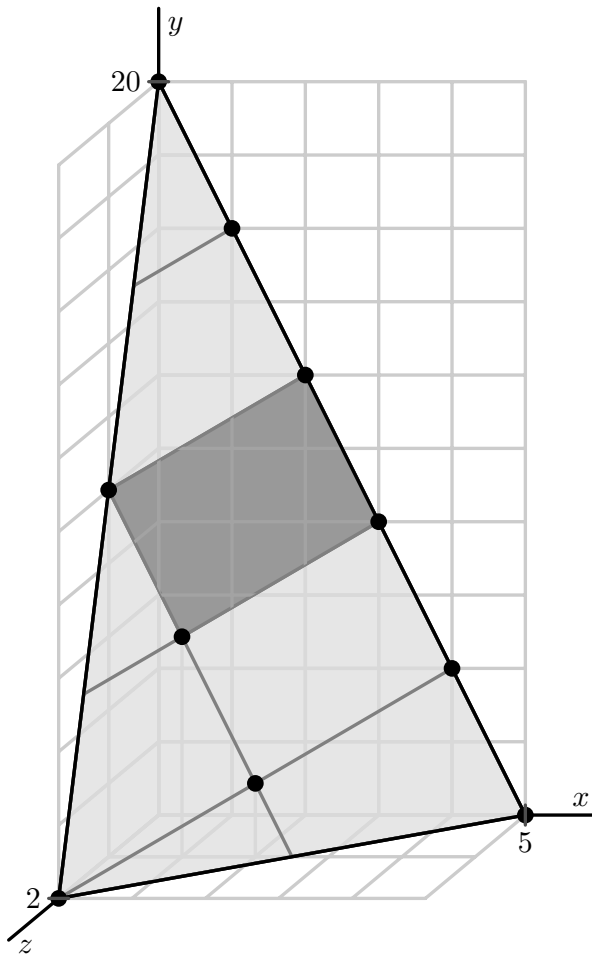


# Ehrhart Polynomials

Day I: Appetizers



Matthias Beck

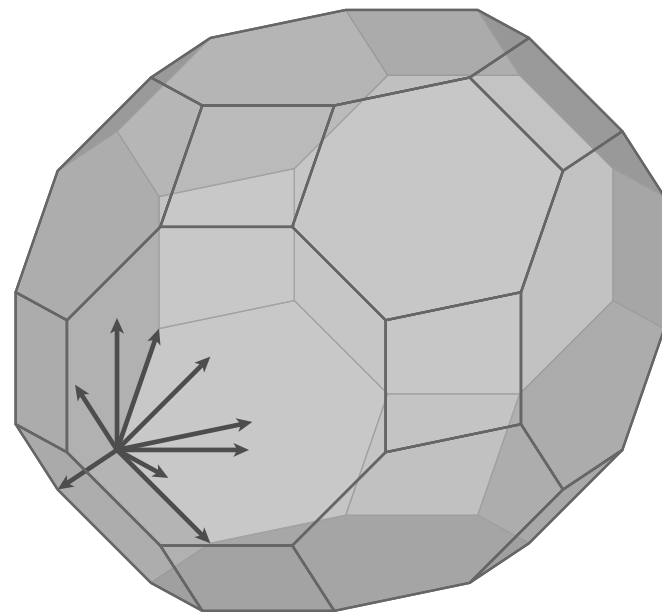
San Francisco State University

<https://matthbeck.github.io/>

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De Combinatoria

“Science is what we understand well enough to explain to a computer, art is all the rest.”

Donald Knuth



# Themes

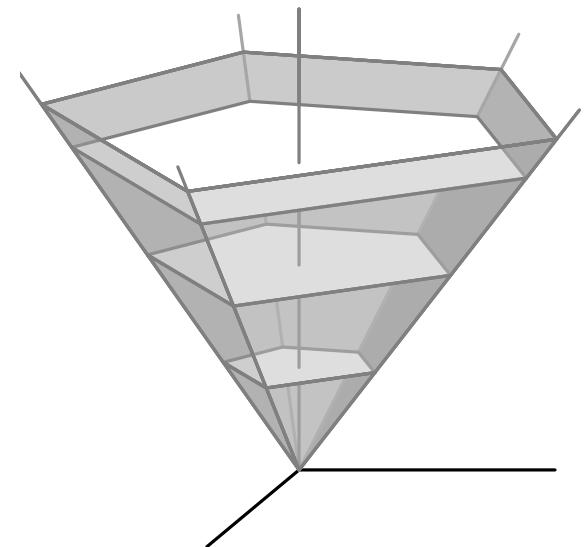
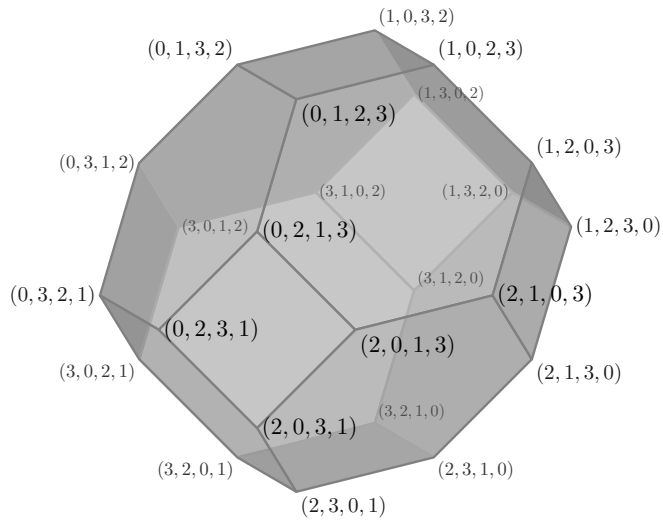
Discrete-geometric  
polynomials

Computation  
(complexity)

Generating  
functions

Combinatorial  
structures

Polyhedra



# A Sample Problem: Birkhoff–von Neumann Polytope

This site is supported by donations to [The OEIS Foundation](#).

## THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[Hints](#)  
(Greetings from [The On-Line Encyclopedia of Integer Sequences!](#))

**A037302** Normalized volume of Birkhoff polytope of  $n \times n$  doubly-stochastic square matrices. If the volume is  $v(n)$ , then  $a(n) = ((n-1)^2)! * v(n) / n^{(n-1)}$ .

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028,  
12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099,  
5091038988117504946842559205930853037841762820367901333706255223000 ([list](#); [graph](#); [refs](#); [listen](#); [history](#);  
[text](#); [internal format](#))

OFFSET 1,3

COMMENTS The Birkhoff polytope is an  $(n-1)^2$ -dimensional polytope in  $n^2$ -dimensional space; its vertices are the  $n!$  permutation matrices.  
Is  $a(n)$  divisible by  $n^2$  for all  $n \geq 4$ ? - [Dean Hickerson](#), Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n^2} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

# Discrete Volumes

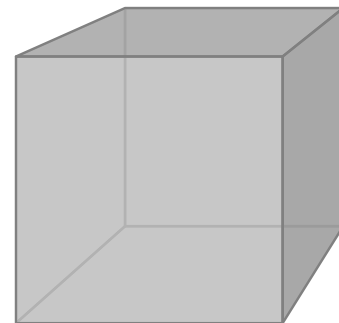
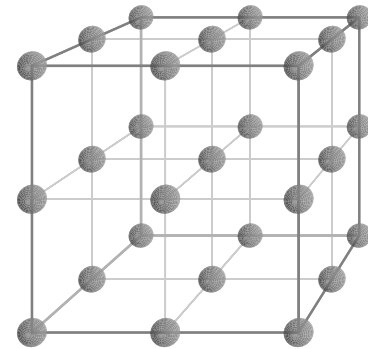
**Rational polyhedron**  $\mathcal{P} \subset \mathbb{R}^d$  – solution set of a system of linear equalities & inequalities with integer coefficients

**Goal:** understand  $\mathcal{P} \cap \mathbb{Z}^d \dots$

► (list) 
$$\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \dots z_d^{m_d}$$

► (count)  $|\mathcal{P} \cap \mathbb{Z}^d|$

► (volume) 
$$\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$$



# Discrete Volumes

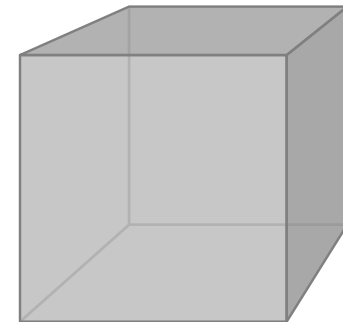
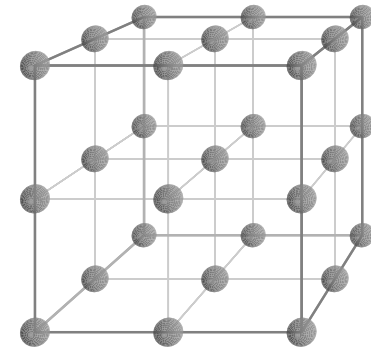
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**Ehrhart function** 
$$L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = |t\mathcal{P} \cap \mathbb{Z}^d| \quad \text{for } t \in \mathbb{Z}_{>0}$$

# Some Motivation

- ▶ Linear systems are *everywhere*, and so polyhedra are everywhere.

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- ▶ Much discrete geometry can be modeled using **polynomials** and, conversely, many combinatorial polynomials can be modeled geometrically.

