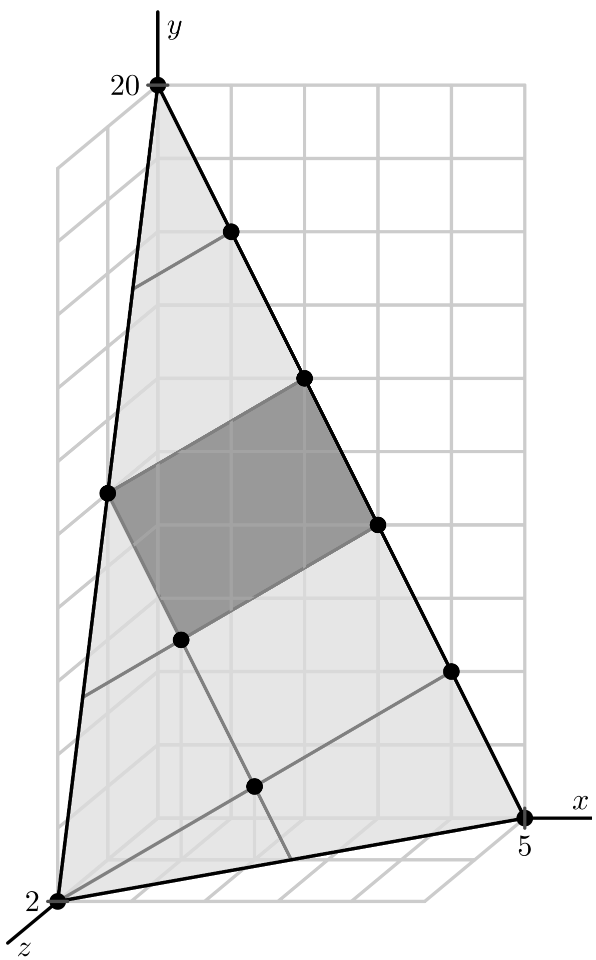
Ehrhart Polynomials

Day I: Appetizers



Matthias Beck

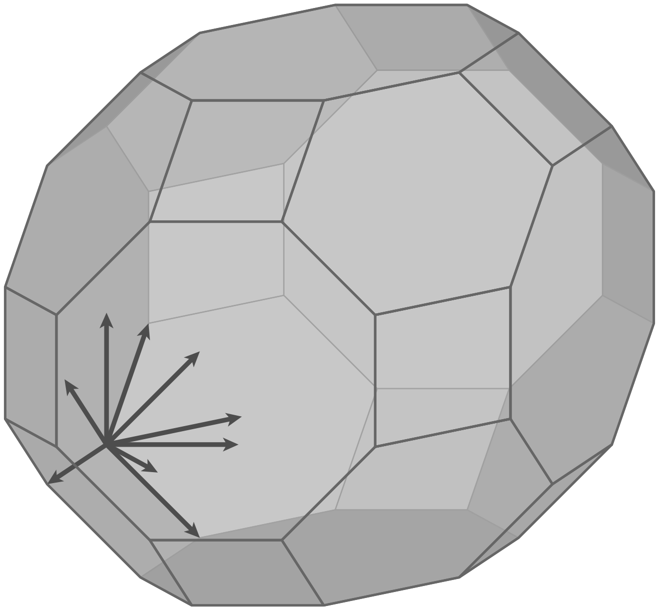
San Francisco State University

https://matthbeck.github.io/

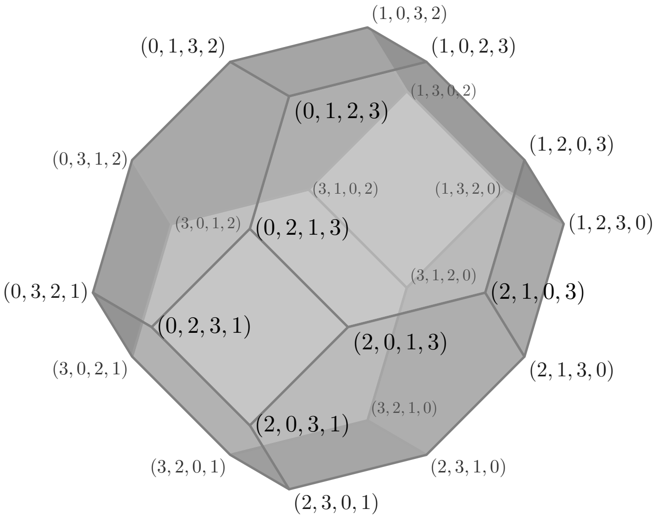
VIII Encuentro Colombiano
De Combinatoria

"Science is what we understand well enough to explain to a computer, art is all the rest."

Donald Knuth



Themes



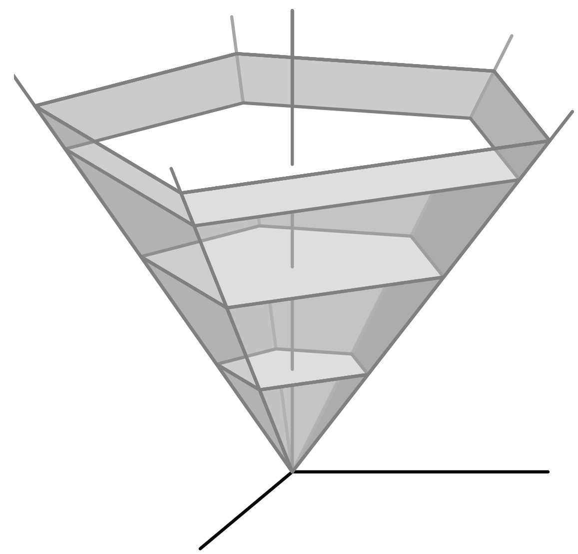
Discrete-geometric polynomials

Computation (complexity)

Generating functions

Combinatorial structures

Polyhedra



A Sample Problem: Birkhoff–von Neumann Polytope

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THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A037302 Normalized volume of Birkhoff polytope of n X n doubly-stochastic square matrices. If the volume is v(n), then a(n) = ((n-1)^2)! * v(n) / n^(n-1). 2

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028, 12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099, 5091038988117504946842559205930853037841762820367901333706255223000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS The Birkhoff polytope is an (n-1)^2-dimensional polytope in n^2-dimensional space; its vertices are the n! permutation matrices.

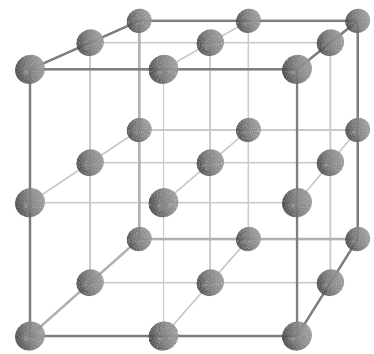
Is a(n) divisible by n^2 for all n>=4? - Dean Hickerson, Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geqslant 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leqslant k \leqslant n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leqslant j \leqslant n \end{array} \right\}$$

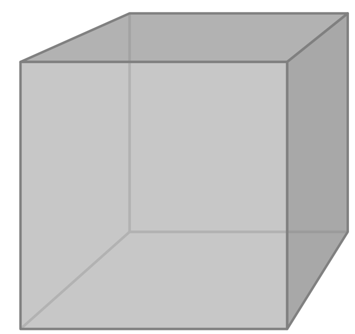
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .



- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$
- (count) $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$
- (volume) $\displaystyle \operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$



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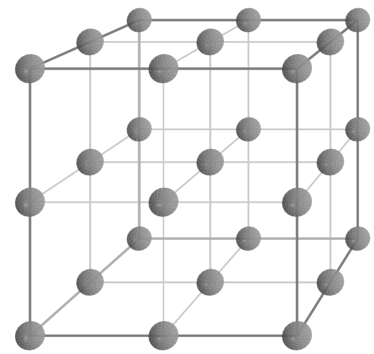

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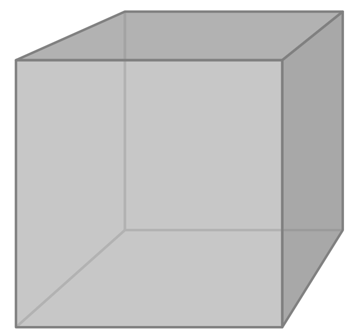

Ehrhart function $\displaystyle L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right| = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ for $t \in \mathbb{Z}_{>0}$

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

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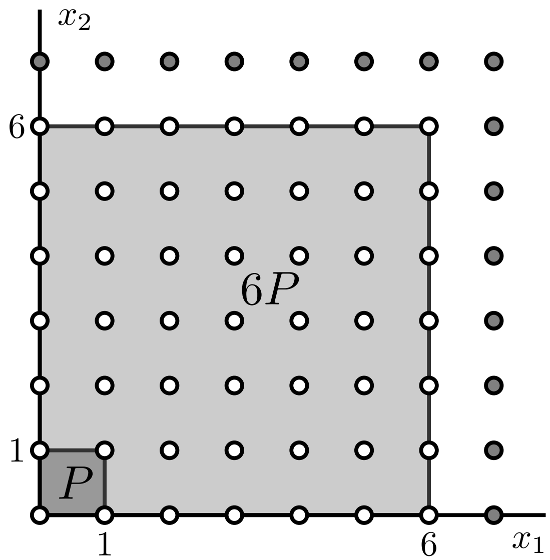
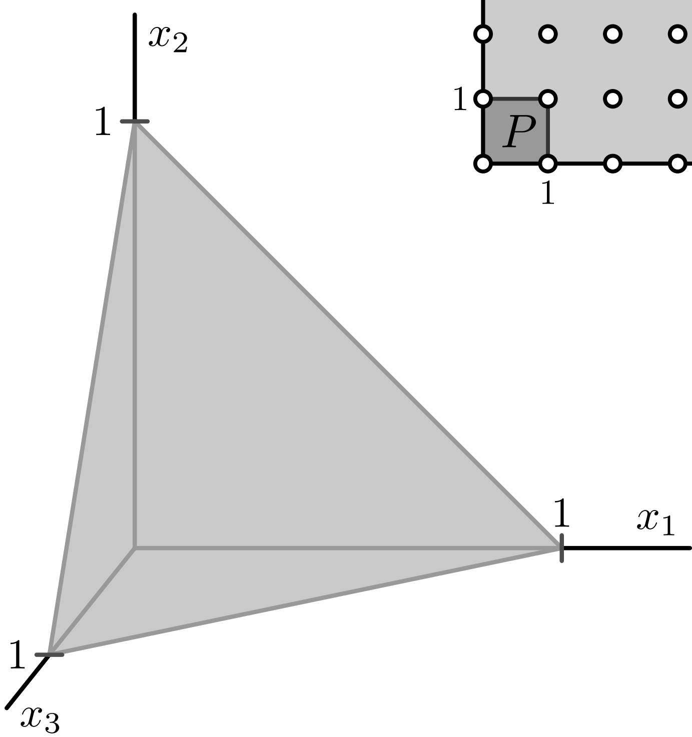
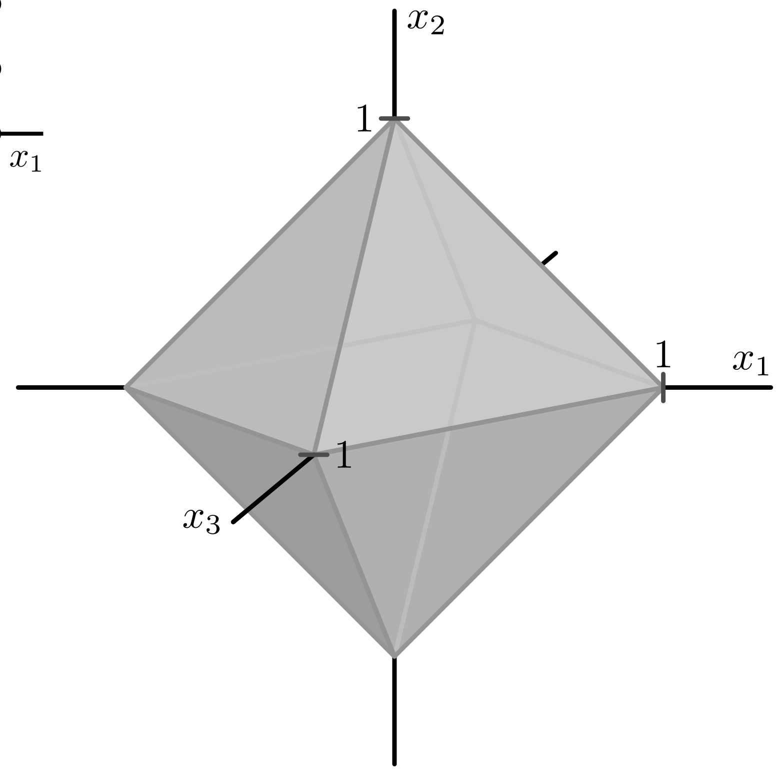
- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Also, polytopes are cool.

Today's Menu: Get Our Hands Dirty





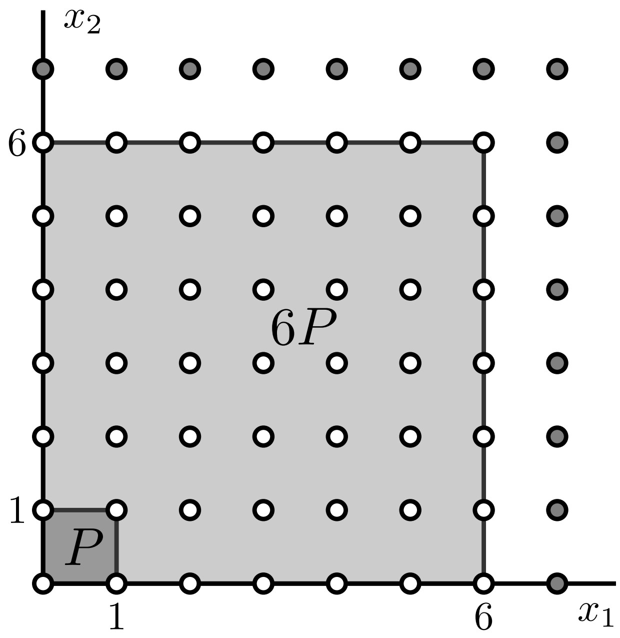


The Unit Cube

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) \; := \; \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

The unit cube in $\mathbb{R}^d$ is $\mathcal{P} \; = \; [0,1]^d \; = \; \left\{ \boldsymbol{x} \in \mathbb{R}^d : \; 0 \leqslant x_j \leqslant 1 \right\}$



$$\longrightarrow \quad L_{\mathcal{P}}(t) \; = \; (t+1)^d$$

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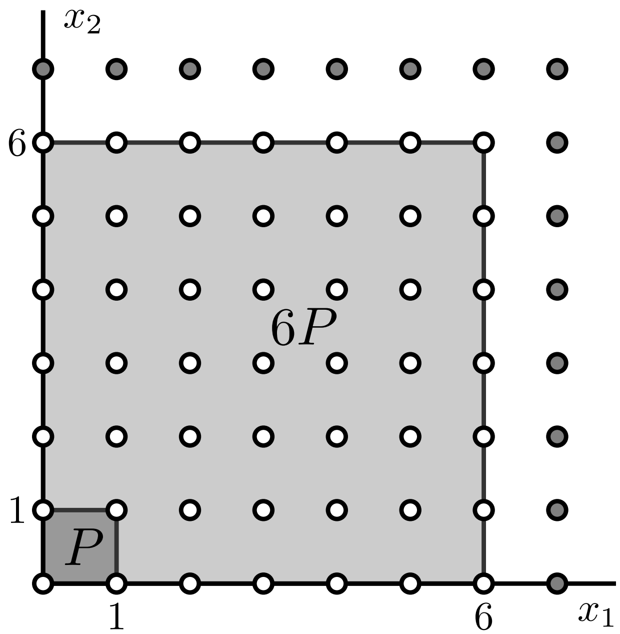


