Chapter 4 Notes

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Integration

1 - Antidifferentiation

In this chapter, we explore *integration*, which is one of the two principal branches of calculus (*differential calculus*, which we have studied in Chapters 1-3, being the other).

With *integral calculus*, we are able to determine accumulation of quantity based on a given rate function, for example:

- Given a function $y = v(t)$, the velocity of an object at time t, we can determine an object's *distance traveled* over an interval of time.
- Given a function $y = P'(x)$, the marginal profit of a business after x units have been sold, we can determine the business's *total profit* after x units have been sold
- Given a function $y = f(t)$, the rate of change of a population after t years, we can determine the *total population growth* after t years

Finding Antiderivatives

One aspect of integral calculus is **Antidifferentiation**, which is the process of differentiation performed in reverse. Given a function f, we find another function F such that

$$\frac{d}{dx}F(x) = f(x)$$

The function F is an **antiderivative** of f. For example, if $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative of f since

$$\frac{d}{dx}(x^2) = 2x$$

Note that functions like $F(x) = x^2 + 5$ and $F(x) = x^2 - 17$ are also antiderivatives of $f(x) = 2x$ since

$$\frac{d}{dx}(x^2 + 5) = 2x + 0 = 2x; \quad \text{and} \quad \frac{d}{dx}(x^2 - 17) = 2x + 0 = 2x$$

Thus, an antiderivative of $f(x) = 2x$ is any function that can be written in the form $F(x) = x^2 + C$, where C is any constant. This leads us to the following theorem.

Theorem 6.1

The antiderivatives of $f(x)$ is the set of functions $F(x) + C$ such that

$$\frac{d}{dx}\left[F(x) + C = f(x)\right]$$

The constant C is called the constant of intergration

If F is an antiderivative of f, we write

$$\int f(x)dx = F(x) + C$$

This equation is read as "the antiderivative of $f(x)$, with respect to x, is $F(x) + C$ "
or as " the integral of $f(x)$, with respect to x if $F(x) + C$". The expression on the left side is called an**indefinite integral**. They symbol $\int$ is the *integral sign*, and $f(x)$ is the *integrand*. The symbol dx can be regarded as indicating that x is the variable of integration, similar to $\frac{d}{dx}$ indicating that the expression that follows it is to be differentiated with respect to x.

Question 1

Determine these indefinite integrals. That is, find the antiderivative of each integrand.

$$a) \int 8\ dx; \quad b) \int 3x^2\ dx; \quad \int e^x\ dx'; \quad \int \frac{1}{x}\ dx, x \neq 0$$

Problem 1.

$$\int 8\ dx$$
$$= 8x + C$$

Problem 2.

$$\int 3x^2\ dx$$
$$= x^3 + C$$

Problem 3.

$$\int e^x\ dx$$
$$= e^x + C$$

Problem 4.

$$\int \frac{1}{x}\ dx$$
$$= \ln|x| + C$$

Note:-

Every antiderivative can be checked by differentiation

The results of Question 1 suggest some rules for antiderivatives, which are summarized in Theorem 2.

Theorem 6.2 Rules for Antiderivatives

A1. Constant Rule:

$$\int k dx = kx + C$$

A2. Power Rule (where $n \neq -1$):

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C.$$

A3. Natural Logarithm Rule:

$$\int \frac{1}{x} dx = \ln|x| + C, \text{ and for } x > 0, \int \frac{1}{x} dx = \ln x + C.$$

A4. Exponential Rule (base e):

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0.$$

The Power Rule for Antiderivaties can be viewed as a two step process:

$$\int x^n \; dx = \frac{1}{n+1} x^{n+1} + C$$

1. Increase the exponent by 1.
2. Divide the term by the new power

> ### Question 2
>
> Find the following indefinite integrals:
>
> $$a) \int x^7 \; dx; \quad b) \int \sqrt{x} \; dx; \quad c) \int \frac{1}{x^3} \; dx;$$

Problem 1.

