Chapter 3 Notes
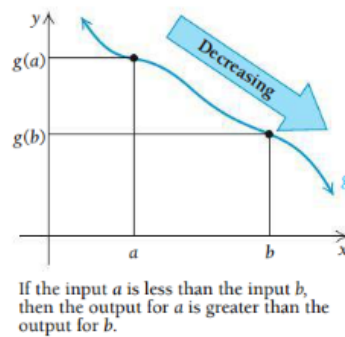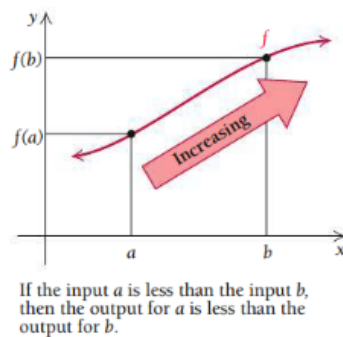
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3.1 - Using First Derivaties to Classify Maximum and Minimum Values

Increasing and Decreasing Functions

If the graph of a function rises from left to right over an interval I, the function is said to be increasing on, or over, I.

If the graph drops from left to right, the function is said to be decreasing on, or over, I.

If the input a is less than the input b, then the output for a is less than the output for b.

If the input a is less than the input b, then the output for a is greater than the output for b.

We can define these concepts as follows.

A function f is **increasing** over I if, for every a and b in I,

$$\text{if } a < b, \quad \text{then } f(a) < f(b)$$
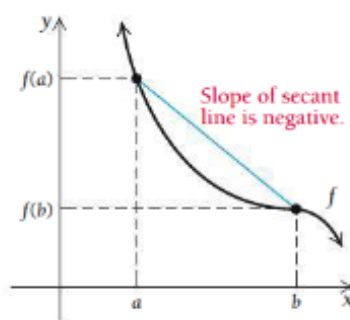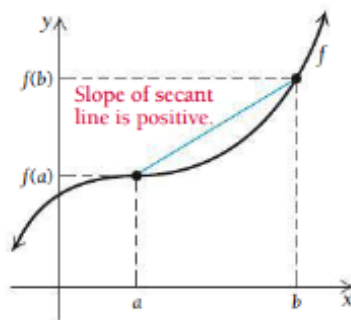
A function f is **decreasing** over I if, for every a and b in I,

$$\text{if } a < b, \quad \text{then } f(a) > f(b)$$

The above definitions can be restated in terms of slopes of secant lines

$$\text{Increasing: } \frac{f(b) - f(a)}{b - a} > 0 \qquad\qquad \text{Decreasing: } \frac{f(b) - f(a)}{b - a} < 0$$

Slope of secant line is positive.

Slope of secant line is negative.

Since the derivative of a function tells us the slope of the tangent line to f at any input x, we can also define an increasing or decreasing function using the derivative
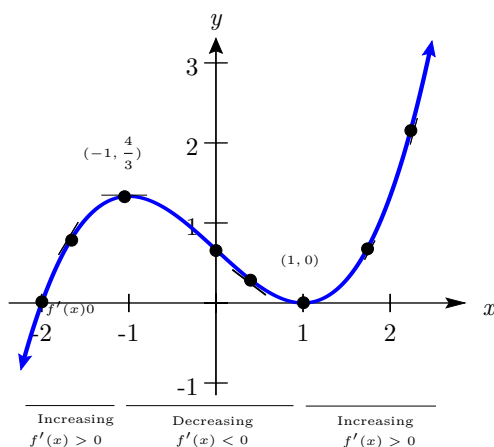
> **Theorem 0.1**
>
> Let f by differentiable over an open interval I
>
> If $f'(x) > 0$ for all x in I, then f is increasing over I
>
> If $f'(x) < 0$ for all x in I, then f is decreasing over I
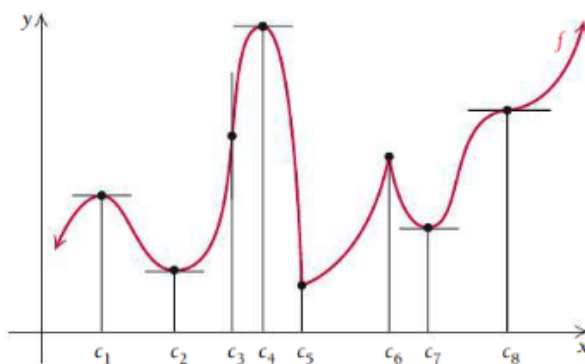
Theorem 0.1 is illustrated in the following graph of

$$f(x) = \frac{1}{3}x^3 - x + \frac{2}{3}$$

Note in the graph above that x = -1 and x = 1 are not included in any interval over which the function is increasing or decreasing. These values are examples of *critical values*

Critical Values

Consider the following graph

Note the following

1. $f'(x) = 0$ for $x = c_1, c_2, c_4, c_7,$ and c_8. That is, the tangent line to the graph is horizontal at these values.

2. $f'(x)$ does not exist for $x = c_3, c_5,$ and C_6. The tangent line is vertical at c_3, and there are corners at both c_5 and c_6.

A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is **horizontal** or for which the derivative does not exist.

That is, c is a critical value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

If c is a critical value of a function f, then $(c, f(c))$ is a **critical point**

Thus, in the graph of f above:

- c_1, c_2, c_4, c_7, c_8 are critical values because $f'(c) = 0$ for each value

- c_3, c_5, c_6 are critical values because $f'(c)$ does not exist for each value

Note:-

A continuous function can change from increasing to decreasing or from decreasing to increasing **only** at a critical value.

