Interest rates project

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Hull white model

Question 1

$$r_t = r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u$$

We want ot prove that :

$$\mathbb{E}[r_t|\mathcal{F}_s] = r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

So let's start with :

$$\mathbb{E}[r_t|\mathcal{F}_s] = \mathbb{E}\left[r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u|\mathcal{F}_s\right]$$

By linearity of the expectation we have :

$$\mathbb{E}[r_t|\mathcal{F}_s] = \mathbb{E}\left[r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}\right] + \sigma\mathbb{E}\left[\int_s^t e^{-a(t-u)} dW_u|\mathcal{F}_s\right]$$

The integral $\int_s^t e^{-a(t-s)} dW_u$ is a well defined stochastic integral. Thus it is a martingale. So it's conditional expectation is equal to the value of the integral at t = 0 which is 0. So the term is null.

We are then left with :

$$\mathbb{E}[r_t|\mathcal{F}_s] = \mathbb{E}\left[r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}|\mathcal{F}_s\right]$$

$$\mathbb{E}[r_t|\mathcal{F}_s] = \mathbb{E}\left[r_s e^{-a(t-s)}|\mathcal{F}_s\right] + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

$$\mathbb{E}[r_t|\mathcal{F}_s] = e^{-a(t-s)}\mathbb{E}\left[r_s|\mathcal{F}_s\right] + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

Since r_s is $\mathcal{F}_f$, measurable we get :

$$\mathbb{E}[r_t|\mathcal{F}_s] = e^{-a(t-s)} r_s + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

For the variance term, we want to show that :

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$

In the short rate model, the first part is not random, the variance is null. So we are left with :

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \mathbb{V}ar\left[\sigma \int_s^t e^{-a(t-u)} dWu|\mathcal{F}_s\right]$$

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2\mathbb{V}ar\left[\int_s^t e^{-a(t-u)} dWu|\mathcal{F}_s\right]$$

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2\left(\mathbb{E}\left[\left(\int_s^t e^{-a(t-u)} dWu\right)^2|\mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t e^{-a(t-u)} dWu|\mathcal{F}_s\right]^2\right)$$

The second term is null, so :

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2 \mathbb{E}\left[\left(\int_s^t e^{-a(t-u)} dWu\right)^2 |\mathcal{F}_s\right]$$

By independence of each increments to past values :

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2 \mathbb{E}\left[\left(\int_s^t e^{-a(t-u)} dWu\right)^2\right]$$

And with the isometry property :

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2 \mathbb{E}\left[\int_s^t e^{-2a(t-u)} du\right]$$

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \sigma^2 \mathbb{E}\left[\frac{1}{2a}\left[e^{-2a(t-u)}\right]_s^t\right]$$

$$\mathbb{V}ar[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$

Question 2

We have :

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma dWt$$

We multiply each side by e^{at}

$$dr_t e^{at} = \left(\theta(t)e^{at} - ar_t e^{at}\right) dt + \sigma dWte^{at}$$

$$dr_t e^{at} + ar_t e^{at} dt = \theta(t)e^{at} dt + \sigma dW_t e^{at}$$

$$d\left(r_t e^{at}\right) = \theta(t)e^{at} + \sigma dW_t e^{at}$$

$$\int_s^t d\left(r_t e^{at}\right) = \int_s^t \theta(u)e^{au} du + \int_s^t \sigma e^{au} dW_u$$

$$r_t e^{at} = r_s e^{as} + \int_s^t \theta(u)e^{au} du + \int_s^t \sigma e^{au} dW_u$$

With $\theta(u) = \frac{\partial f^M(0,u)}{\partial u} + af^M(0,u) + \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right)$ so :

$$r_t = r_s e^{-a(t-s)} + \int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} + af^M(0,u) + \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right)\right) e^{-a(t-u)} du + \int_s^t \sigma e^{-a(t-u)} dW_u$$

let's focus on the second term :

$$\begin{aligned}
&\int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} + af^M(0,u) + \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right)\right) e^{-a(t-u)} du \\
&= \int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} e^{-a(t-u)} + af^M(0,u)e^{-a(t-u)} + \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right) e^{-a(t-u)}\right) du \\
&= \int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} e^{-a(t-u)} + af^M(0,u)e^{-a(t-u)}\right) du + \int_s^t \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right) e^{-a(t-u)} du \\
&= \int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} e^{-a(t-u)} + af^M(0,u)e^{-a(t-u)}\right) du + \int_s^t \frac{\sigma^2}{2a}\left(1 - e^{-2au}\right) e^{-a(t-u)} du
\end{aligned}$$

