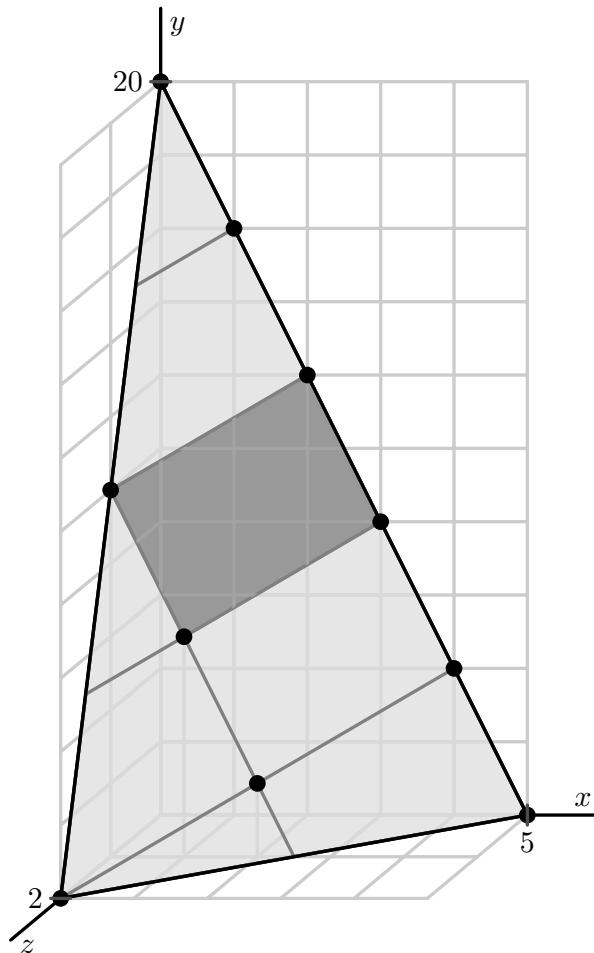


Ehrhart Polynomials

Day I: Appetizers

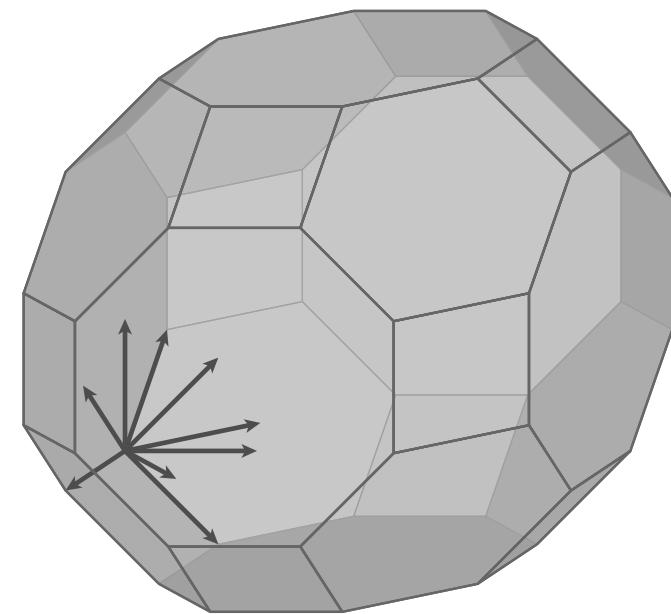


Matthias Beck
San Francisco State University
<https://matthbeck.github.io/>

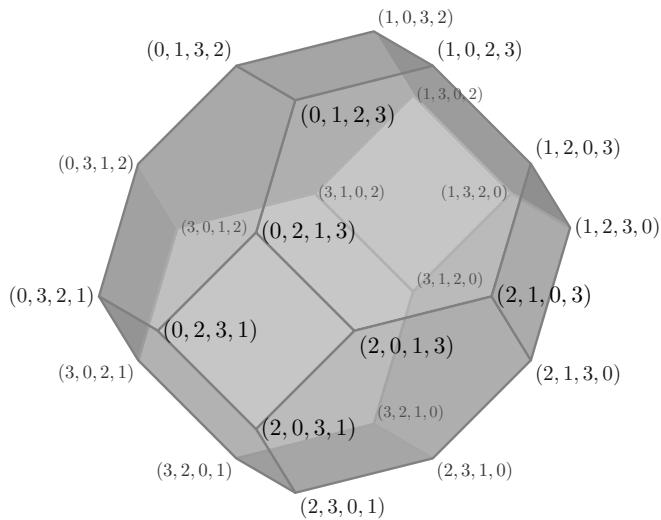
VIII Encuentro Colombiano
De Combinatoria

“Science is what we understand well enough to explain to a computer, art is all the rest.”

Donald Knuth



Themes



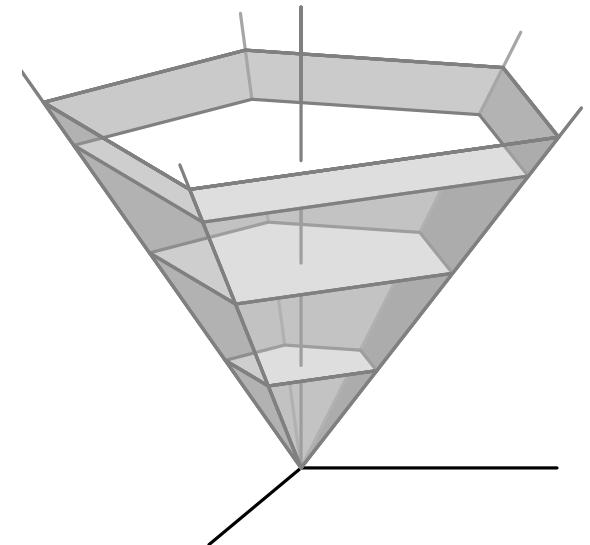
Combinatorial
structures

Discrete-geometric
polynomials

Computation
(complexity)

Generating
functions

Polyhedra



A Sample Problem: Birkhoff–von Neumann Polytope

This site is supported by donations to [The OEIS Foundation](#).

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[Hints](#)
(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

A037302 Normalized volume of Birkhoff polytope of $n \times n$ doubly-stochastic square matrices. If the volume $v(n)$ is $v(n)$, then $a(n) = ((n-1)^2)! * v(n) / n^{(n-1)}$.

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028,
12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099,
5091038988117504946842559205930853037841762820367901333706255223000 ([list](#); [graph](#); [refs](#); [listen](#); [history](#);
[text](#); [internal format](#))

OFFSET 1,3

COMMENTS The Birkhoff polytope is an $(n-1)^2$ -dimensional polytope in n^2 -dimensional space; its vertices are the $n!$ permutation matrices.
Is $a(n)$ divisible by n^2 for all $n \geq 4$? - [Dean Hickerson](#), Nov 27 2002

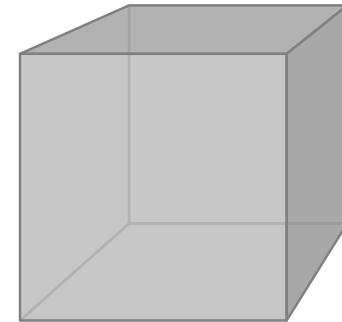
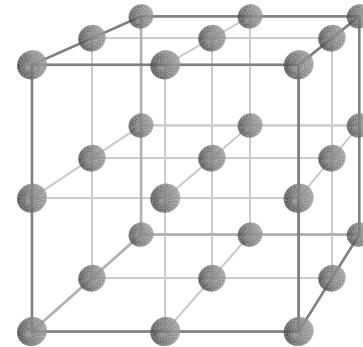
$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d \dots$

- ▶ (list) $\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$
- ▶ (count) $|\mathcal{P} \cap \mathbb{Z}^d|$
- ▶ (volume) $\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$

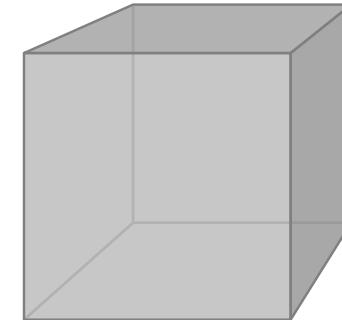
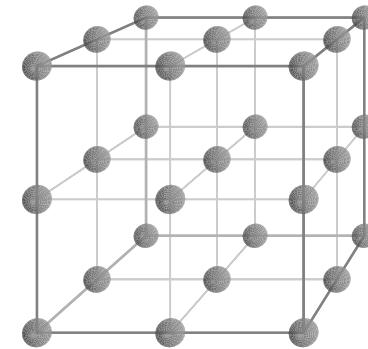


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Ehrhart function $L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = |t\mathcal{P} \cap \mathbb{Z}^d|$ for $t \in \mathbb{Z}_{>0}$

Some Motivation

- ▶ Linear systems are **everywhere**, and so polyhedra are everywhere.

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- ▶ Much discrete geometry can be modeled using **polynomials** and, conversely, many combinatorial polynomials can be modeled geometrically.

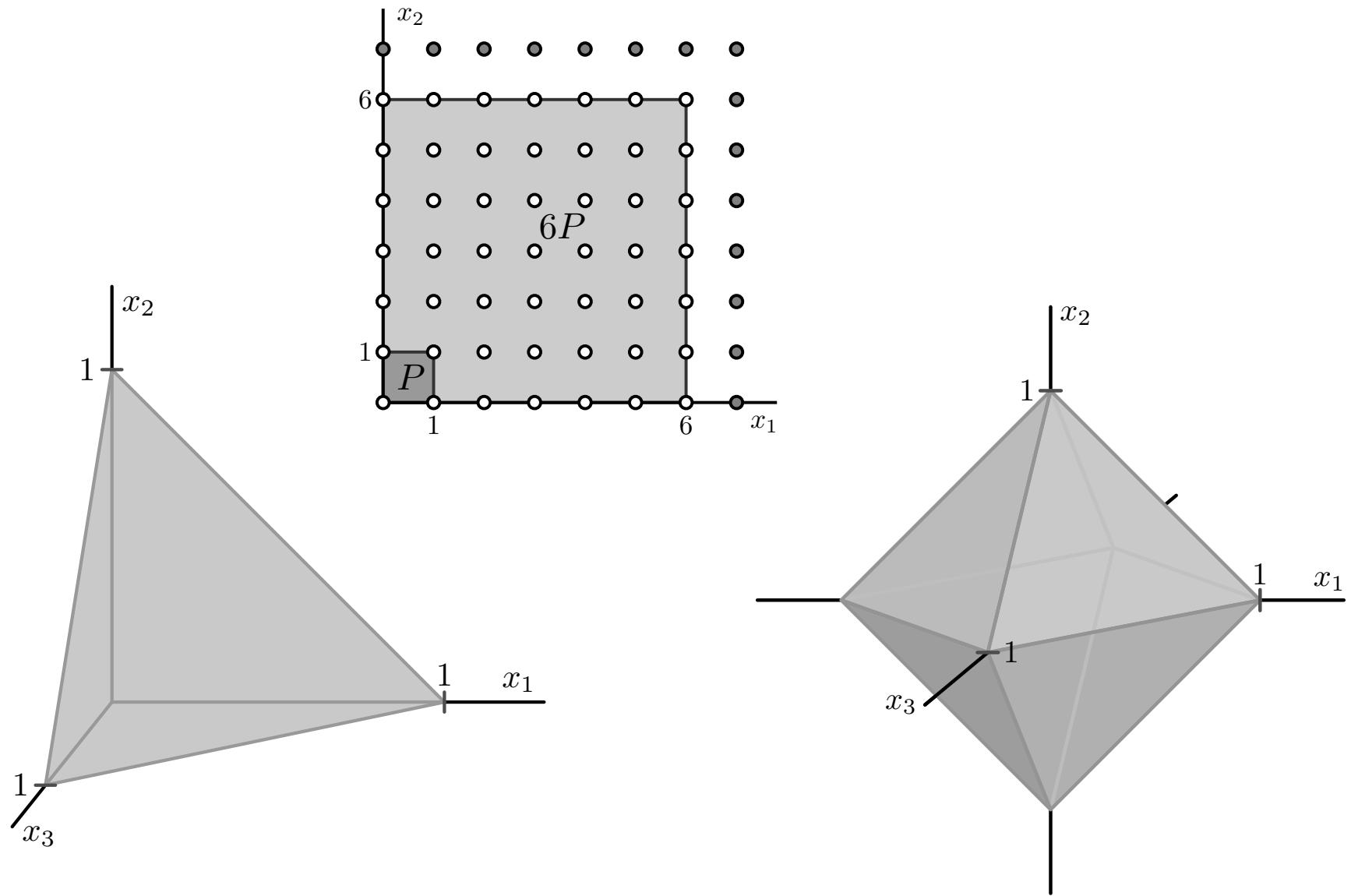
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- ▶ Polytopes are basic geometric objects, yet even for these basic objects volume computation is **hard** and there remain many open problems.
- ▶ Also, polytopes are **cool**.