Finding Relative Maximum and Minimum Values

Now consider a graph with "peaks" and "valleys" at $x = c_1, c_2, c_3, c_4$, and c_5
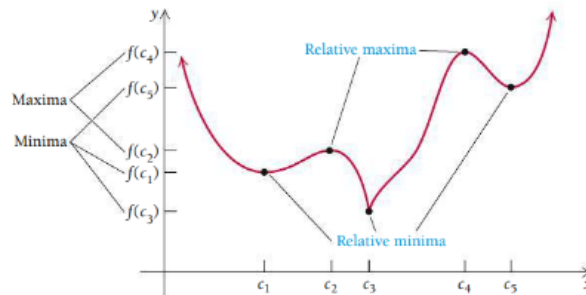
Here, $f(c_2)$ and $f(c_4)$ are each an example of a **relative**, or **local, maximum**, and $f(c_1), f(c_3)$, and $f(c_5)$ are each an example of a **relative**, or **local minimum**.

Collectively, maximum and minimum values are called **extrema**

Note:-

Note that it is possible for a relative minimum to be greater than a relative maximum.

For example, $f(c_5) > f(c_2)$ in the graph in the next page.

Also note that x-values at which a continuous function has relative extrema are those values for which the derivative is 0 or for which the derivative does not exist - the critical values

> **Theorem 0.2**
>
> If a function f has a relative extreme value $f(c)$ on an open interval, then c is a critical value, and
>
> $$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist}$$

Relative Extreme Points

A relative extreme point, $(c, f(c))$, is higher or lower than all other points over an open interval containing c.

Relative Minimum Point

A relative minimum point, $(c, f(c))$, is lower than all other points over an open interval containing c. Such a point has a y-value that is less than those of a neighborhood of points to the left and right of c.

Relative Maximum Point

Similarly, a relative maximum point, $(c, f(c))$, is higher than all other points over an open interval containing c. This maximum point has a y-value that is greater than those of a neighborhood of points to the left and right of c.

Note: In the preceding graph, $(c_1, f(c_1)), (c_3, f(c_3))$, and $(c_5, f(c_5))$ are all relative minimum points. Similarly, $(c_2, f(c_2))$ and $(c_4, f(c_4))$ are both relative maximum points.

Theorem 2

Theorem 2 is useful and important to understand. It states that to find relative extrema, we need only consider inputs for which the derivative is 0 or for which the derivative does not exist. Each critical value is a candidate for a value where a relative extremum might occur.

However, Theorem 2 does not guarantee that every critical value will yield a relative maximum or minimum. For instance, consider the graph of
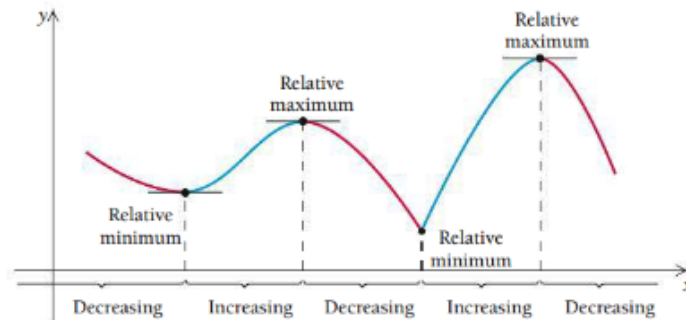
$$f(x) = (x - 1)^3 + 2$$

shown on the left. Note that:

$$f'(x) = 3(x - 1)^2 \quad \text{and} \quad f'(1) = 3(1 - 1)^2 = 0$$

Thus, $c = 1$ is a critical value, but f has no relative maximum or minimum at that value. In fact, this function has no extrema anywhere.
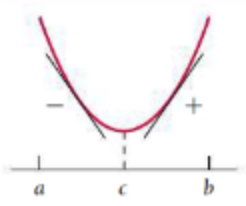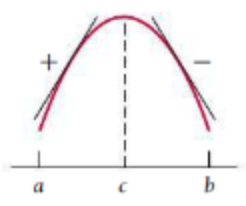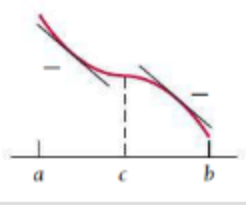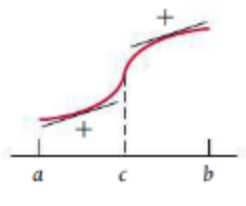
Theorem 2 does guarantee that if a relative maximum or minimum occurs, then the first coordinate of that extremum is a critical value.

The following graph leads us to a test.

Note that at a critical value where there is a relative minimum, the function f is **decreasing** on the left of the critical value and **increasing** on the right.

At a critical value where there is a relative maximum, the function f is **increasing** on the left of the critical value and **decreasing** on the right. In both cases, the derivative changes signs on either side of the critical value.

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	$-$	$+$	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	$+$	$-$	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	$-$	$-$	Decreasing on (a, b)
	No relative maxima or minima	$+$	$+$	Increasing on (a, b)

Derivatives can tell us when a function is increasing or decreasing. This leads us to the First-Derivative Test.

> **Theorem 0.3**
>
> For any continuous function f that has exactly one critical value c in an open interval (a, b)
>
> **F1.** f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c
>
> **F2** f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c
>
> **F3** f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) as on (c, b)
>
> We can use the First-Derivative Test to find relative extrema.