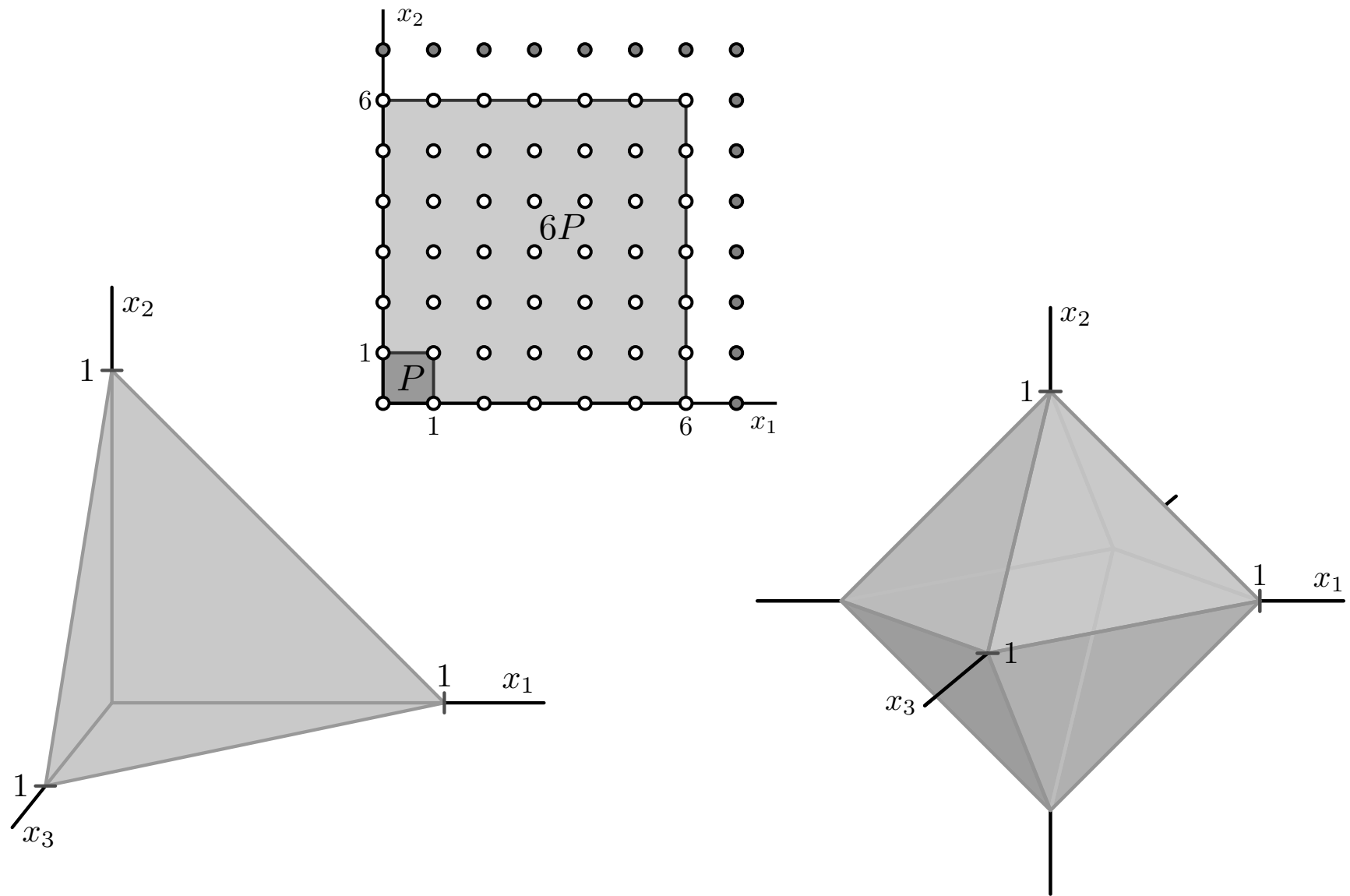
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- ▶ Polytopes are basic geometric objects, yet even for these basic objects volume computation is **hard** and there remain many open problems.
- ▶ Also, polytopes are **cool**.

# Today's Menu: Get Our Hands Dirty

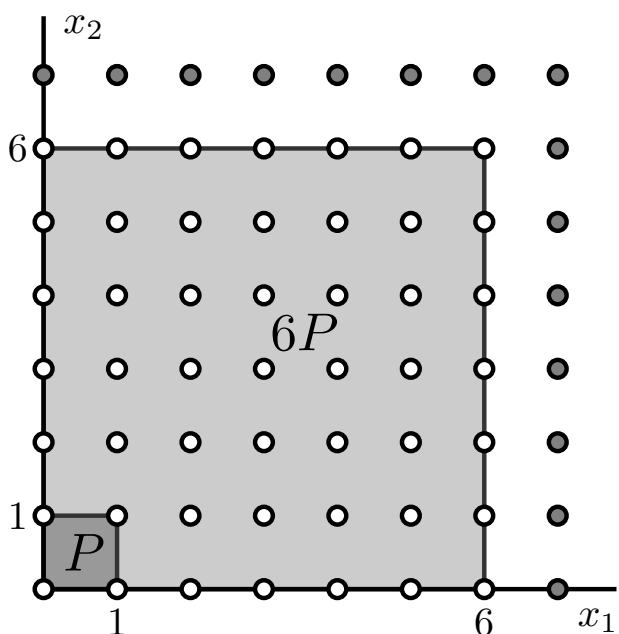


# The Unit Cube

Lattice polytope  $\mathcal{P} \subset \mathbb{R}^d$  – convex hull of finitely points in  $\mathbb{Z}^d$

For  $t \in \mathbb{Z}_{>0}$  let  $L_{\mathcal{P}}(t) := \#(t\mathcal{P} \cap \mathbb{Z}^d)$

The unit cube in  $\mathbb{R}^d$  is  $\mathcal{P} = [0, 1]^d = \{\mathbf{x} \in \mathbb{R}^d : 0 \leq x_j \leq 1\}$



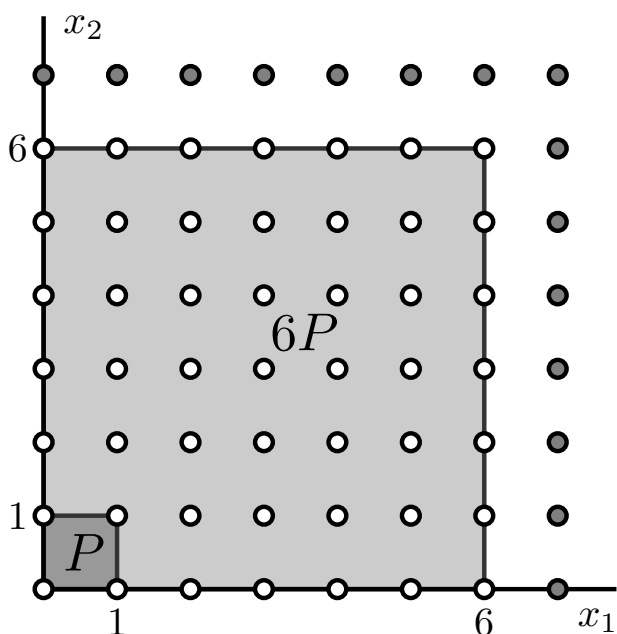
$$\longrightarrow L_{\mathcal{P}}(t) = (t+1)^d$$

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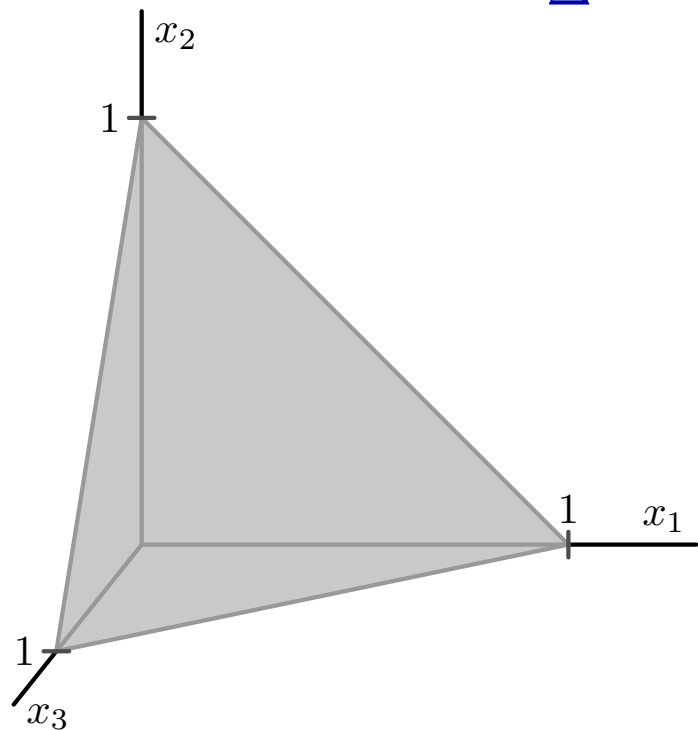
$$\longrightarrow L_{\mathcal{P}}(t) = (t+1)^d$$

$$L_{\mathcal{P}^\circ}(t) = (t-1)^d$$

# The Standard Simplex

The **standard simplex**  $\Delta \in \mathbb{R}^d$  is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta = \{x \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1\}$$





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$$\begin{aligned} L_{\Delta}(t) &= \# \{ (x_1, x_2, \dots, x_d) \in \mathbb{Z}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq t \} \\ &= \# \left\{ (x_1, x_2, \dots, x_d, x_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1} : x_1 + x_2 + \cdots + x_{d+1} = t \right\} \\ &= \binom{d+t}{d} \end{aligned}$$

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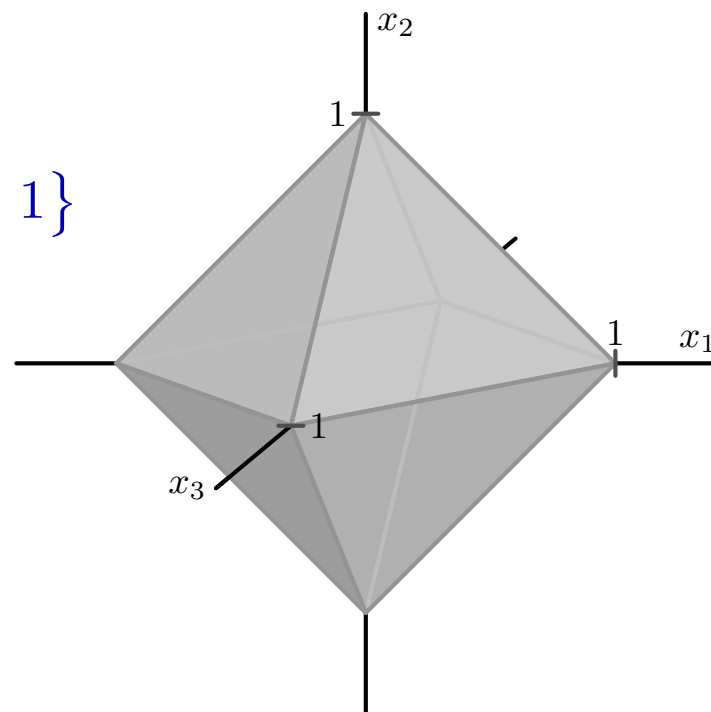
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$$L_{\Delta^{\circ}}(t) = \binom{t-1}{d}$$

