Chapter 3 Notes

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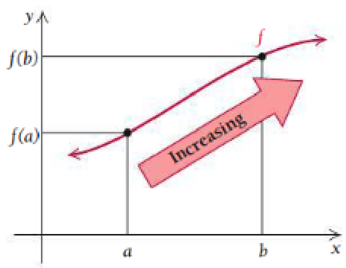
3.1 - Using First Derivaties to Classify Maximum and Minimum Values

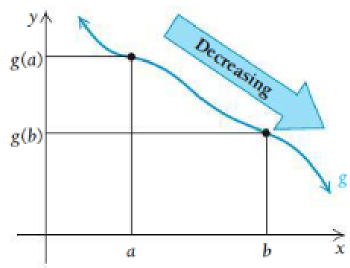
Increasing and Decreasing Functions

If the graph of a function rises from left to right over an interval I, the function is said to be increasing on, or over, I.

If the graph drops from left to right, the function is said to be decreasing on, or over, I.



If the input a is less than the input b, then the output for a is less than the output for b.



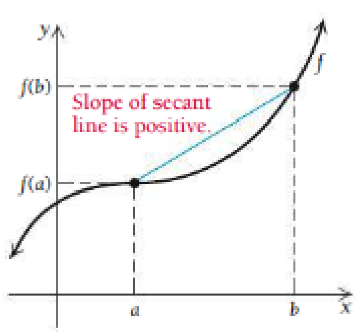
If the input a is less than the input b, then the output for a is greater than the output for b.

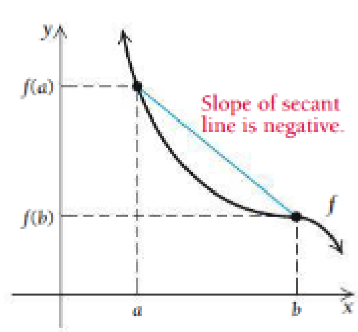
We can define these concepts as follows.

> A function f is **increasing** over I if, for every a and b in I,
>
> $$\text{if } a < b, \quad \text{then } f(a) < f(b)$$
>
> A function f is **decreasing** over I if, for every a and b in I,
>
> $$\text{if } a < b, \quad \text{then } f(a) > f(b)$$

The above definitions can be restated in terms of slopes of secant lines

$$\text{Increasing: } \frac{f(b) - f(a)}{b - a} > 0 \qquad\qquad \text{Decreasing: } \frac{f(b) - f(a)}{b - a} < 0$$





Since the derivative of a function tells us the slope of the tangent line to f at any input x, we can also define an increasing or decreasing function using the derivative

Theorem 0.1

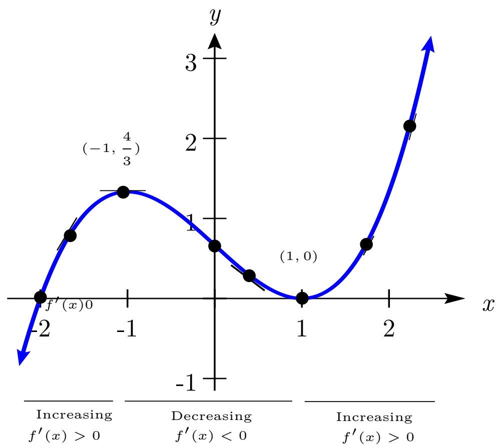
Let f by differentiable over an open interval I

If $f'(x) > 0$ for all x in I, then f is increasing over I

If $f'(x) < 0$ for all x in I, then f is decreasing over I

Theorem 0.1 is illustrated in the following graph of

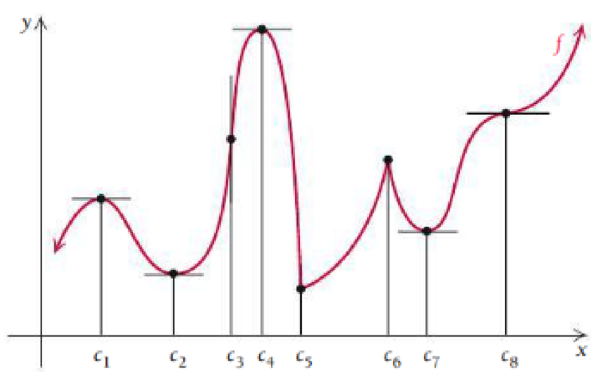
$$f(x) = \frac{1}{3}x^3 - x + \frac{2}{3}$$



