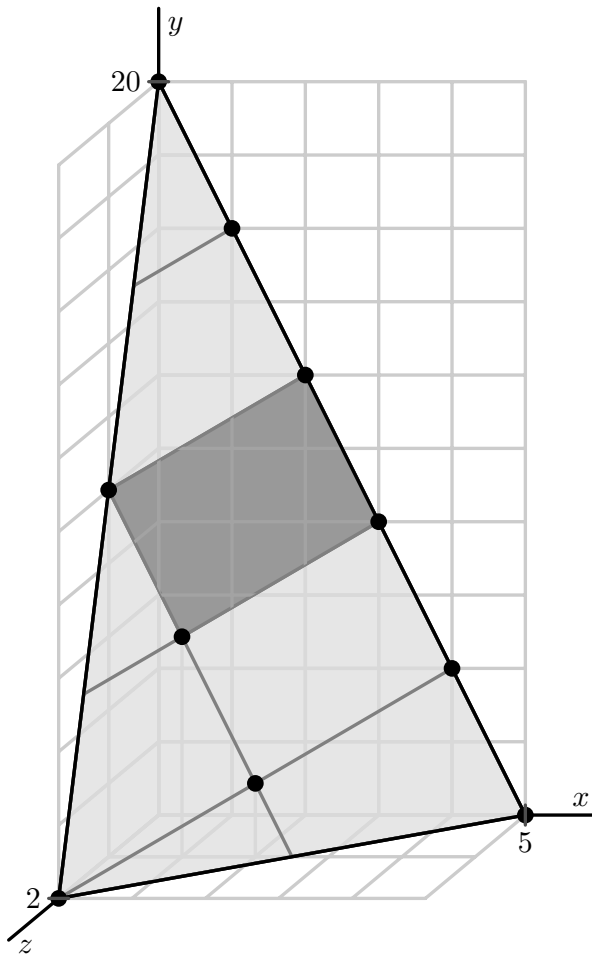


Ehrhart Polynomials

Day I: Appetizers



Matthias Beck

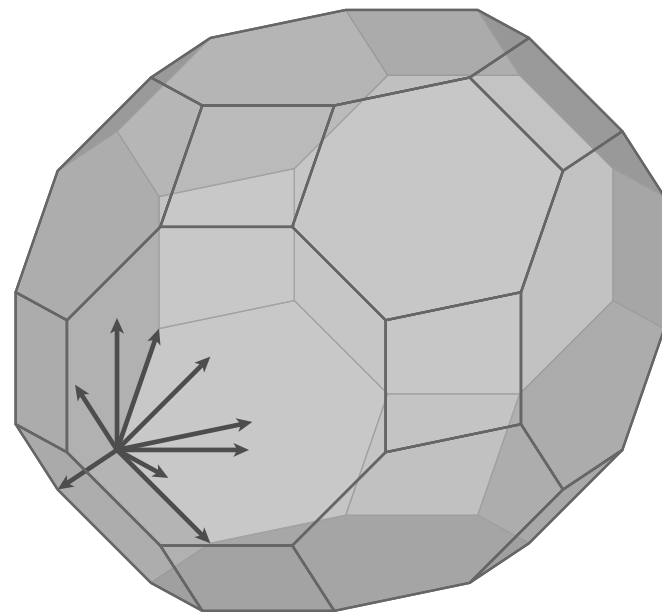
San Francisco State University

<https://matthbeck.github.io/>

VIII Encuentro Colombiano
De Combinatoria

“Science is what we understand well enough to explain to a computer, art is all the rest.”

Donald Knuth



Themes

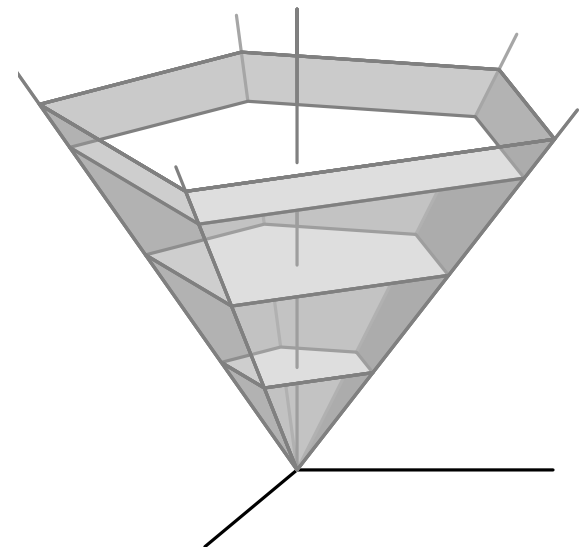
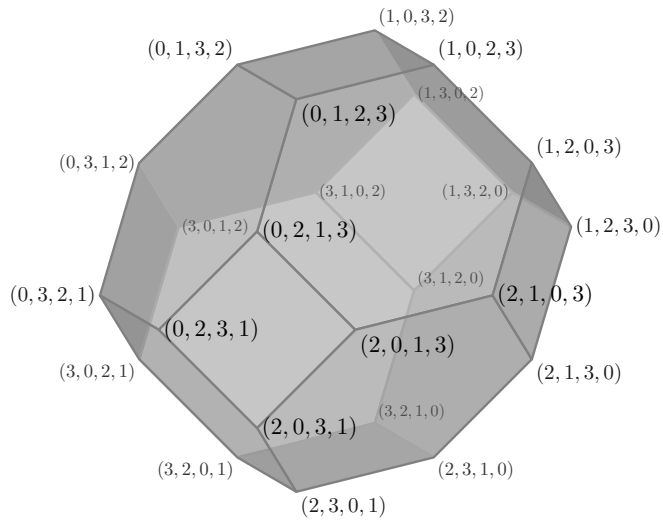
Discrete-geometric
polynomials

Computation
(complexity)

Generating
functions

Combinatorial
structures

Polyhedra



A Sample Problem: Birkhoff–von Neumann Polytope

This site is supported by donations to [The OEIS Foundation](#).

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[Hints](#)
(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

A037302 Normalized volume of Birkhoff polytope of $n \times n$ doubly-stochastic square matrices. If the volume is $v(n)$, then $a(n) = ((n-1)^2)! * v(n) / n^{(n-1)}$.

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028,
12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099,
5091038988117504946842559205930853037841762820367901333706255223000 ([list](#); [graph](#); [refs](#); [listen](#); [history](#);
[text](#); [internal format](#))

OFFSET 1,3

COMMENTS The Birkhoff polytope is an $(n-1)^2$ -dimensional polytope in n^2 -dimensional space; its vertices are the $n!$ permutation matrices.
Is $a(n)$ divisible by n^2 for all $n \geq 4$? - [Dean Hickerson](#), Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

Discrete Volumes

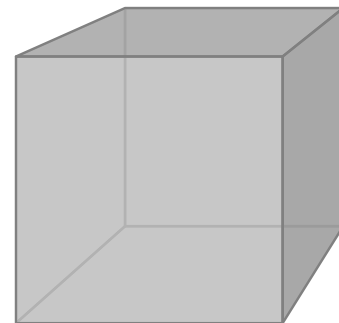
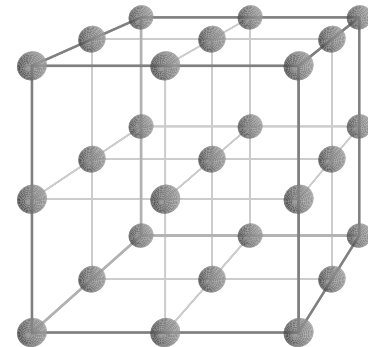
Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d \dots$

► (list)
$$\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \dots z_d^{m_d}$$

► (count) $|\mathcal{P} \cap \mathbb{Z}^d|$

► (volume)
$$\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$$



Discrete Volumes

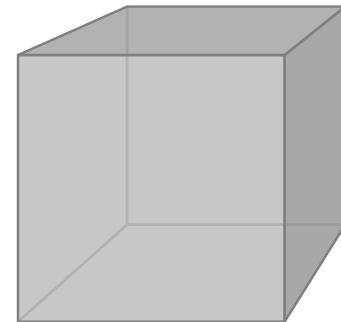
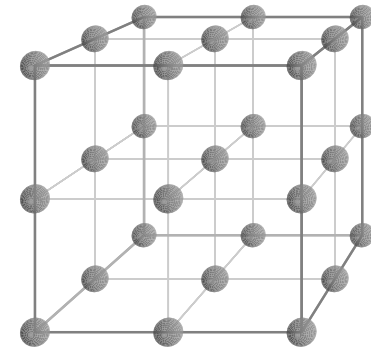
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Ehrhart function
$$L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = |t\mathcal{P} \cap \mathbb{Z}^d| \quad \text{for } t \in \mathbb{Z}_{>0}$$