# The Cross-Polytope

The cross-polytope  $\diamond \in \mathbb{R}^d$  is

$$\diamond = \{x \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \leq 1\}$$

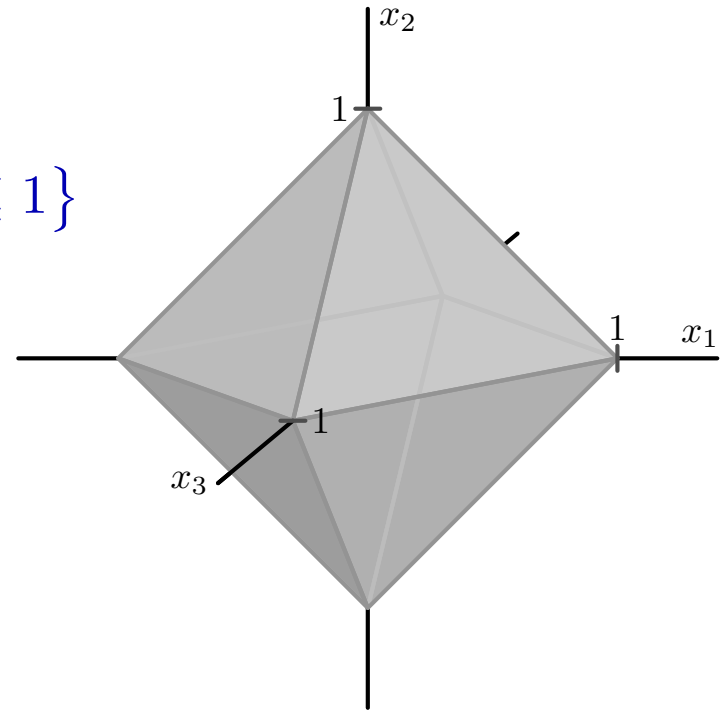


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Let's compute  $L_\diamond(t)$  for  $d = 3 \dots$



- ▶ Triangulation
- ▶ Disjoint triangulation
- ▶ Interpolation
- ▶ Generating function

# The Cross-Polytope

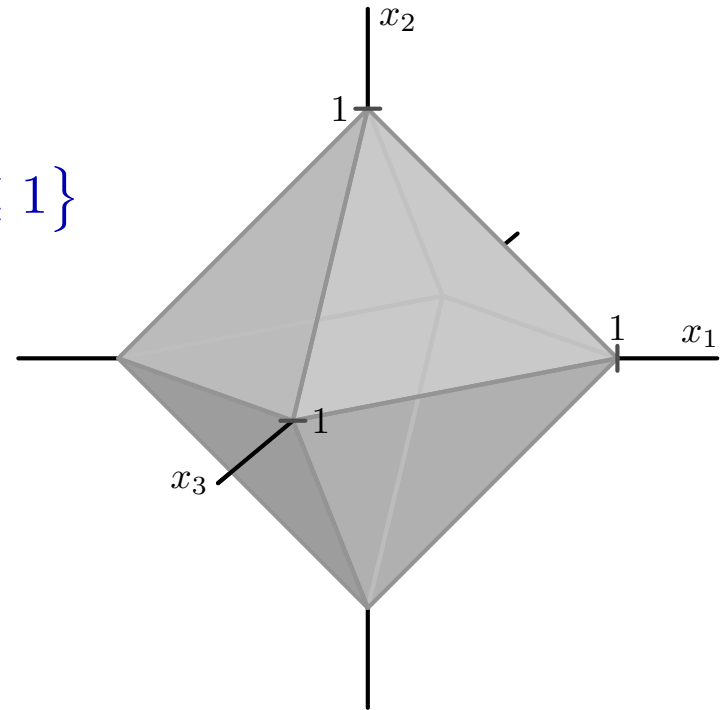
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► Triangulation

Dissect  $\diamond$  into 8 (standard) tetrahedra and use inclusion–exclusion to compute  $L_\diamond(t)$



# The Cross-Polytope

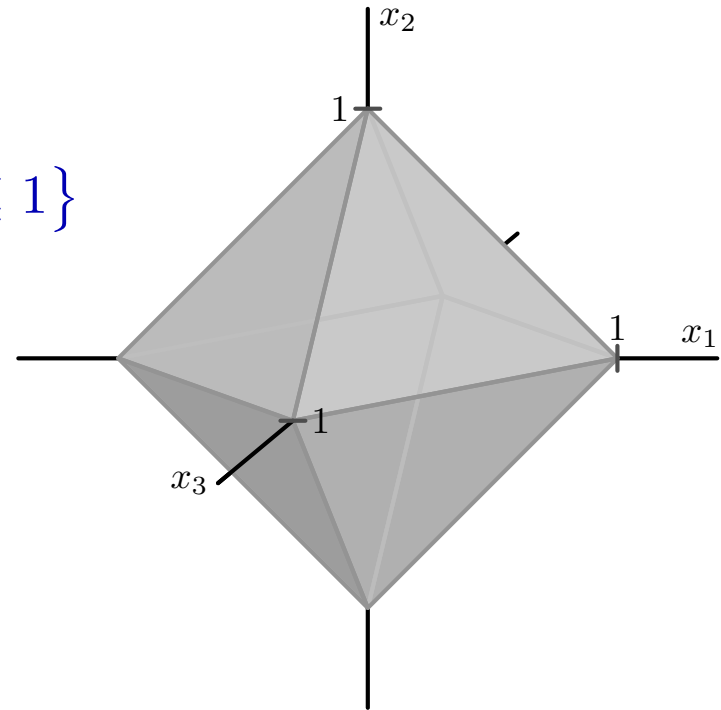
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► Disjoint triangulation