$$\int x^7 \; dx$$

$$= \frac{x^{7+1}}{7+1} + C$$

$$= \frac{1}{8} x^8 + C$$

Problem 2.

$$\int \sqrt{x} \; dx$$

$$= \int x^{\frac{1}{2}} \; dx = \frac{x^{(1/2)+1}}{\frac{1}{2}+1} + C = \frac{x^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} + C, \quad \text{or} \quad \frac{2}{3} x \sqrt{x} + C$$

Problem 3.

$$\int \frac{1}{x^3} \; dx$$

$$= \int x^{-3} = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2} x^{-2} + C$$

$$= -\frac{1}{2x^2} + C$$

The Power Rule for Antiderivatives is valid for all real numbers n, except for $n = -1$. for $n = -1$, we have $x^{-1} = \dfrac{1}{x}$, which is the derivative of the natural logarithm function, $y = \ln |x|$. Therefore,

$$\int \frac{1}{x} \; dx = \ln |x| + C, \text{ and for } x > 0, \int \frac{1}{x} \; dx = \ln x + C$$

In Question 3, we explore the case of $f(x) = e^{ax}$

Question 3

Find

$$\int e^{4x}\ dx$$

Solution:

$$\int e^{4x}\ dx = \frac{1}{4}e^{4x} + C$$

Two useful properties of antiderivatives are presented in Theorem 6.3.

Theorem 6.3

P1. A constant multiplier can be factored to the front of the indefinite integral:

$$\int [c \cdot f(x)]dx = c \cdot \int f(x)dx.$$

P2. The antiderivative of a sum or difference is the sum or difference of the antiderivatives:

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Question 4

Find each antiderivative. Assume $x > 0$

$$a) \int (3x^5 + 7x^2 + 8)\ dx \qquad b) \int \frac{4 + 3x + 2x^4}{x}\ dx$$

Problem 1.

$$\int 3x^5\ dx + \int 7x^2\ dx + \int 8\ dx$$

$$3(\frac{1}{6}x^6) + 7(\frac{1}{3}x^3) + 8x + C$$

$$\frac{1}{2}x^6 + \frac{7}{3}x^3 + 8x + C$$

Problem 2.

$$\int \frac{4 + 3x + 2x^4}{x}\ dx$$

$$= \frac{4}{x} + \frac{3x}{x} + \frac{2x^4}{x} = \frac{4}{x} + 3 + 2x^3$$

Therefore,

$$\int \frac{4 + 3x + 2x^4}{x}\ dx = \int \left(\frac{4}{x} + 3 + 2x^3\right)\ dx$$

$$= 4\ln x + 3x + \frac{1}{2}x^4 + C$$

Initial Conditions

When a point that is a solution of an antiderivative is given, it is possible to solve for C. The given point is called an inital condition

Find

$$\int (2x+3)\ dx$$

Given that $F(1) = -2$

Solution: If we specify that $F'(x) = 2x+3$***, then we have***

$$F(x) = \int (2x+3)\ dx = x^2 + 3x + C$$

Since $F(1) = -2$, we can substiute and solve for C.

$$-2 = (1)^2 + 3(1) + C$$

Simplifiying, we have $-2 = 4 + C$, or $C = -6$

Therefore, the specific antiderivative that satisfies the inital conditions is

$$F(x) = x^2 + 3x - 6$$

Applications

Question 5

A rock is thrown upward with the inital velocity 50 ft/sec from 10 ft above the ground has a velocity modeled by $v(t) = -32t + 50$, where t is the number of seconds after the rock is released and $v(t)$ is in feet per second

a) Determine a distance function h as a function of t (in this case, "distance" is the height of the rock)

b) Find the height and the velocity of the rock after 3 sec.