Today's Menu: Get Our Hands Dirty

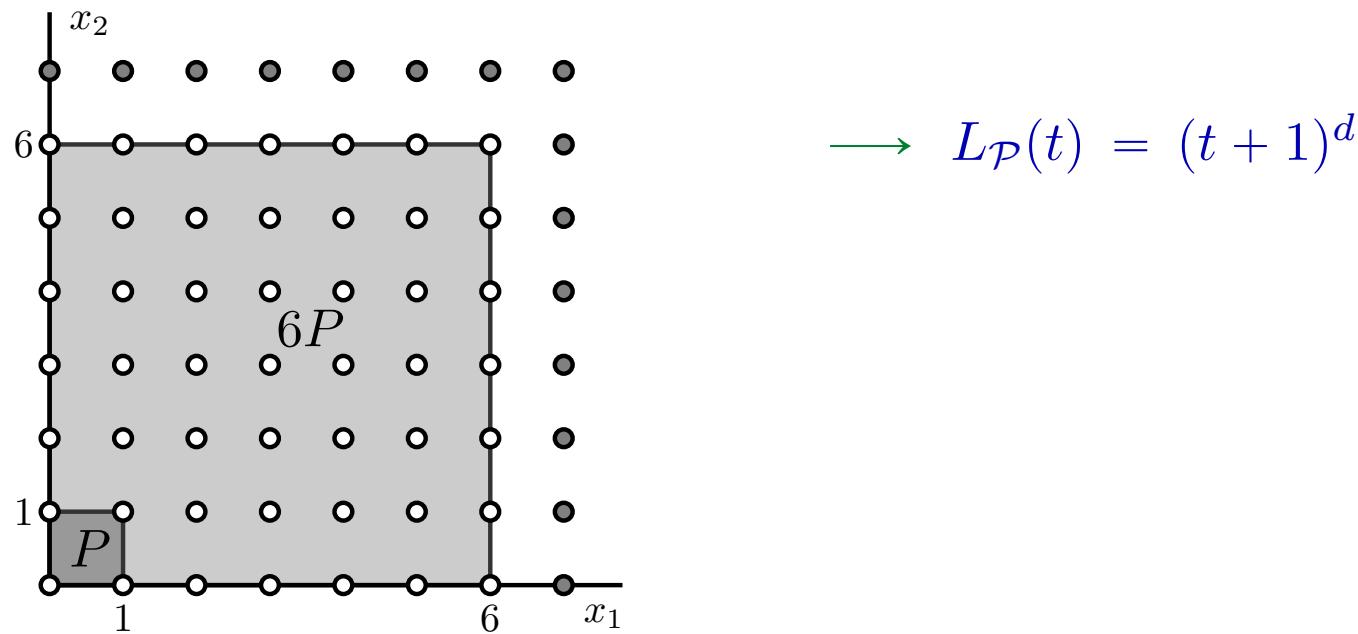


The Unit Cube

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Z}^d

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$

The unit cube in \mathbb{R}^d is $\mathcal{P} = [0, 1]^d = \{x \in \mathbb{R}^d : 0 \leq x_j \leq 1\}$

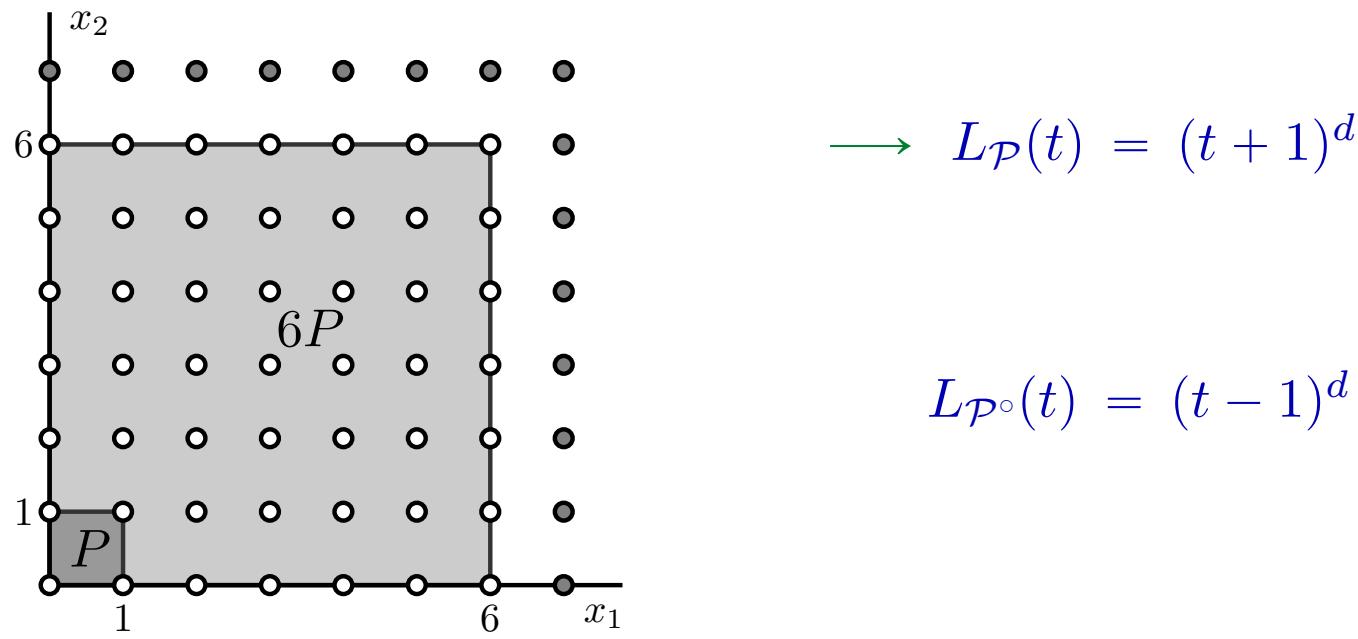


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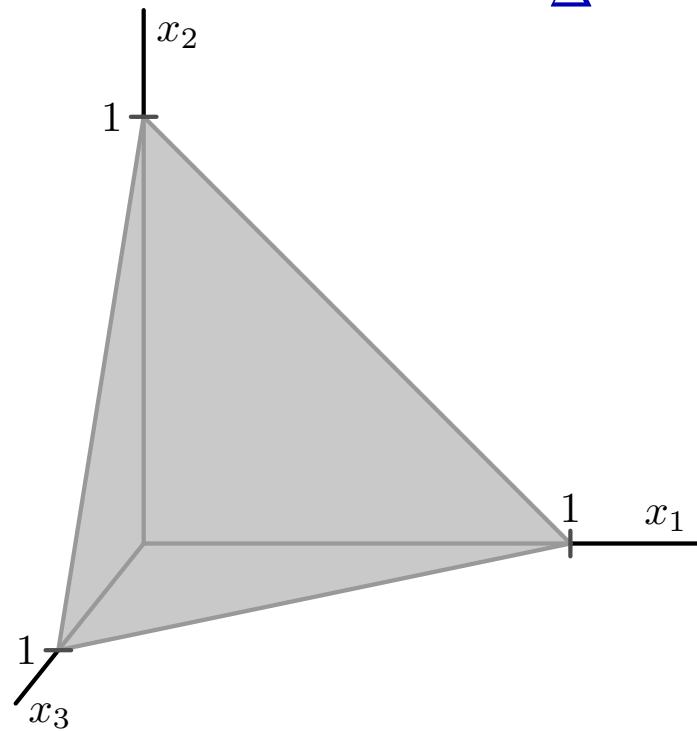
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The Standard Simplex

The **standard simplex** $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta = \{\mathbf{x} \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1\}$$



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$$\begin{aligned} L_\Delta(t) &= \# \{(x_1, x_2, \dots, x_d) \in \mathbb{Z}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq t\} \\ &= \# \{(x_1, x_2, \dots, x_d, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = t\} \\ &= \binom{d+t}{d} \end{aligned}$$

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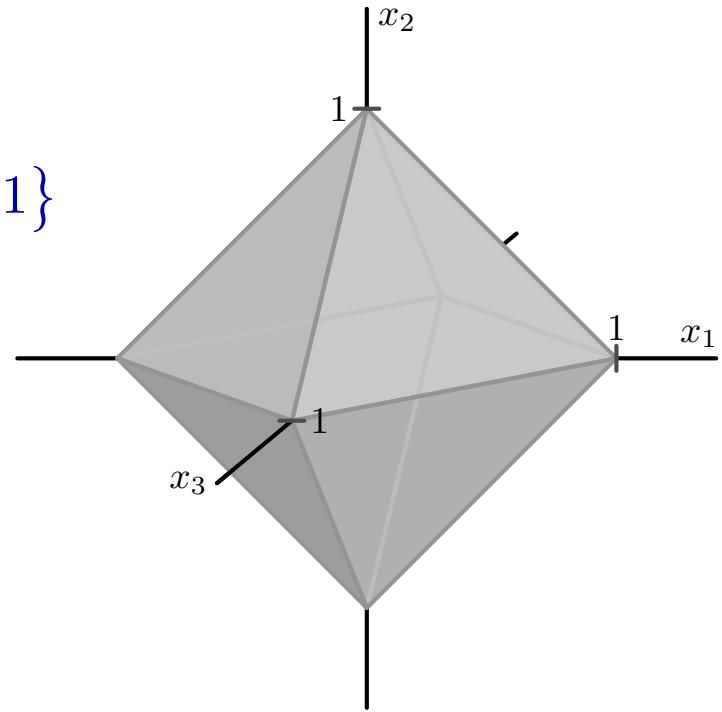
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$$L_{\Delta^\circ}(t) = \binom{t-1}{d}$$

The Cross-Polytope

The cross-polytope $\diamond \in \mathbb{R}^d$ is

$$\diamond = \{\mathbf{x} \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}$$



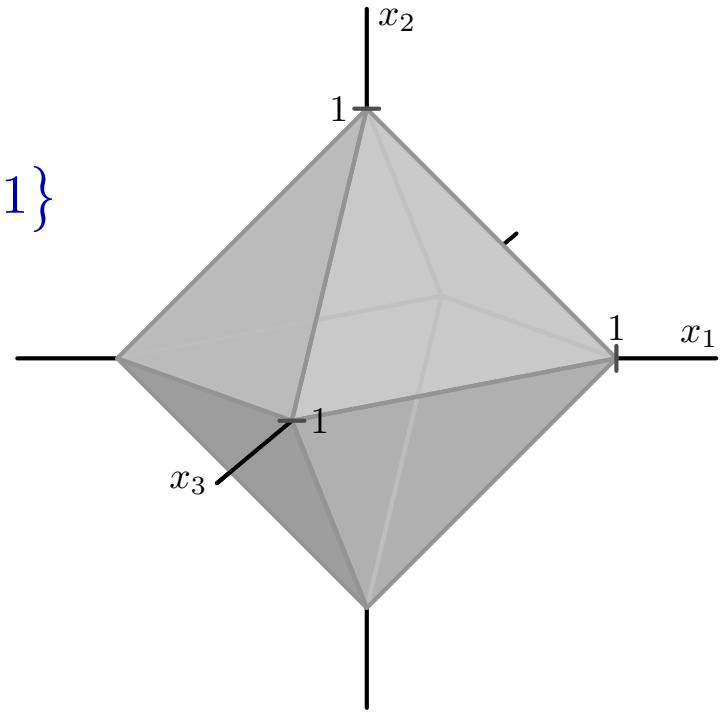
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Let's compute $L_\diamond(t)$ for $d = 3 \dots$

- ▶ Triangulation
- ▶ Disjoint triangulation
- ▶ Interpolation
- ▶ Generating function



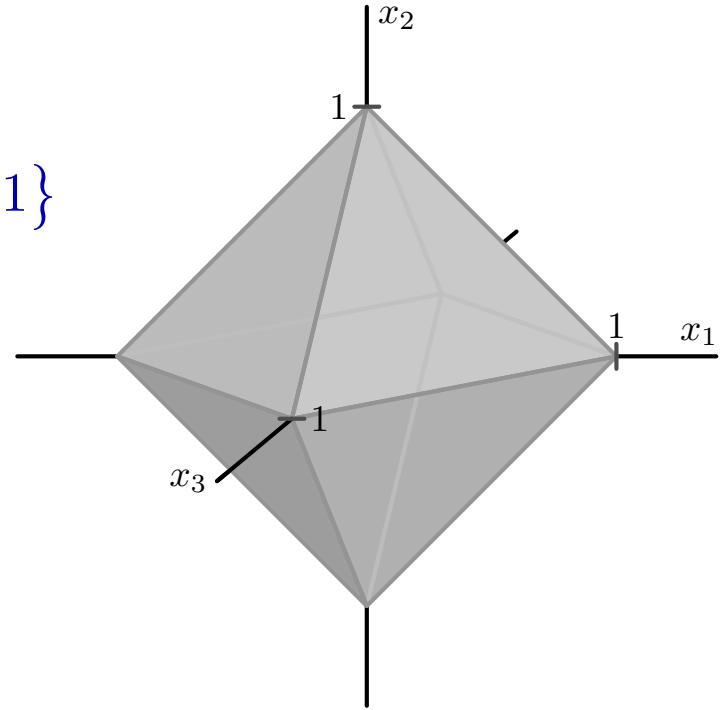
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- ▶ Triangulation



Dissect \diamond into 8 (standard) tetrahedra and use inclusion-exclusion to compute $L_{\diamond}(t)$

The Cross-Polytope

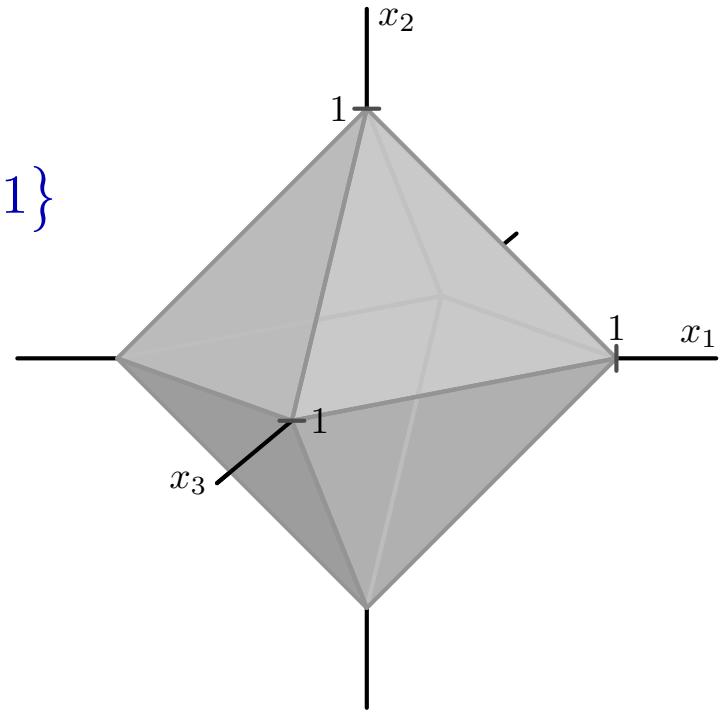
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Let's compute $L_\diamond(t)$ for $d = 3 \dots$