Some Motivation

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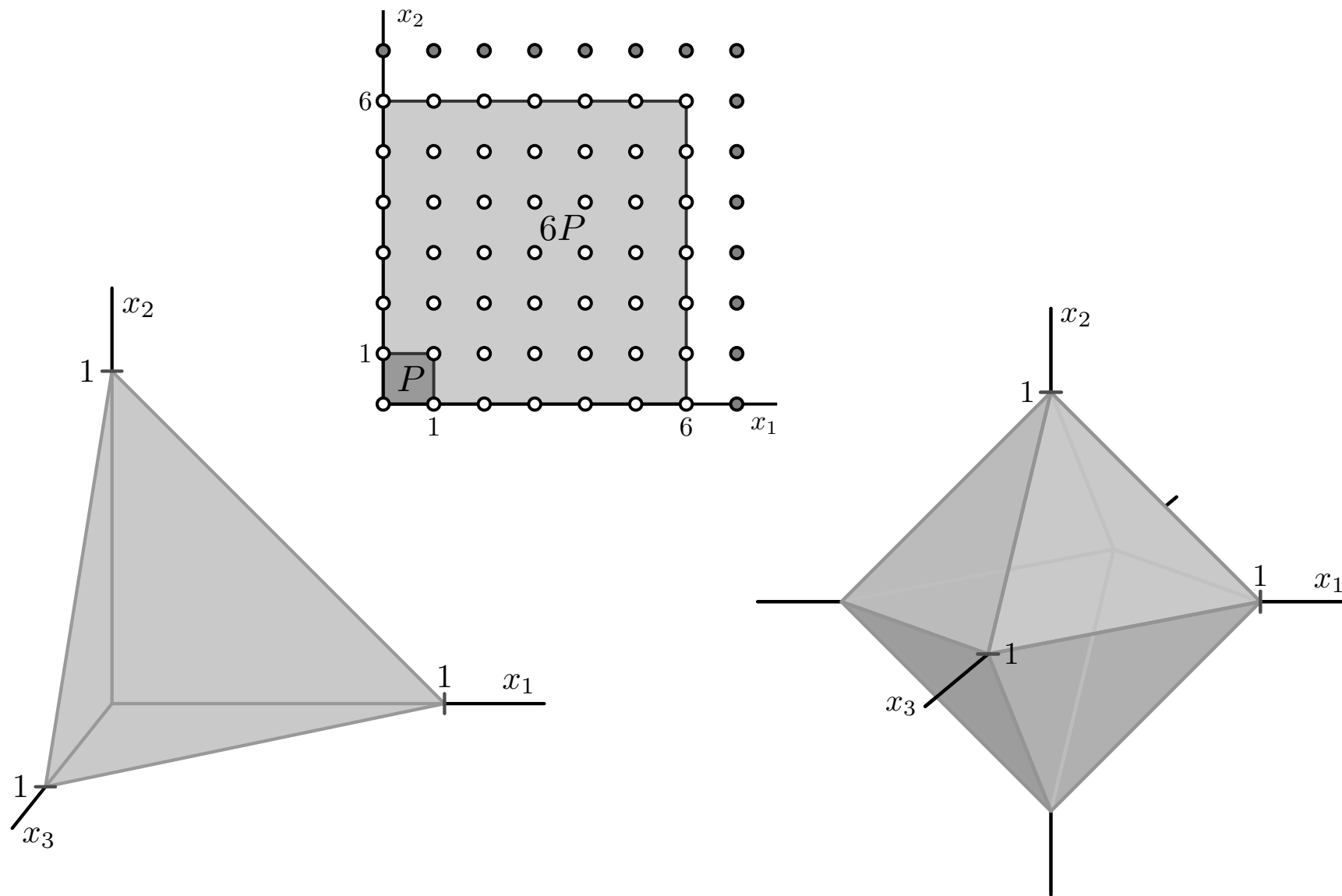
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- ▶ Polytopes are basic geometric objects, yet even for these basic objects volume computation is **hard** and there remain many open problems.
- ▶ Also, polytopes are **cool**.

Today's Menu: Get Our Hands Dirty

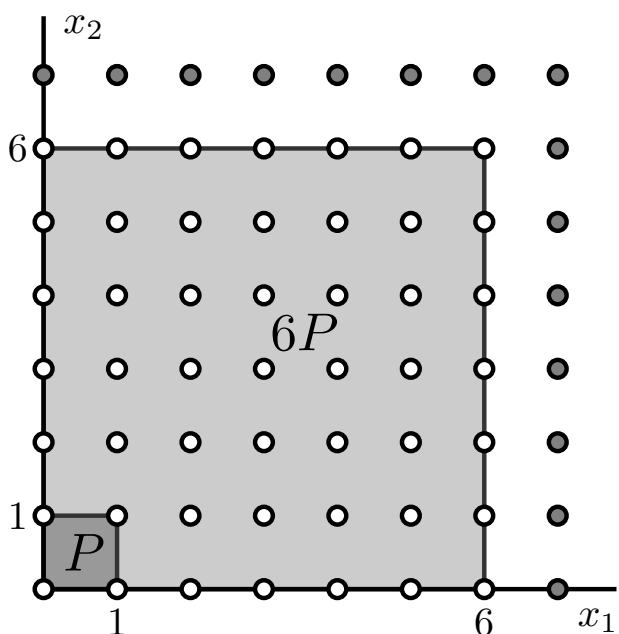


The Unit Cube

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Z}^d

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$

The unit cube in \mathbb{R}^d is $\mathcal{P} = [0, 1]^d = \{\mathbf{x} \in \mathbb{R}^d : 0 \leq x_j \leq 1\}$



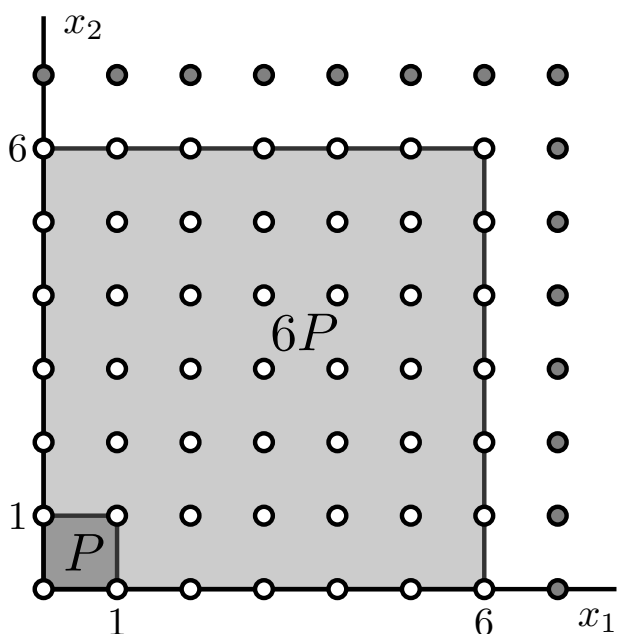
$$\longrightarrow L_{\mathcal{P}}(t) = (t+1)^d$$

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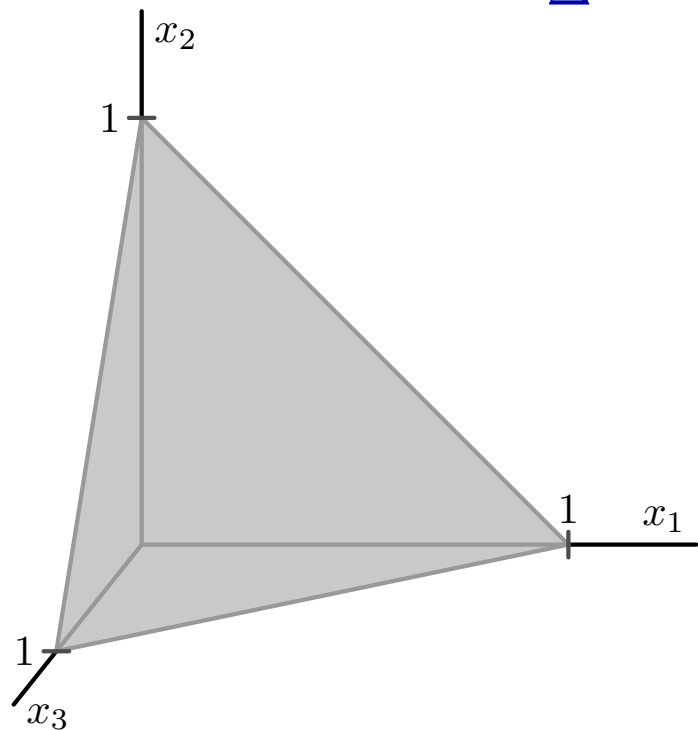
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$$L_{\mathcal{P}^\circ}(t) = (t-1)^d$$

