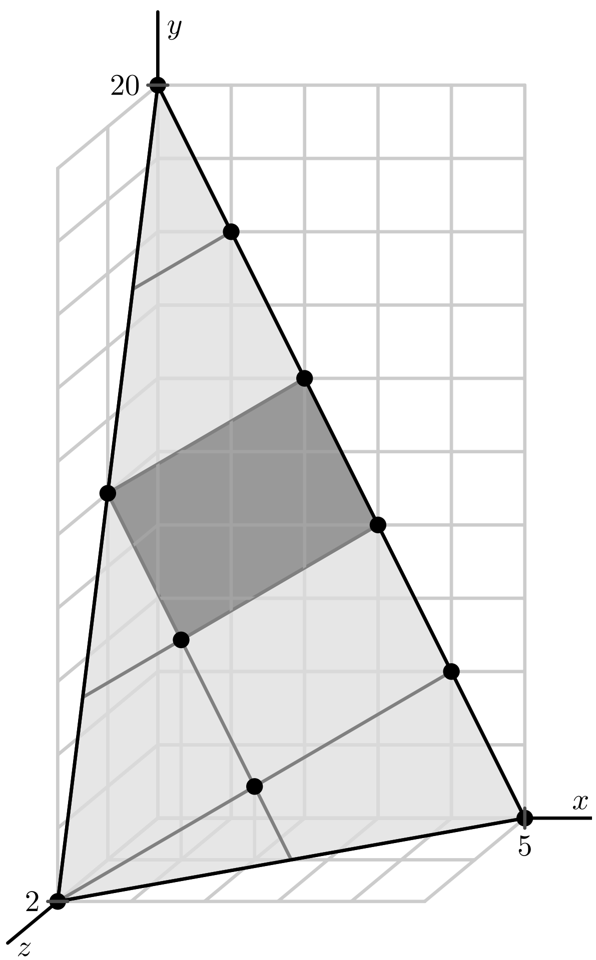
Ehrhart Polynomials

Day I: Appetizers



Matthias Beck

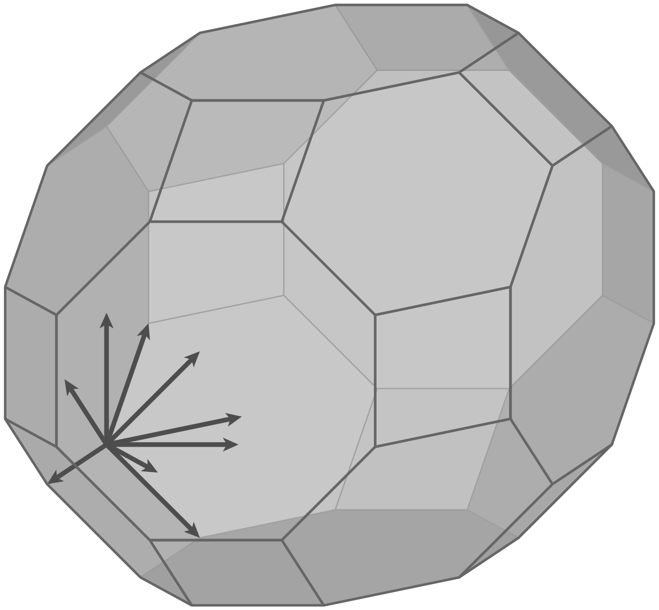
San Francisco State University

https://matthbeck.github.io/

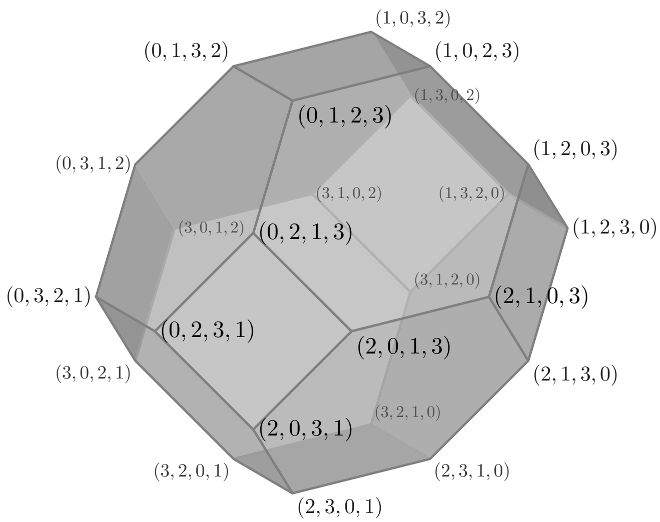
VIII Encuentro Colombiano
De Combinatoria

"Science is what we understand well enough to explain to a computer, art is all the rest."

