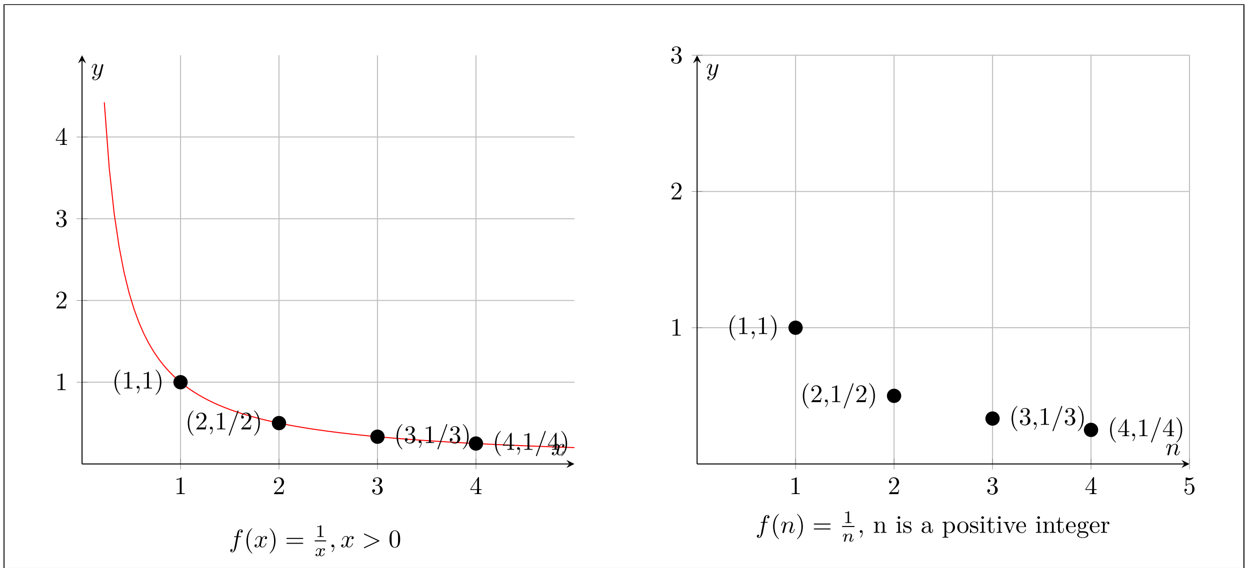
Chapter 11.1 Notes

Matt Warner

SERIES

Definition:

A **sequence** is a function whose domain is the set of positve integers.



Note:-

The graph of the left is a function

The graph on the right is a sequence and does not have a smooth curve, it only containts a series of **points**

A sequence uses curly braces and has subscript notation with the form a_n

Ex 1: Write down the first six terms of the following sequence and graph it.

$$\text{a) } \{a_n\} = \left\{\frac{n^2}{2n+1}\right\}$$

$$a_1 = \frac{1^2}{2(1)+2}$$

$$a_2 = \frac{2^2}{2(2)+1}$$

$$a_3 = \frac{3^2}{2(3)+1}$$

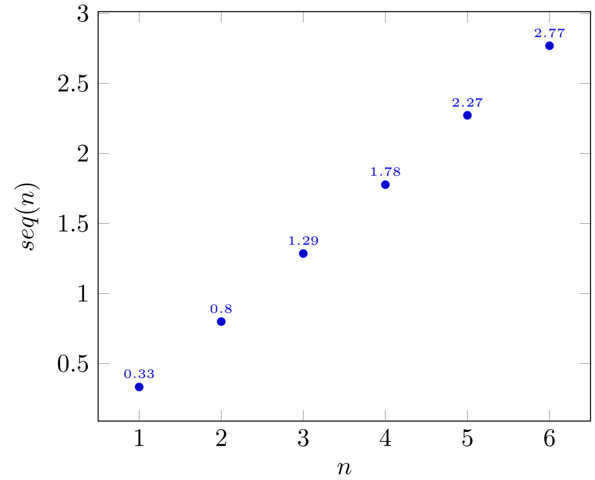
$$a_4 = \frac{4^2}{2(4)+1}$$

$$\ldots$$

So,

The first six terms of the sequence are $\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \frac{25}{11}, \frac{36}{13}$