The Standard Simplex

The **standard simplex** $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta = \{x \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1\}$$



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$$\begin{aligned} L_{\Delta}(t) &= \# \{ (x_1, x_2, \dots, x_d) \in \mathbb{Z}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq t \} \\ &= \# \left\{ (x_1, x_2, \dots, x_d, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = t \right\} \\ &= \binom{d+t}{d} \end{aligned}$$

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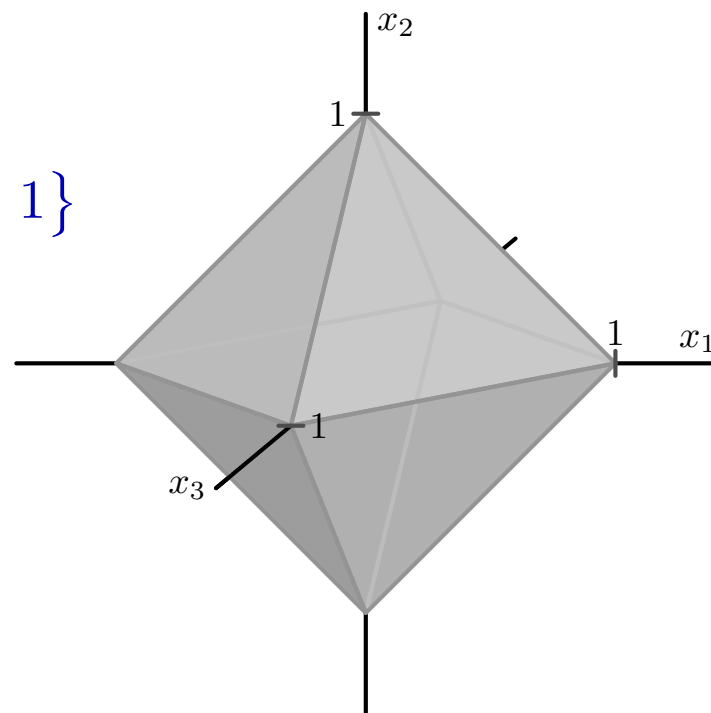
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$$L_{\Delta^{\circ}}(t) = \binom{t-1}{d}$$

The Cross-Polytope

The cross-polytope $\diamond \in \mathbb{R}^d$ is

$$\diamond = \{x \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}$$

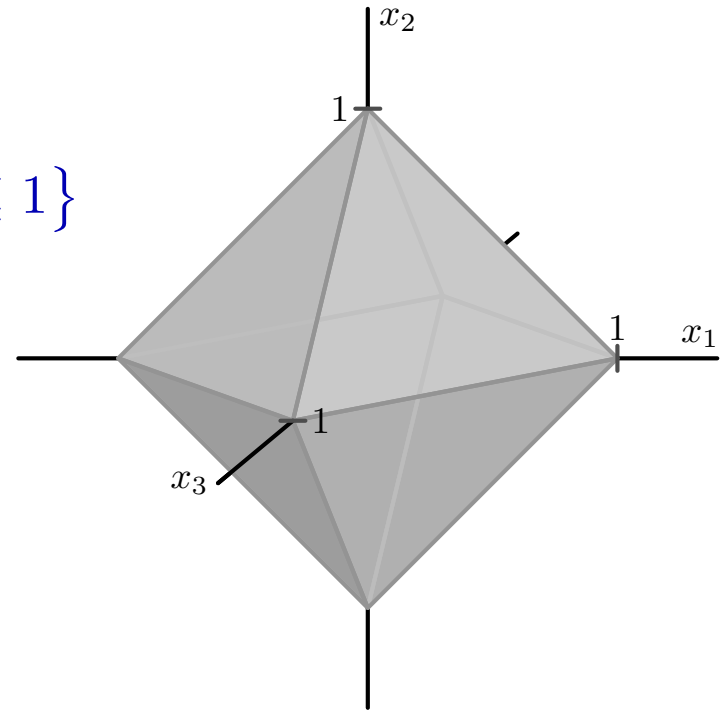


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Let's compute $L_\diamond(t)$ for $d = 3 \dots$



- ▶ Triangulation
- ▶ Disjoint triangulation
- ▶ Interpolation
- ▶ Generating function

The Cross-Polytope

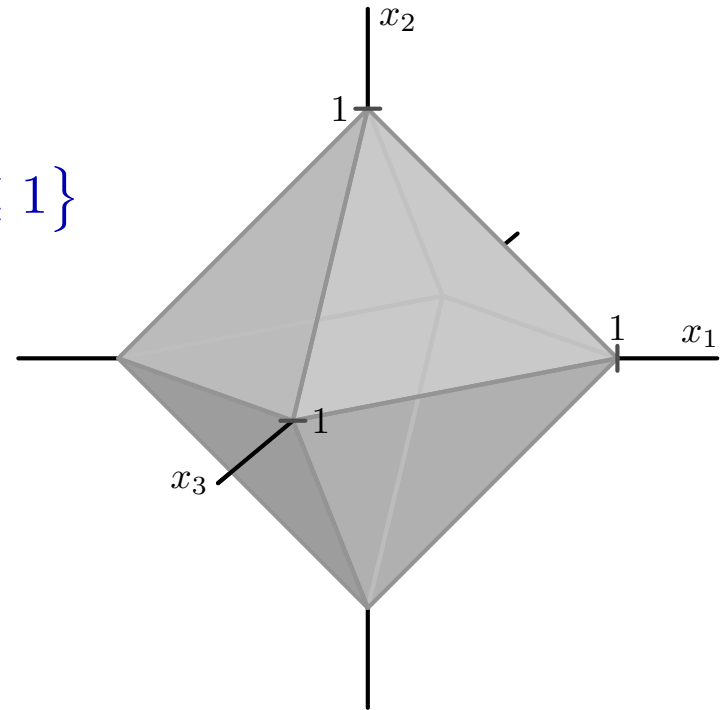
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► Triangulation

Dissect \diamond into 8 (standard) tetrahedra and use inclusion–exclusion to compute $L_\diamond(t)$



The Cross-Polytope

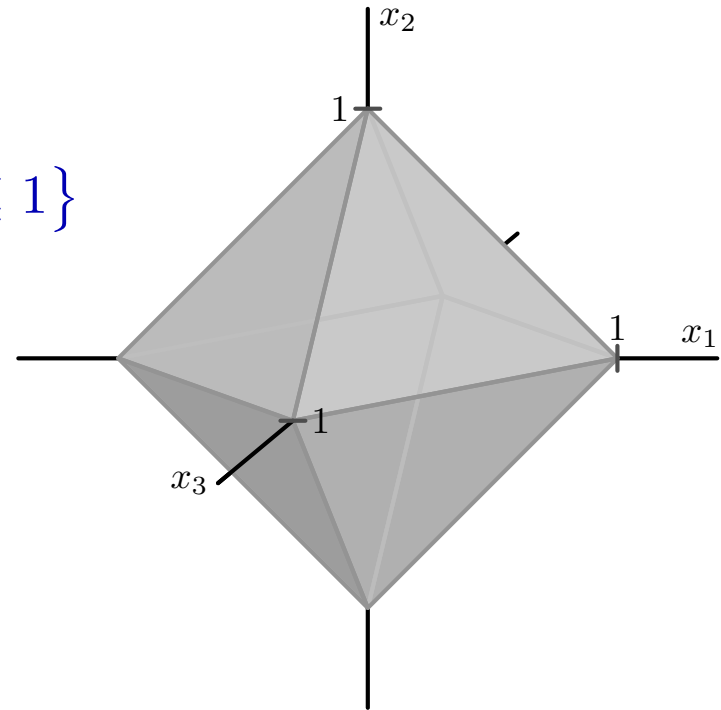
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► Disjoint triangulation