- ▶ Disjoint triangulation

Dissect \diamond into 8 half-open tetrahedra



The Cross-Polytope

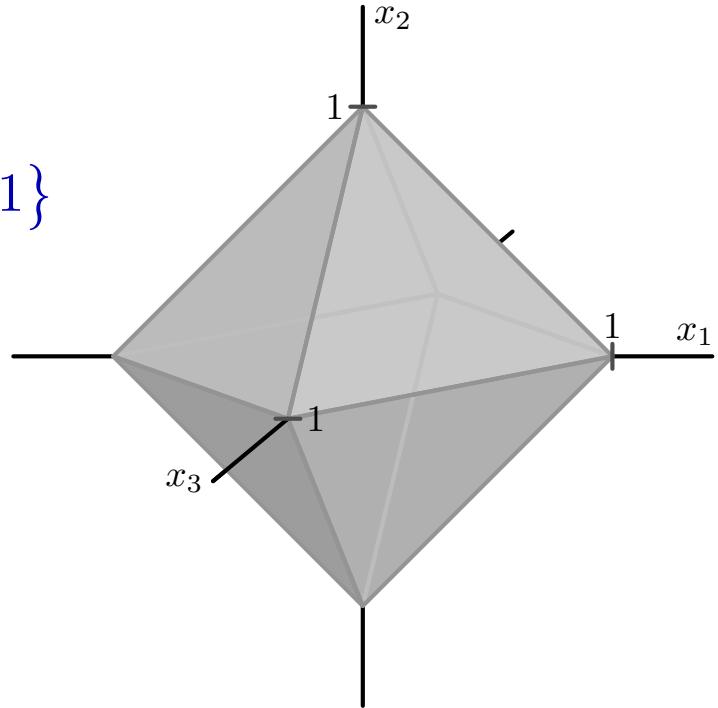
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Let's compute $L_\diamond(t)$ for $d = 3 \dots$

