Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day I: Appetizers

(1) Given integers a, b, c, d, form the line segment $[(a,b),(c,d)] \subset \mathbb{R}^2$ joining the points (a,b) and (c,d). Show that the number of integer points on this line segment is $\gcd(a-c, b-d)+1$.

(2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area $\frac{1}{2}$. (*Hint:* You may begin by "doubling" the triangle to form a parallelogram.)

(3) Pick four points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the Ehrhart polynomial of $\mathcal{P}$. (If you cannot think of a good example, consider the regular tetrahedron with vertices $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$.)

(4) Recall that the standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin. Verify that

$$L_\Delta(t) = \binom{d+t}{d} \qquad \text{and} \qquad L_{\Delta^\circ}(t) = \binom{t-1}{d}.$$

(If you'd like to amuse your colleagues, we can also write $L_{\Delta^\circ}(t) = (-1)^d \binom{d-t}{d}$.)

(5) Given a $(d-1)$-polytope $\mathcal{Q}$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ such that the origin is in $\mathcal{Q}$, we define the bipyramid $\mathrm{BiPyr}(\mathcal{Q})$ over $\mathcal{Q}$ as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \ldots, (\mathbf{v}_m, 0), (0, \ldots, 0, 1), \text{ and } (0, \ldots, 0, -1).$$

Show that $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{Q})}(z) = \dfrac{1+z}{1-z}\,\mathrm{Ehr}_{\mathcal{Q}}(z)$.

(6) Compute the Ehrhart polynomial of the octahedron

$$\Diamond = \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \le 1 \right\}$$

via the four different approaches outlined in the lecture:

(a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);

(b) disjoint triangulation into 8 standard tetrahedra;

(c) [`sage`] interpolation;

(d) [`sage`] generating function.

Generalize.

(7) [`sage`] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What's going on here?

(8) Define the Eulerian number $A\,(d,k)$ through[1]

$$\sum_{j \ge 0} j^d \, z^j \;=\; \frac{\sum_{k=0}^{d} A\,(d,k)\, z^k}{(1-z)^{d+1}} \, .$$

Alternatively, we may think of the polynomial $\sum_{k=0}^{d} A\,(d,k)\, z^k$ is the numerator of the rational function

$$\left(z \frac{d}{dz} \right)^d \left(\frac{1}{1-z} \right) \;=\; \underbrace{z \frac{d}{dz} \cdots z \frac{d}{dz}}_{d \text{ times}} \left(\frac{1}{1-z} \right) .$$

Prove the following:

$$\begin{aligned} A\,(d,k) &= A\,(d, d+1-k) \, , \\ A\,(d,k) &= (d-k+1)\, A\,(d-1,k-1) + k\, A\,(d-1,k) \, , \\ \sum_{k=0}^{d} A\,(d,k) &= d! \, , \\ A\,(d,k) &= \sum_{j=0}^{k} (-1)^j \binom{d+1}{j} (k-j)^d . \end{aligned}$$

(9) The permutahedron $\mathcal{P}_d \in \mathbb{R}^d$ is defined as the convex hull of

$$\{ (\pi(1)-1, \, \pi(2)-1, \ldots, \, \pi(d)-1) : \, \pi \in S_d \} \, ,$$

where S_d is the set of all permutations of $\{1, 2, \ldots, d\}$. Show that P_d is a zonotope:

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d] \, ,$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ are the standard unit vectors.

(10) Prove that $\mathcal{P}_d$ tiles the hyperplane spanned by it.

(11) Show that a sequence $f(n)$ is given by a polynomial of degree $\le d$ if and only if

$$\sum_{n \ge 0} f(n) \, z^n \;=\; \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\le d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

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[1]There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j \ge 0} (j+1)^d \, z^j = \frac{\sum_{k=0}^{d} A(d,k) z^k}{(1-z)^{d+1}}$ instead.

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Day II: Generating Functions & Complexity

(1) Compute the generating functions for

$$P_{\le n} := \{ \lambda \in \mathbb{Z}^n : 0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \} ,$$

partitions into at most n parts. Adjust your computations for partitions into exactly n parts.

(2) Let $n \ge 3$ and

$$T_n := \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \ge \cdots \ge \lambda_1 \ge 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n \} ,$$

the set of all "n-gon partitions."

(a) Compute the generating function for T_3.

(b) What makes your computation more complicated for $n > 3$?

(c) Compute the generating function for

$$\widetilde{T}_n := \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \ge \cdots \ge \lambda_1 \ge 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \le \lambda_n \} ,$$

and conclude from it the generating function for T_n.

(3) Compute the integer-point transform of the cone $\mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

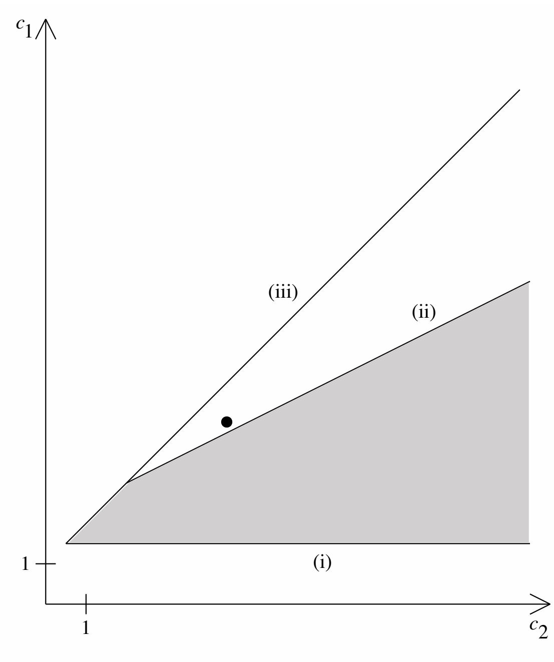
(4) For any polynomial $h(z)$ of degree d, show there exist unique polynomials $a(z)$ and $b(z)$ such that

$$h(z) = a(z) + z\, b(z) \qquad \text{where} \qquad a(z) = z^d\, a(\tfrac{1}{z}) \qquad \text{and} \qquad b(z) = z^{d-1}\, b(\tfrac{1}{z}) .$$

(There are many variations of this; e.g., we could leave out the z factor in front of $b(z)$.)

(5) Derive inequalities for the coefficients of $h(z)$ if we know that both $a(z)$ and $b(z)$ have only nonnegative coefficients.

(6) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



(7) [`research problem`] Give the corresponding classification picture of degree-3 Ehrhart polynomials.

(8) Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.

(9) For a lattice polytope $\mathcal{P}$, the numerator of the generating function is the *h^*-polynomial* of $\mathcal{P}$. Give a non-unimodal example of an h^*-polynomial.

(10) [`research problem`] Now let $\mathcal{P} = \left\{ \mathbf{x} \in [0,1]^d : x_1 + x_2 + \cdots + x_d = k \right\}$, for your favorite integers $2 \le k \le d-2$. (This is the *(d,k)-hypersimplex*.) Prove that the h^*-polynomial of $\mathcal{P}$ is unimodal.

(11) This exercise constructs triangulations. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, randomly choose $h_1, h_2, \ldots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \ldots, (\mathbf{v}_n, h_n)$. The *lower hull* of $\mathcal{Q}$ consists of all points that are *visible from below*: all points $(x_1, x_2, \ldots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \ldots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of $\mathcal{Q}$ is a face of $\mathcal{Q}$ that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of $\mathcal{Q}$ are simplices, and that their projections under π form a triangulation of $\mathcal{P}$.