Dissect \diamond into 8 half-open tetrahedra



The Cross-Polytope

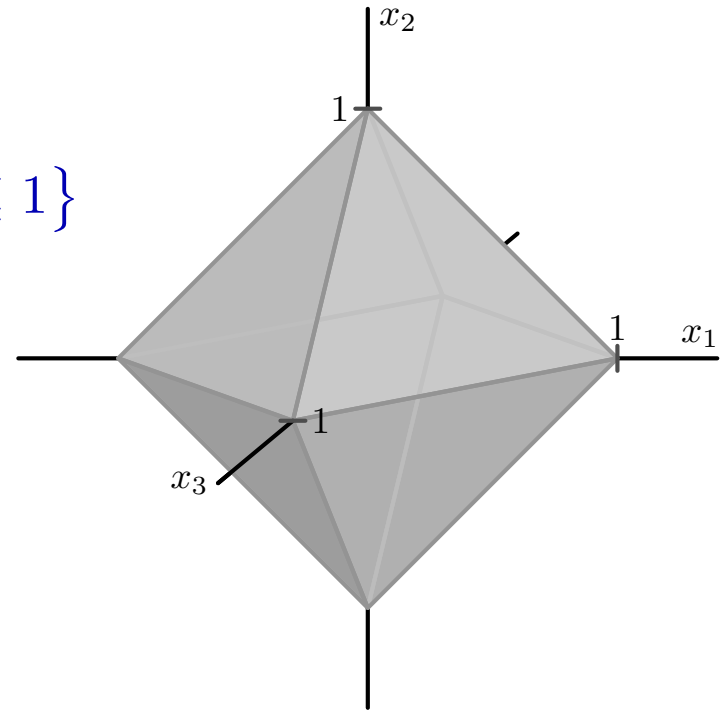
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► Interpolation

```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```

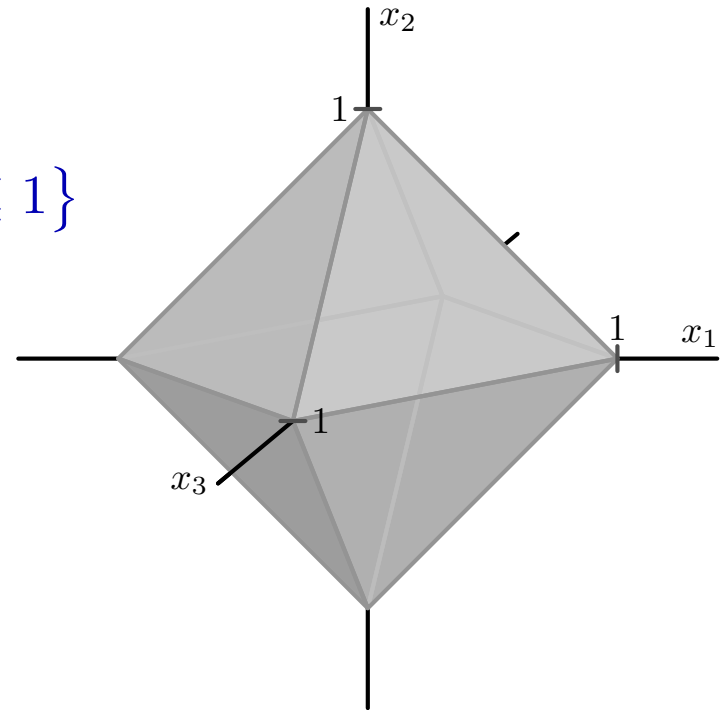


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► Generating function

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$$

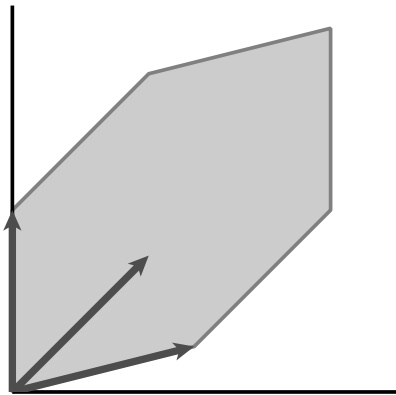
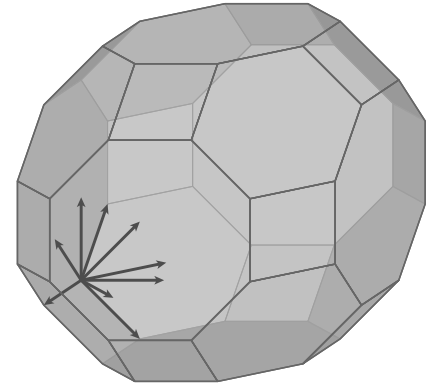
Exercise: $\text{Ehr}_{\text{BiPyr}(\mathcal{P})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{P}}(z)$

\dots for unit cubes \longrightarrow Eulerian polynomials

Zonotopes

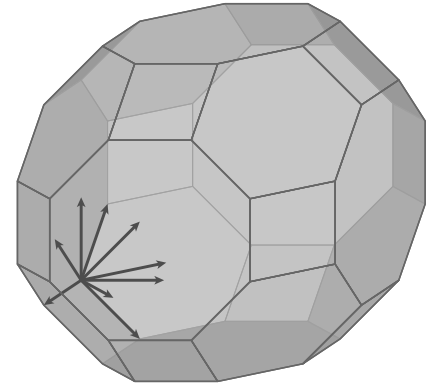
Line segment $[a, b] := \{(1 - \lambda) a + \lambda b : 0 \leq \lambda \leq 1\}$

Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2 := \{p + q : p \in \mathcal{K}_1, q \in \mathcal{K}_2\}$



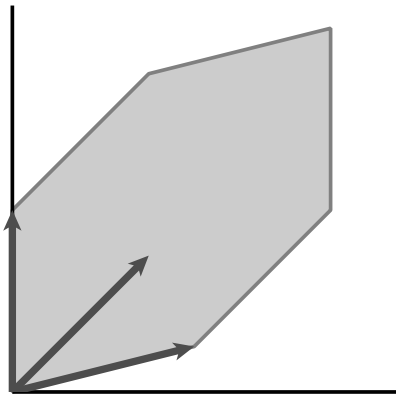
Zonotope $\mathcal{Z} := [a_1, b_1] + [a_2, b_2] + \cdots + [a_m, b_m]$

Zonotopes



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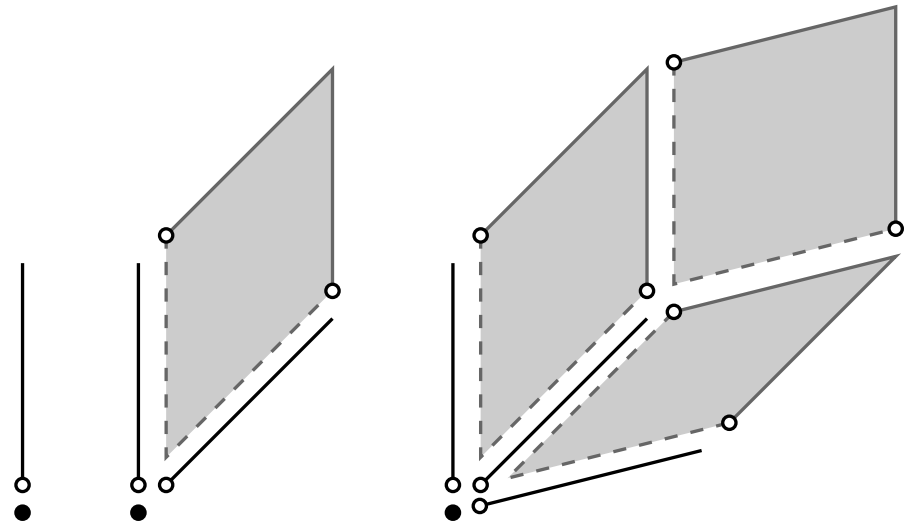


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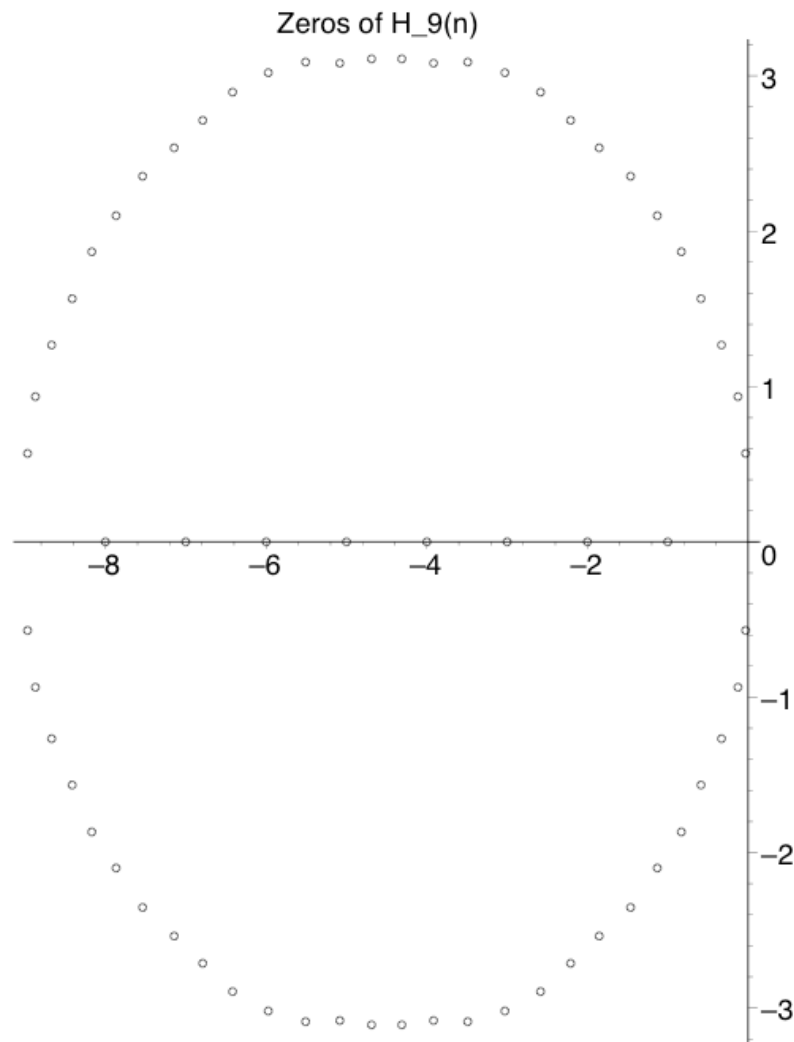
Every zonotope admits a **tiling** into parallelepipeds

\mathcal{P} — half-open d -parallelepiped

$\longrightarrow L_{\mathcal{P}}(t) = t^d$



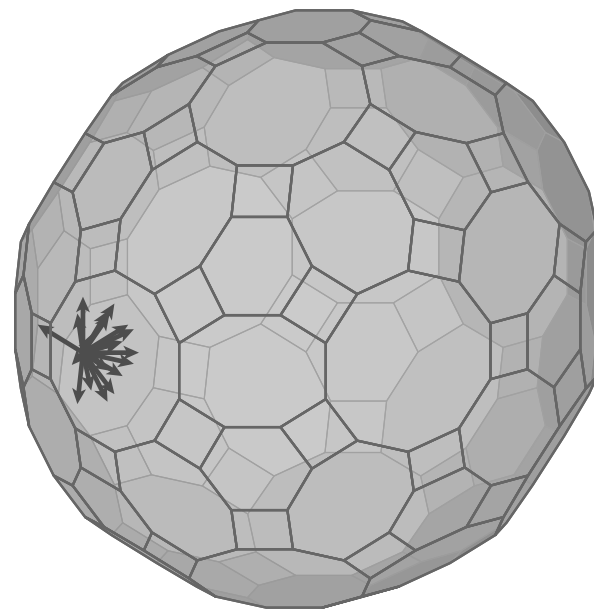
Birkhoff–von Neumann Revisited



For more about roots of
(Ehrhart) polynomials,
see Braun (2008) and
Pfeifle (2010).

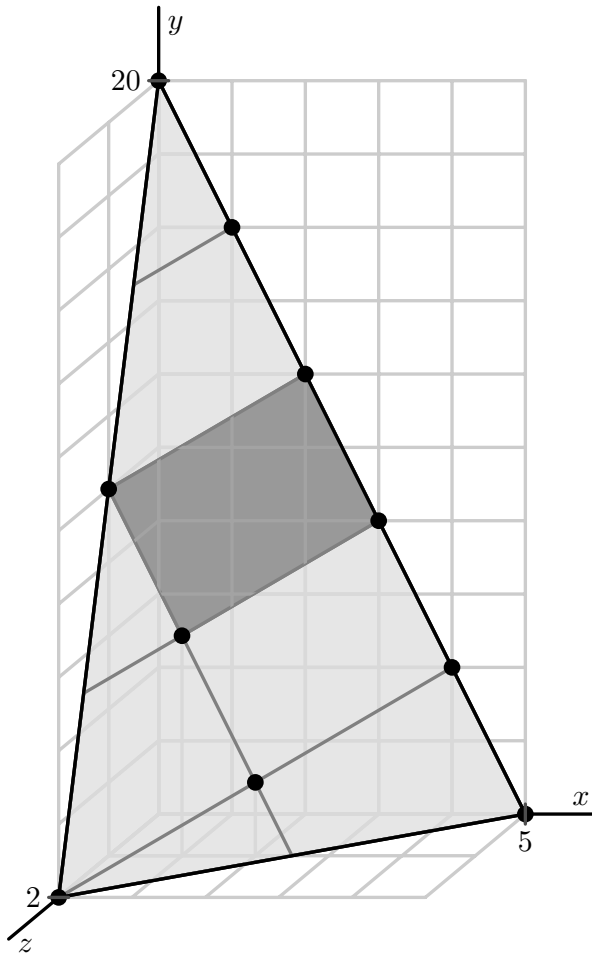
Recap Day I

- ▶ Volume computations \longrightarrow don't agonize, discretize
- ▶ Integer-point counting in dilated polytopes \longrightarrow polynomials
- ▶ Interpolation
- ▶ Generating functions
- ▶ Dissections: triangulations, tilings
- ▶ Tomorrow: enough practice, how does this work in theory?



Ehrhart Polynomials

Day II: Generating Functions & Complexity



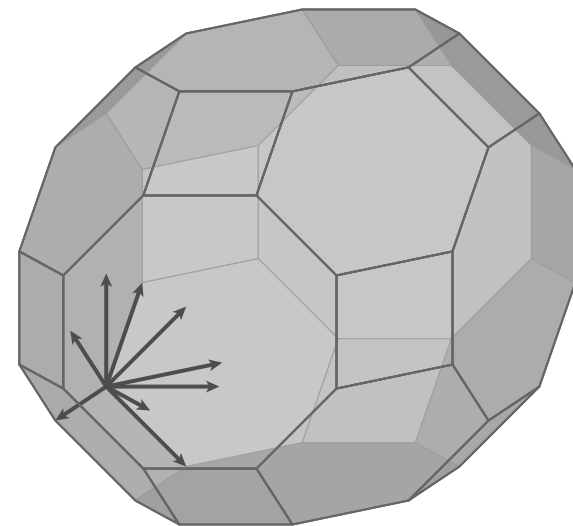
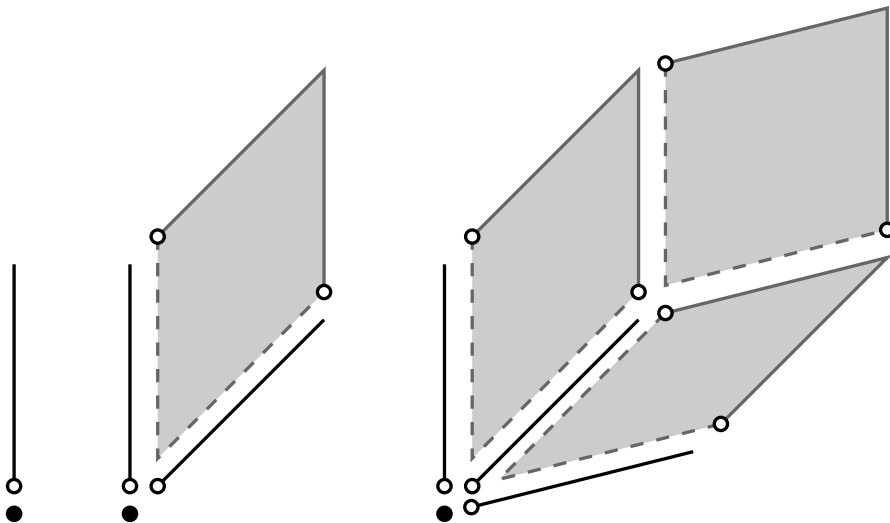
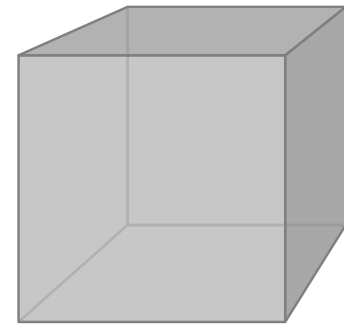
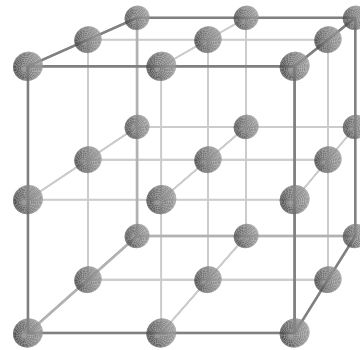
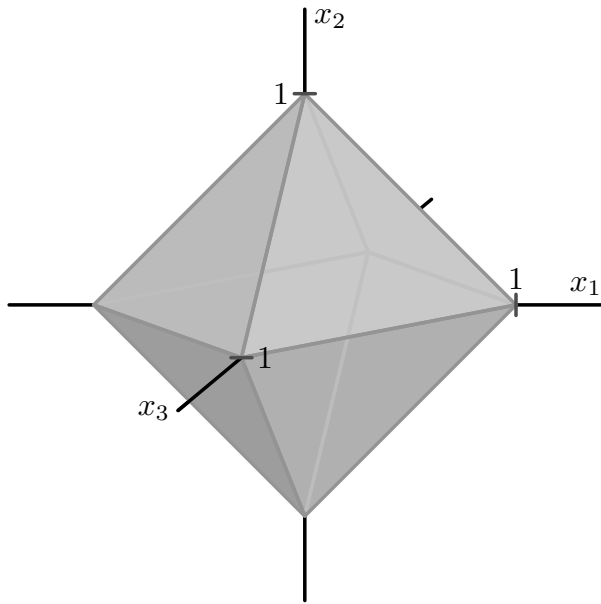