Note in the graph above that x = -1 and x = 1 are not included in any interval over which the function is increasing or decreasing. These values are examples of *critical values*

Critical Values

Consider the following graph



Note the following

1. $f'(x) = 0$ for $x = c_1, c_2, c_4, c_7,$ and c_8. That is, the tangent line to the graph is horizontal at these values.
2. $f'(x)$ does not exist for $x = c_3, c_5$, and C_6. The tangent line is vertical at c_3, and there are corners at both c_5 and c_6.

A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is **horizontal** or for which the derivative does not exist.

That is, c is a critical value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

If c is a critical value of a function f, then $(c, f(c))$ is a **critical point**

Thus, in the graph of f above:

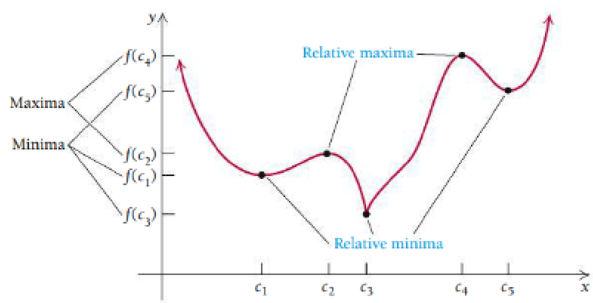
- c_1, c_2, c_4, c_7, c_8 are critical values because $f'(c) = 0$ for each value
- c_3, c_5, c_6 are critical values because $f'(c)$ does not exist for each value

Note:-

A continuous function can change from increasing to decreasing or from decreasing to increasing **only** at a critical value.

Finding Relative Maximum and Minimum Values

Now consider a graph with "peaks" and "valleys" at $x = c_1, c_2, c_3, c_4$, and c_5



Here, $f(c_2)$ and $f(c_4)$ are each an example of a **relative**, or **local, maximum**, and $f(c_1), f(c_3)$, and $f(c_5)$ are each an example of a **relative**, or **local minimum**.

Collectively, maximum and minimum values are called **extrema**

Note:-

Note that it is possible for a relative minimum to be greater than a relative maximum.

For example, $f(c_5) > f(c_2)$ in the graph in the next page.

Also note that x-values at which a continuous function has relative extrema are those values for which the derivative is 0 or for which the derivative does not exist - the critical values

Theorem 0.2

If a function f has a relative extreme value $f(c)$ on an open interval, then c is a critical value, and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

Relative Extreme Points

A relative extreme point, $(c, f(c))$, is higher or lower than all other points over an open interval containing c.

Relative Minimum Point

A relative minimum point, $(c, f(c))$, is lower than all other points over an open interval containing c. Such a point has a y-value that is less than those of a neighborhood of points to the left and right of c.

Relative Maximum Point

Similarly, a relative maximum point, $(c, f(c))$, is higher than all other points over an open interval containing c. This maximum point has a y-value that is greater than those of a neighborhood of points to the left and right of c.

Note: In the preceding graph, $(c_1, f(c_1)), (c_3, f(c_3))$, and $(c_5, f(c_5))$ are all relative minimum points. Similarly, $(c_2, f(c_2))$ and $(c_4, f(c_4))$ are both relative maximum points.

Theorem 2

Theorem 2 is useful and important to understand. It states that to find relative extrema, we need only consider inputs for which the derivative is 0 or for which the derivative does not exist. Each critical value is a candidate for a value where a relative extremum might occur.