- ▶ Interpolation

```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```



The Cross-Polytope

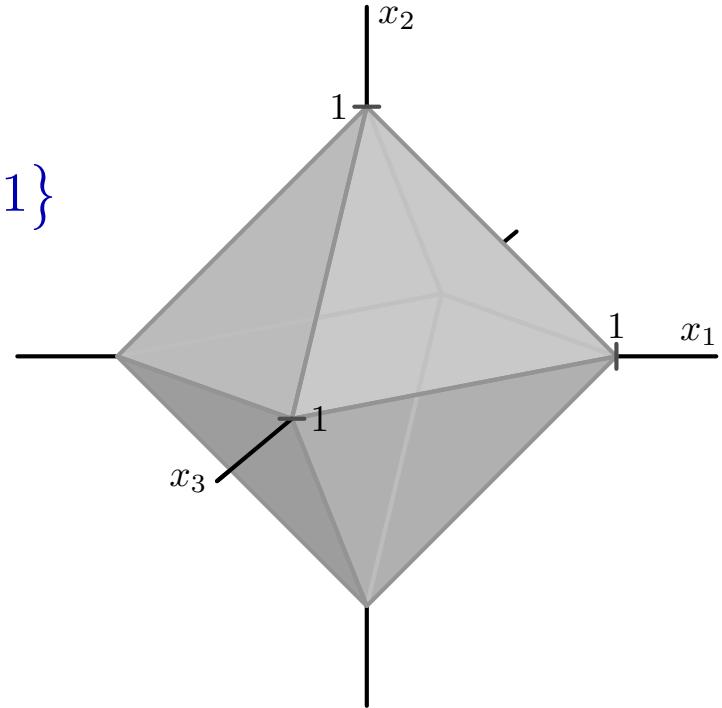
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- ▶ Generating function

$$\text{Ehr}_\diamond(z) := 1 + \sum_{t \geq 1} L_\diamond(t) z^t$$

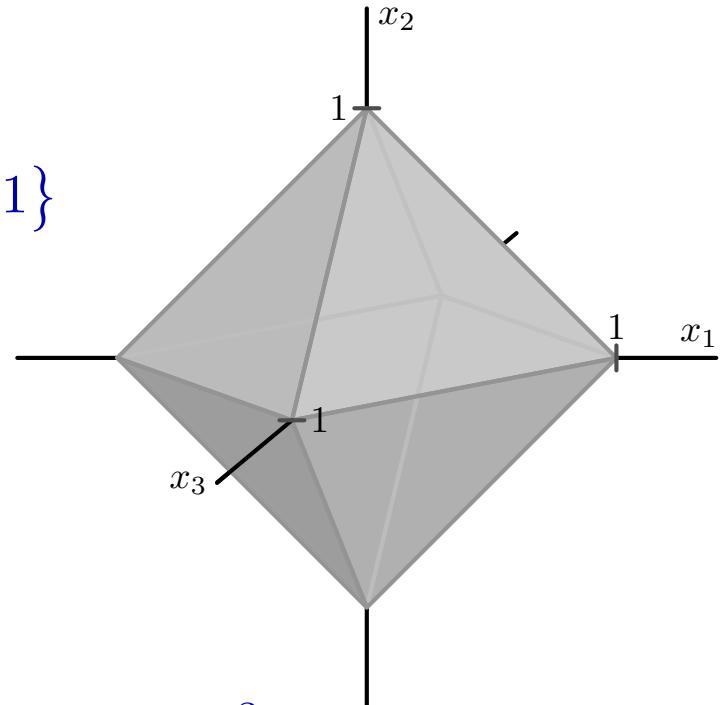


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- ▶ Generating function

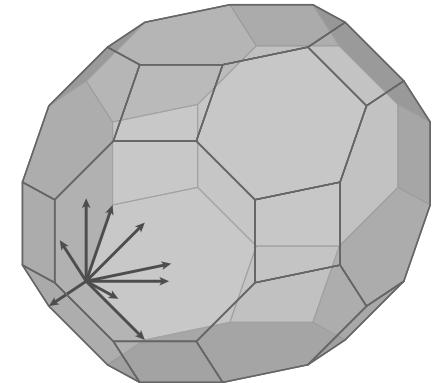
$$\text{Ehr}_\diamond(z) := 1 + \sum_{t \geq 1} L_\diamond(t) z^t = \frac{(1+z)^3}{(1-z)^4}$$

$$\text{Exercise: } \text{Ehr}_{\text{BiPyr}(\mathcal{P})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{P}}(z)$$

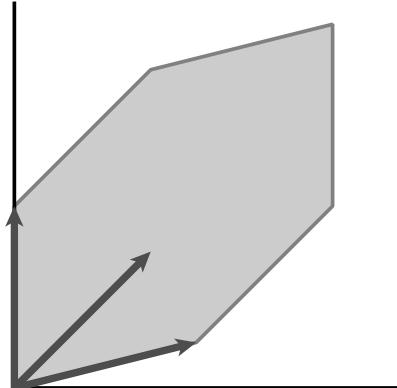
... for unit cubes \longrightarrow Eulerian polynomials

Zonotopes

Line segment $[\mathbf{a}, \mathbf{b}] := \{(1 - \lambda) \mathbf{a} + \lambda \mathbf{b} : 0 \leq \lambda \leq 1\}$

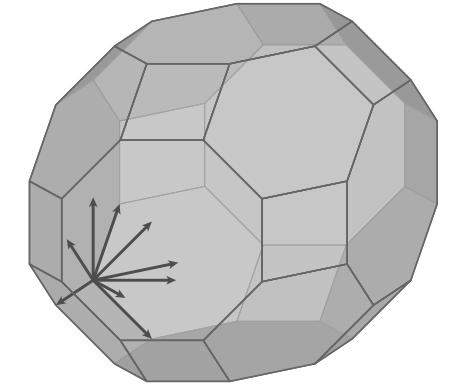


Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2 := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in \mathcal{K}_1, \mathbf{q} \in \mathcal{K}_2\}$



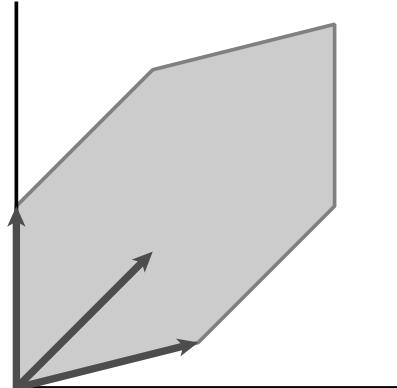
Zonotope $\mathcal{Z} := [\mathbf{a}_1, \mathbf{b}_1] + [\mathbf{a}_2, \mathbf{b}_2] + \cdots + [\mathbf{a}_m, \mathbf{b}_m]$

Zonotopes



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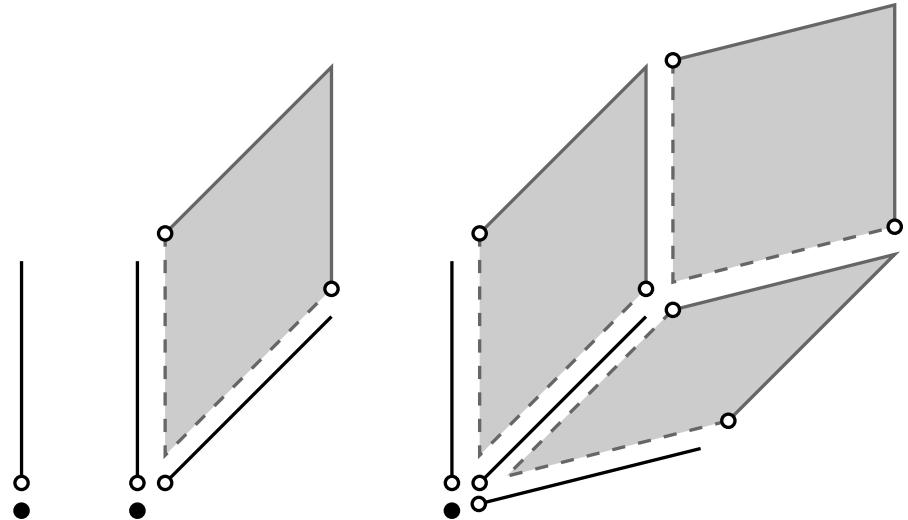


Zonotope $\mathcal{Z} := [\mathbf{a}_1, \mathbf{b}_1] + [\mathbf{a}_2, \mathbf{b}_2] + \cdots + [\mathbf{a}_m, \mathbf{b}_m]$

Every zonotope admits a **tiling** into parallelepipeds

\mathcal{P} — half-open d -parallelepiped

$$\longrightarrow L_{\mathcal{P}}(t) = \text{vol}(\mathcal{P}) t^d$$



Recap Day I

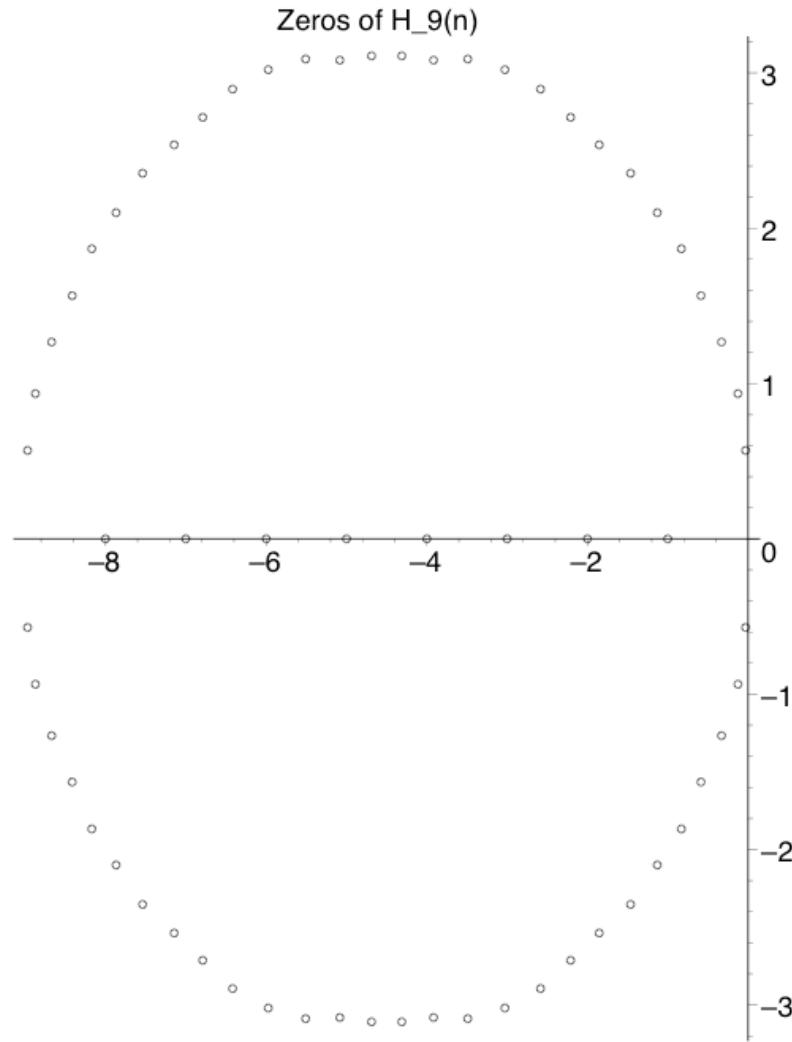
- ▶ Volume computations → don't agonize, discretize
- ▶ Integer-point counting in dilated polytopes → polynomials
- ▶ Interpolation
- ▶ Generating functions
- ▶ Dissections: triangulations, tilings
- ▶ Tomorrow: enough practice, how does this work in theory?



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Dr. Beckcycle

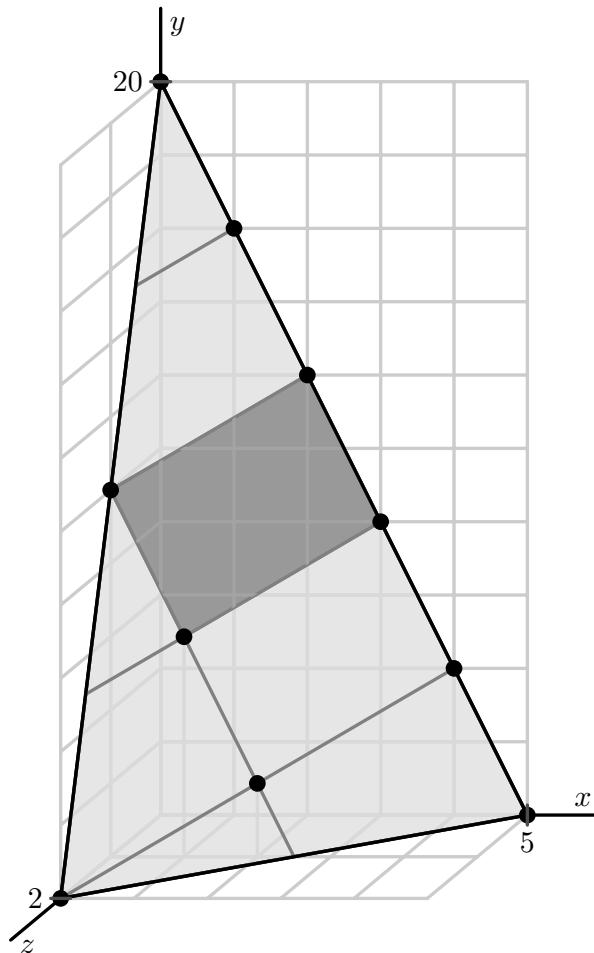
Birkhoff–von Neumann Revisited



For more about roots of
(Ehrhart) polynomials,
see Braun (2008) and
Pfeifle (2010).

Ehrhart Polynomials

Day II: Generating Functions & Complexity



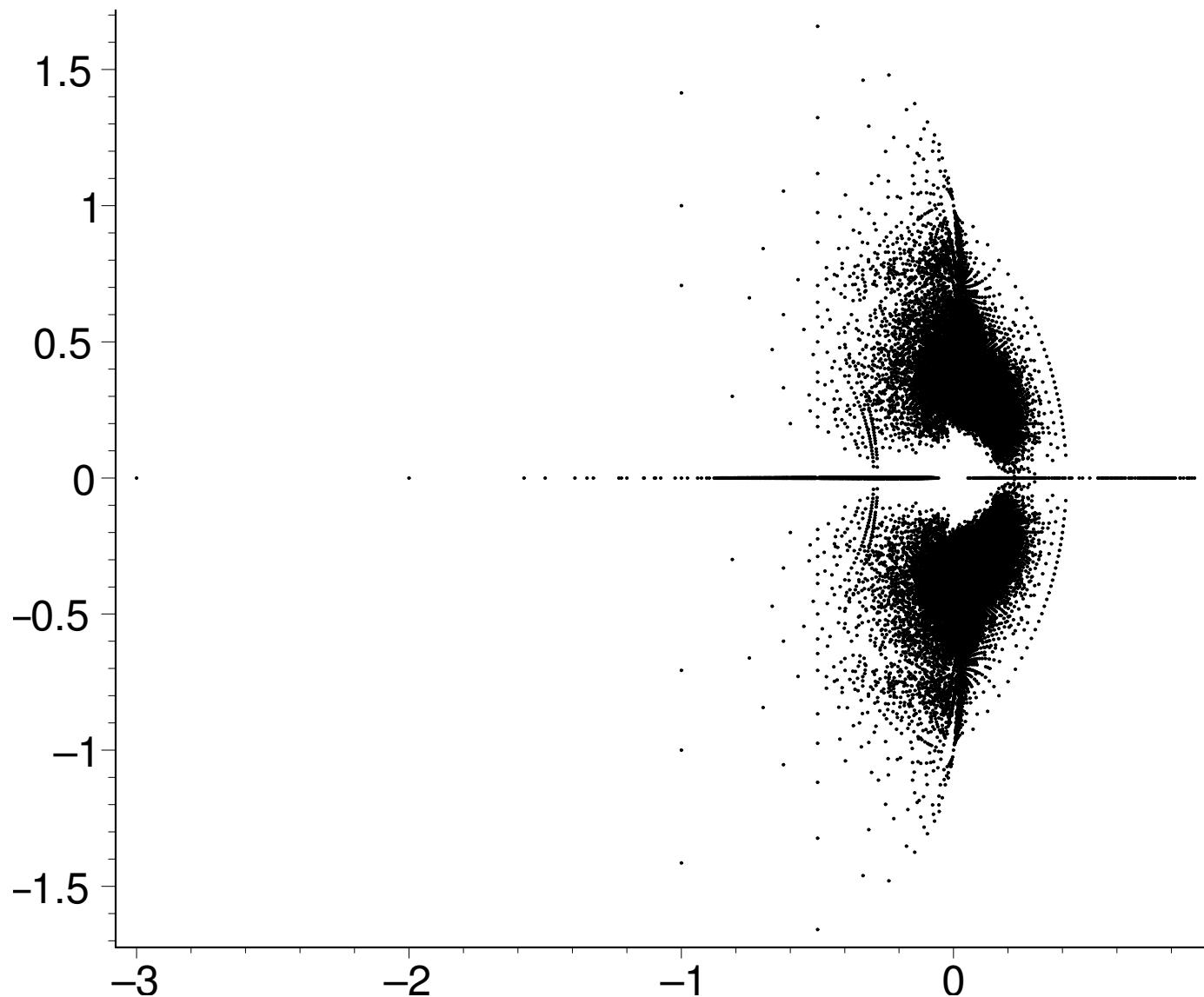
Matthias Beck

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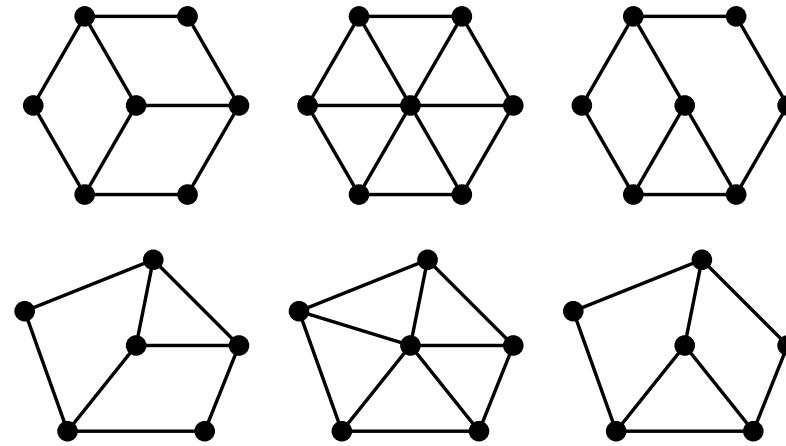
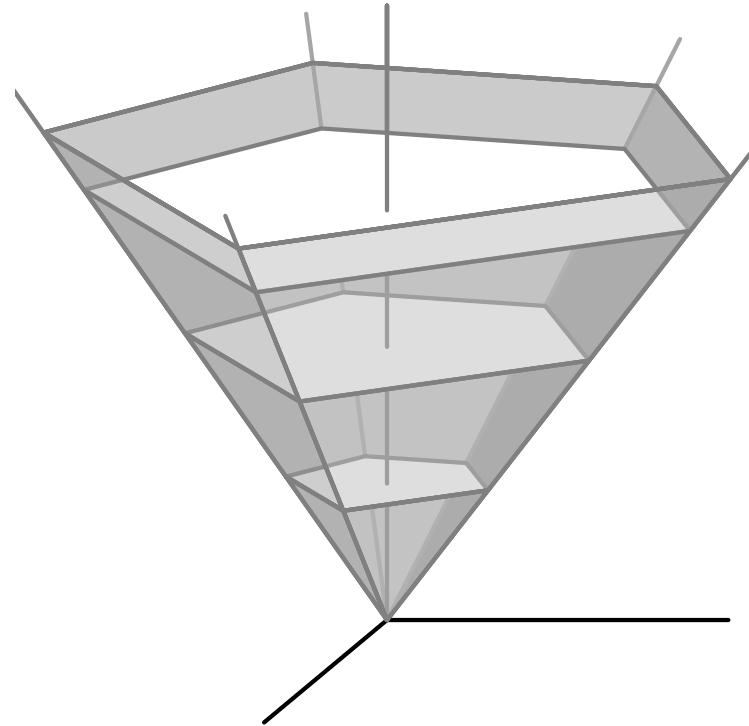
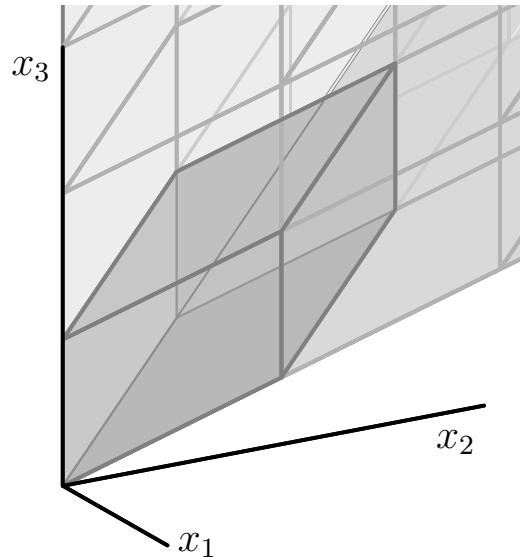
VIII Encuentro Colombiano
De Combinatoria

Any questions about yesterday?



Today's Menu: Theory and Complexity

- ▶ Partition function magic
- ▶ Lots of generating functions
- ▶ Rational cones
- ▶ Triangulations
- ▶ Ehrhart theory



Warm-Up: Partition Generating Functions

A **partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of an integer $k \geq 0$ satisfies

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad \text{and} \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Goal Compute $\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n}$ over your favorite partition family

Example $P_{\leq 3}$ — family of partitions into at most 3 parts

$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

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$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1+\lambda_2+\lambda_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

Idea $P_{\leq 3} = \{\lambda \in \mathbb{Z}^3 : 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \mathcal{K} \cap \mathbb{Z}^3$

$$\mathcal{K} = \{x \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} \leftarrow \text{polyhedral cone} \heartsuit$$

Warm-Up: Partition Generating Functions

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

Warm-Up: Partition Generating Functions

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Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

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$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\mathcal{K}}(q, q, q) = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)}$$

Variations on a Theme

P_3 — family of partitions into exactly 3 parts

$$P_3 = \{\lambda \in \mathbb{Z}^3 : 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \tilde{\mathcal{K}} \cap \mathbb{Z}^3$$

$$\tilde{\mathcal{K}} = \{x \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{> 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Variations on a Theme