Matthias Beck

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VIII Encuentro Colombiano
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Any questions about yesterday?



Warm-Up: Partition Generating Functions

A **partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of an integer $k \geq 0$ satisfies

$$k = \lambda_1 + \lambda_2 + \dots + \lambda_n \quad \text{and} \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

Goal Compute $\sum_{\lambda} q^{\lambda_1 + \dots + \lambda_n}$ over your favorite partition family

Example $P_{\leq 3}$ — family of partitions into at most 3 parts

$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)}$$

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Idea $P_{\leq 3} = \{\lambda \in \mathbb{Z}^3 : 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \mathcal{K} \cap \mathbb{Z}^3$

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} \longleftarrow \text{polyhedral cone } \heartsuit$$

Warm-Up: Partition Generating Functions

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is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

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Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

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$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\mathcal{K}}(q, q, q) = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)}$$

Variations on a Theme

P_3 — family of partitions into **exactly 3** parts

$$P_3 = \{\lambda \in \mathbb{Z}^3 : 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \tilde{\mathcal{K}} \cap \mathbb{Z}^3$$

$$\tilde{\mathcal{K}} = \{\mathbf{x} \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{> 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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P_3 — family of partitions into **exactly 3** parts

$$P_3 = \{\lambda \in \mathbb{Z}^3 : 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \tilde{\mathcal{K}} \cap \mathbb{Z}^3$$

