1}{6}x^3 + \dots] + B[x + \frac{1}{12}x^4 + \dots] \quad \text{as } x \rightarrow 0. \quad (7.12)$$

In this case the terms in all solutions are of similar order, but more generally one of the expansions might be exponentially small relative to another so it might not make much sense to write the total general solution, but instead just talk separately about e.g. “the fastest decaying particular solution”, “the fastest decaying homogeneous solution”, etc.

- Remark(s) 7.1.** • If the highest-order term is not part of the dominant balance, then we won't find all n independent homogeneous solutions. We'll discuss how to find the remaining ones, which do not have algebraic behaviour, in the next subsection.
• Writing a linear differential equation in “equidimensional form”

$$p_n(x) \underbrace{(x^n y^{(n)})}_{\text{ord}(y)} + p_{n-1}(x) \underbrace{(x^{n-1} y^{(n-1)})}_{\text{ord}(y)} + \dots + p_1(x) \underbrace{(xy')}_{\text{ord}(y)} + p_0(x) \underbrace{(y)}_{\text{ord}(y)} = f(x) \quad (7.13)$$

quickly reveals which term(s) on the left-hand side are dominant for solutions with algebraic behaviour, namely the one(s) whose $p_i(x)$ is dominant in the limit. This applies to both $x \rightarrow 0$ and $x \rightarrow \pm\infty$.

- In more complicated problems, identifying the dominant terms still leaves us with an equation we can't solve. Then we'll need to try a suitable ansatz, as in the next example.

Example(s) 7.2. Let's consider the Bessel equation,

$$x^2 y'' + xy' + (x^2 - n^2)y = 0. \quad (7.14)$$

Collecting terms of similar size (assuming algebraic behaviour of y) yields

$$\underbrace{x^2 y'' + xy' - n^2 y}_{\text{ord}(y)} + \underbrace{x^2 y}_{\text{ord}(x^2 y)} = 0. \quad (7.15)$$

For $x \rightarrow 0$ we see that the $x^2 y$ term is negligible and the $\text{ord}(y)$ terms all form part of a dominant balance. We try a **power-law ansatz** and obtain

$$y \approx Ax^\alpha, A \neq 0 \Rightarrow 0 \approx x^2 y'' + xy' - n^2 y \approx Ax^\alpha [\alpha(\alpha-1) + \alpha - n^2] = Ax^\alpha [\alpha^2 - n^2]. \quad (7.16)$$

Thus, the solutions are $\alpha = \pm n$. The coefficient $A \neq 0$ is arbitrary, as usual in linear problems, so we could set $A = 1$ if we wanted to be lazy.

To find the correction, we write $y = Ax^{\pm n} + \tilde{y}_1$ and obtain

$$\underbrace{x^2\tilde{y}_1'' + x\tilde{y}_1' - n^2\tilde{y}_1}_{\text{ord}(\tilde{y}_1)} + \underbrace{x^2\tilde{y}_1}_{\ll\tilde{y}_1} = -Ax^{\pm n+2}, \quad (7.17)$$

which is the Bessel equation again but forced by the term neglected at leading order. Since the forcing is of power-law form, we again try a power-law ansatz

$$\tilde{y}_1 \approx A_1 x^{\alpha_1}, \quad A_1 \neq 0 \quad \Rightarrow \quad A_1 x^{\alpha_1} [\alpha_1^2 - n^2] \approx -Ax^{\pm n+2}. \quad (7.18)$$

In order to have a balance with the forcing term, we then need to equate the exponents and the coefficients,

$$\alpha_1 = \pm n + 2, \quad A_1 = -\frac{A}{\alpha_1^2 - n^2} = -A \frac{1}{4(1 \pm n)}. \quad (7.19)$$

Thus, we obtain the two solutions

$$y = Ax^n \left[1 - \frac{x^2}{4(1+n)} + \dots \right], \quad y = Bx^{-n} \left[1 - \frac{x^2}{4(1-n)} + \dots \right], \quad \text{as } x \rightarrow 0. \quad (7.20)$$

Remark(s) 7.2. Sometimes when we make a power-law ansatz we can determine the exponent but then there is a problem. For example, for a forced problem like the one for \tilde{y}_1 above, the exponent of the forcing term determines the exponent α_1 , but we may obtain an impossible equation for the coefficient A_1 , i.e. $0 \times A_1 = -A$ (which occurs for $n = 1$). The rule of thumb then is to keep the same value of the exponent and multiply the ansatz by a logarithm, i.e. try $\tilde{y}_1 \approx A_1 x^{\alpha_1} \ln x$.

The same fix is needed for the homogeneous solutions if the equation for the exponent has a multiple root – the independent homogeneous solutions would be Ax^α , $Ax^\alpha \ln x$, $Ax^\alpha (\ln x)^2$, etc.

7.1.2 Exponential behaviour

We now consider functions of the form $y = e^{h(x)} f(x)$ where $h(x)$ and $f(x)$ have algebraic behaviour and $h(x) \gg 1$. Differentiating yields

$$y = e^{h(x)} f(x) \quad \Rightarrow \quad \frac{dy}{dx} = e^{h(x)} h'(x) f(x) + e^{h(x)} f'(x) = \underbrace{e^{h(x)} f(x)}_{y} \left[\underbrace{h'(x)}_{\gg 1/x} + \underbrace{\frac{f'(x)}{f(x)}}_{\text{ord}(1/x)} \right]. \quad (7.21)$$

Hence, we can think of the derivative as being of size $d/dx = \text{ord}(h'(x)) \gg 1/x$ instead.

When seeking a particular solution to an equation with exponential forcing $e^{h(x)} f(x)$, a suitable ansatz to try is thus $Ax^\alpha e^{h(x)}$, and the calculation then proceeds as for the algebraic case.

However, solutions with exponential behaviour can also make a perhaps unexpected appearance in homogeneous equations, which will require a special method to solve, as we shall now see.

Example(s) 7.3. Consider the Airy equation $y'' - xy = 0$. The estimate $y'' = \text{ord}(y/x^2)$ means that for $x \rightarrow 0$ (which we solved above) the y'' term dominates and we obtain $y'' \approx 0$ with two independent solutions. However, what happens for $x \nearrow \infty$? Now xy dominates and we immediately run into trouble:

$$-xy \approx 0 \quad \Rightarrow \quad y \approx 0. \quad (7.22)$$

The underlying issue is that when the highest-order derivative is not part of the dominant balance, the leading-order equation is of lower order and has fewer independent solutions. So where are the missing solutions?

The idea is to use an exponential with a large exponent to make the derivatives become “larger than expected” (i.e. $d/dx \gg \text{ord}(1/x)$), thus bringing the highest-order term back into a dominant balance. This method is known as one version of the **WKB (Wentzel–Kramers–Brillouin) method**, or as the **Liouville–Green (LG) method**.

The first step in the method is to make the change of variables

$$y(x) = e^{S(x)} \Rightarrow y' = S'e^S, \quad y'' = (S'^2 + S'')e^S. \quad (7.23)$$

(The use of the name “ S ” for the exponent is standard.) Substitution into the equation allows the exponentials to be cancelled,

$$(S'^2 + S'')e^S - xe^S = 0 \Rightarrow S'^2 + S'' = x. \quad (7.24)$$

This is a nonlinear differential equation for S' , which doesn’t appear to be any simpler than the original, but the key is that the unknown S' is now expected to have algebraic behaviour rather than exponential, which allows us to deduce the dominant balance straightforwardly.

The key point was for the derivative to yield a larger result than dividing by x , so we assume that $S' \gg 1/x$. We can then estimate $S'' = \text{ord}(S'/x) \ll S'^2$, so the leading-order balance must be

$$S'^2 \approx x \Rightarrow S' \approx \pm x^{1/2} \Rightarrow S \approx S_0 = \pm \frac{2}{3}x^{3/2}. \quad (7.25)$$

(Throughout this we can ignore the constant of integration for S as it corresponds to a constant prefactor in y .) Note that since S'' was negligible we were left with an algebraic equation for S' , that we could easily solve. This is the key simplification in the WKB method.

We are not done yet, because $S \sim S_0$ does not imply $e^S \sim e^{S_0}$. We **must** calculate the next correction for S . So we write

$$S = S_0 + \tilde{S}_1, \quad \tilde{S}_1 \ll S_0 \Rightarrow 2S'_0\tilde{S}'_1 + \tilde{S}'^2_1 + S''_0 + \tilde{S}''_1 = 0, \quad (7.26)$$

after cancelling the leading-order result $S'^2_0 = x$. Noting now that $\tilde{S}'^2_1 \ll S'_0\tilde{S}'_1$ and $\tilde{S}''_1 \ll S''_0$, we reduce the system to

$$2S'_0\tilde{S}'_1 + S''_0 \approx 0 \Rightarrow \tilde{S}'_1 \approx -\frac{S''_0}{2S'_0} = -\frac{\pm \frac{1}{2}x^{-1/2}}{\pm 2x^{1/2}} = -\frac{1}{4}x^{-1} \Rightarrow \tilde{S}_1 \approx S_1 = -\frac{1}{4}\ln x. \quad (7.27)$$

This is $\gg 1$, so must be included in the leading-order approximation for y . We could continue to find higher-order corrections, but when seeking a leading-order result we typically just assume that the next correction is algebraically smaller and hence $\ll 1$. (Specifically, the next correction, S_2 , will have $S'_2 \ll 1/x$ so it follows that $S_2 \ll 1$ except for in rare cases like $S_2 = \ln \ln x$ which satisfies $S'_2 \ll 1/x$ but $S_2 \gg 1$.) We conclude that two independent solutions of the Airy equation have the leading-order asymptotic behaviour

$$y = \exp(S_0 + S_1 + o(1)) \sim \exp\left(\pm \frac{2}{3}x^{3/2}\right)x^{-1/4} \quad \text{as } x \nearrow \infty. \quad (7.28)$$

A general solution would be a linear combination of these two, but since the + solution is exponentially dominant, any solution that doesn’t decay will have a full expansion coming from just the + solution.

Now recall that Ai is the decaying solution and Bi is a non-decaying solution, from which we immediately deduce that

$$\text{Ai}(x) \sim A \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}, \quad \text{Bi}(x) \sim B \frac{e^{+\frac{2}{3}x^{3/2}}}{x^{1/4}} \quad \text{as } x \nearrow \infty, \quad (7.29)$$

which is consistent with our previous steepest-descent results.

Further, we did not use the fact that x is real, and hence the WKB result is valid for any complex $x = re^{i\theta}$ with $r \nearrow \infty$ at fixed θ . In particular, for negative $x = -r$, the leading-order asymptotic behaviours are

$$e^{\pm \frac{2}{3}(-r)^{3/2}}(-r)^{-1/4} \propto e^{\pm \frac{2}{3}ir^{3/2}}r^{-1/4}, \quad (7.30)$$

and taking suitable linear combinations of these yields our previous steepest-descent results. However, the coefficients of the linear combination can be different for different values of θ . (This is called the **Stokes phenomenon** – that a function can have different asymptotic behaviour in different regions of the complex plane.)

Remark(s) 7.3. • The WKB method typically requires calculating two terms, with the correction S_1 being a logarithm which becomes a power multiplying the exponential e^{S_0} .

- When the WKB method is applied for small x , the exponential will have a negative power of x .
- The assumption that the exponent S is (algebraically) large yields $S' \gg 1/x$ and hence

$$y = e^S, \quad y' = S'e^S, \quad y'' \sim (S'^2)e^S, \quad \dots, \quad y^{(n)} \sim (S')^n e^S, \quad (7.31)$$

since differentiating the exponent to bring down a factor $S' \gg 1/x$ beats differentiating the algebraic factor in front which is like multiplying by $1/x$. Hence, applying the WKB method to an n th order equation yields a nonlinear differential equation for S' that is approximately an n th degree polynomial equation for S' . As a result, we only need to solve algebraic equations for S' and its corrections. (The correction equations still contain derivatives, but they are derivatives of earlier corrections that are known, and serve as forcing terms in the algebraic equations for the later corrections.)

Solutions S' of the polynomial equation that do not satisfy $S' \gg 1/x$ correspond to solutions of the original equation that have algebraic behaviour, but they are not directly equal to the (logarithm of) those solutions, since the polynomial equation approximates the full differential equation only for $S' \gg 1/x$.

7.2 Nonlinear equations (mastery topic)

Local analysis of nonlinear equations will be the mastery topic for this year. This section of the notes will not be lectured. A separate problem sheet is available for this section.

8 Differential equations with a small parameter

We now turn to solving differential equations with a small parameter ε (or a large parameter $1/\varepsilon$). One consequence of this is that we are no longer dealing with functions $f(\varepsilon)$ of the small parameter only, but instead functions $f(x, \varepsilon)$ depending on another variable x too. Asymptotic expansions of such functions are also called **parameter expansions**.

Definition 8.1. The obvious generalisation of the definition of an asymptotic expansion to have a dependence on x is to allow the coefficients to depend on x ,

$$f(x, \varepsilon) \sim \sum_{n=0}^N a_n(x) \delta_n(\varepsilon) \Leftrightarrow f(x, \varepsilon) - \sum_{n=0}^N a_n(x) \delta_n(\varepsilon) = o(\delta_N(\varepsilon)) \quad (8.1)$$

as $\varepsilon \searrow 0$. This is called a **Poincaré expansion** or a **classical expansion**. If the little- o is uniform for x in the relevant domain, i.e. for all $M > 0$ there exists $\varepsilon_* > 0$ independent of x such that

$$\left| f(x, \varepsilon) - \sum_{n=0}^N a_n(x) \delta_n(\varepsilon) \right| \leq M \delta_N(\varepsilon) \quad \text{for } 0 < \varepsilon < \varepsilon_*, \quad (8.2)$$

then we say that the expansion is **uniformly valid**. (For this module, the main point of this definition won't be to prove when things are uniformly valid, but just to point out when they are not.) If we additionally allow the coefficients to also depend on ε , then we have a **generalised asymptotic expansion**

$$f(x, \varepsilon) \sim \sum_{n=0}^N a_n(x, \varepsilon) \delta_n(\varepsilon) = \sum_{n=0}^N f_n(x, \varepsilon) \Leftrightarrow f(x, \varepsilon) - \sum_{n=0}^N f_n(x, \varepsilon) = o(f_N(x, \varepsilon)). \quad (8.3)$$

Remark(s) 8.1. For a differential equation, the parameter ε can appear in the coefficients of the equation, any forcing terms, and the boundary conditions. If the perturbation is regular in the sense that the solution has a power-series expansion, then we can write

$$y(x) \sim \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \quad (8.4)$$

and substitute into the equation, then solve for $y_n(x)$ at each order. Since it is not at all obvious when this will be the case, a good start is to try it out and see if we can spot any problems.

Example(s) 8.1. Consider the equation $y'' + \varepsilon y = 0$, $y(0) = 0$, $y(1) = e^\varepsilon$.

We make the ansatz

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \dots \quad (8.5)$$

and substitute into the governing equations to find

$$(y_0'' + \varepsilon y_1'' + \dots) + \varepsilon(y_0 + \varepsilon y_1 + \dots) = 0, \quad (8.6)$$

$$y_0(0) + \varepsilon y_1(0) + \dots = 0, \quad y_0(1) + \varepsilon y_1(1) + \dots = 1 + \varepsilon + \dots \quad (8.7)$$

Solving order by order then yields

$$y_0'' = 0, \quad y_0(0) = 0, \quad y_0(1) = 1 \Rightarrow y_0 = ax + b = x \quad (8.8)$$

$$y_1'' = -y_0 = -x, \quad y_1(0) = 0, \quad y_1(1) = 1 \Rightarrow y_1 = -\frac{x^3}{6} + cx + d = -\frac{x^3}{6} + \frac{7x}{6}. \quad (8.9)$$

etc.

So we claim that

$$y \sim x + \varepsilon \left[-\frac{x^3}{6} + \frac{7x}{6} \right], \quad (8.10)$$

and it doesn't look like there will be any problems, so this is probably a uniformly valid Poincaré expansion.

In the next few sections, we will consider different singular perturbation problems and the methods for solving them.

9 Matched asymptotic expansions

9.1 Basic method

Example(s) 9.1. Consider $\varepsilon y'' + y' + y = 0$ with boundary conditions $y(0) = -1$, $y(1) = e^{-1}$, for $\varepsilon \searrow 0$.

The equations and boundary conditions just contain integer powers of ε , so let's try a classical power-series expansion

$$y \sim y_0 + \varepsilon y_1 + \dots \quad (9.1)$$

Solving order by order we obtain

$$O(\varepsilon^0) : y'_0 + y_0 = 0 \Rightarrow y_0 = C_0 e^{-x}, \quad (9.2)$$

$$O(\varepsilon^1) : y'_1 + y_1 = -y''_0 = -C_0 e^{-x} \Rightarrow y_1 = -C_0 x e^{-x} + C_1 e^{-x}. \quad (9.3)$$

We expect the constants C_0 and C_1 to be determined by the boundary conditions, but there are two conditions at each order and they yield conflicting values. For example, at $\text{ord}(\varepsilon^0)$ we have

$$-1 = y_0(0) = C_0 \Rightarrow C_0 = -1, \quad \text{but} \quad e^{-1} = y_0(1) = C_0 e^{-1} \Rightarrow C_0 = +1. \quad (9.4)$$

$$0 = y_1(0) = 0 \Rightarrow C_1 = 0, \quad \text{but} \quad 0 = y_1(1) = -e^{-1}(-C_0 + C_1) \Rightarrow C_1 = C_0. \quad (9.5)$$

The problem is that the small parameter multiplies the highest-derivative term in the equation, so the leading-order problem (and indeed the problem at any order) has lower derivatives than the original – this is a singular perturbation! As a result we can only satisfy one of the two boundary conditions.

It turns out that we should choose to satisfy the condition at $x = 1$, i.e. $C_0 = 1$ and $C_1 = 1$. (We'll discuss why below.) The resulting expansion

$$y(x) = e^{-x} + \varepsilon(1-x)e^{-x} + O(\varepsilon^2) \quad (9.6)$$

will be valid for the **outer region** $x = \text{ord}(1)$, or more specifically for any fixed x in the range $0 < x \leq 1$ as $\varepsilon \searrow 0$. We call this the **outer expansion**.

To satisfy the boundary condition at $x = 0$, we introduce a **boundary layer** near $x = 0$, i.e. a small region in which y varies rapidly so that $\varepsilon y''$ becomes significant. What scaling should it have? Let's set

$$y(x) = Y(X), \quad x = \delta X, \quad \text{where } \delta(\varepsilon) \ll 1 \text{ and } X = \text{ord}(1). \quad (9.7)$$

and try to determine δ . The derivatives then become larger,

$$\frac{d}{dx} = \frac{dX}{dx} \frac{d}{dX} = \frac{1}{\delta} \frac{d}{dX} \quad \Rightarrow \quad \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + Y = 0. \quad (9.8)$$

We now seek a leading-order balance between these terms, similar to how we solved polynomial equations. Balancing the Y' and Y terms leads to $\delta = 1$ (strictly speaking $\delta = \text{ord}(1)$, but we choose the simplest such δ), which is just the outer scaling. Balancing the Y'' and Y terms leads to $\varepsilon/\delta^2 = 1$ and hence $\delta = \varepsilon^{1/2}$ which is inconsistent since the Y' term at $\text{ord}(\varepsilon^{-1/2})$ will be larger. Balancing the Y'' and Y' terms yields the correct balance, $\varepsilon/\delta^2 = 1/\delta$ i.e. $\delta = \varepsilon$.

