Penrose Tilings in Medieval Islamic Culture

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Content

List of Figures

Abbreviation

Fig. Figure

1 Introduction

The original Penrose tiling was proposed in 1974 in a paper entitled the *Role of aesthetics in pure and applied research*.[1]

The specialness of a Penrose tiling is that it is a nonperiodic tiling, which means that it lacks any translational symmetry. More precisely, a shifted copy will never match the original exactly. For all that any finite region in a Penrose tiling appears infinitely many times in that tiling and, in fact, in any other tiling. This property would be trivially true of a tiling with translational symmetry but is non-trivial when applied to the non-periodic Penrose tilings.

A Penrose tiling is also a quasi-crystalline with 5 fold symmetry, what was thought to be impossible till the early 1960s.

A tiling can be generated by an aperiodic set of prototiles named after Sir Roger Penrose, who investigated these sets in the 1970s [1 & 4]. The most important property of a Penrose tiling is that it is inflatable, which is not common for aperiodic patterns. Inflatable means to replace the original tiles of a pattern by bigger tiles of the same shape in a special procedure, so that one gets an augmentation of a part of the original pattern.

Girih tiles are a set of five tiles that were used in the creation of tiling patterns for decoration of buildings in Islamic architecture. They are known to have been used since about the year 1200 and their arrangements found significant improvement starting with the Darb-i Imam shrine in Isfahan in Iran built in 1453.

Recent discoveries of Islamic tessellations from the Darb-i Imam shrine, Isfahan, Iran (1453 C.E.) made by Peter J. Lu & Paul J. Steinhardt (2007), raise the question, if Islamic architects have already been able to construct these complex Penrose patterns in the Middle Ages, over 5 centuries before their discovery in the West? The answer to this question is tried to be given in this work.

2 Work and Discovery of Peter J. Lu and Paul J. Steinhardt (2007)

Peter J. Lu and Paul J. Steinhardt discovered that the Darb-i Imam shrine pattern (Fig.1) can be fragmented in many tiles of a girih tile set (Fig. 3), which consists of the decagon, the bowtie and the rhombus, which are decorated with lines, which are called girih lines [2 & 6].

One can subdivide each of the three girih tiles by the Penrose tiles dart and kite (Fig. 2), as shown in Fig. 3. After these steps one can obtain the pattern shown in Fig. 4, which consists of 3700 Penrose tiles and 11 mismatches which have been determined and corrected by Peter J. Lu and Paul J. Steinhardt. These defects could have resulted by inadvertent constructions or repairing by the architects.

Fig. 1: [2] ***Photograph of the right half of the spandrel.***

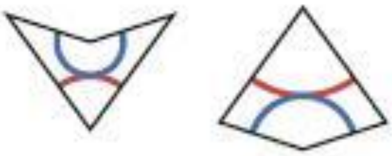


Fig. 2:[2] Penrose dart & kite.

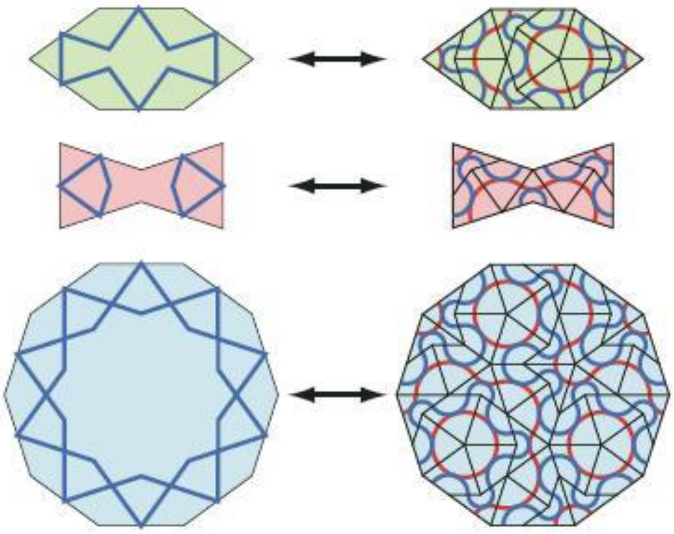


Fig. 3:[2] Generation of rhombus, bowtie & decagon

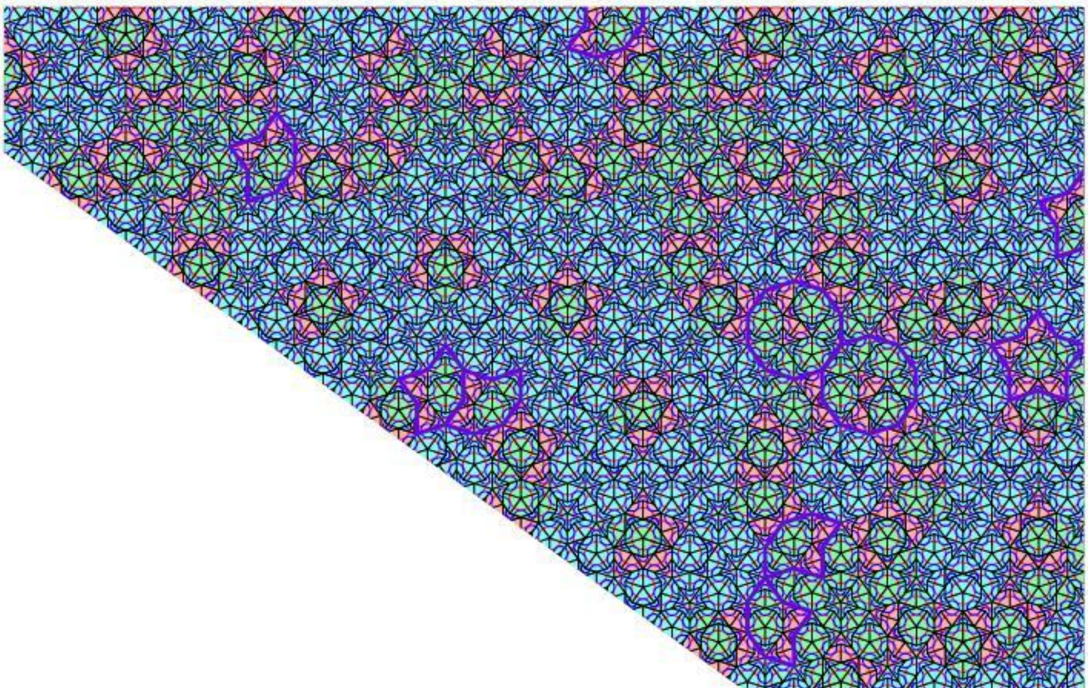


Fig. 4:[2] Mapping of pattern with girih tiles and corrections of missmatches

3 Transfer of Penrose tiles

There are different set of tiles to construct a Penrose tiling. One is kite and dart, as was described in the chapter before, another one is thick and thin rhombus. The next two sections will describe how one may pass from one set to the other and back.