$$\tilde{\mathcal{K}} = \{\mathbf{x} \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{> 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \sigma_{\tilde{\mathcal{K}}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \tilde{\mathcal{K}} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{z_1 z_2 z_3}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

$$\sum_{\lambda \in P_3} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\tilde{\mathcal{K}}}(q, q, q) = \frac{q^3}{(1 - q)(1 - q^2)(1 - q^3)}$$

Integer-point Complexity of a Simplicial Cone

What if \mathcal{K} is (still simplicial and rational but) not unimodular?

Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geq 0} \mathbf{w}_1 + \mathbb{R}_{\geq 0} \mathbf{w}_2 + \mathbb{R}_{\geq 0} \mathbf{w}_3$$

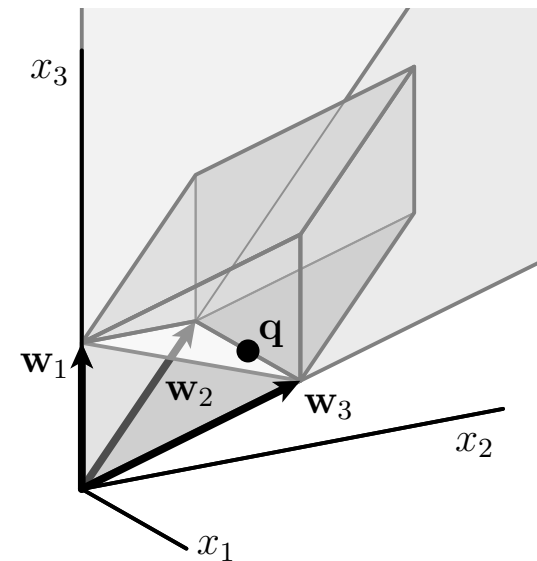
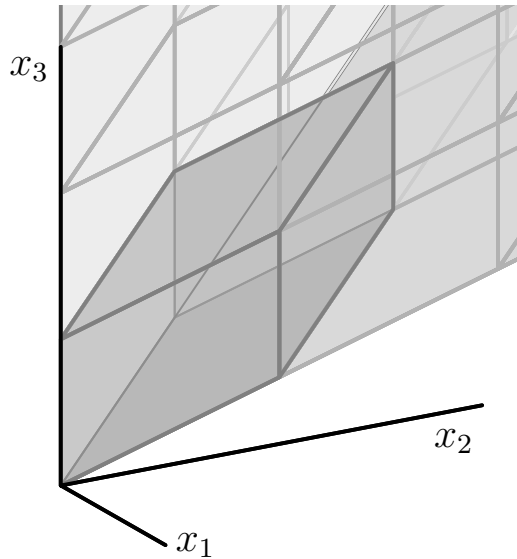
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Idea Tile \mathcal{K} with the half-open parallelepiped $\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$



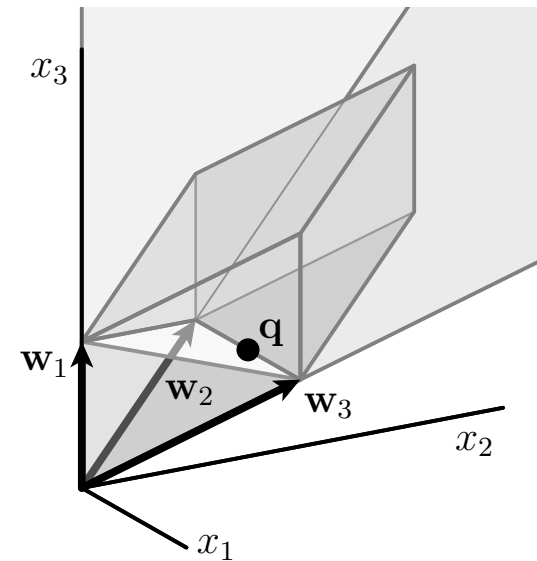
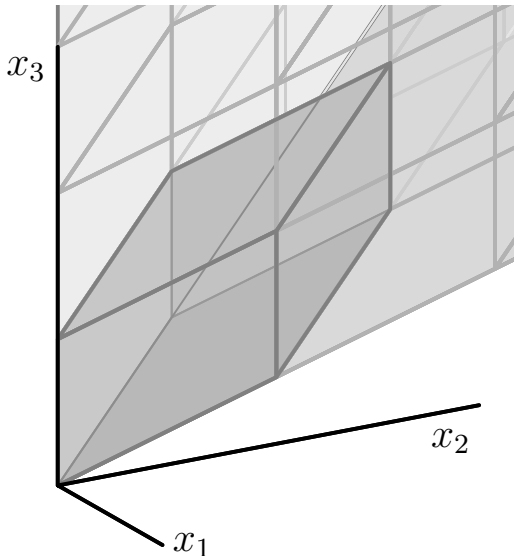
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$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

$$\text{where } \mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$$

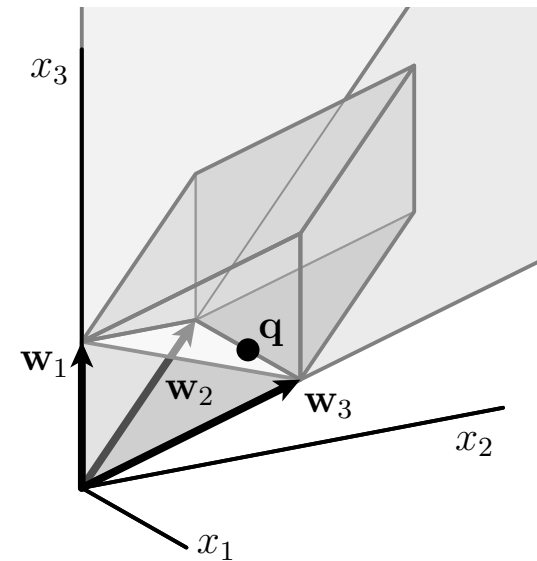
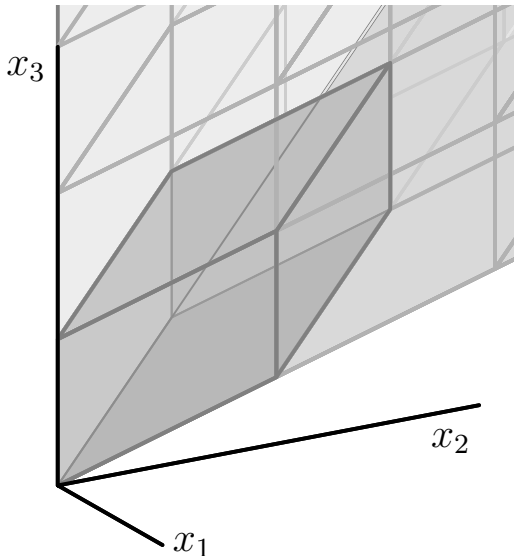
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Complexity: $\sigma_{\Pi}(z_1, z_2, z_3)$ has D terms

Homogenizing Polytopes

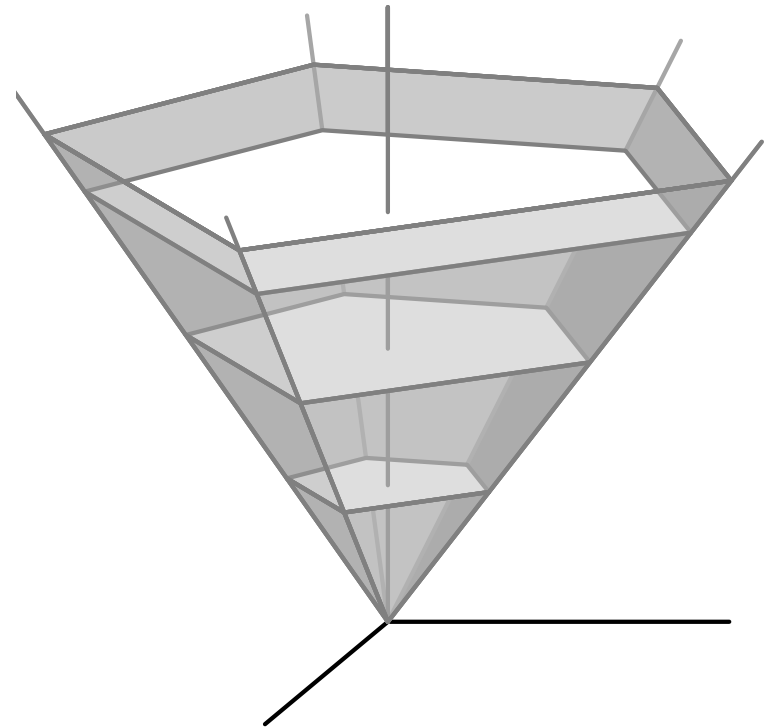
Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

$$\text{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$

$$\text{cone}(\mathcal{P}) \cap \{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

contains a copy of $t\mathcal{P}$



Homogenizing Polytopes

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

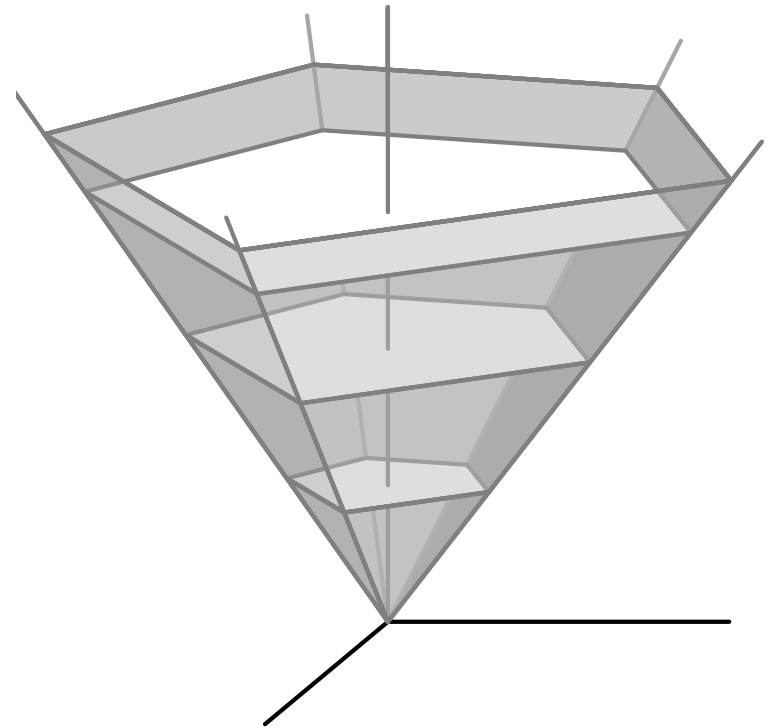
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$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \sigma_{\text{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$$



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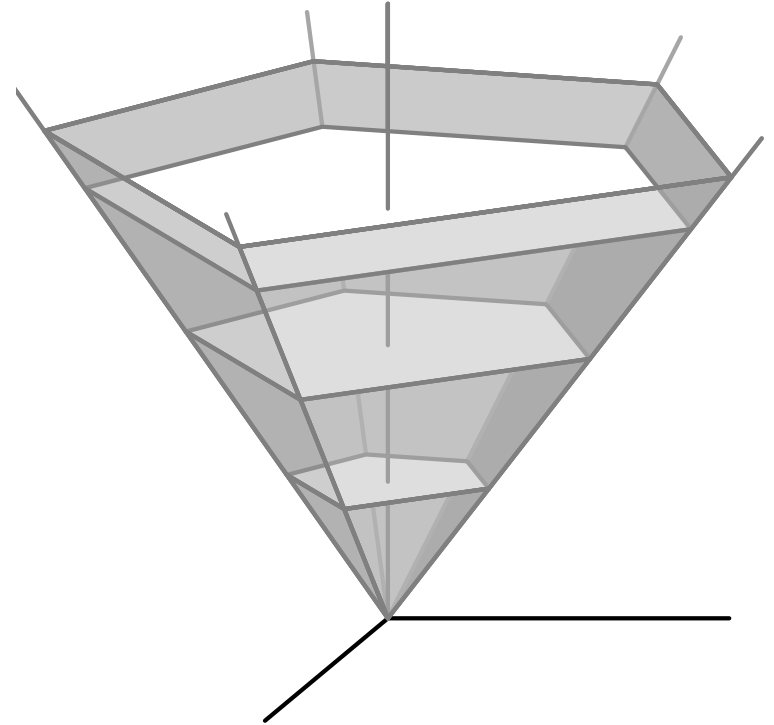
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If \mathcal{P} is a simplex,