Solution:

For part a) Since distance (height) is the antiderivative of velocity, we have

$$h(t) = \int (-32t+50)\ dt = -16t^2 + 50t + C$$

The inital height, 10 ft, gives us the ordered pair (0,10) as an initial condition. We substitute 0 for t and 10 for $h(t)$, and solve for C

$$10 = -16(0)^2 + 50(0) + C$$
$$10 = C$$

Therefore, the distance function is given by

$$h(t) = -16t^2 + 50t + 10$$

For part b) to find the height of the rock after 3 sec, we substitute 3 for t ***in the distance function***

$$h(3) = -16(3)^2 + 50(3) + 10 = 16\text{ft}.$$

The velocity is

$$v(3) = -32(3) + 50 = -46 \text{ ft/sec}.$$

2 Antiderivaties as Areas

Integral calculus studies the *accumulation* of units as the input variables increases. For example, suppose a jogger maintains a constant velocity of 5 mi/hr. As she runs, she "accumulates" distance. After 1 hr, she has run 5 mi. Between the first hour and the second hour, she has run 5 mi, so that for the first 2 hr, she has accumulated a distance of 10 mi.

We can view accumulations graphically, as shown in the following example.

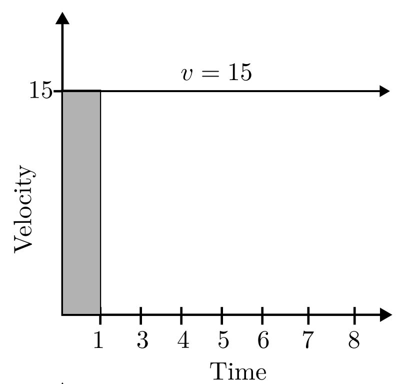
Question 6

Emma drives her motor scooter at 15 mi/hr for an extended period of time

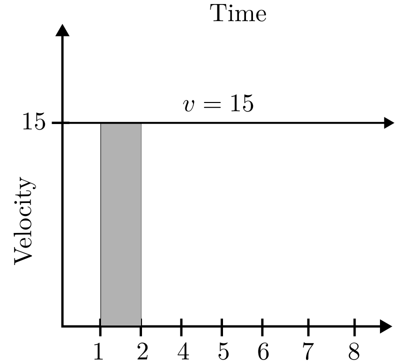
1. How far has she traveled after 1 hr.
2. How far has she traveled between the first hour and the second hour?
3. How far has she traveled cumulatively over the first 2 hr?
4. What function $f(t)$ gives Emma's total distance traveled after t hours?

Solution: Emma's velocity after t hours is given by $v(t) = 15$, where $v(t)$ is in miles per hour. We graph v, noting that its graph is a horizontal line.

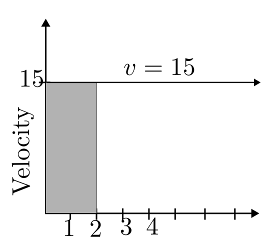
a) On the graph, we shade a rectangular area between $t = 0$ and $t = 1$.
This area represents $(1\text{hr})\left(15\dfrac{\text{mi}}{\text{hr}}\right) = 15$ mi.
Thus, Emma has traveled 15 mi after 1 hr.



b) On the graph, we shade a rectangular area between $t = 1$ and $t = 2$.
This rectangular angle has width 1 and height 15.
Thus, in this 1-hr period, Emma has traveled another 15 mi



c) Emma has accumulated a total of $15 + 15 = 30$ mi traveled over the first 2 hr
Graphically, her total distance traveled after 2 hr is the area of the rectangle between $t = 0$ and $t = 2$, with width 2 and height 15



d) We set up an input-output table for v, including a third column showing distance traveled in each hour and a fourth column showing total *accumulated* (or *cumulative*) distance traveled

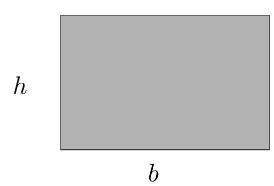
Time t (in hours)	Velocity, $v(t)$ (in miles per hour)	Distance Traveled in Each Hour	Accumulated Distance Traveled
1	15	15	15
2	15	15	30
3	15	15	45
4	15	15	60

The accumulated distances traveled suggest that $f(t) = 15t$ gives Emma's total distance traveled, in miles, after t hours. Note that $f(t) = 15t$ is an antiderivative of $v(t) = 15$

