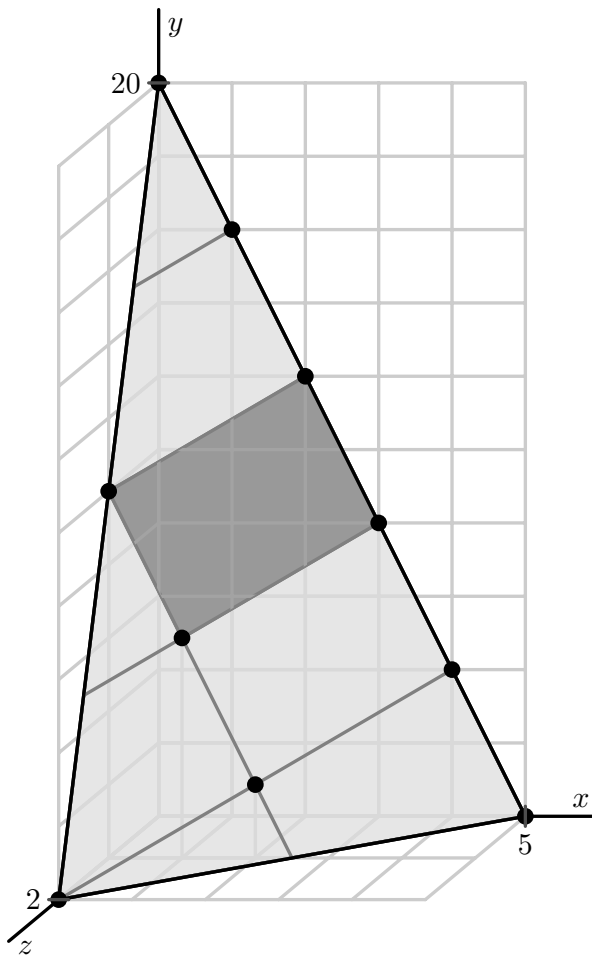


Ehrhart Polynomials



Matthias Beck

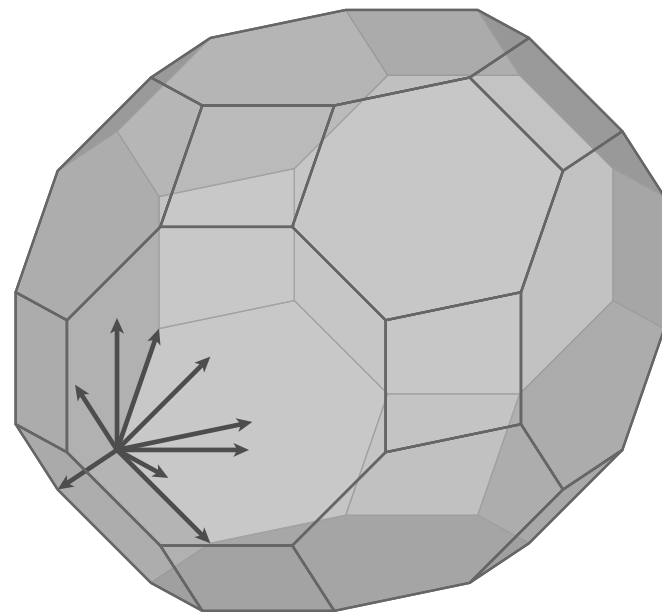
San Francisco State University

<https://matthbeck.github.io/>

VIII Encuentro Colombiano
De Combinatoria

“Science is what we understand well enough to explain to a computer, art is all the rest.”

Donald Knuth



A Sample Problem: Birkhoff–von Neumann Polytope

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES[®]

founded in 1964 by N. J. A. Sloane

[Hints](#)
(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

A037302 Normalized volume of Birkhoff polytope of $n \times n$ doubly-stochastic square matrices. If the volume² is $v(n)$, then $a(n) = ((n-1)^2)! * v(n) / n^{(n-1)}$.

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028,
12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099,
5091038988117504946842559205930853037841762820367901333706255223000 ([list](#); [graph](#); [refs](#); [listen](#); [history](#);
[text](#); [internal format](#))

OFFSET 1,3

COMMENTS The Birkhoff polytope is an $(n-1)^2$ -dimensional polytope in n^2 -dimensional space; its vertices are the $n!$ permutation matrices.
Is $a(n)$ divisible by n^2 for all $n \geq 4$? - [Dean Hickerson](#), Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

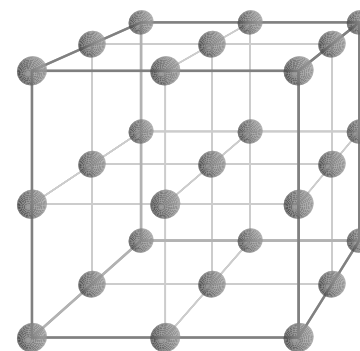
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d \dots$

► (list)
$$\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \dots z_d^{m_d}$$

► (count) $|\mathcal{P} \cap \mathbb{Z}^d|$



Discrete Volumes

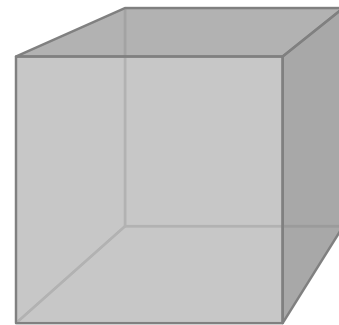
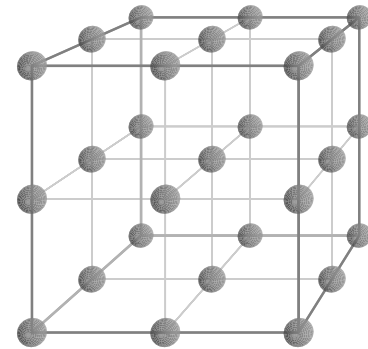
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$$\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$$