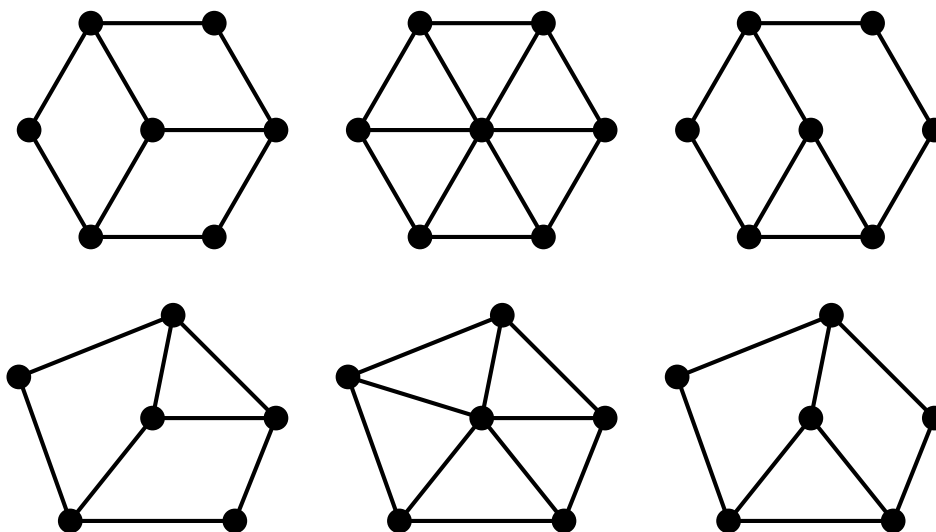
$$\sigma_{\text{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \longrightarrow \text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{d+1}}$$



Trials & Triangulations

Subdivision of a polyhedron \mathcal{P} — finite collection S of polyhedra such that

- ▶ if \mathcal{F} is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- ▶ if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- ▶ $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



If each \mathcal{F} is a simplex \longrightarrow **triangulation** of a polytope

Ehrhart Polynomials



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

Equivalently, $\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$ is rational:

$$\text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{\dim \mathcal{P} + 1}}$$

where the h^* -polynomial $h_{\mathcal{P}}^*(z)$ satisfies $h_{\mathcal{P}}^*(0) = 1$ and $h_{\mathcal{P}}^*(1) = (\dim \mathcal{P})! \text{vol}(\mathcal{P})$.

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Computational bottlenecks:

- ▶ triangulation
- ▶ determinants of resulting simplicial cones

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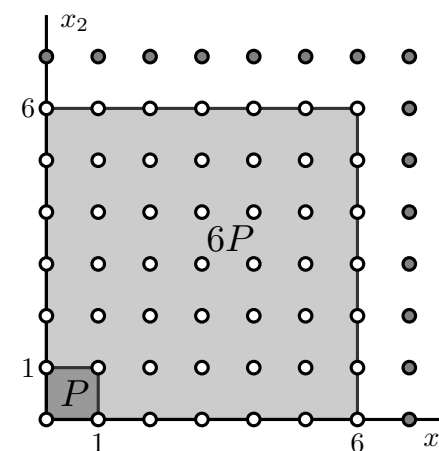
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We saw instances yesterday: $\mathcal{P} = [0, 1]^d$

$$\longrightarrow L_{\mathcal{P}}(t) = (t+1)^d$$

$h_{\mathcal{P}}^*(z)$ — Eulerian polynomial



Ehrhart Polynomials



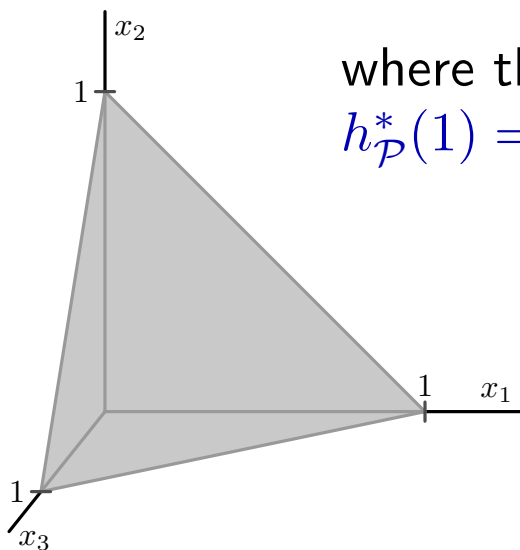
EH
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$$\Delta = \{ \mathbf{x} \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1 \}$$

$$L_{\Delta}(t) = \binom{d+t}{d}$$

$$h_{\mathcal{P}}^*(z) = 1$$

Ehrhart Polynomials



EH
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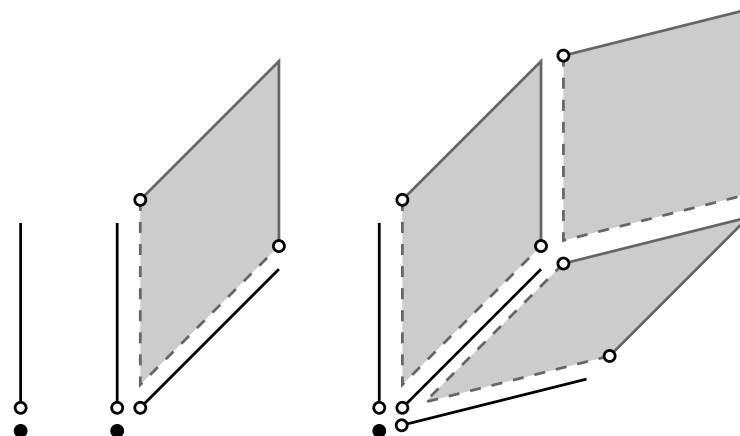
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\mathcal{P} — half-open d -parallelepiped

$$\longrightarrow L_{\mathcal{P}}(t) = t^d$$



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Seeming dichotomy: $\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

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Equivalent descriptions of an Ehrhart polynomial:

