Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day I: Appetizers

(1) Given integers a, b, c, d, form the line segment $[(a,b),(c,d)] \subset \mathbb{R}^2$ joining the points (a,b) and (c,d). Show that the number of integer points on this line segment is $\gcd(a-c, b-d)+1$.

(2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area $\frac{1}{2}$. (*Hint:* You may begin by "doubling" the triangle to form a parallelogram.)

(3) Pick four points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the Ehrhart polynomial of $\mathcal{P}$. (If you cannot think of a good example, consider the regular tetrahedron with vertices $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$.)

(4) Recall that the standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin. Verify that

$$L_\Delta(t) = \binom{d+t}{d} \qquad \text{and} \qquad L_{\Delta^\circ}(t) = \binom{t-1}{d}.$$

(If you'd like to amuse your colleagues, we can also write $L_{\Delta^\circ}(t) = (-1)^d \binom{d-t}{d}$.)

(5) Given a $(d-1)$-polytope $\mathcal{Q}$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ such that the origin is in $\mathcal{Q}$, we define the bipyramid $\mathrm{BiPyr}(\mathcal{Q})$ over $\mathcal{Q}$ as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \ldots, (\mathbf{v}_m, 0), (0, \ldots, 0, 1), \text{ and } (0, \ldots, 0, -1).$$

Show that $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{Q})}(z) = \dfrac{1+z}{1-z}\, \mathrm{Ehr}_{\mathcal{Q}}(z)$.

(6) Compute the Ehrhart polynomial of the octahedron

$$\Diamond = \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \le 1 \right\}$$

via the four different approaches outlined in the lecture:

(a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);
(b) disjoint triangulation into 8 standard tetrahedra;
(c) [`sage`] interpolation;
(d) [`sage`] generating function.

Generalize.

(7) [`sage`] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What's going on here?

(8) [`research problem`] Compute the Ehrhart polynomial of the Birkhoff—von Neumann polytope $\mathcal{B}_{10}$ or the volume of $\mathcal{B}_{11}$.

(9) Define the Eulerian number $A(d,k)$ through[1]

$$\sum_{j\geq 0} j^d\, z^j \;=\; \frac{\sum_{k=0}^{d} A(d,k)\, z^k}{(1-z)^{d+1}}\,.$$

Alternatively, we may think of the polynomial $\sum_{k=0}^{d} A(d,k)\, z^k$ is the numerator of the rational function

$$\left(z\frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) \;=\; \underbrace{z\frac{d}{dz}\cdots z\frac{d}{dz}}_{d\text{ times}}\left(\frac{1}{1-z}\right).$$

Prove the following:

$$\begin{aligned}
A(d,k) &= A(d,d+1-k)\,,\\
A(d,k) &= (d-k+1)\,A(d-1,k-1) + k\,A(d-1,k)\,,\\
\sum_{k=0}^{d} A(d,k) &= d!\,,\\
A(d,k) &= \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d.
\end{aligned}$$

(10) The permutahedron $\mathcal{P}_d \in \mathbb{R}^d$ is defined as the convex hull of

$$\{(\pi(1)-1,\, \pi(2)-1, \dots,\, \pi(d)-1) :\, \pi \in S_d\}\,,$$

where S_d is the set of all permutations of $\{1,2,\dots,d\}$. Show that P_d is a zonotope:

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d]\,,$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ are the standard unit vectors.

(11) Prove that $\mathcal{P}_d$ tiles the hyperplane spanned by it.

(12) Show that a sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n\geq 0} f(n)\, z^n \;=\; \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

MATTHIAS BECK https://matthbeck.github.io/

[1] There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j\geq 0}(j+1)^d\, z^j = \frac{\sum_{k=0}^{d} A(d,k) z^k}{(1-z)^{d+1}}$ instead.

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day II: Generating Functions & Complexity

(1) Compute the generating functions for

$$P_{\le n} := \{\lambda \in \mathbb{Z}^n : 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n\},$$

partitions into at most n parts. Adjust your computations for partitions into exactly n parts.

(2) Compute the integer-point transform of the cone $\mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(3) Let $n \ge 3$ and

$$T_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \ge \cdots \ge \lambda_1 \ge 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\},$$

the set of all "n-gon partitions."

(a) Compute the generating function for T_3.