Dissect  $\diamond$  into 8 half-open tetrahedra



# The Cross-Polytope

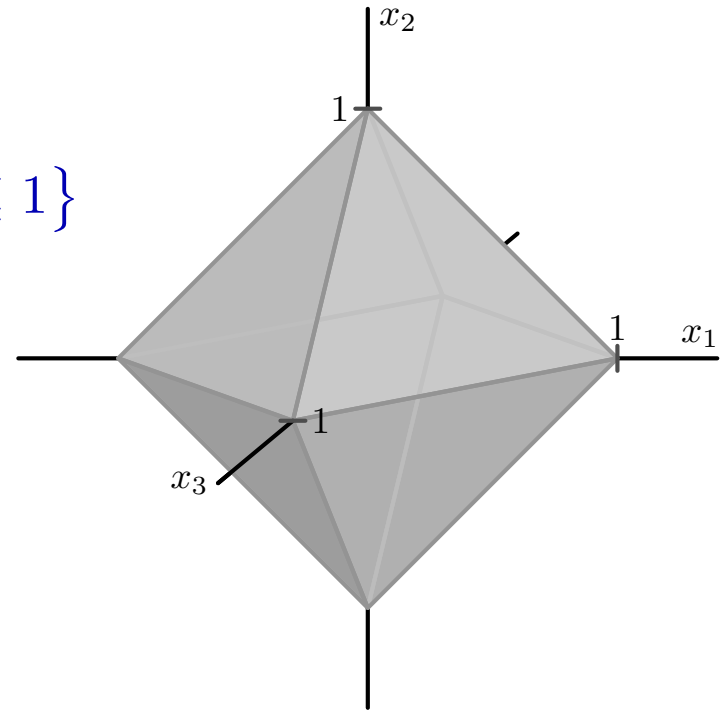
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► Interpolation

```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```



# The Cross-Polytope

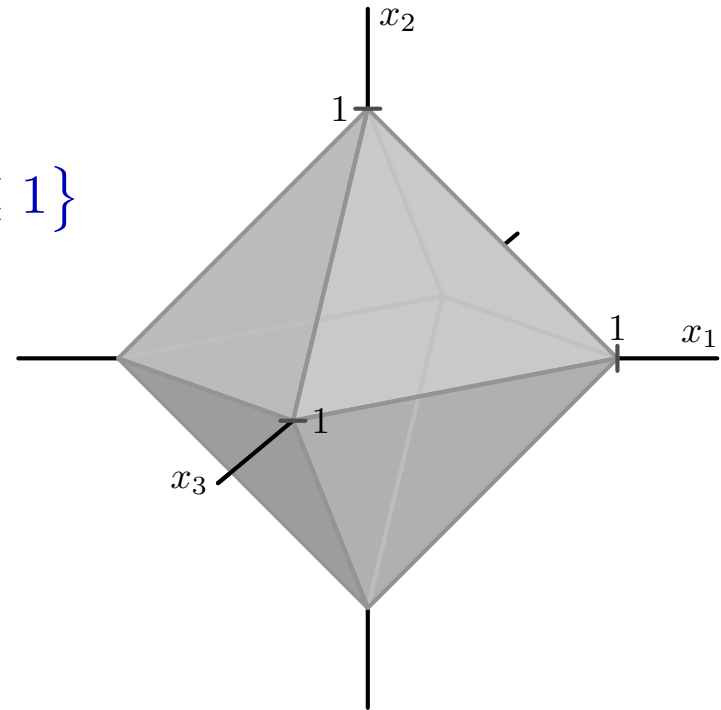
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$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$$



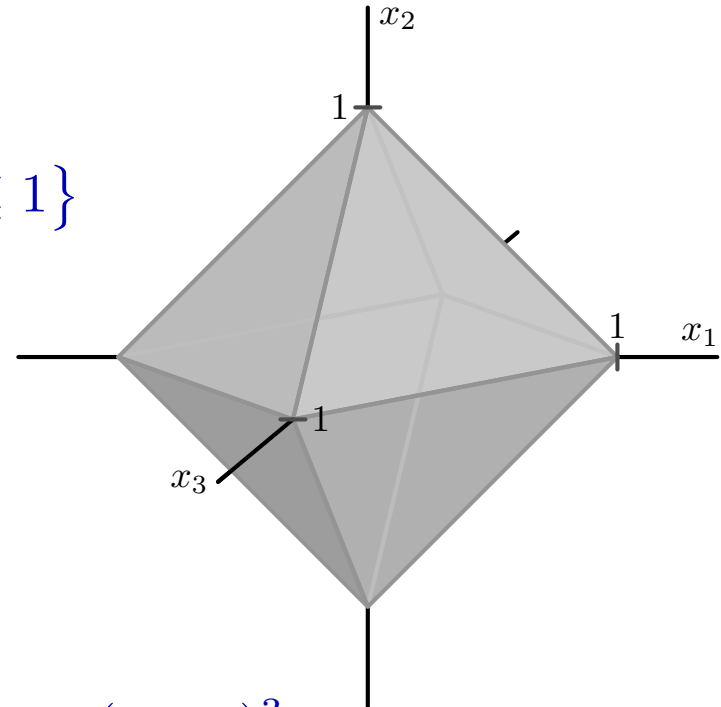


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► Generating function

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{(1+z)^3}{(1-z)^4}$$

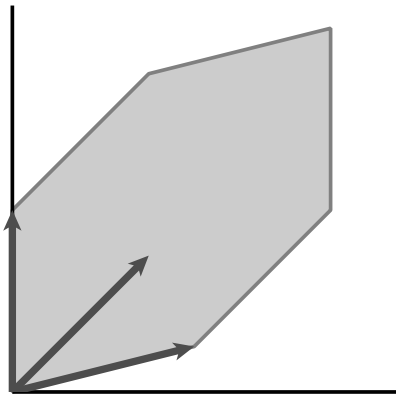
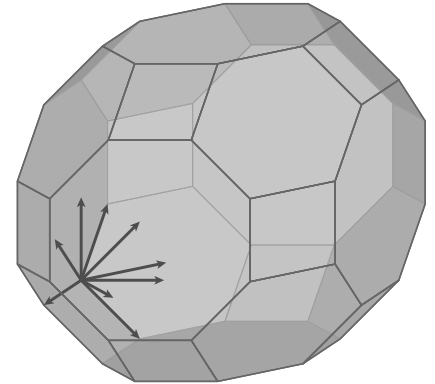
Exercise:  $\text{Ehr}_{\text{BiPyr}(\mathcal{P})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{P}}(z)$

$\dots$  for unit cubes  $\longrightarrow$  Eulerian polynomials

# Zonotopes

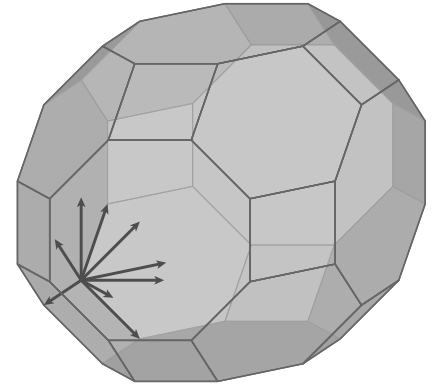
Line segment  $[a, b] := \{(1 - \lambda) a + \lambda b : 0 \leq \lambda \leq 1\}$

Minkowski sum  $\mathcal{K}_1 + \mathcal{K}_2 := \{p + q : p \in \mathcal{K}_1, q \in \mathcal{K}_2\}$



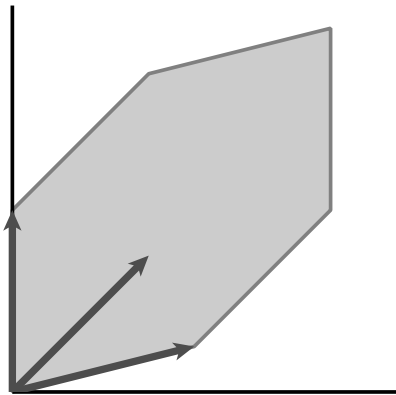
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# Zonotopes



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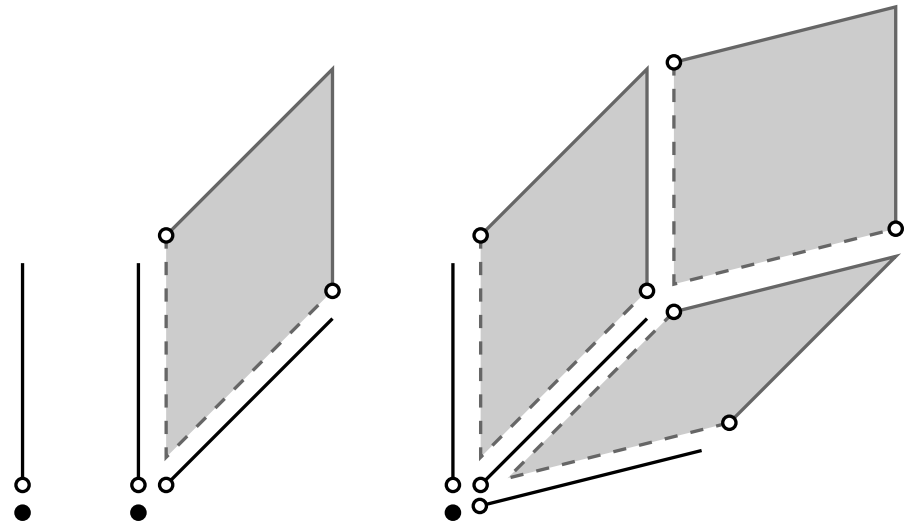


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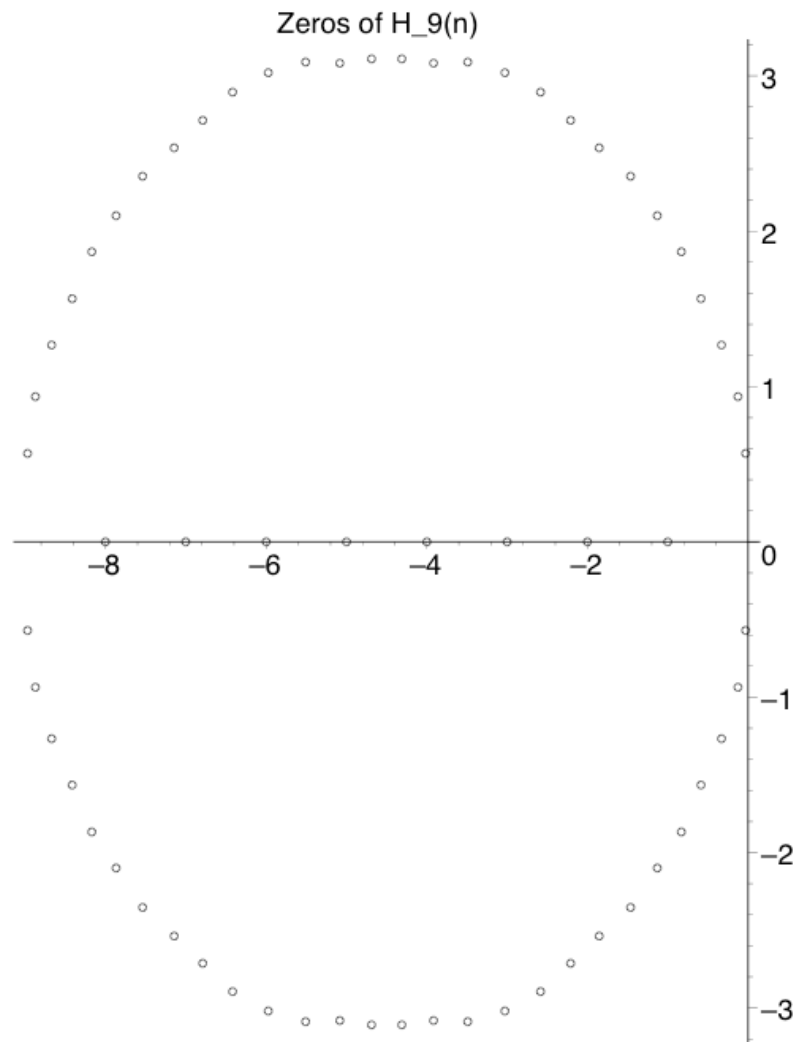
Every zonotope admits a **tiling** into parallelepipeds

$\mathcal{P}$  — half-open  $d$ -parallelepiped

$\longrightarrow L_{\mathcal{P}}(t) = t^d$



# Birkhoff–von Neumann Revisited



For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).

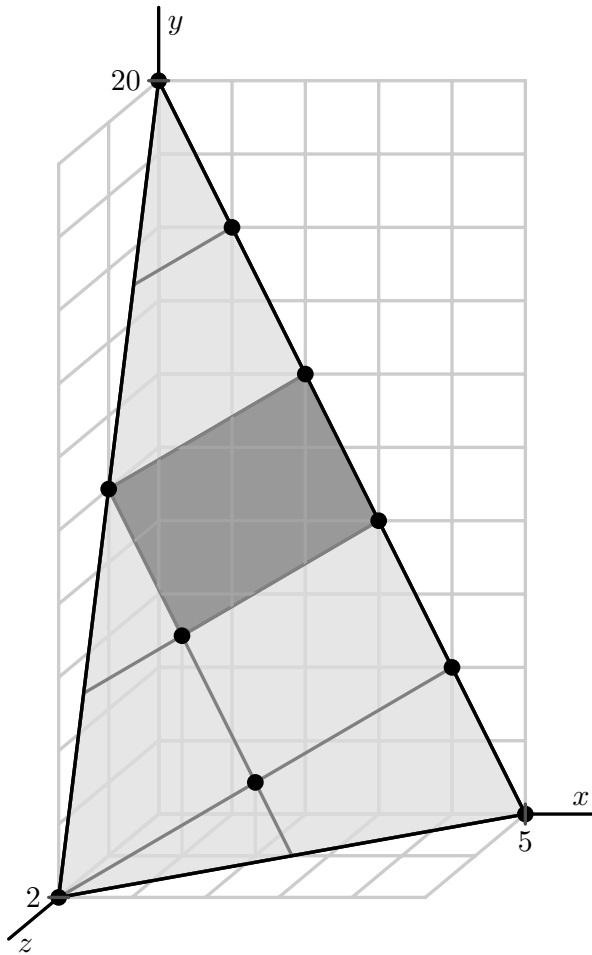
# Recap Day I

- ▶ Volume computations  $\longrightarrow$  don't agonize, discretize
- ▶ Integer-point counting in dilated polytopes  $\longrightarrow$  polynomials
- ▶ Interpolation
- ▶ Generating functions
- ▶ Dissections: triangulations, tilings
- ▶ Tomorrow: enough practice, how does this work in theory?



# Ehrhart Polynomials

Day II: Generating Functions & Complexity



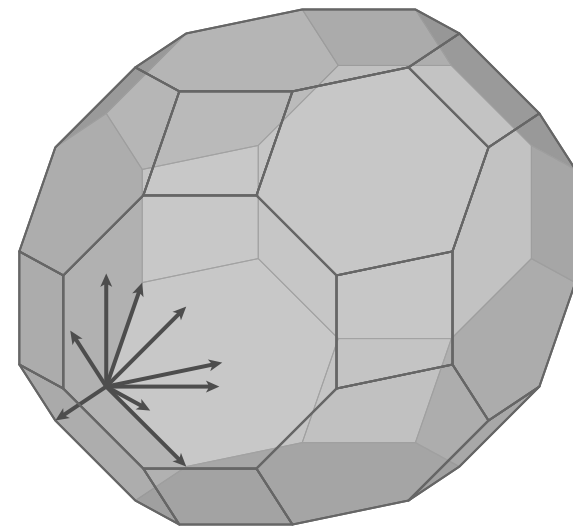
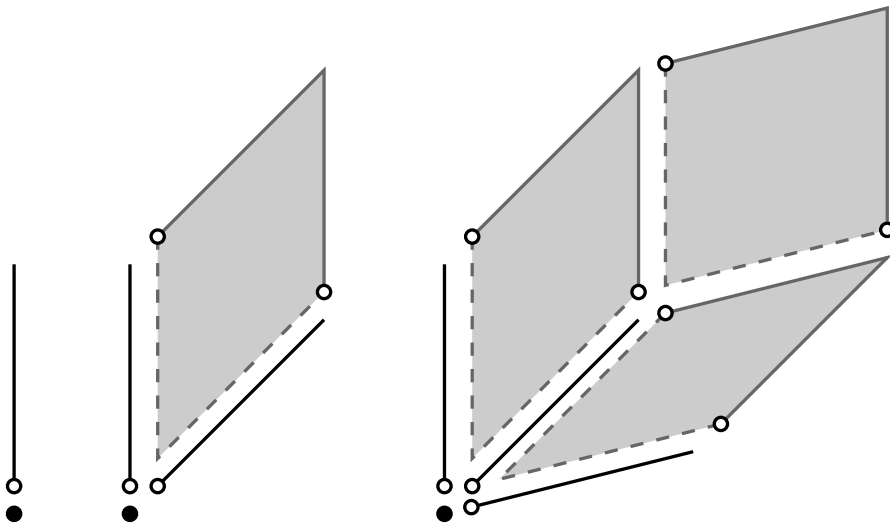
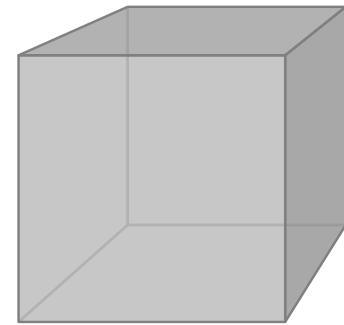
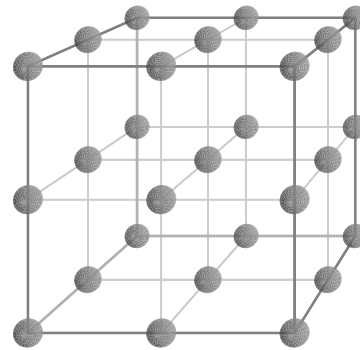
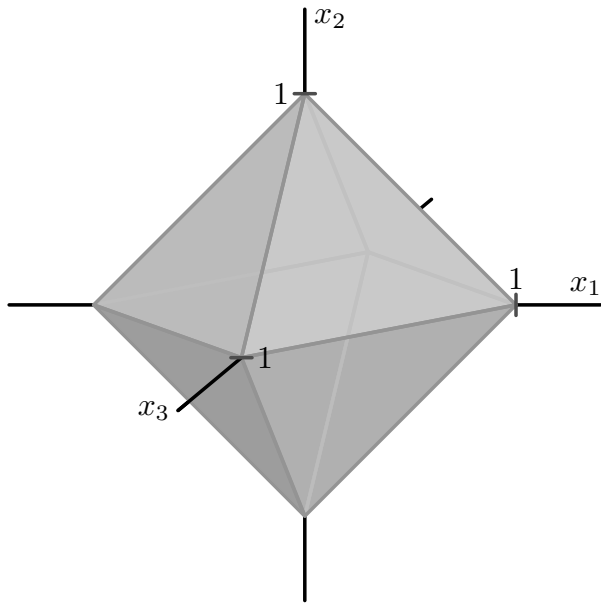
Matthias Beck

San Francisco State University

<https://matthbeck.github.io/>

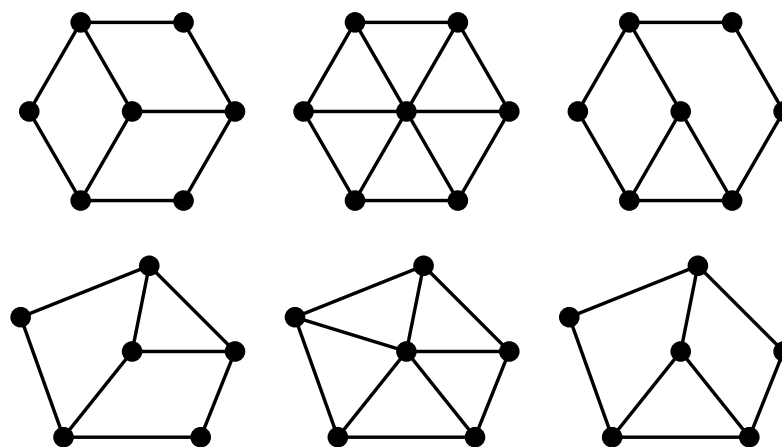
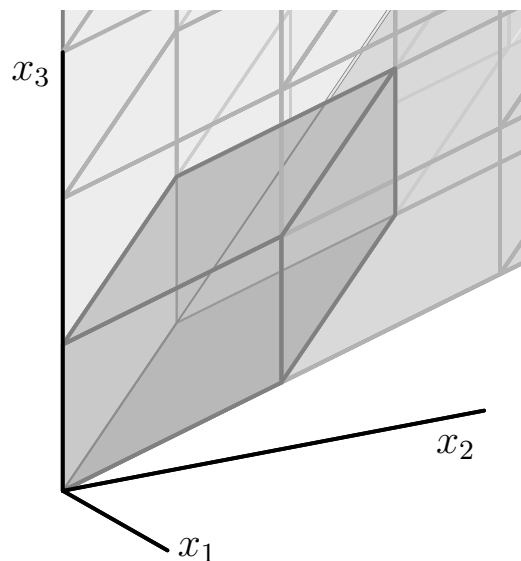
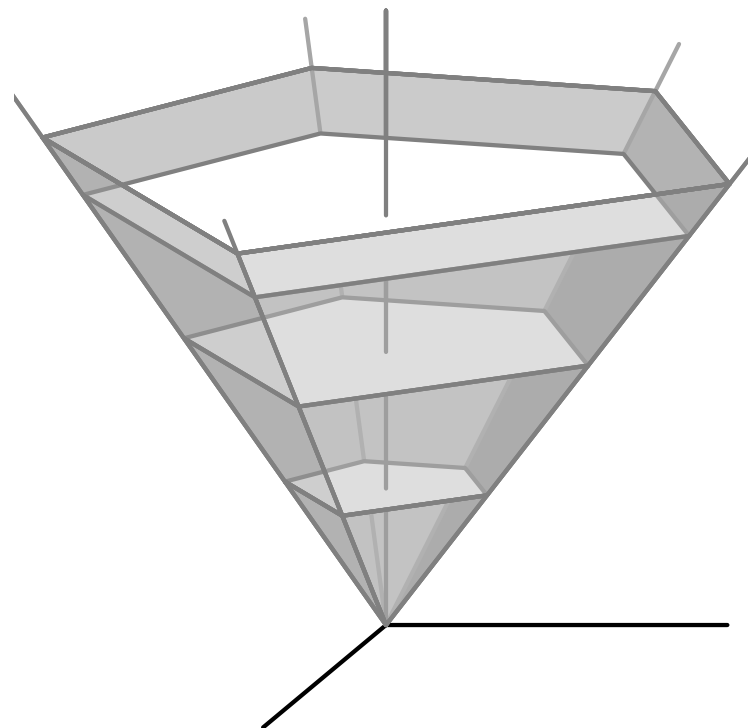
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De Combinatoria

# Any questions about yesterday?



# Today's Menu: Theory and Complexity

- ▶ Partition function magic
- ▶ Lots of generating functions
- ▶ Rational cones
- ▶ Triangulations
- ▶ Ehrhart theory