We have thus identified the scaling for the **inner region**, $x = \varepsilon X$. The governing equations become

$$\varepsilon^{-1} Y'' + \varepsilon^{-1} Y' + Y = 0 \Rightarrow Y'' + Y' = -\varepsilon Y, \quad \text{with } Y(0) = -1. \quad (9.9)$$

We obviously do not try to impose the boundary condition at $x = 1$, which would be at $X = 1/\varepsilon$. We again try a classical power-series expansion, which we call the **inner expansion**, $Y = Y_0 + \varepsilon Y_1 + \dots$, and obtain, to leading order,

$$Y''_0 + Y'_0 = 0, \quad Y_0(0) = -1 \Rightarrow Y_0 = A_0 e^{-X} + B_0 = A_0 e^{-X} - A_0 - 1. \quad (9.10)$$

The unknown constant is determined by **matching** the inner and outer expansions. In this simple case, we can simply equate the limiting values

$$\lim_{X \nearrow \infty} Y_0(X) = \lim_{x \searrow 0} y_0(x) \Rightarrow -A_0 - 1 = 1 \Rightarrow A_0 = -2, \quad Y_0 = 1 - 2e^{-X}. \quad (9.11)$$

Let's now continue to next order in the inner expansion,

$$Y_1'' + Y_1' = -Y_0 = -1 + 2e^{-X}, \quad Y_1(0) = 0 \quad (9.12)$$

$$\Rightarrow Y_1 = -X - 2Xe^{-X} + A_1 e^{-X} + B_1 = -X - 2Xe^{-X} + A_1(e^{-X} - 1). \quad (9.13)$$

To determine A_1 by matching, we need to do something more sophisticated.

There are two methods used for asymptotic matching. The straightforward method is to use **Van Dyke's matching principle**, which states that taking the outer expansion to $\text{ord}(\varepsilon^P)$, substituting in the inner variable and re-expanding to $\text{ord}(\varepsilon^Q)$ should yield the same result as taking the inner expansion to $\text{ord}(\varepsilon^Q)$, substituting in the outer variable and re-expanding to $\text{ord}(\varepsilon^P)$. Taking $P = Q = 1$, and introducing the non-standard notation O_x and O_X (indicating which variable is held fixed in the limit $\varepsilon \searrow 0$), we obtain

$$y(\varepsilon X) = [e^{-\varepsilon X}] + \varepsilon [(1 - \varepsilon X)e^{-\varepsilon X}] + O_x(\varepsilon^2) \quad (9.14)$$

$$= [1 - \varepsilon X + O_X(\varepsilon^2)] + \varepsilon [1 + O_X(\varepsilon)] + O_x(\varepsilon^2) \quad (9.15)$$

$$= 1 - \varepsilon X + \varepsilon + O_X(\varepsilon^2) + O_x(\varepsilon^2) \quad (9.16)$$

$$Y(x/\varepsilon) = [1 - 2e^{-x/\varepsilon}] + \varepsilon [-(x/\varepsilon) - 2(x/\varepsilon)e^{-x/\varepsilon} + A_1(e^{-x/\varepsilon} - 1)] + O_X(\varepsilon^2). \quad (9.17)$$

$$= [1 + O_x(\varepsilon^2)] + \varepsilon [-x/\varepsilon + O_x(\varepsilon) - A_1 + O_x(\varepsilon)] + O_X(\varepsilon^2) \quad (9.18)$$

$$= 1 - x - \varepsilon A_1 + O_x(\varepsilon^2) + O_X(\varepsilon^2). \quad (9.19)$$

(In fact, in the inner expansion the errors are exponentially smaller, but $O_x(\varepsilon^2)$ is sufficient.) Comparing these results, we find that they agree provided that $A_1 = -1$.

We have now determined both the outer and inner expansion to $\text{ord}(\varepsilon)$,

$$y(x) = e^{-x} + \varepsilon(1 - x)e^{-x} + O_x(\varepsilon^2), \quad (9.20)$$

$$Y(X) = [1 - 2e^{-X}] + \varepsilon [1 - X - (1 + 2X)e^{-X}] + O_X(\varepsilon^2), \quad (9.21)$$

which together describe the behaviour of $y(x)$ as $\varepsilon \searrow 0$.

We can form an **(additive) composite approximation** by adding the inner and outer approximations (expressed in terms of x) and subtracting their common overlap behaviour found above,

$$y(x) \approx \{e^{-x} + \varepsilon(1 - x)e^{-x}\} + \left\{1 - 2e^{-x/\varepsilon} - x + \varepsilon - (2x + \varepsilon)e^{-x/\varepsilon}\right\} - \{1 - x + \varepsilon\} \quad (9.22)$$

$$= e^{-x} [1 + \varepsilon(1 - x)] + e^{-x/\varepsilon} [-2 - (2x + \varepsilon)]. \quad (9.23)$$

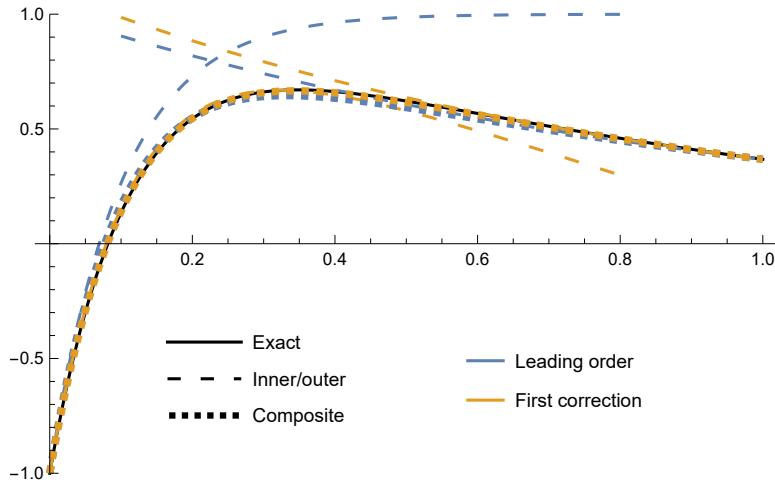


Figure 9.1: Solution of $\varepsilon y'' + y' + y = 0$, $y(0) = -1$, $y(1) = e^{-1}$ for $\varepsilon = 0.1$.

Remark(s) 9.1. • Why did the boundary layer have to be at $x = 0$? If we try a boundary layer at $x = 1$, then we'll find $x = 1 + \varepsilon X$, but the inner expansions will contain e^{-X} which grow exponentially in the matching limit $X \searrow -\infty$ and hence cannot match to the outer. In general, there can be one or more boundary layers, located at the endpoints of the interval as well as in the interior (also known as **interior layers**), and nested within each other.

- In this particular example, the outer solution was fully determined by the boundary conditions, and we only needed matching to determine the unknown constants of the inner solution. In general, it can also be the other way around, or both regions can have unknown constants.
- The Van Dyke matching principle can also be applied to more exotic scale functions, but it does not always work. A more robust method for matching is to use an **intermediate variable** $\xi = x/\varepsilon^\alpha = \varepsilon^{1-\alpha} X$ for some fixed α in the range $0 < \alpha < 1$. We substitute this into the expansions and re-expand each term:

$$y(e^\alpha \xi) = \left[e^{-\varepsilon^\alpha \xi} \right] + \varepsilon \left[(1 - \varepsilon^\alpha \xi) e^{-\varepsilon^\alpha \xi} \right] + O_x(\varepsilon^2) = \quad (9.24)$$

$$= \left[1 - \varepsilon^\alpha \xi + \varepsilon^{2\alpha} \frac{\xi^2}{2} + O_\xi(\varepsilon^{3\alpha}) \right] + [\varepsilon + O_\xi(\varepsilon^{1+\alpha})] + O_x(\varepsilon^2), \quad (9.25)$$

$$Y(\xi/\varepsilon^{1-\alpha}) = \left[1 - 2e^{-\xi/\varepsilon^{1-\alpha}} \right] + \varepsilon \left[-\frac{\xi}{\varepsilon^{1-\alpha}} - 2\frac{\xi}{\varepsilon^{1-\alpha}} e^{-x/\varepsilon^{1-\alpha}} + A_1(e^{-x/\varepsilon^{1-\alpha}} - 1) \right] + O_X(\varepsilon^2) = \quad (9.26)$$

$$= [1 + \text{EST}] + [-\varepsilon^\alpha \xi + \text{EST} - \varepsilon A_1 + \text{EST}] + O_X(\varepsilon^2). \quad (9.27)$$

The two results agree provided that $A_1 = -1$ and that the $\varepsilon^{2\alpha}$ term in the outer is small compared with the $\text{ord}(\varepsilon)$ terms we kept, so we need to choose $1/2 < \alpha < 1$. (This term will match a term coming from the next term in the inner expansion.)

- The outer scaling $x = \text{ord}(1)$ and inner scaling $x = \text{ord}(\varepsilon)$ are **distinguished limits**, in which at least two terms of the governing equation balance. It's possible to solve the governing equations on any scale $x = \text{ord}(\varepsilon^\alpha)$, but the result will be the same as taking the limit of either the outer or the inner expansion (or both).
- The key principle for matching is that the outer and inner expansions represent the same function, and the assumption is that both expansions are valid, in an **overlap region** on an intermediate scale. If the regions of validity do not overlap, this is likely due to an overlooked distinguished limit on an intermediate scale, which then matches correctly to both the smaller and the larger scales.

9.2 Advanced examples

Example(s) 9.2. Consider the equation

$$y'' = -6x + \frac{\varepsilon}{y^4}, \quad y(0) = 1, \quad y'(0) = 0, \quad (9.28)$$

which can be interpreted as the equation for a ball that is pulled down by a steadily increasing gravitational field, but is repelled from the ground by a short-range force.

We seek an outer solution $y = y_0 + \varepsilon y_1 + \dots$ and obtain the leading-order equation, boundary conditions and solution

$$y_0'' = -6x, \quad y_0(0) = 1, \quad y_0'(0) = 0 \quad \Rightarrow \quad y_0 = -x^3 + Ax + B = 1 - x^3. \quad (9.29)$$

At the next order,

$$y_1'' = \frac{1}{y_0^4} = \frac{1}{(1-x^3)^4}, \quad y_1(0) = y_1'(0) = 0. \quad (9.30)$$

We could integrate this twice using partial fractions, but let's settle for noting that the right-hand side has a singularity at $x = 1$, blowing up like $\text{ord}((1-x)^{-4})$, and hence that after integrating twice, y_1 would blow up like $\text{ord}((1-x)^{-2})$.

This means that there's a problem as $x \nearrow 1$, namely that the correction εy_1 eventually becomes larger than the dominant term y_0 , and we expect to need a boundary layer, or more accurately an **interior layer**, near $x = 1$. One way to find the appropriate inner scalings, when we know the behaviour of y_1 , is to argue that **the asymptotic ordering breaks down** when

$$\text{ord}(y_0) = \text{ord}(\varepsilon y_1) \quad \Rightarrow \quad \text{ord}(1-x) = \text{ord}(\varepsilon(1-x)^{-2}) \quad (9.31)$$

$$\Rightarrow \quad 1-x = \text{ord}(\varepsilon^{1/3}), \quad y_0 = \text{ord}(\varepsilon^{1/3}). \quad (9.32)$$

Alternatively, we can seek a general rescaling near $x = 1$,

$$x = 1 + \delta_x(\varepsilon)X \quad \Rightarrow \quad \frac{d}{dx} = \frac{1}{\delta_x} \frac{d}{dX}, \quad y = \delta_y(\varepsilon)Y(X), \quad (9.33)$$

and a dominant balance in the governing equations. However, we now have two unknowns $\delta_{x,y}$, and the dominant balance only gives us one equation. The key is to **use the local behaviour of the outer solution**, assuming that this will be used in the leading-order matching,

$$y_0 = \text{ord}(1-x) \text{ as } x \nearrow 1 \quad \Rightarrow \quad \text{ord}(\delta_y Y) = \text{ord}(\delta_x X) \quad \Rightarrow \quad \delta_y = \delta_x. \quad (9.34)$$

The governing equation then yields

$$\underbrace{\frac{\delta_y}{\delta_x^2} Y''}_{\text{ord}(\delta_x^{-1})} = \underbrace{-6(1+\delta_x X)}_{\text{ord}(1)} + \underbrace{\frac{\varepsilon}{\delta_y^4} Y^{-4}}_{\text{ord}(\varepsilon/\delta_x^4)}. \quad (9.35)$$

The only consistent dominant balance for $\delta_x \ll 1$ is between the first and last terms,

$$\delta_x^{-1} = \frac{\varepsilon}{\delta_x^4} \quad \Rightarrow \quad \delta_x = \varepsilon^{1/3}, \quad \delta_y = \varepsilon^{1/3}. \quad (9.36)$$

Using either method, we end up with the inner equation

$$\frac{1}{\varepsilon^{1/3}} Y'' = -6(1 + \varepsilon^{1/3} X) + \frac{\varepsilon}{\varepsilon^{4/3} Y^{4/3}} \quad \Rightarrow \quad Y'' = -6(\varepsilon^{1/3} + \varepsilon^{2/3} X) + Y^{-4}. \quad (9.37)$$

At leading order, $Y = Y_0 + \dots$, we have

$$Y_0'' = \frac{1}{Y_0^4}, \quad (9.38)$$

but we need two “boundary conditions” by matching with the outer solution. Since the leading-order outer matching behaviour is

$$y \sim y_0 = 1 - (1 + \varepsilon^{1/3} X)^3 \sim -3\varepsilon^{1/3}X \text{ as } x \nearrow 1, \quad (9.39)$$

we expect the inner solution $Y = y/\varepsilon^{1/3}$ to satisfy the leading-order condition

$$Y_0 \sim -3X \text{ as } X \searrow -\infty. \quad (9.40)$$

Indeed, with this condition satisfied, the Van Dyke matching principle yields

$$y = y_0(x) + o_x(1) = -3\varepsilon^{1/3}X + o_X(\varepsilon^{1/3}) + o_x(1), \quad (9.41)$$

$$\varepsilon^{1/3}Y = \varepsilon^{1/3}Y_0(X) + o_X(\varepsilon^{1/3}) = \varepsilon^{1/3}[-3(x-1)/\varepsilon^{1/3} + o_x(1/\varepsilon^{1/3})] + o_X(\varepsilon^{1/3}) = \quad (9.42)$$

$$= -3\varepsilon^{1/3}X + o_x(1) + O_X(\varepsilon^{1/3}), \quad (9.43)$$

which agrees. To obtain a second matching condition, we would need to consider the correction y_1 and use a more detailed matching, but in this case we can make progress without it.

The leading-order inner problem we have is thus

$$Y_0'' = \frac{1}{Y_0^4}, \quad Y_0 \sim -3X \text{ as } X \searrow -\infty. \quad (9.44)$$

We can integrate the governing equation once using the classical trick to multiply by Y_0' , which yields an “energy conservation” equation

$$Y_0'Y_0'' = \frac{Y_0'}{Y_0^4} \Rightarrow \frac{Y_0'^2}{2} = E_0 - \frac{1}{3Y_0^3} = \frac{9}{2} - \frac{1}{3Y_0^3}, \quad (9.45)$$

where we have determined the constant E_0 using the matching condition $Y_0' \rightarrow -3$ (and $Y_0 \nearrow \infty$) as $X \searrow -\infty$.

A key result now is that we can continue matching to the right, as $X \nearrow \infty$, to connect to a second outer region for $x > 1$. Specifically, the energy equation yields $Y_0' = \pm 3$ when $Y_0 \gg 1$, so we must have the far-field behaviour

$$Y_0 \sim +3X \quad \text{as } X \nearrow \infty. \quad (9.46)$$

In other words, our inner solution has shown that, from the perspective of the outer solution, the ball bounces off $y = 0$ with unchanged speed $|y'|$ at leading order. (Physically, we interpret this as being due to conservation of energy during the short $\text{ord}(\varepsilon^{1/3})$ time scale of the bounce, whereas the total energy does change on the $\text{ord}(1)$ time scale due to the term $-6x$ in the equation.) For the next outer solution, the matching with the inner region at $x = 1$ yields the leading-order conditions

$$y_0(1) = 0, \quad y_0'(1) = 3 \Rightarrow y_0 = -x^3 + Ax + B = 6x - x^3 - 5. \quad (9.47)$$

Thus, there will be another bounce at the next solution of $6x - x^3 - 5 = 0$, where the speed will be different but again unchanged by the bounce, and the bouncing continues indefinitely, as shown in figure 9.2.

Remark(s) 9.2. • Overall it is a good idea to start by solving the equation for $\varepsilon = 0$ and attempt to satisfy the boundary conditions.

- For both linear and nonlinear problems, interior layers may appear where the equation has a singularity and/or where a coefficient in the equation vanishes.
- The local behaviour of the outer solution(s) can help give a relation between the scalings for δ_x and δ_y .
- In this example the outer solution turned out to be continuous across the boundary/interior layer while its derivative was discontinuous and hence changed rapidly in the layer, which is consequently also called a **derivative layer**. In particular, this meant that at leading order the inner solution was simply equal to the value of the outer solution at the boundary layer (in this case zero), while the change in derivative occurs at higher order, $\text{ord}(\varepsilon^{1/3})$. When boundary conditions are imposed on the derivative of the function, e.g. $y'(0) = 1$, then the resulting boundary/derivative layer similarly only has a higher-order inner correction.

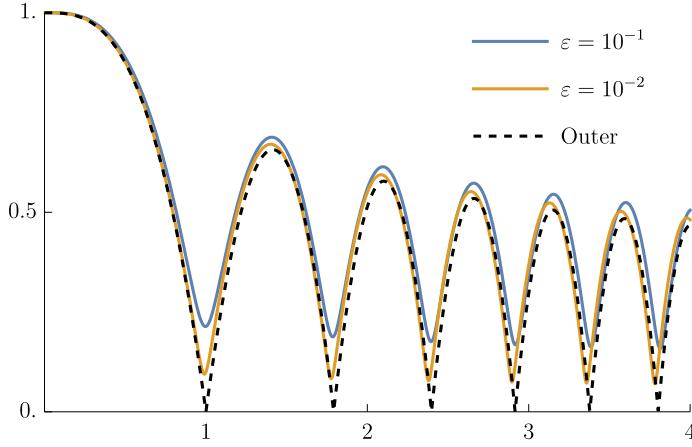


Figure 9.2: Solutions of $y'' = -6x + \varepsilon/y^4$ with $y(0) = 1$ and $y'(0) = 0$.

Example(s) 9.3. Consider

$$(x + \varepsilon)y'' + y' = 0, \quad y(0) = 1, \quad y(1) = 2, \quad \varepsilon \searrow 0. \quad (9.48)$$

(This is easily solved exactly, and in fact the inner problem turns out to be equivalent to the full problem, but the asymptotic calculation is instructive.)