Question 1

Consider the function f given by
$$f(x) = 4x^3 - 9x^2 - 30x + 25$$

Find any relative exterma

Find the derivative

$$\frac{d}{dx}[4x^3] - \frac{d}{dx}[9x^2] - \frac{d}{dx}[30] + \frac{d}{dx}[25]$$

$$f'(x) = 12x^2 - 18x - 30$$

set f'(x) = 0

$$12x^2 - 18x - 30 = 0$$

divide both sides by 6

$$2x^2 - 3x - 5 = 0$$
$$(x + 1)(2x - 5) = 0$$
$$x = -1 \quad or \quad x = \frac{5}{2}$$

The critical values are -1 and $\frac{5}{2}$. Since it is at these values that a relative maximum or minimum might exist, we examine the sign of the derivative on the intervals

$$(-\infty, -1), (-1, \frac{5}{2}), (\frac{5}{2}, \infty)$$

To do so, we select a convenient test value in each interval and evalulate f'(x).

Let's use the values: -2, 0, 4

$$f'(-2) = 54$$
$$f(0) = -30$$
$$f(4) = 90$$

Result:

f is increasing on $(-\infty, -1)$
f is decreasing on $(-1, \frac{5}{2})$
f is increasing on $(\frac{5}{2}, \infty)$

By the First-Derivative Test, f has a relative maximum at $x = -1$ and a relative minimum at $x = \dfrac{5}{2}$
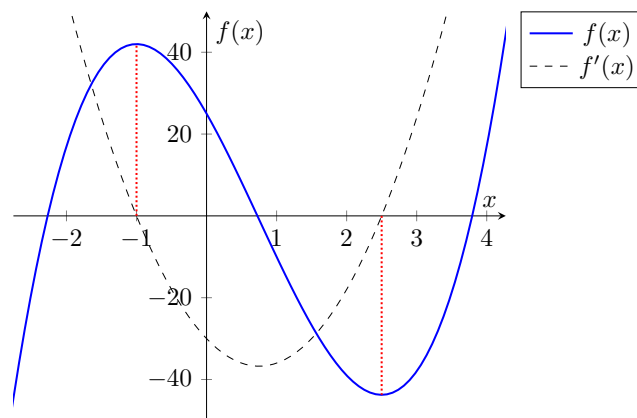
The value of the relative maximum is given by

$$f(-1) = 4(-1)^3 - 9(-1)^2 - 30(-1) + 25$$

$$f(-1) = 42$$

The value of the relative minimum is given by

$$f(\tfrac{5}{2}) = 4(\tfrac{5}{2})^3 - 9(\tfrac{5}{2})^2 - 30(\tfrac{5}{2}) + 25$$

$$f(\tfrac{5}{2}) = -\frac{175}{4}$$

Thus, there is a relative maximum point at $(-1, 42)$ and a relative minimum point at $(\frac{5}{2}, -\frac{175}{4})$

Note:-

Note that $f'(x) = 0$ where $f(x)$ has relative extrema. We summarize the behavior of this function by noting where it is increasing or decreasing and by characterizing its critical points

- f is increasing over the interval $(-\infty, -1)$

- f has a relative maximum point at $(-1, 42)$

- f is decreasing over the interval $(-1, \dfrac{5}{2})$

- f has a relative minimum point at $(\dfrac{5}{2}, -\dfrac{175}{4})$

- f is increasing over the interval $(\dfrac{5}{2}, \infty)$

To use the first derivative for graphing a function f

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but $f(x)$ is defined). Find $f(x)$ for each critical value

2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval

3. Find the sign of $f'(x)$ for each test value chosen in step 2, and use this information to determine where f is increasing or decreasing and to classify any extrema as relative maxima or minima

4. Plot some additional points and sketch the graph

Question 2

Find the relative extrema and sketch the graph of the function f given by

$$f(x) = 2x^3 - x^4$$

We first need to take the derivative

$$f'(x) = \frac{d}{dx}2x^3 - \frac{d}{dx}x^4$$

$$f'(x) = 6x^2 - 4x^3$$

Find the critical values

$$6x^2 - 4x^3$$
$$= 2x(3 - 2x)$$
$$2x^2 = 0 \quad 3 - 2x = 0$$
$$x = 0 \quad x = \frac{3}{2}$$

So, the intervals are

$$(-\infty, 0), \quad (0, \frac{3}{2}), \quad (\frac{3}{2}, \infty)$$

Choose test values within the intervals to see where its increasing / decreasing

$$(-\infty, 0): \text{ Test } -1, \quad f'(-1) = 6(-1)^2 - 4(-1)^3 = 6 + 4 = 10 > 0$$

$$(0, \frac{3}{2}): \text{ Test } 1 \quad f'(1) = 6(1)^2 - 4(1)^3 = 6 - 4 = 2 > 0$$

$$(\frac{3}{2}): \text{ Test } 2 \quad f'(2) = 6(2)^2 - 4(2)^3 = 24 - 32 = -8 < 0$$

The relative maxima / minima would be at the critical values: $0, \frac{3}{2}$

Since f in increasing on both sides of 0, there is no extremum there. There is however, a maximum at $x = \frac{3}{2}$
Thus,

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4$$

$$= 2 \cdot \frac{27}{8} - \frac{81}{16}$$

$$\frac{108}{16} - \frac{81}{16} = \frac{27}{16}$$

So, there is a Relative maximum at

$$\left(\frac{3}{2}, \frac{27}{16}\right)$$

So,

- f is increasing over the interval $(-\infty, 0)$

- f has a critical value at x = 0, but the critical point $(0,0)$ is neither a minimum nor a maximum

- f is increasing over the interval $(0, \frac{3}{2})$

- f has a relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$

- f is decreasing over the interval $(\frac{3}{2}, \infty)$

3.2 - Using Second Derivatives to Classify Maximum and Minimum Values and Sketch Graphs

The graphs of two continous functions are shown Below. The graph of f bends upwards and the graph of g bends downwards. Let's relate these observations to each function's derivative.

