Calculating the Concentration Ratio

To calculate the concentration ratio, r_t, of the plant's current contents, we measure (for any time t):

- The volume of product in the system, v_t
- The flow of permeate, f_2, t
- The flow of retentate, f_3, t

The volume of permeate, $v_{2,t}$ can be calculated from the flow rate and the duration of time, Δt:

$$v_{2,t} = f_{2,t} \cdot \Delta t \tag{1}$$

In a similar way, we can calculate the volume of retentate:

$$v_{3,t} = f_{3,t} \cdot \Delta t \tag{2}$$

From those measurements we can calculate the volume going into the system

$$v_{1,t} = v_t - v_{t-1} + v_{2,t} + v_{3,t} \tag{3}$$

To make the calculation for the concentration ratio, we must consider the mass of solids.

We begin by saying that product going into the tank has a concentration ratio of one (i.e. it is full-strength product). This product has a solids content of C. It is unimportant what units are used for C, but for instance it could be measured as grams per litre.

We wish to know the mass of solids in the system, m_t, at time t. This can be calculated from the volume of liquid in the system, its current concentration ratio and the solids content of full-strength product:

$$m_t = v_t \cdot r_t \cdot C \tag{4}$$

Equally, the mass of the system at time $t-1$ can be calculated:

$$m_{t-1} = v_{t-1} \cdot r_{t-1} \cdot C \tag{5}$$

The mass of solids going into the system (m_1) is the volume going in ($v_{1,t}$), multiplied by the ratio of full-strength product (one), multiplied by the solids content of full-strength product (C):

$$m_{1,t} = v_{1,t} \cdot C \tag{6}$$

The mass of solids going out via the retentate line, m_3 is the product of the volume out of the retentate line, the concentration ratio of the system and the solids content of full-strength product. We use r_{t-1} as a convenient approximation for the concentration ratio.

$$m_{3,t} = v_{3,t} \cdot r_{t-1} \cdot C \tag{7}$$

The mass of solids going out the permeate line, m_2, is the product of the volume out of the permeate line, the concentration ratio of the system, the solids content of full-strength product and a k factor which designates the proportion of solids that pass via the permeate line. A k factor of 0 would mean that no solids permeate the membrane, while a k factor of one would indicate the membrane was not filtering the product. It is possible the k-factor is dependent on the current concentration ratio, but this effect will not be considered.

$$m_{2,t} = v_{2,t} \cdot r_{t-1} \cdot C \cdot k \tag{8}$$

The solids content of the system at time t is the solids content of the system at time $t-1$ plus the solids added into the system, $m_{1,t}$, less the solids removed via the permeate line $m_{2,t}$, less the solids removed via the retentate line $m_{3,t}$:

$$m_t = m_{t-1} + m_{1,t} - m_{2,t} - m_{3,t} \tag{9}$$

Replacing the masses with their respective volume-based equivalents gives:

$$m_t = v_t \cdot r_t \cdot C = v_{t-1} \cdot r_{t-1} \cdot C + v_{1,t} \cdot C - v_{2,t} \cdot r_{t-1} \cdot C \cdot k - v_{3,t} \cdot r_{t-1} \cdot C \tag{10}$$

Eliminating C gives:

$$v_t \cdot r_t = v_{t-1} \cdot r_{t-1} + v_{1,t} - v_{2,t} \cdot r_{t-1} \cdot k - v_{3,t} \cdot r_{t-1} \tag{11}$$

We then replace $v_{1,t}$ with what we are measuring (from equation 3):

$$v_t \cdot r_t = v_{t-1} \cdot r_{t-1} + v_t - v_{t-1} + v_{2,t} + v_{3,t} - v_{2,t} \cdot r_{t-1} \cdot k - v_{3,t} \cdot r_{t-1} \tag{12}$$

And then factor out the repeating expressions to get:

$$v_t \cdot r_t = v_{t-1} \cdot (r_{t-1} - 1) + v_t + v_{2,t}(1 - r_{t-1} \cdot k) + v_{3,t}(1 - r_{t-1}) \tag{13}$$

Divide both sides by v_t gives:

$$r_t = 1 + \frac{v_{t-1} \cdot (r_{t-1} - 1) + v_{2,t}(1 - r_{t-1} \cdot k) + v_{3,t}(1 - r_{t-1})}{v_t} \tag{14}$$