3.1 Transfer from kite and dart to thick and thin rhombus

The first step for the transformation is to erase the decorating lines of kite and dart. After that you have to put a thin rhombus in the kite, in the way that it becomes a thin rhombus and a dart. The third step is, to bisect each dart symmetrically with a straight line segment. As fourth and last step the dart halves have to be combined, so that a thick rhombus appears. For easier understanding the four steps are visualized in Fig. 5.

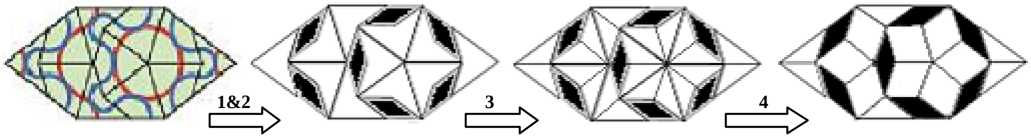


Fig. 5: Tile transformation in rhombus as an example

At the end of the transformation, black respectively white rhombuses represent thin respectively thick rhombuses.

3.2 Transfer from thick and thin rhombus to kite and dart

The transformation from thick and thin rhombus to kite and dart starts with bisecting the thick rhombuses along their longer diagonal line. The next step is to combine two thick rhombus halves, in the way that you get a dart. Now every thin rhombus, which lies beyond a dart synthesizes with it and becomes a kite. The last step cannot generate fewer darts as there were at the beginning of the first transformation, because of the arrangements of the thin rhombuses. This transfer can be seen in Fig. 5 from right to left.

3.3 Transfer of Penrose tiles starting with triangles

Another transfer of tiles can be seen if one takes a pentagon and intersects it along two diagonals, which start at the same brink. Another intersection is made along the parallel of the brink, so that one gets a point of intersection, where three lines get-together. After this, one gets two big distinct equal-sided triangles (colored green respectively red), which include two respectively three small triangles (colored blue). The three small triangles within the big triangle (colored red) can be deflated to smaller ones by replacing each triangle with its corresponding one [3]. If one takes a closer look at this generation (Fig. 6) one can easily recognize the other two sets of Penrose tiles, thick and thin rhombus and dart and kite [5].

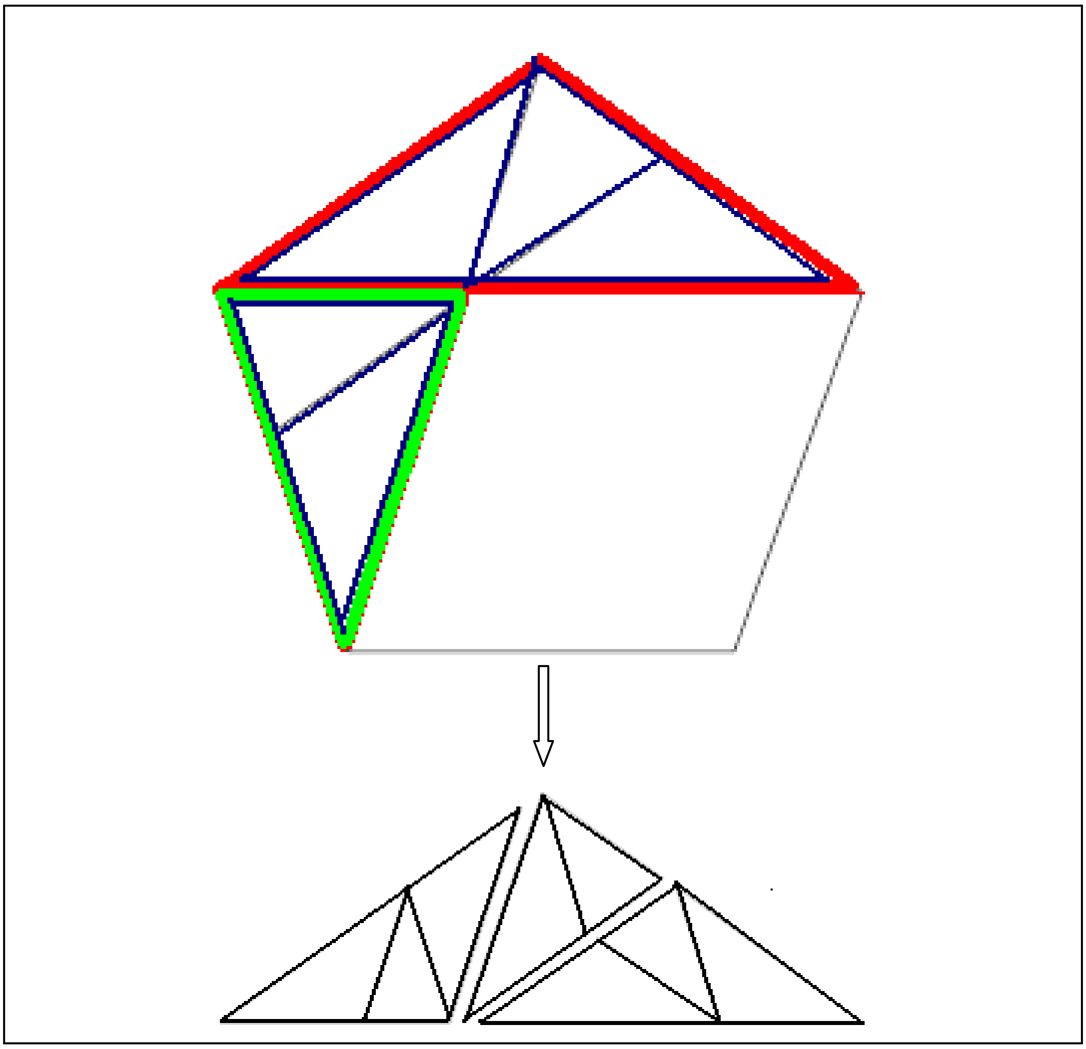


Fig. 6:[3] Generation of triangles

4 Analysis of the Darb-i Imam shrine pattern

4.1 Inflation of the pattern

A safe way to generate a real Penrose tiling is using deflation of kite or dart. In Fig. 7 one can see a deflation using a kite half, a dart half, a sun consisting of kites and a star consisting of darts, as an example.