In the outer region, we expand $y = y_0 + \varepsilon y_1 + \dots$ and obtain

$$xy_0'' + y_0' = 0 \quad \Rightarrow \quad xy_0' = A_0 \quad \Rightarrow \quad y_0 = A_0 \ln x + B_0. \quad (9.49)$$

This time we have two unknown constants, but the solution has a singularity at $x = 0$ so imposing $y(0) = 0$ would require $A_0 = B_0 = 0$, which is incompatible with the other condition $y(1) = B_0 = 2$. Again it turns out that the boundary layer must be at $x = 0$, so we impose the condition at $x = 1$ and obtain

$$y_0 = A_0 \ln x + 2. \quad (9.50)$$

The boundary layer again has the scaling $x = \varepsilon X$, so

$$(X + 1)Y'' + Y' = 0, \quad Y(0) = 1. \quad (9.51)$$

Expanding $Y = Y_0 + \varepsilon Y_1 + \dots$ yields

$$(X + 1)Y_0'' + Y_0' = 0, \quad Y_0(0) = 1 \quad \Rightarrow \quad (X + 1)Y_0' = C_0 \quad (9.52)$$

$$\Rightarrow \quad Y_0 = C_0 \ln(1 + X) + D_0 = C_0 \ln(1 + X) + 1. \quad (9.53)$$

We now try to match the outer and inner solutions using Van Dyke's matching principle:

$$y(\varepsilon X) = A_0 \ln(\varepsilon X) + 2 + O_x(\varepsilon) = A_0 \ln \varepsilon + A_0 \ln X + 2 + O_x(\varepsilon), \quad (9.54)$$

$$= A_0 \ln x + 2 + O_x(\varepsilon), \quad (9.55)$$

$$Y(x/\varepsilon) = C_0 \ln(1 + x/\varepsilon) + 1 + O_X(\varepsilon) = 1 + C_0 \left[\ln \frac{x}{\varepsilon} + \ln \left(\frac{\varepsilon}{x} + 1 \right) \right] + O_X(\varepsilon) \quad (9.56)$$

$$= 1 + C_0 \ln x - C_0 \ln \varepsilon + O_x(\varepsilon) + O_X(\varepsilon). \quad (9.57)$$

Here, we have rewritten the first result in terms of x again to facilitate the comparison of the two results. In order for the $\ln x$ terms to match, we need $A_0 = C_0$, but we are left trying to match 2 with $1 - C_0 \ln \varepsilon$.

So we wish to effectively have $-C_0 \ln \varepsilon = 1$, i.e. $A_0 = C_0 = -1/\ln(\varepsilon)$, which is strictly speaking achieved by setting $C_0 = 0$ but introducing new terms in the inner and outer that are $\text{ord}(1/\ln \varepsilon)$. Thus, we obtain the results

$$y(x) = 2 - \frac{1}{\ln \varepsilon} \ln x + O_x(\varepsilon), \quad Y(X) = 1 - \frac{1}{\ln \varepsilon} \ln(1 + X) + O_X(\varepsilon). \quad (9.58)$$

Remark(s) 9.3. • Logarithmic terms are special because they unexpectedly jump order when the variables are rescaled, e.g.

$$\ln(x) = \text{ord}(1) \quad \text{but} \quad \ln(\varepsilon X) = \ln \varepsilon + \ln X = \text{ord}(\ln \varepsilon). \quad (9.59)$$

This has two important consequences:

- The matching process may require us to introduce terms at an unexpected order, in the sense that the governing equations have no forcing terms at that order. This phenomenon is sometimes called **switchbacking**. In the example above, the governing equations imply corrections at $\text{ord}(\varepsilon)$, but we had to introduce $\text{ord}(1/\ln \varepsilon)$ corrections (which are thus necessarily homogeneous solutions of the equations).
- When matching expansions involving logarithms, **you must not cut between logarithmic orders**. In this example, we had both outer and inner terms of $\text{ord}(1)$ and $\text{ord}(1/\ln \varepsilon)$. If we try to keep only the $\text{ord}(1)$ terms in the matching and discard the $\text{ord}(1/\ln \varepsilon)$ terms then we obtain

$$y(x) = 2 + O_x(1/\ln \varepsilon), \quad Y(X) = 1 + O_X(1/\ln \varepsilon), \quad (9.60)$$

and after trivially substituting $x = \varepsilon X$ or $X = x/\varepsilon$ and re-expanding we are still left with the constants 2 and 1 that don't match – we need the $\ln x/\ln \varepsilon$ term to jump order and contribute at $\text{ord}(1)$. Thus, when matching for example

$$[\text{ord}(\ln \varepsilon) + \text{ord}(1)] + \varepsilon [\text{ord}((\ln \varepsilon)^2) + \text{ord}(\ln \varepsilon) + \text{ord}(1)] + \varepsilon^2 [\dots] + \dots, \quad (9.61)$$

you should only cut between the algebraic orders and thus either keep or neglect all terms within the same square bracket.

9.3 Relaxation oscillations – Van der Pol oscillator (extra)

We finish this section with a famous nonlinear example, the Van der Pol oscillator. The full analysis is somewhat tedious, so we will only cover some highlights here, and even less in the lecture. (See e.g. Hinch for more details.)

Example(s) 9.4. Consider the following equation for $x(t)$:

$$x'' = -\mu(x^2 - 1)x' - x, \quad \mu > 0. \quad (9.62)$$

The idea is that we have a standard harmonic oscillator $x'' = -x$, such as a particle attached to a spring that provides a restoring force towards equilibrium, but with a nonlinear friction term. For $|x| > 1$, the friction term has the opposite sign to x' so provides a force that slows down the particle as usual, but for $|x| < 1$ the friction term has the same sign as x' so provides a force in the same direction as the particle is already moving. This means in particular that the trivial solution $x = 0$ is unstable, and instead the particle settles into a unique periodic oscillation (a “limit cycle”) for each value of $\mu > 0$, that crosses between $|x| < 1$ where the friction is putting energy into the system and $|x| > 1$ where the friction is drawing energy out of the system.

We consider the limit $\mu = \varepsilon^{-1} \nearrow \infty$, and start by rescaling $t \rightarrow t/\varepsilon$ to obtain an $\text{ord}(1)$ balance between the friction and the restoring force on the right-hand side,

$$\varepsilon^2 x'' = -(x^2 - 1)x' - x. \quad (9.63)$$

A numerical solution is plotted in figure 9.3(a), and we can see that the limit cycle consists of alternating slow “relaxation” phases (outer solutions) where x decreases from ± 2 to ± 1 on an $\text{ord}(1)$ timescale and fast phases (interior layers) where x almost instantaneously jumps from ± 1 to ∓ 2 . Let’s try to describe these oscillations using the method of matched asymptotic expansions.

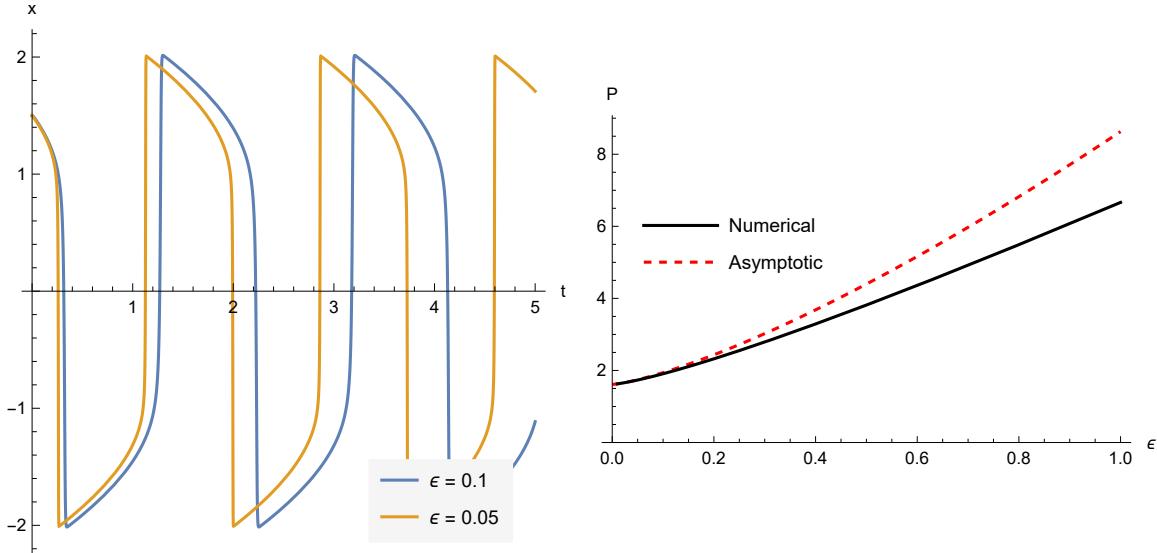


Figure 9.3: Numerical results for the Van der Pol oscillator. (a) Evolution of $x(t)$. (b) Dependence of (rescaled) period on ε .

Let’s start with a positive branch of the outer solution. The leading-order outer equation can be solved using separation of variables,

$$(x_0^2 - 1)x'_0 + x_0 = 0 \quad \Rightarrow \quad (x_0 - x_0^{-1})x'_0 = -1 \quad \Rightarrow \quad \frac{x_0^2}{2} - \ln x_0 = t_0 - t, \quad (9.64)$$

where the constant of integration t_0 is irrelevant. It’s possible to express x_0 explicitly in terms of t using the Lambert W -function, but we won’t need it.

Where is the boundary layer? The coefficient of x'_0 vanishes at $x_0 = 1$ so that’s a good guess. Looking more closely, the left-hand side $\frac{1}{2}x_0^2 - \ln x_0$ is minimal for $x_0 = 1$, taking the value $1/2$, so the outer

solution ceases to exist as $t \nearrow t_0 - 1/2$ and $x_0 \searrow 1$, which is thus where we expect to find a boundary layer. We choose $t_0 = 1/2$ for convenience and further expand near this point to find

$$\frac{1}{2} + (x_0 - 1)^2 + O((x_0 - 1)^3) = \frac{1}{2} - t \Rightarrow x_0 = 1 + (-t)^{1/2} + O(t) \text{ as } t \nearrow 0. \quad (9.65)$$

What is the scaling for the boundary layer? Since the problem is nonlinear the scaling for x matters as well as the scaling for t , and we need to use the local expansion of the outer solution to link the scales δ_x and δ_t for the local changes in x and t . The outer solution comes in with a square-root behaviour, so $\delta_x = \delta_t^{1/2}$ and we write

$$t = \delta_t T, \quad x = 1 + \delta_t^{1/2} X \Rightarrow x^2 - 1 = (x - 1)(x + 1) = \delta_t^{1/2} X (2 + \delta_t^{1/2} X) \quad (9.66)$$

$$\Rightarrow \varepsilon^2 \frac{\delta_t^{1/2}}{\delta_t^2} X'' = -\delta_t^{1/2} X (2 + \delta_t^{1/2} X) \frac{\delta_t^{1/2}}{\delta_t} X' - 1 - \delta_t^{1/2} X. \quad (9.67)$$

We see that the square-root coupling between the scales of x and t from the outer solution has resulted in the two right-hand side terms remaining $\text{ord}(1)$, and so the inner distinguished scaling is when all three terms are in balance, i.e. $\varepsilon^2 \delta_t^{1/2} / \delta_t^2 = 1$. We thus obtain

$$\delta_t = \varepsilon^{4/3}, \quad t = \varepsilon^{4/3} T, \quad x = 1 + \varepsilon^{2/3} X, \quad X'' = -(2 + \varepsilon^{2/3} X) X X' - 1 - \varepsilon^{2/3} X. \quad (9.68)$$

(An alternative way of finding the scaling is to simply ask when the neglected term $\varepsilon^2 x_0'' = \text{ord}(\varepsilon^2 (-t)^{-3/2})$ becomes comparable to the kept terms $x_0 = \text{ord}(1)$ and $(x_0^2 - 1)x_0' = \text{ord}((-t)^{1/2}(-t)^{-1/2}) = \text{ord}(1)$, which yields $(-t) = \text{ord}(\varepsilon^{4/3})$.)

The leading-order boundary layer equations are then

$$X_0'' = -2X_0 X_0' - 1 \Rightarrow X_0' = -X_0^2 - T + T_0, \quad (9.69)$$

where T_0 is a constant of integration. This equation can be transformed into the Airy equation using $X_0(T) = Z'(T)/Z(T)$, but we shall simply proceed by local analysis. As $T \searrow -\infty$ it turns out that the dominant balance is $-X_0^2 - T \approx 0 \Rightarrow X_0 \sim (-T)^{1/2}$ which matches the outer solution. As T increases, the right-hand side must become negative eventually, and then X_0 becomes increasingly negative and decreases increasingly faster until we obtain a dominant balance

$$X_0' \approx -X_0^2 \Rightarrow \frac{X_0'}{X_0^2} \sim -1 \Rightarrow -\frac{1}{X} \sim T_* - T \Rightarrow X \sim -\frac{1}{T_* - T}. \quad (9.70)$$

Hence, the solution blows up negatively at a finite T , indicating that there is a further nested boundary layer, and this $\varepsilon^{4/3}$ layer was just an intermediate transition layer.

We again use the limiting behaviour of the current layer, $X = \text{ord}((T_* - T)^{-1})$, to inform the scaling of the next layer,

$$t = \varepsilon^{4/3} T = \varepsilon^{4/3} (T_* + \delta_T \tau), \quad x = 1 + \varepsilon^{2/3} X = 1 + \varepsilon^{2/3} \delta_T^{-1} \xi \quad (9.71)$$

$$\Rightarrow \delta_T^{-3} \xi'' = -\delta_T^{-3} (2 + \varepsilon^{2/3} \delta_T^{-1} \xi) \xi \xi' - 1 - \varepsilon^{2/3} \delta_T^{-1} \xi \quad (9.72)$$

We see that the ξ'' and ξ' terms remain in balance, provided $\varepsilon^{2/3} \delta_T^{-1} = O(1)$, so seeking a distinguished limit we decrease δ_T until more terms come into balance, which occurs at $\delta_T = \varepsilon^{2/3}$.

On this scale, the rescalings from x to X and from X to ξ cancel out so that $x = 1 + \xi$. Hence instead of ξ we use $\zeta(\tau) = 1 + \xi(\tau) = x(t)$, and obtain

$$t = \varepsilon^{4/3} T = \varepsilon^{4/3} T_* + \varepsilon^2 \tau, \quad x = \zeta, \quad \zeta'' = -(\zeta^2 - 1) \zeta' - \varepsilon^2 \zeta. \quad (9.73)$$

We recognise this as the fast scale that one might have obtained from naively seeking a boundary-layer scaling in the original governing equations by balancing $\varepsilon^2 x''$ with x' , which would have missed out the intermediate transition layer caused by the coefficient of x' vanishing at $x = 1$.

The leading-order equation can be integrated once, with the constant determined by the matching condition $\zeta - 1 \sim \tau^{-1}$ from the intermediate layer, and then integrated again using separation of variables and partial fractions,

$$\zeta'_0 = -\frac{1}{3} \zeta_0^3 + \zeta_0 - \frac{2}{3} = -\frac{1}{3} (\zeta_0 - 1)^2 (\zeta_0 + 2) \quad (9.74)$$

$$\Rightarrow \tau - \tau_0 = \int \frac{-3 d\zeta_0}{(\zeta_0 - 1)^2 (\zeta_0 + 2)} = \int \frac{-1}{(\zeta_0 - 1)^2} + \frac{1/3}{\zeta_0 - 1} - \frac{1/3}{\zeta_0 + 2} d\zeta_0 = \frac{1}{\zeta_0 - 1} + \frac{1}{3} \ln \frac{\zeta_0 - 1}{\zeta_0 + 2}. \quad (9.75)$$

We see how $\zeta_0 \nearrow 1$ yields $\tau \sim (\zeta_0 - 1)^{-1}$ which matches the intermediate layer, while

$$\zeta_0 \searrow -2 \quad \text{yields} \quad \tau \sim \frac{1}{3} \ln(\zeta_0 + 2) \quad \Rightarrow \quad \zeta_0 = -2 + \text{EST}. \quad (9.76)$$

We conclude that the next outer solution must start at $x = -2$, and then go through the process with the opposite sign, returning to the $x > 0$ outer solution at $x = 2$.

We can now approximate the period of the oscillation to leading order, by just taking the time spent in the outer regions,

$$P = 2 \left[-\frac{x_0^2}{2} + \ln x_0 \right]_{x_0=2}^1 = 2 \left(\frac{3}{2} - \ln 2 \right) \approx 1.614. \quad (9.77)$$

To calculate the correction to this result, we need to track the corrections to the leading-order solutions more carefully. If we assume that we start with the smallest possible correction, $\text{ord}(\varepsilon^2)$, in the outer, then it is possible to calculate the $\text{ord}(\varepsilon^{4/3})$ additional time spent in the transition region from slow to fast. It is tempting to conclude that this yields the correction to P , but there is another effect: The local expansion in the transition region generates an $\text{ord}(\varepsilon^{4/3})$ correction in the fast region, which propagates into an $\text{ord}(\varepsilon^{4/3})$ correction in the next slow region. This correction turns out to be of the form $C\varepsilon^{4/3}x'_0(t)$, which represents a shift in time that we can undo,

$$x_0(t) + C\varepsilon^{4/3}x'_0(t) + \text{ord}(\varepsilon^2) = x_0(t + C\varepsilon^{4/3}) + \text{ord}(\varepsilon^2), \quad (9.78)$$

in order to truly return to the (mirror of the) original slow solution with $\text{ord}(\varepsilon^2)$ corrections that we started with. This time shift must be added to the estimate of the period, and hence produces another $\text{ord}(\varepsilon^{4/3})$ correction.

In the end, the result (which is plotted in figure 9.3(b)) turns out to be

$$P = 2 \left(\frac{3}{2} - \ln 2 \right) + \varepsilon^{4/3} 3a + \text{ord}(\varepsilon^2 \ln \varepsilon) \approx 1.614 + 7.014\varepsilon^{4/3}, \quad (9.79)$$

where $-a = 2.338$ is the smallest zero of the Airy function $\text{Ai}(x)$.

10 WKB method

We now encounter the WKB (Wentzel–Kramers–Brillouin) method again, and like before it is used in linear differential equations to balance a higher-order derivative term that appears to be small with a lower-order term that appears to be large. The difference is that instead of looking at local expansions of $y(x)$ as x tends to a limit, we're looking at parameter expansions of $y(x, \varepsilon)$ as ε tends to a limit.

10.1 WKB solution away from turning points

Example(s) 10.1. The most classical WKB example is the equation

$$\varepsilon^2 y'' - q(x)y = 0, \quad \varepsilon \searrow 0, \quad (10.1)$$

for $y(x)$, where $q(x)$ is a given real function of x .

If we wanted to balance the two terms in the equation, then we could rescale $x = \varepsilon X$, and let $y(x) = Y(X)$, to obtain

$$Y'' - q(\varepsilon X)Y = 0, \quad (10.2)$$

which is a differential equation whose coefficient $q(\varepsilon X)$ varies slowly, relative to the natural length scale $X = \text{ord}(1)$ of the equation. Thus, we might expect to be able to naively approximate q as being a constant, resulting in two independent solutions of the form

$$Y = A_{\pm}(\varepsilon X) e^{\pm \sqrt{q(\varepsilon X)} X} = A_{\pm}(x) e^{\pm \varepsilon^{-1} \sqrt{q(x)} x}, \quad (10.3)$$

where the coefficients $A_{\pm}(\varepsilon X)$ vary slowly on the X -scale. However, as we shall see, this would not remain valid throughout a long interval up to $X = \text{ord}(1/\varepsilon)$, corresponding to $x = \text{ord}(1)$, due to errors accumulating in the exponent. Instead, we need the exponent to still be $\text{ord}(X) = \text{ord}(x/\varepsilon)$ but of a more general form.

This is the motivation for the WKB method, which starts with the (exact) change of variables

$$y = e^{S(x)/\delta} \Rightarrow y' = e^{S/\delta} \frac{S'}{\delta}, \quad y'' = e^{S/\delta} \left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta} \right), \quad (10.4)$$

where the scaling for $\delta \ll 1$ in terms of ε needs to be determined (although in this example we can already suspect it's going to be $\delta = \varepsilon$). Substitution into the governing equation yields

$$\varepsilon^2 e^{S/\delta} \left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta} \right) - q(x) e^{S/\delta} = 0 \Rightarrow \varepsilon^2 \left(\frac{S'^2}{\delta^2} + \frac{S''}{\delta} \right) - q(x) = 0. \quad (10.5)$$

We must first determine the correct scaling of δ , by identifying the dominant balance in this equation. Since $1/\delta^2 \gg 1/\delta$, the only non-trivial balance is between the first and last terms, so we choose $\delta = \varepsilon$ and obtain

$$y = e^{S(x)/\varepsilon}, \quad S'^2 + \varepsilon S'' = q(x). \quad (10.6)$$

We have thus exactly converted the singularly perturbed linear homogeneous second-order differential equation for $y(x)$ into a regularly perturbed non-linear inhomogeneous first-order differential equation for $S'(x)$.