However, Theorem 2 does not guarantee that every critical value will yield a relative maximum or minimum. For instance, consider the graph of

$$f(x) = (x-1)^3 + 2$$

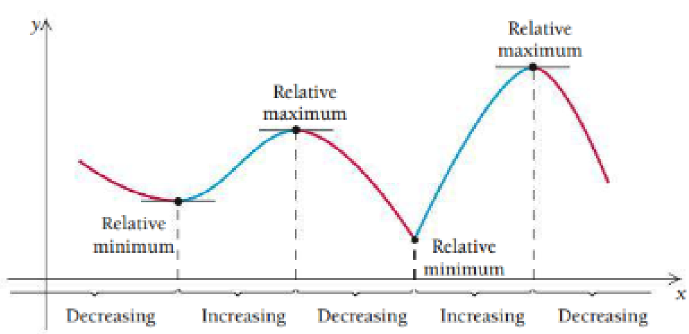
shown on the left. Note that:

$$f'(x) = 3(x-1)^2 \quad \text{and} \quad f'(1) = 3(1-1)^2 = 0$$

Thus, $c = 1$ is a critical value, but f has no relative maximum or minimum at that value. In fact, this function has no extrema anywhere.

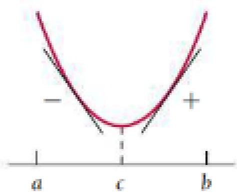
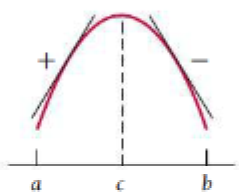
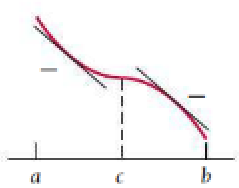
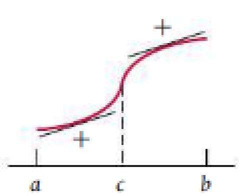
Theorem 2 does guarantee that if a relative maximum or minimum occurs, then the first coordinate of that extremum is a critical value.

The following graph leads us to a test.



Note that at a critical value where there is a relative minimum, the function f is **decreasing** on the left of the critical value and **increasing** on the right.

At a critical value where there is a relative maximum, the function f is **increasing** on the left of the critical value and **decreasing** on the right. In both cases, the derivative changes signs on either side of the critical value.

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	−	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	−	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	−	−	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

Derivatives can tell us when a function is increasing or decreasing. This leads us to the First-Derivative Test.

Theorem 0.3

For any continuous function f that has exactly one critical value c in an open interval (a, b)

F1. f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c

F2 f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c

F3 f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) as on (c, b)

We can use the First-Derivative Test to find relative extrema.

Question 1

Consider the function f given by

$$f(x) = 4x^3 - 9x^2 - 30x + 25$$

Find any relative exterma

Find the derivative

$$\frac{d}{dx}[4x^3] - \frac{d}{dx}[9x^2] - \frac{d}{dx}[30] + \frac{d}{dx}[25]$$

$$f'(x) = 12x^2 - 18x - 30$$

set f'(x) = 0

$$12x^2 - 18x - 30 = 0$$

divide both sides by 6

$$2x^2 - 3x - 5 = 0$$

$$(x + 1)(2x - 5) = 0$$

$$x = -1 \quad or \quad x = \frac{5}{2}$$

The critical values are -1 and $\frac{5}{2}$. Since it is at these values that a relative maximum or minimum might exist, we examine the sign of the derivative on the intervals

$$(-\infty, -1), (-1, \frac{5}{2}), (\frac{5}{2}, \infty)$$

To do so, we select a convenient test value in each interval and evalulate f'(x).

Let's use the values: -2, 0, 4

$$f'(-2) = 54$$

$$f(0) = -30$$

$$f(4) = 90$$

Result:

f is increasing on $(-\infty, -1)$
f is decreasing on $(-1, \frac{5}{2})$
f is increasing on $(\frac{5}{2}, \infty)$

By the First-Derivative Test, f has a relative maximum at $x = -1$ and a relative minimum at $x = \dfrac{5}{2}$

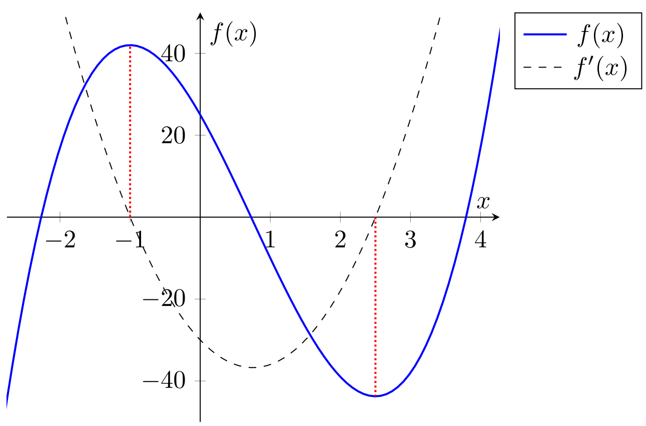
The value of the relative maximum is given by