Donald Knuth



Themes



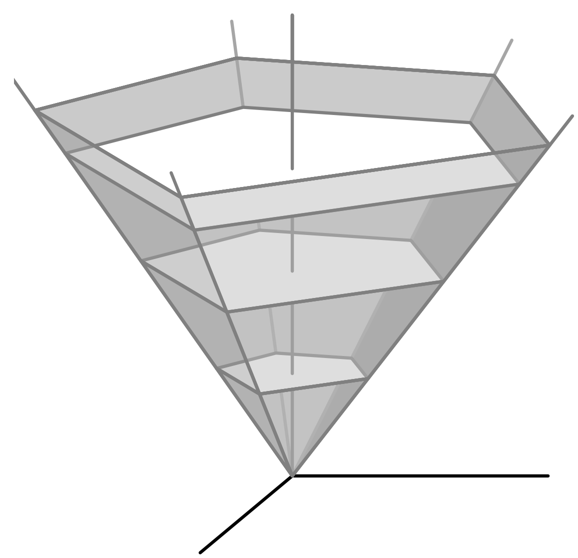
Discrete-geometric polynomials

Computation (complexity)

Generating functions

Combinatorial structures

Polyhedra



A Sample Problem: Birkhoff–von Neumann Polytope

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THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A037302 Normalized volume of Birkhoff polytope of n X n doubly-stochastic square matrices. If the volume is v(n), then a(n) = ((n-1)^2)! * v(n) / n^(n-1). 2

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028, 12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099, 5091038988117504946842559205930853037841762820367901333706255223000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

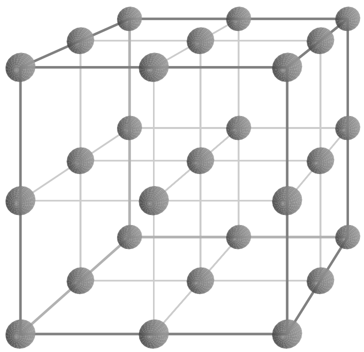
COMMENTS The Birkhoff polytope is an (n-1)^2-dimensional polytope in n^2-dimensional space; its vertices are the n! permutation matrices.
Is a(n) divisible by n^2 for all n>=4? - Dean Hickerson, Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geq 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

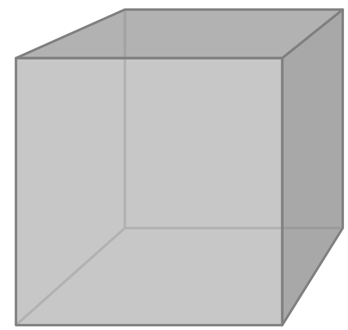
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .



- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$
- (count) $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$
- (volume) $\displaystyle \mathrm{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$



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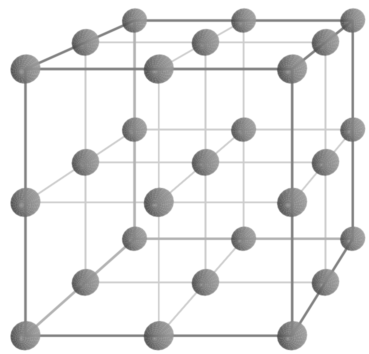

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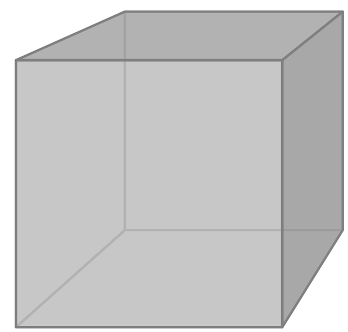

Ehrhart function $\displaystyle L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d \right| = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ for $t \in \mathbb{Z}_{>0}$