It is now clear that we should expand

$$S = S_0 + \varepsilon S_1 + O(\varepsilon^2) \Rightarrow S_0'^2 + \varepsilon 2S_0' S_1' + \varepsilon S_0'' + O(\varepsilon^2) = q \quad (10.7)$$

$$\Rightarrow S_0'^2 = q, \quad 2S_0' S_1' + S_0'' = 0. \quad (10.8)$$

$$\Rightarrow S_0' = \pm q^{1/2}, \quad S_1' = -\frac{S_0''}{2S_0'} = -\left(\frac{1}{2} \ln S_0'\right)' = -\left(\frac{1}{4} \ln q\right)'. \quad (10.9)$$

We see that there are two different solutions, depending on the sign choice of S_0' , which is good since the original equation for y is of second order so should have two independent solutions. (Coincidentally, the value of S_1' is the same for both choices.) We conclude that the original solutions are

$$y_{\pm} = \exp \left[\pm \frac{1}{\varepsilon} \int_{x_0}^x q^{1/2} dx - \frac{1}{4} \ln q + O(\varepsilon) \right] = \frac{1}{q^{1/4}} \exp \left[\pm \frac{1}{\varepsilon} \int_{x_0}^x q^{1/2} dx \right] [1 + O(\varepsilon)]. \quad (10.10)$$

The general leading-order solution would be a linear combination of these, i.e.

$$y \sim A_+ y_+ + A_- y_-. \quad (10.11)$$

As before, adding a constant of integration is equivalent to changing A_{\pm} , so any lower limit x_0 can be used.

For $q(x) < 0$ the result above is valid using any branch for the roots, but it is nicer to rewrite the result as

$$y_{\pm} = \exp \left[\pm \frac{i}{\varepsilon} \int_{x_0}^x (-q)^{1/2} dx - \frac{1}{4} \ln(-q) + O(\varepsilon) \right] = \quad (10.12)$$

$$= \frac{1}{(-q)^{1/4}} \exp \left[\pm \frac{i}{\varepsilon} \int_{x_0}^x (-q)^{1/2} dx \right] [1 + O(\varepsilon)], \quad (10.13)$$

and the general solution can be written in either of the two forms

$$y \sim \frac{B_+ e^{+iS_0/\varepsilon} + B_- e^{-iS_0/\varepsilon}}{(-q)^{1/4}} = \frac{C \cos(S_0/\varepsilon) + D \sin(S_0/\varepsilon)}{(-q)^{1/4}}, \quad S_0 = \int_{x_0}^x (-q)^{1/2} dx. \quad (10.14)$$

Remark(s) 10.1. • The WKB method requires a linear equation (as we need to be able to cancel the $e^{S/\delta}$ to cancel) and, roughly speaking, either $\text{ord}(1)$ terms with coefficients that vary slowly or coefficients varying at $\text{ord}(1)$ but a small parameter multiplying a higher-order derivative.

- The key results are that the phase, i.e. the imaginary part of the exponent, varies like $\varepsilon^{-1} \int (-q)^{1/2} dx$ rather than just $\varepsilon^{-1} (-q)^{1/2} x$, and that the amplitude has a $|q|^{-1/4}$ variation.
- Sometimes the alternative ansatz $y(x) = A(x) e^{\psi(x)/\varepsilon}$ is used (possibly with a factor of i in the exponent), which is equivalent to our $e^{S_0(x)/\varepsilon + S_1(x)} = e^{S_1(x)} e^{S_0(x)/\varepsilon}$.

Example(s) 10.2. One classical application of the WKB method is to wave propagation problems. Here is some background setup: Consider the one-dimensional wave equation in a medium for $u(X, t)$ where the wave speed $c(\varepsilon X)$ varies slowly with position (i.e. on a length scale that is long compared with the wavelength),

$$\partial_t^2 u - c(\varepsilon X)^2 \partial_X^2 u = 0. \quad (10.15)$$

Assuming a time-harmonic solution (or equivalently taking a Fourier transform in time), and rescaling $X = x/\varepsilon$ for convenience, yields the 1D Helmholtz equation

$$u(X, t) = e^{-i\omega t} y(x) \Rightarrow e^{-i\omega t} [-\omega^2 y - \varepsilon^2 c(x)^2 \partial_x^2 y] = 0 \Rightarrow \varepsilon^2 \partial_x^2 y + k(x)^2 y = 0, \quad (10.16)$$

with a wavenumber $k(x) = \omega/c(x) > 0$.

When $k(x)$ is a constant, the solutions are given by

$$y = A_+ e^{+i\varepsilon^{-1} kx} + A_- e^{-i\varepsilon^{-1} kx} \Rightarrow u = A_+ e^{ik(\varepsilon^{-1} x - ct)} + A_- e^{-ik(\varepsilon^{-1} x + ct)}, \quad (10.17)$$

which we identify as rightward and leftward propagating waves with speed $dx/dt = \pm \varepsilon c$ (i.e. $dX/dt = \pm c$), respectively. (For example, the first term is constant for $x - \varepsilon ct = \text{const}$, i.e. $x = \text{const} + \varepsilon ct$.)

For varying $k(x)$, we have the standard WKB equation with $q = -k^2 < 0$ and can thus write down

$$y \sim \frac{A_+}{k(x)^{1/2}} e^{i\varepsilon^{-1} \int k(x) dx} + \frac{A_-}{k(x)^{1/2}} e^{-i\varepsilon^{-1} \int k(x) dx}, \quad (10.18)$$

where the coefficients A_+ and A_- would be determined by boundary conditions. Again, the two terms represent rightward and leftward propagating waves, as the imaginary part of the exponents are increasing and decreasing, respectively. For example, for the first term, if we follow a trajectory $x = x(t)$, then the imaginary part of the exponent of $u = e^{-i\omega t} y$ is constant if we travel at the local wave speed, $x'(t) = \varepsilon^{-1} c$, as

$$\frac{d}{dt} \left[\varepsilon^{-1} \int^{x(t)} k(x) dx - \omega t \right] = \varepsilon^{-1} x' k(x) - \omega = \varepsilon^{-1} k(x) [x' - \varepsilon c(x)] = 0. \quad (10.19)$$

As a specific problem, we consider the example of a wave with frequency $\omega = 1$ travelling in a mostly uniform medium, say $c = 1$ (i.e. $k = 1$), except that there is a finite region in which the speed varies, i.e. c and k not necessarily equal to 1. Assuming the wave is incident from the left, the far-field boundary conditions are

$$y \sim A_i e^{+i\varepsilon^{-1}x} + A_r e^{-i\varepsilon^{-1}x} \text{ as } x \searrow -\infty, \quad y \sim A_t e^{+i\varepsilon^{-1}x} + 0 e^{-i\varepsilon^{-1}x} \text{ as } x \nearrow \infty. \quad (10.20)$$

Here, the complex amplitudes are the incident amplitude A_i which is assumed known, and the reflected amplitude A_r and the transmitted amplitude A_t which are unknown. Note that the incidence condition imposes two boundary conditions, namely the incident amplitude A_i from the left, and the incident amplitude 0 from the right.

Arbitrarily taking the phase integrals to have lower limit $x = 0$, we write down the general solution

$$y \sim \frac{A_+}{k^{1/2}} e^{i\varepsilon^{-1} \int_0^x k dx} + \frac{A_-}{k^{1/2}} e^{-i\varepsilon^{-1} \int_0^x k dx}. \quad (10.21)$$

Comparing with the conditions at $x \nearrow \infty$, recalling that $k \rightarrow 1$, we immediately obtain

$$A_- = 0, \quad A_+ = \lim_{x \nearrow \infty} \frac{A_t e^{i\varepsilon^{-1}x}}{e^{i\varepsilon^{-1} \int_0^x k dx}} = \lim_{x \nearrow \infty} A_t e^{i\varepsilon^{-1}(x - \int_0^x k dx)} = A_t e^{i\varepsilon^{-1} \int_0^\infty 1-k dx}. \quad (10.22)$$

Then, comparing with the conditions at $x \searrow -\infty$, again with $k \rightarrow 1$, we obtain

$$A_r = 0, \quad A_i = \lim_{x \searrow -\infty} \frac{A_+ e^{i\varepsilon^{-1} \int_0^x k dx}}{e^{i\varepsilon^{-1}x}} = \lim_{x \searrow -\infty} A_+ e^{i\varepsilon^{-1}(\int_0^x k dx - x)} = A_+ e^{i\varepsilon^{-1}(\int_{-\infty}^0 1-k dx)}. \quad (10.23)$$

Recalling that this is a leading-order calculation, we thus conclude that actually

$$A_r = o(1), \quad A_t \sim A_i e^{i\varepsilon^{-1}(-\int_0^\infty 1-k dx - \int_{-\infty}^0 1-k dx)} = A_i e^{i\varepsilon^{-1}(\int_{-\infty}^\infty k-1 dx)}. \quad (10.24)$$

This shows that, to leading order, the wave passes through the varying region without reflection, and although its amplitude varies inside that region, in the end it returns to its original amplitude ($|A_t| \sim |A_i|$) but picks up a phase shift $\varepsilon^{-1} \int_{-\infty}^\infty k(x) - 1 dx$.

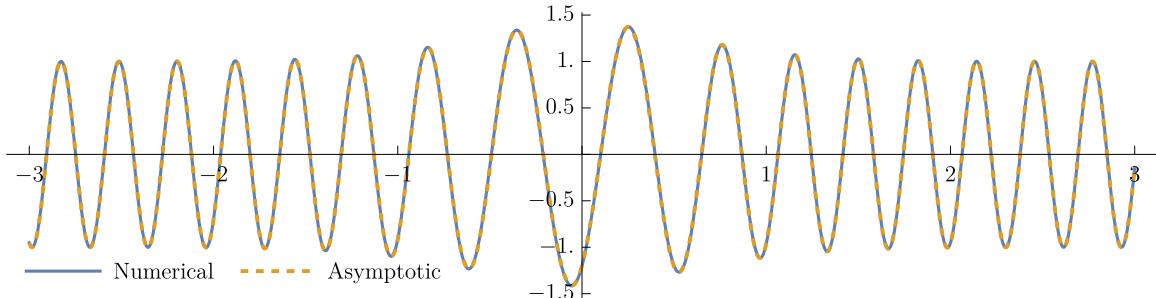


Figure 10.1: Real part of solution $y(x)$ for a wave incident from the left ($A_i = 1$) in a medium with $k = 1 - \frac{1}{2} e^{-x^2}$, for $\varepsilon = 0.05$.

10.2 WKB solution near turning points

Example(s) 10.3. A famous application of the WKB method is to quantum mechanics. The steady-state wavefunction $y(x)$ for a particle with energy E in a given potential $V(x)$ satisfies the time-independent Schrödinger equation, which in rescaled form is

$$-\varepsilon^2 y'' + V(x)y = E y \quad \Rightarrow \quad \varepsilon^2 y'' - q(x)y = 0, \quad q(x) = V(x) - E. \quad (10.25)$$

(The first term in the original equation represents kinetic energy, because velocity is like $-\varepsilon i \partial_x$ so velocity squared is like $-\varepsilon^2 \partial_x^2$, and the other two terms represent potential energy and total energy.) The WKB approximation applies in the limit $\varepsilon \searrow 0$, when $q(x)$ varies on a much larger scale than the natural length scale of the system.

In regions where $E > V(x)$ (which correspond to the regions where a classical, i.e. non-quantum, particle can exist), we have $q < 0$ so the solutions are oscillatory. Specifically, the solution with $+i$ in the exponent corresponds to a rightward propagating wave, and the $-i$ solution corresponds to a leftward one. In regions where $E < V(x)$ (where the classical particle cannot exist), we have $q > 0$ so the solutions are exponential.

Let's first consider a particle reflecting off a potential barrier at $x = 0$, i.e. suppose that $q < 0$ (oscillatory solutions) for $x < 0$ and $q > 0$ (exponential solutions) for $x > 0$. We thus have the leading-order WKB solutions, valid in the two separate regions,

$$x < 0 : \quad y \sim \frac{A_-}{(-q)^{1/4}} e^{-i \int_x^0 (-q)^{1/2} dx/\varepsilon} + \frac{A_+}{(-q)^{1/4}} e^{i \int_x^0 (-q)^{1/2} dx/\varepsilon} \quad (10.26)$$

$$x > 0 : \quad y \sim \frac{B_-}{q^{1/4}} e^{-\int_0^x q^{1/2} dx/\varepsilon} + \frac{B_+}{q^{1/4}} e^{+\int_0^x q^{1/2} dx/\varepsilon}. \quad (10.27)$$

(Note here that we have chosen to place the integral limits at the turning point $x = 0$, which will make the asymptotic matching easier, and flipped the signs of the $x < 0$ integrals so that the integrals themselves are positive, which is just a convention to make it easy to translate the method to other cases.)

The conditions describing a particle (wave) incident from the left and bouncing against the potential barrier is that the incident coefficient A_- (which corresponds to the exponent whose imaginary part increases with x) is known as $x \searrow -\infty$, and that $y \rightarrow 0$ as $x \nearrow +\infty$ so that $B_+ = 0$. The reflection coefficient A_+ is to be determined, as well as the coefficient B_- of the exponential decay inside the barrier.

The point $x = 0$ where $q = 0$ is called a **turning point**, and the WKB approximations blow up there as the denominators $|q|^{1/4}$ vanish. This means that the WKB approximations are not valid at $x = 0$, and in particular we cannot just equate the left-hand and right-hand solutions to determine the correct relationship between A_{\pm} and B_{\pm} . Instead, we need to obtain an inner asymptotic expansion of y near $x = 0$. We assume that q can be Taylor expanded with a non-zero, and hence positive, derivative $m = q'(0)$, so that

$$q(x) = mx + O(x^2) \approx mx \quad \text{as } x \rightarrow 0. \quad (10.28)$$

We will only do a leading-order approximation, so neglect the higher-order term here. Substituting this into the original equation yields

$$\varepsilon^2 y'' - mxy \approx 0. \quad (10.29)$$

We seek a rescaling of x to balance the two terms in the equation, and it is convenient to include the factor m in the rescaling as well, so we define

$$x = \varepsilon^{2/3} m^{-1/3} X, \quad y(x) = Y(X) \quad \Rightarrow \quad Y'' - XY = 0 \quad (10.30)$$

We recognise this as the Airy equation, with the general solution given by

$$Y \sim C \text{Ai}(X) + D \text{Bi}(X). \quad (10.31)$$

Since the outer solution on the right decays exponentially with x , we expect it to successfully match to $\text{Ai}(X)$ which also decays exponentially as $X \nearrow \infty$, but not $\text{Bi}(X)$ which grows exponentially. Hence, we

set $D = 0$ and expand the Ai to find

$$X \nearrow \infty : Y \sim C \frac{e^{-\frac{2}{3}X^{3/2}}}{2\sqrt{\pi} X^{1/4}}, \quad (10.32)$$

$$X \searrow -\infty : Y \sim C \frac{\sin(\frac{2}{3}(-X)^{3/2} + \frac{\pi}{4})}{\sqrt{\pi} (-X)^{1/4}} = C \frac{e^{i\frac{2}{3}(-X)^{3/2}-i\pi/4} + e^{-i\frac{2}{3}(-X)^{3/2}+i\pi/4}}{2\sqrt{\pi} (-X)^{1/4}}. \quad (10.33)$$

(Although this result is tricky to memorize, a trick for determining the relative values of the coefficients is to remember that one side is proportional to the exponentially decaying WKB solution, while the other side consists of two terms that are just the analytical continuation of the exponentially decaying term to the negative real axis counterclockwise and clockwise, respectively, in the complex plane.)

We now match with the inner limit of the outer WKB solutions. Using $q \sim mx$ as $x \rightarrow 0$, we expand, for $x > 0$,

$$\int_0^x q^{1/2} dx \approx \int_0^x (mx)^{1/2} dx = \frac{2}{3} m^{1/2} x^{3/2} = \varepsilon \frac{2}{3} X^{3/2}, \quad (10.34)$$

$$q^{1/4} \approx m^{1/4} x^{1/4} = \varepsilon^{1/6} m^{1/6} X^{1/4}, \quad (10.35)$$

and for $x < 0$,

$$\int_x^0 (-q)^{1/2} dx \approx \int_x^0 (-mx)^{1/2} dx = \left[-\frac{2}{3} m^{1/2} (-x)^{3/2} \right]_x^0 = \frac{2}{3} m^{1/2} (-x)^{3/2} = \varepsilon \frac{2}{3} (-X)^{3/2}, \quad (10.36)$$

$$(-q)^{1/4} \approx (-mx)^{1/4} = \varepsilon^{1/6} m^{1/6} (-X)^{1/4}. \quad (10.37)$$

Thus, we have

$$x \searrow 0 : y \sim \frac{B_-}{\varepsilon^{1/6} m^{1/6} X^{1/4}} e^{-\frac{2}{3} X^{3/2}}, \quad x \nearrow 0 : y \sim \frac{A_- e^{-i\frac{2}{3}(-X)^{3/2}} + A_+ e^{+i\frac{2}{3}(-X)^{3/2}}}{\varepsilon^{1/6} m^{1/6} (-X)^{1/4}}. \quad (10.38)$$

Comparison of the expansions of the inner and outer solutions then yields

$$B_- = \frac{\varepsilon^{1/6} m^{1/6} C}{2\sqrt{\pi}}, \quad A_- = \frac{\varepsilon^{1/6} m^{1/6} C e^{-i\pi/4}}{2\sqrt{\pi}}, \quad A_+ = \frac{\varepsilon^{1/6} m^{1/6} C e^{i\pi/4}}{2\sqrt{\pi}} \quad (10.39)$$

$$\Rightarrow A_- = B_- e^{i\pi/4}, \quad A_+ = B_- e^{-i\pi/4}. \quad (10.40)$$

This final result simply relates the outer coefficients on either side of the inner region, and is called a **connection formula**.

We conclude that $A_+ = A_- e^{-i\pi/2}$, i.e. the reflected coefficient is equal to the incident coefficient in amplitude but has a phase change.

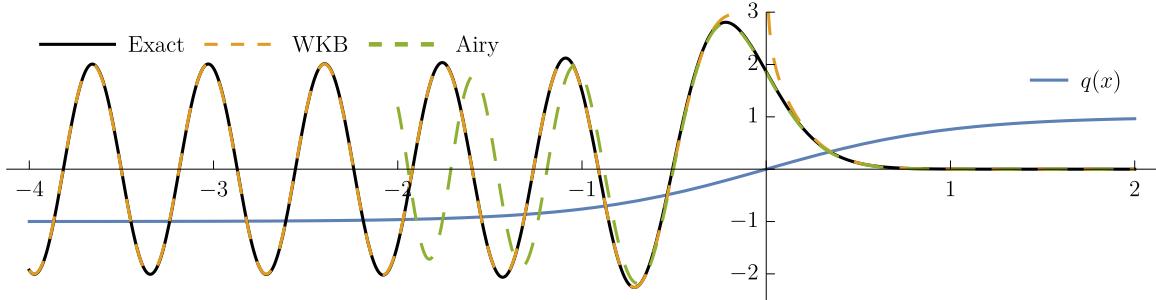


Figure 10.2: Reflection against a potential barrier with $q(x) = \tanh(x)$, $\varepsilon = 0.1$.