P_3 — family of partitions into exactly 3 parts

$$P_3 = \{\lambda \in \mathbb{Z}^3 : 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \tilde{\mathcal{K}} \cap \mathbb{Z}^3$$

$$\tilde{\mathcal{K}} = \{x \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{> 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \sigma_{\tilde{\mathcal{K}}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \tilde{\mathcal{K}} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{z_1 z_2 z_3}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

$$\sum_{\lambda \in P_3} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\tilde{\mathcal{K}}}(q, q, q) = \frac{q^3}{(1 - q)(1 - q^2)(1 - q^3)}$$

Integer-point Complexity of a Simplicial Cone

What if \mathcal{K} is (still simplicial and rational but) not unimodular?

Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geq 0} \mathbf{w}_1 + \mathbb{R}_{\geq 0} \mathbf{w}_2 + \mathbb{R}_{\geq 0} \mathbf{w}_3$$

Integer-point Complexity of a Simplicial Cone

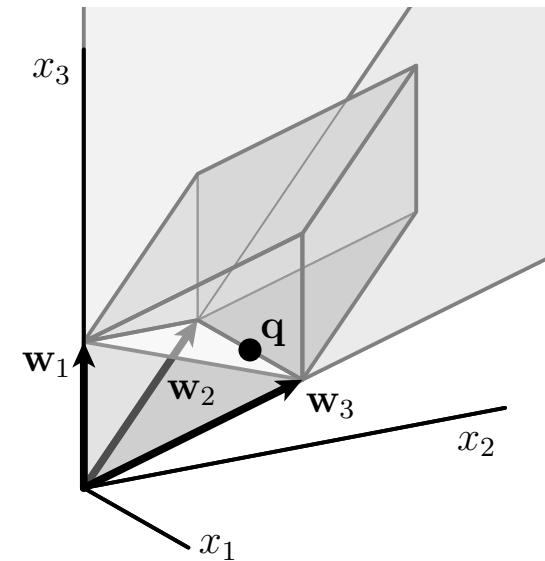
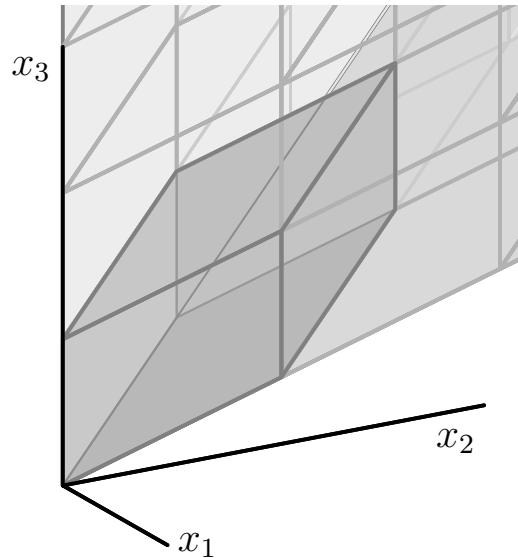
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Idea Tile \mathcal{K} with the half-open parallelepiped

$$\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$$



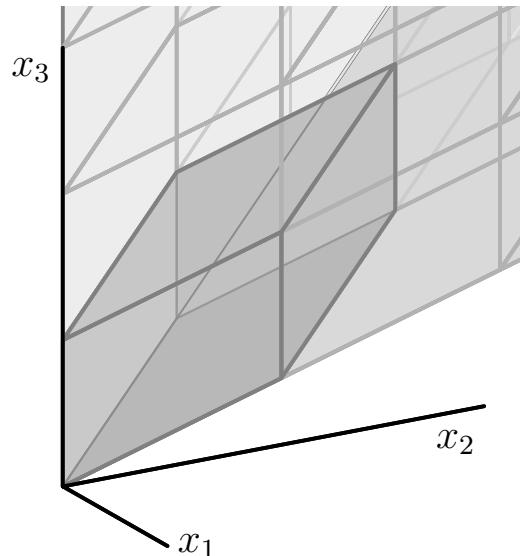
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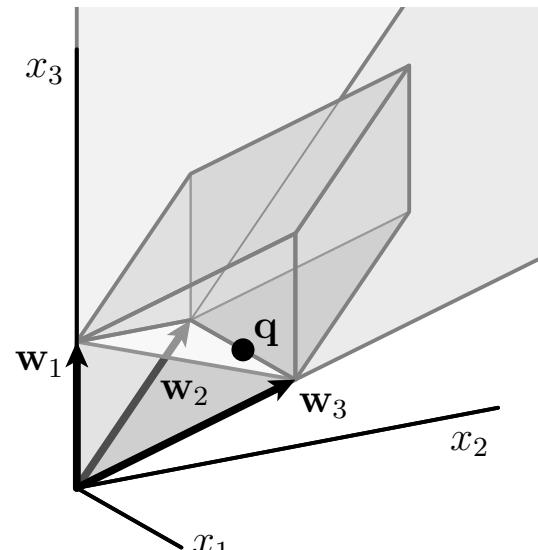
Idea Tile \mathcal{K} with the half-open parallelepiped
 $\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$



$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

$$\text{where } \mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$$



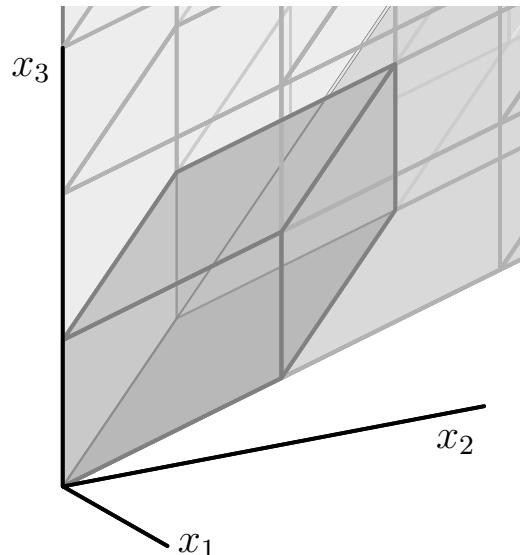
Integer-point Complexity of a Simplicial Cone

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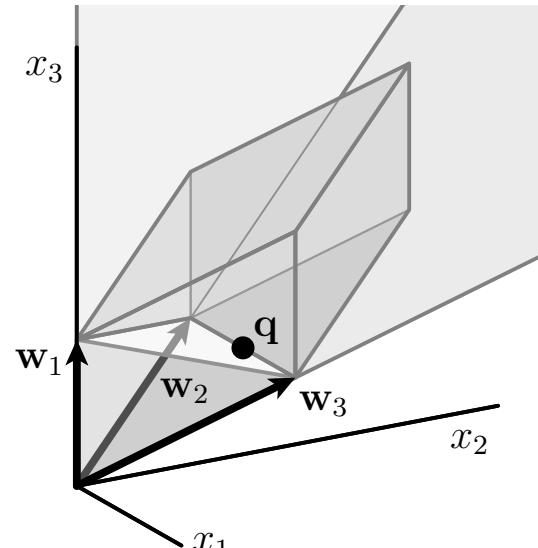
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$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$



Complexity: $\sigma_{\Pi}(z_1, z_2, z_3)$ has D terms

Homogenizing Polytopes

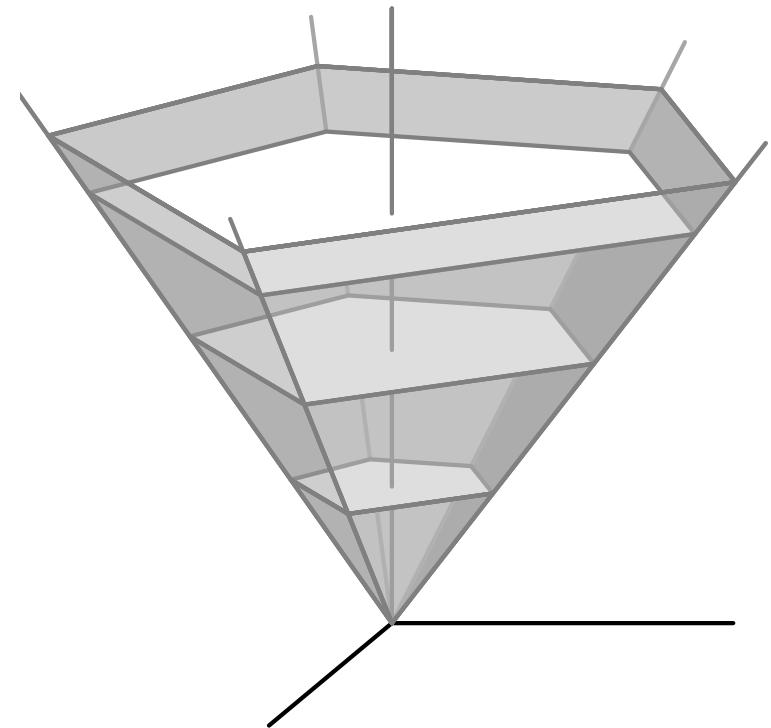
Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

$$\text{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$

$$\text{cone}(\mathcal{P}) \cap \{x \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

contains a copy of $t\mathcal{P}$



Homogenizing Polytopes

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

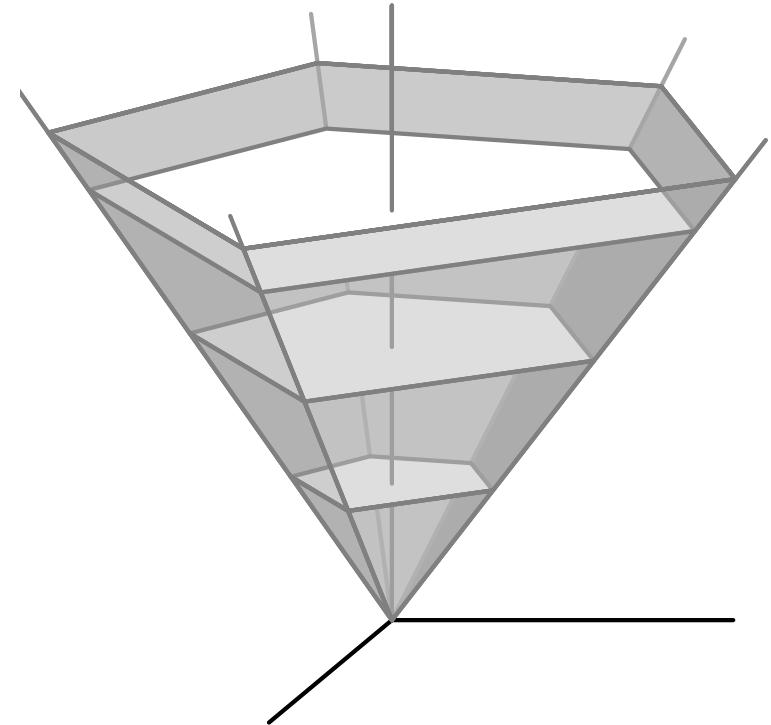
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contains a copy of $t\mathcal{P}$ \longrightarrow

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \sigma_{\text{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$$



Homogenizing Polytopes

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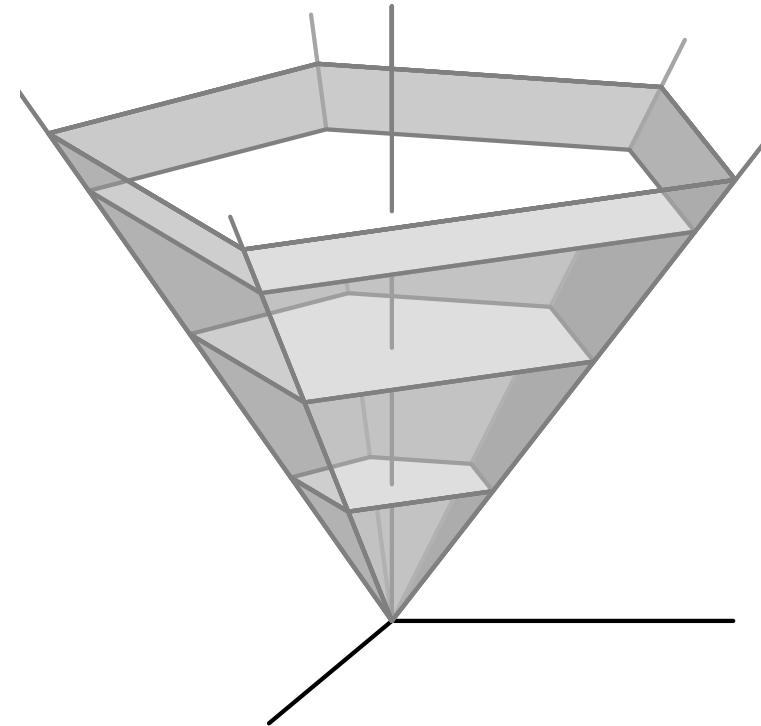
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If \mathcal{P} is a simplex,

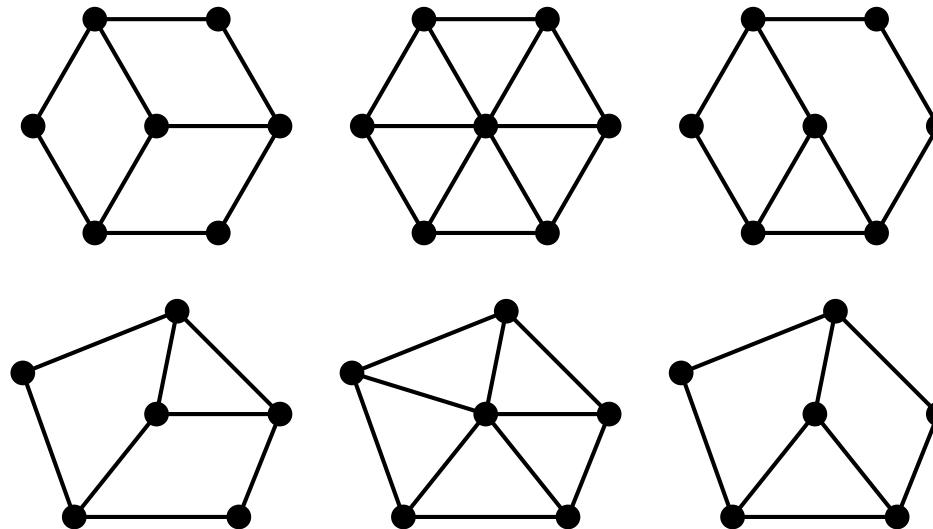
$$\sigma_{\text{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \longrightarrow \text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{d+1}}$$



Trials & Triangulations

Subdivision of a polyhedron \mathcal{P} — finite collection S of polyhedra such that

- ▶ if \mathcal{F} is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- ▶ if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- ▶ $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



If each \mathcal{F} is a simplex \longrightarrow triangulation of a polytope

Ehrhart Polynomials



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

Equivalently, $\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$ is rational:

$$\text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the **h^* -polynomial** $h_{\mathcal{P}}^*(z)$ satisfies $h_{\mathcal{P}}^*(0) = 1$ and $h_{\mathcal{P}}^*(1) = (\dim \mathcal{P})! \text{vol}(\mathcal{P})$.

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Computational bottlenecks:

- ▶ triangulation
- ▶ determinants of resulting simplicial cones

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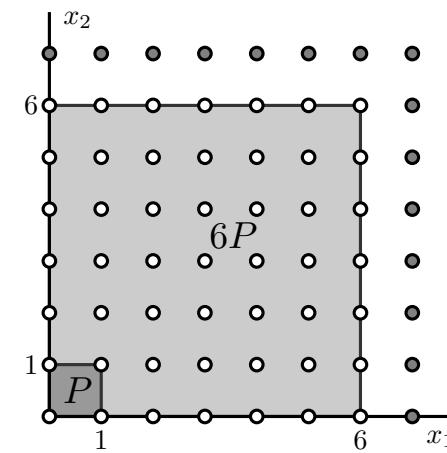
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We saw instances yesterday: $\mathcal{P} = [0, 1]^d$

$$\rightarrow L_{\mathcal{P}}(t) = (t+1)^d$$

$h_{\mathcal{P}}^*(z)$ — Eulerian polynomial



Ehrhart Polynomials

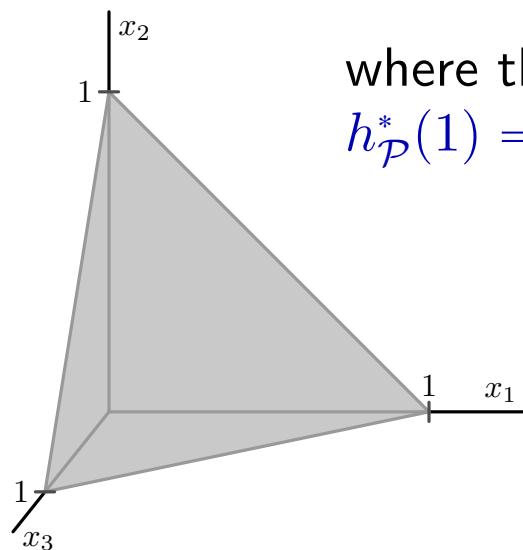


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$$\Delta = \{x \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1\}$$

$$L_{\Delta}(t) = \binom{d+t}{d} \quad h_{\mathcal{P}}^*(z) = 1$$