# Warm-Up: Partition Generating Functions

A **partition**  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  of an integer  $k \geq 0$  satisfies

$$k = \lambda_1 + \lambda_2 + \dots + \lambda_n \quad \text{and} \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

**Goal** Compute  $\sum_{\lambda} q^{\lambda_1 + \dots + \lambda_n}$  over your favorite partition family

**Example**  $P_{\leq 3}$  — family of partitions into at most 3 parts

$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)}$$

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**Idea**  $P_{\leq 3} = \{\lambda \in \mathbb{Z}^3 : 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \mathcal{K} \cap \mathbb{Z}^3$

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} \longleftarrow \text{polyhedral cone } \heartsuit$$

## Warm-Up: Partition Generating Functions

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

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Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

## Warm-Up: Partition Generating Functions

$$\mathcal{K} = \{x \in \mathbb{R}^3 : 0 \leq x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\mathcal{K}}(q, q, q) = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)}$$

## Variations on a Theme

$P_3$  — family of partitions into **exactly 3** parts

$$P_3 = \{\lambda \in \mathbb{Z}^3 : 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3\} = \tilde{\mathcal{K}} \cap \mathbb{Z}^3$$

$$\tilde{\mathcal{K}} = \{\mathbf{x} \in \mathbb{R}^3 : 0 < x_1 \leq x_2 \leq x_3\} = \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{> 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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$$\begin{aligned} \sigma_{\tilde{\mathcal{K}}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \tilde{\mathcal{K}} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{z_1 z_2 z_3}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

$$\sum_{\lambda \in P_3} q^{\lambda_1 + \lambda_2 + \lambda_3} = \sigma_{\tilde{\mathcal{K}}}(q, q, q) = \frac{q^3}{(1 - q)(1 - q^2)(1 - q^3)}$$

# Integer-point Complexity of a Simplicial Cone

What if  $\mathcal{K}$  is (still simplicial and rational but) not unimodular?

Say  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$  are linearly independent,  $\det[\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geq 0} \mathbf{w}_1 + \mathbb{R}_{\geq 0} \mathbf{w}_2 + \mathbb{R}_{\geq 0} \mathbf{w}_3$$



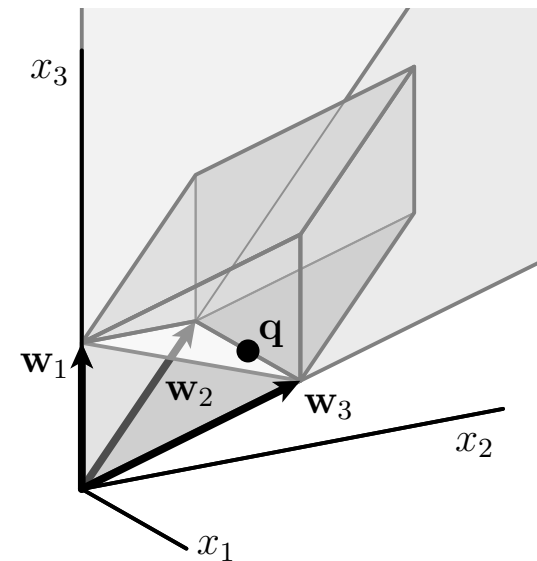
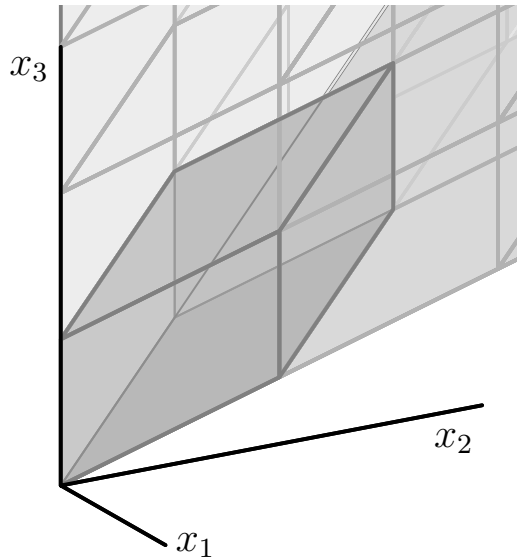
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# Integer-point Complexity of a Simplicial Cone

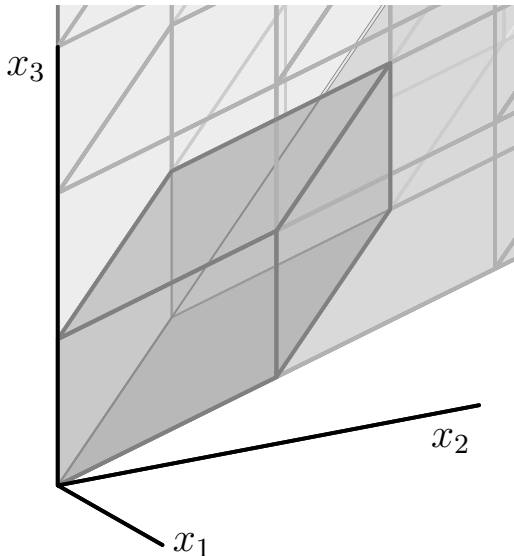
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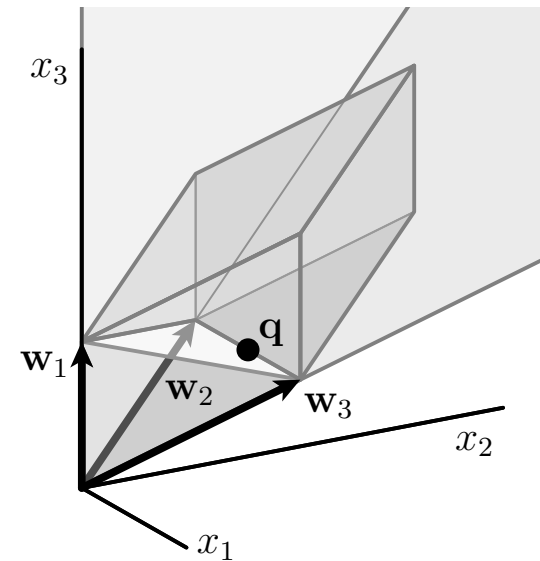