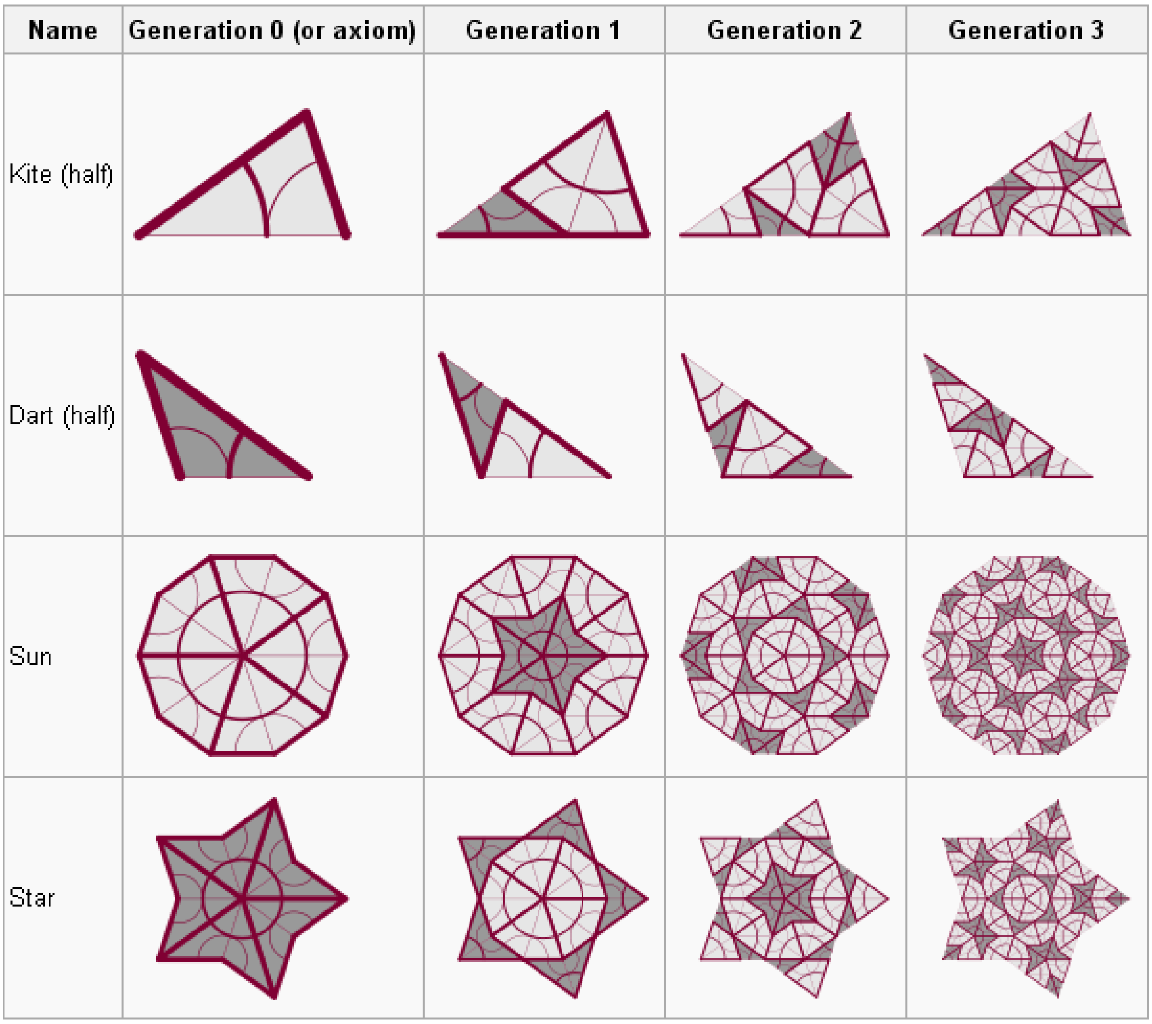


Fig. 7:[1] Generation of Penrose tiling using deflation

One gets to the first order generation by subdividing a kite half into a small kite and small dart half, respectively by subdividing a dart half into a small kite half and a small dart half. Hence every Penrose tiling must also be inflatable. This is shown by generating the first, second, third and fourth order inflation of the Darb-i Imam pattern exposed in Fig. 8. Bigger pictures of the first and second order inflation are attached at the end of my work.

To get to an inflation of the pattern each dart has to be bisected. The next step is to combine two dart halves which were not a dart before, plus one respectively two kites, so that one gets a bigger dart respectively a bigger kite. This process can be seen from right to left at the last two rows of Fig 7. Inflation can be seen as a subdivision of tiles into smaller tiles of the same shape so it is the exact opposite of a deflation.