We draw tangent lines moving along the graph of f from left to right. What happens to the slopes of the tangent lines? We do the same for the graph of g. Is there a relationship between the changing slopes and the way the graph bends?
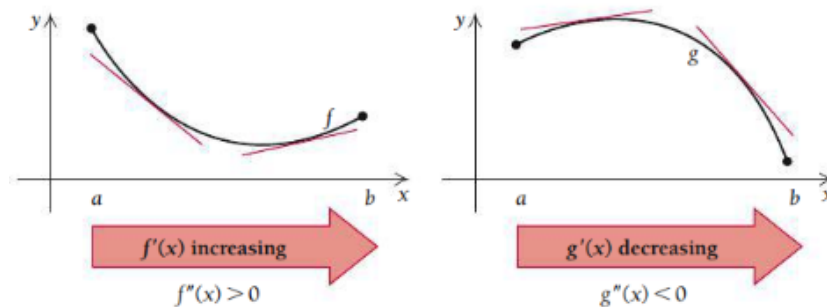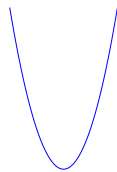
Concavity: Increasing and Decreasing Derivatives

For the graph of f, the slopes of the tangent lines are increasing. That is, f' is increasing over the interval. This can be determined by noting that $f''(x)$ is positive, since the relationship between f' and f'' is like the relationship between f and f'.
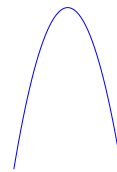
For the graph of g, the slopes are decreasing. This can be determined by noting that g' is decreasing whenever $g''(x)$ is negative. The bending of each curve is called the graphs **concavity**

Suppose that f is a function whose derivative f' exists at every point in an open interval I. Then

f is **concave up** on I if	f is **concave down on I if**
f' is increasing over I	f' is decreasing over I

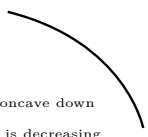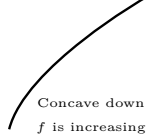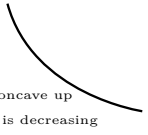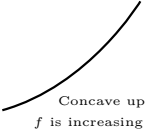The following theorem states how concavity and the second derivative are related

Theorem 0.4

1. If $f''(x) > 0$ for all x on an open interval I, then the graph of f is concave up on I

2. If $f''(x) < 0$ for all x on an open interval I, then the graph of f is concave down on I

A function can be

- decreasing and concave up,

- Decreasing and concave down,

- Increasing and concave up

- Or, increasing and concave down.

That is, **concavity** and increasing/decreasing are seperate concepts. It is the increasing or decreasing aspect of the derivative that tells us about the functions concavity.

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) < 0$	Concave down f is decreasing	Concave down f is increasing
$f''(x) > 0$	Concave up f is decreasing	Concave up f is increasing

Classifying Relative Extrema Using Second Derivatives

Theorem 0.5

Suppose that f is differentiable for every x in an open interval (a,b) and that there is a critical value c in (a,b) for which $f'(c) = 0$. Then:

1. $f(c)$ is a relative minimum if $f''(c) > 0$

2. $f(c)$ is a relative maximum if $f''(c) < 0$

For $f''(c) = 0$, the First-Derivative Test can be used to determine whether $f(x)$ is a relative extremum
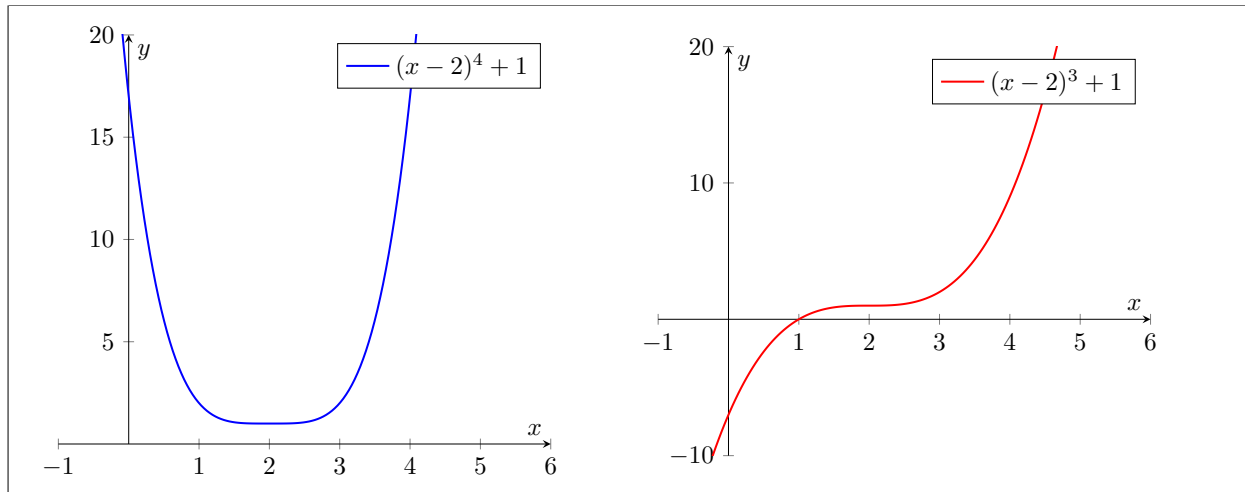
Notice that the graph on the left has an extremum at c = 2 and the other function does not. In cases like this, An approach other than the Second-Derivative Test must be used to determine whether $f(c)$ is an extremum