The first term gives :

$$
\begin{aligned}
&\int_s^t \left(\frac{\partial f^M(0,u)}{\partial u} + af^M(0,u) + \frac{\sigma^2}{2a}\left(1-e^{-2au}\right)\right) e^{-a(t-u)} du \\
&= \int_s^t d\left(f^M(0,u)e^{-a(t-u)}\right) du \\
&= \left[f^M(0,u)e^{-a(t-u)}\right]_s^t \\
&= f^M(0,t) + f^M(0,s)e^{-a(t-s)}
\end{aligned}
$$

We are left with :

$$
r_t = r_s e^{-a(t-s)} + f^M(0,t) - f^M(0,s)e^{-a(t-s)} + \int_s^t \frac{\sigma^2}{2a}\left(1-e^{-2au}\right) e^{-a(t-u)} du + \int_s^t \sigma e^{-a(t-u)} dW_u
$$

The third term :

$$
\begin{aligned}
&\int_s^t \frac{\sigma^2}{2a}\left(1-e^{-2au}\right) e^{-a(t-u)} du \\
&= \frac{\sigma^2}{2a}\int_s^t \left(e^{-a(t-u)} - e^{-a(t+u)}\right) du \\
&= \frac{\sigma^2}{2a}\left[\frac{e^{-a(t-u)}}{a} + \frac{e^{-a(t+u)}}{a}\right]_s^t \\
&= \frac{\sigma^2}{2a^2}\left(1 - e^{-a(t-s)} + e^{-2at} - e^{-a(t+s)}\right) \\
&= \frac{\sigma^2}{2a^2}\left(1 + e^{-2at} - e^{-a(t+s)} - e^{-a(t-s)}\right) \\
&= \frac{\sigma^2}{2a^2}\left(1 - 2e^{-at} + e^{-2at} + 2e^{-at} - e^{-a(t+s)} - e^{-a(t-s)}\right) \\
&= \frac{\sigma^2}{2a^2}\left(1 - 2e^{-at} + e^{-2at}\right) + \frac{\sigma^2}{2a^2}\left(2e^{-at} - e^{-a(t+s)} - e^{-a(t-s)}\right) \\
&= \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 + \frac{\sigma^2}{2a^2}\left(2e^{-at} - e^{-a(t+s)} - e^{-a(t-s)}\right) \\
&= \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 + \frac{\sigma^2}{2a^2}\left(2e^{-as} - e^{-2as} - 1\right) e^{-a(t-s)} \\
&= \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 - \frac{\sigma^2}{2a^2}\left(1 - e^{-2s}\right)^2 e^{-a(t-s)}
\end{aligned}
$$

At the end :

$$
r_t = r_s e^{-a(t-s)} + f^M(0,t) - f^M(0,s)e^{-a(t-s)} + \frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2 - \frac{\sigma^2}{2a^2}\left(1-e^{-2s}\right)^2 e^{-a(t-s)} + \int_s^t \sigma e^{-a(t-u)} dW_u
$$

$$
r_t = r_s e^{-a(t-s)} + f^M(0,t) + \frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2 - f^M(0,s)e^{-a(t-s)} - \frac{\sigma^2}{2a^2}\left(1-e^{-2s}\right)^2 e^{-a(t-s)} + \int_s^t \sigma e^{-a(t-u)} dW_u
$$

by defining $\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2$

We get :

$$
r_t = r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \int_s^t \sigma e^{-a(t-u)} dW_u
$$