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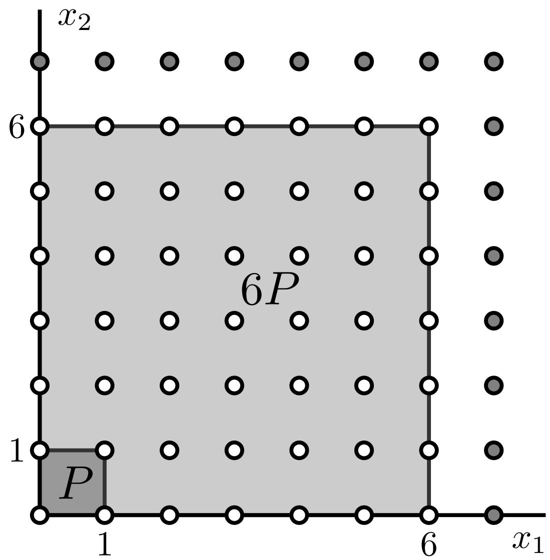
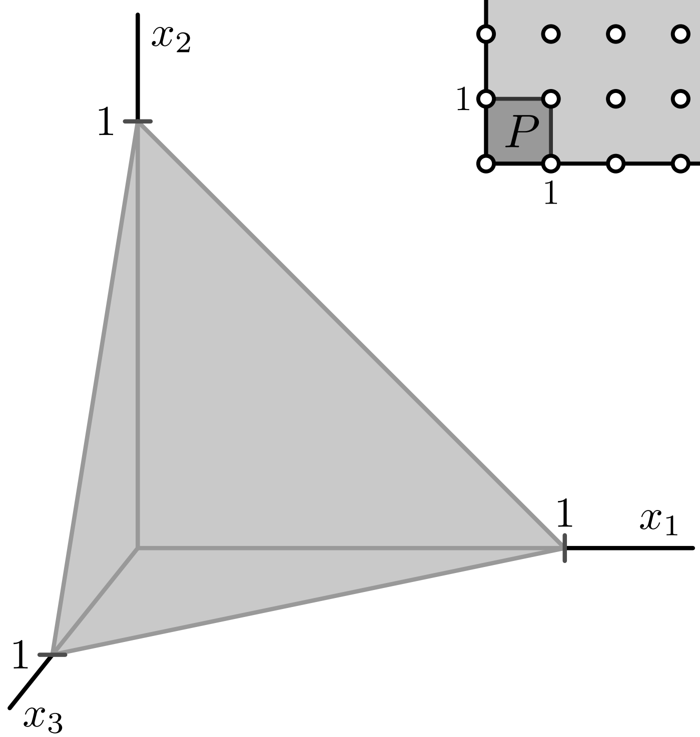
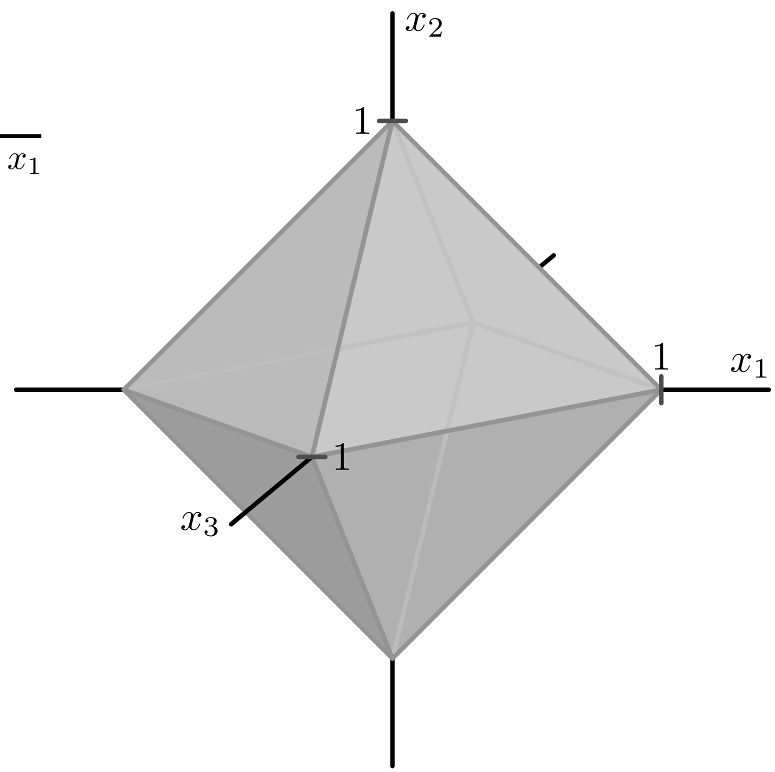
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.

