

# Graph and Tree Theory Notes

Matt Warner

## 0.1 Trails, Paths, and Circuits

### Definition

Let  $G$  be a graph, and let  $v$  and  $w$  be vertices in  $G$ .

A walk from  $v$  to  $w$  is a finite alternating sequence of adjacent vertices and edges of  $G$ . Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

where the  $v$ 's represent vertices, the  $e$ 's represent edges,  $v_0 = v, v_n = w$ , and for each  $i = 1, 2, \dots, n$ ,  $v_{i-1}$  and  $v_i$  are the endpoints of  $e_i$ . The trivial walk from  $v$  to  $v$  consists of the single vertex  $v$ .

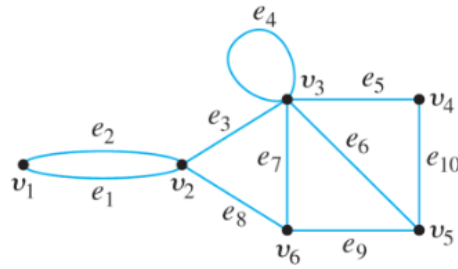
- A trail from  $v$  to  $w$  is a walk from  $v$  to  $w$  that does not contain a repeated edge.
- A path from  $v$  to  $w$  is a trail that does not contain a repeated vertex.
- A closed walk is a walk that starts and ends at the same vertex.
- A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.
- A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

	Repeated Edge?	Repeated Vertex?	Starts and Ends at Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed Walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	yes	yes

### Example 0.1.1

Determine which of the following walks are trails, paths, circuits, or simple circuits.

1.  $v_1, e_1, v_2, e_3, v_3, e_4, v_3, e_5, v_4$
2.  $e_1, e_3, e_5, e_5, e_6$
3.  $v_2 v_3 v_4 v_5 v_3 v_6 v_2$
4.  $v_1 e_1 v_2 e_1 v_1$
5.  $v_2, v_3, v_4, v_5, v_6, v_2$
6.  $v_1$



*Soultion:*



1. this walk is a Trail, since it does not contain any repeted edges.
2. This is just a walk and nothing else.
3. This walk is a Closed walk and is also a circuit, but not a simple circuit.
4. This is just a closed walk, it cant be a circuit since there is a repeated edge.
5. This is a closed walk, as well as a simple circuit.
6. This is a closed walk, as well as a trail, not a circuit becuase the walk does not contain any edges.

## 0.2 Subgraphs

### Defintion

A graph  $H$  is said to be a subgraph of a graph  $G$  if, and only if, every vertex in  $H$  is also a vertex in  $G$ , every edge in  $H$  is also an edge in  $G$ , and every edge in  $H$  has the same endpoints as it has in  $G$ .

### Example 0.2.1

List all subgraphs of the graph  $G$  with vertex set  $\{v_1, v_2\}$  and edge set  $\{e_1, e_2, e_3\}$ , where the endpoints of  $e_1$  are  $v_1$  and  $v_2$ , the endpoints of  $e_2$  are  $v_1$  and  $v_2$ , and  $e_3$  is a loop at  $v_1$ .

$G$  can be drawn as shown below.

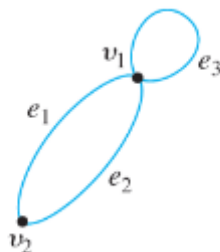


Figure 1:  $G$

There are 11 subgraphs of  $G$ , which can be grouped according to those that do not have any edges, those that have one edge, those that have two edges, and those that have three edges.

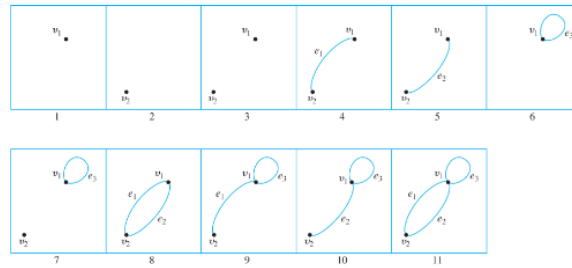


Figure 2: Subgraphs of  $G$

### 0.3 Connectedness

#### Definition

Let  $G$  be a graph. Two **vertices,  $v$  and  $w$  of  $G$  are connected** if, and only if, there is a walk from  $v$  to  $w$ .