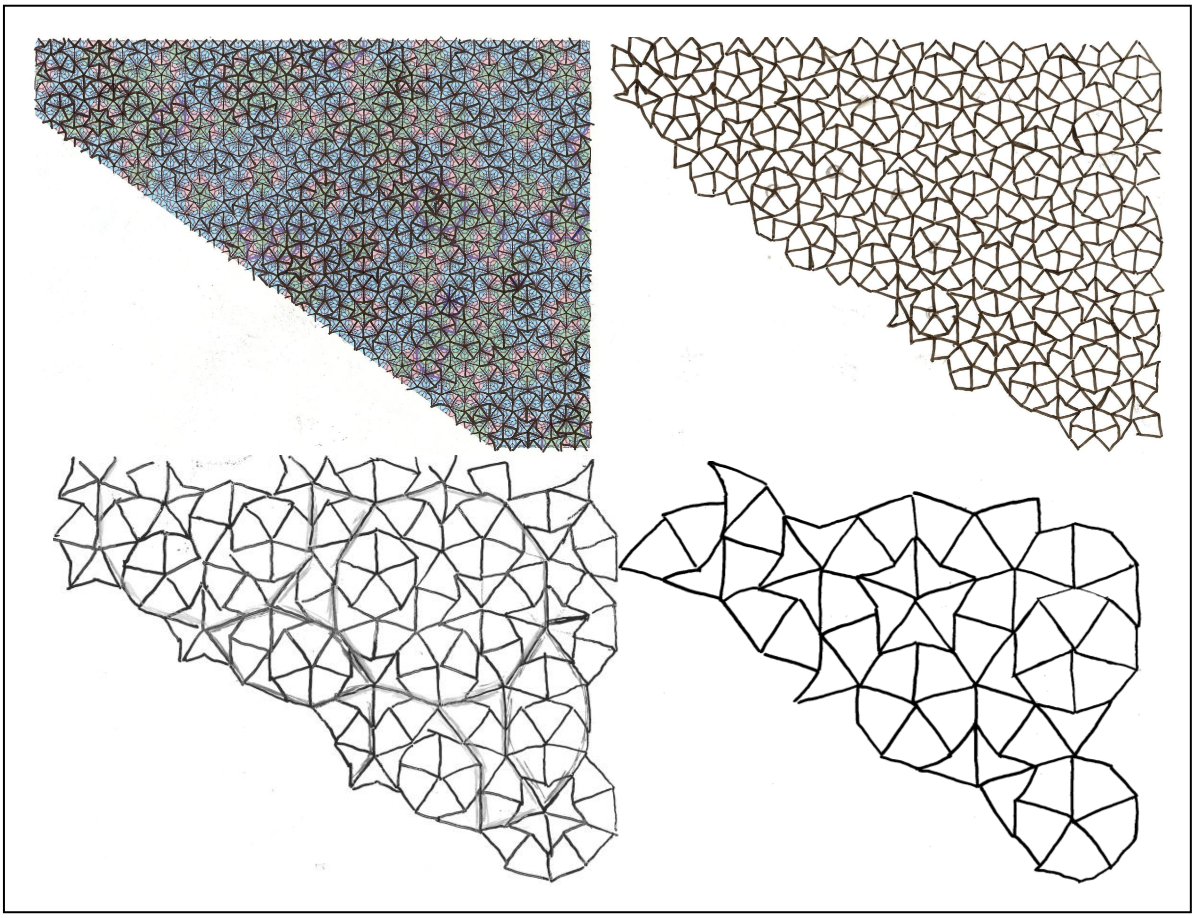


Fig. 8: 1st, 2nd, 3rd and 4th order inflations of the Darb-i Imam pattern

In each of these inflations one can map the 3 girih tiles, which can be detected in the original shrine pattern. With the inflation of the pattern it is shown that the inflatability of the pattern is given which is an important feature for a Penrose tiling. Bigger figures of the first and second deflation for better recognition are attached at the end of this work.

4.2 Mapping the pattern with other Penrose tiles

If a Penrose tiling consists of rhombuses it is much easier to detect infringements of the pattern. Hence if the pattern consists of other tiles one should make a transformation. Starting with the transformation of the decagons, by mapping three thick rhombuses at the location where the two bowties are lying beyond each-other. Then one has to map an additional thick rhombus in the center below so one gets a decagon, which consists of 5 thick and 4 thin rhombuses (Fig. 9). Where crossings of five bowties occur one gets a star (Fig. 10). Combining this transformations for the whole pattern one gets to the pattern of Fig. 11, which looks very much penroselike. This transformation is equal to an inflation of second order of the pattern combined with a transfer from kite and dart to thick and thin rhombuses.