- Also, polytopes are cool.

Today's Menu: Get Our Hands Dirty





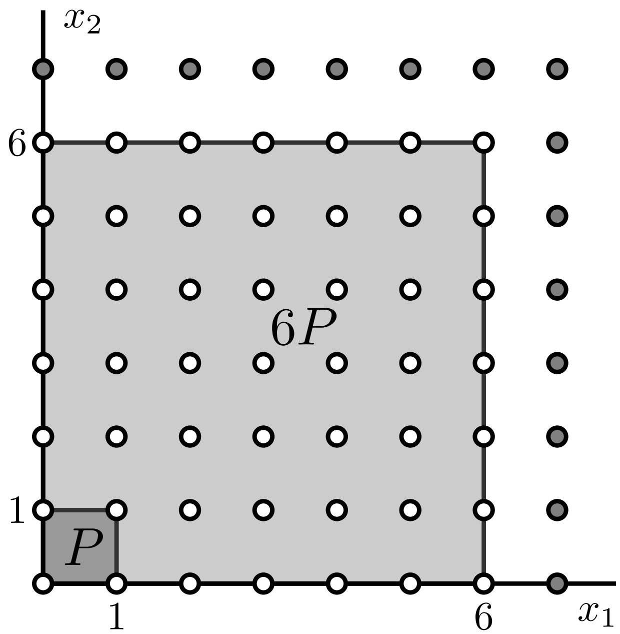


The Unit Cube

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$

The unit cube in $\mathbb{R}^d$ is $\mathcal{P} = [0,1]^d = \left\{\boldsymbol{x} \in \mathbb{R}^d : 0 \le x_j \le 1\right\}$



$$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$$

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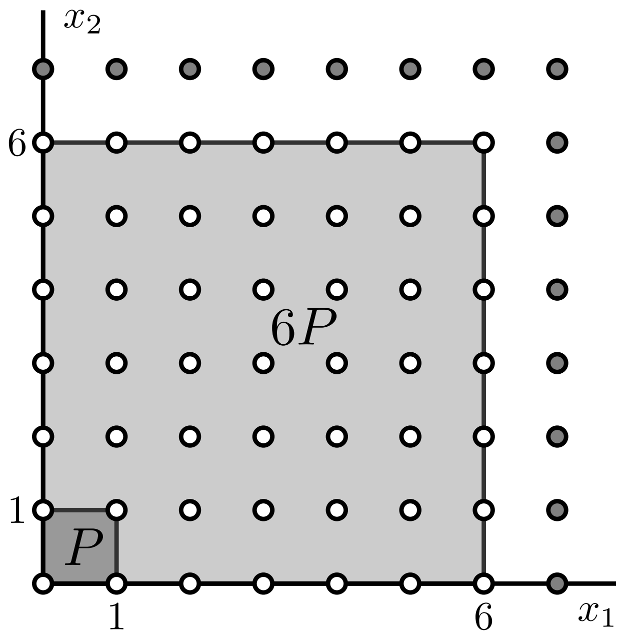


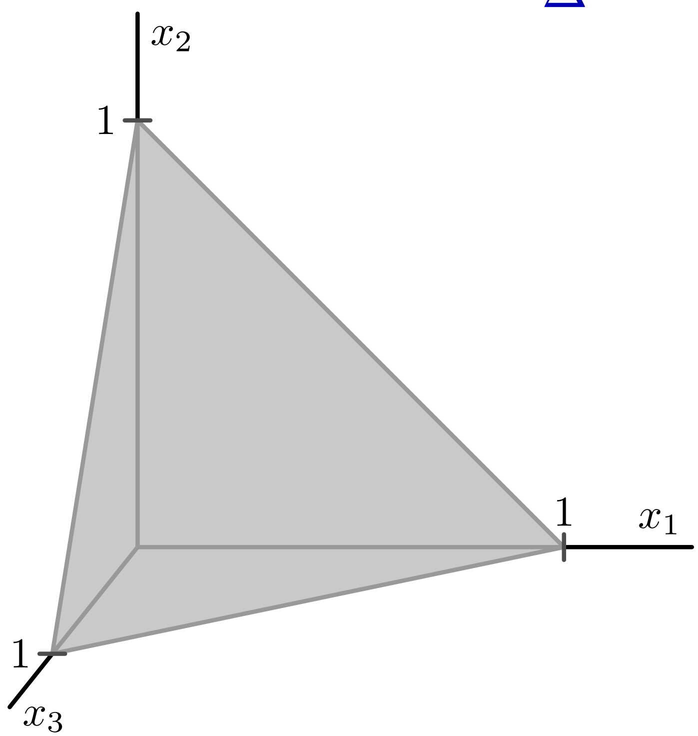
$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$

