<u>Pset 3: Kinematics II</u> Matt Gardner 6.4212 FA24

3.8 b)

1. Since we are only producing joint velocities, integration is essential to understand the current position of the end effector and all of the arm's links.
2. Initial state of the integrator is the constant term during the integration The integrator only calculates _relative_ position from the sum of the velocities commanded, so the integrator must be set to the beginning of the initial position of the arm's joints before moves are commanded.
3. State is set using the `robot_state` input port of the DiffInverseIK integrator. According to the documentation, _If the robot_state port is connected, then the initial state of the integrator is set to match the positions from this port (the port accepts the state vector with positions and velocities for easy of use with MultibodyPlant, but only the positions are used)._ This means that when we tie the input port to the robot state, the integrator assumes the initial state from the existing robot state at t0 automatically.

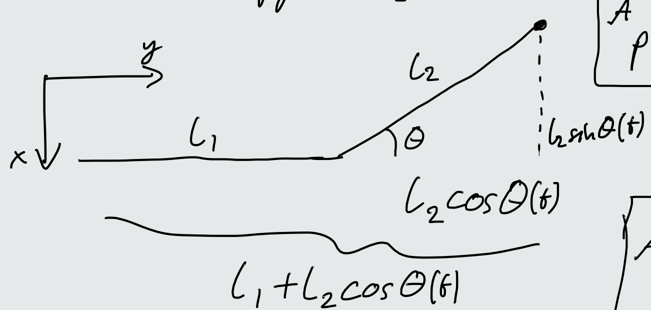
3.12 a) ${}^{B}p^{C}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow$ write ${}^{A}p^{C}(0)$.

${}^{A}p^{C} = {}^{A}R^{B}(t)\,{}^{B}p^{C}_{B} + {}^{A}p^{B}(t)$ $\quad f(x)$ of $l_0, l_1, l_2, \theta(t)$

${}^{A}R^{B}(t)$: 2D version $\Rightarrow$ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$p_x = -l_2 \sin\theta$ $\qquad +90°$

$p_y = l_2 \cos\theta$



$$\boxed{{}^{A}p^{B}(t) = [-l_2 \sin\theta(t),\ l_1 + l_2\cos\theta(t),\ l_0]}$$

$$\boxed{{}^{A}R^{B}(\theta(t)) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\theta(t) \rightarrow$ rotation about z

$$\begin{bmatrix} -l_2\sin\theta & 0 & 0 \\ l_2\cos\theta & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3.12 b) posit that $\omega = \dot{R}R^{-1}$

$\omega^T = (\dot{R}R^{-1})^T$

$= R^{-1T} \cdot \dot{R}$

because R is square and invertible, $R^{-1} = R^T$

so $(R^{-1})^T = (R^T)^T = R$. Therefore,

$\omega^T = R \cdot \dot{R}^T \rightarrow$ remember that $RR^T = I$.

take derivative... $d/dt\ R \cdot R^T = \dot{R} R^T + R \cdot \dot{R}^T = 0$

so $\dot{R} R^T = -R\dot{R}^T$

$\dot{R} R^T = -\omega^T$

since $R^T = R^{-1}$, $\dot{R} R^{-1} = -\omega^T$. $\boxed{\text{So } \hat{\omega} = -\hat{\omega}^T.}$

$$R = {}^A R^B(t) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{R} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{R} R^{-1} = \begin{bmatrix} -\sin\theta\cos\theta + \sin\theta\cos\theta & -\sin^2\theta - \cos^2\theta & 0 \\ \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\omega}^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\hat{\omega} = -\hat{\omega}^T$.

$$-\hat{\omega}^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3.12c) $${}^A V^B(t) = \frac{d}{dt}\, {}^A p^B(t) + \frac{d}{dt}\left({}^A R^B(t)\right)$$

$$= \frac{d\theta}{dt}\begin{bmatrix} -l_2 \sin\theta(t) & l_1 + l_2\cos\theta(t) & l_0 \end{bmatrix} + \dot{R} \text{ (from above)}$$

$$\boxed{{}^A V^B(t) = \underbrace{\begin{bmatrix} l_2\cos\theta(t)\cdot\dot{\theta}(t), & -l_2\sin\theta(t)\cdot\dot{\theta}(t), & 0 \end{bmatrix}}_{\text{position}} + \underbrace{\begin{bmatrix} -\sin\theta(t)\cdot\dot{\theta}(t) & -\cos\theta(t)\cdot\dot{\theta}(t) & 0 \\ \cos\theta(t)\cdot\dot{\theta}(t) & -\sin\theta(t)\cdot\dot{\theta}(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{velocity}}}$$