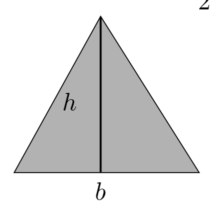
Geometry and Areas

Example 1 suggests that the antiderivative plays a role in determining area under a graph. For linear functions, we can use geometry to find the area under the graph of a function. Two formulas, where b = base and h = height, are useful:

Area of a rectangle: $A = bh$



Area of a Triangle: $A = \frac{1}{2}bh$



Question 7

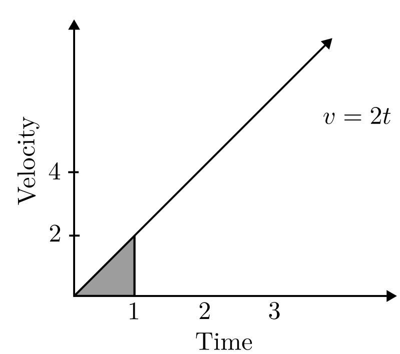
A toy drone flies in a straight line, and its velocity t seconds after takeoff is given by $v(t) = 2t$, where $v(t)$ is in meters per second.

1. Find the distance the drone has flown after 1 sec.
2. Find the distance the drone has flown between $t = 1$ sec and $t = 2$ sec.
3. Find the cumulative distance the drone has flown over the first 2 sec.

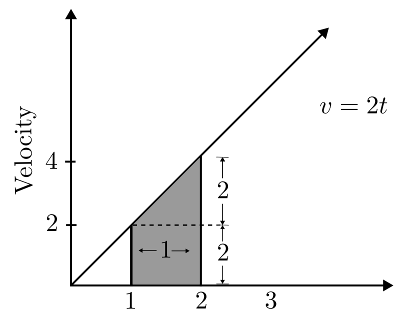
Solution: We graph $v(t) = 2t$, noting that it is a linear function. Thus, we can use geometry to find the areas under its graph

Problem 1. To find the distance flown after 1 sec, we form a triangular area under v from $t = 0$ to $t = 1$
The area is given by $A = \frac{1}{2}bh$. Here, $b = 1$ and $h = 2$, so after 1 sec, the drone has flown

$$A = \frac{1}{2}\,(1 \text{ sec})\left(2\frac{\text{m}}{\text{sec}}\right) = 1 \text{ m}$$

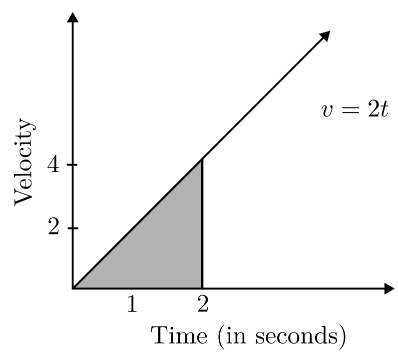


Problem 2. To find the distance traveled between $t = 1$ sec and $t = 2$ sec, we form a trapezoidal area under v over $[1, 2]$
This can be regarded as a rectangle and a triangle, as shown in the figure to the right.
The rectangle has area $(1)(2) = 2$, and the triangle has area $\frac{1}{2}(1)(2) = 1$.
Thus, the drone has flown a distance of $2 + 1 = 3$ m between $t = 1$ sec and $t = 2$ sec.



Problem 3. The cummulative distance flown after 2 sec is 1 m [from part problem 1] plus 3 m [from problem 2], or a total of 4 m.
We can also veiw the cummulative distance as the area of the entire triangle over [0,2], or

