# Graph and Tree Theory Notes

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# 1 Trails, Paths, and Circuits

**Definition**

Let $G$ be a graph, and let $v$ and $w$ be vertices in $G$.

A walk from $v$ to $w$ is a finite alternating sequence of adjacent vertices and edges of $G$. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

where the $v$ 's represent vertices, the $e$ 's represent edges, $v_0 = v, v_n = w$, and for each $i = 1, 2, \ldots n, v_{i-1}$ and $v_i$ are the endpoints of $e_i$. The trivial walk from $v$ to $v$ consists of the single vertex $v$.

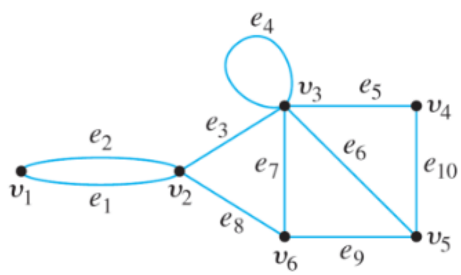
- A trail from $v$ to $w$ is a walk from $v$ to $w$ that does not contain a repeated edge.
- A path from $v$ to $w$ is a trail that does not contain a repeated vertex.
- A closed walk is a walk that starts and ends at the same vertex.
- A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.
- A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

| | **Repeated Edge?** | **Repeated Vertex?** | **Starts and Ends at Same Point?** | **Must Contain at Least One Edge?** |
|---|---|---|---|---|
| Walk | allowed | allowed | allowed | no |
| Trail | no | allowed | allowed | no |
| Path | no | no | no | no |
| Closed Walk | allowed | allowed | yes | no |
| Circuit | no | allowed | yes | yes |
| Simple Circuit | no | first and last only | yes | yes |

**Example 1.1**

Determine which of the following walks are trails, paths, circuits, or simple circuits.

1. $v_1, e_1, v_2, e_3, v_3, e_4, v_3, e_5, v_4$
2. $e_1, e_3, e_5, e_5, e_6$
3. $v_2 v_3 v_4 v_5 v_3 v_6 v_2$
4. $v_1 e_1 v_2 e_1 v_1$
5. $v_2, v_3, v_4, v_5, v_6, v_2$
6. $v_1$




***Soultion:*** 😄

1. this walk is a Trail, since it does not contain any repeted edges.
2. This is just a walk and nothing else.
3. This walk is a Closed walk and is also a circuit, but not a simple circuit.
4. This is just a closed walk, it cant be a circuit since there is a repeated edge.
5. This is a closed walk, as well as a simple circuit.
6. This is a closed walk, as well as a trail, not a circuit becuase the walk does not contain any edges.

## 2 Subgraphs

**Definition**

A graph $H$ is said to be a subgraph of a graph $G$ if, and only if, every vertex in $H$ is also a vertex in $G$, every edge in $H$ is also an edge in $G$, and every edge in $H$ has the same endpoints as it has in $G$.

**Example 2.1**

List all subgraphs of the graph $G$ with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$, where the endpoints of $e_1$ are $v_1$ and $v_2$, the endpoints of $e_2$ are $v_1$ and $v_2$, and $e_3$ is a loop at $v_1$.

$G$ can be drawn as shown below.

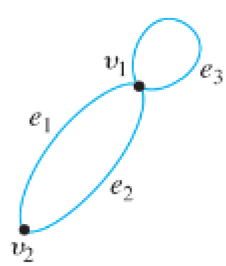



Figure 1: $G$

There are 11 subgraphs of $G$, which can be grouped according to those that do not have any edges, those that have one edge, those that have two edges, and those that have three edges.