$$f(-1) = 4(-1)^3 - 9(-1)^2 - 30(-1) + 25$$
$$f(-1) = 42$$

The value of the relative minimum is given by

$$f(\frac{5}{2}) = 4(\frac{5}{2})^3 - 9(\frac{5}{2})^2 - 30(\frac{5}{2}) + 25$$
$$f(\frac{5}{2}) = -\frac{175}{4}$$

Thus, there is a relative maximum point at $(-1, 42)$ and a relative minimum point at $(\frac{5}{2}, -\frac{175}{4})$



Note:-

Note that $f'(x) = 0$ where $f(x)$ has relative extrema. We summarize the behavior of this function by noting where it is increasing or decreasing and by characterizing its critical points

- f is increasing over the interval $(-\infty, -1)$
- f has a relative maximum point at $(-1, 42)$
- f is decreasing over the interval $(-1, \dfrac{5}{2})$
- f has a relative minimum point at $(\dfrac{5}{2}, -\dfrac{175}{4})$
- f is increasing over the interval $(\dfrac{5}{2}, \infty)$

To use the first derivative for graphing a function f

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but $f(x)$ is defined). Find $f(x)$ for each critical value
2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval
3. Find the sign of $f'(x)$ for each test value chosen in step 2, and use this information to determine where f is increasing or decreasing and to classify any extrema as relative maxima or minima
4. Plot some additional points and sketch the graph

Question 2

Find the relative extrema and sketch the graph of the function f given by

$$f(x) = 2x^3 - x^4$$

We first need to take the derivative

$$f'(x) = \frac{d}{dx}2x^3 - \frac{d}{dx}x^4$$
$$f'(x) = 6x^2 - 4x^3$$

Find the critical values

$$6x^2 - 4x^3$$
$$= 2x(3 - 2x)$$
$$2x^2 = 0 \quad 3 - 2x = 0$$
$$x = 0 \quad x = \frac{3}{2}$$

So, the intervals are

$$(-\infty, 0), \quad (0, \frac{3}{2}), \quad (\frac{3}{2}, \infty)$$

Choose test values within the intervals to see where its increasing / decreasing

$$(-\infty, 0): \text{ Test } -1, \quad f'(-1) = 6(-1)^2 - 4(-1)^3 = 6 + 4 = 10 > 0$$
$$(0, \frac{3}{2}): \text{ Test } 1 \quad f'(1) = 6(1)^2 - 4(1)^3 = 6 - 4 = 2 > 0$$
$$(\frac{3}{2}): \text{ Test } 2 \quad f'(2) = 6(2)^2 - 4(2)^3 = 24 - 32 = -8 < 0$$

The relative maxima / minima would be at the critical values: 0, $\frac{3}{2}$

Since f in increasing on both sides of 0, there is no extremum there. There is however, a maximum at $x = \frac{3}{2}$ Thus,

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4$$
$$= 2 \cdot \frac{27}{8} - \frac{81}{16}$$
$$\frac{108}{16} - \frac{81}{16} = \frac{27}{16}$$

So, there is a Relative maximum at

$$\left(\frac{3}{2}, \frac{27}{16}\right)$$

So,

- f is increasing over the interval $(-\infty, 0)$
- f has a critical value at x = 0, but the critical point (0,0) is neither a minimum nor a maximum
- f is increasing over the interval $(0, \frac{3}{2})$
- f has a relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$
- f is decreasing over the interval $(\frac{3}{2}, \infty)$

3.2 - Using Second Derivatives to Classify Maximum and Minimum Values and Sketch Graphs

The graphs of two continous functions are shown Below. The graph of f bends upwards and the graph of g bends downwards. Let's relate these observations to each function's derivative.

We draw tangent lines moving along the graph of f from left to right. What happens to the slopes of the tangent lines? We do the same for the graph of g. Is there a relationship between the changing slopes and the way the graph bends?