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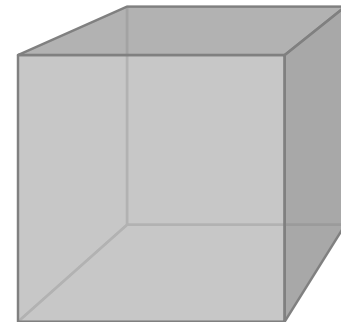
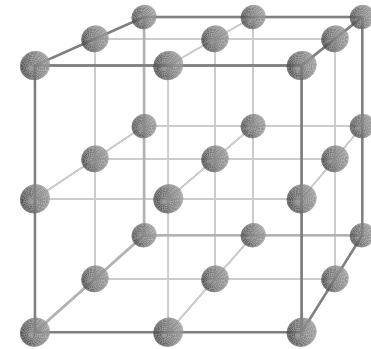
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Ehrhart function
$$L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = |t\mathcal{P} \cap \mathbb{Z}^d| \quad \text{for } t \in \mathbb{Z}_{>0}$$

Some Motivation

- ▶ Linear systems are **everywhere**, and so polyhedra are everywhere.

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- ▶ In applications, the **volume** of the polytope represented by a linear system measures some fundamental data of this system (“average”).
- ▶ Polytopes are basic geometric objects, yet even for these basic objects volume computation is **hard** and there remain many open problems.
- ▶ Many **discrete problems** in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

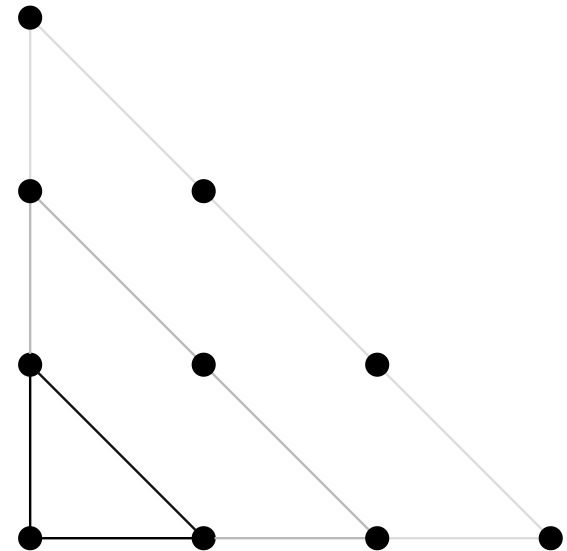
A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Z}^d

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$

Example 1:

$$\begin{aligned}\Delta &= \text{conv} \{(0, 0), (1, 0), (0, 1)\} \\ &= \{(x, y) \in \mathbb{R}_{\geq 0}^2 : x + y \leq 1\}\end{aligned}$$



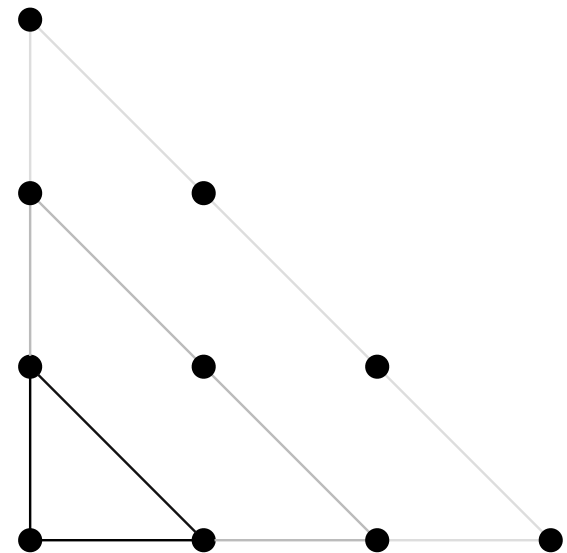
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Example 2:

$$\square = [0, 1]^d \text{ (the unit cube in } \mathbb{R}^d \text{)}$$

Ehrhart Polynomials



EH
1959

Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

Equivalently, $\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$ is rational:

$$\text{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1 - z)^{\dim \mathcal{P} + 1}}$$

where the **Ehrhart h-vector** $h(z)$ satisfies $h(0) = 1$ and $h(1) = (\dim \mathcal{P})! \text{vol}(\mathcal{P})$.

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Seeming dichotomy: $\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

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Equivalent descriptions of an Ehrhart polynomial:

- ▶ $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$
- ▶ via roots of $L_{\mathcal{P}}(t)$
- ▶ $\text{Ehr}_{\mathcal{P}}(z) \longrightarrow L_{\mathcal{P}}(t) = h_0 \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_d \binom{t}{d}$

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Open Problem Classify Ehrhart polynomials.

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Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the **interior** lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) = h_d \binom{t+d-1}{d} + h_{d-1} \binom{t+d-2}{d} + \cdots + h_0 \binom{t-1}{d}$$

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Corollary If $h_{d+1-k} > 0$ then $k\mathcal{P}^\circ$ contains an integer point.