Question 3

Find any relative extrema and graph the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13$$

Take the first derivative

$$f'(x) = 3x^2 + 6x - 9$$

To find any critical values, we solve $f'(x) = 0$

$$3x^2 + 6x - 9 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3 \quad or \quad x = 1$$

Sub the critical values in the original function

$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14$$

$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18$$

Next, we find the second derivative
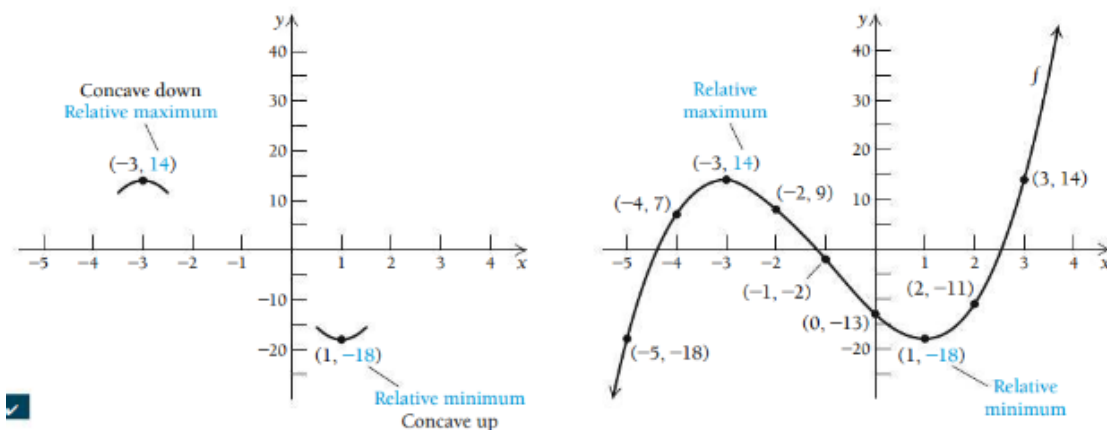$$f''(x) = 6x + 6$$

We use the Second-Derivative Test with the critical values

$$f''(-3) = 6(-3) + 6 = -12 < 0$$

$$f''(1) = 6(1) + 6 = 12 > 0$$

Thus, $f(-3) = 14$ is a relative maximum and $f(1) = -18$ is a relative minimum.

We plot both $(-3, 14)$ and $(1, -18)$, including short arcs at each point to indicate the concavity

Question 4

Find any relative extrema and graph the function k given by

$$k(x) = e^{2x} - e^x$$

Take the first derivative

$$k'(x) = 2e^{2x} - e^x$$

$$e^x(2e^x - 1) = 0$$

$$2e^x - 1 = 0$$

$$e^x = \frac{1}{2}$$

Take the natural log of both sides

$$\ln e^x = \ln \frac{1}{2}$$

$$x = \ln 0.5$$

Sub in the critical value into the original function

$$k(\ln 0.5) = e^{2(\ln 0.5)} - e^{\ln 0.5}$$

$$e^{\ln 0.5^2} - e^{\ln 0.5}$$

$$= 0.5^2 - 0.5$$

$$0.25 - 0.5$$

$$= -0.25$$

Find the second derivative
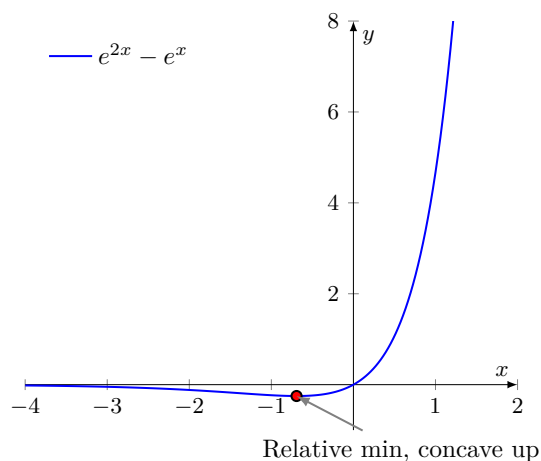
$$k''(x) = 4e^{2x} - e^x$$

$$k''(\ln 0.5) = 4e^{2(\ln 0.5)} - e^{\ln 0.5}$$

$$4e^{\ln 0.5^2} - e^{\ln 0.5}$$

$$= 4(0.25) - 0.5$$

$$= 0.5 > 0$$

Since the second derivative is positive at $\ln 0.5$, we conclude that $k(\ln 0.5) = -0.25$ is a relative minimum