Ehrhart Polynomials



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

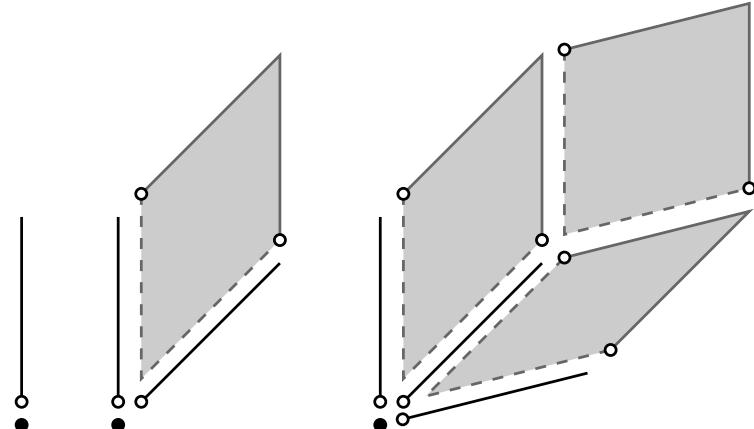
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\mathcal{P} — half-open d -parallelepiped

$$\rightarrow L_{\mathcal{P}}(t) = \text{vol}(\mathcal{P}) t^d$$



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Seeming dichotomy: $\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

Ehrhart Polynomials



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

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Equivalent descriptions of an Ehrhart polynomial:

- ▶ $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$
- ▶ via roots of $L_{\mathcal{P}}(t)$
- ▶ $\text{Ehr}_{\mathcal{P}}(z) \quad \longrightarrow \quad L_{\mathcal{P}}(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$

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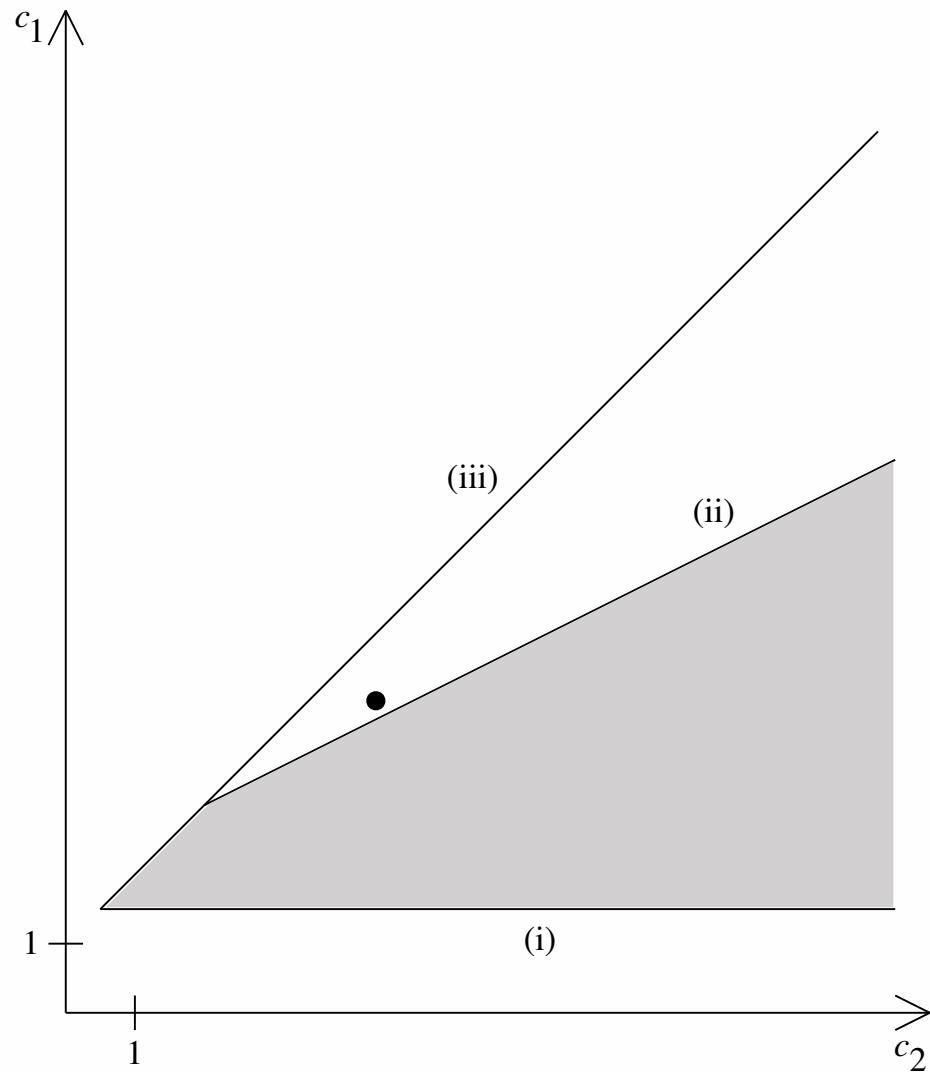
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Open Problem Classify Ehrhart polynomials.

Ehrhart Polynomials in Dimension 2



\mathcal{P} — lattice polygon

$$\rightarrow L_{\mathcal{P}}(t) = c_2 t^2 + c_1 t + 1$$

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in \mathbb{Q}^d

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in t :

$$L_{\mathcal{P}}(t) = c_d(t) t^d + c_{d-1}(t) t^{d-1} + \cdots + c_0(t)$$

where $c_0(t), \dots, c_d(t)$ are periodic functions.

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely many points in \mathbb{Q}^d

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where $c_0(t), \dots, c_d(t)$ are periodic functions. Equivalently,

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$$

for some (minimal) $p \in \mathbb{Z}_{>0}$ (the period of $L_{\mathcal{P}}(t)$).

Open Problem Study periods of Ehrhart quasipolynomials.

Partitions Revisited

Definition n -gon partitions

$$T_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \dots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \dots + \lambda_{n-1} > \lambda_n\}$$

Theorem (Andrews–Paule–Riese 2001)

$$\begin{aligned} \sum_{\lambda \in T_n} q^{\lambda_1 + \dots + \lambda_n} &= \frac{q}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &- \frac{q^{2n-2}}{(1-q)(1-q^2)(1-q^4)\cdots(1-q^{2n-2})} \end{aligned}$$

Geometric Philosophy The following cone is arithmetically nicer:

$$\{x \in \mathbb{R}^n : x_n \geq \dots \geq x_1 > 0 \text{ and } x_1 + \dots + x_{n-1} \leq x_n\}$$

Partitions Revisited

Definition Lecture-hall partitions

$$\text{LH}_n := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_n}{n} \right\}$$

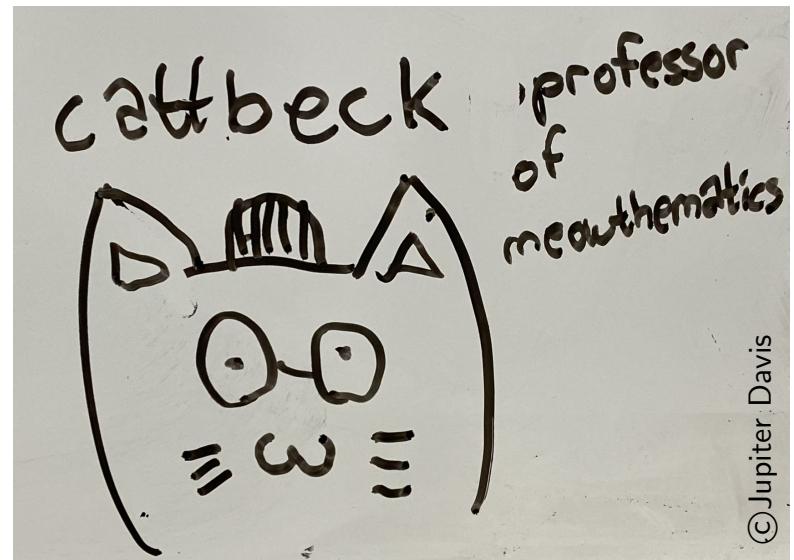
Lecture-Hall Theorem (Bousquet-Mélou–Eriksson 1997)

$$\sum_{\lambda \in \text{LH}_n} q^{\lambda_1 + \dots + \lambda_n} = \frac{1}{(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$

Open Problem Explain this geometrically. (Caveat: the lecture-hall cone has determinant $(n-1)!$).

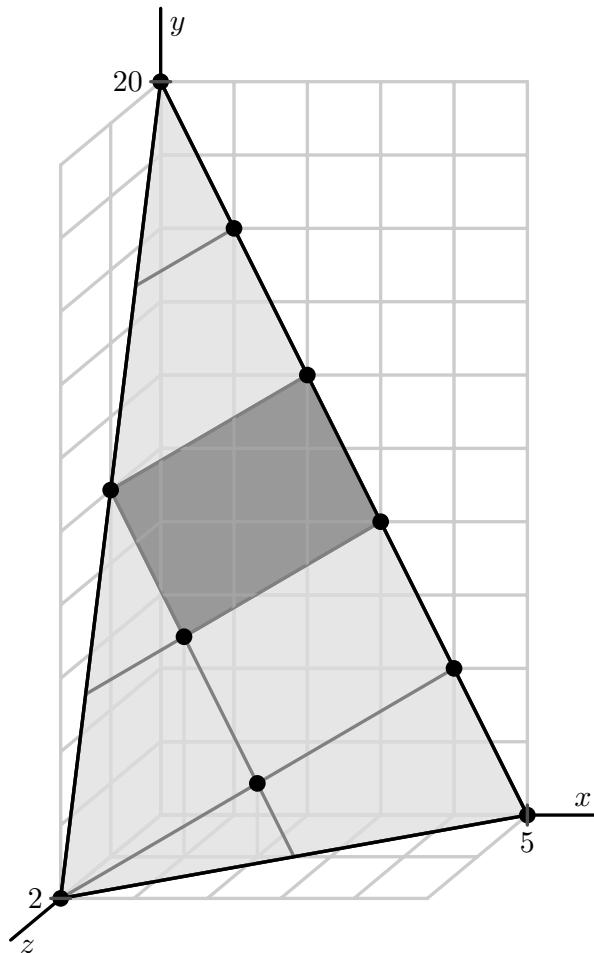
Recap Day II

- ▶ Generating functions son cheveres
- ▶ Integer-point transforms of rational polyhedra \longrightarrow rational functions
- ▶ Arithmetic complexity of a simplicial cone: determinant of its generators
- ▶ Homogenize polytopes
- ▶ Triangulations
- ▶ Polynomial data
- ▶ Thursday: positivity, reciprocity & friends



Ehrhart Polynomials

Day III: Positivity, Reciprocity & Friends



Matthias Beck

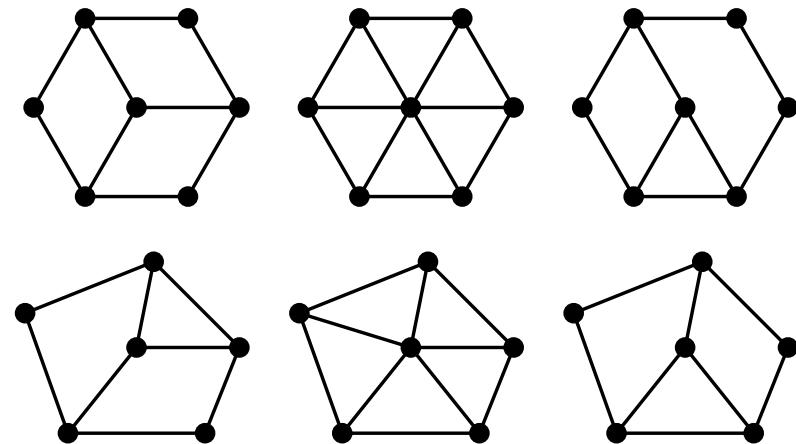
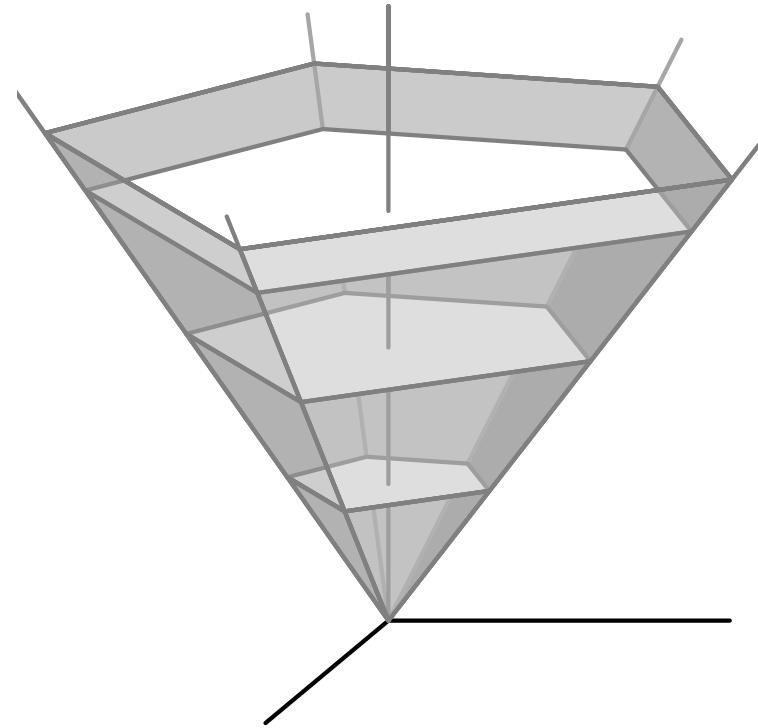
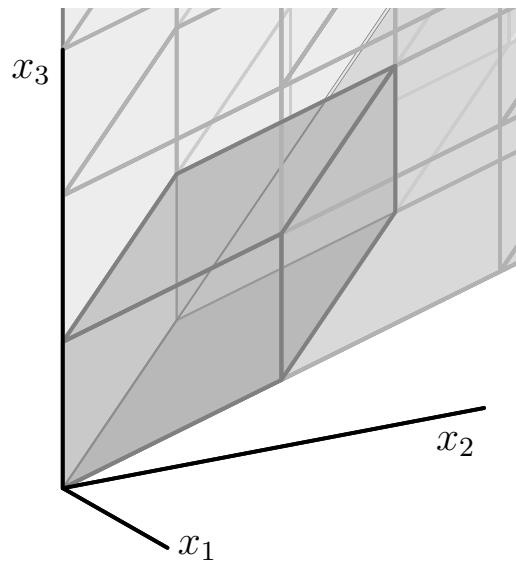
San Francisco State University

<https://matthbeck.github.io/>

VIII Encuentro Colombiano
De Combinatoria

Any questions about yesterday?

$$\sum_{\lambda \in P_3} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\tilde{\mathcal{K}}}(q, q, q)$$
$$= \frac{q^3}{(1-q)(1-q^2)(1-q^3)}$$



Today's Menu: Positivity, Reciprocity & Friends

- ▶ Graph coloring
- ▶ Half-open triangulations
- ▶ Ehrhart positivity
- ▶ Ehrhart–Macdonald reciprocity

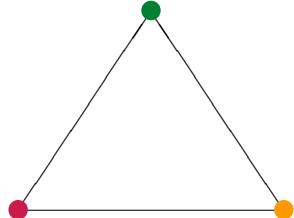
Warm-Up: Chromatic Polynomials of Graphs

$\Gamma = (V, E)$ — graph (without loops)

Proper k -coloring of Γ — $x \in \{1, 2, \dots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_\Gamma(k) := \#$ (proper k -colorings of Γ)

Example:



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

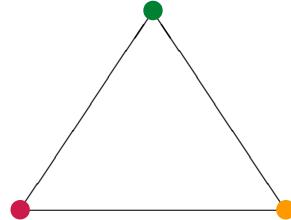
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$\chi_\Gamma(k) := \# (\text{proper } k\text{-colorings of } \Gamma)$ ← polynomial ❤

Example:



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