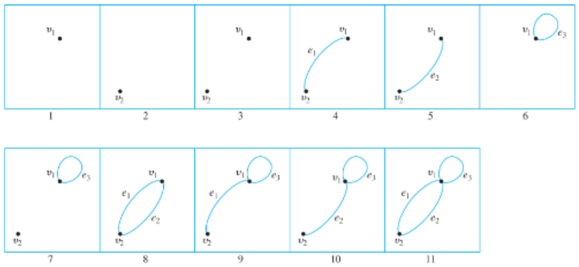


Figure 2: Subgraphs of $G$

## 3 Connectedness

**Definition**

Let $G$ be a graph. Two **vertices,** $v$ **and** $w$ **of** $G$ **are connected** if, and only if, there is a walk from $v$ to $w$.

The **graph** $G$ **is connected** if, and only if, given *any* two vertices $v$ and $w$ in $G$, there is a walk from $v$ to $w$. Symbolically:

$$G \text{ is connected} \longleftrightarrow \forall \text{ vertices } v \text{ and } w \text{ in } G, \exists \text{ a walk from } v \text{ to } w.$$

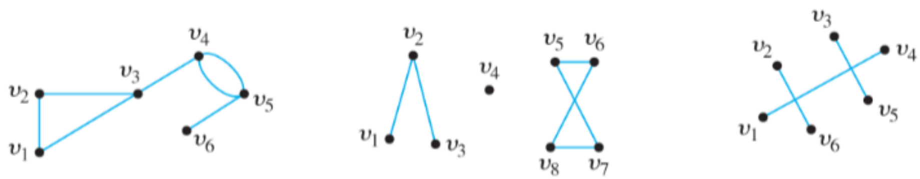
**Note:-**

If you take the negation of this definition, you will see that a graph $G$ is *not connected* if, and only if, there exists two vertices of $G$ that are not connected by any walk.

**Example 3.1**

Which of the following graphs are connected?

The graphs are listed below




(a) (b) (c)

***Solution:***

Graph $A$ is connected | Graph $B$ is not connected | Graph $C$ is also not connected

Some useful facts relating circuits and connectedness are collected in the following lenma.

**Lenma 3.1**

Let $G$ be a graph.

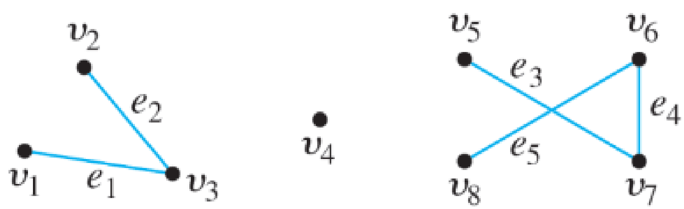
a. If $G$ is connected, then any two distinct vertices of $G$ can be connected by a path.

b. If vertices $v$ and $w$ are part of a circuit in $G$ and one edge is removed from the circuit, then there still exists a trail from $v$ to $w$ in $G$.

c. If $G$ is connected and $G$ contains a circuit, then an edge of the circuit can be removed without disconnecting $G$.

**Example 3.2**

Find all connected components of the following graph $G$.




***Solution:*** ☺

$G$ has three connected components: $H_1, H_2$ and $H_3$ with vertex sets $V_1, V_2$, and $V_3$ and edge sets $E_1, E_2$, and $E_3$

$$\begin{array}{ll} V_1 = \{v_1, v_2, v_3\}, & E_1 = \{e_1, e_2\}, \\ V_2 = \{v_4\}, & E_2 = \varnothing, \\ V_3 = \{v_5, v_6, v_7, v_8\}, & E_3 = \{e_3, e_4, e_5\}. \end{array}$$

## 4 Euler Circuits

**Definition**

Let $G$ be a graph. An Euler circuit for $G$ is a circuit that contains every vertex and every edge of $G$. That is, an Euler circuit for $G$ is a sequence of adjacent vertices and edges in $G$ that has at least one edge, starts and ends at the same vertex, uses every vertex of $G$ at least once, and uses every edge of $G$ exactly once.