Relative min, concave up

Points of Inflection

A point at which concavity changes from down to up or from up to down is called a point of inflection, or an inflection point. In the figures below, point P is an inflection point. The figures display the sign of $f''(x)$ to indicate the concavity on either side of P
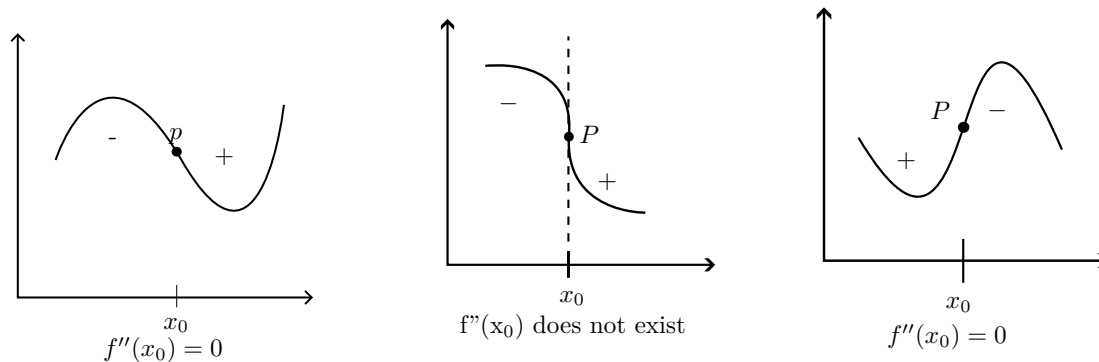
As we move along each curve, the concavity changes at P. Since the sign of $f''(x)$ changes, the value of $f''(x_0)$ at P either must be 0 or must not exist.

Theorem 0.6

If f has a point of inflection, it must occur at a value x_0, in the domain of f, where

$$f''(x_0) = 0 \quad or \quad f''(x_0) \text{ does not exist}$$

Question 5

Find the points of inflection on the graphs of

$$f(x) = x^3 + 3x^2 - 9x - 13$$

Find the second derivative

$$f''(x) = 6x + 6$$

Set equal to 0

$$6x + 6 = 0$$

$$x = -1$$

Test values on both intervals

$$\text{Interval } (-\infty, -1): \quad \text{Test } -2: \quad f''(-2) = 6(-2) + 6 = -6 < 0$$

$$\text{Interval } (-1, \infty): \quad \text{Test } 0: \quad f''(0) = 6(0) + 6 = 6 > 0$$

f is concave down on $(-\infty, -1)$

f is concave up on $(-1, \infty)$

The sign of $f''(x)$ changes on either side of $x = -1$, so $(-1, -2)$ is a point of inflection

> **Note:-**
>
> The first and second derivatives enhance our ability to sketch curves. We use the following strategy:

Strategy for Sketching Graphs

(a) **Derivatives and domain.** Find $f''(x)$ and $f'(x)$. Note the domain of f.

(b) **Critical values of f.** Find the critical values by solving $f''(x) = 0$ and by finding where $f''(x)$ does not exist. These numbers yield candidates for relative maxima or minima. Find the function values at these points.

(c) **Increasing and/or decreasing; relative extrema.** Substitute each critical value, x_0, from step (b) into $f'(x)$. If $f'(x_0) > 0$, then $f(x_0)$ is a relative maximum and f is increasing on the left of x_0 and decreasing on the right. If $f'(x_0) < 0$, then $f(x_0)$ is a relative minimum and f is decreasing on the left of x_0 and increasing on the right.

(d) **Inflection points.** Determine candidates for inflection points by finding where $f'(x) = 0$ or where $f'(x)$ does not exist. Find the function values at any such points.

(e) **Concavity.** Use any candidates for inflection points from step (d) to define intervals. Substitute test values into $f'(x)$ to determine where the graph is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$). Step (c) may have provided some of this information.

(f) **Sketch the graph.** Sketch the graph using the information from steps (a)–(e), plotting extra points as needed.

Question 6

Find any relative extrema and inflection points and graph

Take the first and second derivative and find the domains

$$f'(x) = 4x^3 - 4x$$

$$f''(x) = 12x^2 - 4$$

The domain of f f' and f'' is $\mathbb{R}$

Find the critical values

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$4x = 0 \quad or \quad x^2 - 1 = 0$$

$$x = 0 \quad or \quad x = \pm 1$$

find the critical points

$$f(-1) = (-1, -1)$$

$$f(0) = (0, 0)$$

$$f(1) = (1, -1)$$

So, we have the intervals

$$(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$$

Now we can find where the function is increasing and decreasing, as well as concavity and relative max and min by substituting the critical points into f"

$$f''(-1) = 8 > 0$$

Since $f''(-1) = 8 > 0$, we see that f is decreasing on the left over then interval $(-\infty, -1)$ and increasing on $(-1, 0)$, This shows us that $(-1, -1)$ is a relative min. And because the value we obtained from $f''(-1)$ is positive, it is also concave up ***Using the same logic for the other 2 critical values we get***

$$f''(0) = -4 < 0$$

So, f is increasing over the interval $(-1, 0)$, and decreasing over the interval $(0, 1)$. It is a relative max, and concave down.

$$f''(1) = 8 > 0$$

So, f is decreasing over the interval $(0, 1)$, and increasing over the interval $(1, \infty)$. So, the critical point $(1, -1)$ is a relative min, and concave up