$$|\chi_{K_3}(-1)| = 6 \dots$$

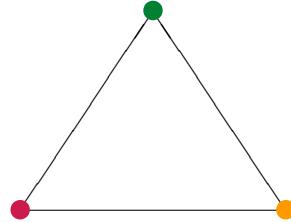
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Proper k -coloring of Γ — $x \in \{1, 2, \dots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

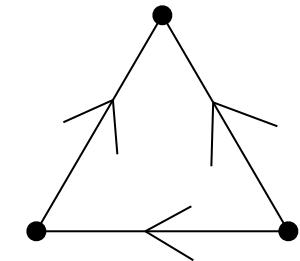
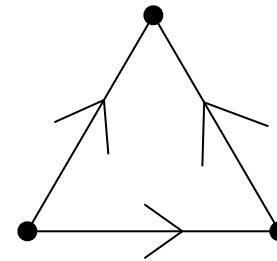
$\chi_\Gamma(k) := \#$ (proper k -colorings of Γ) ← polynomial ❤

Example:



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

$|\chi_{K_3}(-1)| = 6$ counts the number
of acyclic orientations of K_3



Warm-Up: Chromatic Polynomials of Graphs

$\Gamma = (V, E)$ — graph (without loops)

Proper k -coloring of Γ — $x \in \{1, 2, \dots, k\}^V$ such that $x_i \neq x_j$ if $ij \in E$

$\chi_\Gamma(k) := \#$ (proper k -colorings of Γ) ← polynomial 

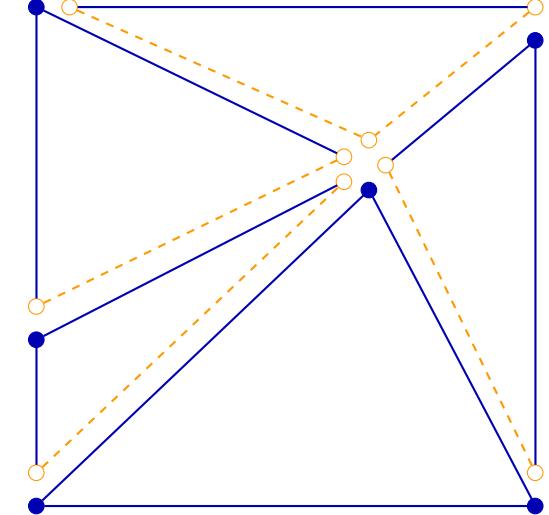
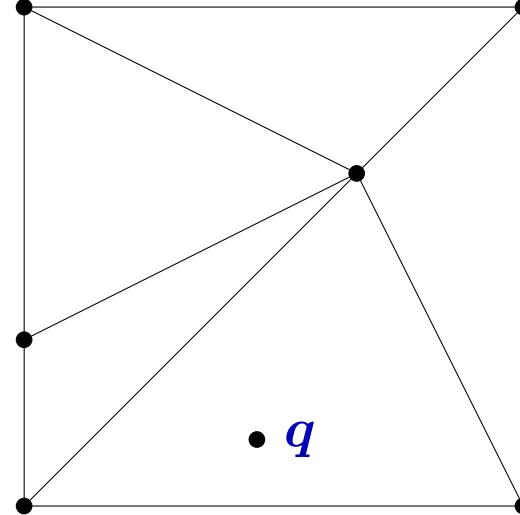
Theorem (Stanley 1973) $(-1)^{|V|}\chi_\Gamma(-k)$ equals the number of pairs (α, x) consisting of an acyclic orientation α of Γ and a compatible k -coloring x . In particular, $(-1)^{|V|}\chi_\Gamma(-1)$ equals the number of acyclic orientations of Γ

(An orientation α of Γ and a k -coloring x are compatible if $x_j \geq x_i$ whenever there is an edge oriented from i to j . An orientation is acyclic if it has no directed cycles.)

Half-open Triangulations

Triangulation of a polytope \mathcal{P} — finite collection S of simplices such that

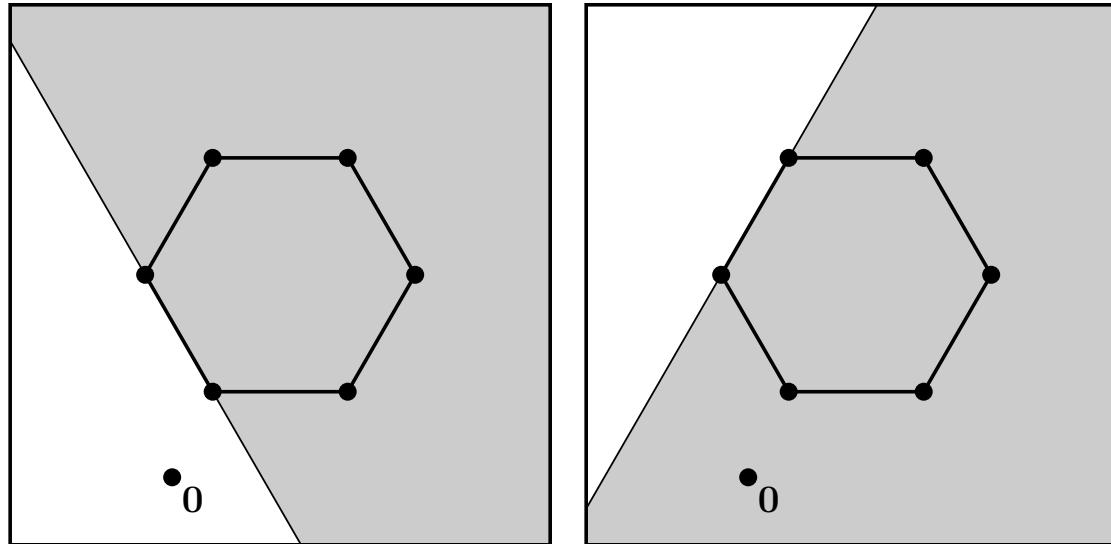
- ▶ if \mathcal{F} is a face of $\mathcal{G} \in S$ then $\mathcal{F} \in S$
- ▶ if $\mathcal{F}, \mathcal{G} \in S$ then $\mathcal{F} \cap \mathcal{G}$ is a face of both
- ▶ $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



Tangent Cones & Visibility

\mathcal{F} facet of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ with defining halfspace H

\mathcal{F} is **visible** from $q \in \mathbb{R}^d$ if $q \notin H$



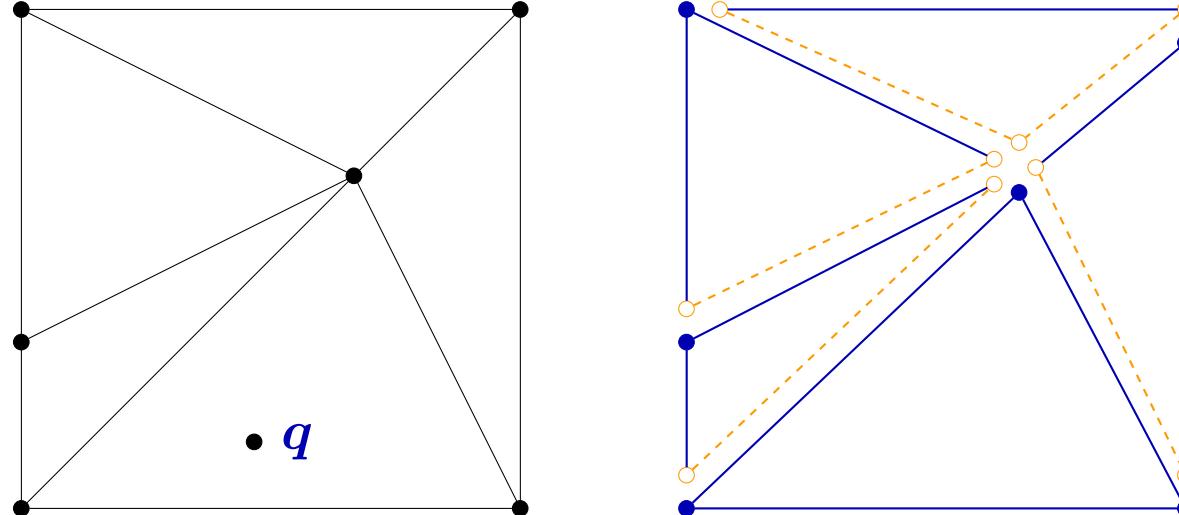
Equivalent lingo: $q \in \mathbb{R}^d$ is **beyond** \mathcal{F} (and **beneath** otherwise)

Half-open Triangulations

Define $\mathbb{H}_q(\mathcal{Q})$ to be \mathcal{Q} minus facets that are visible from q

Exercise $\mathcal{P} \subset \mathbb{R}^d$ — full-dimensional polyhedron with dissection $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_m$, $q \in \mathbb{R}^d$ generic relative to each \mathcal{P}_j \longrightarrow

$$\mathbb{H}_q \mathcal{P} = \mathbb{H}_q \mathcal{P}_1 \uplus \mathbb{H}_q \mathcal{P}_2 \uplus \dots \uplus \mathbb{H}_q \mathcal{P}_m$$



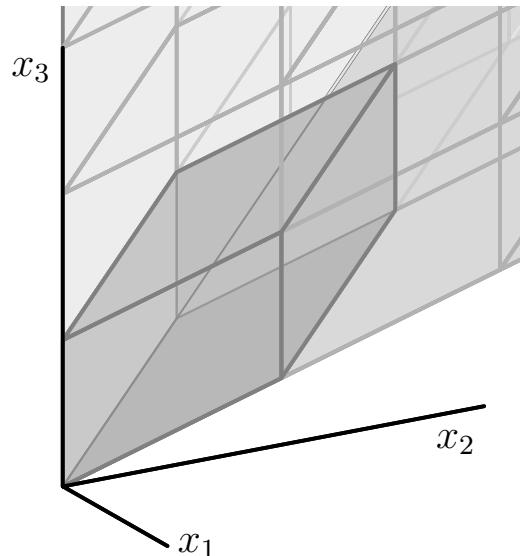
Recall: Integer-point Complexity of a Simplicial Cone

What if \mathcal{K} is (still simplicial and rational but) not unimodular?

Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geq 0} \mathbf{w}_1 + \mathbb{R}_{\geq 0} \mathbf{w}_2 + \mathbb{R}_{\geq 0} \mathbf{w}_3$$

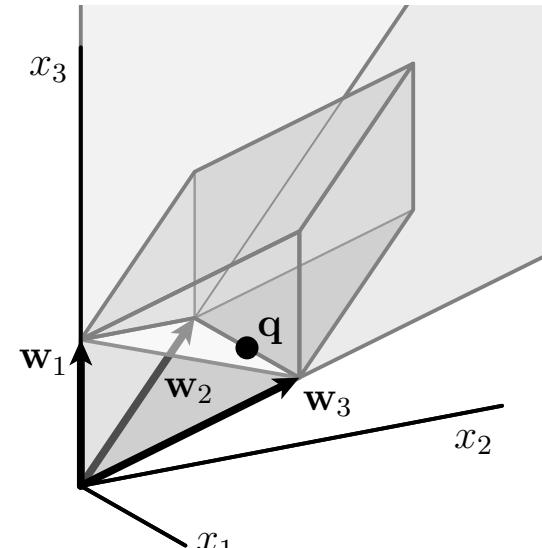
Idea Tile \mathcal{K} with the half-open parallelepiped
 $\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$



$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

$$\text{where } \mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$$



Recall: Homogenizing Polytopes

Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ let

$$\text{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$

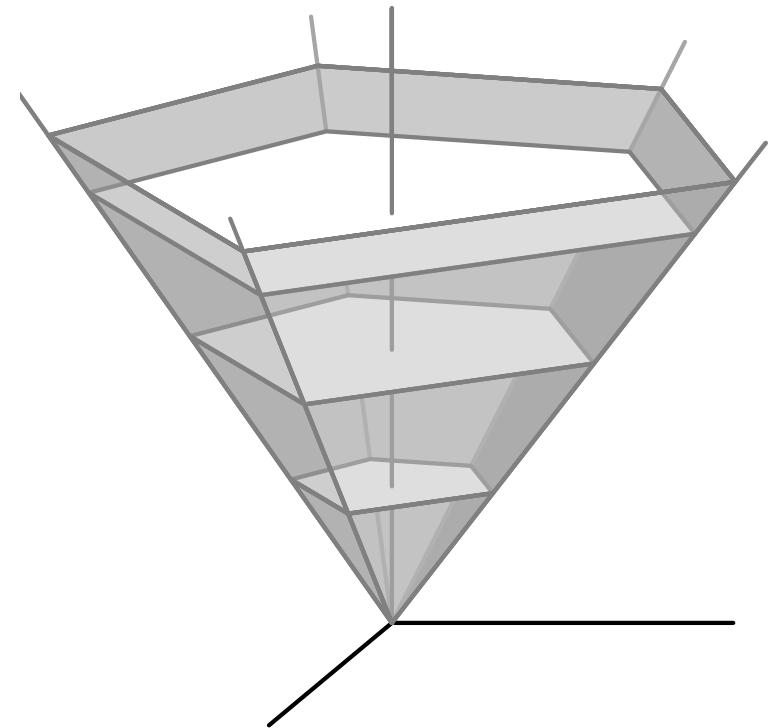
$$\text{cone}(\mathcal{P}) \cap \{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

contains a copy of $t\mathcal{P}$ \longrightarrow

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \sigma_{\text{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$$

If \mathcal{P} is a simplex,

$$\sigma_{\text{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \quad \longrightarrow \quad \text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{d+1}}$$



Ehrhart Positivity



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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Open Problem Prove that the h^* -polynomial of

- ▶ hypersimplices
- ▶ polytopes admitting a unimodular triangulation
- ▶ polytope with the integer decomposition property are **unimodal**

- ✓ Gorenstein polytopes with regular unimodular triangulation (Bruns–Römer 2007)
- ✓ Zonotopes (MB–Jochemko–McCullough 2019)

Recall:² Integer-point Complexity of a Simplicial Cone

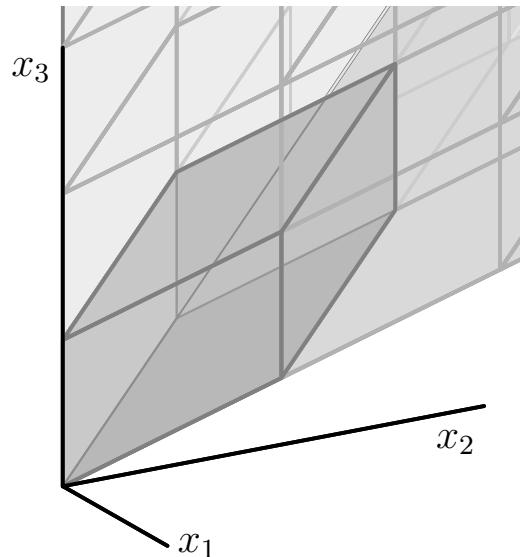
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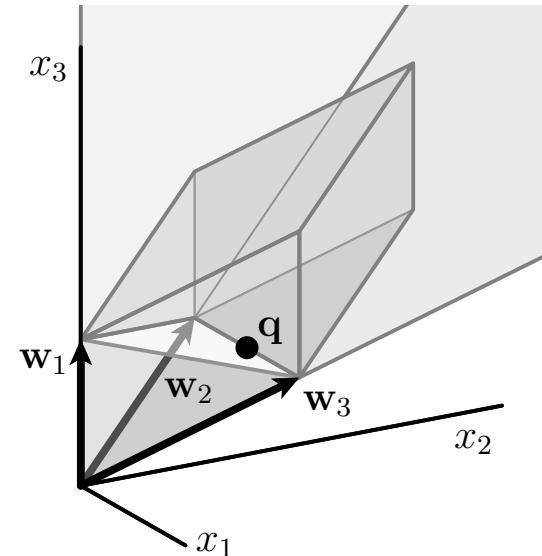
$$\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$$



$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