$$\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$$



$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

$$\text{where } \mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$$



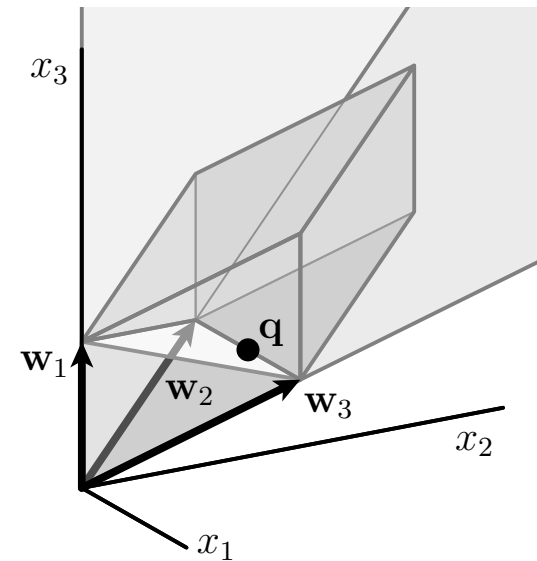
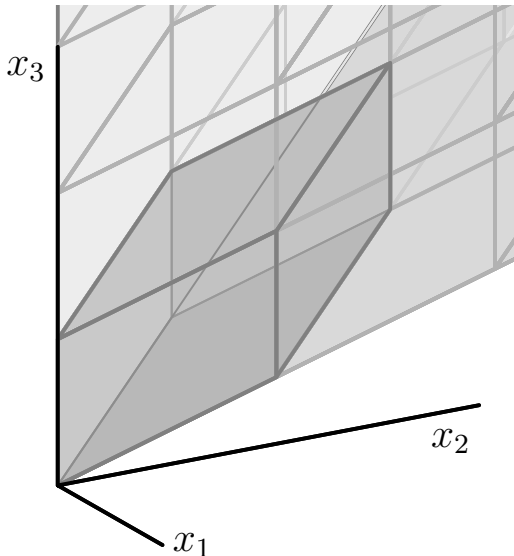
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**Idea** Tile  $\mathcal{K}$  with the half-open parallelepiped  $\Pi = [0, 1) \mathbf{w}_1 + [0, 1) \mathbf{w}_2 + [0, 1) \mathbf{w}_3$



$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

**Complexity:**  $\sigma_{\Pi}(z_1, z_2, z_3)$  has  $D$  terms

# Homogenizing Polytopes

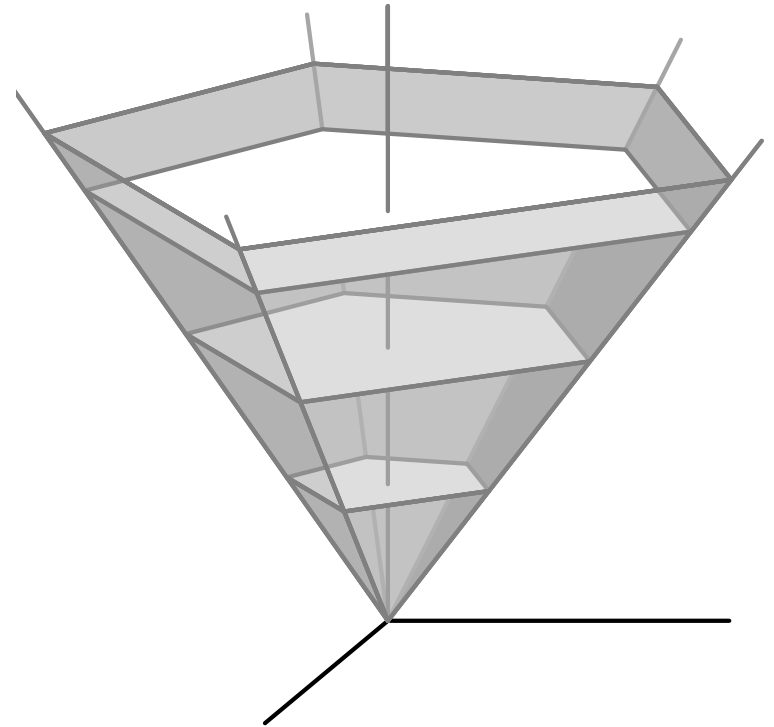
Given a polytope  $\mathcal{P} \subset \mathbb{R}^d$  let

$$\text{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1}$$

$$= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix}$$

$$\text{cone}(\mathcal{P}) \cap \{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

contains a copy of  $t\mathcal{P}$



# Homogenizing Polytopes

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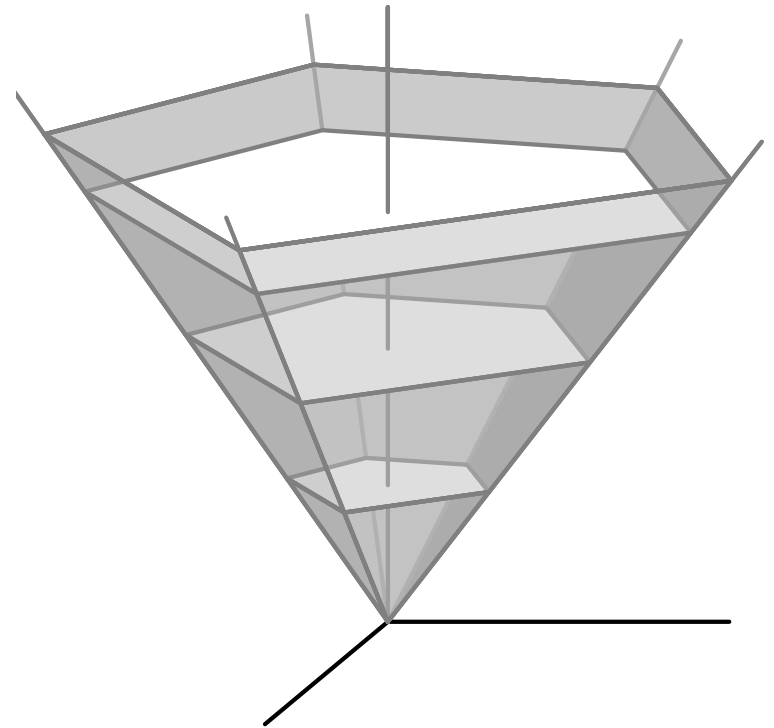
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contains a copy of  $t\mathcal{P}$   $\longrightarrow$

$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \sigma_{\text{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$$



# Homogenizing Polytopes

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$$\begin{aligned} \text{cone}(\mathcal{P}) &:= \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\}) \subset \mathbb{R}^{d+1} \\ &= \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + \mathbb{R}_{\geq 0} \begin{bmatrix} \mathbf{v}_n \\ 1 \end{bmatrix} \end{aligned}$$

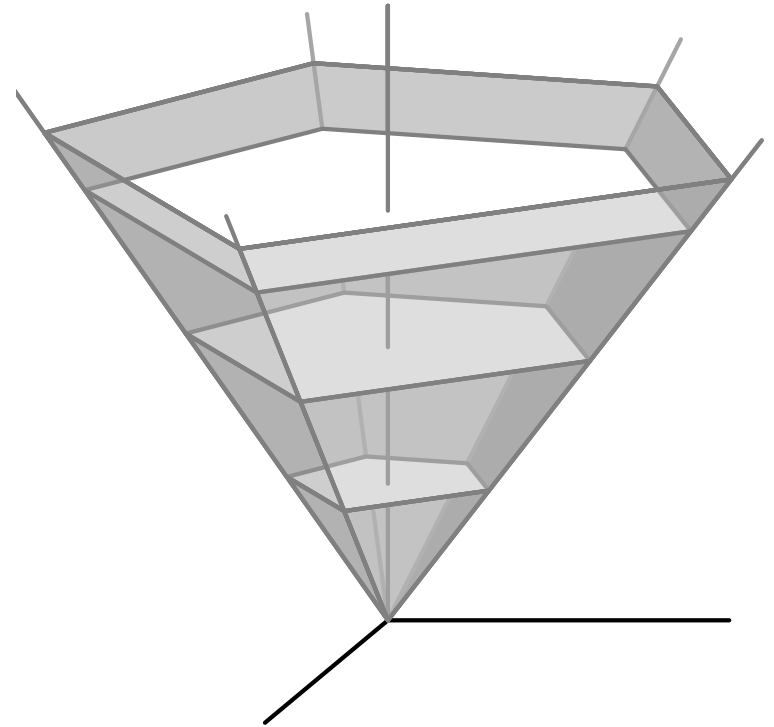
$$\text{cone}(\mathcal{P}) \cap \{\mathbf{x} \in \mathbb{R}^{d+1} : x_{d+1} = t\}$$

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$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \sigma_{\text{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$$

If  $\mathcal{P}$  is a simplex,

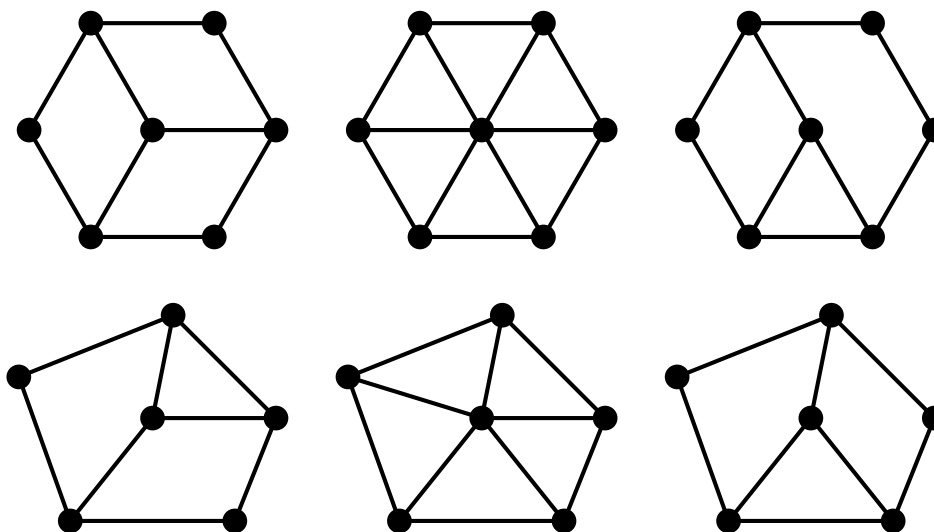
$$\sigma_{\text{cone}(\mathcal{P})}(\mathbf{z}) = \frac{\sigma_{\Pi}(\mathbf{z})}{\prod_{\mathbf{v} \text{ vertex}} (1 - \mathbf{z}^{\mathbf{v}})} \longrightarrow \text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{d+1}}$$



# Trials & Triangulations

**Subdivision** of a polyhedron  $\mathcal{P}$  — finite collection  $S$  of polyhedra such that

- ▶ if  $\mathcal{F}$  is a face of  $\mathcal{G} \in S$  then  $\mathcal{F} \in S$
- ▶ if  $\mathcal{F}, \mathcal{G} \in S$  then  $\mathcal{F} \cap \mathcal{G}$  is a face of both
- ▶  $\mathcal{P} = \bigcup_{\mathcal{F} \in S} \mathcal{F}$



If each  $\mathcal{F}$  is a simplex  $\longrightarrow$  **triangulation** of a polytope