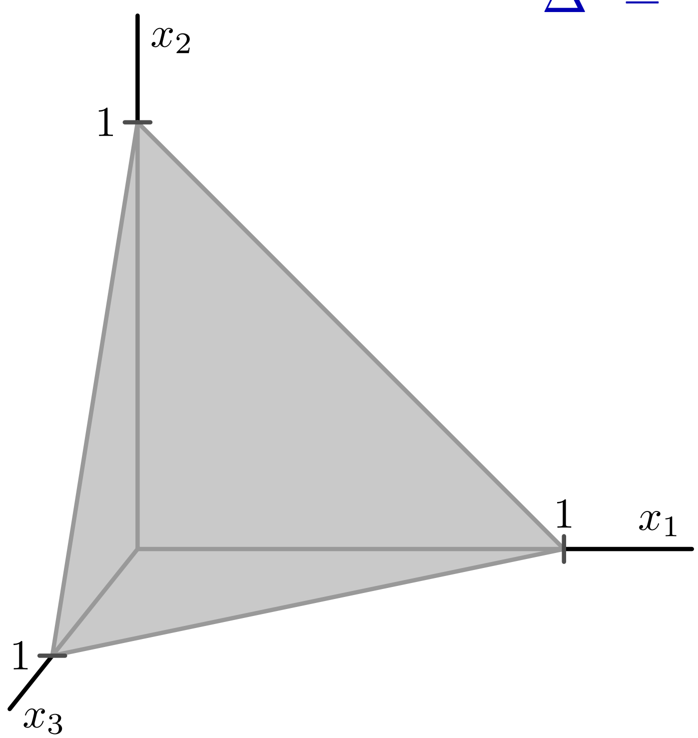
$$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$$

$$L_{\mathcal{P}^\circ}(t) = (t-1)^d$$

The Standard Simplex

The standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\geqslant 0} : \, x_1 + x_2 + \cdots + x_d \leqslant 1 \right\}$$



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$$\begin{aligned} L_\Delta(t) &= \# \left\{ (x_1, x_2 \dots, x_d) \in \mathbb{Z}^d_{\geqslant 0} : \, x_1 + x_2 + \cdots + x_d \leqslant t \right\} \\ &= \# \left\{ (x_1, x_2 \dots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1}_{\geqslant 0} : \, x_1 + x_2 + \cdots + x_{d+1} = t \right\} \\ &= \binom{d+t}{d} \end{aligned}$$

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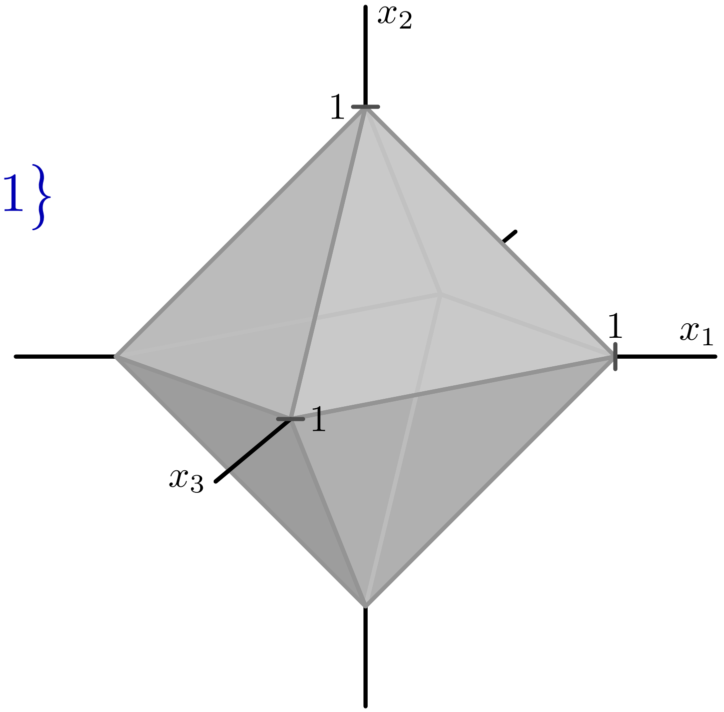
$$L_{\Delta^\circ}(t) \;=\; \binom{t-1}{d}$$

The Cross-Polytope

The cross-polytope $\Diamond \in \mathbb{R}^d$ is

$$\Diamond \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^d : \; |x_1| + |x_2| + \cdots + |x_d| \leqslant 1 \right\}$$



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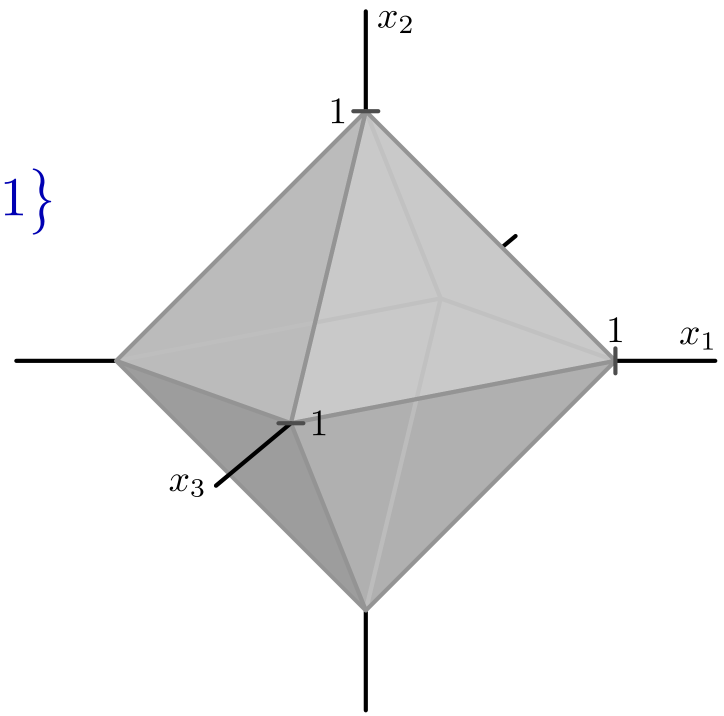

Let's compute $L_\Diamond(t)$ for $d = 3$ . . .

- Triangulation
- Disjoint triangulation
- Interpolation
- Generating function

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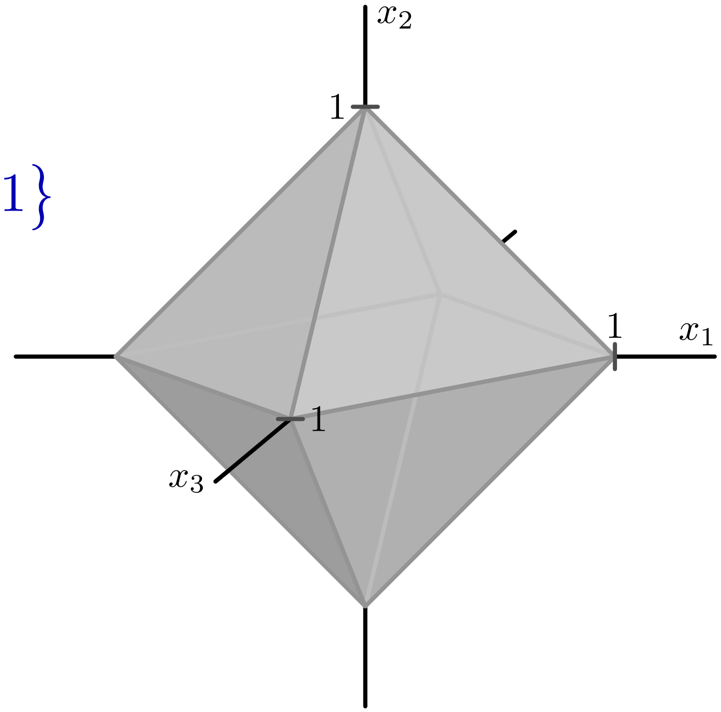

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- Triangulation

Dissect $\Diamond$ into 8 (standard) tetrahedra and use inclusion–exclusion to compute $L_\Diamond(t)$

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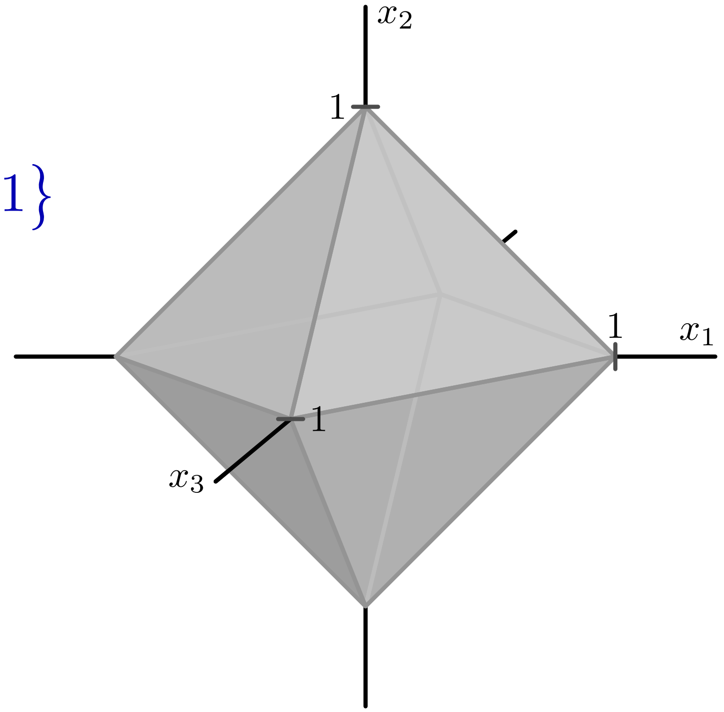

Let's compute $L_\Diamond(t)$ for $d = 3$ . . .

- Disjoint triangulation

Dissect $\Diamond$ into 8 half-open tetrahedra

The Cross-Polytope

The cross-polytope $\diamond \in \mathbb{R}^d$ is

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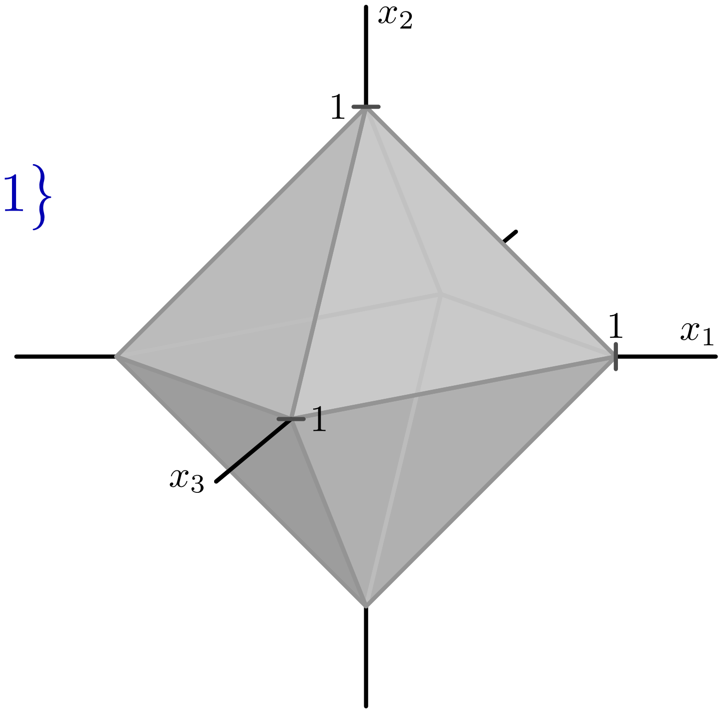

Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Interpolation

```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```

The Cross-Polytope

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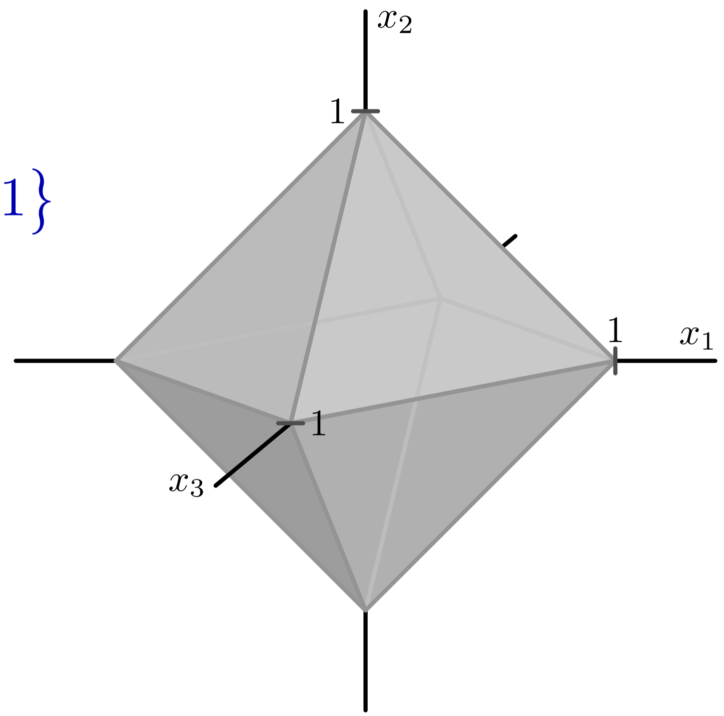

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- Generating function

$$\mathrm{Ehr}_\diamond(z) \;:=\; 1 + \sum_{t \geqslant 1} L_\diamond(t)\, z^t$$

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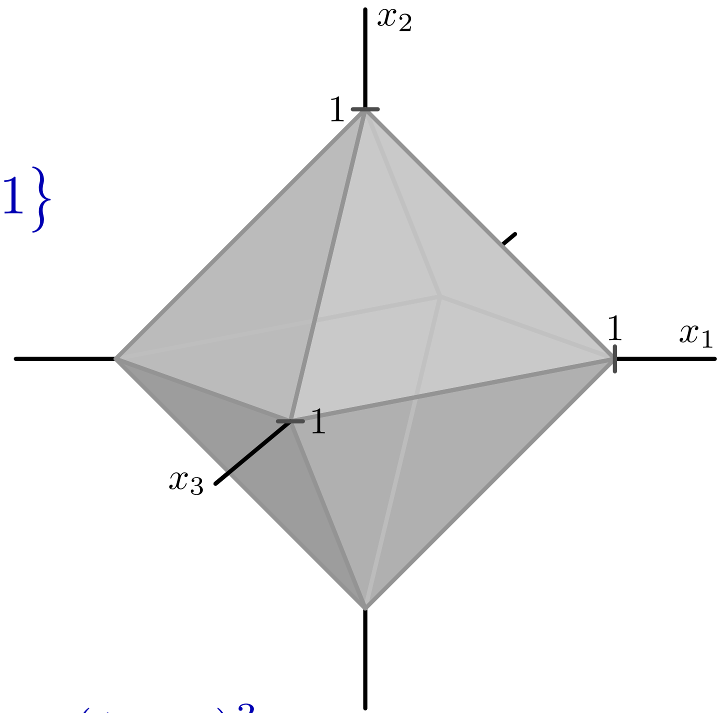

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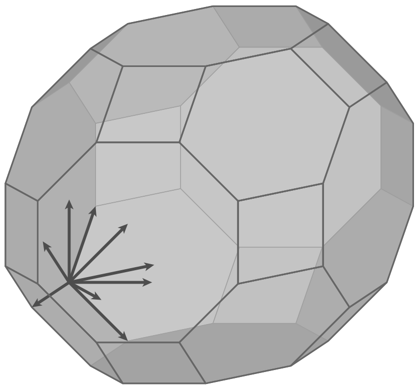
- Generating function

$$\mathrm{Ehr}_\diamond(z) \;:=\; 1 + \sum_{t \geqslant 1} L_\diamond(t)\, z^t \;=\; \frac{(1+z)^3}{(1-z)^4}$$

Exercise: $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{P})}(z) \;=\; \dfrac{1+z}{1-z}\,\mathrm{Ehr}_{\mathcal{P}}(z)$

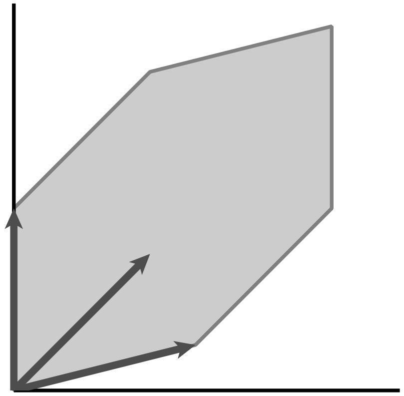
. . . for unit cubes $\longrightarrow$ Eulerian polynomials

Zonotopes



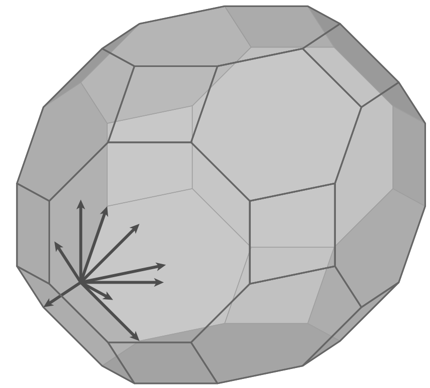
Line segment $[\boldsymbol{a},\boldsymbol{b}] := \{(1-\lambda)\,\boldsymbol{a} + \lambda\,\boldsymbol{b} :\ 0 \leqslant \lambda \leqslant 1\}$

Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2 \ := \ \{\boldsymbol{p} + \boldsymbol{q} :\ \boldsymbol{p} \in \mathcal{K}_1,\ \boldsymbol{q} \in \mathcal{K}_2\}$



Zonotope $\mathcal{Z} := [\boldsymbol{a}_1, \boldsymbol{b}_1] + [\boldsymbol{a}_2, \boldsymbol{b}_2] + \cdots + [\boldsymbol{a}_m, \boldsymbol{b}_m]$

Zonotopes



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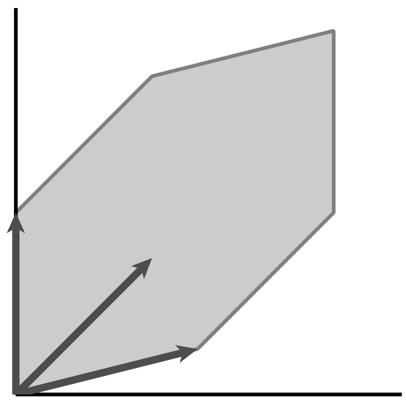

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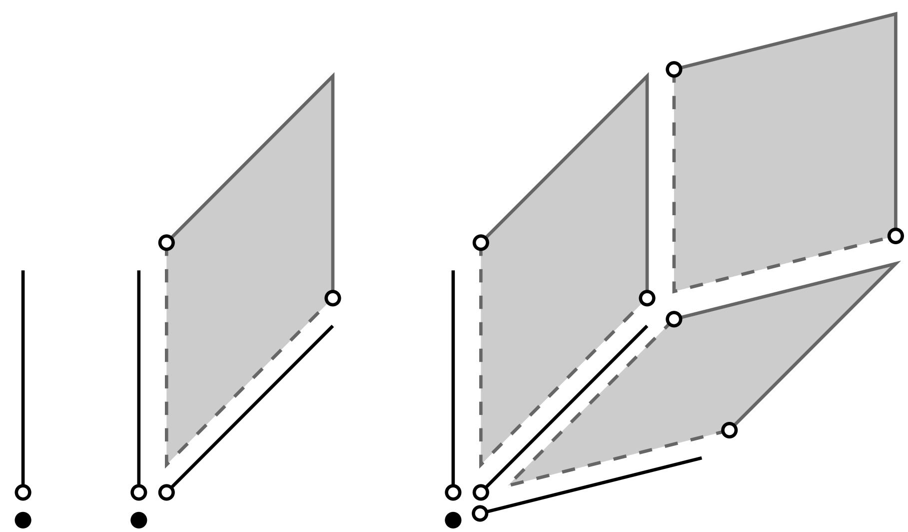
Every zonotope admits a tiling into parallelepipeds



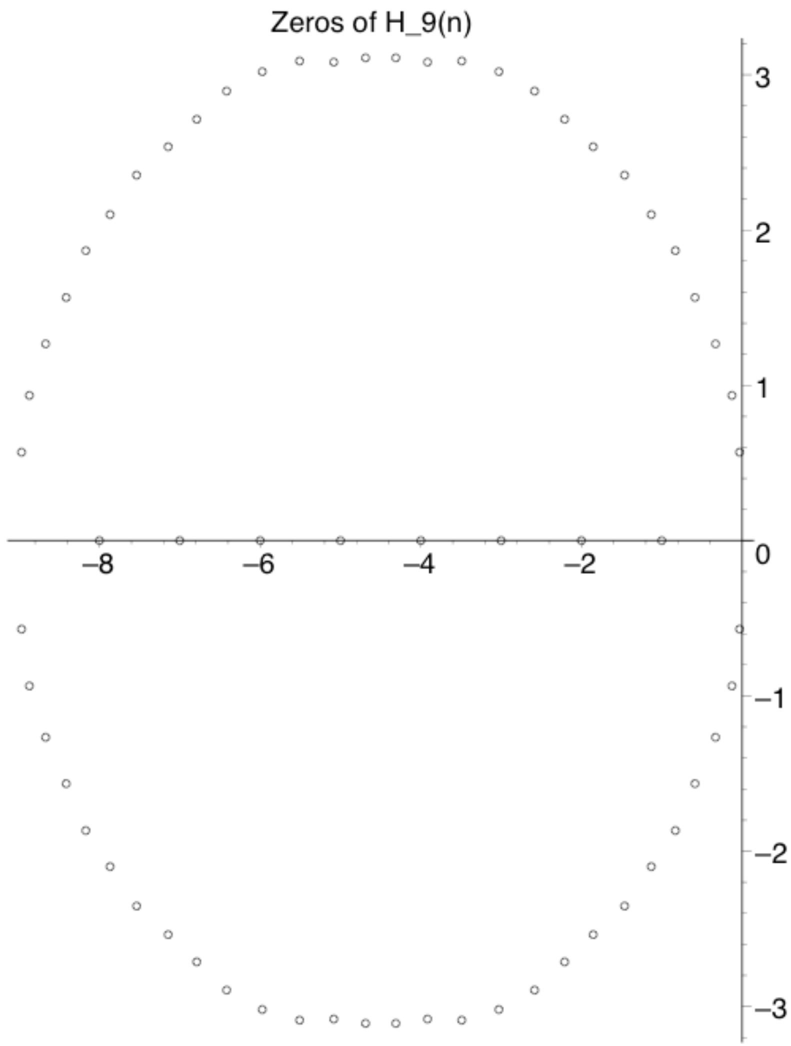
$\mathcal{P}$ — half-open d-parallelepiped

$\longrightarrow \quad L_{\mathcal{P}}(t) \ = \ \operatorname{vol}(\mathcal{P})\, t^d$

Recap Day I

- Volume computations $\longrightarrow$ don't agonize, discretize
- Integer-point counting in dilated polytopes $\longrightarrow$ polynomials
- Interpolation
- Generating functions
- Dissections: triangulations, tilings
- Tomorrow: enough practice, how does this work in theory?

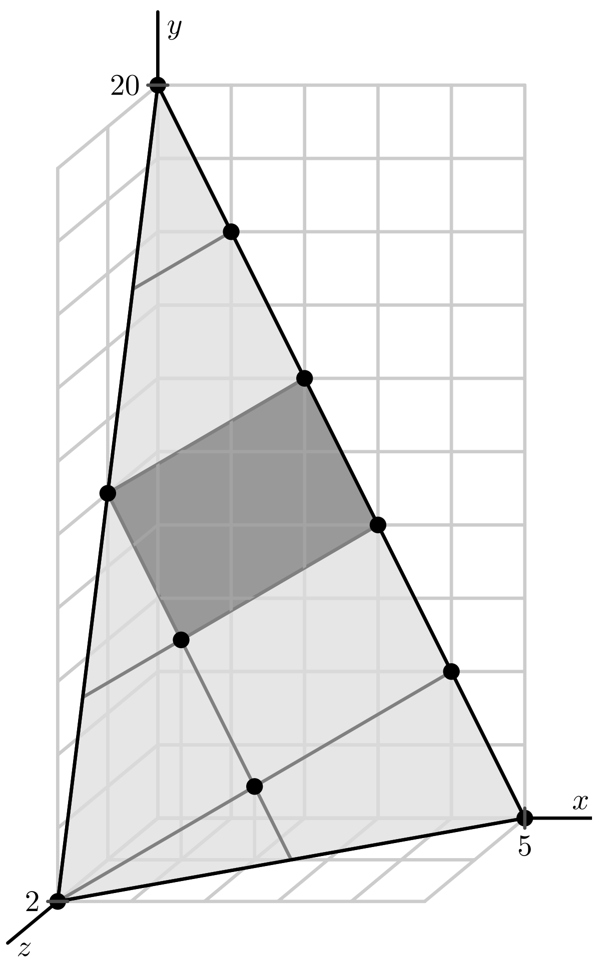
Birkhoff–von Neumann Revisited



For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).

Ehrhart Polynomials

Day II: Generating Functions & Complexity



Matthias Beck

San Francisco State University

https://matthbeck.github.io/

VIII Encuentro Colombiano
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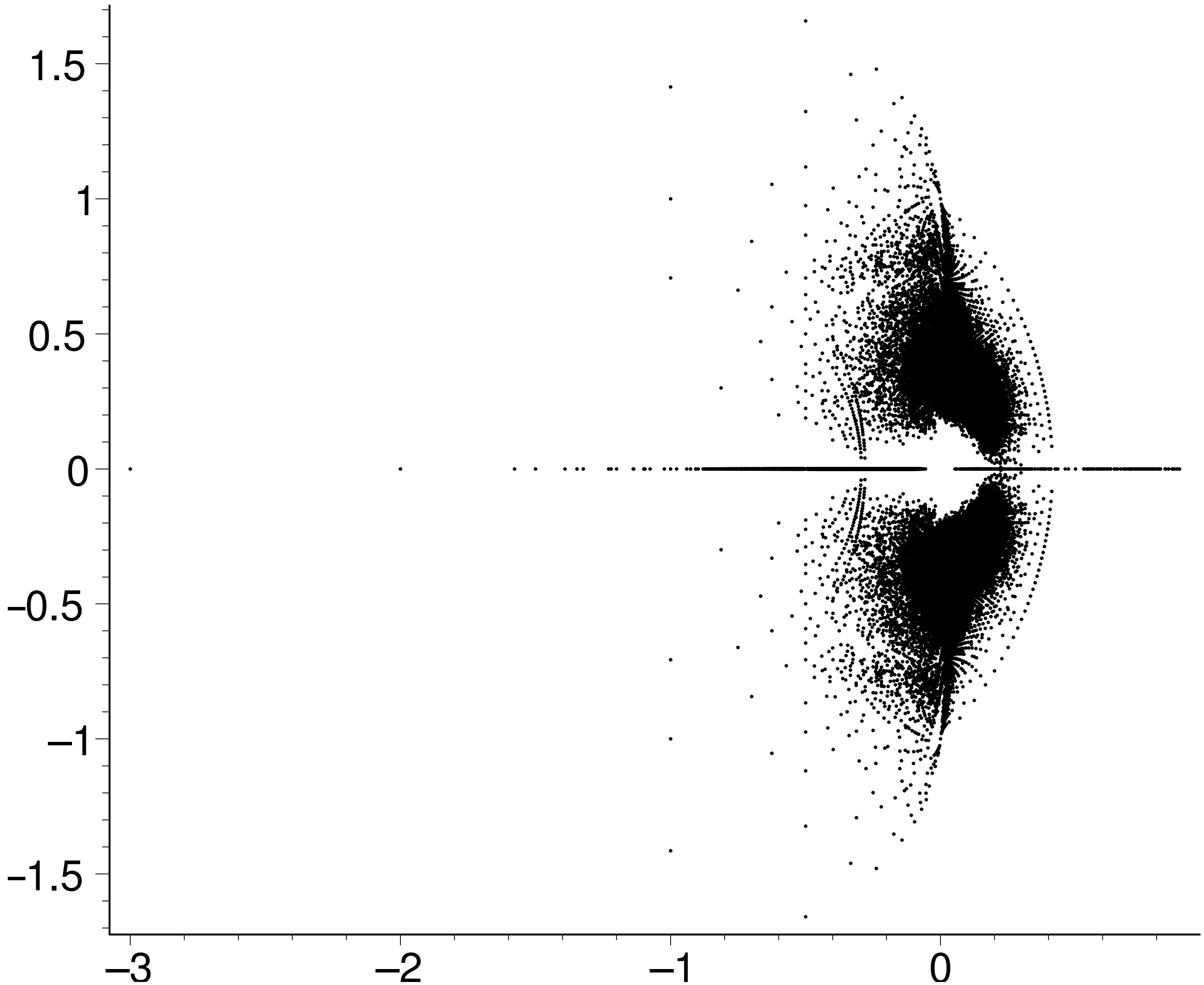
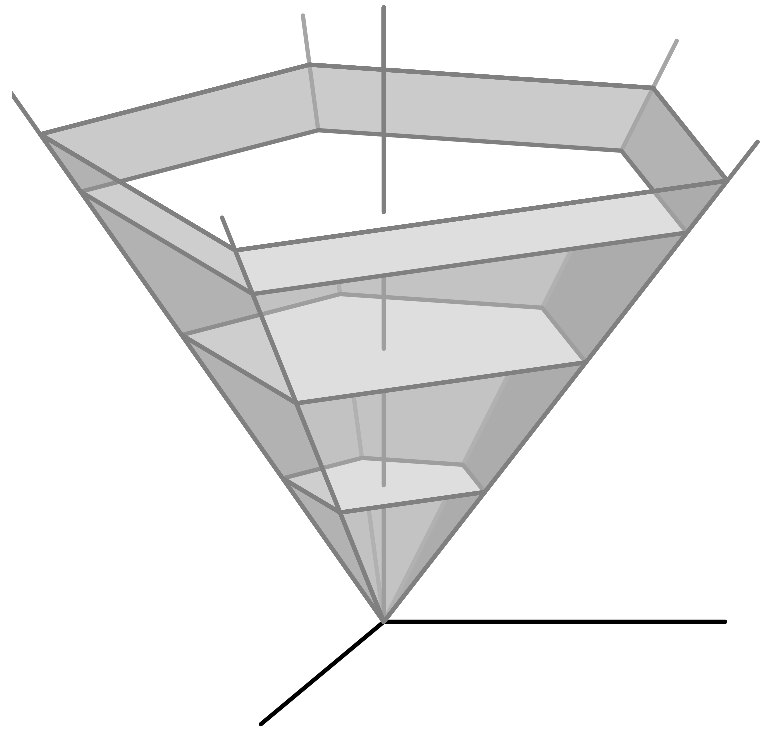
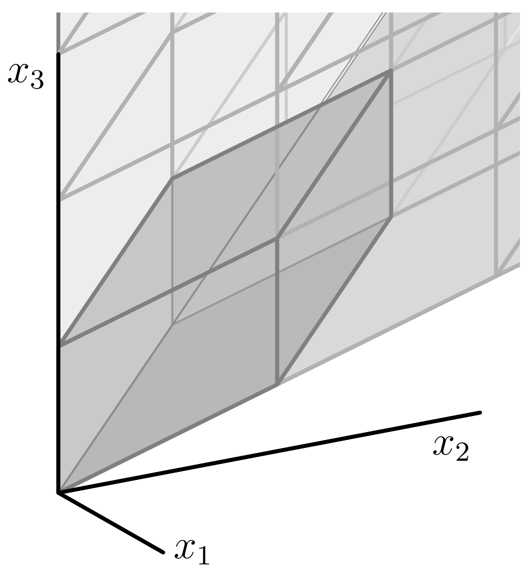
Any questions about yesterday?

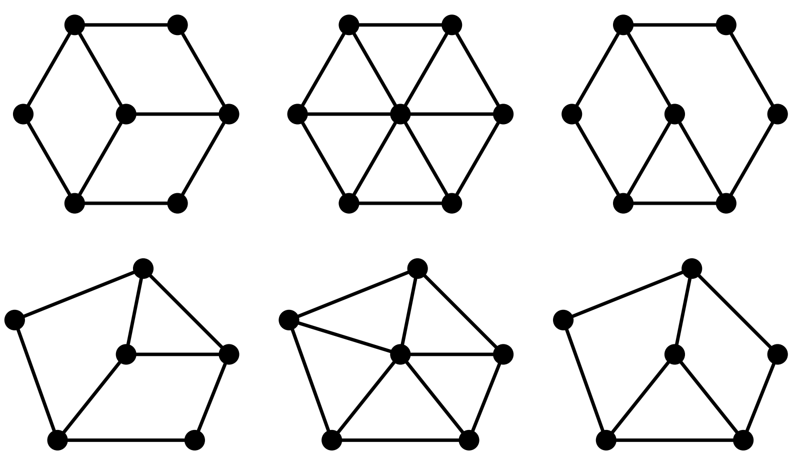


Today's Menu: Theory and Complexity

- Partition function magic
- Lots of generating functions
- Rational cones
- Triangulations
- Ehrhart theory







Warm-Up: Partition Generating Functions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of an integer $k \geqslant 0$ satisfies

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \text{and} \qquad 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$$

Goal Compute $\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n}$ over your favorite partition family

Example $P_{\leqslant 3}$ — family of partitions into at most 3 parts

$$\sum_{\lambda \in P_{\leqslant 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

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Idea $P_{\leqslant 3} = \left\{ \lambda \in \mathbb{Z}^3 : \, 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \right\} = \mathcal{K} \cap \mathbb{Z}^3$

$$\mathcal{K} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \, 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \right\} \longleftarrow \text{ polyhedral cone } \heartsuit$$

Warm-Up: Partition Generating Functions

$$\mathcal{K} \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \, 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \right\} \;=\; \mathbb{R}_{\geqslant 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geqslant 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geqslant 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

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Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) \quad &= \quad \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \quad \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

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$$\sum_{\lambda \in P_{\leqslant 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\mathcal{K}}(q, q, q) = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

Variations on a Theme

P_3 — family of partitions into exactly 3 parts

$$P_3 \;=\; \left\{ \lambda \in \mathbb{Z}^3 : \, 0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \right\} \;=\; \widetilde{\mathcal{K}} \cap \mathbb{Z}^3$$

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$$\begin{aligned} \sigma_{\widetilde{\mathcal{K}}}(z_1, z_2, z_3) \;&=\; \sum_{\mathbf{m} \in \widetilde{\mathcal{K}} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &=\; \frac{z_1 z_2 z_3}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