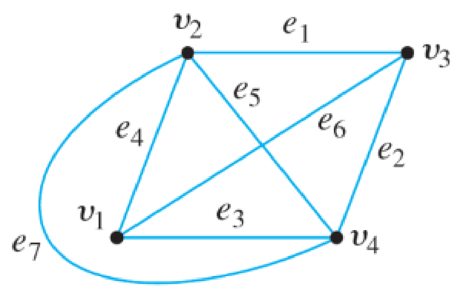
**Theorem 4.1**

If a graph has a Euler circuit, then every vertex of the graph has positive even degree.

If some vertex of a graph has odd degree, then the graph does not have a Euler circuit

**Example 4.1**

Show that the graph below does not have a Euler circuit.




***Solution:*** 😄

The vertices $v_1$, and $v_3$ both have odd degrees (degree 3). So the graph cannot be a Euler circuit.

**Note:-**

If a graph $G$ is connected and the degree of every vertex of $G$ is a postive even integer, then $G$ has a Euler circuit.

## 4.1 Euler Trails

**Definition**

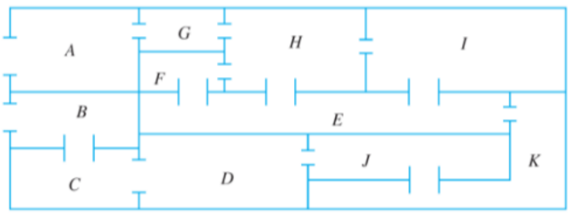
Let $G$ be a graph, and let $v$ and $w$ be two distinct vertices of $G$. An Euler trail from $v$ to $w$ is a sequence of adjacent edges and vertices that starts at $v$, ends at $w$, passes through every vertex of $G$ at least once, and traverses every edge of $G$ exactly once.

**Corollary 4.1**

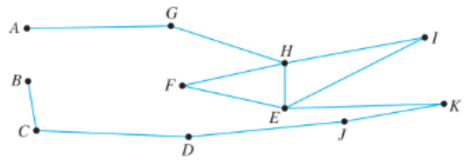
Let $G$ be a graph, and let $v$ and $w$ be two distinct vertices of $G$. There is an Euler trail from $v$ to $w$ if, and only if, $G$ is connected, $v$ and $w$ have odd degree, and all other vertices of $G$ have positive even degree.

**Example 4.2**

The floor plan shown below is for a house that is open for public viewing. Is it possible to find a trail that starts in room $A$, ends in room $B$, and passes through every interior doorway of the house exactly once? If so, find such a trail.




We can represent this floor plan as a graph:




Each vertex of this graph has an even degree except for $A$ and $B$, hence by Corollary 0.4.1 there is an Euler trail from $A$ to $B$ one such trail is

$$A - G - H - F - E - I - H - E - K - J - D - C - B$$

# 5 Hamiltonian Circuits

**Definition**

Given a graph $G$, a **Hamiltonian circuit** for $G$ is a simple circuit that includes every vertex of $G$. That is, a Hamiltonian circuit for $G$ is a sequence of adjacent vertices and distinct edges in which every vertex of $G$ appears exactly once, except for the first and the last, which are the same.

**Proposition 5.1**

If a graph $G$ has a Hamiltonian circuit, then $G$ has a subgraph $H$ with the following properties:

1. $H$ contains every vertex of $G$.
2. $H$ is connected.
3. $H$ has the same number of edges as vertices.
4. Every vertex of $H$ has degree 2 .

# 6 Matrices

**Definition**

Matrices are two-dimensional analogues of sequences.

They also are called two-dimensional arrays

An $m \times n$ (read " $m$ by $n$ ") matrix A over a set $S$ is a rectangular array of elements of $S$ arranged into $m$ rows and $n$ columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{ ith row of A}$$

We write $\mathbf{A} = (a_{ij})$

The $i$th row of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix}$$

and the $j$th column of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

A matrix for which the numbers of row and columns are equal is called a **square matrix**

If $\mathbf{A}$ is a square matrix of size $n \times n$, then the main diagonal of $\mathbf{A}$ consists of all the entries $a_{11}, a_{22}, \ldots, a_{nn}$