3.13 a) Full rpy rotation matrix is the following (multiplying the matrices given in Drake RPY docs)

$R = R_z(y)R_y(p)R_x(r) =$

$$\begin{bmatrix} \cos(p)\cos(y) & \sin(r)\sin(p)\cos(y)-\cos(r)\sin(y) & \cos(r)\sin(p)\cos(y)+\sin(r)\sin(y) \\ \cos(p)\sin(y) & \sin(r)\sin(p)\sin(y)+\cos(r)\cos(y) & \cos(r)\sin(p)\sin(y)-\sin(r)\cos(y) \\ -\sin(p) & \sin(r)\cos(p) & \cos(r)\cos(p) \end{bmatrix}$$

apply $p = -\pi/2 \Rightarrow \cos(-\pi/2) = 0, \ \sin(-\pi/2) = -1$

so the matrix becomes:
$$\begin{bmatrix} 0 & -\sin(r)\cos(y)-\cos(r)\sin(y) & -\cos(r)\cos(y)+\sin(r)\sin(y) \\ 0 & -\sin(r)\sin(y)+\cos(r)\cos(y) & -\cos(r)\sin(y)-\sin(r)\cos(y) \\ 1 & 0 & 0 \end{bmatrix}$$

simplifying w/ identities:
$$\boxed{R = \begin{bmatrix} 0 & -\sin(r+y) & -\cos(r+y) \\ 0 & \cos(r+y) & -\sin(r+y) \\ 1 & 0 & 0 \end{bmatrix}}$$

In all of the terms, we are taking the sum of r and y, so the effects of a change in r are equivalent to those of a change in y. RPY represents rotations properly in the forward kinematic direction, but we are unable to distinguish a given rotation's roll and yaw components (in this condition). Conceptually, this is similar to the gimbal lock condition, where we can no longer distinguish rotations in one axis from those of another when attempting to understand a vehicle's state in 3D space.

3.13 b)

The axis-angle coordinate frame does not produce a unique orientation based on the inputs from each of the different components.
In addition to the baseline rotation (0, 0, pi/2), the following rotations produce the same output rotation:
1. (-pi/2, -pi/2, pi/2)
2. (0, 0, 5*pi/2) or any multiple of 2*pi thereafter

3.13 c) quaternion: (w, x, y, z) s.t. $w^2+x^2+y^2+z^2=1$

$$\theta = 2\cos^{-1}(w)$$

$$a_x, a_y, a_z = \frac{\theta}{\sin(\theta/2)}(x, y, z) \quad \theta \neq 0 \ \text{(assume nontrivial case)}$$

trying with $-w$:

$\theta = 2\cos^{-1}(-w) \rightarrow$ applying hint 1, $\cos^{-1}(w) = \pi - \cos^{-1}(w)$

$\theta = 2(\pi - \cos^{-1}(w)) = 2\pi - 2\cos^{-1}(w) \rightarrow$ applying hint 2, we can drop the 2π since $\theta = \theta \pm 2\pi$

so $\theta = -2\cos^{-1}(w)$ $\rightarrow$ for any θ, there is $-\theta = \pi - \theta$

$-\theta = 2\cos^{-1}(w)$

$\pi - \theta = 2\cos^{-1}(w)$

so now $\theta = \pi - 2\cos^{-1}(w)$

plugging in

$$a_x, a_y, a_z = \frac{\pi - 2\cos^{-1}(w)}{\sin\left(\frac{\pi - 2\cos^{-1}(w)}{2}\right)}(-x, -y, -z)$$

Since we know that the representation of the angle 2*pi - theta = theta (b/c theta is periodic), we know that the value of the function theta / (sin(theta/2)) is the same because theta is the same.
Applying pi - theta in the numerator produces the same output as the original theta since cos(x) = -cos(-x). Likewise, the denominator of the above function simplifies to w. Since sine is a symmetric function, sin(x) = sin (-x), so the equation is not dependent on the sign of w. The vector (-x, -y, -z) is multiplied by the same factor as (x, y, z), so we can expect the value of the quaternions to be the same when expressed in the axis-angle coordinate frame. This is equivalent to rotating 360˚ across the -x, -y, and -z axes.