

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day I: Appetizers

- (1) Given integers a, b, c, d , form the line segment $[(a, b), (c, d)] \subset \mathbb{R}^2$ joining the points (a, b) and (c, d) . Show that the number of integer points on this line segment is $\gcd(a - c, b - d) + 1$.
- (2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area $\frac{1}{2}$. (*Hint*: You may begin by “doubling” the triangle to form a parallelogram.)
- (3) Pick four points in \mathbb{Z}^3 and let \mathcal{P} be their convex hull (in \mathbb{R}^3). Compute the Ehrhart polynomial of \mathcal{P} . (If you cannot think of a good example, consider the regular tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)$.)

- (4) Recall that the standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin. Verify that

$$L_{\Delta}(t) = \binom{d+t}{d} \quad \text{and} \quad L_{\Delta^{\circ}}(t) = \binom{t-1}{d}.$$

(If you'd like to amuse your colleagues, we can also write $L_{\Delta^{\circ}}(t) = (-1)^d \binom{d-t}{d}$.)

- (5) Given a $(d-1)$ -polytope \mathcal{Q} with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ such that the origin is in \mathcal{Q} , we define the bipyramid $\text{BiPyr}(\mathcal{Q})$ over \mathcal{Q} as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0), (0, \dots, 0, 1), \text{ and } (0, \dots, 0, -1).$$

Show that $\text{Ehr}_{\text{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{Q}}(z)$.

- (6) Compute the Ehrhart polynomial of the octahedron

$$\diamond = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$$

via the four different approaches outlined in the lecture:

- (a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);
- (b) disjoint triangulation into 8 standard tetrahedra;
- (c) [sage] interpolation;
- (d) [sage] generating function.

Generalize.

- (7) [sage] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What's going on here?

(8) Define the Eulerian number $A(d, k)$ through¹

$$\sum_{j \geq 0} j^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}.$$

Alternatively, we may think of the polynomial $\sum_{k=0}^d A(d, k) z^k$ is the numerator of the rational function

$$\left(z \frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) = \underbrace{z \frac{d}{dz} \cdots z \frac{d}{dz}}_{d \text{ times}} \left(\frac{1}{1-z}\right).$$

Prove the following:

$$A(d, k) = A(d, d+1-k),$$

$$A(d, k) = (d-k+1) A(d-1, k-1) + k A(d-1, k),$$

$$\sum_{k=0}^d A(d, k) = d!,$$

$$A(d, k) = \sum_{j=0}^k (-1)^j \binom{d+1}{j} (k-j)^d.$$

(9) [research problem] Choose $d+1$ of the 2^d vertices of the unit d -cube, and let Δ be the simplex defined by their convex hull.

(a) Which choice of vertices maximizes $\text{vol } \Delta$?

(b) What is the maximum volume of such a Δ ?

(10) Show that a sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n \geq 0} f(n) z^n = \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

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¹There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j \geq 0} (j+1)^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}$ instead.

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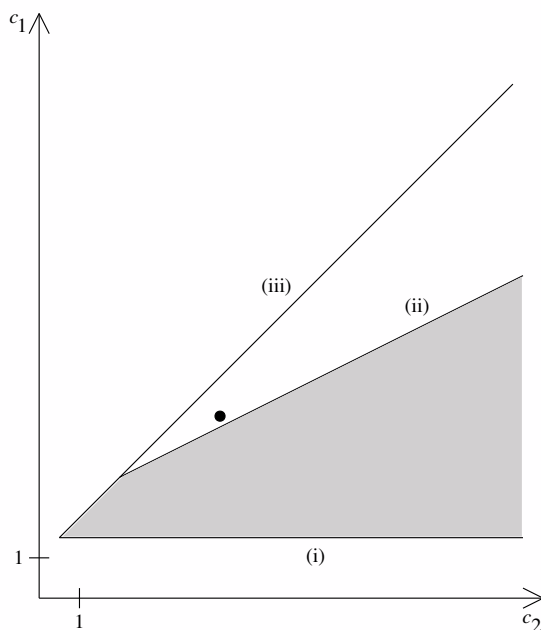
Day II: Ehrhart Theory

- (1) For any polynomial $h(z)$ of degree d , show there exist unique polynomials $a(z)$ and $b(z)$ such that

$$h(z) = a(z) + z b(z) \quad \text{where} \quad a(z) = z^d a\left(\frac{1}{z}\right) \quad \text{and} \quad b(z) = z^{d-1} b\left(\frac{1}{z}\right).$$

(There are many variations of this; e.g., we could leave out the z factor in front of $b(z)$.)

- (2) Derive inequalities for the coefficients of $h(z)$ if we know that both $a(z)$ and $b(z)$ have only nonnegative coefficients.
- (3) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



- (4) [research problem] Give the corresponding classification picture of degree-3 Ehrhart polynomials.
- (5) Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.
- (6) For a lattice polytope \mathcal{P} , the numerator of the generating function is the h^* -polynomial of \mathcal{P} . Give a non-unimodal example of an h^* -polynomial.

- (7) [research problem] Now let $\mathcal{P} = \{\mathbf{x} \in [0,1]^d : x_1 + x_2 + \cdots + x_d = k\}$, for your favorite integers $2 \leq k \leq d-2$. (This is the (d,k) -hypersimplex.) Prove that the h^* -polynomial of \mathcal{P} is unimodal.
- (8) Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, randomly choose $h_1, h_2, \dots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_n, h_n)$. The *lower hull* of \mathcal{Q} consists of all points that are *visible from below*: all points $(x_1, x_2, \dots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \dots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of \mathcal{Q} is a face of \mathcal{Q} that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of \mathcal{Q} are simplices, and that their projections under π form a triangulation of \mathcal{P} .