(b) What makes your computation more complicated for $n > 3$?

(c) Compute the generating function for

$$\widetilde{T}_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \ge \cdots \ge \lambda_1 \ge 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \le \lambda_n\},$$

and conclude from it the generating function for T_n.

(4) Recall the lecture-hall partitions

$$\mathrm{LH}_n := \left\{\lambda \in \mathbb{Z}^n : 0 \le \frac{\lambda_1}{1} \le \frac{\lambda_2}{2} \le \cdots \le \frac{\lambda_n}{n}\right\}.$$

Compute the generators of the underlying cone, and verify the first few instances of the Lecture-Hall Theorem:

$$\sum_{\lambda \in \mathrm{LH}_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}.$$

(5) [`sage`] Pick five points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the Ehrhart polynomial of $\mathcal{P}$.

(6) [`research problem`] Choose $d+1$ of the 2^d vertices of the unit d-cube, and let Δ be the simplex defined by their convex hull.

(a) Which choice of vertices maximizes $\operatorname{vol}\Delta$?

(b) What is the maximum volume of such a Δ?

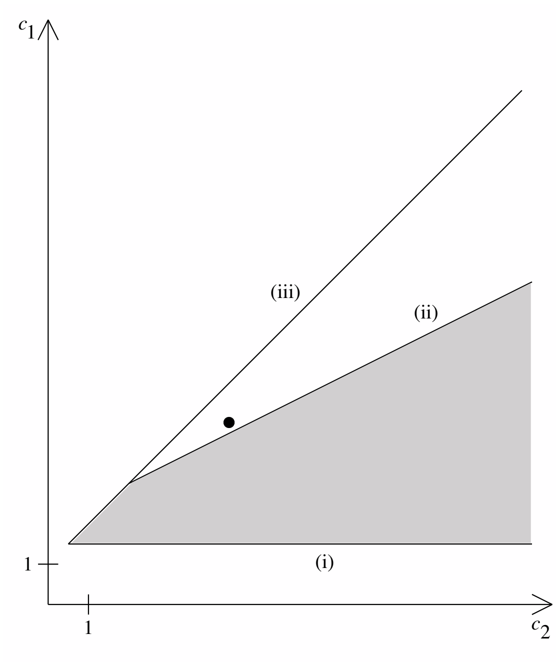
(7) Give an explicit bijection between the faces (including $\varnothing$) of a given polytope $\mathcal{P}$ and the faces (excluding $\varnothing$) of its homogenization $\operatorname{cone}(\mathcal{P})$.

(8) Suppose $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$ are lattice polytopes. Prove that the *convolution* of their Ehrhart polynomials,

$$L(t) := \sum_{s=0}^{t} L_{\mathcal{P}}(s)\, L_{\mathcal{Q}}(t-s)$$

equals the Ehrhart quasipolynomial of the polytope given by the convex hull of $\mathcal{P} \times \{\mathbf{0}_n\} \times \{0\}$ and $\{\mathbf{0}_m\} \times \mathcal{Q} \times \{1\}$. Here $\mathbf{0}_d$ denotes the origin in $\mathbb{R}^d$.

(9) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



(10) [`research problem`] Give the corresponding classification picture of degree-3 Ehrhart polynomials.

(11) This exercise constructs triangulations. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, randomly choose $h_1, h_2, \ldots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \ldots, (\mathbf{v}_n, h_n)$. The *lower hull* of $\mathcal{Q}$ consists of all points that are *visible from below*: all points $(x_1, x_2, \ldots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \ldots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of $\mathcal{Q}$ is a face of $\mathcal{Q}$ that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of $\mathcal{Q}$ are simplices, and that their projections under π form a triangulation of $\mathcal{P}$.

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day III: Positivity, Reciprocity & Friends

(1) Let $\mathcal{P} \subset \mathbb{R}^d$ be a full-dimensional polytope. Show that there is no $\mathbf{q} \in \mathbb{R}^d$ such that $\mathbb{H}^{\mathbf{q}}\mathcal{P} = \mathcal{P}$.