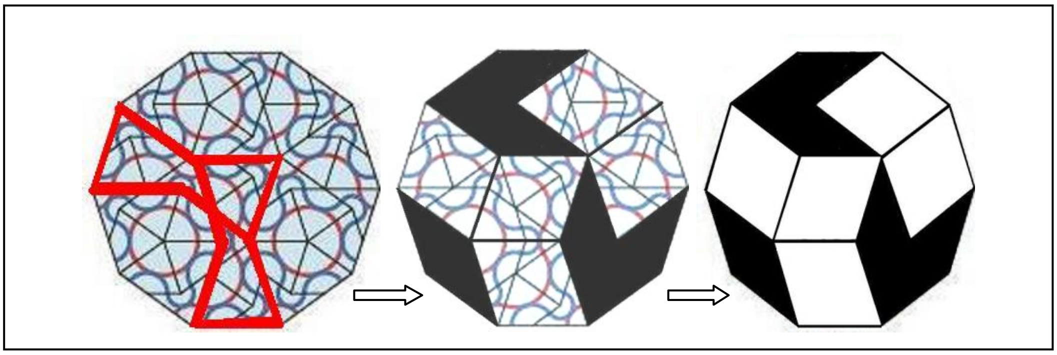


Fig. 9: Transformation of decagon with two intersecting bowties

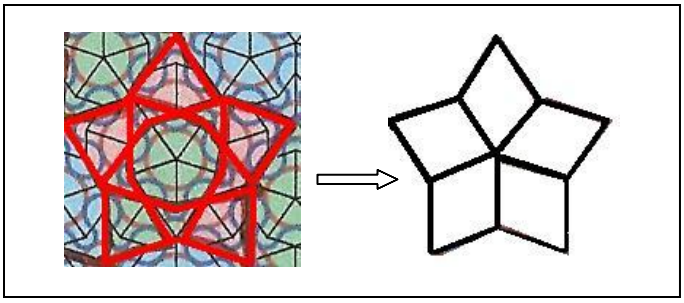


Fig. 10: Transformation of a star with five crossing bowties

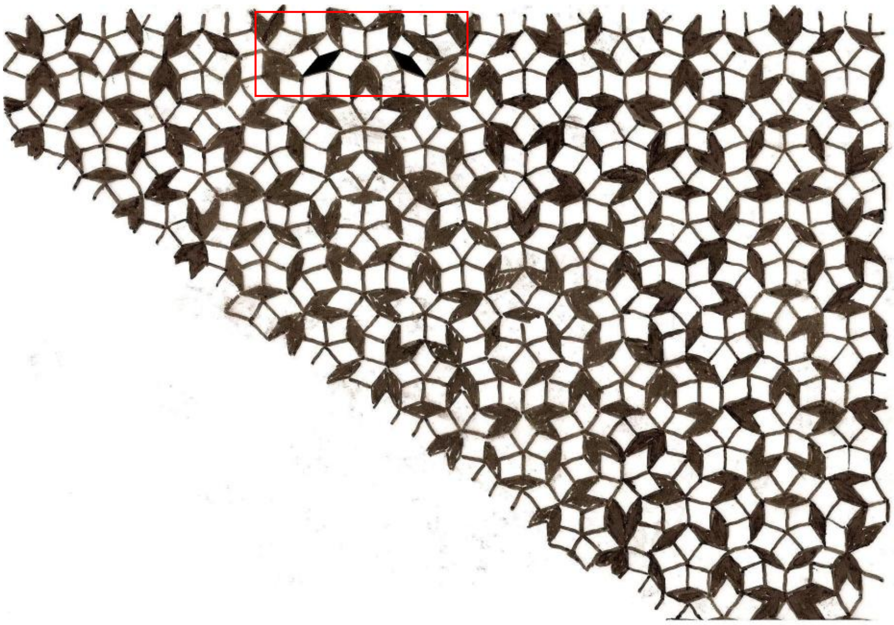


Fig. 11: Transformation of the whole pattern of Fig. 4

By comparing the Darb-i Imam pattern with a Penrose pattern, you can easily realize that the three white decagons at the middle-top of the pattern (Fig. 1) cannot exist in a Penrose pattern. Some proves why such a combination is not allowed will be shown later.

First, after mapping the pattern with thick and thin rhombuses, one can detect that the pattern is not a Penrose tiling, because there are two thin rhombuses standing orthogonal to each other (encircled in red). This can be detected twice (Fig. 12) and does not coincide with a real Penrose tiling. Moreover one gets a new tile which lies between the two thin rhombuses which has the shape of a trapezoid. This tile neither is included in the set of thick and thin rhombus nor in another one.

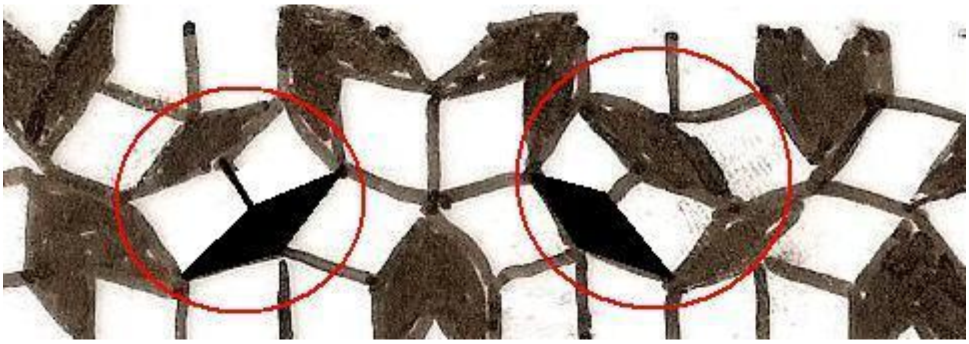


Fig. 12: Pattern mismatch

The mismatch comes from the three decagons lying in a straight line side by side which never occurs in a correct Penrose tiling, but this should occur if the pattern is penroselike, because the repetition of each part of a pattern, no matter how big or small, is an important feature of a Penrose tiling.

If one replaces the decagons with rhombuses and bowties one also will not get a global correct Penrose tiling because the tilings of kite and dart will not be affected of it.

The conclusion form this discovery is that even if one uses the “matching rules” correctly one will sometimes come to a global incorrect Penrose tiling.

5 Possible construction of the Darb-i Imam pattern

5.1 Penroselike construction

It was assumed that a possible construction to generate an original Penrose tiling by using the three girih tiles decagon, bowtie and rhombus, is to map these tiles like in Fig. 13.

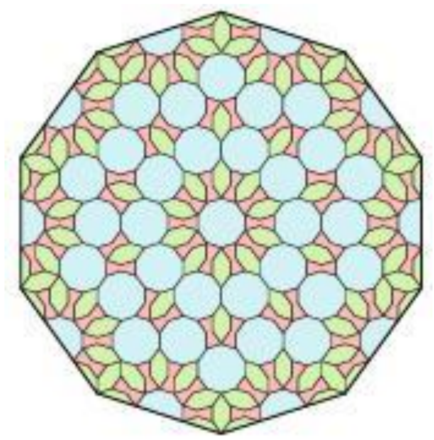


Fig. 13:[2] Graphical depiction of the subdivision rules transforming the large decagon

First, I sketched the arrangement of the girih tiles located like in the figure above and then charted the girih lines in this originated pattern (Fig. 14).

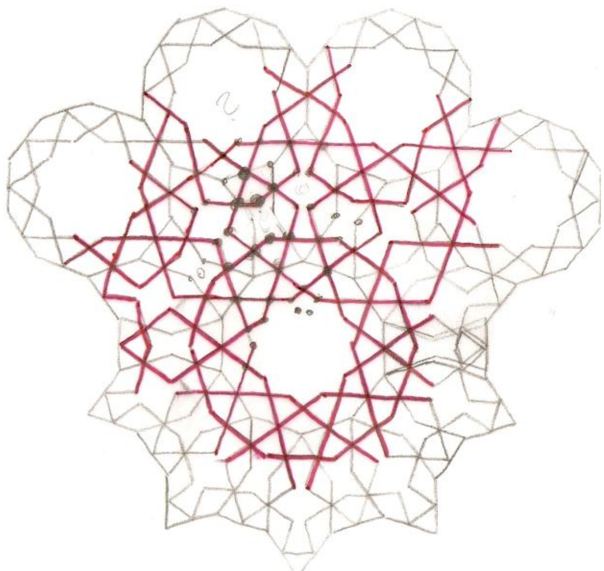


Fig. 14: Girih lines colored in red

Secondly, I took a given Penrose tiling [3], in which I detected the center as a symmetric decagon like there is one in the center of Fig. 13. After this step I assumed that the decagon must be encircled by bowties and rhombuses as one can see in Fig. 13. A further assumption I made is, that every edge or crossing of girih lines defines the middle of small decagons (Fig. 14). That results in Fig. 15.