The **graph  $G$  is connected** if, and only if, given *any* two vertices  $v$  and  $w$  in  $G$ , there is a walk from  $v$  to  $w$ . Symbolically:

$$G \text{ is connected} \iff \forall \text{ vertices } v \text{ and } w \text{ in } G, \exists \text{ a walk from } v \text{ to } w.$$

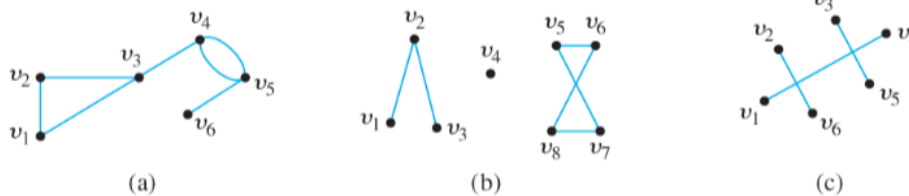
#### Note:-

If you take the negation of this definition, you will see that a graph  $G$  is *not connected* if, and only if, there exists two vertices of  $G$  that are not connected by any walk.

#### Example 0.3.1

Which of the following graphs are connected?

The graphs are listed below



**Solution:**

Graph  $A$  is connected

Graph  $B$  is not connected

Graph  $C$  is also not connected

Some useful facts relating circuits and connectedness are collected in the following lemma.

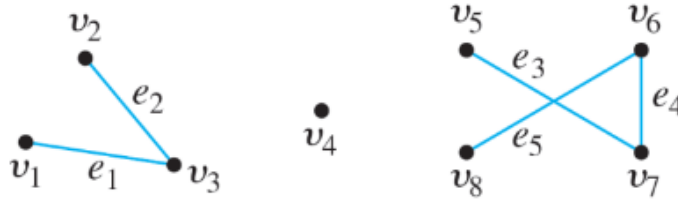
**Lemma 0.3.1**

Let  $G$  be a graph.

- If  $G$  is connected, then any two distinct vertices of  $G$  can be connected by a path.
- If vertices  $v$  and  $w$  are part of a circuit in  $G$  and one edge is removed from the circuit, then there still exists a trail from  $v$  to  $w$  in  $G$ .
- If  $G$  is connected and  $G$  contains a circuit, then an edge of the circuit can be removed without disconnecting  $G$ .

**Example 0.3.2**

Find all connected components of the following graph  $G$ .



**Solution:**



$G$  has three connected components:  $H_1, H_2$  and  $H_3$  with vertex sets  $V_1, V_2$ , and  $V_3$  and edge sets  $E_1, E_2$ , and  $E_3$

$$\begin{aligned} V_1 &= \{v_1, v_2, v_3\}, & E_1 &= \{e_1, e_2\}, \\ V_2 &= \{v_4\}, & E_2 &= \emptyset, \\ V_3 &= \{v_5, v_6, v_7, v_8\}, & E_3 &= \{e_3, e_4, e_5\}. \end{aligned}$$

## 0.4 Euler Circuits

**Definition**

Let  $G$  be a graph. An Euler circuit for  $G$  is a circuit that contains every vertex and every edge of  $G$ . That is, an Euler circuit for  $G$  is a sequence of adjacent vertices and edges in  $G$  that has at least one edge, starts and ends at the same vertex, uses every vertex of  $G$  at least once, and uses every edge of  $G$  exactly once.

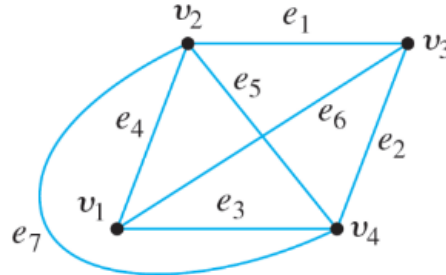
**Theorem 0.4.1**

If a graph has a Euler circuit, then every vertex of the graph has positive even degree.

If some vertex of a graph has odd degree, then the graph does not have a Euler circuit

**Example 0.4.1**

Show that the graph below does not have a Euler circuit.



**Solution:**



The vertices  $v_1$ , and  $v_3$  both have odd degrees (degree 3). So the graph cannot be a Euler circuit.

**Note:-**

If a graph  $G$  is connected and the degree of every vertex of  $G$  is a positive even integer, then  $G$  has a Euler circuit.

**0.4.1 Euler Trails**

**Defintion**

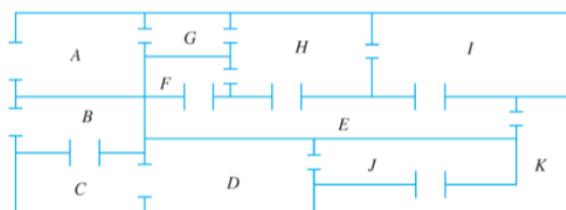
Let  $G$  be a graph, and let  $v$  and  $w$  be two distinct vertices of  $G$ . An Euler trail from  $v$  to  $w$  is a sequence of adjacent edges and vertices that starts at  $v$ , ends at  $w$ , passes through every vertex of  $G$  at least once, and traverses every edge of  $G$  exactly once.