Remark(s) 10.2. We can restate the connection formula in terms of the outer WKB solutions, for a

turning point at some $x = a$ with $q'(a) > 0$, as

$$y \sim \frac{A}{q^{1/4}} e^{-\int_a^x q^{1/2} dx/\varepsilon} \quad \text{for } x > a \quad (10.41)$$

$$\Rightarrow y \sim \frac{A}{(-q)^{1/4}} \left[e^{i \int_x^a (-q)^{1/2} dx / \varepsilon - i\pi/4} + e^{-i \int_x^a (-q)^{1/2} dx / \varepsilon + i\pi/4} \right] = \quad (10.42)$$

$$= \frac{A}{(-q)^{1/4}} 2 \sin \left[\frac{1}{\varepsilon} \int_x^a (-q)^{1/2} dx + \frac{\pi}{4} \right] \quad \text{for } x < a. \quad (10.43)$$

For the left-right mirrored case, where $q'(a) < 0$ so that the exponential side is $x < a$ and the oscillatory side is $x > a$, we simply need to reverse the inequality signs and flip the integral limits $\int_a^x \leftrightarrow \int_x^a$ so that again each integral is positive.

Example(s) 10.4. We again consider the Schrödinger equation

$$-\varepsilon^2 y'' + V(x)y = E y \Rightarrow \varepsilon^2 y'' - q(x)y = 0, \quad q(x) = V(x) - E, \quad (10.44)$$

but suppose that $V(x)$ has a typical potential well shape, similar to a parabola. Then we expect to have some interval $a < x < b$ in which $E > V(x)$ and the solution is oscillatory, while the outer regions $x < a$ and $x > b$ have $E < V(x)$ and exponential solutions. We seek a bound-state solution, i.e. one in which $y \rightarrow 0$ as $x \rightarrow \pm\infty$. This is an eigenvalue problem for E , meaning that a nontrivial ($y \not\equiv 0$) solution only exists for specific values of E , and we would like to determine these values of E .

We already have all the building blocks needed to solve this problem. Considering the decay in $x > b$ matching via a turning point to the region $a < x < b$, we find that, for some constant A_R ,

$$y \sim \frac{A_R}{q^{1/4}} e^{-\int_b^x q^{1/2} dx/\varepsilon} \quad \text{for } x > b \quad (10.45)$$

$$\Rightarrow y \sim \frac{A_R}{(-q)^{1/4}} 2 \sin \left[\frac{1}{\varepsilon} \int_x^b (-q)^{1/2} dx + \frac{\pi}{4} \right] \quad \text{for } a < x < b. \quad (10.46)$$

Similarly, the decay in $x < a$ matching via a turning point to $a < x < b$ yields

$$y \sim \frac{A_L}{q^{1/4}} e^{-\int_a^x q^{1/2} dx/\varepsilon} \quad \text{for } x < a \quad (10.47)$$

$$\Rightarrow y \sim \frac{A_L}{(-q)^{1/4}} 2 \sin \left[\frac{1}{\varepsilon} \int_a^x (-q)^{1/2} dx + \frac{\pi}{4} \right] \quad \text{for } a < x < b. \quad (10.48)$$

We now have two different expressions for the leading-order behaviour in the middle range $a < x < b$, so we require that they be equal,

$$A_R \sin \left[\frac{1}{\varepsilon} \int_x^b (-q)^{1/2} dx + \frac{\pi}{4} \right] = A_L \sin \left[\frac{1}{\varepsilon} \int_a^x (-q)^{1/2} dx + \frac{\pi}{4} \right] \quad \text{for } a < x < b. \quad (10.49)$$

The arguments of the sines can't be equal (up to a constant phase shift), because one increases and the other decreases with x . Instead, one must be π minus the other, up to a phase shift of $n\pi$, and the amplitudes must be equal too, up to a sign difference of $(-1)^n$. Hence,

$$\left[\frac{1}{\varepsilon} \int_x^b (-q)^{1/2} dx + \frac{\pi}{4} \right] = \pi - \left[\frac{1}{\varepsilon} \int_a^x (-q)^{1/2} dx + \frac{\pi}{4} \right] + n\pi, \quad A_L = A_R(-1)^n \quad (10.50)$$

$$\Rightarrow \frac{1}{\varepsilon} \int_a^b (-q)^{1/2} dx = \left(n + \frac{1}{2} \right) \pi. \quad (10.51)$$

Since the left-hand side is positive, the possible values of the integer n are $n = 0, 1, 2, \dots$. This quantisation condition is a key result in quantum mechanics, and (in principle) allows us to calculate the allowed values of E by substituting in the expression for q and integrating. Note that the $1/\varepsilon$ on the left-hand side implies that the approximation is only valid (i.e. $\varepsilon \ll 1$) for $n \gg 1$.

As a concrete example, the harmonic oscillator with $V(x) = x^2$ yields

$$q = V(x) - E = x^2 - E \Rightarrow a = -\sqrt{E}, b = \sqrt{E}, \quad (10.52)$$

$$\varepsilon(n + \frac{1}{2})\pi = \int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E - x^2} dx \stackrel{x=\sqrt{E}u}{=} E \int_{-1}^1 \sqrt{1 - u^2} du = E\pi/2, \quad (10.53)$$

using the area of a semicircle or a substitution $u = \sin v$ and the double angle formula. We conclude that $E = \varepsilon(2n + 1)$ for $\varepsilon \ll 1$ and $n \gg 1$. Famously, for the harmonic oscillator this approximation happens to be exact for all ε and n , and the exact solution is given by $y(x) \propto H_n(x/\sqrt{\varepsilon})e^{-x^2/2\varepsilon}$ where H_n is the Hermite polynomial of degree n .

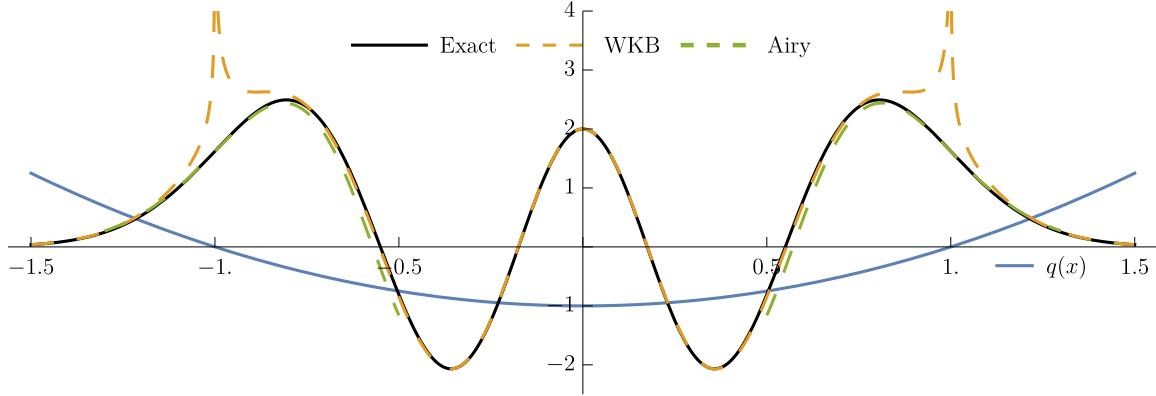


Figure 10.3: Solution for a potential well $V = x^2$ with $n = 4$, $\varepsilon = 1/9$, $E = \varepsilon(2n + 1) = 1$, $q = E - V = 1 - x^2$.

11 Perturbed harmonic oscillators

In this section we study ordinary differential equations that are small perturbations to the harmonic oscillator, which after a suitable rescaling takes the form

$$y''(t) + y(t) = 0. \quad (11.1)$$

As we shall see, the small perturbations can accumulate over long time scales to affect the leading-order oscillating solution $y = A \cos t + B \sin t$.

11.1 Poincaré–Lindstedt method / method of strained coordinates

Example(s) 11.1. The equation for the angle $\theta(\tau)$ of a pendulum can be written as

$$m\ell^2\theta''(\tau) + mgl \sin \theta(\tau) = 0. \quad (11.2)$$

We consider small-amplitude oscillations of the pendulum, $\theta \ll 1$, for which we can Taylor expand the \sin . For convenience, we write $\theta(\tau) = (6\varepsilon)^{1/2}y(t)$, where $t = (\ell/g)^{1/2}\tau$, $\varepsilon \ll 1$ and $y = \text{ord}(1)$. The governing equation then simplifies to

$$0 = y'' + (6\varepsilon)^{-1/2} \sin [(6\varepsilon)^{1/2}y] = y'' + y - \varepsilon y^3 + \varepsilon^2 \frac{3y^5}{10} + O(\varepsilon^3). \quad (11.3)$$

We consider just the terms down to $O(\varepsilon)$,

$$y'' + y - \varepsilon y^3 = 0. \quad (11.4)$$

This equation is a form of Duffing's oscillator with no damping, no forcing, and a restoring force that has a weak nonlinearity. How does the small nonlinear term change the oscillation?

We first attempt a naive expansion

$$y(t) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2), \quad (11.5)$$

and find the leading-order solution

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0 e^{it} + \text{cc}. \quad (11.6)$$

Here, in order to simplify later calculations, we have chosen to use a complex representation with a complex constant of integration A_0 , and “+ cc” denotes adding the complex conjugate, resulting in the real expression

$$A_0 e^{it} + A_0^* e^{-it} = 2 \operatorname{Re}[A_0 e^{it}] = 2 [\operatorname{Re} A_0 \cos t - (\operatorname{Im} A_0) \sin t]. \quad (11.7)$$

Note that since $\operatorname{Re} A_0$ and $\operatorname{Im} A_0$ represent two independent real constants, this is indeed a most general form of a solution y_0 if we assume that it must be real, which is the only physically relevant case.

Carefully taking the cube of the complex expression,

$$(A_0 e^{it} + \text{cc})^3 = (A_0^3 e^{it})^3 + 3(A_0 e^{it})^2 (A_0^* e^{-it}) + 3(A_0 e^{it})(A_0^* e^{-it})^3 + (A_0^* e^{-it})^3, \quad (11.8)$$

then yields the forcing for the $O(\varepsilon)$ corrections,

$$y_1'' + y_1 = -y_0^3 = A_0^3 e^{3it} + 3A_0 |A_0|^2 e^{it} + \text{cc} \quad \Rightarrow \quad y_1 = -\frac{A_0^3}{8} e^{3it} - \underbrace{\frac{3A_0 |A_0|^2 i}{2} t e^{it}}_{\text{secular}} + A_1 e^{it} + \text{cc}. \quad (11.9)$$

We thus obtain the result

$$y(t) = A_0 e^{it} + \varepsilon \left[-\frac{A_0^3}{8} e^{3it} - \underbrace{\frac{3A_0 |A_0|^2 i}{2} t e^{it}}_{\text{secular}} + A_1 e^{it} \right] + O(\varepsilon^2) + \text{cc}. \quad (11.10)$$

Here, the $t e^{it}$ term yields oscillations that grow in amplitude, and we refer to it as a “secular” term. We also refer to the original forcing term, that was proportional to e^{it} , as “secular”. Thus, although the

expansion is valid for $t = O(1)$, it becomes invalid for large $t = \text{ord}(1/\varepsilon)$, due to the secular term no longer being a small correction to the leading-order term.

However, we expect the exact solutions to the original equation to oscillate periodically and remain bounded, because we can multiply the equation by y' and integrate to obtain an energy conservation equation,

$$0 = y' (y'' + y - \varepsilon y^3) = \left(\frac{y'^2}{2} + \frac{y^2}{2} - \varepsilon \frac{y^4}{4} \right)' \Rightarrow \frac{y'^2}{2} + \frac{y^2}{2} - \varepsilon \frac{y^4}{4} = \text{const.} \quad (11.11)$$

The problem with our naive expansion comes from the fact that its oscillation frequency is wrong. To illustrate how an error in frequency generates a secular term, we consider naively expanding an oscillation with angular frequency $(1 + \varepsilon)$:

$$\cos((1 + \varepsilon)t) = \cos(t + \varepsilon t) = \cos(t) - \varepsilon t \sin(t) + O(\varepsilon^2). \quad (11.12)$$

Again, we see a secular term appearing, which invalidates the expansion when $t = \text{ord}(1/\varepsilon)$. In essence, although in $\cos(t + \varepsilon t)$ the correction εt is always small relative to t , since \cos is an exponential-type function the expansion is only valid if εt is small relative to 1.

With the insight that the oscillation frequency needs adjustment, we arrive at the **Poincaré–Lindstedt method**. The idea is to use the exact, but unknown, angular frequency ω of the oscillator (with $\omega = 1 + o(1)$) to introduce a “strained” time coordinate

$$s = \omega t, \quad \frac{d}{dt} = \omega \frac{d}{ds}, \quad \omega = 1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + O(\varepsilon^3). \quad (11.13)$$

We thus obtain the equation (where primes now denote differentiation with respect to s) and condition

$$\omega^2 y'' + y - \varepsilon y^3 = 0, \quad y \text{ is } 2\pi\text{-periodic in } s. \quad (11.14)$$

The expansion $y(s) = y_0(s) + \varepsilon y_1(s) + \dots$ then yields

$$y_0'' + y_0 = 0 \Rightarrow y_0(s) = A_0 e^{is} + \text{cc}, \quad (11.15)$$

$$y_1'' + y_1 = y_0^3 - 2\omega_1 y_0'' = A_0^3 e^{3is} + 3A_0 |A_0|^2 e^{is} + 2\omega_1 A_0 e^{is} + \text{cc}. \quad (11.16)$$

The condition that y_1 is 2π -periodic in s requires that the secular terms (i.e. the ones proportional to e^{is}) cancel out, which determines the frequency correction ω_1 and yields the general solution for y_1 :

$$3A_0 |A_0|^2 + 2\omega_1 A_0 = 0 \Rightarrow \omega_1 = -\frac{3}{2} |A_0|^2, \quad y_1 = -\frac{A_0^3}{8} e^{3is} + A_1 e^{is} + \text{cc}. \quad (11.17)$$

We conclude that the answer is

$$y = [A_0 + \varepsilon A_1] e^{is} - \varepsilon \frac{3|A_0|^2}{8} e^{3is} + O(\varepsilon^2) + \text{cc}, \quad s = \left(1 - \frac{3}{2}\varepsilon |A_0|^2 + O(\varepsilon^2)\right) t, \quad (11.18)$$

where the constants of integration A_0 , A_1 , \dots are determined by, for example, initial conditions. Dropping the $O(\varepsilon^2)$ terms yields a result that is valid up to $t = O(1/\varepsilon)$, but again fails at $t = \text{ord}(1/\varepsilon^2)$ due to the $\text{ord}(\varepsilon^2)$ correction to the frequency multiplying with the $\text{ord}(1/\varepsilon^2)$ time resulting in a $\text{ord}(1)$ change in phase of the oscillation. Note that expanding the result naively for $t = \text{ord}(1)$ recovers the secular term from before, since

$$A_0 e^{i(t - \frac{3}{2}\varepsilon |A_0|^2 t + O(\varepsilon^2))} = A_0 e^{it} - \varepsilon \frac{3i|A_0|^2 A_0 t}{2} e^{it} + O(\varepsilon^2) \text{ for } t = \text{ord}(1). \quad (11.19)$$

As seen in figure 11.1 which shows a range of times $t = O(1/\varepsilon)$, the leading-order result $y_0 = \cos(t)$ has a slightly wrong period. The naive correction looks like it has a corrected period to begin with, but grows without bound due to the secular term. The Poincaré–Lindstedt result agrees excellently with the exact result, both in period and amplitude, over the range shown.

Remark(s) 11.1. • It is also possible to do the calculation in real form, using trigonometric functions.

To keep the number of terms small we rewrite the classical general solution using an amplitude and a phase instead, $y_0 = a_0 \cos s + b_0 \sin s = a_0 \cos(s + \phi_0)$, and we further write $\hat{s} = s + \phi_0$ for simplicity.

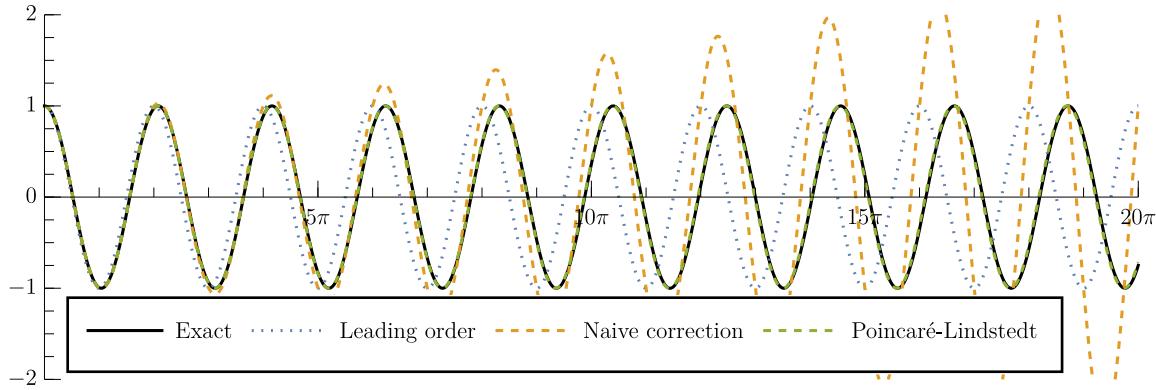


Figure 11.1: Solution of the duffing oscillator $y'' + y - \varepsilon y^3 = 0$, $y(0) = 1$, $y'(0) = 0$, with $\varepsilon = 0.1$.

We must then use a trigonometric multiple-angle formula to convert the nonlinear term into a sum of linear ones,

$$y_1'' + y_1 = y_0^3 - 2\omega_1 y_0'' = a_0^3(\cos \hat{s})^3 + 2\omega_1 a_0 \cos \hat{s} = a_0^3 \frac{\cos 3\hat{s} + 3 \cos \hat{s}}{4} + 2\omega_1 a_0 \cos \hat{s} \quad (11.20)$$

$$\Rightarrow 2\omega_1 a_0 + \frac{3a_0^3}{4} = 0 \Rightarrow \omega_1 = -\frac{3a_0^2}{8}. \quad (11.21)$$

- When comparing complex and real forms, don't forget the factor two in $A_0 e^{it} + cc = 2|A_0| \cos(t + \arg A_0)$, i.e. $|A_0| = a_0/2$.
- We see that nonlinearities in general generate “harmonics”, i.e. oscillations at integer multiples of the original frequency.
- In general, we end up with $y_n'' + y_n = f_n(t)$ (but with $t = s$) where the forcing $f_n(t)$ is 2π -periodic, and it may be difficult to identify the secular terms in $f_n(t)$ if it has a complicated expression. If we are able to determine the Fourier series of f_n in any of the equivalent forms

$$f_n(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mt + b_m \sin mt) = \frac{a_0}{2} + \sum_{m=1}^{\infty} c_m \cos(mt + \phi_m) = \sum_{m=0}^{\infty} A_m e^{imt} + cc, \quad (11.22)$$

then the coefficients corresponding to secular terms are a_1, b_1, c_1 and A_1 , respectively. Equivalently, using the integral expression for the Fourier coefficients, the secularity condition on f_n is that

$$\int_0^{2\pi} \cos t f_n(t) dt = \int_0^{2\pi} \sin t f_n(t) dt = 0, \quad \text{or} \quad \int_0^{2\pi} e^{-it} f_n(t) dt = \int_0^{2\pi} e^{+it} f_n(t) dt = 0. \quad (11.23)$$

These relationships can also be obtained from the governing equation by integrating by parts, e.g.

$$\int_0^{2\pi} e^{-it} f_n(t) dt = \int_0^{2\pi} e^{-it} (y_n'' + y_n) dt = [e^{-it} (y_n' + iy_n)]_0^{2\pi} = 0, \quad (11.24)$$

since y_n is required to be periodic. (In fact, for more general linear equations $\mathcal{L}y_n = f_n$, you may be able to find an “adjoint” function y^\dagger such that the equation has appropriate solutions only if $\int y^\dagger f_n dt = 0$, where y^\dagger satisfies an “adjoint equation” $\mathcal{L}^\dagger y^\dagger = 0$, which is different unless \mathcal{L} is “self-adjoint”, i.e. $\mathcal{L} = \mathcal{L}^\dagger$.)