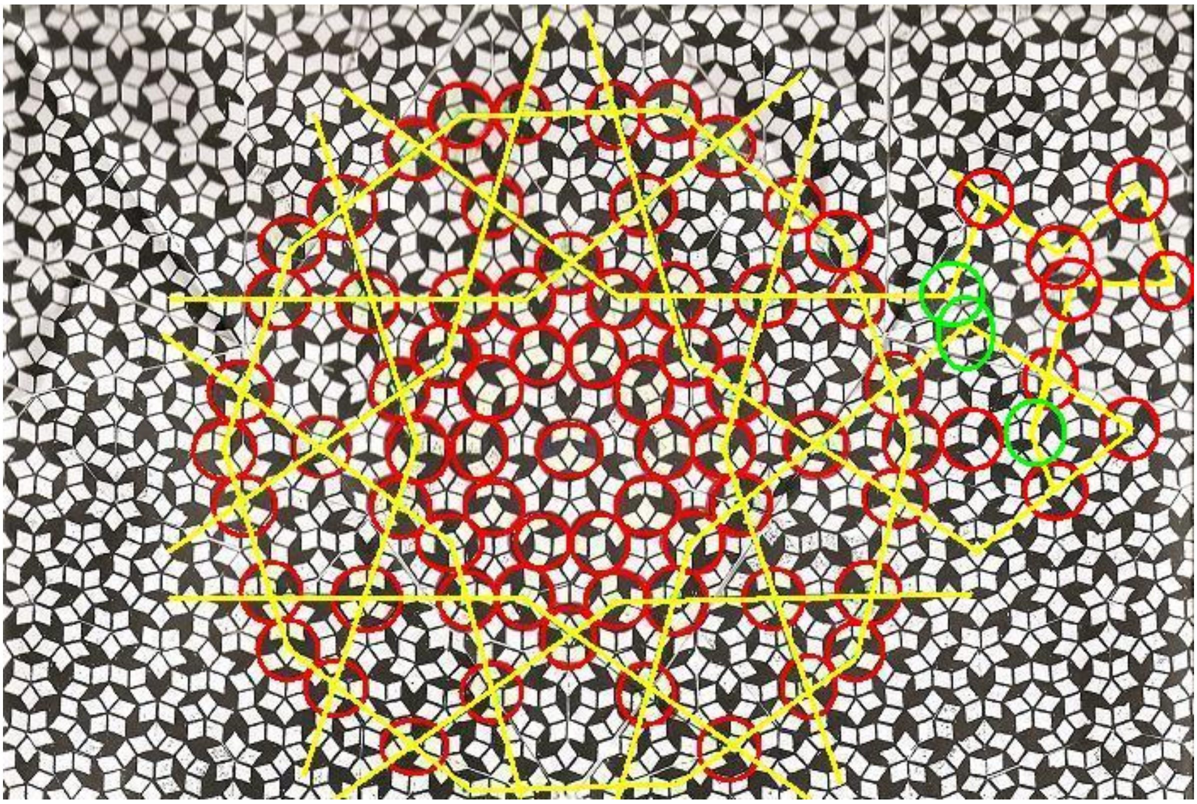


Fig. 15: Girih line mapping in Penrose pattern

In the picture one is able to see that my assumption was wrong, because where I drew the green circles there should have been, according to my assumption, red circles which include a decagon, as seen in Fig. 8.

If one instead subdivides the decagon of Fig. 13 in a bowtie and three rhombuses (Fig. 16), which is possible because a decagon mapped with kites and darts can be subdivided like this, one is also not able to detect this asymmetric pattern of bowtie and rhombus in the Penrose tiling of Fig. 15.

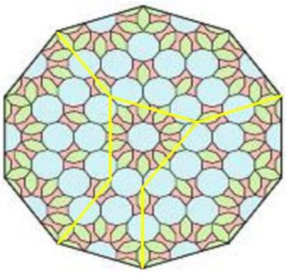


Fig. 16: Asymmetric creation of rhombus and bowtie

Due to the lack of a known construction of the girih pattern, my wrong assumption of a generation and further inconclusive tries, I must admit, that I have no plan to construct a shrine pattern, which has all the properties a real Penrose tiling should have.

5.2 Alternative girih pattern construction

The failure to find a way to construct a Penrose pattern with girih tiles brought me to the point that every architect has a plan to construct his creations or buildings. In this part I will take the role of the ancient architects of the Darb-i Imam shrine and construct my very own Girih tile pattern, which could be similar to the plan the Islamic architects might have had.

Beginning with the two tiles, decagon (Fig. 13) and bowtie (Fig. 17), which Peter Lu and Paul J. Steinhardt created, I thought of adding a rhombus as a third element, because the original pattern consists of these three tiles.

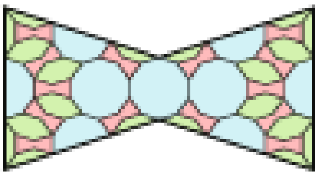


Fig. 17:[2] Graphical depiction of the subdivision rules transforming the large bowtie

Because every edge in the large bowtie and decagon look equal, the big rhombus must have the same arrangements at the edges. Another property is that at each girih line edge there must lie small decagons. Hence I came to a first sketch of the rhombus shown in Fig. 18.