Next, we find the points of inflection by setting f" equal to 0

$$12x^2 - 4 = 0$$

$$12x^2 = 4$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}}$$

plug in these x values into the original function to find the y values

$$f\left(\sqrt{\frac{1}{3}}\right) = \left(\sqrt{\frac{1}{3}}\right)^4 - 2\left(\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$

$$f\left(-\sqrt{\frac{1}{3}}\right) = \left(-\sqrt{\frac{1}{3}}\right)^4 - 2\left(-\sqrt{\frac{1}{3}}\right)^2 = \frac{1}{9} - \frac{2}{3} = -\frac{5}{9}$$

So, our possible inflection points are

$$\left(\sqrt{\frac{1}{3}}, -\frac{5}{9}\right)\left(-\sqrt{\frac{1}{3}}, -\frac{5}{9}\right)$$

So, Since the points of inflection are where the concavity switches, using the information obtained when we found concavity. We can conclude that f is

$$\text{Concave up on } \left(-\infty, -\sqrt{\frac{1}{3}}\right)$$

$$\text{Concave down on } \left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$$

$$\text{Concave up on } \left(\sqrt{\frac{1}{3}}, \infty\right)$$
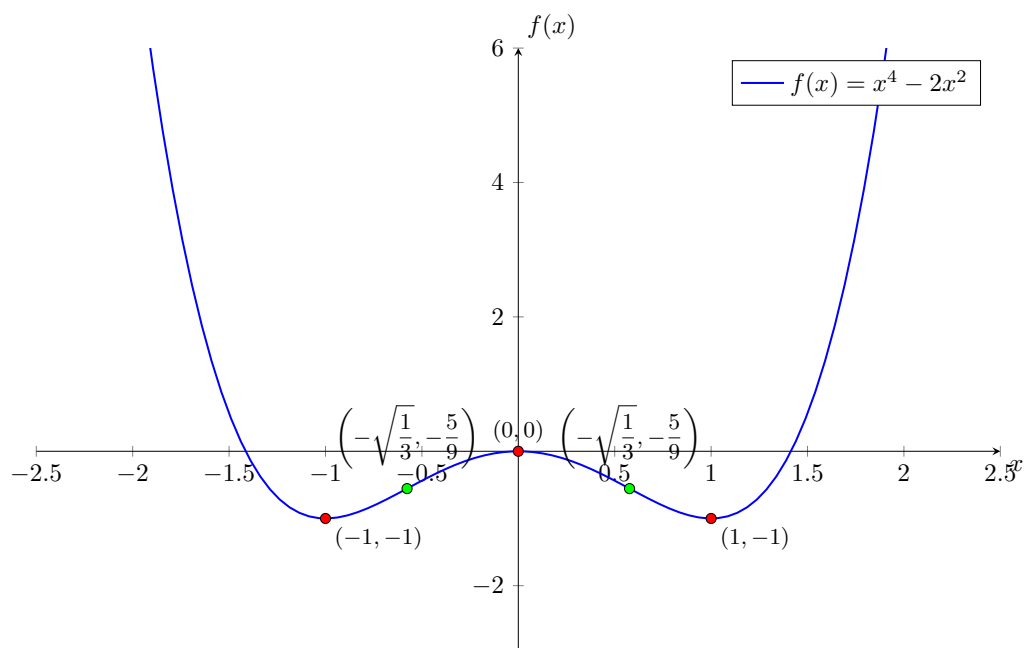
3.5 - Optimization: Business, Economics, and General Applications

Question 7

Maximizing Area. The Hobby Farm has 20 ft of fencing to fence off a rectangular area for an electric train in one corner of its display room. The two sides up against the wall require no fence. What dimensions of the rectangle will maximize the area? What is the maximum area?

We let x be the length of one side of the rectangle and y the length of the other side. The area A of the triangle formed by the fencing is given by the objective function.

$$A = xy$$

We are limited in the amount of fencing available: This is the constaint. Since 20 ft of fencing is available, the two side lengths x and y are related by the equation

$$x + y = 20, \quad \text{or} \quad y = 20 - x$$

x

$$y = 20 - x$$
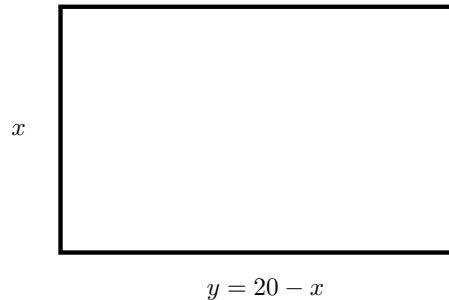
The practical domain is $0 \leqslant x \leqslant 20$, since the amount of fencing used for one side cannot be negative or greater than 20ft.

$$8ft \cdot 12ft = 96ft^2$$
$$9ft \cdot 11ft = 99ft^2$$
$$10ft \cdot 10ft = 100ft^2$$
$$11ft \cdot 9ft = 99ft^2$$
$$12ft \cdot 8ft = 96ft^2$$

Based on this, it appears that letting $x = 10$ ft will result in a maxium area of 100ft^2. To be sure, we use calculus.