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Integer-point Complexity of a Simplicial Cone

What if $\mathcal{K}$ is (still simplicial and rational but) not unimodular?
Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1\, \mathbf{w}_2\, \mathbf{w}_3] = D > 1$

$\mathcal{K} = \mathbb{R}_{\geqslant 0}\, \mathbf{w}_1 + \mathbb{R}_{\geqslant 0}\, \mathbf{w}_2 + \mathbb{R}_{\geqslant 0}\, \mathbf{w}_3$

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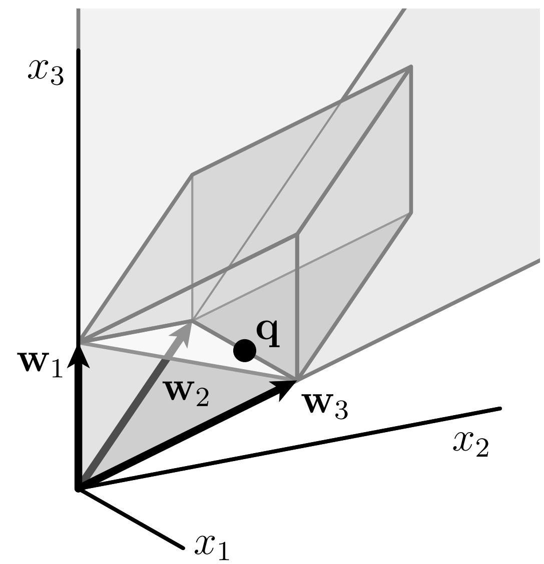
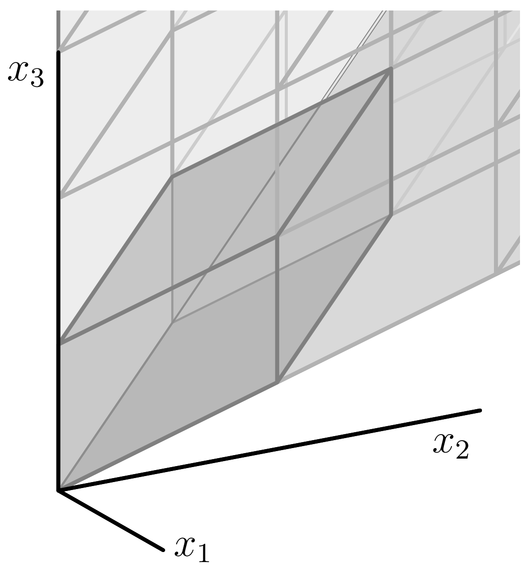


Idea Tile $\mathcal{K}$ with the half-open parallelepiped
$\Pi = [0,1) \, \mathbf{w}_1 + [0,1) \, \mathbf{w}_2 + [0,1) \, \mathbf{w}_3$



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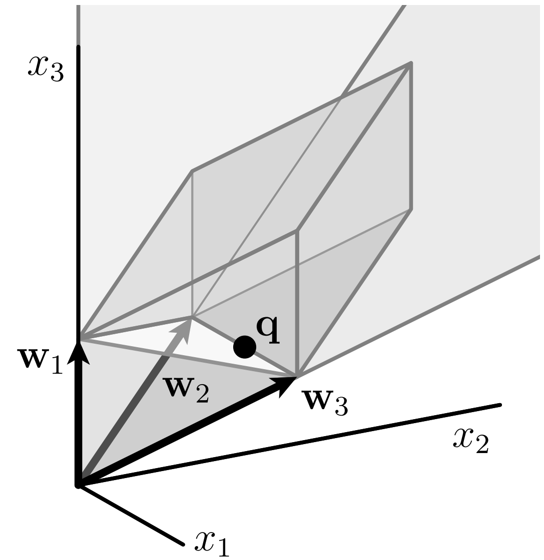

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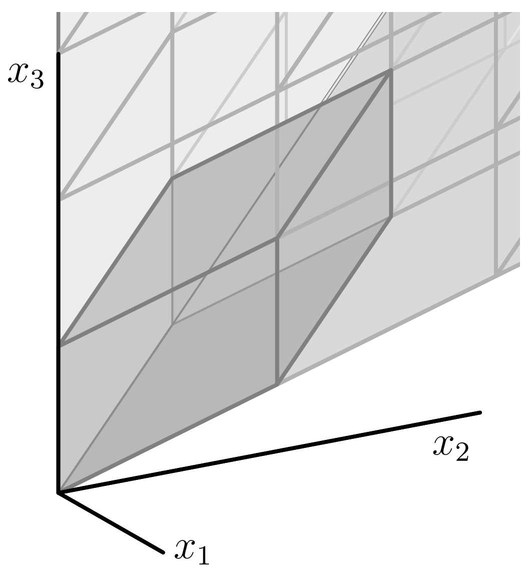

$\downarrow$

$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{\sigma_\Pi(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

where $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$

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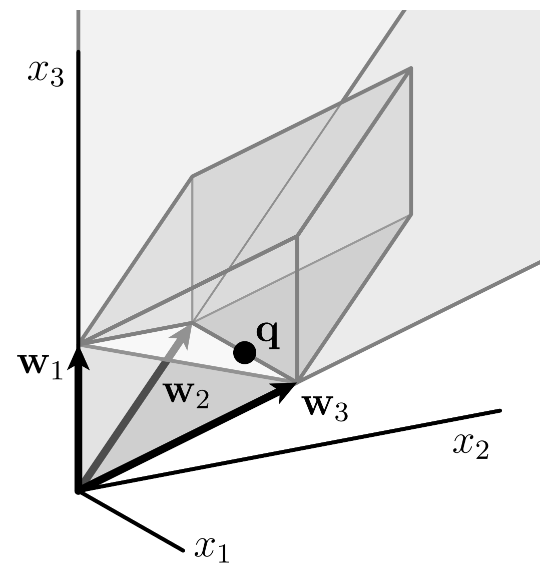

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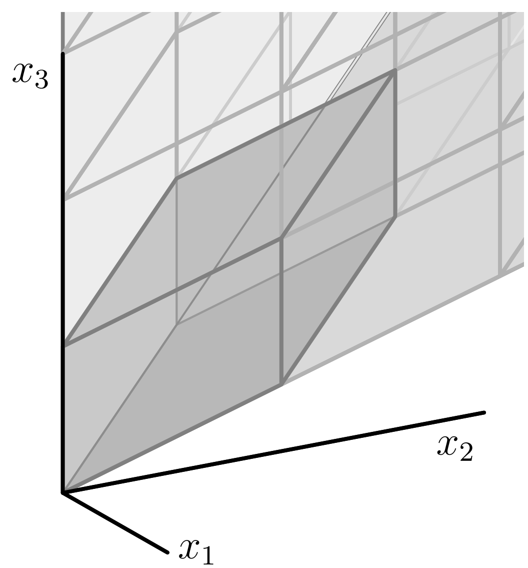

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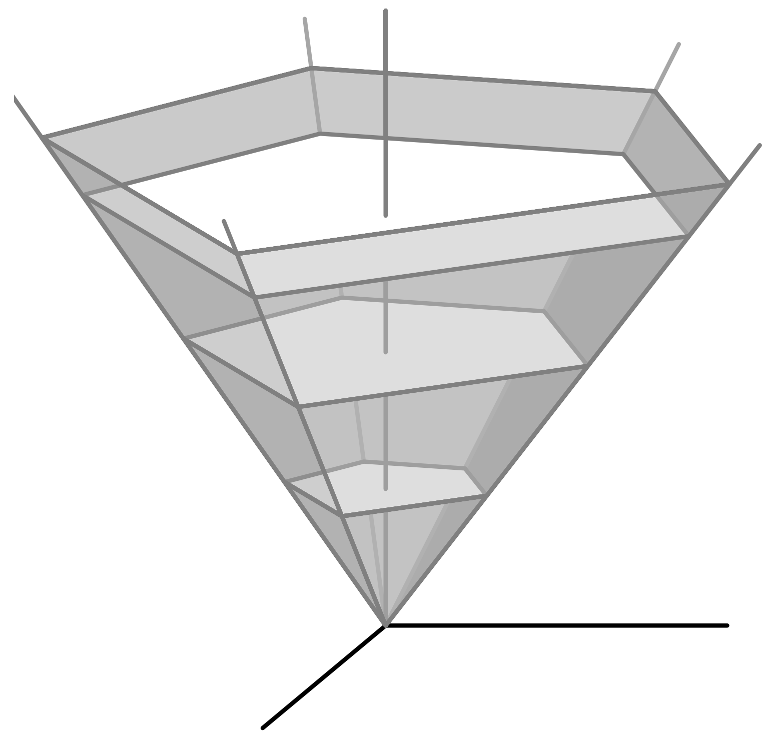
Complexity: $\sigma_{\Pi}(z_1, z_2, z_3)$ has D terms

Homogenizing Polytopes

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

$$\operatorname{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$



$$\operatorname{cone}(\mathcal{P}) \cap \{\boldsymbol{x} \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

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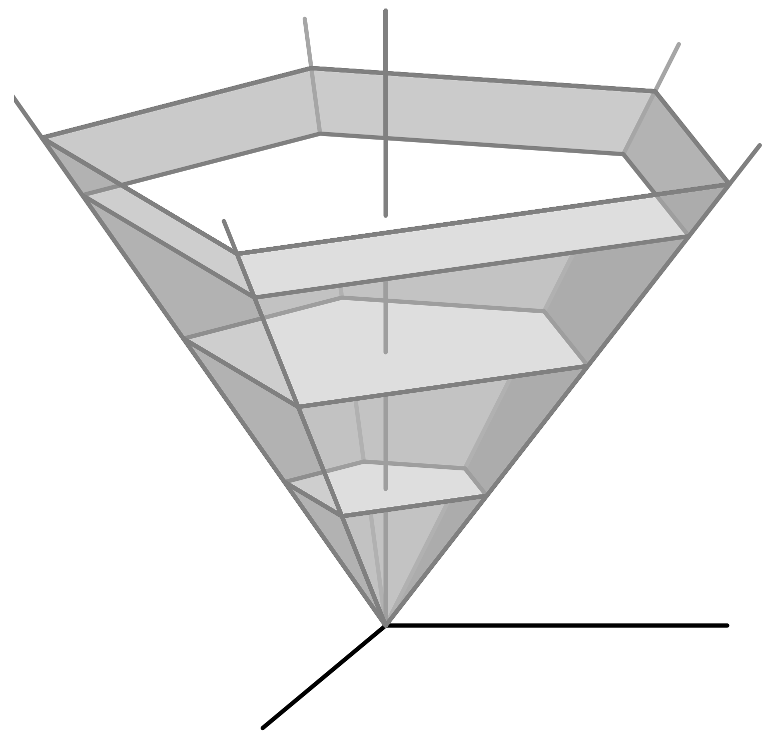

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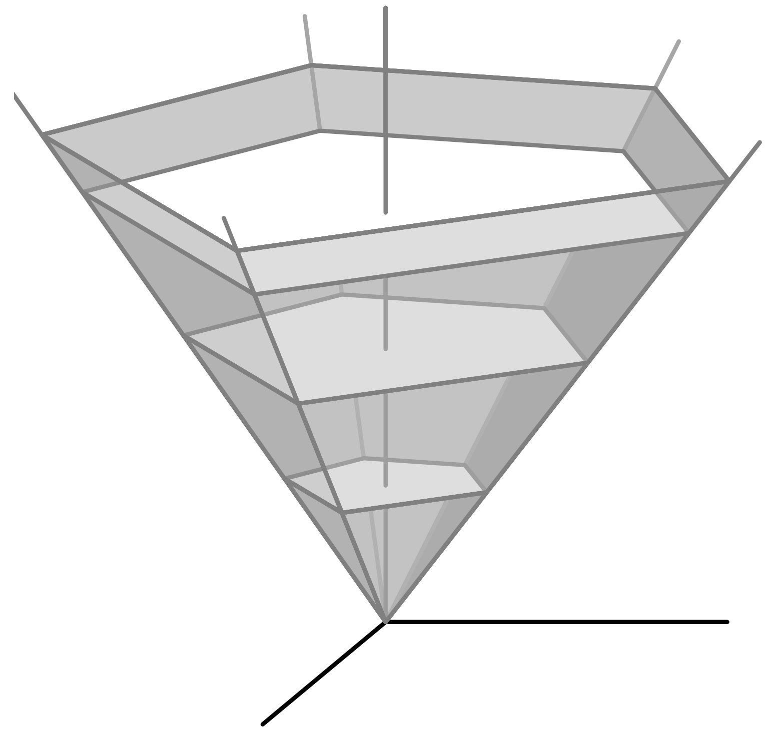
$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, z)$$

Homogenizing Polytopes



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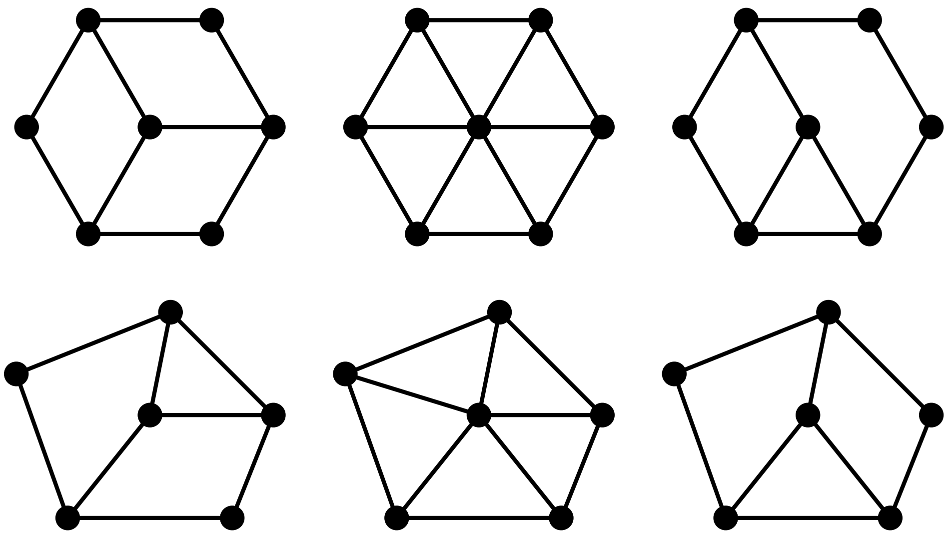
If $\mathcal{P}$ is a simplex,

$$\sigma_{\operatorname{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \quad \longrightarrow \quad \operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{d+1}}$$

Trials & Triangulations

Subdivision of a polyhedron $\mathcal{P}$ — finite collection S of polyhedra such that

- if $\mathcal{F}$ is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



If each $\mathcal{F}$ is a simplex $\longrightarrow$ triangulation of a polytope

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t)\, z^t$ is rational:

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the h^*-polynomial $h^*_{\mathcal{P}}(z)$ satisfies $h^*_{\mathcal{P}}(0) = 1$ and $h^*_{\mathcal{P}}(1) = (\dim \mathcal{P})! \operatorname{vol}(\mathcal{P})$.

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Computational bottlenecks:

- triangulation
- determinants of resulting simplicial cones

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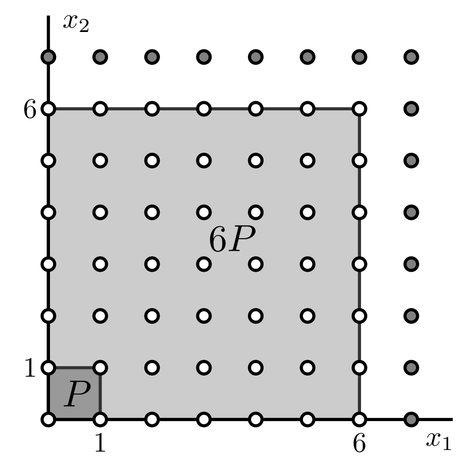
We saw instances yesterday: $\mathcal{P} = [0,1]^d$

$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$

$h^*_{\mathcal{P}}(z)$ — Eulerian polynomial



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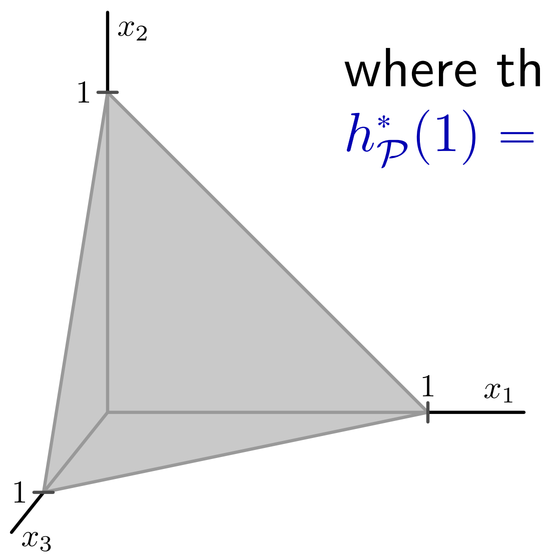

$$\Delta \ = \ \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\geqslant 0} : \, x_1 + x_2 + \cdots + x_d \leqslant 1 \right\}$$

$$L_\Delta(t) = \binom{d+t}{d} \qquad\qquad h^*_{\mathcal{P}}(z) = 1$$

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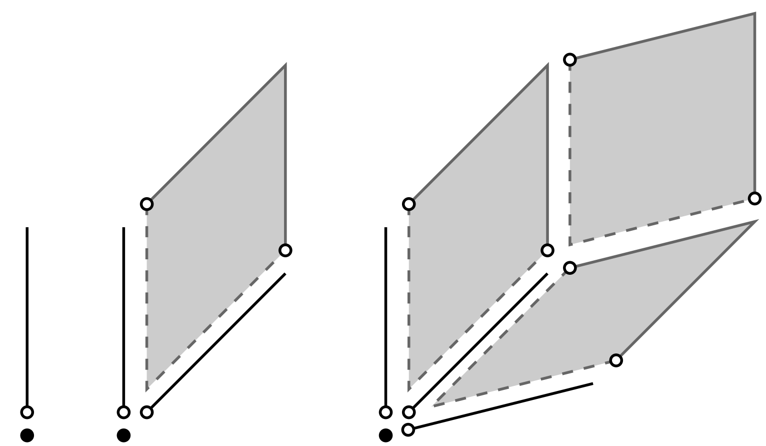
$\mathcal{P}$ — half-open d-parallelepiped

$\longrightarrow \quad L_{\mathcal{P}}(t) = \operatorname{vol}(\mathcal{P})\, t^d$



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Seeming dichotomy: $\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

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Equivalent descriptions of an Ehrhart polynomial:

- $L_\mathcal{P}(t) \ = \ c_d \, t^d + c_{d-1} \, t^{d-1} + \cdots + c_0$
- via roots of $L_\mathcal{P}(t)$
- $\operatorname{Ehr}_\mathcal{P}(z) \quad \longrightarrow \quad L_\mathcal{P}(t) \ = \ h^*_0 \binom{t+d}{d} + h^*_1 \binom{t+d-1}{d} + \cdots + h^*_d \binom{t}{d}$

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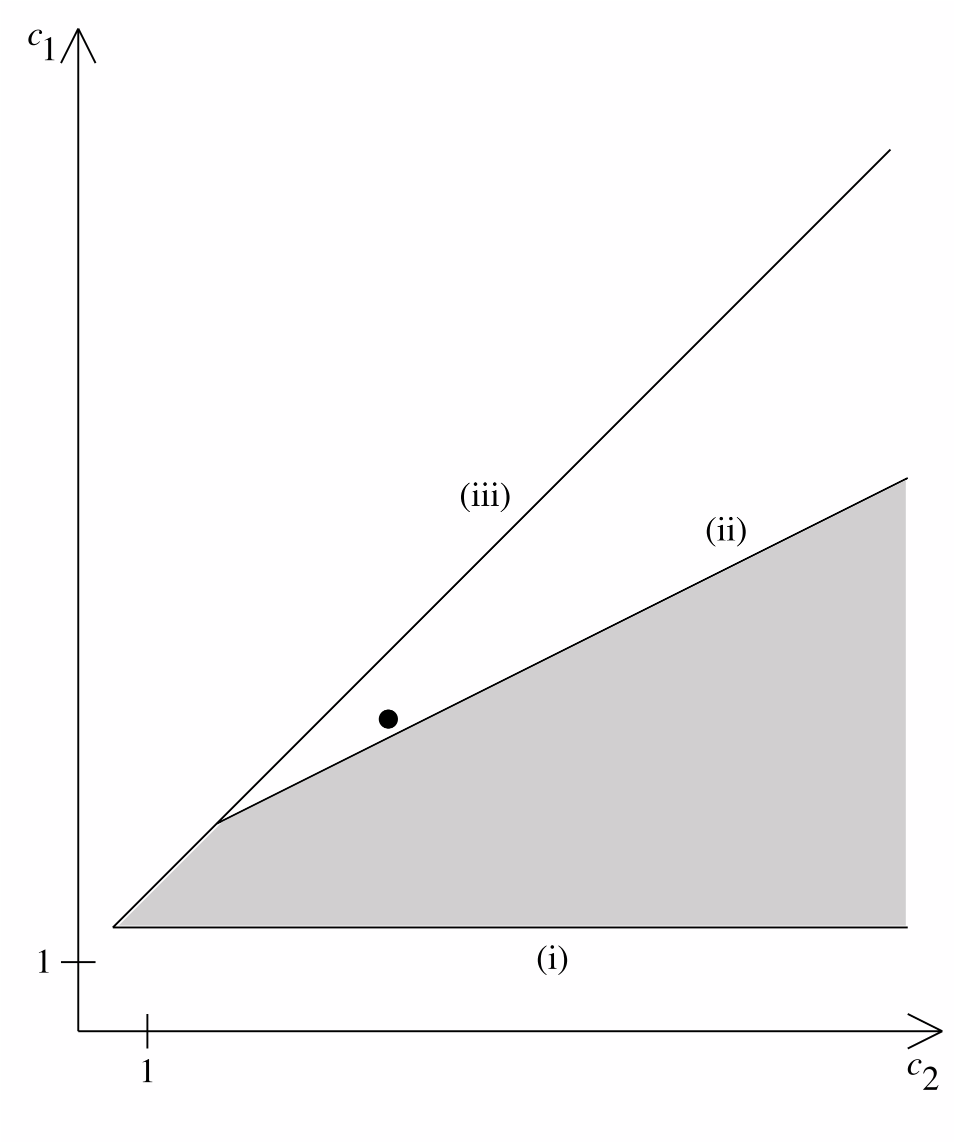
Open Problem Classify Ehrhart polynomials.

Ehrhart Polynomials in Dimension 2



$\mathcal{P}$ — lattice polygon

$\longrightarrow \quad L_{\mathcal{P}}(t) = c_2\, t^2 + c_1\, t + 1$

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in t:

$$L_{\mathcal{P}}(t) \ = \ c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$$

where $c_0(t), \dots, c_d(t)$ are periodic functions.

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$$\mathrm{Ehr}_\mathcal{P}(z) \ := \ 1 + \sum_{t \geqslant 1} L_\mathcal{P}(t)\, z^t \ = \ \frac{h(z)}{(1-z^p)^{\dim \mathcal{P}+1}}$$