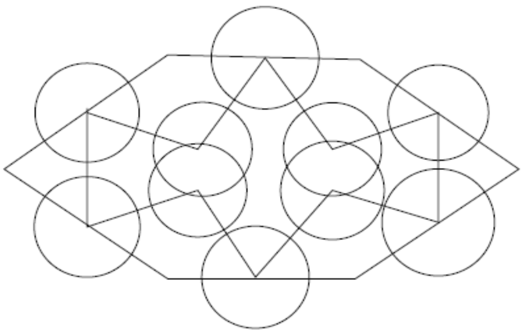


Fig. 18: Sketch of big rhombus

Having had a closer look at the big decagon, I assumed that the two small decagons at the central edges must be encircled with small bowties and rhombuses. As a last property I determined that the rhombus must have two symmetry axis, a horizontal one and a vertical one. After this I came to the big rhombus shown in Fig. 19.

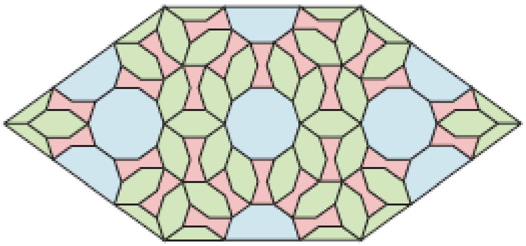


Fig. 19: Final sketch of big rhombus

In Fig. 18 one is able to see two decagons which intersect themselves. Because this is not possible, I transformed these two decagons in bowties and rhombuses.

To construct a whole girih pattern with these big three tiles, one starts with a normal decagon and deflates it in the way that one gets to a decagon as the one in Fig. 13. The three girih tiles in this decagon can be replaced by the decagon of Fig. 13, bowtie of Fig. 17 and rhombus of Fig. 18. This replacing process can be repeated infinite times so one can get to any order of deflation of a girih pattern, with ten-fold symmetry, which looks very similar to the one of the Darb-i Imam shrine. After replacing the tiles of the decagon (Fig. 13) by the bigger tiles one gets to the girih pattern shown in Fig. 20.

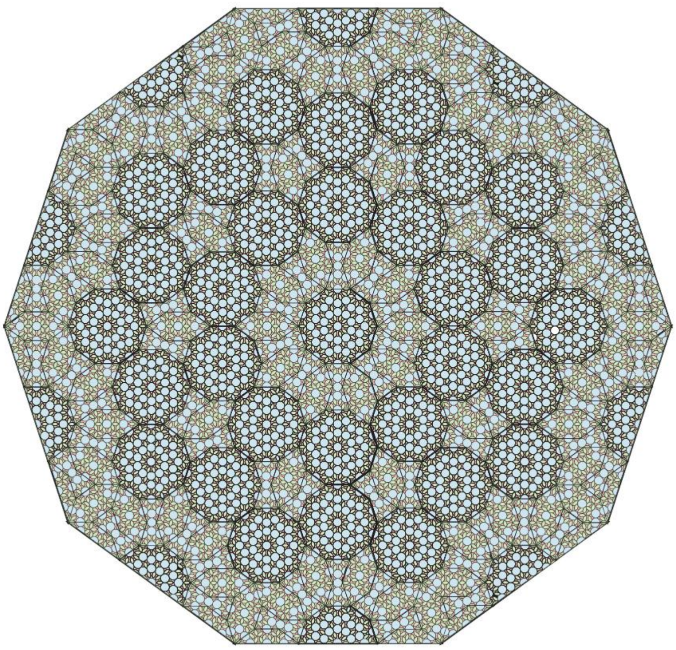


Fig. 20: Self-made girih pattern

6 Conclusion

By my investigations I come to the conclusion that medieval Islamic architects were not able to generate a perfect quasi-crystalline Penrose pattern. At first sight the pattern seems to be penroselike, but the fact that one cannot find a construction plan which maintains all features of a Penrose tiling is a proof, that there exists no real generation of a Penrose pattern with girih tiles. A transmission of a construction plan would emphasis this fact.
A special feature of this girih pattern is the ten-fold symmetry which exists only locally and even than only imperfectly in a real Penrose tiling. This characteristic is also the reason for mismatches.
On one hand this work proves the correctness of the discoveries that Peter J. Lu and Paul J. Steinhardt. If one neglects the fringe effect of the three decagons at the middle-top of the pattern, the pattern could be mapped almost totally in an existing Penrose. On the other hand, relating to the construction plan of the girih pattern, there is no allowed continuation of a true Penrose tiling for this pattern.
Peter J. Lu and Paul J. Steinhardt also tried to find a construction plan for this girih pattern, but their idea does not lead to a global correct Penrose tiling. The reason for this is the fact of three decagons lying side by side in a straight line which is not allowed if one wants to get to a globally correct Penrose tiling.

Neither my given construction plan nor other attempts led to a globally correct pattern. This was when I realized that there is no construction plan for a globally correct Penrose pattern with girih tiles.

Since every architect has a construction plan I thought that there must have been such a plan to realize the idea of generating this girih pattern. Hence I took the two tiles, decagon (Fig. 13) and bowtie (Fig. 17), which Peter J. Lu and Paul J. Steinhardt found plus my generated rhombus (Fig. 19) and developed a construction plan which the ancient architects might have used to construct this pattern.

7 Bibliography

[1] - Penrose tilings: http://en.wikipedia.org/wiki/Penrose_tilings (30.06.2009).

[2] - Peter J. Lu & Paul J. Steinhardt.: **Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture.-** Science, **315**: 1106-1110 (2007).

