

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day I: Appetizers

- (1) Given integers a, b, c, d , form the line segment $[(a, b), (c, d)] \subset \mathbb{R}^2$ joining the points (a, b) and (c, d) . Show that the number of integer points on this line segment is $\gcd(a - c, b - d) + 1$.
- (2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area $\frac{1}{2}$. (*Hint*: You may begin by “doubling” the triangle to form a parallelogram.)
- (3) Pick four points in \mathbb{Z}^3 and let \mathcal{P} be their convex hull (in \mathbb{R}^3). Compute the Ehrhart polynomial of \mathcal{P} . (If you cannot think of a good example, consider the regular tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)$.)

- (4) Recall that the standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin. Verify that

$$L_{\Delta}(t) = \binom{d+t}{d} \quad \text{and} \quad L_{\Delta^{\circ}}(t) = \binom{t-1}{d}.$$

(If you'd like to amuse your colleagues, we can also write $L_{\Delta^{\circ}}(t) = (-1)^d \binom{d-t}{d}$.)

- (5) Given a $(d-1)$ -polytope \mathcal{Q} with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ such that the origin is in \mathcal{Q} , we define the bipyramid $\text{BiPyr}(\mathcal{Q})$ over \mathcal{Q} as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0), (0, \dots, 0, 1), \text{ and } (0, \dots, 0, -1).$$

Show that $\text{Ehr}_{\text{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{Q}}(z)$.

- (6) Compute the Ehrhart polynomial of the octahedron

$$\diamond = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$$

via the four different approaches outlined in the lecture:

- (a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);
- (b) disjoint triangulation into 8 standard tetrahedra;
- (c) [sage] interpolation;
- (d) [sage] generating function.

Generalize.

- (7) [sage] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What's going on here?
- (8) [research problem] Compute the Ehrhart polynomial of the Birkhoff–von Neumann polytope \mathcal{B}_{10} or the volume of \mathcal{B}_{11} .

(9) Define the Eulerian number $A(d, k)$ through¹

$$\sum_{j \geq 0} j^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}.$$

Alternatively, we may think of the polynomial $\sum_{k=0}^d A(d, k) z^k$ is the numerator of the rational function

$$\left(z \frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) = \underbrace{z \frac{d}{dz} \cdots z \frac{d}{dz}}_{d \text{ times}} \left(\frac{1}{1-z}\right).$$

Prove the following:

$$\begin{aligned} A(d, k) &= A(d, d+1-k), \\ A(d, k) &= (d-k+1) A(d-1, k-1) + k A(d-1, k), \\ \sum_{k=0}^d A(d, k) &= d!, \\ A(d, k) &= \sum_{j=0}^k (-1)^j \binom{d+1}{j} (k-j)^d. \end{aligned}$$

(10) The permutahedron $\mathcal{P}_d \in \mathbb{R}^d$ is defined as the convex hull of

$$\{(\pi(1)-1, \pi(2)-1, \dots, \pi(d)-1) : \pi \in S_d\},$$

where S_d is the set of all permutations of $\{1, 2, \dots, d\}$. Show that \mathcal{P}_d is a zonotope:

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d],$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$ are the standard unit vectors.

(11) Prove that \mathcal{P}_d tiles the hyperplane spanned by it.

(12) Show that a sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n \geq 0} f(n) z^n = \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

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¹There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j \geq 0} (j+1)^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}$ instead.

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Day II: Generating Functions & Complexity

- (1) Compute the generating functions for

$$P_{\leq n} := \{\lambda \in \mathbb{Z}^n : 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\},$$

partitions into at most n parts. Adjust your computations for partitions into exactly n parts.

- (2) Compute the integer-point transform of the cone $\mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

- (3) Let $n \geq 3$ and

$$T_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\},$$

the set of all “ n -gon partitions.”

- (a) Compute the generating function for T_3 .
(b) What makes your computation more complicated for $n > 3$?
(c) Compute the generating function for

$$\tilde{T}_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \leq \lambda_n\},$$

and conclude from it the generating function for T_n .