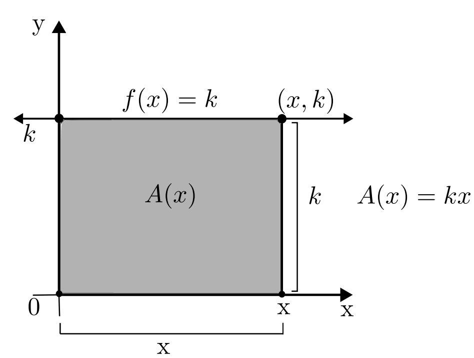
$$\frac{1}{2}(2)(4) = 4 \text{ m}$$



Examples 1 and 2 in this section suggest a pattern

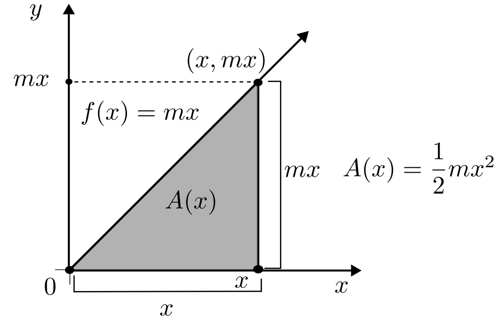
The graph of $f(x) = k$, where k is a constant, is a horizontal line of height k. The region under this graph over the interval [0,x] is a rectangle, and its area is

$$A = kx \text{ (height times base)}$$



The graph of $f(x) = mx$ is a line with slope m, passing through the origin.
The region under this graph over the interval [0,x] is a triangle, and its area is

$$A = \frac{1}{2}(x)(mx) = \frac{1}{2}mx^2.$$



Question 8

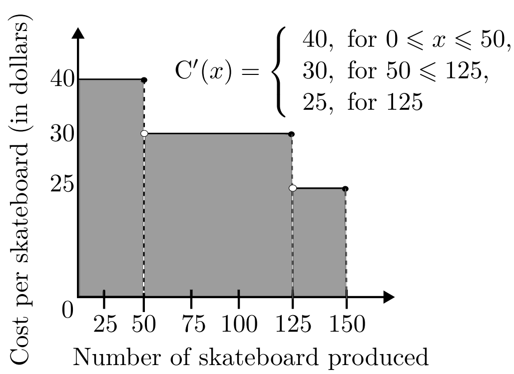
Green Leaf Skateboards has the following marginal-cost function for producing skateboards: For up to 50 skateboards, the cost is $40 per skateboard. For quantities from 51 through 125 skateboards, the cost drops to $30 per skateboard. After 125 skateboard, it drops to $25 per skateboard. If x represents the number of skateboards produced, we have

$$C'(x) = \begin{cases} 40, & \text{for } 0 \leqslant x \leqslant 50, \\ 30, & \text{for } 50 < x \leqslant 125, \\ 25, & \text{for } 125 < x \end{cases}$$

Where $C'(x)$ is the cost per skateboard, in dollars. Find the total cost of producing 150 skateboards.

Solution: We calculate the areas of the rectangles formed under the graph of C' over the intervals

$$[0, 50], [50, 125], \text{ and } [125, 150]$$

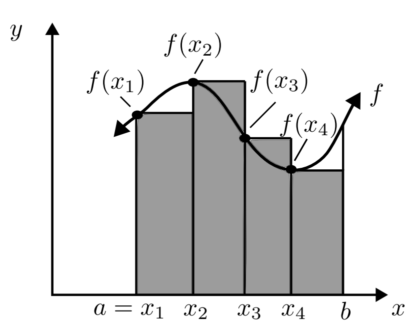


Riemann Summation

In **Riemann summation**, rectangles can be used to approximate the area under the graph of a continuous function.

In the following figure, [a,b] is divided into four subintervales, each having width

$$\Delta x = \frac{(b-a)}{4}$$



The heights of the rectangles shown are $f(x_1), f(x_2), f(x_3)$, and $f(x_4)$ and the area of the region under the curve is approximately the sum of the areas of the four rectangles

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \qquad \text{(This is the riemann sum)}$$

We can denote this sum with **summation**, or **sigma, notation**, which uses the Greek capital letter sigma, $\sum$:

$$\sum_{i=1}^{4} f(x_i)\Delta x$$

Question 9

Express $\sum_{i=1}^{5} h(x_i)\Delta x$ without using summation notation

Solution: We have

$$\sum_{i=1}^{5} h\left(x_i\right)\Delta x = h\left(x_1\right)\Delta x + h\left(x_2\right)\Delta x + h\left(x_3\right)\Delta x + h\left(x_4\right)\Delta x + h\left(x_5\right)\Delta x$$