**Corollary 0.4.1**

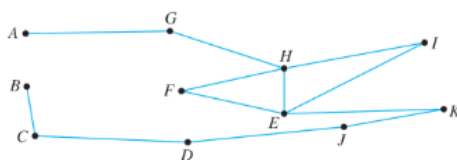
Let  $G$  be a graph, and let  $v$  and  $w$  be two distinct vertices of  $G$ . There is an Euler trail from  $v$  to  $w$  if, and only if,  $G$  is connected,  $v$  and  $w$  have odd degree, and all other vertices of  $G$  have positive even degree.

**Example 0.4.2**

The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room  $A$ , ends in room  $B$ , and passes through every interior doorway of the house exactly once? If so, find such a trail.



We can represent this floor plan as a graph:



Each vertex of this graph has an even degree except for  $A$  and  $B$ , hence by Corollary 0.4.2 there is an Euler trail from  $A$  to  $B$  one such trail is

$$A - G - H - F - E - I - H - E - K - J - D - C - B$$

## 0.5 Hamiltonian Circuits

### Definition

Given a graph  $G$ , a **Hamiltonian circuit** for  $G$  is a simple circuit that includes every vertex of  $G$ . That is, a Hamiltonian circuit for  $G$  is a sequence of adjacent vertices and distinct edges in which every vertex of  $G$  appears exactly once, except for the first and the last, which are the same.

### **Proposition 0.5.1**

If a graph  $G$  has a Hamiltonian circuit, then  $G$  has a subgraph  $H$  with the following properties:

1.  $H$  contains every vertex of  $G$ .
2.  $H$  is connected.
3.  $H$  has the same number of edges as vertices.
4. Every vertex of  $H$  has degree 2 .

## 0.6 Matrices

### Defintion

Matrices are two-dimensional analogues of sequences.

They also are called two-dimensional arrays

An  $m \times n$  (read "  $m$  by  $n$  ") matrix  $\mathbf{A}$  over a set  $S$  is a rectangular array of elements of  $S$  arranged into  $m$  rows and  $n$  columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{ith row of } \mathbf{A}$$

We write  $\mathbf{A} = (a_{ij})$

The  $i$ th row of  $\mathbf{A}$  is

$$[ a_{i1} \quad a_{i2} \quad \cdots \quad a_{in} ]$$

and the  $j$ th column of  $\mathbf{A}$  is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

A matrix for which the numbers of row and columns are equal is called a **square matrix**

If  $\mathbf{A}$  is a square matrix of size  $n \times n$ , then the main diagonal of  $\mathbf{A}$  consists of all the entries  $a_{11}, a_{22}, \dots, a_{nn}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}$$

← main diagonal of  $\mathbf{A}$

Figure 3: Square matrix diagonal



**Example 0.6.1**

The following is a 3 x 3 matrix over the set of integers.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -1 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

1. What is  $a_{23}$ , the entry in row 2 , column 3 ?
2. What is the second column of  $\mathbf{A}$  ?
3. What are the entries in the main diagonal of  $\mathbf{A}$  ?

**Solution:**

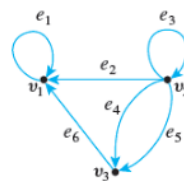


1.  $a_{23} = 5$

2.  $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

3.  $1, -1, 0$

### 0.6.1 Matrices and Directed Graphs



Directed Graph  $G$   
(a)

$$\mathbf{A} = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

Adjacency Matrix of  $G$   
(b)

A Directed Graph and Its Adjacency Matrix

**Figure 10.2.1**

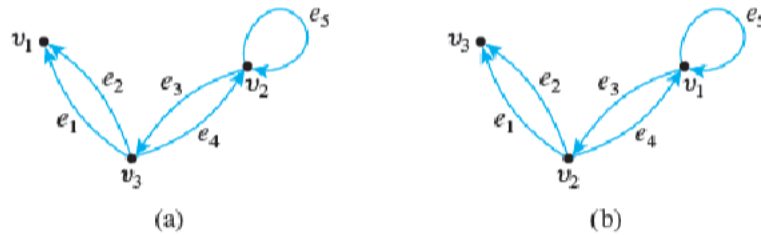
**Note:-**

As shown by the figure above, the adjacency matrix holds all the data for the directed graph, each entry in the matrix is either a 1 or 0, the entry is a 0 if the two vertices are not adjacent, and a 1 if they are adjacent, with the number increasing by the amount of edges that connect the vertices.

**Example 0.6.2**

The two graphs show below are identical and differ only in the ordering of their vertices.

Find their adjacency matrices



*Solution:*



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

## 0.6.2 Matrices and Undirected Graphs