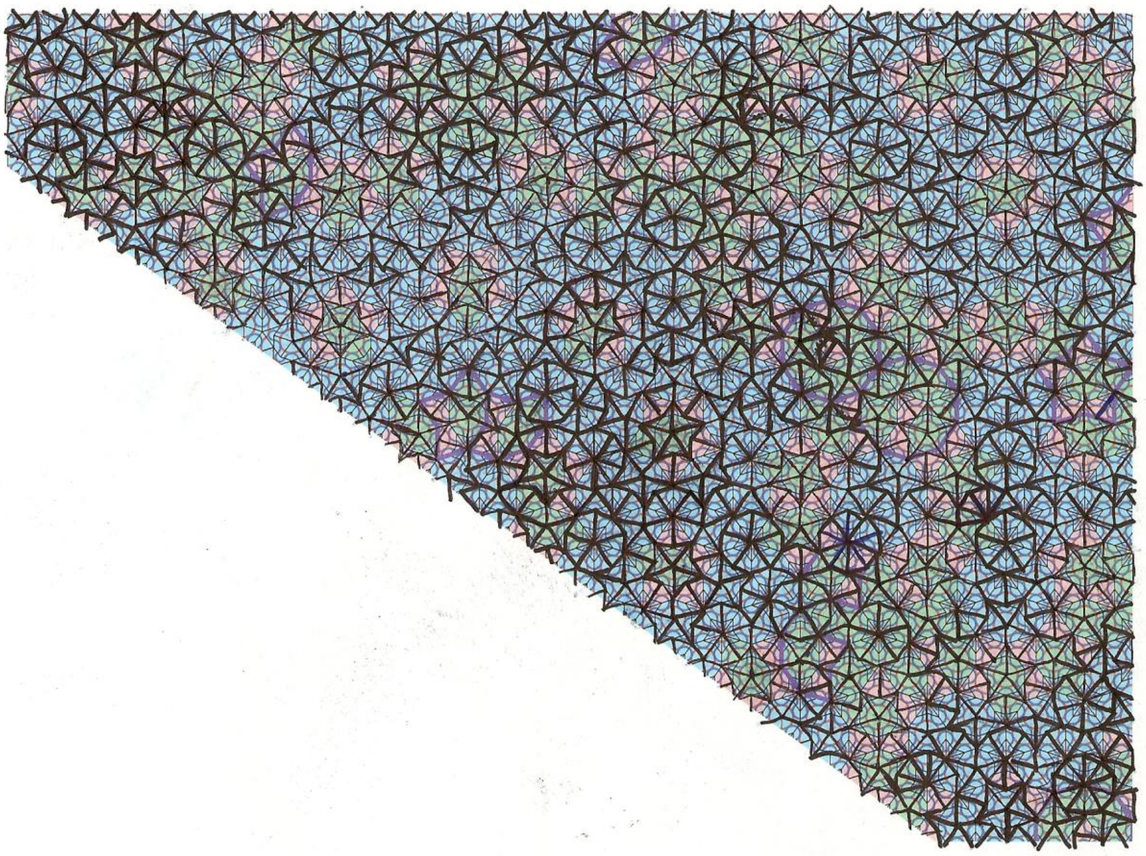
[3] - J-H Eschenburg: **Die Zahl Fünf und die Quasikristalle.-** http://www.math.uni-augsburg.de/~eschenbu/penrose.pdf (10.08.08), pp. 18 (2004).

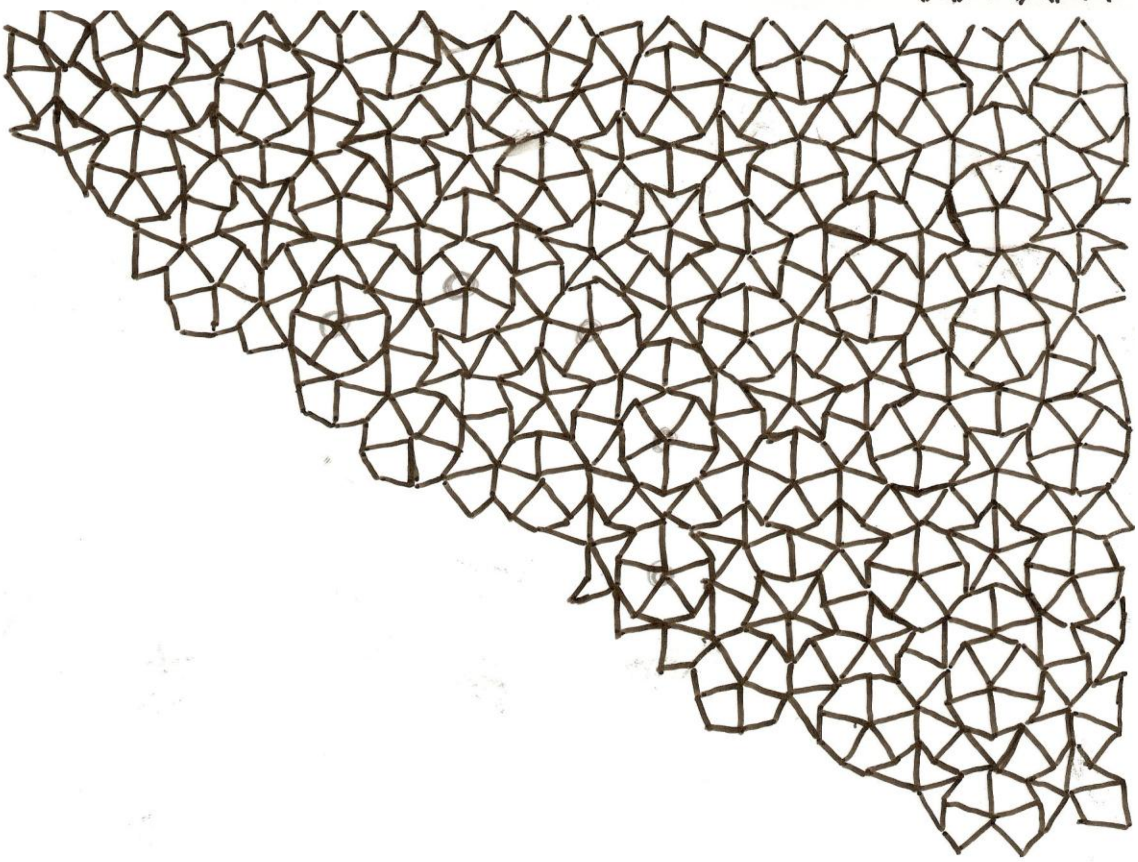
[4] - Roger Penrose: **Pentaplexity - a class of nonperiodic tilings of the plane**.- Mathematical Intelligencer, **2:** 32 – 37 (1979).

[5] - Penrose Tiling: http://goldennumber.net/penrose.htm (4.08.2009).

[6] - Girih tiles: http://en.wikipedia.org/wiki/Girih (30.06.2009).

8 Attachments





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Augsburg den, 8.08.2009

Benjamin Schleich