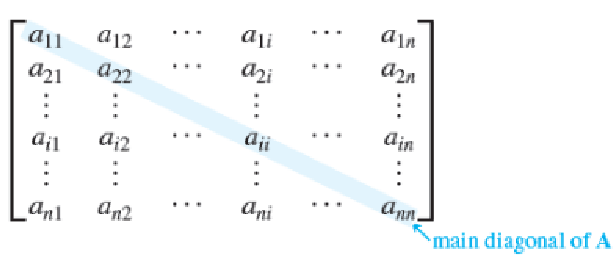



Figure 3: Square matrix diagonal

**Example 6.1**
The following is a 3 x 3 matrix over the set of integers.

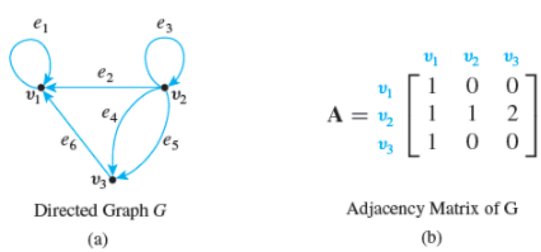
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 \\ 4 & -1 & 5 \\ -2 & 2 & 0 \end{bmatrix}$$

1. What is $a_{23}$, the entry in row 2 , column 3 ?
2. What is the second column of $\boldsymbol{A}$ ?
3. What are the entries in the main diagonal of $\mathbf{A}$ ?

***Solution:***

1. $a_{23} = 5$
2. $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$
3. $1, -1, 0$

## 6.1 Matrices and Directed Graphs




A Directed Graph and Its Adjacency Matrix

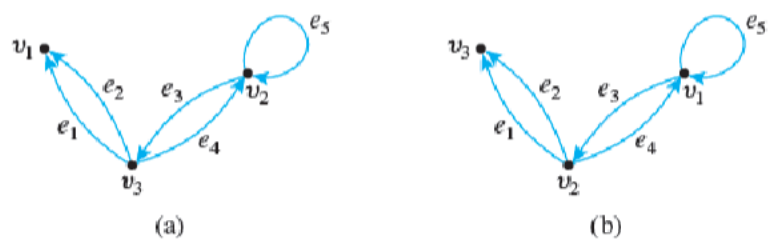
**Figure 10.2.1**

**Note:-**

As shown by the figure above, the adjacency matrix holds holds all the data for the directed graph, each entry in the matrix is either a 1 or 0, the entry is a 0 if the two vertices are not adjacent, and a 1 if they are adjacent, with the number increasing by the amount of edges that connect the vertices.

**Example 6.2**
The two graphs show below are identical and differ only in the ordering of their vertices.

Find their adjacency matrices




***Solution:*** ☺

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

## 6.2 Symmetric Matrices

**Definition**

An $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is called symmetric if, and only if, for every $i$ and $j = 1, 2, \ldots, n$,

$$a_{ij} = a_{ji}$$

**Example 6.3**
Which of the following matrices are symmetric?

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

***Soultion:*** ☺

Only graph $B$ is symmetric

In matrix $A$ the entry in the first row and the second column differs from the entry in the second row and the first column; the matrix, $C$, is not even square.

## 6.3 Matrix Multiplication

The product of two matrices is built up of *scalar* or *dot* products of their individual rows and columns.

**Definition**

Suppose that all entries in matrices **A** and **B** are real numbers. If the number of element, $n$, in the $i$ th row of **A** equals the number of elements in the $j$ th column of **B**, then the scalar product or dot product of the $i$ th row of **A** and $j$ th column of **B** is the real number obtained as follows:

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

**Example 6.4**

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix}$$

***solution:*** 😊

A has size $2 \times 3$ and **B** has size $3 \times 2$, so the number of columns of **A** equals the number of rows of **B** and the matrix product of **A** and **B** can be computed. Then