- (4) Recall the lecture-hall partitions

$$\text{LH}_n := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n} \right\}.$$

Compute the generators of the underlying cone, and verify the first few instances of the Lecture-Hall Theorem:

$$\sum_{\lambda \in \text{LH}_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{(1-q)(1-q^3) \cdots (1-q^{2n-1})}.$$

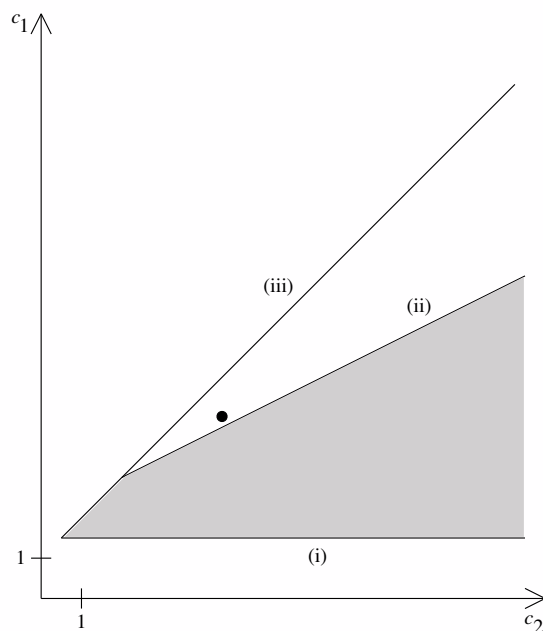
- (5) [sage] Pick five points in \mathbb{Z}^3 and let \mathcal{P} be their convex hull (in \mathbb{R}^3). Compute the Ehrhart polynomial of \mathcal{P} .
(6) [research problem] Choose $d+1$ of the 2^d vertices of the unit d -cube, and let Δ be the simplex defined by their convex hull.
(a) Which choice of vertices maximizes $\text{vol } \Delta$?
(b) What is the maximum volume of such a Δ ?

- (7) Give an explicit bijection between the faces (including \emptyset) of a given polytope \mathcal{P} and the faces (excluding \emptyset) of its homogenization cone(\mathcal{P}).
- (8) Suppose $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$ are lattice polytopes. Prove that the *convolution* of their Ehrhart polynomials,

$$L(t) := \sum_{s=0}^t L_{\mathcal{P}}(s) L_{\mathcal{Q}}(t-s)$$

equals the Ehrhart quasipolynomial of the polytope given by the convex hull of $\mathcal{P} \times \{\mathbf{0}_n\} \times \{0\}$ and $\{\mathbf{0}_m\} \times \mathcal{Q} \times \{1\}$. Here $\mathbf{0}_d$ denotes the origin in \mathbb{R}^d .

- (9) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



- (10) [research problem] Give the corresponding classification picture of degree-3 Ehrhart polynomials.
- (11) This exercise constructs triangulations. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, randomly choose $h_1, h_2, \dots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_n, h_n)$. The *lower hull* of \mathcal{Q} consists of all points that are *visible from below*: all points $(x_1, x_2, \dots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \dots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of \mathcal{Q} is a face of \mathcal{Q} that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of \mathcal{Q} are simplices, and that their projections under π form a triangulation of \mathcal{P} .

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Day III: Positivity, Reciprocity & Friends

- (1) Let $\mathcal{P} \subset \mathbb{R}^d$ be a full-dimensional polyhedron with dissection $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$. If $\mathbf{q} \in \mathbb{R}^d$ is generic relative to each \mathcal{P}_i , then

$$\mathbb{H}_{\mathbf{q}}\mathcal{P} = \mathbb{H}_{\mathbf{q}}\mathcal{P}_1 \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_2 \uplus \cdots \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_m.$$

- (2) Prove Ehrhart's theorem for half-open lattice polytopes.
- (3) [sage] Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.
- (4) [sage] For a lattice polytope \mathcal{P} , the numerator of the generating function is the h^* -polynomial of \mathcal{P} . Give a non-unimodal² example of an h^* -polynomial.
- (5) [research problem] Now let $\mathcal{P} = \{\mathbf{x} \in [0, 1]^d : x_1 + x_2 + \cdots + x_d = k\}$, for your favorite integers $2 \leq k \leq d - 2$. (This is the (d, k) -hypersimplex.) Prove that the h^* -polynomial of \mathcal{P} is unimodal.
- (6) For any polynomial $h(z)$ of degree d , show there exist unique polynomials $a(z)$ and $b(z)$ such that

$$h(z) = a(z) + z b(z) \quad \text{where} \quad a(z) = z^d a\left(\frac{1}{z}\right) \quad \text{and} \quad b(z) = z^{d-1} b\left(\frac{1}{z}\right).$$

(There are many variations of this; e.g., we could leave out the z factor in front of $b(z)$.)

- (7) Derive inequalities for the coefficients of $h(z)$ if we know that both $a(z)$ and $b(z)$ have only nonnegative coefficients.

²A polynomial is *unimodal* if its coefficients increase up to some point and then decrease.