Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day II: Generating Functions & Complexity

(1) Compute the generating functions for

$$P_{\leq n} := \{\lambda \in \mathbb{Z}^n : 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\},$$

partitions into at most n parts. Adjust your computations for partitions into exactly n parts.

(2) Compute the integer-point transform of the cone $\mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

(3) Let $n \geq 3$ and

$$T_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\},$$

the set of all "n-gon partitions."

(a) Compute the generating function for T_3.

(b) What makes your computation more complicated for $n > 3$?

(c) Compute the generating function for

$$\widetilde{T}_n := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \leq \lambda_n\},$$

and conclude from it the generating function for T_n.

(4) Recall the lecture-hall partitions

$$\mathrm{LH}_n := \left\{\lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n}\right\}.$$

Compute the generators of the underlying cone, and verify the first few instances of the Lecture-Hall Theorem:

$$\sum_{\lambda \in \mathrm{LH}_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}.$$

(5) Pick five points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the Ehrhart polynomial of $\mathcal{P}$.

(6) [`research problem`] Choose $d+1$ of the 2^d vertices of the unit d-cube, and let Δ be the simplex defined by their convex hull.

(a) Which choice of vertices maximizes $\operatorname{vol}\Delta$?

(b) What is the maximum volume of such a Δ?

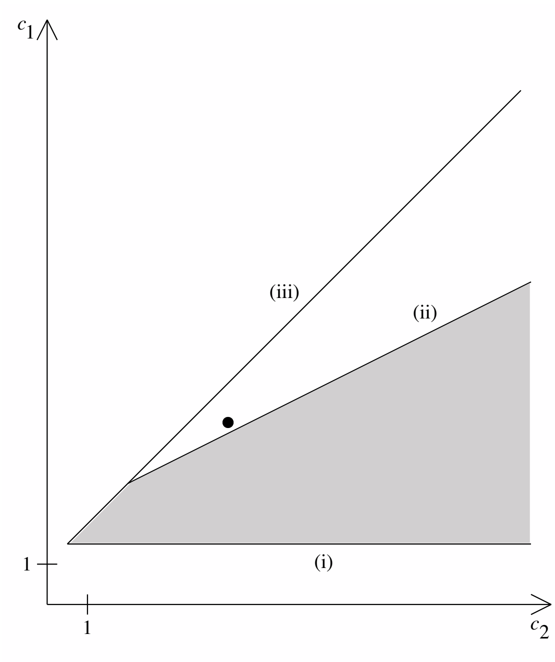
(7) Give an explicit bijection between the faces (including $\varnothing$) of a given polytope $\mathcal{P}$ and the faces (excluding $\varnothing$) of its homogenization $\operatorname{cone}(\mathcal{P})$.

(8) Suppose $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$ are lattice polytopes. Prove that the *convolution* of their Ehrhart polynomials,

$$L(t) := \sum_{s=0}^{t} L_{\mathcal{P}}(s)\, L_{\mathcal{Q}}(t-s)$$

equals the Ehrhart quasipolynomial of the polytope given by the convex hull of $\mathcal{P} \times \{\mathbf{0}_n\} \times \{0\}$ and $\{\mathbf{0}_m\} \times \mathcal{Q} \times \{1\}$. Here $\mathbf{0}_d$ denotes the origin in $\mathbb{R}^d$.

(9) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2t^2 + c_1t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



(10) [`research problem`] Give the corresponding classification picture of degree-3 Ehrhart polynomials.

(11) This exercise constructs triangulations. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, randomly choose $h_1, h_2, \ldots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \ldots, (\mathbf{v}_n, h_n)$. The *lower hull* of $\mathcal{Q}$ consists of all points that are *visible from below*: all points $(x_1, x_2, \ldots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \ldots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of $\mathcal{Q}$ is a face of $\mathcal{Q}$ that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of $\mathcal{Q}$ are simplices, and that their projections under π form a triangulation of $\mathcal{P}$.