for some (minimal) $p \in \mathbb{Z}_{>0}$ (the period of $L_\mathcal{P}(t)$).

Open Problem Study periods of Ehrhart quasipolynomials.

Partitions Revisited

Definition n-gon partitions

$$T_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geqslant \cdots \geqslant \lambda_1 \geqslant 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\}$$

Theorem (Andrews–Paule–Riese 2001)

$$\sum_{\lambda \in T_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{q}{(1-q)(1-q^2)\cdots(1-q^n)} - \frac{q^{2n-2}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2n-2})}$$

Geometric Philosophy The following cone is arithmetically nicer:

$$\{\boldsymbol{x} \in \mathbb{R}^n : x_n \geqslant \cdots \geqslant x_1 > 0 \text{ and } x_1 + \cdots + x_{n-1} \leqslant x_n\}$$

Partitions Revisited

Definition Lecture-hall partitions

$$\mathrm{LH}_n \; := \; \left\{ \lambda \in \mathbb{Z}^n : \; 0 \leqslant \frac{\lambda_1}{1} \leqslant \frac{\lambda_2}{2} \leqslant \cdots \leqslant \frac{\lambda_n}{n} \right\}$$

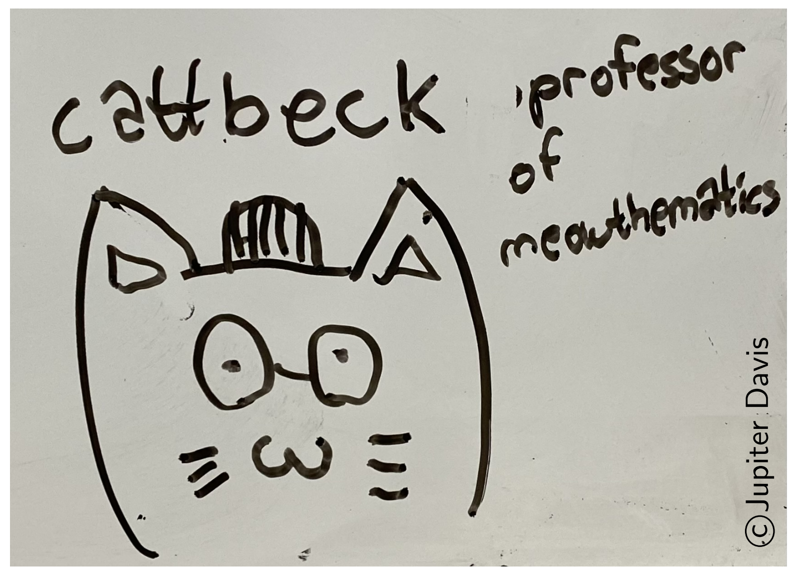
Lecture-Hall Theorem (Bousquet-Mélou–Eriksson 1997)

$$\sum_{\lambda \in \mathrm{LH}_n} q^{\lambda_1 + \cdots + \lambda_n} \; = \; \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

Open Problem Explain this geometrically. (Caveat: the lecture-hall cone has determinant $(n-1)!$).

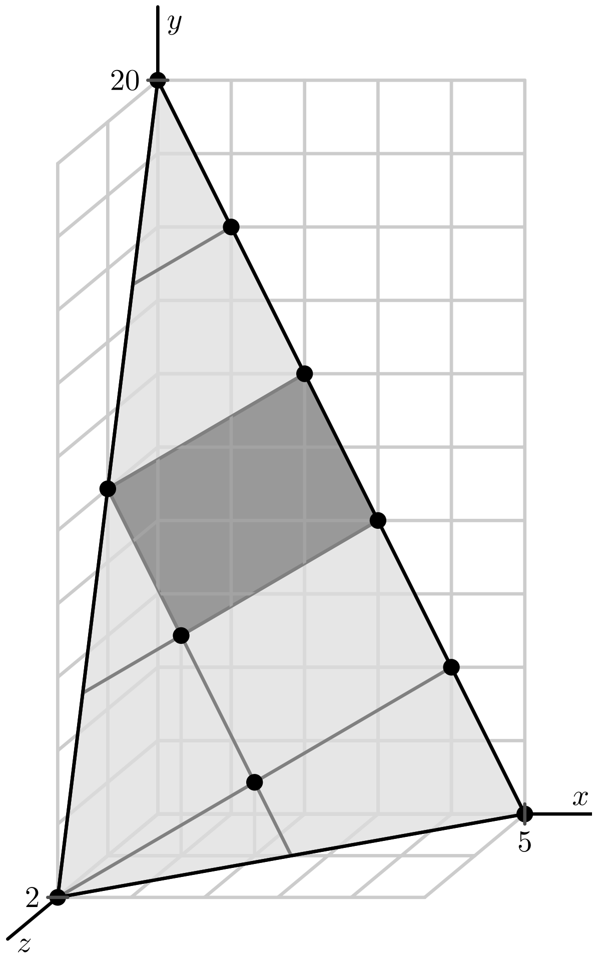
Recap Day II

- Generating functions son cheveres
- Integer-point transforms of rational polyhedra $\longrightarrow$ rational functions
- Arithmetic complexity of a simplicial cone: determinant of its generators
- Homogenize polytopes
- Triangulations
- Polynomial data
- Thursday: positivity, reciprocity & friends



Ehrhart Polynomials

Day III: Positivity, Reciprocity & Friends



Matthias Beck

San Francisco State University

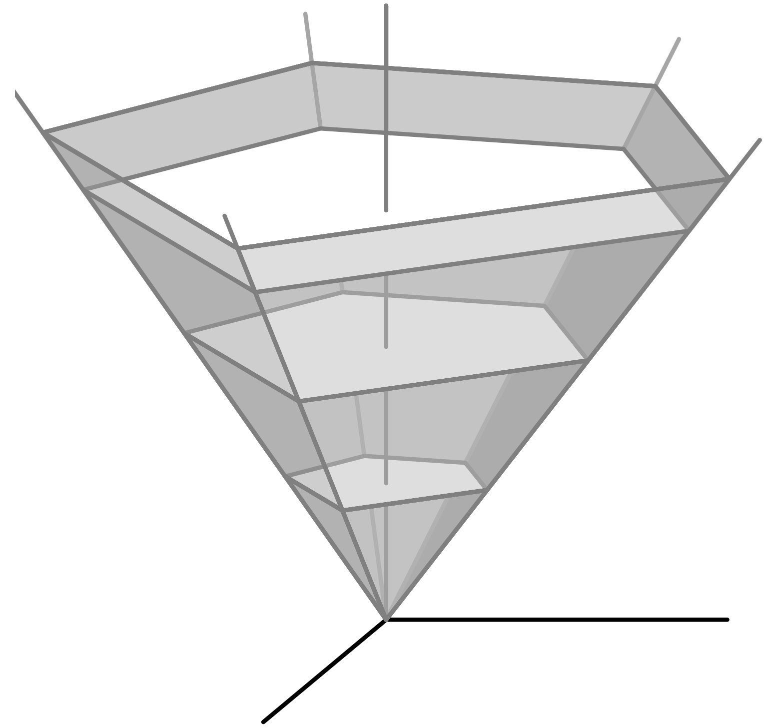
https://matthbeck.github.io/

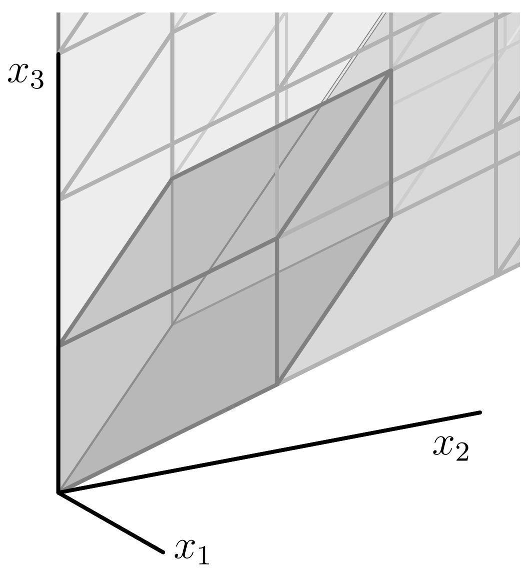
VIII Encuentro Colombiano
De Combinatoria

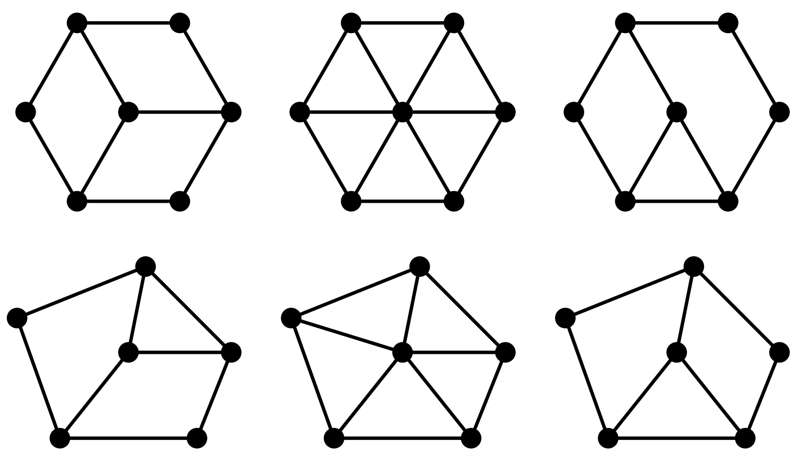
Any questions about Tuesday?

$$\sum_{\lambda \in P_3} q^{\lambda_1+\lambda_2+\lambda_3} = \sigma_{\widetilde{\mathcal{K}}}(q,q,q)$$

$$= \frac{q^3}{(1-q)(1-q^2)(1-q^3)}$$







Today's Menu: Positivity, Reciprocity & Friends

- Graph coloring
- Half-open triangulations
- Ehrhart positivity
- Ehrhart–Macdonald reciprocity

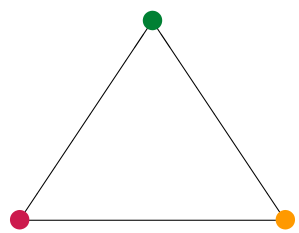
Warm-Up: Chromatic Polynomials of Graphs

$\Gamma = (V, E)$ — graph (without loops)

Proper k-coloring of Γ — $\boldsymbol{x} \in \{1, 2, \dots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_\Gamma(k) := \#\,(\text{proper } k\text{-colorings of } \Gamma)$

Example:



$\chi_{K_3}(k) = k(k-1)(k-2)$

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$\chi_\Gamma(k) := \#\,$(proper k-colorings of Γ) $\longleftarrow$ polynomial ♡

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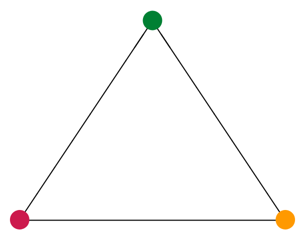

$\chi_{K_3}(k) = k(k-1)(k-2)$

$|\chi_{K_3}(-1)| = 6 \ \ldots$

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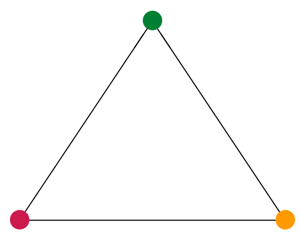
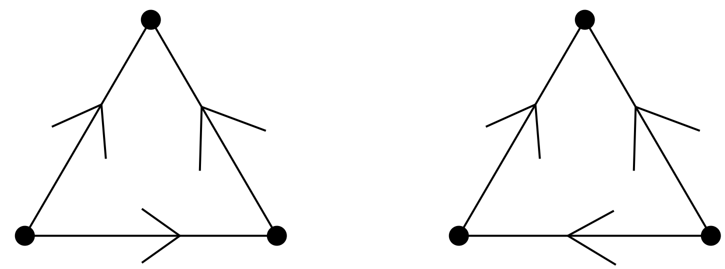


$$\chi_{K_3}(k) = k(k-1)(k-2)$$

$|\chi_{K_3}(-1)| = 6$ counts the number of acyclic orientations of K_3



Warm-Up: Chromatic Polynomials of Graphs

$\Gamma = (V, E)$ — graph (without loops)

Proper k-coloring of Γ — $\boldsymbol{x} \in \{1, 2, \dots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_\Gamma(k) := \#\,(\text{proper } k\text{-colorings of } \Gamma)$ $\longleftarrow$ polynomial ♡

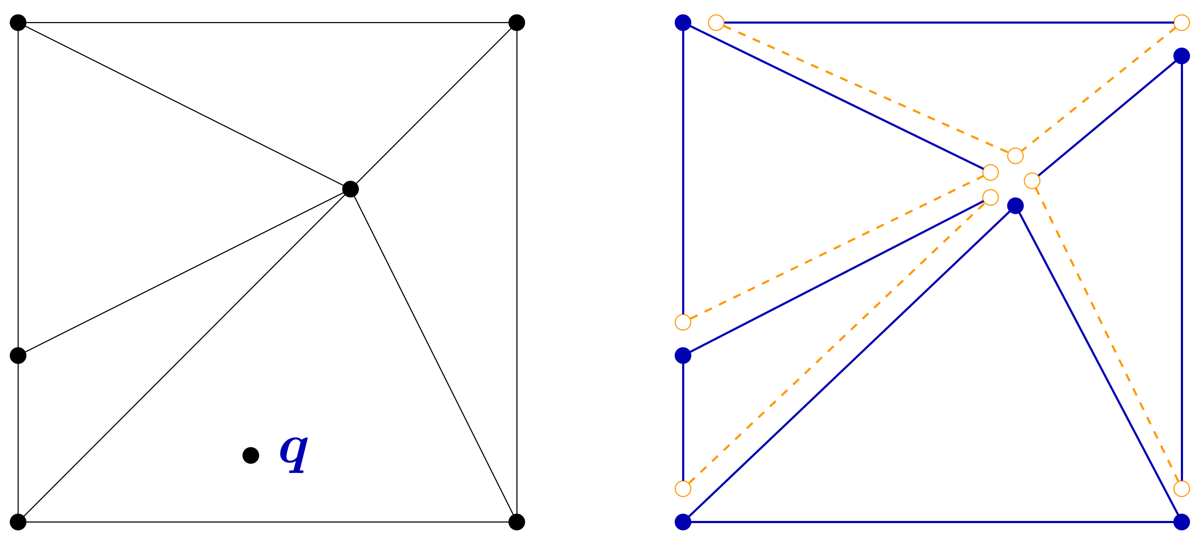
Theorem (Stanley 1973) $(-1)^{|V|}\chi_\Gamma(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation α of Γ and a compatible k-coloring $\boldsymbol{x}$. In particular, $(-1)^{|V|}\chi_\Gamma(-1)$ equals the number of acyclic orientations of Γ

(An orientation α of Γ and a k-coloring x are compatible if $x_j \geqslant x_i$ whenever there is an edge oriented from i to j. An orientation is acyclic if it has no directed cycles.)

Half-open Triangulations

Triangulation of a polytope $\mathcal{P}$ — finite collection S of simplices such that

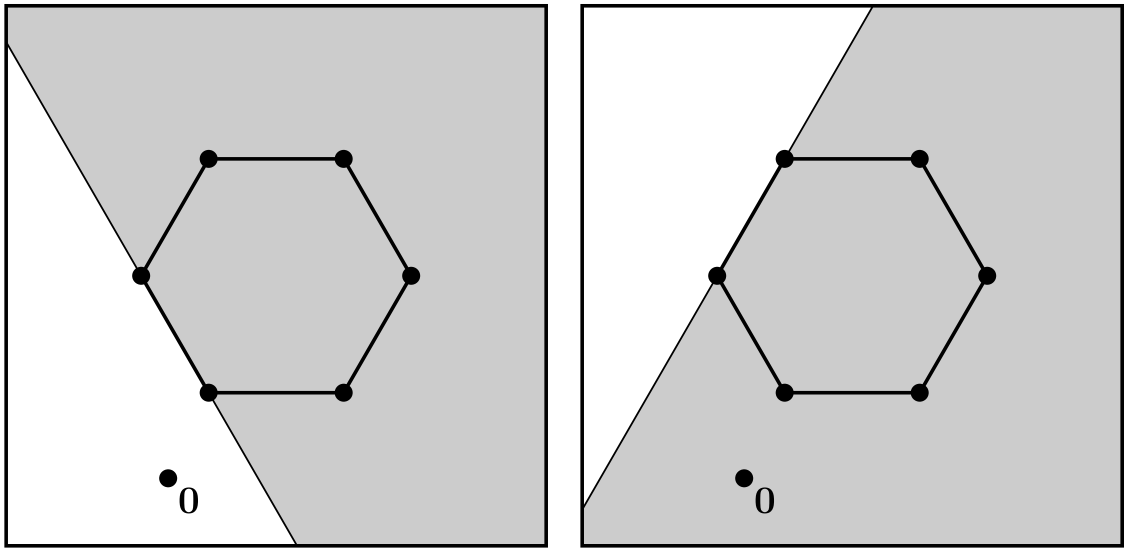
- if $\mathcal{F}$ is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



Tangent Cones & Visibility

$\mathcal{F}$ facet of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ with defining halfspace H

$\mathcal{F}$ is visible from $\boldsymbol{q} \in \mathbb{R}^d$ if $\boldsymbol{q} \notin H$



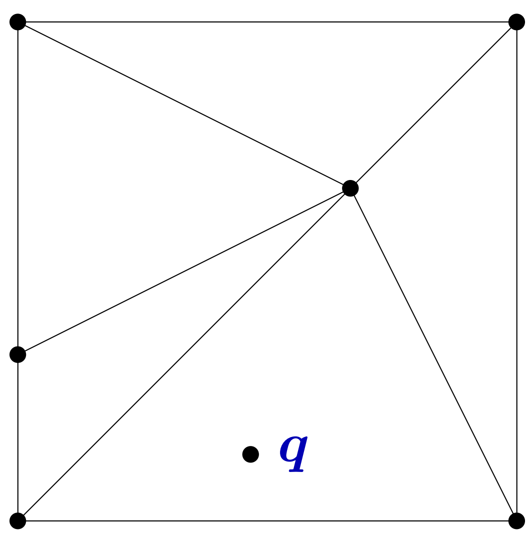
Equivalent lingo: $\boldsymbol{q} \in \mathbb{R}^d$ is beyond $\mathcal{F}$ (and beneath otherwise)

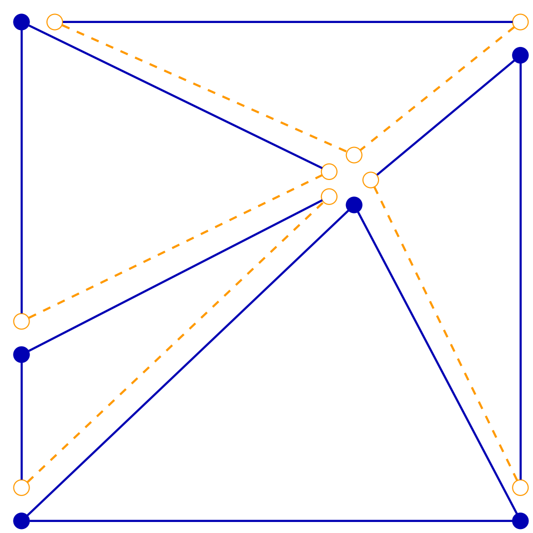
Half-open Triangulations

Define $\mathbb{H}_{\boldsymbol{q}}(\mathcal{Q})$ to be $\mathcal{Q}$ minus facets that are visible from $\boldsymbol{q}$

Exercise $\mathcal{P} \subset \mathbb{R}^d$ — full-dimensional polyhedron with dissection $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$, $\boldsymbol{q} \in \mathbb{R}^d$ generic relative to each $\mathcal{P}_j$ $\longrightarrow$