Question 10

Consider the graph of $f(x) = \sqrt{4-x^2}$ over the interval [0,2]. This is a quarter-circle radius 2. USe a Riemann sum to approximate the area under the graph using 4 equally sized subintervals and then 8 equally sized subintervals. Then use geometry to find the area under f over [0,2], and compare this value to your approximations

Solution:

Dividing [0,2] into 4 subintervals of equal width, we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

So, we have

$$\sum_{i=1}^{4} f(x_i)\Delta x$$
$$= f(0)\cdot\frac{1}{2} + f\left(\frac{1}{2}\right)\cdot\frac{1}{2} + f(1)f\left(1\right)\cdot\frac{1}{2} + f\left(\frac{3}{2}\right)\cdot\frac{1}{2}$$
$$= 3.49571$$

Thus, the area under f over [0,2] is approximately 3.49571 square units. Note that this is approximation is greater than the actual area of the quarter-circle

Divding [0,2] into 8 subintervals of equal width, we have

$$\Delta x = \frac{2-0}{8} = \frac{1}{4}$$

So, we have

$$\sum_{i=1}^{8} f(x_i)\Delta x$$
$$= 3.33982$$

Using 8 subintervals, we have refined the estimate of the area under f over [0,2] to 3.33982 square units.

Definite Integrals

Defintion

Let y = $f(x)$ be continuous and nonnegative over an interval [a,b]. A **definite integral** is the limit as $n \to \infty$ (equivalently $\Delta x \to 0$) of the Riemann sum of the areas of rectangles under the graph of $y = f(x)$ over [a,b]

$$\text{Exact area} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \int_a^b f(x)\ dx$$

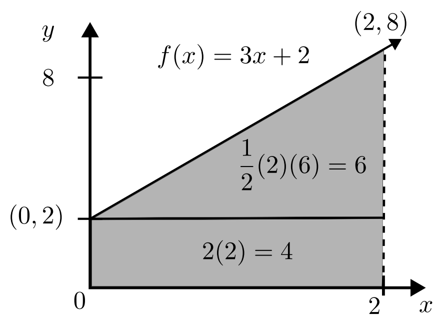
Question 11

Find the value of

$$\int_0^2 (3x+2)\ dx$$

Solution:

We sketch the graph over the interval [0,2] and note that the region is a trapezoid. Thus, we can use geometry to determine this area. So,



$$\int_0^2 (3x+2)\ dx = 2 \cdot 2 + \frac{1}{2} \cdot 2 \cdot 6$$
$$= 10$$

Area and Definite integrals

Question 12

Find the area under the graph of

$$f(x) = \frac{1}{5}x^2 + 3 \text{ over } [2, 5]$$

Solution:

Although making a drawing is not required, doing so helps visualize the problem. The interval is [2,5], so we have $a = 2$ ***and*** $b = 5$

$$\int \left(\frac{1}{5}x^2 + 3\right) = F(x)$$
$$F(x) = \frac{1}{15}x^3 + 3x + C$$

For simplicity, we set C = 0, so that $F(x) = \frac{1}{15}x^3 + 3x$

$$\begin{aligned} \text{Area over } [2,5] &= F(5) - F(2) \\ &= \frac{1}{15}(5)^3 + 3(5) - \left[\frac{1}{15}(2)^3 + 3(2)\right] \\ &= \left(\frac{125}{15} + 15\right) - \left(\frac{8}{15} + 6\right) \\ &= 16\frac{4}{5} \end{aligned}$$

Question 13

Find the area under the graph of

$$y = x^2 + 1 \text{ over } [-1, 2]$$

Solution:

Find the antiderivative of f

$$F(x) = \frac{x^3}{3} + x$$

Note:-

For simplicity, we set C = 0

Now we substitute the endpoints, 2 and -1, and find the difference F(2) - F(-1)

$$\begin{aligned} F(2) - F(-1) &= \left[\frac{2^3}{3} + 2\right] - \left[\frac{(-1)^3}{3} + (-1)\right] \\ &= \frac{8}{3} + 2 - \left[\frac{-1^3}{3} + (-1)\right] \\ &= \frac{8}{3} + 2 + \frac{1}{3} + 1 \\ &= 6 \end{aligned}$$

Defintion

Let f be any continuous function over [a,b] and F be any antiderivative of f.