# Ehrhart Polynomials



**Theorem** (Ehrhart 1962) For any lattice polytope  $\mathcal{P}$ ,  $L_{\mathcal{P}}(t)$  is a polynomial in  $t$  of degree  $\dim \mathcal{P}$  with leading coefficient  $\text{vol } \mathcal{P}$  and constant term 1.

Equivalently,  $\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t$  is rational:

$$\text{Ehr}_{\mathcal{P}}(z) = \frac{h_{\mathcal{P}}^*(z)}{(1 - z)^{\dim \mathcal{P} + 1}}$$

where the  $h^*$ -polynomial  $h_{\mathcal{P}}^*(z)$  satisfies  $h_{\mathcal{P}}^*(0) = 1$  and  $h_{\mathcal{P}}^*(1) = (\dim \mathcal{P})! \text{vol}(\mathcal{P})$ .



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Computational bottlenecks:

- ▶ triangulation
- ▶ determinants of resulting simplicial cones

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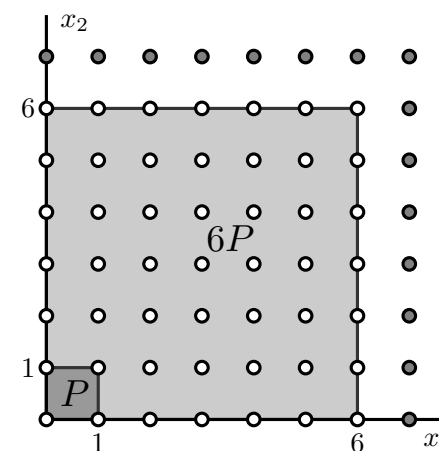
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We saw instances yesterday:  $\mathcal{P} = [0, 1]^d$

$$\longrightarrow L_{\mathcal{P}}(t) = (t + 1)^d$$

$h_{\mathcal{P}}^*(z)$  — Eulerian polynomial



# Ehrhart Polynomials



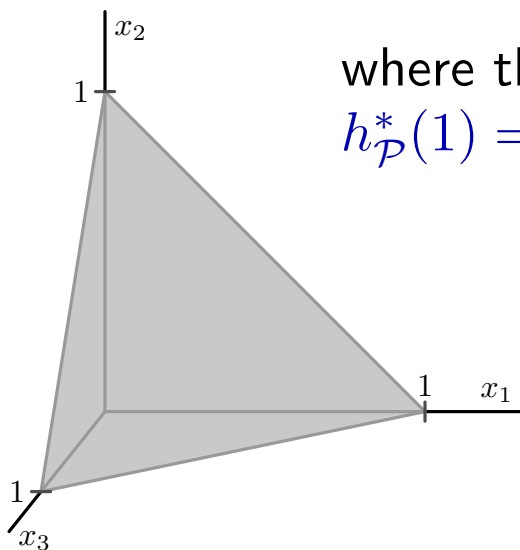
EH  
1959

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$$\Delta = \{ \mathbf{x} \in \mathbb{R}_{\geq 0}^d : x_1 + x_2 + \cdots + x_d \leq 1 \}$$

$$L_{\Delta}(t) = \binom{d+t}{d}$$

$$h_{\mathcal{P}}^*(z) = 1$$

# Ehrhart Polynomials



EH  
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**Theorem** (Ehrhart 1962) For any lattice polytope  $\mathcal{P}$ ,  $L_{\mathcal{P}}(t)$  is a polynomial in  $t$  of degree  $\dim \mathcal{P}$  with leading coefficient  $\text{vol } \mathcal{P}$  and constant term 1.

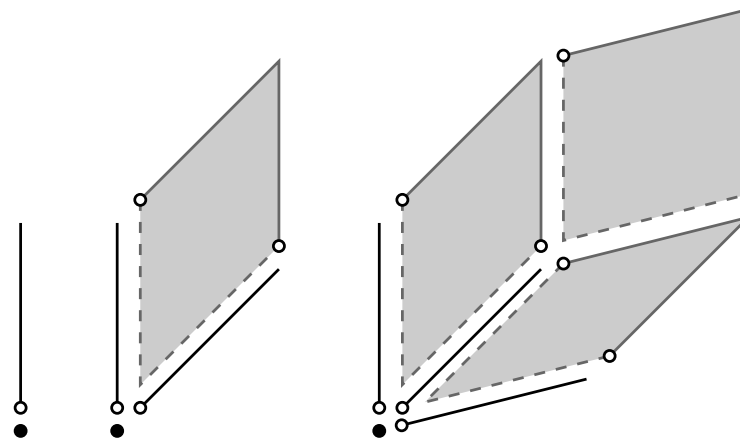
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$\mathcal{P}$  — half-open  $d$ -parallelepiped

→  $L_{\mathcal{P}}(t) = t^d$



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**Seeming dichotomy:**  $\text{vol}(\mathcal{P}) = \lim_{t \rightarrow \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$  can be computed discretely via a finite amount of data.

# Ehrhart Polynomials



**Theorem** (Ehrhart 1962) For any lattice polytope  $\mathcal{P}$ ,  $L_{\mathcal{P}}(t)$  is a polynomial in  $t$  of degree  $d := \dim \mathcal{P}$  with leading coefficient  $\text{vol } \mathcal{P}$  and constant term 1.

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Equivalent descriptions of an Ehrhart polynomial:

- ▶  $L_{\mathcal{P}}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$
- ▶ via roots of  $L_{\mathcal{P}}(t)$
- ▶  $\text{Ehr}_{\mathcal{P}}(z) \longrightarrow L_{\mathcal{P}}(t) = h_0^*\binom{t+d}{d} + h_1^*\binom{t+d-1}{d} + \cdots + h_d^*\binom{t}{d}$

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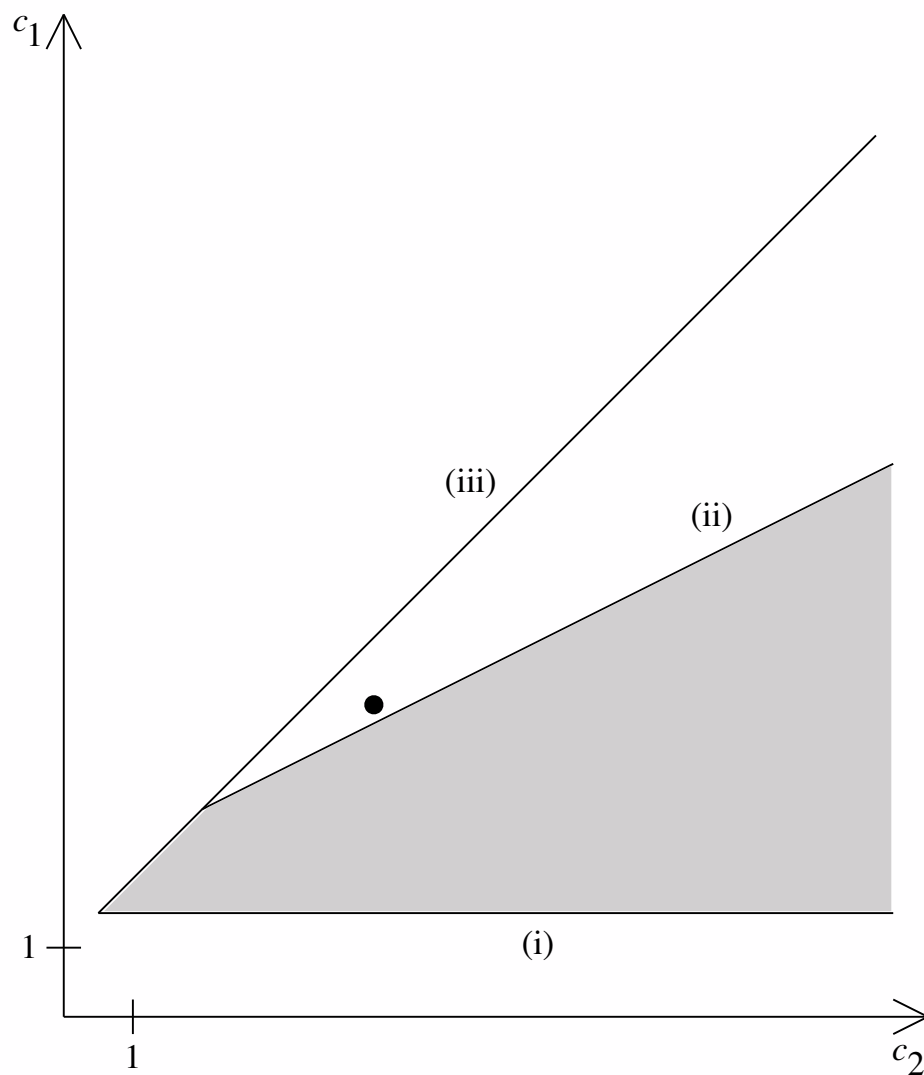