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Where,

$$c_{11} = 2 \cdot 4 + 0 \cdot 2 + 3 \cdot (-2) = 2$$
$$c_{12} = 2 \cdot 3 + 0 \cdot 2 + 3 \cdot (-1) = 3$$
$$c_{21} = (-1) \cdot 4 + 1 \cdot 2 + 0 \cdot (-2) = -2$$
$$c_{22} = (-1) \cdot 3 + 1 \cdot 2 + 0 \cdot (-1) = -1$$

Hence,

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

# 7 Isomorphisms of Graphs

**Definition**

Two graphs that are the same except for the labeling of their vertices and edges are called *isomorphic*. The word isomorphism comes from the Greek, meaning "same form." Isomorphic graphs are those that have essentially the same form

Let $G$ and $G'$ be graphs with vertex sets $V(G)$ and $V(G')$ and edge sets $E(G)$ and $E(G')$, respectively. $\boldsymbol{G}$ is isomorphic to $\boldsymbol{G'}$ if, and only if, there exist one-to-one correspondences $g : V(G) \to V(G')$ and $h : E(G) \to E(G')$ that preserve the edge-endpoint functions of $G$ and $G'$ in the sense that for each $v \in V(G)$ and $e \in E(G)$, $v$ is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

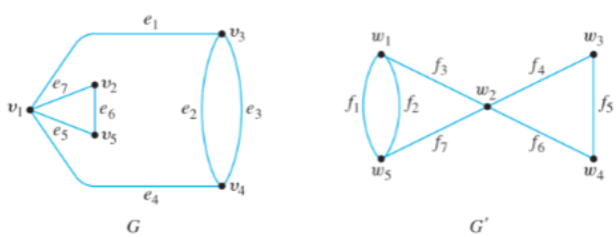
$$v \text{ is an endpoint of } e \Leftrightarrow g(v) \text{ is an endpoint of } h(e).$$

In other words,

$G$ is isomorphic to $G'$ if, and only if, the vertices and edges of $G$ and $G'$ can be matched up by one-to-one, onto functions in such a way that the edges between corresponding vertices correspond to each other.

**Example 7.1**

Show that the following two graphs are isomorphic.




***Solution:***

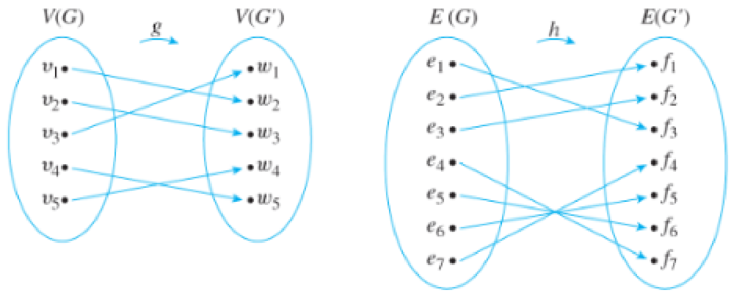
To solve this problem, you must find functions g: $V(G) \to V(G')$ and h: $E(G) \to E(G')$ such that for each $v \in V(G)$ and $e \in E(G)$, v is an endpoint of e if, and only if, $g(v)$ is an endpoint of $h(e)$

**Note:-**

Setting up such functions is partly a matter of trial and error and partly a matter of deduction.

For instance, since $e_2$ and $e_3$ are parallel (*have the same endpoint*), $h(e_2)$ and $h(e_3)$ must be parallel also.

One pair of functions for showing isomorphism between the two graphs is shown below




# 8 Trees

**Definition**

In mathematics, a tree is a connected graph that does not contain any circuits. Mathematical trees are similar in certain ways to their botanical namesakes.

A graph is said to be circuit-free if, and only if, it has no circuits. A graph is called a tree if, and only if, it is circuit-free and connected. A **trivial tree** is a graph that consists of a single vertex. A graph is called a forest if, and only if, it is circuit-free and not connected.