$$\mathbb{H}_{\boldsymbol{q}}\mathcal{P} \;=\; \mathbb{H}_{\boldsymbol{q}}\mathcal{P}_1 \uplus \mathbb{H}_{\boldsymbol{q}}\mathcal{P}_2 \uplus \cdots \uplus \mathbb{H}_{\boldsymbol{q}}\mathcal{P}_m$$

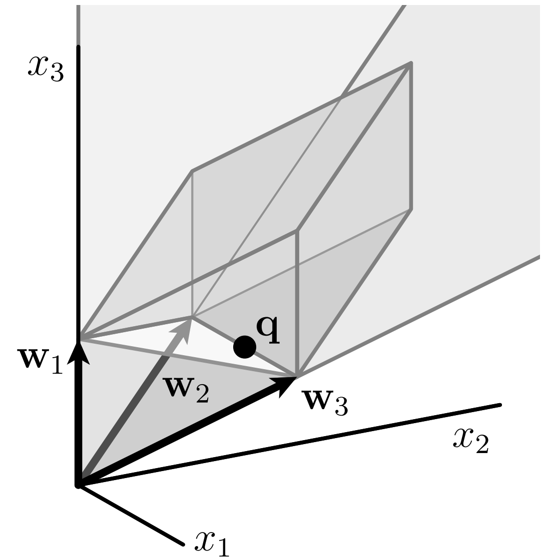




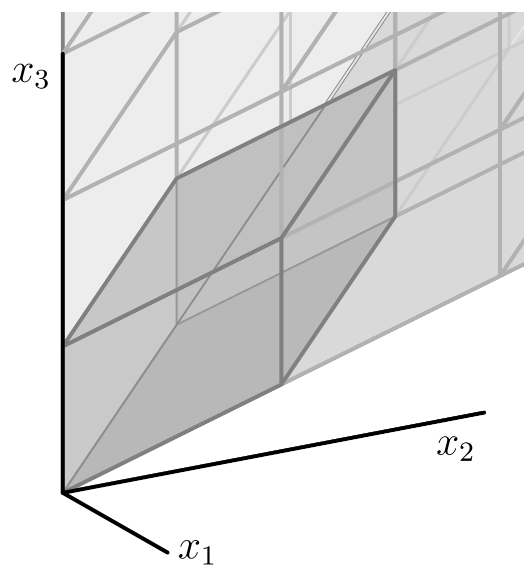
Recall: Integer-point Complexity of a Simplicial Cone

What if $\mathcal{K}$ is (still simplicial and rational but) not unimodular?
Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1\, \mathbf{w}_2\, \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geqslant 0}\, \mathbf{w}_1 + \mathbb{R}_{\geqslant 0}\, \mathbf{w}_2 + \mathbb{R}_{\geqslant 0}\, \mathbf{w}_3$$



Idea Tile $\mathcal{K}$ with the half-open parallelepiped
$\Pi = [0,1)\, \mathbf{w}_1 + [0,1)\, \mathbf{w}_2 + [0,1)\, \mathbf{w}_3$

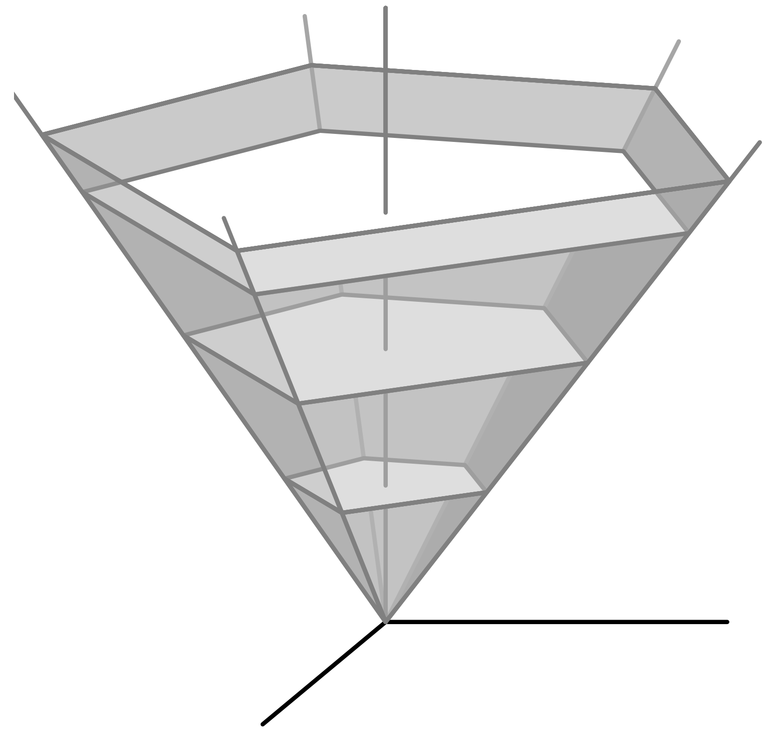


$\downarrow$

$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

where $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$

Recall: Homogenizing Polytopes



Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

$$\operatorname{cone}(\mathcal{P}) := \mathbb{R}_{\geqslant 0}\,(\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geqslant 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geqslant 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geqslant 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$

$$\operatorname{cone}(\mathcal{P}) \cap \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \, x_{d+1} = t\}$$

contains a copy of $t\mathcal{P}$ $\longrightarrow$

$$\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t)\, z^t = \sigma_{\operatorname{cone}(\mathcal{P})}(1, 1, \ldots, 1, z)$$

If $\mathcal{P}$ is a simplex,

$$\sigma_{\operatorname{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \quad \longrightarrow \quad \operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{d+1}}$$

Ehrhart Positivity

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

Open Problem Prove that the h^*-polynomial of

- hypersimplices
- polytopes admitting a unimodular triangulation
- polytope with the integer decomposition property are unimodal

✓ Gorenstein polytopes with regular unimodular triangulation (Bruns–Römer 2007)

✓ Zonotopes (MB–Jochemko–McCullough 2019)

Recall:^{2} Integer-point Complexity of a Simplicial Cone

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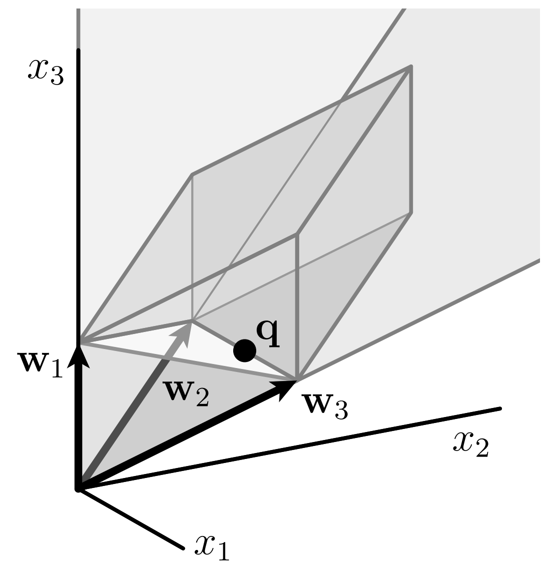

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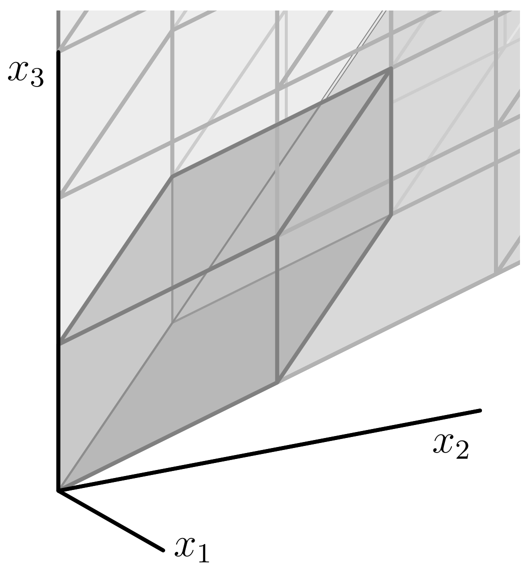

$\downarrow$

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where $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$

Simplicial Cone Reciprocity

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{Z}^d$ linearly independent

$$\begin{aligned}
\widehat{\mathcal{K}} &:= \mathbb{R}_{\geqslant 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{\geqslant 0}\mathbf{v}_{m-1} + \mathbb{R}_{>0}\mathbf{v}_m + \cdots + \mathbb{R}_{>0}\mathbf{v}_k \\
\breve{\mathcal{K}} &:= \mathbb{R}_{>0}\mathbf{v}_1 + \cdots + \mathbb{R}_{>0}\mathbf{v}_{m-1} + \mathbb{R}_{\geqslant 0}\mathbf{v}_m + \cdots + \mathbb{R}_{\geqslant 0}\mathbf{v}_k
\end{aligned}$$

$$\downarrow$$

$$\sigma_{\widehat{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\widehat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})} \qquad \sigma_{\breve{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\breve{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})}$$

where

$$\begin{aligned}
\widehat{\Pi} &:= [0,1)\,\mathbf{v}_1 + \cdots + [0,1)\,\mathbf{v}_{m-1} + (0,1]\,\mathbf{v}_m + \cdots + (0,1]\,\mathbf{v}_k \\
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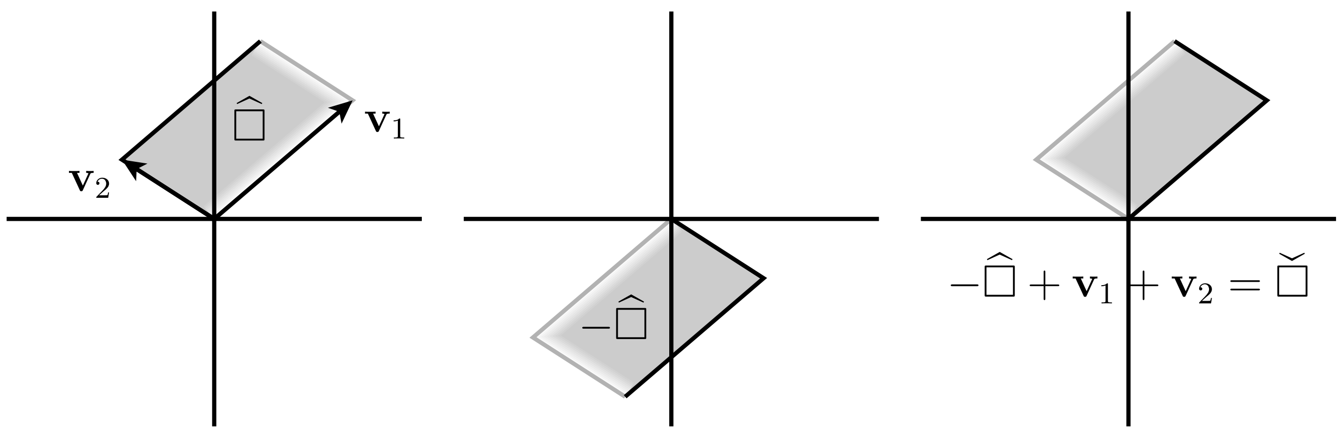
$$\check{\Pi} := (0,1]\,\mathbf{v}_1 + \cdots + (0,1]\,\mathbf{v}_{m-1} + [0,1)\,\mathbf{v}_m + \cdots + [0,1)\,\mathbf{v}_k$$

Fun Fact $\widehat{\Pi} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k - \check{\Pi}$



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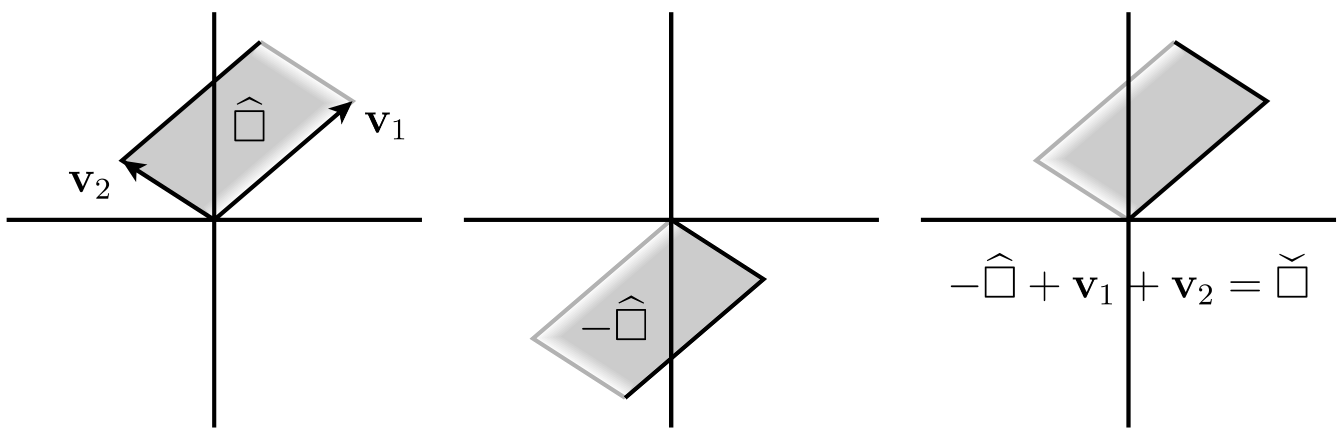

$$\longrightarrow \quad \sigma_{\widehat{\Pi}}(\mathbf{z}) \;=\; \mathbf{z}^{\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_k}\, \sigma_{\check{\Pi}}\left(\frac{1}{\mathbf{z}}\right)$$

Stanley Reciprocity

$$\sigma_{\hat{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\hat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})} \qquad \sigma_{\check{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\check{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})}$$

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$$\begin{aligned}\sigma_{\check{\mathcal{K}}}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{\check{\Pi}}(\frac{1}{\mathbf{z}})}{(1-\mathbf{z}^{-\mathbf{v}_1})\cdots(1-\mathbf{z}^{-\mathbf{v}_k})} = \frac{\mathbf{z}^{-\mathbf{v}_1-\mathbf{v}_2-\cdots-\mathbf{v}_k}\,\sigma_{\hat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{-\mathbf{v}_1})\cdots(1-\mathbf{z}^{-\mathbf{v}_k})} \\ &= (-1)^k \frac{\sigma_{\hat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})} = (-1)^k\,\sigma_{\hat{\mathcal{K}}}(\mathbf{z})\end{aligned}$$