- ▶ $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$
- ▶ via roots of $L_{\mathcal{P}}(t)$
- ▶ $\text{Ehr}_{\mathcal{P}}(z) \longrightarrow L_{\mathcal{P}}(t) = h_0^*\binom{t+d}{d} + h_1^*\binom{t+d-1}{d} + \cdots + h_d^*\binom{t}{d}$

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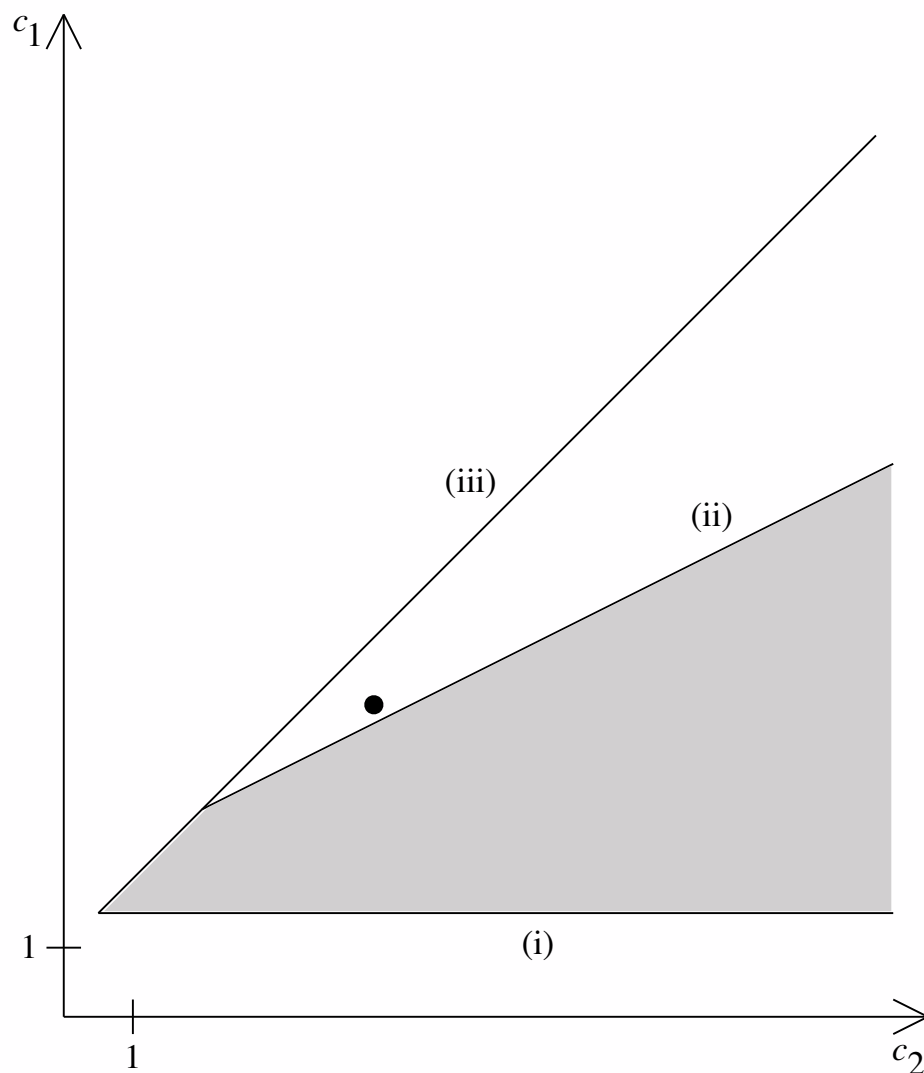
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Open Problem Classify Ehrhart polynomials.

Ehrhart Polynomials in Dimension 2



\mathcal{P} — lattice polygon

→ $L_{\mathcal{P}}(t) = c_2 t^2 + c_1 t + 1$

Ehrhart Polynomials



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Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the **interior** lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) = h_d^* \binom{t+d-1}{d} + h_{d-1}^* \binom{t+d-2}{d} + \cdots + h_0^* \binom{t-1}{d}$$

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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Corollary If $h_{d+1-k}^* > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

Positivity Among Ehrhart Polynomials



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Theorem (Betke–McMullen 1985, Stapledon 2009) If $h_d^* > 0$ then

$$h(z) = a(z) + z b(z)$$

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Open Problem Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Q}^d

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a **quasipolynomial** in t :

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where $c_0(t), \dots, c_d(t)$ are periodic functions.

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$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$$

for some (minimal) $p \in \mathbb{Z}_{>0}$ (the **period** of $L_{\mathcal{P}}(t)$).

Open Problem Study periods of Ehrhart quasipolynomials.

Recap Day II

- ▶ Generating functions are cool
- ▶ Integer-point transforms of rational polyhedra \rightarrow rational functions
- ▶ Complexity of a simplicial cone: determinant of its generators
- ▶ Homogenize polytopes
- ▶ Triangulations
- ▶ Polynomial data
- ▶ Thursday: structural results about Ehrhart polynomials