**Example 8.1** (Trees and Non-trees)
All the graphs shown in figure 4 are trees, whereas those in figure 5 are not.

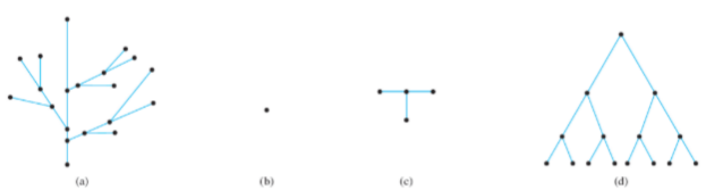



Figure 4: Trees

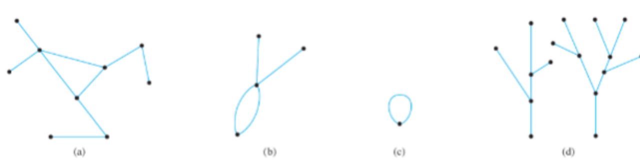



Figure 5: Non-Trees

# Examples of Trees

**Example 8.2** (A Decision Tree)
During orientation week, a college administers a mathematics placement exam to all entering students. The exam consists of two parts, and placement recommendations are made as indicated by the tree shown below in figure 6.

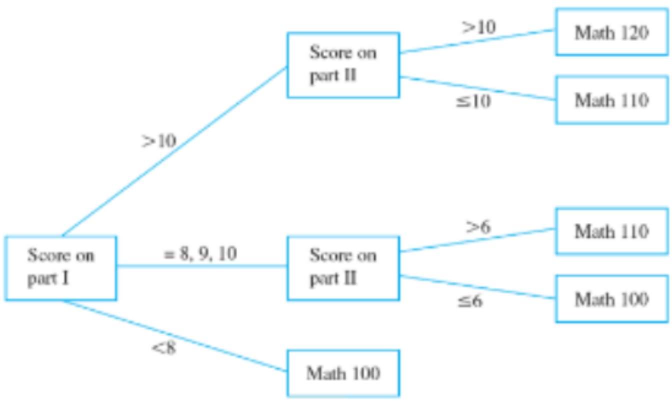



Figure 6

We read the tree from left to right to decide what course should be recommended for a student who scored 9 on part I and 7 on part II.

Since the student socred 9 on part I, the score on part II is checked.

Since it is greater than 6, the student should be advised to take Math 110.

## 8.1 Characterizing Trees

**Defintion**

There is a somewhat suprising relation between the number of vertices and the number of edges of a tree.

It turns out that if $n$ is a postive integer, then any tree with $n$ vertices (no matter what its shape) has $n-1$ edges.

Perhaps even more surprisingly, a partial converse to this fact is also true - namely, any connected graph with $n$ vertices and $n-1$ edges is a tree.

It follows from these facts that if even one new edge (but no new vertex) is added to a tree, the resulting graph must contain a circuit.

Also, from the fact that removing an edge from a circuit does not disconnect a graph, it can be shown that every connected graph has a subgraph that is a tree.

It follows that if $n$ is a positive integer, any graph with $n$ vertices and *fewer* than $n-1$ edges is not connected.

**Theorem 8.1**
Any tree that has more than one vertex has at least one vertex of degree 1.

**Theorem 8.2**
For any positive integer $n$, any tree with $n$ vertices has $n-1$ edges.

**Terminal Vertices**

Let $T$ be a tree. If $T$ has at least two vertices, then a vertex of degree 1 in $T$ is called a **leaf** (or a **terminal vertex**), and a vertex of degree greater than 1 in $T$ is called an **internal vertex** (or a **branch vertex**). The unique vertex in a trivial tree is also called a **leaf** or **terminal vertex**.

**Example 8.3**
Find all leaves (or terminal vertices) and all internal (or branch) vertices in the following tree.


***Solution:***

The terminal vertices are

$$v_0, v_2, v_4, v_5, v_7, v_8$$

The internal vertices are

$$v_6, v_1, v_3$$

**Example 8.4**
A graph $G$ has ten vertices and twelve edges. Is it a tree?

***Solution:***

No, By definition of Theorem 8.2, any tree with ten vertices has nine edges, not twelve.

### 8.2 Rooted Trees