- When defining a strained time, instead of the angular frequency ω it is also possible to use the “angular period” $\chi = \omega^{-1}$, to equivalently obtain

$$t = \chi s, \quad \frac{d}{dt} = \chi^{-1} \frac{d}{ds}, \quad \chi = 1 + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + O(\varepsilon^3), \quad (11.25)$$

and proceed as usual.

- The “**method of strained coordinates**” sometimes refers to using the more general transformation

$$t = s + \varepsilon t_1(s) + \varepsilon^2 t_2(s) + O(\varepsilon^3), \quad (11.26)$$

where $t_1(s), t_2(s)$, etc. are general functions of s which are determined (although not uniquely) during the calculation, and the method can be applied to non-periodic solutions as well.

11.2 Method of multiple scales

In the previous subsection we saw how to use the Poincaré–Lindstedt method of strained coordinates to calculate the solutions of weakly perturbed oscillators that remain periodic. We now turn to more general weak perturbations of oscillators, which result in non-periodic motion. In fact, the motion turns out to be quasi-periodic, in the sense that there is a fast oscillation on an $\text{ord}(1)$ time scale, whose amplitude and phase varies slowly with time.

Example(s) 11.2. Let's revisit the Van der Pol oscillator, but now consider the limit of small friction,

$$y''(t) + \varepsilon(y(t)^2 - 1)y'(t) + y(t) = 0. \quad (11.27)$$

As before, a naive expansion $y(t) = y_0(t) + \varepsilon y_1(t) + \dots$ results in an oscillatory leading-order term

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0 e^{it} + \text{cc}. \quad (11.28)$$

Then the correction

$$y_1'' + y_1 = (1 - y_0^2)y_0' = [1 - 2|A_0|^2 - A_0^2 e^{2it} - (A_0^*)^2 e^{-2it}] [iA_0 e^{it} - iA_0^* e^{-it}] = \quad (11.29)$$

$$= (1 - 2|A|^2)iA_0 e^{it} + A_0^2 e^{2it} iA_0^* e^{-it} - A_0^2 e^{2it} iA_0 e^{it} + \text{cc} \quad (11.30)$$

$$= \underbrace{i(1 - |A_0|^2)A_0 e^{it}}_{\text{secular}} - iA_0^3 e^{3it} + \text{cc}. \quad (11.31)$$

has a secular term that yields an oscillation with amplitude growing like t , resulting in the expansion breaking down at $t = \text{ord}(1/\varepsilon)$.

The **method of multiple scales** proceeds as follows: We introduce a “**slow time**” $T = \varepsilon t$ and treat it as a separate independent variable to the “**fast time**” t , with $y = y(t, T)$. The derivatives of y become

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}, \quad (11.32)$$

which converts the ordinary differential equation into a partial differential equation, which we solve while ensuring that no secular terms arise, and at the end we set $T = \varepsilon t$ to obtain the final result.

The introduction of slow time $T = \varepsilon t$ converts the ODE into the PDE

$$\partial_t^2 y + 2\varepsilon \partial_t \partial_T y + \varepsilon^2 \partial_T^2 y + \varepsilon(y^2 - 1)(\partial_t y + \varepsilon \partial_T y) + y = 0. \quad (11.33)$$

We expand $y(t, T) = y_0(t, T) + \varepsilon y_1(t, T) + \dots$ and recover the same leading-order behaviour as before

$$\partial_t^2 y_0 + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0(T) e^{it} + \text{cc}, \quad (11.34)$$

with the difference that the amplitude $A_0(T)$ is now a function of the slow time T . The point is that the oscillator oscillates on the fast time scale $t = \text{ord}(1)$, with its (complex) amplitude evolving on the slower time scale $t = \text{ord}(1/\varepsilon)$ represented by T .

The slow derivative of A_0 comes in at next order, resulting in the same equation as before but with an extra derivative term

$$\partial_t^2 y_1 + y_1 = -2\partial_t \partial_T y_0 + (1 - y_0^2)\partial_t y_0 = i[-2\partial_T A_0 + (1 - |A_0|^2)A_0] e^{it} - iA_0^3 e^{3it} + \text{cc}. \quad (11.35)$$

The condition that there is no secular term, so that the expansion remains valid for $t = \text{ord}(1/\varepsilon)$, yields a condition on the time derivative, i.e. an evolution equation for $A_0(T)$,

$$\partial_T A_0 = \frac{1 - |A_0|^2}{2} A_0. \quad (11.36)$$

To solve this, we write $A_0(T) = r(T)e^{i\phi(T)}$ where the magnitude $r = |A_0| > 0$ and argument $\phi = \arg A_0$ are real, resulting in

$$A'_0 = r'e^{i\phi} + r i\phi' e^{i\phi} = (r' + ir\phi')e^{i\phi} \quad \Rightarrow \quad r' + r i\phi' = \frac{1 - r^2}{2} r \quad \Rightarrow \quad r' = \frac{1 - r^2}{2} r, \quad \phi' = 0. \quad (11.37)$$

Hence the argument ϕ is constant while the magnitude equation can be integrated using separation of variables and partial fractions,

$$T = \int \frac{2 dr}{(1-r^2)r} = \int \frac{2}{r} + \frac{1}{1-r} - \frac{1}{1+r} dr = \ln \frac{r^2}{1-r^2} + \text{const} \quad (11.38)$$

$$\Rightarrow C e^T = \frac{r^2}{1-r^2} = \frac{1}{r^{-2}-1} \quad \Rightarrow \quad r = \frac{1}{\sqrt{1+C^{-1}e^{-T}}} = \frac{1}{\sqrt{1+De^{-T}}}. \quad (11.39)$$

We conclude that the leading-order behaviour, valid up to $t = O(1/\varepsilon)$, is

$$y(t) = \frac{e^{i\phi}}{\sqrt{1+De^{-\varepsilon t}}} e^{it} + \text{cc} + O(\varepsilon), \quad (11.40)$$

where the constants ϕ and D are determined by initial conditions. We could further calculate the $\text{ord}(\varepsilon)$ correction,

$$\partial_t^2 y_1 + y_1 = -i A_0^3 e^{3it} + \text{cc} \quad \Rightarrow \quad y_1 = i \frac{A_0^3}{8} e^{3it} + A_1(T) e^{it} + \text{cc}, \quad (11.41)$$

but the evolution of $A_1(T)$ is only determined by considering the secularity condition at $\text{ord}(\varepsilon^2)$. This turns out to yield a T -secular term in A_1 resulting in εA_1 no longer being small relative to A_0 at $T = \text{ord}(1/\varepsilon)$, i.e. $t = \text{ord}(1/\varepsilon^2)$. We would then need to introduce a slow-slow time $T_2 = \varepsilon^2 t$, and seek an expansion for $y(t, T, T_2)$ with

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} + \varepsilon^2 \frac{\partial}{\partial T_2}. \quad (11.42)$$

Figure 11.2 shows that $A_0(T)$ correctly captures the slow variation of the amplitude of the oscillator. (There is no change in phase/frequency at this order, since ϕ is constant.)

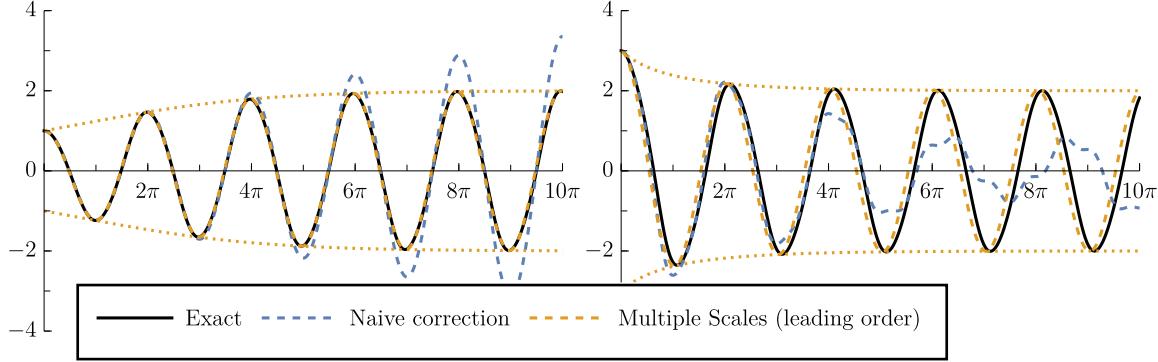


Figure 11.2: Solution of the Van der Pol oscillator $y'' + \varepsilon(y^2 - 1)y' + y = 0$, (a) $y(0) = 1$ or (b) $y(0) = 3$, and $y'(0) = 0$, with $\varepsilon = 0.2$.

Remark(s) 11.2. • The evolution equation for $A_0(T)$ is typically of the form $A'_0(T) = f(|A_0|^2)A_0$, where f is a polynomial, because if the terms in the equation are of the form

$$\alpha [A_0 e^{it}]^p [A_0^* e^{-it}]^q + \text{cc} = \alpha A_0^p (A_0^*)^q e^{i(p-q)t} + \text{cc} \quad (11.43)$$

then the secular terms with e^{+it} have $(p-q) = 1$ and hence are of the form $\alpha |A_0|^{2q} A_0 e^{it}$.

- The evolution equation for $A_0(T)$ is sometimes simplified by splitting A_0 into magnitude and argument. In particular, for the above form,

$$A'_0(T) = f(|A_0|^2)A_0, \quad A_0(T) = r(T)e^{i\phi(T)} \quad \Rightarrow \quad r' + ir\phi' = f(r^2)r. \quad (11.44)$$

If f is purely real then the argument is constant and the magnitude evolves. If f is purely imaginary then the magnitude is constant while the argument evolves (grows linearly with time).

- For more general f , the evolution equation for $A_0(T)$ might not have an explicit solution. In that case, we'd have to solve the equation for $A_0(T)$ numerically, but it's still an improvement to only have to resolve the evolution of the amplitude and phase on the slow time scale, corresponding to $t = \text{ord}(1/\varepsilon)$, compared with solving the original equation where we would have to resolve the oscillations on the $t = \text{ord}(1)$ timescale.

- Instead of the complex amplitude $A_0(T)$ it is also possible to use the real form

$$y_0 = a_0(T) \cos(t + \phi(T)) \quad \Rightarrow \quad \partial_T y_0 = a'_0 \cos \hat{t} - a_0 \phi' \sin \hat{t}, \quad \text{where } \hat{t} = t + \phi(T). \quad (11.45)$$

This would yield, with $\hat{t} = t + \phi(T)$,

$$\partial_t^2 y_1 + y_1 = -2\partial_t \partial_T y_0 + (1 - y_0^2) \partial_t y_0 = 2(a'_0 \sin \hat{t} + a_0 \phi' \cos \hat{t}) - a_0 \sin \hat{t} + a_0^3 \underbrace{(\cos \hat{t})^2 \sin \hat{t}}_{=(\sin 3\hat{t} + \sin \hat{t})/4} \quad (11.46)$$

$$= [2a_0 \phi'] \cos \hat{t} + [2a'_0 - a_0 + \frac{1}{4}a_0^3] \sin \hat{t} + [\frac{1}{4}a_0^3] \sin 3\hat{t} \quad (11.47)$$

$$\Rightarrow \phi' = 0, \quad a'_0 = \frac{1 - \frac{1}{4}a_0^2}{2} a_0. \quad (11.48)$$

These agree with our previous equations for $\arg A_0 = \phi$ and $|A_0| = a_0/2$.

11.3 Resonance

Resonance occurs when a time-dependent term in the equation excites oscillations at the natural frequency of the harmonic oscillator. We study this phenomenon using the method of multiple scales, focusing on a harmonic oscillator with weak damping to consume any energy generated by the resonance, so as to avoid the amplitude diverging to infinity,

$$y''(t) + 2\varepsilon y'(t) + y(t) = 0, \quad (11.49)$$

which we will perturb in various ways.

Example(s) 11.3. We illustrate the basic principle of resonance by considering a forcing term with some constant frequency ω and amplitude $2F > 0$:

$$y''(t) + 2\varepsilon y'(t) + y(t) = 2F \cos(\omega t) = F e^{i\omega t} + \text{cc}. \quad (11.50)$$

This equation is simple enough that we can solve it explicitly, by seeking a solution oscillating with the same frequency as the forcing,

$$y(t) = A e^{i\omega t} + \text{cc} \Rightarrow (-\omega^2 + 2i\omega\varepsilon + 1)A = F \Rightarrow A = \frac{F}{1 - \omega^2 + 2i\omega\varepsilon}. \quad (11.51)$$

(We ignore the homogeneous solution $C e^{-\varepsilon t + i\sqrt{1-\varepsilon^2}t} + \text{cc}$, which is simply damped oscillations that decay as $t \nearrow \infty$.) We note that the amplitude of the resulting oscillations is

$$|A| = \left| \frac{F}{1 - \omega^2 + 2i\omega\varepsilon} \right| = \frac{|F|}{\sqrt{(1 - \omega^2)^2 + 4\varepsilon^2\omega^2}}, \quad (11.52)$$

and thus identify two different regimes of behaviour.

In the “off-resonance” regime, where $\omega - 1 \gg \varepsilon$ (which is equivalent to $\omega^2 - 1 \gg \varepsilon$), the amplitude of the response is approximately $A \sim F/(1 - \omega^2)$. This corresponds to a leading-order balance between the oscillator terms and the forcing, with damping being negligible:

$$y_0'' + y_0 = F e^{i\omega t} + \text{cc} \Rightarrow y_0 = \frac{F}{1 - \omega^2} e^{i\omega t} + \text{cc}. \quad (11.53)$$

However, in the “on-resonance” regime, where $\omega - 1 = O(\varepsilon)$, the amplitude becomes much larger, $A = \text{ord}(1/\varepsilon)$. Importantly, if we look back at the terms in the governing equation, the terms that dominate are y'' and y which are both $\text{ord}(1/\varepsilon)$, so the resonant solution satisfies the homogeneous equation to leading order while the forcing and damping terms that drive the leading-order oscillation only come in at next order.

Example(s) 11.4. Let’s now consider a resonance example that we can’t solve exactly, by adding in a nonlinear term. For convenience, instead of having an $\text{ord}(1)$ forcing resulting in an $\text{ord}(1/\varepsilon)$ solution, we make the forcing be $\text{ord}(\varepsilon)$ and expect to obtain an $\text{ord}(1)$ solution. We thus consider the equation

$$y'' + 2\varepsilon y' + y + k\varepsilon y^3 = \varepsilon F e^{i\omega t} + \text{cc}, \quad (11.54)$$

where k is a given real $\text{ord}(1)$ constant. (This is the Duffing equation again, now including damping and forcing.) We have chosen the coefficient of the nonlinear term to be $\text{ord}(\varepsilon)$ so that it comes in at that order together with the forcing and damping terms. Further, we represent the closeness of the forcing frequency ω to the resonant frequency 1 by substituting

$$\omega = 1 + \varepsilon\omega_1, \quad (11.55)$$

and using the **detuning parameter** $\omega_1 = \text{ord}(1)$ as the parameter instead of ω . (Note that this is an exact relation between the parameter ω and ω_1 , and not an expansion of some solution in powers of ε .)

We now use the method of multiple scales,

$$T = \varepsilon t, \quad \frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \quad (11.56)$$

$$\Rightarrow \partial_t^2 y + 2\varepsilon \partial_t \partial_T y + \varepsilon^2 \partial_T^2 y + 2\varepsilon (\partial_t y + \varepsilon \partial_T y) + y + k\varepsilon y^3 = \varepsilon F e^{it} e^{i\omega_1 T} + \text{cc}. \quad (11.57)$$

Here, we have made the key substitution $\omega t = (1 + \varepsilon\omega_1)t = t + \omega_1 T$, in order to capture the long-term effect of the small frequency difference $\omega - 1 = \varepsilon\omega_1$ at leading order.

We then expand $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ and obtain $y_0 = A_0 e^{it} + \text{cc}$ followed by

$$\partial_t^2 y_1 + y_1 = [F e^{it} e^{i\omega_1 T} + \text{cc}] - 2\partial_t \partial_T y_0 - 2\partial_t y_0 - k y_0^3 = \quad (11.58)$$

$$= F e^{it} e^{i\omega_1 T} - 2i\partial_T A_0 e^{it} - 2iA_0 e^{it} - 3k|A_0|^2 A_0 e^{it} - kA_0^3 e^{3it} + \text{cc}. \quad (11.59)$$

We thus identify the secularity condition

$$2iA'_0 + 2iA_0 + 3k|A_0|^2 A_0 = F e^{i\omega_1 T} \Rightarrow A'_0 + (1 - \frac{3}{2}ik|A_0|^2)A_0 = -\frac{1}{2}iF e^{i\omega_1 T}. \quad (11.60)$$

The forcing term comes in with T -dependence $e^{i\omega_1 T}$, due to the forcing frequency $\omega = 1 + \varepsilon\omega_1$ being slightly different from the natural frequency 1, so we change variables to remove this,

$$A_0 = \bar{A}_0 e^{i\omega_1 T} \Rightarrow \bar{A}'_0 = (\bar{A}'_0 + i\omega_1 \bar{A}_0) e^{i\omega_1 T} \Rightarrow \bar{A}'_0 + (1 + i\omega_1 - \frac{3}{2}ki|\bar{A}_0|^2)\bar{A}_0 = -\frac{1}{2}iF. \quad (11.61)$$

We can again decompose $\bar{A}_0(T) = r(T)e^{i\phi(T)}$ into magnitude and argument, but this time we are unable to solve for the evolution r and ϕ explicitly,

$$\hat{A}'_0 = (r' + ir\phi')e^{i\phi} \Rightarrow r' + ir\phi' + (1 + i\omega_1 - \frac{3}{2}ikr^2)r = -\frac{1}{2}iF e^{-i\phi} = \frac{1}{2}F(-\sin\phi - i\cos\phi) \quad (11.62)$$

$$\Rightarrow r' + r = -\frac{1}{2}F \sin\phi, \quad r\phi' + (\omega_1 - \frac{3}{2}kr^2)r = -\frac{1}{2}F \cos\phi. \quad (11.63)$$

We instead seek just the steady-state solutions $r' = \phi' = 0$ (corresponding to steady periodic oscillations at this order), which yields

$$r = -\frac{1}{2}F \sin\phi, \quad (\omega_1 - \frac{3}{2}kr^2)r = -\frac{1}{2}F \cos\phi \quad (11.64)$$

$$\Rightarrow r^2 [1 + (\omega_1 - \frac{3}{2}kr^2)^2] = \frac{1}{4}F^2, \quad \cot\phi = \omega_1 - \frac{3}{2}kr^2. \quad (11.65)$$

This is the amplitude–frequency response equation. The dependence of r on ω_1 can be plotted parametrically by solving for $\omega_1 = \frac{3}{2}kr^2 \pm \sqrt{F^2/(4r^2) - 1}$ as a function of r instead (figure 11.3). For sufficiently large nonlinearity $|k|$, there is a range of frequencies ω_1 for which three steady amplitudes r are possible, and it turns out that the top and bottom branches are stable while the middle branch is unstable. With two stable branches coexisting, the value selected by the system depends on the initial conditions, and the system can exhibit hysteresis (i.e. the state of the system is dependent not only on the parameter values but also on its history): If the frequency ω_1 is slowly increased, the amplitude r will follow one solution branch until it ceases to exist, at which point r will jump onto the other solution branch. If the frequency is then slowly decreased, the amplitude r stays on the other branch and does not jump back until the other branch ceases to exist, resulting in a “hysteresis loop”.