(2) Let $\mathcal{P} \subset \mathbb{R}^d$ be a full-dimensional polyhedron with dissection $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$. If $\mathbf{q} \in \mathbb{R}^d$ is generic relative to each $\mathcal{P}_i$, then

$$\mathbb{H}_{\mathbf{q}}\mathcal{P} \ = \ \mathbb{H}_{\mathbf{q}}\mathcal{P}_1 \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_2 \uplus \cdots \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_m$$

and

$$\mathbb{H}^{\mathbf{q}}\mathcal{P} \ = \ \mathbb{H}^{\mathbf{q}}\mathcal{P}_1 \uplus \mathbb{H}^{\mathbf{q}}\mathcal{P}_2 \uplus \cdots \uplus \mathbb{H}^{\mathbf{q}}\mathcal{P}_m \, .$$

(3) Fix linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in \mathbb{Z}^d$ and consider the simplicial cone

$$\mathcal{K} \ := \ \mathbb{R}_{\ge 0}\mathbf{v}_1 + \mathbb{R}_{\ge 0}\mathbf{v}_2 + \cdots + \mathbb{R}_{\ge 0}\mathbf{v}_d \, .$$

Prove that, for

$$\widehat{\mathcal{K}} \ := \ \mathbb{R}_{\ge 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{\ge 0}\mathbf{v}_{m-1} + \mathbb{R}_{> 0}\mathbf{v}_m + \cdots + \mathbb{R}_{> 0}\mathbf{v}_d \, ,$$

there exists $\mathbf{q} \in \mathbb{R}^d$ (generic relative to $\mathcal{K}$) such that

$$\widehat{\mathcal{K}} \ = \ \mathbb{H}_{\mathbf{q}}\mathcal{K} \, .$$

Conversely, show that, for every generic $\mathbf{q} \in \mathbb{R}^d$ relative to $\mathcal{K}$, the half-open cone $\mathbb{H}_{\mathbf{q}}\mathcal{K}$ is of the form $\widehat{\mathcal{K}}$ for some reordering of the $\mathbf{v}_j$s and some m.

(4) Prove Ehrhart's theorem for half-open lattice polytopes.

(5) Let S be an m-dimensional subset of $\mathbb{R}^d$ (i.e., the affine span of S has dimension m). Then we define the *relative volume* of S to be

$$\operatorname{vol} S \ := \ \lim_{n \to \infty} \frac{1}{n^m} \left| n\, S \cap \mathbb{Z}^d \right| .$$

   (a) Convince yourself that $\operatorname{vol} S$ is the usual volume if $m = d$.

   (b) Show that, if $\Delta \subset \mathbb{R}^d$ is an m-dimensional lattice simplex, then the leading coefficient of $L_\Delta(n)$ (i.e., the coefficient of n^m) equals $\operatorname{vol} \Delta$.

(6) [`sage`] Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.

(7) [`sage`] For a lattice polytope $\mathcal{P}$, the numerator of the generating function is the *h^*-polynomial* of $\mathcal{P}$. Give a non-unimodal[2] example of an h^*-polynomial.

[2] A polynomial is *unimodal* if its coefficients increase up to some point and then decrease.

(8) [`research problem`] Now let $\mathcal{P} = \left\{ \mathbf{x} \in [0,1]^d : x_1 + x_2 + \cdots + x_d = k \right\}$, for your favorite integers $2 \le k \le d-2$. (This is the *(d,k)-hypersimplex*.) Prove that the h^*-polynomial of $\mathcal{P}$ is unimodal.

(9) Let $\mathcal{P}$ be a lattice d-polytope and write

$$\mathrm{Ehr}_{\mathcal{P}}(z) \;=\; \frac{h_d^* \, z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}} \, .$$

Prove that:

(a) $h_d^* = \left| \mathcal{P}^\circ \cap \mathbb{Z}^d \right|$.

(b) $h_1^* = \left| \mathcal{P} \cap \mathbb{Z}^d \right| - d - 1$.

(c) $h_0^* + h_1^* + \cdots + h_d^* = d! \, \mathrm{vol}(\mathcal{P})$.