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Theorem (Stanley) Let $\mathcal{K} \subset \mathbb{R}^d$ be a full-dimensional pointed rational cone, and let $\boldsymbol{q} \in \mathbb{R}^d$ be generic relative to $\mathcal{K}$. Then

$$\sigma_{\mathbb{H}_{\boldsymbol{q}}\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d\,\sigma_{\mathbb{H}^{\boldsymbol{q}}\mathcal{K}}(\mathbf{z})$$

Stanley Reciprocity

$$\sigma_{\widehat{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\widehat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})} \qquad \sigma_{\check{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\check{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})}$$

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$$= (-1)^k \frac{\sigma_{\widehat{\Pi}}(\mathbf{z})}{(1-\mathbf{z}^{\mathbf{v}_1})\cdots(1-\mathbf{z}^{\mathbf{v}_k})} = (-1)^k\,\sigma_{\widehat{\mathcal{K}}}(\mathbf{z})$$

Theorem (Stanley) Let $\mathcal{K} \subset \mathbb{R}^d$ be a full-dimensional pointed rational cone, and let $\boldsymbol{q} \in \mathbb{R}^d$ be generic relative to $\mathcal{K}$. Then

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Corollary $\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d\,\sigma_{\mathcal{K}^\circ}(\mathbf{z})$

Ehrhart–Macdonald Reciprocity

Corollary (Stanley) Let $\mathcal{K} \subset \mathbb{R}^d$ be a full-dimensional pointed rational cone, and let $\boldsymbol{q} \in \mathbb{R}^d$ be generic relative to $\mathcal{K}$. Then

$$\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{\mathcal{K}^\circ}(\mathbf{z})$$

$$\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t) \, z^t = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{d+1}} = \sigma_{\operatorname{cone}(\mathcal{P})}(1, 1, \ldots, z)$$

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Ehrhart–Macdonald Reciprocity

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$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) := \sum_{t \geqslant 1} L_{\mathcal{P}^\circ}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})^\circ}(1, 1, \ldots, z)$$

Corollary^2 Let $\mathcal{P}$ be a lattice d-polytope. Then

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = \frac{z^{d+1}\, h^*_{\mathcal{P}}(\frac{1}{z})}{(1-z)^{d+1}} = (-1)^{d+1}\, \mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right)$$

Ehrhart–Macdonald Reciprocity

Corollary (Stanley) Let $\mathcal{K} \subset \mathbb{R}^d$ be a full-dimensional pointed rational cone, and let $\boldsymbol{q} \in \mathbb{R}^d$ be generic relative to $\mathcal{K}$. Then

$$\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \, \sigma_{\mathcal{K}^\circ}(\mathbf{z})$$

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geqslant 1} L_{\mathcal{P}}(t) \, z^t = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{d+1}} = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \dots, z)$$

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) := \sum_{t \geqslant 1} L_{\mathcal{P}^\circ}(t) \, z^t = \sigma_{\mathrm{cone}(\mathcal{P})^\circ}(1, 1, \dots, z)$$

Corollary^2 Let $\mathcal{P}$ be a lattice d-polytope. Then

$$\mathrm{Ehr}_{\mathcal{P}^\circ}(z) = \frac{z^{d+1} \, h^*_{\mathcal{P}}(\frac{1}{z})}{(1-z)^{d+1}} = (-1)^{d+1} \, \mathrm{Ehr}_{\mathcal{P}}\left(\frac{1}{z}\right)$$

Corollary^3 (Ehrhart–Macdonald) $L_{\mathcal{P}}(-t) = (-1)^d \, L_{\mathcal{P}^\circ}(t)$

Order Polytopes

$(\Pi, \preceq)$ — finite partially ordered set (poset)

$$\mathcal{O}_\Pi \ := \ \left\{ \phi \in \mathbb{R}^\Pi : \begin{array}{ll} 0 \leqslant \phi(p) \leqslant 1 & \text{for all } p \in \Pi \\ \phi(a) \leqslant \phi(b) & \text{whenever } a \preceq b \end{array} \right\}$$

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Integer points in $t\,\mathcal{O}_\Pi$ correspond to order preserving maps $\Pi \to \{0, 1, \ldots, t\}$

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those in $t\,\mathcal{O}_\Pi^\circ$ correspond to strictly order preserving maps $\Pi \to \{1, \dots, t-1\}$

$$\phi(a) < \phi(b) \qquad \text{whenever } a \prec b$$

Ehrhart–Macdonald Reciprocity $\longrightarrow$ $L_{\mathcal{O}_\Pi}(-t) \ = \ (-1)^{|\Pi|}\, L_{\mathcal{O}_\Pi^\circ}(t)$

Back to Graph Colorings

$\Gamma = (V, E)$ — graph (without loops)

Proper n-coloring of Γ — $\boldsymbol{x} \in \{1, 2, \ldots, n\}^V$ such that $x_i \neq x_j$ if $ij \in E$

An orientation α of Γ is acyclic if it has no directed cycles $\longrightarrow$ poset Π_α

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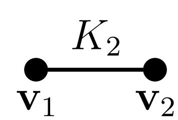
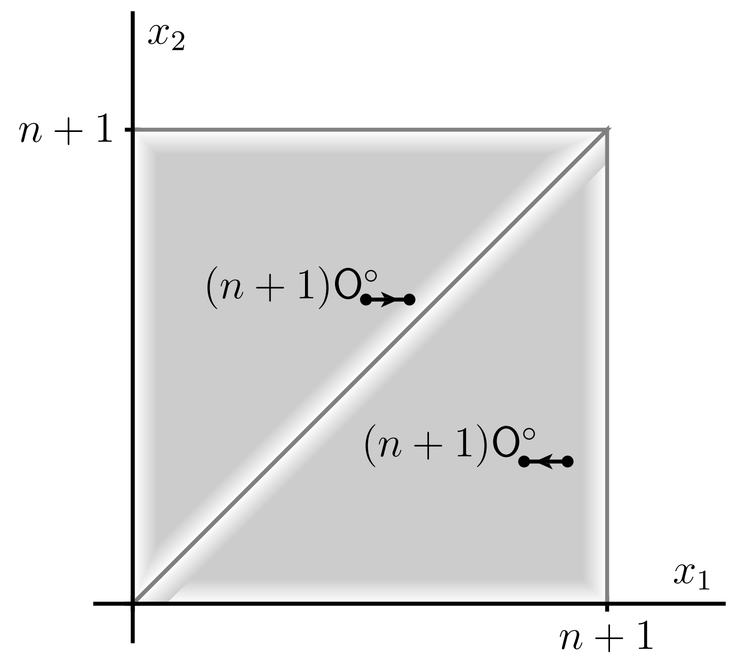
Graph Coloring a la Ehrhart:

$$\chi_\Gamma(n) = \sum_\alpha L_{\mathcal{O}^\circ_{\Pi_\alpha}}(n+1)$$





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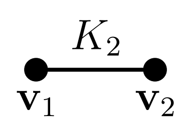

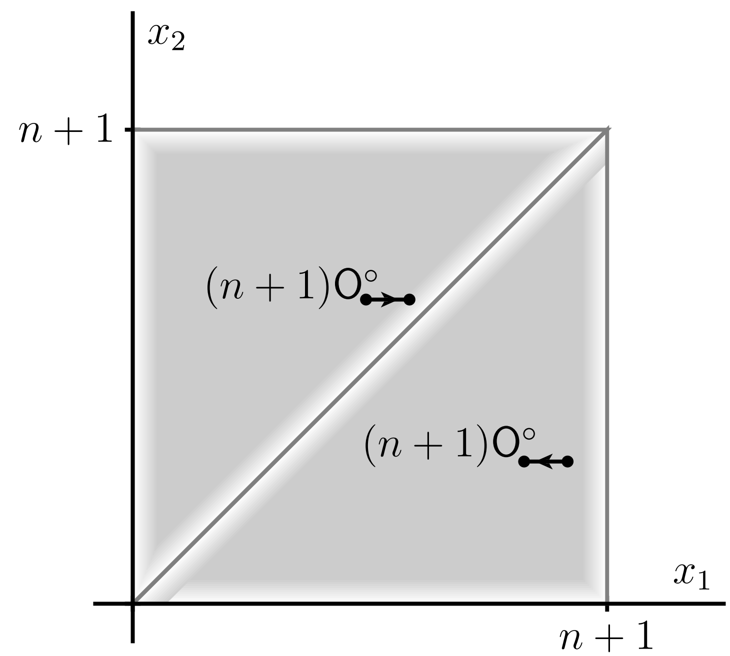

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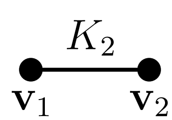
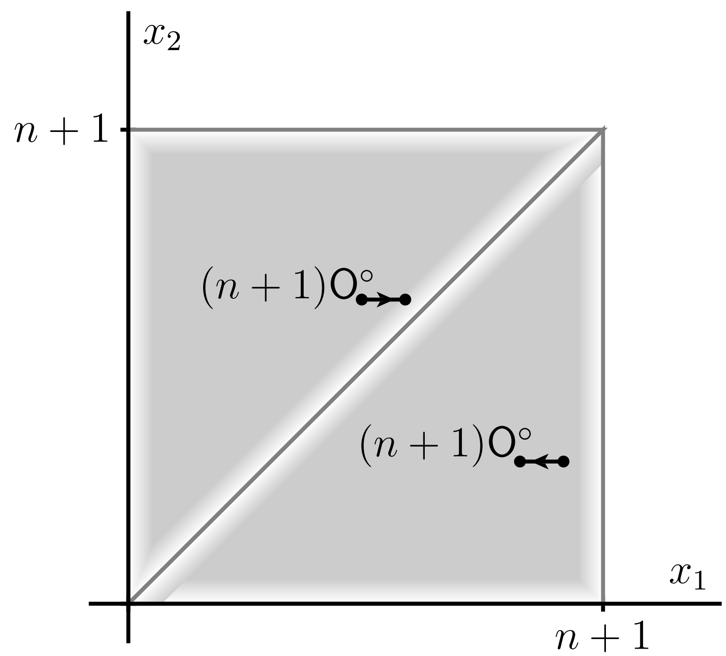
$$\chi_\Gamma(-n) = (-1)^{|V|} \sum_\alpha L_{\mathcal{O}_{\Pi_\alpha}}(n-1)$$