Graph of $\left(\frac{n^2}{2(n)+1}\right)$



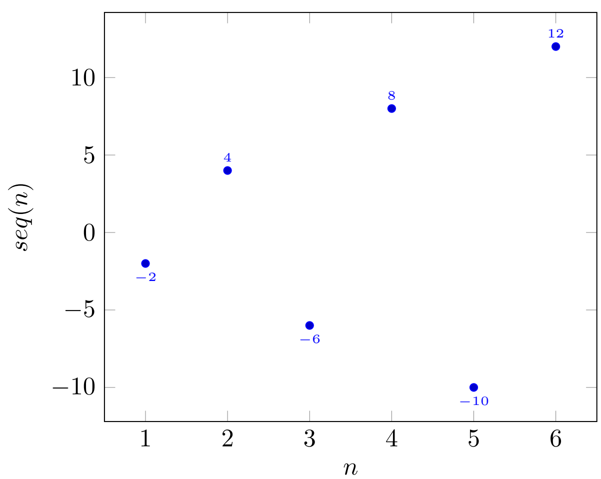
---

b) $\{b_n\} = \{(-1)^n \cdot 2n\}$

$$\begin{aligned} b_1 &= (-1)^1(2 \cdot 1) \\ &= -2 \\ b_2 &= (-1)^2(2 \cdot 2) \\ &= 4 \\ b_3 &= (-1)^3(2 \cdot 3) \\ &= -6 \end{aligned}$$

So,

Continuing the sequence, the first 6 terms are $-2, 4, -6, 8, -10, 12$



Ex 2: Write down the nth term of the sequence suggested by the pattern.

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$$a_1 = \tfrac{1}{2^1}$$
$$a_2 = \tfrac{1}{2^2}$$
$$a_3 = \tfrac{1}{2^3}$$

So, it follows that the equation of the sequence is,

$$\{a_n\} = \{\tfrac{1}{2^n}\}$$

b) $5, 7, 9, 11, \ldots$

$$\begin{aligned} b_1 &= 2(1) + 3 \\ &= 5 \\ b_2 &= 2(2) + 3 \\ &= 7 \\ b_3 &= 2(3) + 3 \\ &= 9 \end{aligned}$$

So, the equation of the sequence is,

$$\boxed{\{b_n\} = \{2n + 3\}}$$

The Factorial Symbol

If $n \geq 0$ is an integer, the factorial symbol n! is defined as follows:

$$0! = 1$$
$$1! = 1$$
$$n! = n(n-1) \cdot \ldots \cdot 2 \cdot 2 \cdot 1 \quad \text{if } n \geq 2$$

Example 3: Find the value of the following expressions.

a) $5!$

$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 120 \end{aligned}$$

b) $6!$

$$\begin{aligned} 6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 720 \end{aligned}$$

Note:-

Write the Terms of a Sequence Defined by a Recursive Formula

A second way of defining a sequence is to assign a value to the first (or the first few) terms(s) and specify the *nth* term by a formula that involves one or more of the terms preceding it.

Example 4: Write down the first six terms of the following recursively defined sequence.

$$s_1 = 5, \quad s_n = 2 \cdot s_{n-1}$$

$$s_1 = 5$$

$$s_2 = 2 \cdot 5 = 10$$

$$s_3 = 2 \cdot 10 = 20$$

$$s_4 = 2 \cdot 20 = 40$$

$$s_5 = 2 \cdot 40 = 80$$

$$s_6 = 2 \cdot 80 = 160$$

So,

The first six terms of the sequence are:

$$\boxed{5, 10, 20, 40, 80, 160}$$

Summation Noation

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Note:-

The summation of a_k from $k = 1$ to n is given by:

$$\sum_{k=1}^{n} a_k$$

Example 5: Write out each sum

a) $\sum_{k=1}^{n} \frac{k}{k+1}$

$$\sum_{k=1}^{n} \frac{k}{k+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \cdots + \frac{n}{n+1}$$

(b) $\sum_{k=0}^{n} \left(k^2 - 1\right)$

$$(0)^2 - 1 + (1)^2 - 1 + (2)^2 - 1 + (3)^2 - 1 + \ldots$$

$$-1 + 0 + 3 + 8 + \ldots + (n)^2 - 1$$

Example 6: Express each sum using summation notation.

a) $\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{5}\right)^2$

the numerator remains constant in each interval, while the denominator increases by a factor of one.

So,

We can conclude that the Summation notation is going to be:

$$\sum_{k=1}^{5} \left(\frac{1}{k}\right)^2$$

b) $1 + 3 + 5 + \cdots + 2n - 1$

Given that the equation for each value of n is $2n - 1$, the inital value of k is one.

We can conclude that the Summation notation is going to be:

$$\sum_{k=1}^{n} 2k - 1$$

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then

$$\sum_{k=1}^{n} c = ca_1 + ca_2 + \cdots + ca_n = c\left(a_1 + a_2 + \cdots + a_n\right) = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} \left(a_k + b_k\right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} \left(a_k - b_k\right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{j=1}^{n} a_k \quad \text{where} \quad 0 < j < n$$

Forumlas for Sums of Sequences

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]$$

Example 7: Find the sum of the sequence

$$\textstyle\sum_{k=1}^{10} \left(k^2 - k\right)$$

Using the formula $\rightarrow \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for k^2

And $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for k,

Our equation for finding the sum of the sequence is going to be:

$$\frac{10(10+1)(2 \cdot 10 + 1)}{6} - \frac{10(10+1)}{2}$$

$$= \frac{2310}{6} - 55$$

$$= \boxed{330}$$