Question 4

We have :

$$y(\tau) = \int_0^\tau \frac{f(s)}{\tau} ds$$

$$\tau y(\tau) = \int_0^\tau f(s) ds$$

$$\frac{d}{d\tau} \tau y(\tau) = f(\tau)$$

Where $y(\tau)$ is :

$$y(\tau) = \beta_0 + \beta_1 \left[\frac{1 - e^{\frac{-\tau}{\lambda}}}{\frac{\tau}{\lambda}} \right] + \beta_2 \left[\frac{1 - e^{\frac{-\tau}{\lambda}}}{\frac{\tau}{\lambda}} - e^{\frac{-\tau}{\lambda}} \right]$$

$$y(\tau) = \beta_0 + \beta_1 \frac{\lambda}{\tau} \left[1 - e^{\frac{-\tau}{\lambda}} \right] + \beta_2 \frac{\lambda}{\tau} \left[1 - e^{\frac{-\tau}{\lambda}} - \frac{\tau}{\lambda} e^{\frac{-\tau}{\lambda}} \right]$$

$$y(\tau) = \frac{\lambda}{\tau} \left(\beta_0 \frac{\tau}{\lambda} + \beta_1 \left[1 - e^{\frac{-\tau}{\lambda}} \right] + \beta_2 \left[1 - e^{\frac{-\tau}{\lambda}} - \frac{\tau}{\lambda} e^{\frac{-\tau}{\lambda}} \right] \right)$$

$$y(\tau) \frac{\tau}{\lambda} = \beta_0 \frac{\tau}{\lambda} + \beta_1 \left[1 - e^{\frac{-\tau}{\lambda}} \right] + \beta_2 \left[1 - e^{\frac{-\tau}{\lambda}} - \frac{\tau}{\lambda} e^{\frac{-\tau}{\lambda}} \right]$$

By computing the derivative :

$$\begin{aligned}
&\frac{1}{\lambda} \frac{d}{d\tau} \tau y(\tau) \\
&= \frac{d}{d\tau} \left(\beta_0 \frac{\tau}{\lambda} + \beta_1 \left[1 - e^{\frac{-\tau}{\lambda}} \right] + \beta_2 \left[1 - e^{\frac{-\tau}{\lambda}} - \frac{\tau}{\lambda} e^{\frac{-\tau}{\lambda}} \right] \right) \\
&= \frac{\beta_0}{\lambda} + \beta_1 \frac{e^{\frac{-\tau}{\lambda}}}{\lambda} + \beta_2 \frac{e^{\frac{-\tau}{\lambda}}}{\lambda} - \beta_2 \frac{e^{\frac{-\tau}{\lambda}}}{\lambda} + \beta_2 \tau \frac{e^{\frac{-\tau}{\lambda}}}{\lambda^2} \\
&= \frac{1}{\lambda} \left(\beta_0 + e^{\frac{-\tau}{\lambda}} \left[\beta_1 + \frac{\tau \beta_2}{\lambda} \right] \right)
\end{aligned}$$

So we are left with :

$$f(\tau) = \frac{d}{d\tau} \tau y(\tau) = \beta_0 + e^{\frac{-\tau}{\lambda}} \left[\beta_1 + \frac{\tau \beta_2}{\lambda} \right]$$

Question 5

$\theta(t)$ is defined as :

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial t} + a f^M(0,t) + \frac{\sigma^2}{2a} \left(1 - e^{-2at} \right)$$

Where $f^M(0,t) = \beta_0 + e^{\frac{-t}{\lambda}} \left(\beta_1 + \frac{\beta_2 t}{\lambda} \right)$
And :

$$\frac{\partial f^M(0,t)}{\partial t} = \frac{e^{\frac{-t}{\lambda}}}{\lambda} \left(\beta_2 - \beta_1 - \frac{\beta_2 t}{\lambda} \right)$$

Which gives :

$$\theta(t) = \frac{e^{\frac{-t}{\lambda}}}{\lambda} \left(\beta_2 - \beta_1 - \frac{\beta_2 t}{\lambda} \right) + a\beta_0 + a \frac{e^{\frac{-t}{\lambda}}}{\lambda} (\lambda \beta_1 + \beta_2 t) + \frac{\sigma^2}{2a} \left(1 - e^{-2at} \right)$$