Then the **definite integral** of f from a to b is

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Where $F(x)$ is an antiderivative of $f(x)$

Question 14

Evalulate each of the following

$$\text{a)} \int_{-1}^{4} (x^2 - x)\ dx; \quad \text{b)} \int_{0}^{2} e^x\ dx; \quad \text{c)} \int_{2}^{5} \frac{1}{x}\ dx;$$

Problem 1.

$$\int_{-1}^{4} (x^2 - x)\ dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_{-1}^{4}$$

Remember that we dont care about C (C = 0)

$$\left(\frac{(4)^3}{3} - \frac{(4)^2}{2} \right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^3}{2} \right)$$

$$= \left(\frac{64}{3} - \frac{16}{2} \right) - \left(\frac{-1}{3} - \frac{1}{2} \right)$$

$$= \frac{64}{3} - 8 + \frac{1}{3} + \frac{1}{2} = 14\frac{1}{6}$$

Problem 2.

$$\int_{0}^{2} e^x\ dx$$

$$= [e^x]_0^2$$

$$e^2 - e^0$$

$$\approx 6.389$$

Problem 3.

$$\int_{2}^{5} \frac{1}{x}\ dx$$

$$= [\ln|x|]_2^5$$

$$= \ln|5| - \ln|2|$$

$$\approx 0.916$$

More on Area

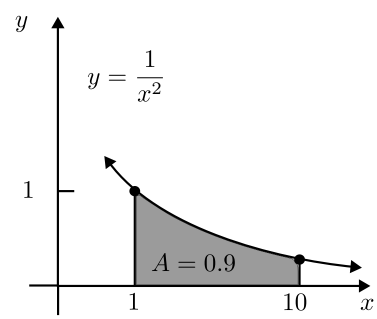
When we evalulate the definite integral of a nonnegative function f over [a,b], we get the area under the graph f over that interval

Question 15

Find the area under the graph of

$$y = \frac{1}{x^2} \text{ over } [1, 10]$$

$$\begin{aligned}\int_1^{10} \frac{dx}{x^2} &= \int_1^{10} x^{-2} dx \\ &= \left[\frac{x^{-2+1}}{-2+1}\right]_1^{10} \\ &= \left[\frac{x^{-1}}{-1}\right]_1^{10} = \left[-\frac{1}{x}\right]_1^{10} \\ &= \left(-\frac{1}{10}\right) - \left(-\frac{1}{1}\right) \\ &= 1 - \frac{1}{10} = 0.9\end{aligned}$$



4.5 - Integration Techniques: Substitution

The following formulas provide a basis for an integration technique called **substitution**, a process that is, as we will see, the reverse of differentiation using the Chain Rule.

A. $\int u^r du = \frac{u^{r+1}}{r+1} + C$, assuming $r \neq -1$

B. $\int e^u du = e^u + C$

C. $\int \frac{1}{u} du = \ln|u| + C;$ and $\int \frac{1}{u} du = \ln u + C, \quad u > 0$

In the above formulas, the variable u represents some function of x and du is the derivative of u with respect to x. Recall that we solve $\int_x^7 \ dx$ using the Power Rule for Antiderivaties:

$$\int x^7 \ dx = \frac{x^7+1}{7+1} + C = \frac{x^8}{8} + C. \text{ or } \frac{1}{8}x^8 + C$$