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**Open Problem** Classify Ehrhart polynomials.

# Ehrhart Polynomials in Dimension 2



$\mathcal{P}$  — lattice polygon

→  $L_{\mathcal{P}}(t) = c_2 t^2 + c_1 t + 1$



# Ehrhart Polynomials



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$$\longrightarrow L_{\mathcal{P}}(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$$

**Theorem** (Macdonald 1971)  $(-1)^d L_{\mathcal{P}}(-t)$  enumerates the **interior** lattice points in  $t\mathcal{P}$ . Equivalently,

$$L_{\mathcal{P}^\circ}(t) = h_d^* \binom{t+d-1}{d} + h_{d-1}^* \binom{t+d-2}{d} + \cdots + h_0^* \binom{t-1}{d}$$

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**Theorem** (Stanley 1980)  $h_0^*, h_1^*, \dots, h_d^*$  are nonnegative integers.

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**Theorem** (Stanley 1980)  $h_0^*, h_1^*, \dots, h_d^*$  are nonnegative integers.

**Corollary** If  $h_{d+1-k}^* > 0$  then  $k\mathcal{P}^\circ$  contains an integer point.

# Positivity Among Ehrhart Polynomials



**Theorem** (Ehrhart 1962) For any lattice polytope  $\mathcal{P}$ ,  $L_{\mathcal{P}}(t)$  is a polynomial in  $t$  of degree  $d := \dim \mathcal{P}$  with leading coefficient  $\text{vol } \mathcal{P}$  and constant term 1.

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**Theorem** (Betke–McMullen 1985, Stapledon 2009) If  $h_d^* > 0$  then

$$h(z) = a(z) + z b(z)$$

where  $a(z) = z^d a(\frac{1}{z})$  and  $b(z) = z^{d-1} b(\frac{1}{z})$  with nonnegative coefficients.

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**Open Problem** Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

# Ehrhart Quasipolynomials

**Rational polytope**  $\mathcal{P} \subset \mathbb{R}^d$  – convex hull of finitely points in  $\mathbb{Q}^d$

**Theorem** (Ehrhart 1962)  $L_{\mathcal{P}}(t)$  is a **quasipolynomial** in  $t$ :

$$L_{\mathcal{P}}(t) = c_d(t) t^d + c_{d-1}(t) t^{d-1} + \cdots + c_0(t)$$

where  $c_0(t), \dots, c_d(t)$  are periodic functions.

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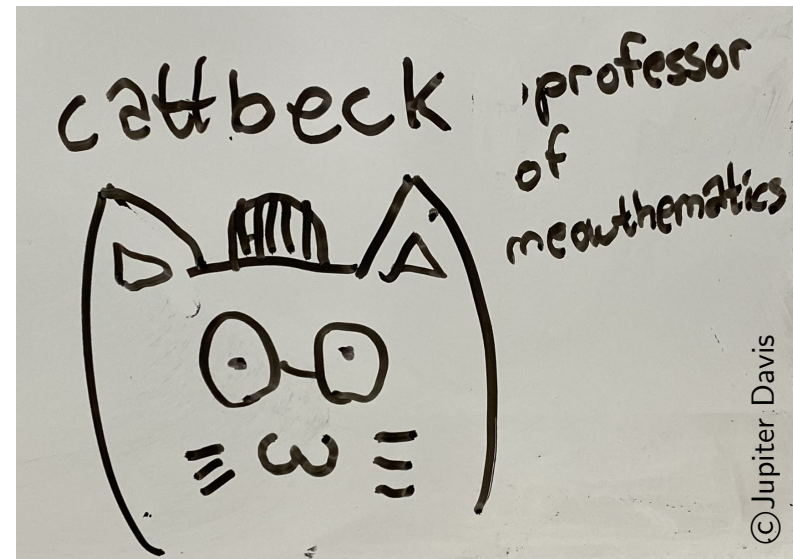
$$\text{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$$

for some (minimal)  $p \in \mathbb{Z}_{>0}$  (the **period** of  $L_{\mathcal{P}}(t)$ ).

**Open Problem** Study periods of Ehrhart quasipolynomials.

# Recap Day II

- ▶ Generating functions son chevere
- ▶ Integer-point transforms of rational polyhedra  $\longrightarrow$  rational functions
- ▶ Complexity of a simplicial cone: determinant of its generators
- ▶ Homogenize polytopes
- ▶ Triangulations
- ▶ Polynomial data
- ▶ Thursday: structural results about Ehrhart polynomials



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