So :

$$\theta(t) = a\beta_0 + \frac{e^{\frac{-t}{\lambda}}}{\lambda} \left(\beta_2 + \beta_1(\lambda a - 1) + \beta_2(1 + at - \frac{t}{\lambda}) \right) + \frac{\sigma^2}{2a} \left(1 - e^{-2at} \right)$$

Question 6

Mean reversion in interest rate models is based on the idea that interest rates tend to return to a long-term average over time due to fundamental economic forces. Central banks (e.g., the Federal Reserve, ECB, BoJ) influence short-term interest rates through monetary policy tools like open market operations, interest rate targets, and forward guidance.
When rates are too high, central banks may lower them to stimulate borrowing and investment.
When rates are too low, they may raise them to prevent excessive inflation.
This creates a tendency for interest rates to revert to a long-run equilibrium level (natural rate of interest).

Question 7

So we have :

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t$$

By applying Ito :

$$dP(t,T) = \left(\frac{\partial P}{\partial t} + (\theta(t) - ar_t) \frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} \right) dt$$

With the no arbitrage condition, we must have :

$$\frac{\partial P}{\partial t} + (\theta(t) - ar_t) \frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} = r_t P(t,T)$$

By replacing in the formula above :

$$dP(t,T) = r_t P(t,T)dt + \sigma \frac{\partial P}{\partial r^2} dW_t$$

In the hull white model, we have :

$$P(t,T) = A(t,T)e^{-\Phi(t,T)r_t}$$

so :

$$\frac{\partial P}{\partial r} = -\Phi(t,T)P(t,T)$$

Thus :

$$dP(t,T) = r_t P(t,T)dt - \sigma\Phi(t,T)P(t,T)dW_t$$

$$dP(t,T) = P(t,T)\left[r_t dt - \sigma\Phi(t,T)dW_t\right]$$

Let : $Z_t = \frac{P(t,T)}{B_t}$ with $B_t = e^{\int_0^t r_s ds}$

$$\frac{dZ_t}{Z_t} = d\left(\frac{P(t,T)}{B_t}\right)\frac{B_t}{P(t,T)} = \frac{dP(t,T)}{P(t,T)} - \frac{dB_t P(t,T) B_t}{B_t^2 P(t,T)}$$

So :

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma\Phi(t,T)dW_t$$

$$\frac{dB_t}{B_t} = dln(B_t) = r_t dt$$

Finally, $\frac{dZ_t}{Z_t} = (r_t - r_t)dt - \sigma\Phi(t,T)dW_t = -\sigma\Phi(t,T)dW_t$

With Girsanov theorem :

$$dW_t^T = dW_t + \sigma\Phi(t,T)dt$$

With the expression of r_t :

$$dr_t = (\theta(t) - ar_t)\, dt + \sigma dW_t^T - \sigma^2 \Phi(t,T) dt$$

The Hull White under T - forward measure :

$$r_t = \big(\theta(t) - ar_t - \sigma^2 \Phi(t,T)\big)\, dt + \sigma dW_t$$

With $\sigma^2 \Phi(t,T)$ being the drift adjustment.

Question 10

To price a swap on an amortizing loan, we need to adjust the notional at each payment date according to the amortization schedule. Fixed leg payments are then calculated using the reduced notionals, and their present value reflects this change. The floating leg may also need adjustment if it follows the same notional reduction.

Question 16

Correlation among interest rates is important in swaption valuation because a swap's value depends on multiple forward rates. These rates move together to varying degrees, and their correlation affects the volatility of the swap rate. Higher correlation increases swap rate volatility, making the swaption more valuable.

Question 17

The 1-factor Hull-White (HW) model does not capture the full correlation among interest rates because it uses only a single source of randomness (i.e., one Brownian motion) to drive the entire term structure. This means all forward rates are perfectly correlated in their movements — they all respond in the same way to the single factor. In reality, different forward rates can move differently based on various market influences, leading to imperfect (less than 100%) correlations. Because the 1-factor HW model cannot represent these more complex, imperfect co-movements between rates, it lacks the flexibility to accurately capture the full correlation structure observed in the market. Multi-factor models are needed to reflect this behavior.