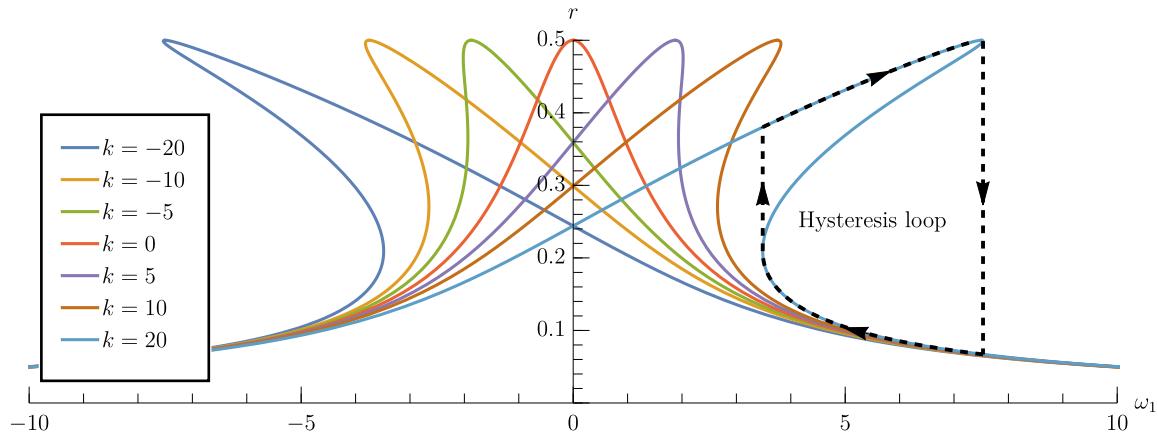


Figure 11.3: Dependence of steady oscillation amplitude $r = |A_0|$ on detuning frequency $\omega_1 = (\omega - 1)/\varepsilon$ for the Duffing equation $\varepsilon y'' + 2\varepsilon y' + y + k\varepsilon y^3 = \varepsilon 2F \cos(\omega t)$, with $F = 1$ and various values of k .

Remark(s) 11.3. We considered the example when the forcing resonates with the fundamental mode, i.e. the forcing frequency ω_f equals the frequency ω_0 of the oscillator (which we have taken to be $\omega_0 = 1$), but with nonlinearities we can also have resonance due to the forcing nonlinearly generating a harmonic with frequency $n\omega_f$ that equals ω_0 , and, perhaps more surprisingly, a forcing of frequency $\omega_f = n\omega_0$ interacting nonlinearly with a natural ω_0 oscillation to provide the driving force that sustains it.

Example(s) 11.5. We finally consider an example of **parametric resonance**, which is when the imposed oscillation does not appear in the classical “forcing term” in the harmonic oscillator equation, but rather in its coefficients (i.e. “parameters”).

The angle $\theta(\tau)$ of a pendulum whose pivot point is oscillated up and down with amplitude $2a$ and frequency Ω satisfies the equation

$$m\ell^2\theta''(\tau) + m(g + 2a\Omega^2 \cos(\Omega\tau))\ell \sin\theta(\tau) = 0. \quad (11.66)$$

Rescaling time by $\tau = (\ell/g)^{1/2}t$, assuming a small angle $\theta = \delta y \ll 1$ (this time ignoring the nonlinearity completely) yields the Mathieu equation

$$y'' + (1 + \varepsilon 2 \cos(\omega t))y = 0, \quad \varepsilon = a\Omega^2/g, \quad \omega = (\ell/g)^{1/2}\Omega, \quad (11.67)$$

and we consider the limit $\varepsilon \searrow 0$ where the oscillations are small. We note that the oscillation in the equation causes the instantaneous frequency $\sqrt{1 + \varepsilon 2 \cos(\omega t)}$ to oscillate. Unlike the case when there is an oscillating forcing term, this equation is linear and homogeneous, so does have a trivial solution $y = 0$, but there are also nontrivial solutions which may be amplified due to the resonance.

Let's start with a naive expansion $y(t) = y_0(t) + \varepsilon y_1(t) + \dots$, which yields

$$y_0'' + y_0 = 0 \quad \Rightarrow \quad y_0 = A_0 e^{it} + \text{cc}, \quad (11.68)$$

$$y_1'' + y_1 = -2 \cos(\omega t) y_0 = -(\text{e}^{i\omega t} + \text{cc}) (A_0 e^{it} + \text{cc}) = -A_0 e^{i(\omega+1)t} - A_0^* e^{i(\omega-1)t} - \text{cc}. \quad (11.69)$$

We obtain a secular term if $\omega + 1 = \pm 1$ or $\omega - 1 = \pm 1$, for which the only positive solution is $\omega = 2$. (The case $\omega = 0$ also yields a secular term, but looking back at the original equation we recognise that this just changes the oscillator frequency to a new constant $\sqrt{1 + 2\varepsilon}$.)

We conclude that this parametric resonance occurs when the driving frequency is (nearly) double the oscillator frequency. To investigate this phenomenon in more detail, we use the method of multiple scales and introduce a slow time $T = \varepsilon t$, and further we allow for driving frequency to be slightly different from the resonant value 2,

$$\frac{d}{dt} \rightarrow \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}, \quad \omega = 2 + \varepsilon \omega_1 \quad \Rightarrow \quad \partial_t^2 y + \varepsilon 2 \partial_t \partial_T y + \varepsilon^2 \partial_T^2 y + [1 + \varepsilon (\text{e}^{2it} \text{e}^{i\omega_1 T} + \text{cc})] y = 0. \quad (11.70)$$

We expand $y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$ and obtain $y_0 = A_0(T) e^{it} + \text{cc}$ followed by

$$y_1'' + y_1 = -2 \partial_t \partial_T y_0 - (\text{e}^{2it} \text{e}^{i\omega_1 T} + \text{cc}) y_0 = \text{e}^{it} [-2iA'_0(T) - A_0^* e^{i\omega_1 T}] - \text{e}^{3it} A_0 e^{i\omega_1 T} + \text{cc}. \quad (11.71)$$

The secularity condition then yields

$$2iA'_0 + A_0^* e^{i\omega_1 T} = 0 \quad \Rightarrow \quad A'_0 = \frac{i}{2} A_0^* e^{i\omega_1 T}. \quad (11.72)$$

This equation can be solved as follows. We eliminate the T -variation by defining

$$A_0 = \bar{A}_0 e^{i\omega_1 T/2} \quad \Rightarrow \quad A'_0 = (\bar{A}'_0 + \frac{i}{2} \omega_1 \bar{A}_0) e^{i\omega_1 T/2} \quad \Rightarrow \quad \bar{A}'_0 + \frac{i}{2} \omega_1 \bar{A}_0 = \frac{i}{2} \bar{A}_0^*. \quad (11.73)$$

Splitting $\bar{A}_0 = \bar{A}_r + i\bar{A}_i$ into real and imaginary parts then yields

$$\bar{A}'_r + i\bar{A}'_i = -\frac{i}{2} \omega_1 (\bar{A}_r + i\bar{A}_i) + \frac{i}{2} (\bar{A}_r - i\bar{A}_i) = \frac{\omega + 1}{2} \bar{A}_i + i \frac{-\omega_1 + 1}{2} \bar{A}_r \quad (11.74)$$

$$\Rightarrow \quad \bar{A}''_r = \frac{\omega + 1}{2} \frac{-\omega_1 + 1}{2} \bar{A}_r = \frac{1 - \omega_1^2}{4} \bar{A}_r \quad \Rightarrow \quad \bar{A}_r \propto e^{\pm \sqrt{(1 - \omega_1^2)/4} T}. \quad (11.75)$$

We see that if the frequency is sufficiently close to the resonant one, $|\omega_1| < 1$, then there is an exponentially growing solution, while if the frequency is farther away, $|\omega_1| > 1$, then the solutions oscillate.

Since the original equation was linear, the amplitude of the solution is arbitrary, and it is not surprising that the solution that grows does so exponentially without bound. Including a linear damping term $+2\mu\varepsilon y'$ would simply modify the growth criterion and rate, but a suitable nonlinear term could cause the amplitude to saturate at a finite value instead of growing indefinitely.

Part IV

Bonus content

This material is outside the scope of the module and will not be on the exam.

12 Homogenisation

Solids materials that appear to be homogeneous (i.e. have spatially uniform material properties) on large scales can sometimes have inhomogeneities (i.e. spatially varying material properties) that are only apparent when zoomed in to small scales. The method of **homogenisation** is used to transform equations with coefficients that vary rapidly (but quasi-periodically) on a “fast” lengthscale x into approximate equations with “effective coefficients” that vary only on a “slow” lengthscale $X = \varepsilon x$.

Example(s) 12.1. Consider the heat equation for the temperature $u(X, t)$ in a block of material,

$$\partial_t u = \partial_X [D(X/\varepsilon) \partial_X u] + Q(X/\varepsilon) \quad (12.1)$$

where the diffusivity coefficient D (which must be > 0) and source term Q are periodic functions with some period L .

We proceed as in the method of multiple scales, by treating the fast scale $x = X/\varepsilon$ and the slow scale X as independent variables. This yields

$$\partial_X \rightarrow \partial_X + \varepsilon^{-1} \partial_x \quad \Rightarrow \quad \varepsilon^2 \partial_t u = (\partial_x + \varepsilon \partial_X) [D(x)(\partial_x + \varepsilon \partial_X) u] + \varepsilon^2 Q(x), \quad (12.2)$$

and we seek an expansion $u \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2$.

At $\text{ord}(\varepsilon^0)$, we obtain

$$0 = \partial_x [D(x) \partial_x u_0] \quad \Rightarrow \quad D(x) \partial_x u_0 = A_0 \quad \Rightarrow \quad u_0 = A_0 \int \frac{1}{D(x)} dx + B_0. \quad (12.3)$$

Here, the constants of integration A_0 and B_0 are functions of the slow length X and of time t . Since the integrand $1/D$ is positive and periodic, the integral grows like $\text{ord}(x)$. We impose as a secular condition that u_0 must grow slower than $\text{ord}(x)$, which turns out to be equivalent to requiring that u_0 is periodic in the fast variable x , and we will also require that the higher-order terms are periodic in x . Hence we have $A_0 = 0$, and by renaming $B_0 = U_0$ for convenience, we conclude that the leading-order solution is

$$u_0 = U_0(X, t). \quad (12.4)$$

At $\text{ord}(\varepsilon^1)$, we obtain (using the notation $U'_0 = \partial_X U_0(X, T)$)

$$0 = \partial_x [D(x) \partial_x u_1] + \partial_X [D(x) \underbrace{\partial_x u_0}_{=0}] + \partial_x [D(x) \underbrace{\partial_X u_0}_{=U'_0}] \quad (12.5)$$

$$\Rightarrow \quad \partial_x [D(x) \partial_x u_1] = -\partial_x [D(x) U'_0(X, t)] \quad \Rightarrow \quad D(x) \partial_x u_1 = -D(x) U'_0(X, t) + A_1 \quad (12.6)$$

$$\Rightarrow \quad u_1 = \int -U'_0(X, t) + \frac{A_1(X, t)}{D(x)} dx + B_1(X, t). \quad (12.7)$$

The integrand is again periodic in x , and we again do not want it to have a net increase over each period as that would lead to u_1 growing like $\text{ord}(x)$. Thus, the requirement this time is that the integral over a period vanishes, i.e.

$$0 = \int_{x=0}^L -U'_0(X, t) + \frac{A_1(X, t)}{D(x)} dx = -L U'_0 + A_1 \int_0^L D(x)^{-1} dx \quad \Rightarrow \quad A_1 = \frac{U'_0}{\langle D^{-1} \rangle}, \quad (12.8)$$

where

$$\langle \cdot \rangle = \frac{1}{L} \int_0^L \cdot dx \quad (12.9)$$

denotes the average over a period. The solution is then

$$u_1 = U'_0(X, t) \int -1 + \frac{D(x)^{-1}}{\langle D^{-1} \rangle} dx + B_1(X, t). \quad (12.10)$$

At $\text{ord}(\varepsilon^2)$, we finally obtain

$$\partial_t U_0 = \partial_x [D(x) \partial_x u_2] + \partial_x [D(x) \partial_X u_1] + \partial_X [D(x) \partial_x u_1] + \partial_X [D(x) \partial_X u_0] + Q(x). \quad (12.11)$$

The trick now is to avoid solving the equation and simply consider the overall integral, or equivalently taking the average. For any L -periodic function $f(x)$, we have

$$\langle \partial_x [f(x)] \rangle = \frac{1}{L} \int_0^L \partial_x [f(x)] dx = \frac{1}{L} [f(L) - f(0)] = 0, \quad (12.12)$$

so we obtain

$$\langle \partial_t U_0 \rangle = 0 + 0 + \langle \partial_X [D(x) \partial_x u_1] \rangle + \langle \partial_X [D(x) U'_0] \rangle + \langle Q(x) \rangle. \quad (12.13)$$

We note that

$$D(x) \partial_x u_1 = D(x) U'_0(X, t) \left[-1 + \frac{D(x)^{-1}}{\langle D^{-1} \rangle} \right] = U'_0(X, t) [-D(x) + \langle D^{-1} \rangle^{-1}] \quad (12.14)$$

and that the averaging over x can be passed through derivatives with respect to X and t and functions of X and t alone, to obtain

$$\partial_t U_0 = \partial_X [\langle -D(x) + \langle D^{-1} \rangle^{-1} U'_0 \rangle] + \partial_X [\langle D(x) U'_0 \rangle] + \langle Q(x) \rangle = [\langle D^{-1} \rangle^{-1} U'_0]' + \langle Q \rangle. \quad (12.15)$$

We conclude that the homogenised equation is of the same form as the original equation, with the source term Q replaced by its average, and the diffusivity coefficient replaced by its ‘‘harmonic average’’ (i.e. the inverse of the average of the inverse). The above calculation and result are also valid if D and Q are allowed to vary slowly in space, i.e. have a separate dependence on the slow variable X .

Example(s) 12.2. Now let’s consider the same equation but in two spatial dimensions,

$$\partial_t u = \nabla_X \cdot [D(\mathbf{x}) \nabla_X u] + Q(\mathbf{x}), \quad (12.16)$$

where D and Q are periodic in both x and y with periods L_x and L_y . The multiple-scales ansatz yields

$$\varepsilon^2 \partial_t u = (\nabla_x + \varepsilon \nabla_X) \cdot [D(\mathbf{x}) (\nabla_x + \varepsilon \nabla_X) u] + \varepsilon^2 Q(\mathbf{x}), \quad (12.17)$$

and we again seek a solution $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$ with the secularity condition that the solution is periodic in x and y .

The $\text{ord}(\varepsilon^0)$ equation becomes

$$0 = \nabla_x \cdot [D(\mathbf{x}) \nabla_x u_0]. \quad (12.18)$$

Now we can’t write down the general solution, but we will rely on the fact that for periodic $D > 0$ any periodic solution f to the equation $\nabla_x \cdot [D \nabla_x f] = 0$ must be constant. As an aside, this little lemma can be shown by multiplying the equation by f , integrating over a periodic cell in both directions, and applying the divergence theorem:

$$0 = \int_A f \nabla_x \cdot [D \nabla_x f] dA = \int_A \nabla_x \cdot [f D \nabla_x f] - D |\nabla_x f|^2 dA = \int_{\partial A} D f \mathbf{n} \cdot \nabla_x f d\ell - \int_A D |\nabla_x f|^2 dA. \quad (12.19)$$

The boundary terms cancel due to periodicity in x and y , and we are left a non-negative quantity $D |\nabla_x f|^2$ integrating to zero, so we must have $|\nabla_x f| = 0$ everywhere. Using this fact, we conclude that

$$u_0 = U_0(X, t). \quad (12.20)$$

The $\text{ord}(\varepsilon^1)$ equation also becomes more tricky to solve,

$$0 = \nabla_x \cdot [D(\mathbf{x}) \nabla_x u_1] + \nabla_X \underbrace{[D(\mathbf{x}) \nabla_x u_0]}_{=0} + \nabla_x [D(\mathbf{x}) \nabla_X U_0] \quad (12.21)$$

$$\Rightarrow \nabla_x \cdot [D(\mathbf{x}) \nabla_x u_1] = -(\nabla_x D) \cdot (\nabla_X U_0(\mathbf{X}, t)) \quad (12.22)$$

In fact, we won't solve it at all, but we will note that due to linearity of the equations, if we can find two periodic solutions $a_{x,y}(\mathbf{x})$ to the equations

$$\nabla_x \cdot [D(\mathbf{x}) \nabla_x a_x] = -\partial_x D, \quad \nabla_x \cdot [D(\mathbf{x}) \nabla_x a_y] = -\partial_y D, \quad (12.23)$$

then the solution we seek is given by

$$u_1 = a_x \partial_X U_0 + a_y \partial_Y U_0 = \mathbf{a} \cdot \nabla_X U_0. \quad (12.24)$$

The equations for $a_{x,y}$ are called the **cell problem**, and although we can't solve them in general, we can still proceed to see what the structure of the homogenised equation will be. (The lemma above shows that for given $D(\mathbf{x})$, the solutions $a_{x,y}$ are determined up to an additive constant, which corresponds to A_1 in the previous one-dimensional calculation.)

Finally, at $\text{ord}(\varepsilon^2)$ we again take the average over a periodic cell,

$$\langle \cdot \rangle = \frac{1}{L_x L_y} \int_{x=0}^{L_x} \int_{y=0}^{L_y} \cdot \, dx \, dy, \quad (12.25)$$

noting that the integral of any x -divergence cancels out by the divergence theorem to obtain

$$\langle \partial_t U_0 \rangle = 0 + 0 + \langle \nabla_X [D(\mathbf{x}) \nabla_x u_1] \rangle + \langle \nabla_X [D(\mathbf{x}) \nabla_X U_0] \rangle + \langle Q(\mathbf{x}) \rangle. \quad (12.26)$$

We calculate

$$\nabla_x u_1 = \begin{pmatrix} \partial_x a_x & \partial_x a_y \\ \partial_y a_x & \partial_y a_y \end{pmatrix} \cdot \nabla_X U_0, \quad (12.27)$$

and hence obtain

$$\partial_t U_0 = \nabla_X \cdot [\mathbf{D} \cdot \nabla_X U_0] + \langle Q \rangle, \quad \mathbf{D} = \begin{pmatrix} \langle D \partial_x a_x \rangle + \langle D \rangle & \langle D \partial_x a_y \rangle \\ \langle D \partial_y a_x \rangle & \langle D \partial_y a_y \rangle + \langle D \rangle \end{pmatrix}. \quad (12.28)$$

Here, we see that the diffusivity coefficient $D(\mathbf{x})$ has been replaced by a diffusivity tensor \mathbf{D} which may have different behaviour in different directions. For example, if the material consists of alternating stripes of conducting and insulating material, then heat will conduct more easily in the direction parallel to the stripes than in the direction perpendicular to them. Again, we can allow D and Q to also vary slowly in space with \mathbf{X} .

Example(s) 12.3. A classical application of homogenisation is to flow in porous media such as a sponge or a layer of sand. Here is a brief summary of the calculation.