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$$(-1)^{|V|} \chi_\Gamma(-n) = \sum_\alpha L_{\mathcal{O}_{\Pi_\alpha}}(n-1)$$

counts colorings with colors in $\{0, 1, \dots, n-1\}$ with multiplicity coming from compatible acyclic orientations.

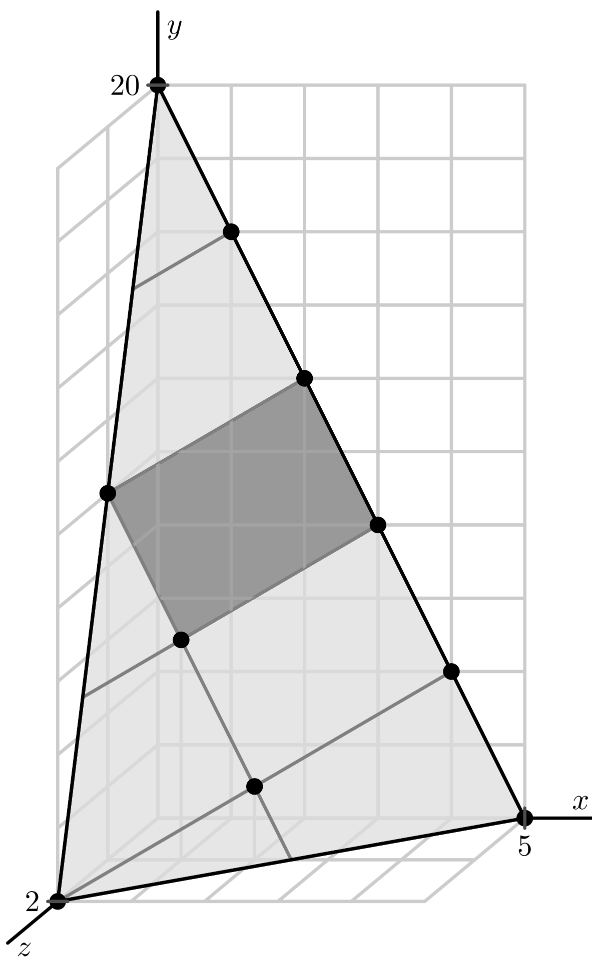
Stanley: "told you."

Recap Day III

- Combinatorial reciprocity theorems
- Visibility constructions & half-open triangulations
- h^*-polynomials are nonnegative
- Stanley reciprocity for integer-point transforms of cones
- Ehrhart–Macdonald reciprocity for Ehrhart polynomials
- Order polytopes & order-preserving maps
- Chromatic polynomials
- Tomorrow: why h^* is called h^*

Ehrhart Polynomials

Day IV: From h to h^*



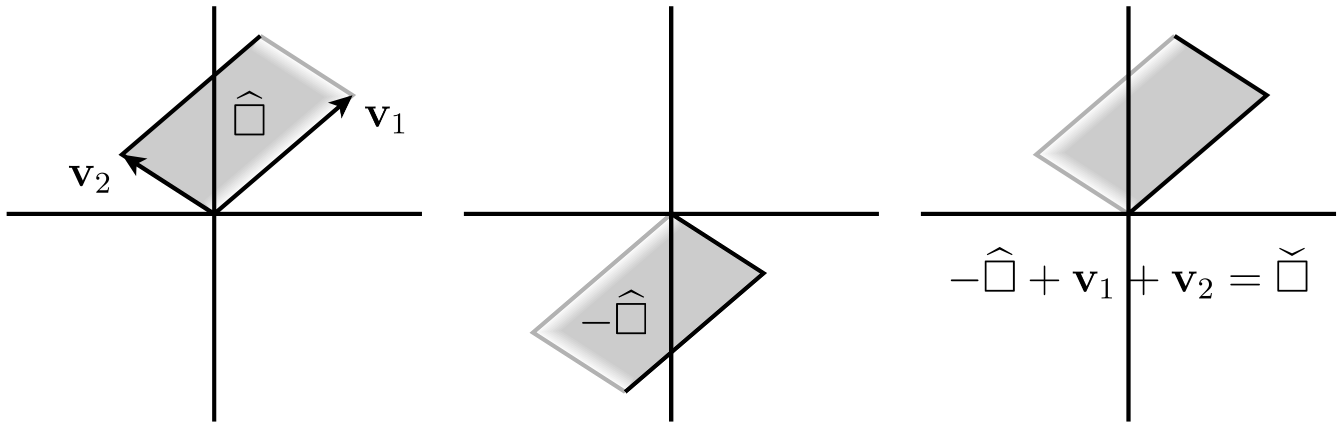
Matthias Beck

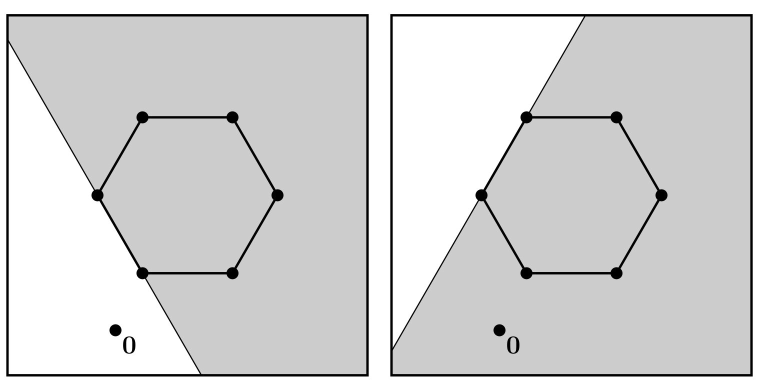
San Francisco State University

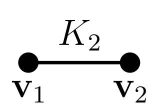
https://matthbeck.github.io/

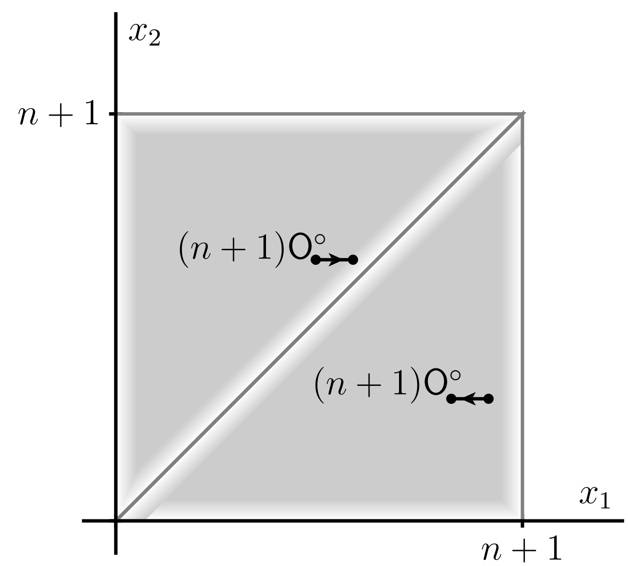
VIII Encuentro Colombiano
De Combinatoria

Any questions about yesterday?









Today's Menu: connect with Bella's Minicourse

- Unimodular triangulations
- f- and h-vectors of triangulations

Unimodular Triangulations

A lattice d-simplex with volume $\frac{1}{d!}$ is unimodular

Alternative description: if the simplex has vertices $v_0, v_1, \dots, v_d$, the vectors $v_1 - v_0, \dots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

Every lattice polygon admits a unimodular triangulation, the regular tetrahedron with vertices $(0,0,0)\,, (1,1,0)\,, (1,0,1)\,, (0,1,1)$ does not.

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Theorem (Kempf–Knudsen–Mumford–Saint-Donat–Waterman 1970's)
For every lattice polytope $\mathcal{P}$ there exists an integer m such that $m\mathcal{P}$ admits a regular unimodular triangulation.

Theorem (Liu 2024+) For every lattice polytope $\mathcal{P}$ there exists an integer m such that $k\mathcal{P}$ admits a regular unimodular triangulation for $k \geqslant m$.

Conjecture There exists an integer m_d such that, if $\mathcal{P}$ is a d-dimensional lattice polytope, then $m_d\mathcal{P}$ admits a regular unimodular triangulation.

f- and h-vectors of triangulation

f_k — number of k-simplices in a given triangulation T of a polytope

$f_{-1} := 1$

h-polynomial of T $$h_T(z) := \sum_{k=-1}^{d} f_k \, z^{k+1} \, (1-z)^{d-k}$$

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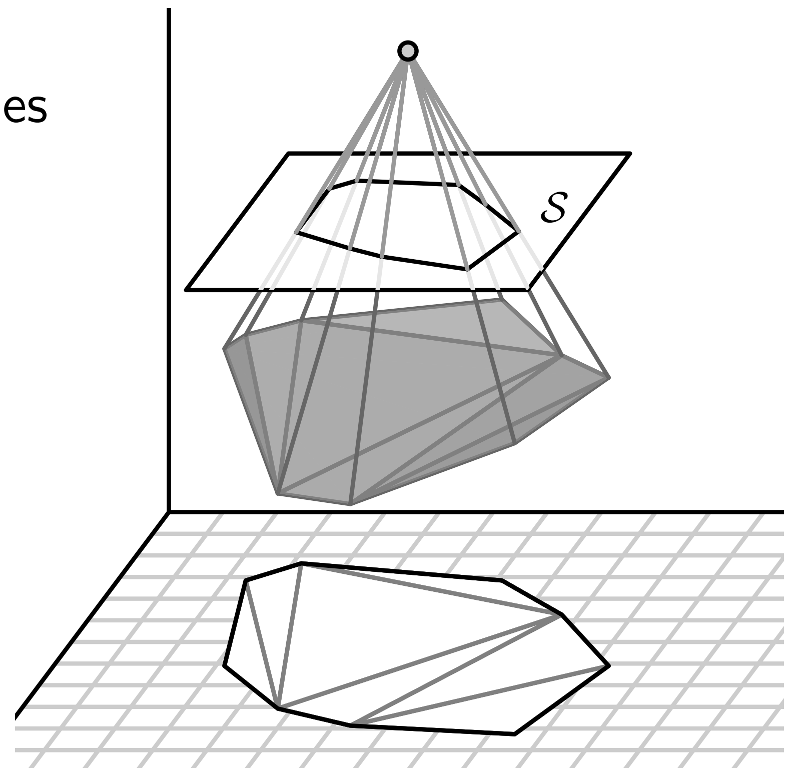
For a boundary triangulation T one defines

$$h_T(z) := \sum_{k=-1}^{d-1} f_k \, z^{k+1} \, (1-z)^{d-1-k}$$

and if this triangulation is regular,
Dehn–Sommerville holds.



Unimodular Triangulations and h^*

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If Δ is a unimodular k-simplex then $\quad \mathrm{Ehr}_\Delta(z) = \frac{1}{(1-z)^{k+1}}$

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Ehrhart–Macdonald Reciprocity $\longrightarrow \quad \mathrm{Ehr}_{\Delta^\circ}(z) = \left(\dfrac{z}{1-z}\right)^{k+1}$

The Point These Ehrhart series can help us count things.

Unimodular Triangulations and h^*

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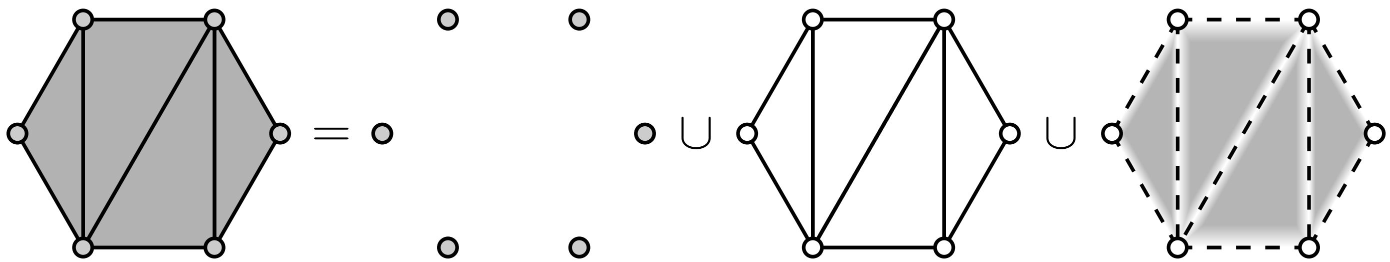

If $\mathcal{P}$ admits a unimodular triangulation T then

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{k=0}^{d} f_k \left(\frac{z}{1-z}\right)^{k+1} = \frac{\sum_{k=-1}^{d} f_k \, z^{k+1}(1-z)^{d-k}}{(1-z)^{d+1}}$$

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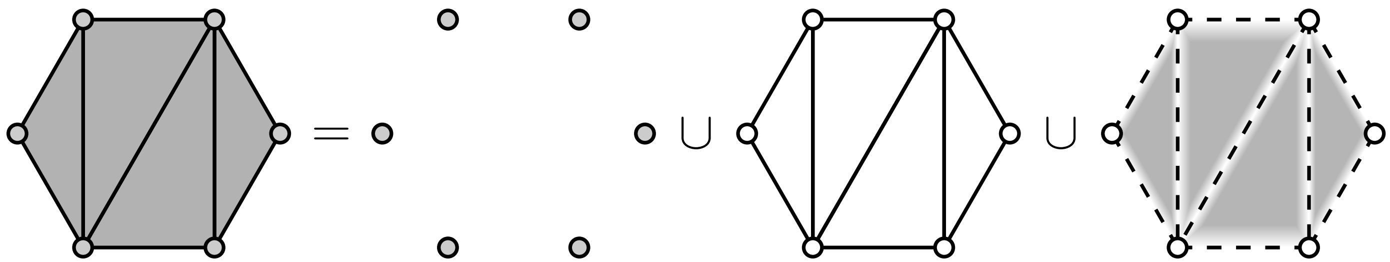

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$$\mathrm{Ehr}_{\mathcal{P}}(z) \ = \ \frac{\sum_{k=-1}^{d} f_k \, z^{k+1} (1-z)^{d-k}}{(1-z)^{d+1}} \ = \ \frac{h_T(z)}{(1-z)^{d+1}}$$

that is, $\ h^*_{\mathcal{P}}(z) = h_T(z)$

Stapledon Decompositions

If $\mathcal{P}$ admits a unimodular triangulation T then $h^*_{\mathcal{P}}(z) = h_T(z)$

What if not?

Stapledon Decompositions

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What if not?

The degree s of a lattice polytope $\mathcal{P}$ is the degree of $h^*_{\mathcal{P}}(z)$

Codegree $d+1-s$ $\longleftarrow$ smallest integer ℓ such that $\ell\mathcal{P}^\circ \cap \mathbb{Z}^d \neq \varnothing$

Theorem (Stapledon 2009) If $\mathcal{P}$ is a lattice d-polytope with codegree ℓ then

$$\left(1 + z + \cdots + z^{\ell-1}\right) h^*_{\mathcal{P}}(z) \; = \; a(z) + z^\ell \, b(z)$$

where $a(z) = z^d \, a(\frac{1}{z})$, $b(z) = z^{d-\ell} \, b(\frac{1}{z})$ and $a(z)$ and $b(z)$ are nonnegative.

The case $\ell = 1$ was proved by Betke & McMullen (1985). There is a version for rational polytopes (MB–Braun–Vindas-Meléndez 2022).

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Topological story $a(z)$ and $b(z)$ can be written in terms of h-polynomials of links of a given triangulation of $\mathcal{P}$ and associated arithmetic datat ("box polynomials").

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Corollary Inequalities for h^*-coefficients $\longleftarrow$ Exercises

Open Problem Try to prove an analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

Boundary h^*-polynomials