Derivatives pricing under the Hull White Model

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1 The Hull White Model

We intend to value interest rate derivatives using the Hull White Model. Recall that the evolution of the short rate process is determined as follows:

$$dr_t = (\theta(t) - a\, r_t)\, dt + \sigma dW_t. \tag{1}$$

Recall here that the introduction of a time-variant θ aims to perfectly fit the initial term structure. Let $f^M(0,T)$ denote the observed instantaneous forward rate at the current time :

$$f^M(0,t) = -\frac{\partial \ln P^M(0,T)}{\partial T}. \tag{2}$$

The HW model reproduces the current yield curve if :

$$\theta(t) = \frac{\partial f^M}{\partial T}(0,T) + af^M(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \tag{3}$$

Assuming that the previous condition is fulfilled one can show that :

$$\begin{aligned} r_t &= r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u \\ \alpha(t) &= f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 \end{aligned} \tag{4}$$

$$\begin{aligned} \mathbb{E}[r_t|\mathcal{F}_s] &= r_s e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} \\ Var[r_t|\mathcal{F}_s] &= \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right) \end{aligned} \tag{5}$$

The price of the Zero coupon bond is given by :

$$P(t,T) = A(t,T)e^{-\Phi(t,T)r(t)} \tag{6}$$

$$\begin{cases} A(t,T) = \dfrac{P^M(0,T)}{P^M(0,t)} exp\left[\Phi(t,T)f^M(0,T) - \dfrac{\sigma^2}{4a}(1 - e^{-2at})\Phi^2(t,T)\right] \\ \Phi(t,T) = \dfrac{1}{a}\left(1 - e^{-a(T-t)}\right) \end{cases} \tag{7}$$

1. Prove the two relations in Equation (5).

2. Assuming that equation (3) is satisfied show, using Itô Lemma, prove equation (4).

3. Download the ZC yield curve from the ECB website and use it to calibrate the Nelson Siegel yield curve model. Under this model, the yield for maturity τ is given by :

$$y(\tau) = \beta_0 + \beta_1 \left[\frac{1 - exp\left(\frac{-\tau}{\lambda}\right)}{\frac{\tau}{\lambda}} \right] + \beta_2 \left[\frac{1 - exp\left(\frac{-\tau}{\lambda}\right)}{\frac{\tau}{\lambda}} - exp\left(\frac{-\tau}{\lambda}\right) \right] \tag{8}$$

   Determine the set of optimal parameters to fit the initial yield curve.

4. Using the definition of the instantaneous forward rate $y(\tau) = \int_0^\tau \frac{f(s)}{\tau} ds$, derive its expression under the Nelson Siegel characterization of the yield curve.

5. Determine $\theta(t)$ in order to fit the model to the initial yield curve using equation (3).

6. What is the economic foundation behind mean reversion in interest rate models ?

7. Define the T-forward measure and derive the Hull and White Model dynamics under this measure.

2 Swap Valuation

8. Given the term structure of interest rates, construct a swap pricer that calculates the forward swap interest rate for an in fine bond.

9. Compare the obtained valuation with the theoretical value (using the exact formula for the swap rate),

10. How to extend the swap pricer to an amortizing loans ?

3 Cap Valuation

From now on, we assume that $r_0 = 0.03$, $a = 0.1$ and $\sigma = 0.01$. In this section, we aim to value a cap with a 10-year maturity with an annual settlement frequency. The strike is set to 2%

11. Determine the cap value with a Monte Carlo simulation under the risk neutral measure.

12. Conduct the valuation under the T- forward measure

13. Determine a closed-form value for the cap. Recall that the price of a European put option with strike K and maturity T and written on a zero-coupon bond with maturity S at time $t \in [0, T]$ is given by :

$$ZBP(t, T, S, K) = -P(t, S)\Phi(-h) + KP(t, T)\Phi(-h + \tilde{\sigma})$$

with :

$$\tilde{\sigma} = \sigma\sqrt{\frac{1-e^{-2a(T-t))}}{2a}}B(T,S)$$
$$B(T,S) = \frac{1-e^{-a(S-T)}}{a}$$
$$h = \frac{1}{\tilde{\sigma}}\ln\left(\frac{P(t,S)}{P(t,T)K}\right) + \frac{\tilde{\sigma}}{2}$$

4 Swaption valuation

In this section, we aim to value of swaption with a 10-year maturity and an annual settlement frequency. The strike is set to 2%

14. Propose a Monte Carlo simulation for the pricing of the European Swaption.

15. Propose a Monte Carlo simulation for the pricing of the Bermudan Swaption.

16. Why is the correlation among interest rates important in the valuation of swaptions ?

17. Explain why the 1-factor HW model does not capture the correlation among interest rate models.

5 Practical information

- This project should be done before the 03^{rd} of April 2025.

- You are expected to send a full report of your work along all the codes (you are free to select the programming language of your choice)

- You can work in groups however each student should submit an individual report explaining the methodology.

References

D. Brigo and F. Mercurio. *Interest rate models-theory and practice: with smile, inflation and credit*. Springer Science & Business Media, 2007.

J. Hull and A. White. One-factor interest-rate models and the valuation of interest-rate derivative securities. *Journal of financial and quantitative analysis*, 28(2):235–254, 1993.