The fluid satisfies the Stokes equation in the pores of the material with no-slip boundary conditions on the solid boundaries,

$$\nabla_X p = \nabla_X^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } V, \quad \mathbf{u} = \mathbf{0} \quad \text{on } S. \quad (12.29)$$

We assume that V is a periodic shape (e.g. a repeating cube obstructed by a solid sphere in the centre) on the fast scale $\mathbf{x} = \mathbf{X}/\varepsilon$, and use the multiple-scales ansatz to obtain

$$\varepsilon(\nabla_x + \varepsilon \nabla_X)p = (\nabla_x^2 + 2\varepsilon \nabla_x \cdot \nabla_X + \varepsilon^2 \nabla_X^2)\mathbf{u}, \quad (\nabla_x + \varepsilon \nabla_X) \cdot \mathbf{u} = 0. \quad (12.30)$$

It turns out that the correct expansion is

$$p \sim p_0 + \varepsilon p_1, \quad \mathbf{u} \sim \varepsilon^2 \mathbf{u}_2 + \varepsilon^3 \mathbf{u}_3. \quad (12.31)$$

The equation at $\text{ord}(\varepsilon)$ then yields

$$\nabla_x p_0 = 0 \quad \Rightarrow \quad p_0 = P_0(\mathbf{X}). \quad (12.32)$$

At $\text{ord}(\varepsilon^2)$, we obtain

$$\nabla_x p_1 + \nabla_X P_0 = \nabla_x^2 \mathbf{u}_2, \quad \nabla_x \cdot \mathbf{u}_2 = 0. \quad (12.33)$$

Thus, the cell problem is to find periodic solutions satisfying

$$-\nabla_x \hat{p}_i + \nabla_x^2 \hat{\mathbf{u}}_i = -\mathbf{e}_i, \quad \nabla_x \cdot \hat{\mathbf{u}}_i = 0, \quad (12.34)$$

and the required solution is then

$$\mathbf{u}_2 = - \sum_i \hat{\mathbf{u}}_i \partial_{X_i} P_0, \quad p_2 = - \sum_i \hat{p}_i \partial_{X_i} P_0. \quad (12.35)$$

In particular, we identify the so-called Darcy velocity as

$$\mathbf{U}_0 = \langle \mathbf{u}_2 \rangle = - \underbrace{\sum_i \frac{1}{L_x L_y L_z} \int_{x=0}^{L_x} \int_{y=0}^{L_y} \int_{z=0}^{L_z} \hat{\mathbf{u}}_i dx dy dz}_{\mathbf{k}} \mathbf{e}_i \cdot \nabla_X P_0, \quad (12.36)$$

where \mathbf{k} is the permeability tensor.

Finally, at $\text{ord}(\varepsilon^3)$, we consider the conservation equation,

$$\nabla_x \cdot \mathbf{u}_3 + \nabla_X \cdot \mathbf{u}_2 = 0. \quad (12.37)$$

The integral of the first term over the fluid in the periodic cell is

$$\int_{V_1} \nabla_x \cdot \mathbf{u}_3 dV = \int_{\partial V_1} \mathbf{n} \cdot \mathbf{u}_3 dS, \quad (12.38)$$

which vanishes due to the assumption of periodicity of \mathbf{u}_3 and due to the no-slip condition $\mathbf{u}_3 = \mathbf{0}$ on the solid surfaces. Hence, we obtain the equation

$$0 = \langle \nabla_X \cdot \mathbf{u}_2 \rangle = \nabla_X \cdot \mathbf{U}_0. \quad (12.39)$$

We have thus recovered Darcy's equations,

$$\mathbf{U}_0 = -\mathbf{k} \cdot \nabla_X P_0, \quad \nabla_X \cdot \mathbf{U}_0 = 0. \quad (12.40)$$

13 A closer look at the Airy functions in the complex plane

(Trust me, it won't be quite as boring as it sounds!)

Let's revisit the behaviour in the complex plane (so naming the independent variable z instead of x) of the solutions $y(z)$ of the Airy equation,

$$y'' - zy = 0, \quad (13.1)$$

as $|z| \nearrow \infty$ for various fixed values of $\theta = \arg z$.

We already know from the WKB method that there are two independent asymptotic behaviours

$$y_0^\pm(z) \sim \frac{\exp(\pm\frac{2}{3}z^{3/2})}{z^{1/4}}, \quad (13.2)$$

and it is possible to continue to higher order to work out the corrections, which turn out to be descending powers of $z^{3/2}$. It happens to be easier to calculate the corrections if we instead make the change of variables

$$y = \frac{\exp(\pm\frac{2}{3}z^{3/2})}{z^{1/4}} f(w = z^{3/2}) \Rightarrow f'' \pm \frac{4}{3}f' + \frac{5}{36w^2}f = 0. \quad (13.3)$$

Assuming algebraic behaviour of f yields the estimates $\text{ord}(f/w^2)$, $\text{ord}(f/w)$ and $\text{ord}(f/w^2)$ for the three terms, so at leading order (as $|w| \nearrow \infty$) we obtain

$$f' \approx 0 \Rightarrow f \approx f_0 = A_0 = 1, \quad (13.4)$$

(choosing the constant to be 1 for simplicity). Then,

$$f_0 = 1 + \tilde{f}_1 \Rightarrow \tilde{f}_1'' \pm \underbrace{\frac{4}{3}\tilde{f}_1'}_{\text{dominant}} + \frac{5}{36w^2}\tilde{f}_1 = -f_0'' - \frac{5}{36w^2}f_0 = -\frac{5}{36w^2}f_0 \quad (13.5)$$

$$\Rightarrow \tilde{f}_1 \approx f_1 = A_1 w^{-1}, \quad \pm\frac{4}{3}(-1)A_1 = -\frac{5}{36} \Rightarrow A_1 = \pm\frac{3}{4} \times \frac{5}{6} \times \frac{1}{6}. \quad (13.6)$$

Continuing like this, we obtain

$$f = \sum_{n=0}^N f_n(w) + \tilde{f}_{N+1}(w), \quad f_n = A_n w^{-n}, \quad \pm\frac{4}{3}(-n)A_n = -[(1-n)(-n) - \frac{5}{36}] A_{n-1} \quad (13.7)$$

$$\Rightarrow A_n = \pm\frac{3}{4} \times \frac{(n-\frac{1}{6})(n-\frac{5}{6})}{n} A_{n-1} \Rightarrow A_n = \left(\pm\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})}{\Gamma(\frac{1}{6})} \frac{\Gamma(n+\frac{1}{6})}{\Gamma(\frac{5}{6})} \frac{1}{n!}, \quad (13.8)$$

using the Gamma recurrence formula, and we can further simplify the denominators using the reflection formula $\Gamma(\frac{1}{6})\Gamma(\frac{5}{6}) = 2\pi$.

We conclude that the full expansions are (recalling that $w = z^{3/2}$)

$$y_\infty^\pm(z) \sim \frac{\exp(\pm\frac{2}{3}z^{3/2})}{z^{1/4}} \sum_{n=0}^{\infty} \frac{(\pm 3/4)^n \Gamma(n+\frac{5}{6}) \Gamma(n+\frac{1}{6})}{2\pi n!} z^{-3n/2}, \quad (13.9)$$

and note that the series are divergent since the magnitude of the ratio of each term to the previous is $\text{ord}(n/z^{3/2})$ which is > 1 for sufficiently large n . This is a version of the classical “factorial over power” behaviour that is common for divergent asymptotic expansions.

Optimal truncation

Although we have a full asymptotic expansion, as can be seen from the definition

$$\tilde{f}_{N+1}(w) = f(w) - \sum_{n=0}^N f_n(w) + o(f_N(w)) \text{ as } |w| \nearrow \infty, \quad (13.10)$$

this only gives us information about the behaviour of the error in the limit $|w| \nearrow \infty$, but doesn't reveal any information about its value at any fixed finite $|w|$. Since the expansion is divergent, we know that no matter how large w is, taking more and more terms will initially make the error smaller, but eventually the terms and the error start to grow again. How many terms should we keep to get the most accurate approximation, i.e. the smallest error?

It turns out that in practice a reasonable estimate for the remainder is given by the full next term including the prefactor,

$$|\tilde{f}_{N+1}(w)| \approx |f_{N+1}(w)|. \quad (13.11)$$

The principle of **optimal truncation** states that to get the best approximation at a fixed w , we should choose the smallest $\tilde{f}_{N+1}(w)$, i.e. keep all terms **up to but not including the smallest term** in the expansion.

For large values of w , we expect to be able to keep a large number of terms, so we can estimate the magnitude of the terms for $n \gg 1$ using Stirling's approximation $\Gamma(m+1) \sim \sqrt{2\pi m} (m/e)^n$. For the Airy terms, we obtain

$$|f_n(w)| \sim \left(\frac{3}{4}\right)^n \frac{\sqrt{2\pi/n}(n/e)^n}{2\pi|w|^n} = \frac{1}{\sqrt{2\pi}} \left(\frac{3}{4e|w|}\right)^n n^{n-1/2} \quad \text{as } n \nearrow \infty. \quad (13.12)$$

We find the minimum by taking the logarithm and differentiating,

$$0 = \ln \frac{3}{4e|w|} + \ln n + \left(1 - \frac{1}{2n}\right) \Rightarrow n \sim \frac{4w}{3}. \quad (13.13)$$

Taking $N+1 = \frac{4|w|}{3} + O(1)$ yields the optimal error

$$|\tilde{f}_N(w)| = |f_N(w)| \sim \frac{1}{\sqrt{2\pi(4|w|/3)}} e^{-4|w|/3}. \quad (13.14)$$

Even though each individual error $\tilde{f}_N(w)$ is algebraically small in w , the optimal truncation with N dependent on w results in an error that is exponentially small in $|w|$. This is the case in general for asymptotic expansions that diverge like factorials divided by powers.

Stokes phenomenon

Now let's look more closely at the WKB solutions for the Airy equation in the complex plane,

$$y_0^\pm(z) \sim \frac{\exp(\pm\frac{2}{3}z^{3/2})}{z^{1/4}}. \quad (13.15)$$

These expressions involve fractional powers so we will need to place a branch cut somewhere. Without a cut, if we walk a full lap around the origin anticlockwise, i.e. increase $\arg z$ by 2π , then

$$y_0^\pm = \frac{\exp(\pm\frac{2}{3}z^{3/2})}{z^{1/4}} \quad \text{change to} \quad \frac{\exp(\pm\frac{2}{3}e^{2\pi i(3/2)}z^{3/2})}{e^{2\pi i(1/4)}z^{1/4}} = \frac{\exp(\mp\frac{2}{3}z^{3/2})}{iz^{1/4}}, \quad (13.16)$$

i.e. the two solutions trade places, and also the coefficients are multiplied by an imaginary factor.

However, any solution $y(z)$ of the original equation is analytic everywhere so its value must remain unchanged after completing a full lap. We conclude that in the general asymptotic relation

$$y \sim A_+ y_0^+ + A_- y_0^- \quad \text{as } |z| \nearrow \infty \text{ with fixed } \theta = \arg z, \quad (13.17)$$

(by which we mean that the error is $o(y_0^+, y_0^-)$) the coefficients A_+ and A_- cannot be the same in all directions (i.e. for all values of θ). This result, that the function $y(z)$ has different asymptotic behaviour in different directions, is called the **Stokes phenomenon**.

Since $y_0^\pm(z)$ are approximate solutions of the differential equation, if a solution $y(z)$ satisfies for example

$$y \sim Ay_0^+ \text{ as } |z| \nearrow \infty \text{ with fixed } \theta = \arg z, \quad (13.18)$$

then we expect it to also satisfy this relation with the same value of A in some neighbourhood of this direction (i.e. $-\delta < |\arg z - \theta| < \delta$, where $\delta > 0$). This means that the coefficient cannot change when it “visible” as part of a dominant term, and so **the coefficients can only change when they are hidden by an exponentially larger term**.

The magnitude of the solutions depends on the real part of the exponent,

$$|y_0^\pm(z)| \sim \frac{\exp(\pm\frac{2}{3}\operatorname{Re}(z^{3/2}))}{|z|^{1/4}}, \quad (13.19)$$

so in the interval $-\pi/3 < \arg z < \pi/3$ where $\operatorname{Re}(z^{3/2}) > 0$ we have $y_0^+ \gg y_0^-$ (exponentially), but as we cross either line $\arg z = \pm\pi/3$ the two solutions exchange dominance and we instead obtain $y_0^- \gg y_0^+$. If we go further around and cross the negative real axis $\arg z = \pm\pi$ then the solutions swap again. Note that we obtain opposite results when going clockwise or anticlockwise, due to the need for a branch cut that flips the signs in the exponents.

These (half-)lines where the different terms exchange dominance are called anti-Stokes lines. For a classical pair of WKB solutions with exponents $\pm S_0$, we find the anti-Stokes lines by requiring that the exponents have the same real part, i.e. $\operatorname{Re} S_0 = 0$. For the Airy function with $S_0 = \frac{2}{3}z^{3/2}$, this yields

$$\operatorname{Re}(z^{3/2}) = 0 \Rightarrow \arg z = \pi/3, \pi, 5\pi/3. \quad (13.20)$$

The lines in between, where the terms are maximally different in size are called Stokes lines. These are found by requiring that the exponents have maximally different real part, i.e.

$$\operatorname{Im}(z^{3/2}) = 0 \Rightarrow \arg z = 0, 2\pi/3, 4\pi/3. \quad (13.21)$$

(Confusingly, the definitions of “Stokes lines” and “anti-Stokes lines” are sometimes swapped, and there is no overall consensus on which naming convention is the correct one.)

We have argued earlier that for any given solution $y(z) \sim A_+ y_0^+ + A_- y_0^-$ the coefficients A_\pm must change as we go around the origin, but each coefficient can change value when it’s hidden behind the other (non-zero) coefficient with an exponentially larger term. So the coefficient A_- can change in the region where $y_0^+ \gg y_0^-$, and it is customary to imagine that it changes on the Stokes line, where y_0^- is the smallest, and hence changing the value of the coefficient causes the smallest discontinuity.

Stokes line smoothing

What more can we say about the coefficient A_- in a region where $y_0^+ \gg y_0^-$ (and $A_+ \neq 0$)? The term y_0^- is exponentially smaller than y_0^+ and hence **beyond all orders** in the expansion y_∞^+ where the terms are only algebraically smaller than y_0^+ . Hence, we would think the value of the coefficient A_- is completely inaccessible (or indeed meaningless), because it is consumed by the error term in

$$y = A_+ y_0^+ \left[\sum_{n=0}^N f_n + o(f_N) \right] + \underbrace{A_- y_0^- [\dots]}_{=o(y_0^+ f_N)} \quad (13.22)$$

for any finite N , and we cannot somehow eliminate the $o(f_N)$ by summing up all the terms since the series is divergent.

But there is a way to make the error term smaller – use optimal truncation! We estimated earlier that this reduces the error inside the square brackets to $O(e^{-4w/3}w^{-1/2})$, which is just below what we need for the coefficient, since $y_0^-/y_0^+ = e^{-4w/3}$.

From the calculation at the start of this section, for the + solution the error term \tilde{f}_N satisfies the equation

$$\tilde{f}_N'' + \frac{4}{3}\tilde{f}_N' + \frac{5}{36w^2}\tilde{f}_N = -f_{N-1}'' - \frac{5}{36w^2}f_{N-1} = -\frac{4}{3}NA_N w^{-N-1}. \quad (13.23)$$

We make the change of variables from $\tilde{f}_N(w)$ to the “Stokes multiplier” $g(w)$ via

$$\tilde{f}_N(w) = e^{-4w/3}g(w) \Rightarrow g'' - \frac{4}{3}g' + \frac{5}{36w^2}g = -\frac{4}{3}NA_N w^{-N-1}e^{+4w/3}. \quad (13.24)$$

Here, the exponential factor is chosen to be the one corresponding to the exponentially smaller solution $f = y_0^-/y_0^+$, or equivalently to the homogeneous solution of the equation. As a result, $g(w)$ is equal to A_- at leading order, and we now investigate how it changes with $\theta = \arg w = \frac{3}{2} \arg z$ for fixed $r = |w| = |z|^{3/2}$.

For fixed r , we have

$$w = r e^{i\theta} \Rightarrow \frac{d}{d\theta} = \frac{dw}{d\theta} \frac{d}{dw} = i r e^{i\theta} \frac{d}{dw} = i w \frac{d}{dw} \Rightarrow \frac{d}{dw} = \frac{1}{i w} \frac{d}{d\theta}. \quad (13.25)$$

We substitute this into the equation for g , neglecting g'' and g/w^2 in favour of g' , and use the optimal truncation estimates for $A_N|w|^{-N}$ with $N = 4r/3$ to obtain

$$-\frac{4}{3} \frac{1}{i w} \frac{dg}{d\theta} \approx -\frac{4}{3} \frac{4r}{3} \left[\frac{1}{\sqrt{2\pi(4r/3)}} e^{-4r/3} \right] \frac{|w|^r}{w^{r+1}} e^{+4w/3} \Rightarrow \frac{dg}{d\theta} = i \sqrt{\frac{2r}{3\pi}} \frac{e^{4(w-r)/3}}{e^{i\theta(4r/3)}}. \quad (13.26)$$

We write the term in the exponent as

$$w - r = r(e^{i\theta} - 1) = -r(1 - \cos(\theta)) + ir \sin \theta, \quad (13.27)$$

and note that for large r (similarly to in the Laplace method) the right-hand side is exponentially small except in a small neighbourhood of $\theta = 0$, so we expand around there to find

$$\frac{dg}{d\theta} = i \sqrt{\frac{2r}{3\pi}} \exp \left[\frac{4}{3}(-r\theta^2/2 + O(r\theta^3)) \right] \Rightarrow \Delta g \sim i \sqrt{\frac{2r}{3\pi}} \int e^{-2r\theta^2/3} d\theta. \quad (13.28)$$

Evaluating the Gaussian integral from $-\infty$ to ∞ , we conclude that the coefficient A_- changes by a total value i across a boundary layer of angular width $\theta = O(r^{-1/2})$ around the Stokes line $\theta = 0$.

To check the result, we consider the particular solution

$$y(z) = \sqrt{\pi} \operatorname{Bi}(z) \sim \frac{\exp(+\frac{2}{3}z^{3/2})}{z^{1/4}} \quad \text{as } z \nearrow +\infty. \quad (13.29)$$

For $z < 0$, we have

$$y(z) \sim \frac{\cos(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}) + \sin(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4})}{(-z)^{1/4}} = \frac{e^{+i\frac{2}{3}(-z)^{3/2}} + e^{-i\frac{2}{3}(-z)^{3/2}}}{\sqrt{2}(-z)^{1/4}}. \quad (13.30)$$

Rotating z clockwise through the upper half plane (decreasing its argument by π) to $z > 0$ while retaining the coefficient of y_0^- yields

$$y(z) \sim A_+ y_0^+ + e^{-\frac{2}{3}z^{3/2}} \sqrt{2} e^{-i\pi/4} z^{1/4} \Rightarrow A_- = e^{i\pi/4}/\sqrt{2} \quad \text{for } 0 < \arg z < \pi/6. \quad (13.31)$$

Rotating z anticlockwise through the lower half plane to $z > 0$ yields

$$y(z) \sim A_+ y_0^+ + e^{-\frac{2}{3}z^{3/2}} \sqrt{2} e^{+i\pi/4} z^{1/4} \Rightarrow A_- = e^{-i\pi/4}/\sqrt{2} \quad \text{for } -\pi/6 < \arg z < 0. \quad (13.32)$$

Hence, the net change in A_- across the Stokes line at $\arg z = 0$ is

$$\frac{e^{i\pi/4} - e^{-i\pi/4}}{\sqrt{2}} = \frac{(1+i) - (1-i)}{2} = i. \quad (13.33)$$

Exponential asymptotics

Above we calculated exponentially small corrections in a local analysis, but the same can be done for perturbation expansions, e.g. of solutions to differential equations with a small parameter ε . The main steps are the same: Find the behaviour of the late terms as $n \rightarrow \infty$ (typically just the leading order), observe a factorial-divided-by-power divergence and perform an optimal truncation to look at how exponentially small terms switch on across Stokes lines. In such problems, the exponentially small corrections can sometimes be properly important, e.g. by appearing at certain points where the algebraic terms all vanish.