(10) A *reflexive polytope* is a lattice polytope $\mathcal{P}$ such that the origin is the unique interior lattice point of $\mathcal{P}$ and[3]

$$L_{\mathcal{P}^\circ}(n) \;=\; L_{\mathcal{P}}(n-1) \qquad \text{for all } n \in \mathbb{Z}_{>0} \, . \tag{1}$$

Prove that if $\mathcal{P}$ is a lattice d-polytope that contains the origin in its interior and that has the Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) \;=\; \frac{h_d^* \, z^d + h_{d-1}^* \, z^{d-1} + \cdots + h_1^* \, z + h_0^*}{(1-z)^{d+1}} \, ,$$

then $\mathcal{P}$ is reflexive if and only if $h_k^* = h_{d-k}^*$ for all $0 \le k \le \frac{d}{2}$.

(11) For any polynomial $h(z)$ of degree d, show there exist unique polynomials $a(z)$ and $b(z)$ such that

$$h(z) = a(z) + z \, b(z) \qquad \text{where} \qquad a(z) = z^d \, a(\tfrac{1}{z}) \qquad \text{and} \qquad b(z) = z^{d-1} \, b(\tfrac{1}{z}) \, .$$

(There are many variations of this; e.g., we could leave out the z factor in front of $b(z)$.)

(12) Derive inequalities for the coefficients of $h(z)$ if we know that both $a(z)$ and $b(z)$ have only nonnegative coefficients.

[3] More generally, if the 1 on the right-hand side of (1) is replaced by an arbitrary fixed positive integer, we call $\mathcal{P}$ *Gorenstein*. You may think about how this exercise can be extended to Gorenstein polytopes.

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day IV: From h to h^*

(1) If you haven't done so already, work on the last two exercises from yesterday.

(2) Let $\mathbb{H}_{\mathbf{q}}\Delta$ be a unimodular simplex with r facets missing. Show that $h^*_{\mathbb{H}_{\mathbf{q}}\Delta}(z) = z^r$. Use this to give an alternative description of the h^*-polynomial of a lattice polytope that admits a unimodular triangulation.

(3) Given a triangulation T of the boundary of a d-polytope $\mathcal{P}$ and a point $\mathbf{v} \in \mathcal{P}^\circ$, construct a triangulation K of $\mathcal{P}$ consisting of T with the simplices $\operatorname{conv}(\Delta, \mathbf{v})$ for all $\Delta \in T$ appended; i.e., the new triangulation K comes from coning over T. Prove that

$$h_K(z) = h_T(z) \, .$$

(4) A lattice polytope is polytope $\mathcal{P}$ has the *integer decomposition property (IDP)* if for every positive integer k and every integer point $\mathbf{x} \in k\mathcal{P}$, there exist integer points $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k \in \mathcal{P}$ such that $\mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_k = \mathbf{x}$. Prove that every polytope that admits a unimodular triangulation is IDP.

(5) Let $\mathcal{P}$ be a d-dimensional lattice polytope. Show that if T is a unimodular triangulation of $\mathcal{P}$, then the f-vector of T can be computed via

$$f_{k-1} \ = \ \sum_{r=0}^{k} \binom{d+1-r}{k-r} h_r^*(\mathcal{P})$$

and conclude that $f_T(z) = h^*_{\mathcal{P}^\circ}(z+1)$.

(6) Under the same conditions as in the previous exercise, let $f_T^{\text{int}}(z)$ be the polynomial whose kth coefficient counts the number of k-simplices that are not on the boundary of P. Prove that $f_T^{\text{int}}(z) = h^*_{\mathcal{P}}(z+1)$.

(7) Let $\Delta = \operatorname{conv}(\mathbf{u}_0, \dots, \mathbf{u}_d) \subset \mathbb{R}^d$, $\Delta' = \operatorname{conv}(\mathbf{v}_0, \dots, \mathbf{v}_e) \subset \mathbb{R}^e$ be two unimodular simplices and let $\mathcal{P} = \Delta \times \Delta'$ be their Cartesian product.

   (a) Show that any $(d+e)$-simplex spanned by the vertices of $\mathcal{P}$ is unimodular.

   (b) We can identify the vertices of $\mathcal{P}$ with the nodes of the square grid $\{0, \dots, d\} \times \{0, \dots, e\}$. A *lattice path* from $(0,0)$ to (d,e) is a path on the grid that uses only unit steps $\rightarrow$ and $\uparrow$. Show that any such path encodes a unique $(d+e)$-simplex of $\mathcal{P}$.

   (c) Show that the collection of all such simplices yields a triangulation of $\mathcal{P}$.

   (d) Compute the h^*-vector of $\mathcal{P}$.