Substituting the constraint $y = 20 - x$ into the formula for area, we have

$$A = xy$$
$$A = x(20 - x)$$
$$A(x) = 20x - 2x$$

Thus, we want to maximize

$$A(x) = 20x - x^2 \text{ over } [0, 20]$$

We find $A'(x)$

$$A'(x) = 20 - 2x$$

This derivative exists for all x in $(0, 20)$. The only critical values are where $A'(x) = 0$

$$20 - 2x = 0$$
$$x = 10$$
$$\boxed{A(10) = 100}$$

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Thus, the point $(10, 100)$ is a critical point on the graph of A. The student can confirm that, the endpoints $x = 0$ and $x = 20$ both result in a minimum area of 0ft^2. We use the Second-Derivative Test to classify $(10, 100)$ as a relative maximum or a relative minimum. The second derivative is

$$A''(x) = -2$$

This means that the graph of A is concave down for all x, so $(10, 100)$ is a relative maximum point. By Maximum-Minimum Principle 1, we conclude that the *absolute* maximum value of A over the interval $[0, 20]$ occurs at $x = 10$

The Hobby Farm should use 10ft of fencing on one side and 10ft on the other, to achieve the largest possible area of 100ft^2. Using calculus has shown that this time, our original intuition of using 10 feet of fencing to a side was correct

Here is a general strategy for solving maximum-minimum problems.

A Strategy for Solving Maximum-Minimum Problems

1. Read the problem carefully. Make a drawing, and label any dimensions, if possible.

2. Make a list of appropriate variables and constraints, noting what varies, what stays fixed, and what units are being used. Label the measurements on your drawing.

3. Translate the problem to an objective function f that expresses the quantity to be maximized or minimized. Represent f in terms of the varibles of step 2.

4. Use algebra and Substitution as needed to express f as a function of one variable

5. Use the procedures developed in Sections 3.1-3.4 to determine the maximum or minimum values and the points at which they occur

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Maximizing Volume. From a sheet of cardboard, 8in. by 11in., squares are cut out at the corners so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?
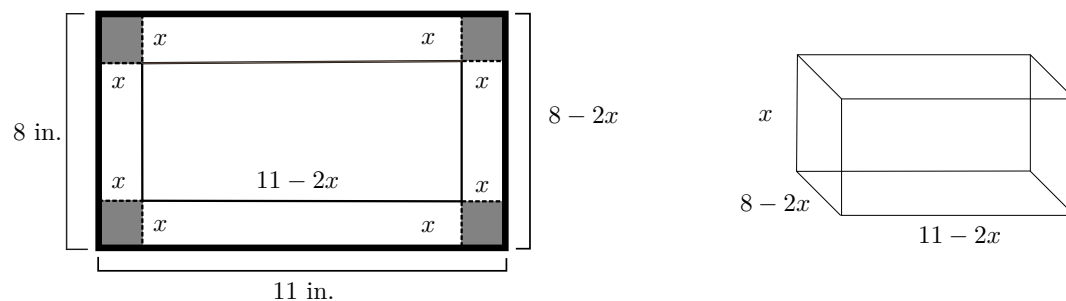
After the four squares are removed and the sides are folded up, the volume V of the resulting box is

$$V = l \cdot w \cdot h = (11 - 2x) \cdot (8 - 2x) \cdot x$$

We have $V = l \cdot w \cdot h = (11 - 2x) \cdot (8 - 2x) \cdot x$. Multiplying gives us the objective function.

$$V(x) = 4x^3 - 38x^2 + 88x$$

The amount cut out at each corner cannot be negative, nor can it be more than half of the 8-in. side length. Thus, the domain is

$$0 \leqslant x \leqslant 4$$

Note that for x = 0 or x = 4, the volume is 0: In the first case, the box has no height, and in the second case, the box has no width.

To determine critical values, we find $V'(x)$:

$$V'(x) = 12x^2 - 76x + 88$$

Since $V'(x)$ is defined for all x in the interval (0,4), we set $V'(x) = 0$ to find any critical values. Using the quadratic formula, we have

$$x = \frac{-(-76) \pm \sqrt{(-76)^2 - 4(12)(88)}}{2(12)} = \frac{76 \pm \sqrt{1552}}{24}$$

Using a calculator and rounding to three decimal places, we have

$$x = \frac{76 - \sqrt{1552}}{24} \approx 1.525 \quad \text{and} \quad x = \frac{76 + \sqrt{1552}}{24} \approx 4.808$$

However, the second value is not in the domain of the objective function, so the only critical value is 1.525. Thus, the point

$$(1.525, V(1.525))$$

Is a critical point.

To classify this critical point, we use the Second-Derivative Test. The second derivative is

$$V''(x) = 24x - 76$$

Evaluating at x = 1.525, we obtain

$$V''(1.525) = 24(1.525) - 76 = -39.4 < 0$$

Since $V''(1.525)$ is negative, the graph of V is concave down, indicating that $(1.525, V(1.525))$ is a relative maximum point over the interval [0,4]. Since we have already determined that both x = 0, and x = 4 (the endpoints) give zero volume, by Maximum-Minimum Principle 1, we conclude that $(1.525, V(1.525))$ must also be the **absolute** maximum point over [0,4]

Thus, to maximize the volume of the box, we should cut a square of side length x = 1.525 in. from each corner, then fold up the flaps. The box will have a maximum volume of

$$V(1.525) = 4(1.525)^3 - 38(1.525)^2 + 88(1.525) = 60.0126\text{i n}^3$$

Question 8

Minimizing Cost of Material. Mendoza Manufacturers specializes in food-storage containers and makes a cylindrical soup can with a volume of 250 cm^3. The cost of material for the two circular ends is \$0.0008/cm^2, and the cost of material for the side is \$0.0015/cm^2. What dimensions minimize the cost of material for the soup can? What is the minimum cost?