But, what about an integral like $\int (3x-4)^7 \ dx$? Suppose we thought the antiderivative was

$$\frac{(3x-4)^8}{8} + C$$

If we check by differentiating, we get

$$8 \cdot \frac{1}{8} \cdot (3x-4)^7 \cdot 3 \cdot dx$$

This simplifies to

$$3(3x-4)^7, \text{ not } (3x-4)^7$$

To correct our antiderivative, let's make this substitution:

$$u = 3x - 4$$

Then $\dfrac{du}{dx} = 3$, and recalling our work with differentials, we have

$$du = 3 \cdot dx, \text{ and } \frac{du}{3} = dx$$

With *substitution*, our original integral, $\int (3x-4)^7\ dx$, takes the form

$$\int (3x-4)^7\ dx = \int u^7 \cdot \frac{du}{3}$$

$$= \frac{1}{3} \cdot \int u^7\ du$$

$$= \frac{1}{3} \cdot \frac{u^8}{8} + C$$

$$= \frac{1}{3 \cdot 8} \cdot (3x-4)^8 + C = \frac{1}{24}(3x-4)^8 + C$$

Question 16

Find dy for each function

a) $y = f(x) = x^3$

b) $y = f(x) = x^{2/3}$;

c) $y = g(x) = \ln x$;

d) $y = f(x) = e^{x^2}$

Problem 1.

$$y = f(x) = x^3$$

$$= \frac{dy}{dx} x^3$$

$$= 3x^2$$

Problem 2.

$$y = f(x) = x^{\frac{2}{3}}$$

$$dy = f'(x)\ dx = \frac{2}{3} x^{\frac{-1}{3}}\ dx$$

Problem 3.

We have

$$\frac{dy}{dx} = g'(x) = \frac{1}{x}$$

So,

$$dy = g'(x)\ dx = \frac{1}{x}\ dx, \quad \text{or } \frac{dx}{x}$$

Question 17

Evalulate

$$\int 3x^2(x^3+1)^{10}\ dx$$

Solution:

We let

$$u = x^3 + 1$$

So, we have

$$u = x^3 + 1$$

$$du = 3x^2\ dx$$

We now solve for dx

$$dx = \frac{du}{3x^2}$$

Now we have

$$\int 3x^2(u)^{10} \cdot \frac{du}{3x^2}$$

$3x^2$ ***cancels so we are left with***

$$\int u^{10}\ du$$

Now integrate

$$\int u^{10} = \frac{u^{11}}{11} + C$$

Reversing the substitution we get

$$\frac{1}{11}\left(x^3+1\right)^{11} + C$$

Note:-

to find du, we take the derivative of u

Integration Techniques: Integration by Parts

Let $y = u(x)$ and $y = v(x)$ be two functions. Applying the Product Rule, we have

$$\frac{d}{dx}(u(x)) \cdot v(x)) = u(x) \cdot \frac{d}{dx}v(x) + v(x) \cdot \frac{d}{dx}u(x)$$

Integrating both sides with respect to x, we have

$$\int \left[\frac{d}{dx}(u(x) \cdot v(x))\right] dx = \int \left[u(x) \cdot \frac{d}{dx}v(x)\right] dx + \int \left[v(x) \cdot \frac{d}{dx}u(x)\right] dx,$$

Note that

$$\int \left[\frac{d}{dx}(u(x) \cdot v(x))\right]\ dx = u(x) \cdot v(x)$$

We simplify by writing u for $u(x)$ v for $v(x), du$ for $\frac{d}{dx}u(x)$ and dv for $\frac{d}{dx}v(x)\ dx$:

$$uv = \int u\ dv + \int v\ du$$

Solving for $\int u\ dv$, we obtain the following theorem

Theorem 6.4

$$\int u\ dv = uv - \int v\ du$$

Question 18

Evaluate:

$$\int xe^x\ dx$$

Solution: We let

$$u = x \quad \text{and} \quad dv = e^x\ dx$$

In this case, differentiating u gives

$$du = dx$$

and integrating dv gives

$$v = e^x$$

The integrating-by-Parts Formula gives us

$$\int (x)(e^x\ dx) = (x)(e^x) - \int (e^x)(dx)$$

$$= xe^x - e^x + C$$