$$\text{where } \mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$$



Simplicial Cone Reciprocity

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{Z}^d$ linearly independent

$$\hat{\mathcal{K}} := \mathbb{R}_{\geq 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{\geq 0}\mathbf{v}_{m-1} + \mathbb{R}_{> 0}\mathbf{v}_m + \cdots + \mathbb{R}_{> 0}\mathbf{v}_k$$

$$\check{\mathcal{K}} := \mathbb{R}_{> 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{> 0}\mathbf{v}_{m-1} + \mathbb{R}_{\geq 0}\mathbf{v}_m + \cdots + \mathbb{R}_{\geq 0}\mathbf{v}_k$$



$$\sigma_{\hat{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\hat{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{\mathbf{v}_k})} \quad \sigma_{\check{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\check{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{\mathbf{v}_k})}$$

where

$$\hat{\Pi} := [0, 1) \mathbf{v}_1 + \cdots + [0, 1) \mathbf{v}_{m-1} + (0, 1] \mathbf{v}_m + \cdots + (0, 1] \mathbf{v}_k$$

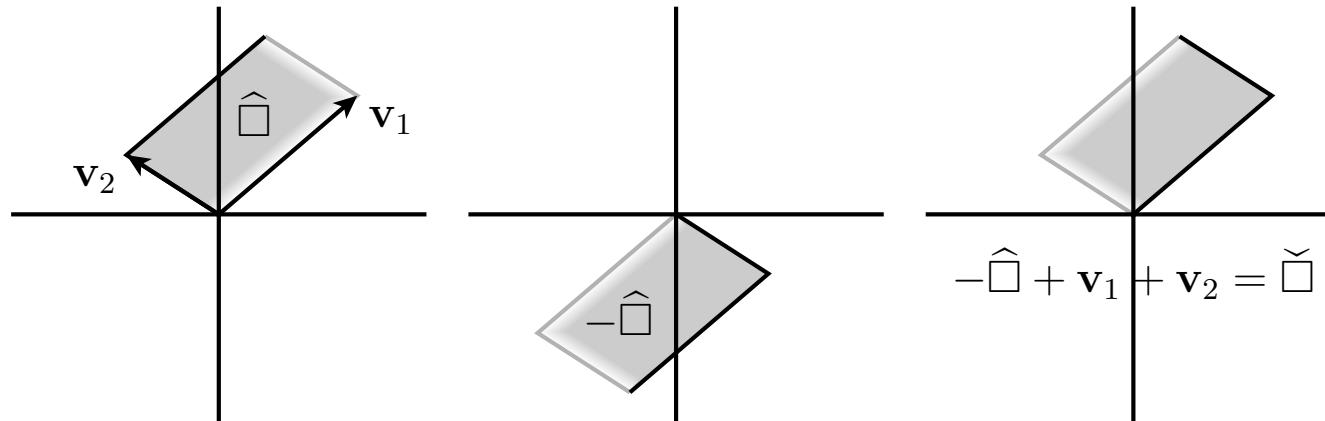
$$\check{\Pi} := (0, 1] \mathbf{v}_1 + \cdots + (0, 1] \mathbf{v}_{m-1} + [0, 1) \mathbf{v}_m + \cdots + [0, 1) \mathbf{v}_k$$

Simplicial Cone Reciprocity

$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{Z}^d$ linearly independent

$$\begin{aligned}\hat{\Pi} &:= [0, 1) \mathbf{v}_1 + \cdots + [0, 1) \mathbf{v}_{m-1} + (0, 1] \mathbf{v}_m + \cdots + (0, 1] \mathbf{v}_k \\ \breve{\Pi} &:= (0, 1] \mathbf{v}_1 + \cdots + (0, 1] \mathbf{v}_{m-1} + [0, 1) \mathbf{v}_m + \cdots + [0, 1) \mathbf{v}_k\end{aligned}$$

Fun Fact $\hat{\Pi} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k - \breve{\Pi}$



$$\longrightarrow \sigma_{\hat{\Pi}}(\mathbf{z}) = \mathbf{z}^{\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_k} \sigma_{\breve{\Pi}}\left(\frac{1}{\mathbf{z}}\right)$$

Stanley Reciprocity

$$\sigma_{\hat{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\hat{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{\mathbf{v}_k})}$$

$$\sigma_{\check{\mathcal{K}}}(\mathbf{z}) = \frac{\sigma_{\check{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{\mathbf{v}_k})}$$

$$\sigma_{\hat{\Pi}}(\mathbf{z}) = \mathbf{z}^{\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k} \sigma_{\check{\Pi}}\left(\frac{1}{\mathbf{z}}\right) \quad \longrightarrow$$

$$\begin{aligned} \sigma_{\check{\mathcal{K}}}\left(\frac{1}{\mathbf{z}}\right) &= \frac{\sigma_{\check{\Pi}}\left(\frac{1}{\mathbf{z}}\right)}{(1 - \mathbf{z}^{-\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{-\mathbf{v}_k})} = \frac{\mathbf{z}^{-\mathbf{v}_1 - \mathbf{v}_2 - \cdots - \mathbf{v}_k} \sigma_{\hat{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{-\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{-\mathbf{v}_k})} \\ &= (-1)^k \frac{\sigma_{\hat{\Pi}}(\mathbf{z})}{(1 - \mathbf{z}^{\mathbf{v}_1}) \cdots (1 - \mathbf{z}^{\mathbf{v}_k})} = (-1)^k \sigma_{\hat{\mathcal{K}}}(\mathbf{z}) \end{aligned}$$

Theorem (Stanley) Let $\mathcal{K} \subset \mathbb{R}^d$ be a full-dimensional pointed rational cone, and let $\mathbf{q} \in \mathbb{R}^d$ be generic relative to \mathcal{K} . Then

$$\sigma_{\mathbb{H}\mathbf{q}\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \sigma_{\mathbb{H}\mathbf{q}\mathcal{K}}(\mathbf{z})$$

Corollary $\sigma_{\mathcal{K}}\left(\frac{1}{\mathbf{z}}\right) = (-1)^d \sigma_{\mathcal{K}^\circ}(\mathbf{z})$

Ehrhart Polynomials



Theorem (Ehrhart 1962) For any lattice polytope \mathcal{P} , $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\text{vol } \mathcal{P}$ and constant term 1.

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h(z)}{(1-z)^{d+1}}$$

$$\longrightarrow L_{\mathcal{P}}(t) = h_0^*(t+d) + h_1^*(t+d-1) + \cdots + h_d^*(t)$$

Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the **interior** lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) = h_d^*(t+d-1) + h_{d-1}^*(t+d-2) + \cdots + h_0^*(t-1)$$

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

Corollary If $h_{d+1-k}^* > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

Positivity Among Ehrhart Polynomials



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Theorem (Betke–McMullen 1985, Stapledon 2009) If $h_d^* > 0$ then

$$h(z) = a(z) + z b(z)$$

where $a(z) = z^d a(\frac{1}{z})$ and $b(z) = z^{d-1} b(\frac{1}{z})$ with nonnegative coefficients.

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Open Problem Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.