$L_{\mathcal{P}^\circ}(t) = (t-1)^d$

The Standard Simplex

The standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta = \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d \leq 1 \right\}$$



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$$\begin{aligned} L_\Delta(t) &= \# \left\{ (x_1, x_2 \ldots, x_d) \in \mathbb{Z}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d \leq t \right\} \\ &= \# \left\{ (x_1, x_2 \ldots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1}_{\geq 0} : \, x_1 + x_2 + \cdots + x_{d+1} = t \right\} \\ &= \binom{d+t}{d} \end{aligned}$$

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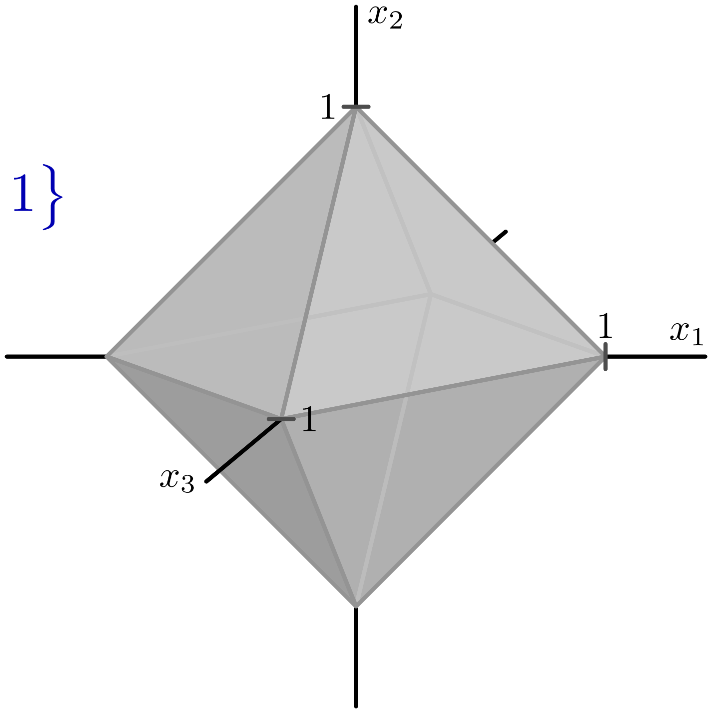
&=\; \# \left\{(x_1, x_2 \dots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1}_{\geq 0} : \, x_1 + x_2 + \cdots + x_{d+1} = t\right\} \\
&=\; \binom{d+t}{d}
\end{aligned}$$

$$L_{\Delta^\circ}(t) \;=\; \binom{t-1}{d}$$

The Cross-Polytope

The cross-polytope $\diamond \in \mathbb{R}^d$ is

$$\diamond \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^d : \, |x_1| + |x_2| + \cdots + |x_d| \leq 1 \right\}$$



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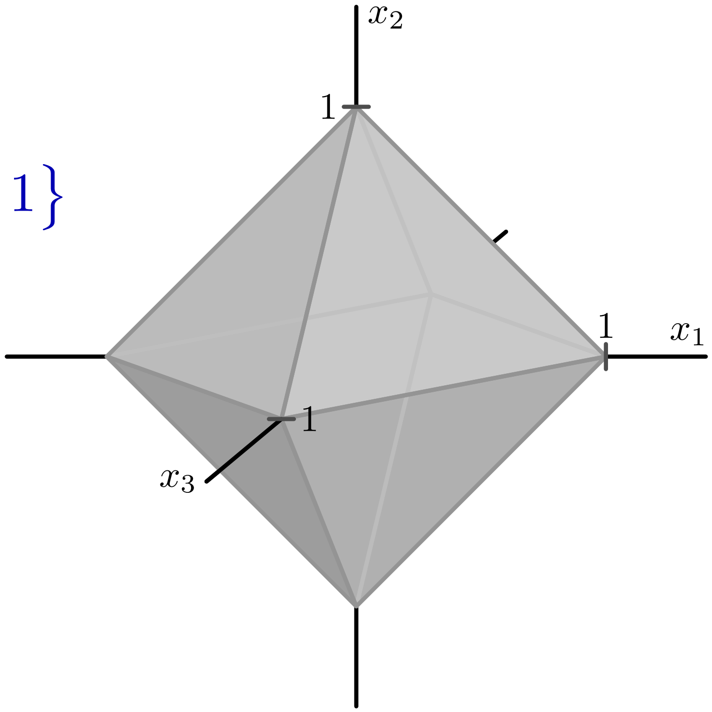

Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Triangulation
- Disjoint triangulation
- Interpolation
- Generating function

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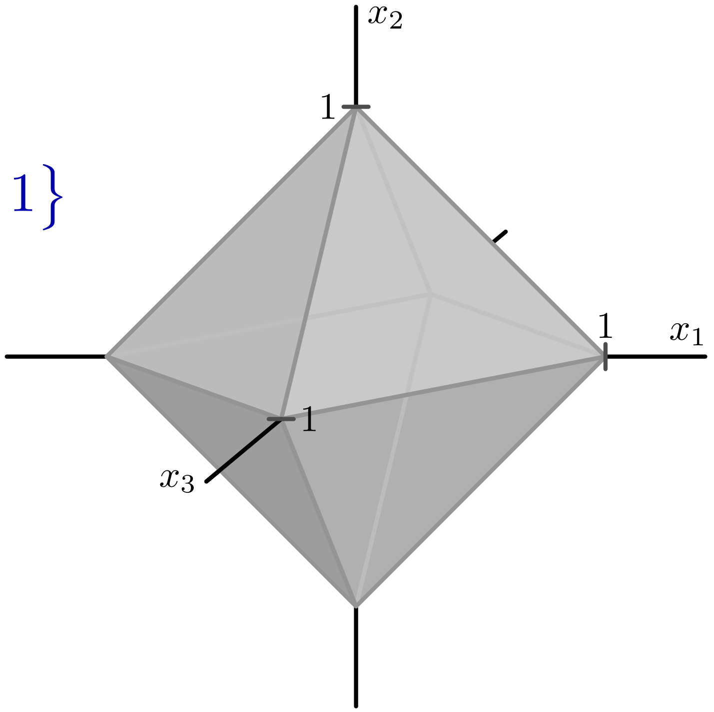

Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Triangulation

Dissect $\diamond$ into 8 (standard) tetrahedra and use inclusion–exclusion to compute $L_\diamond(t)$

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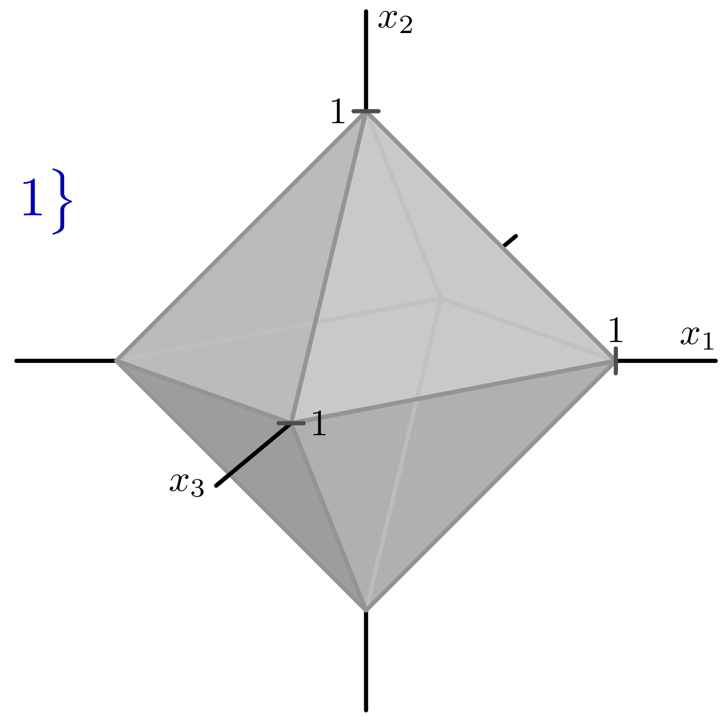

Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Disjoint triangulation

Dissect $\diamond$ into 8 half-open tetrahedra

The Cross-Polytope

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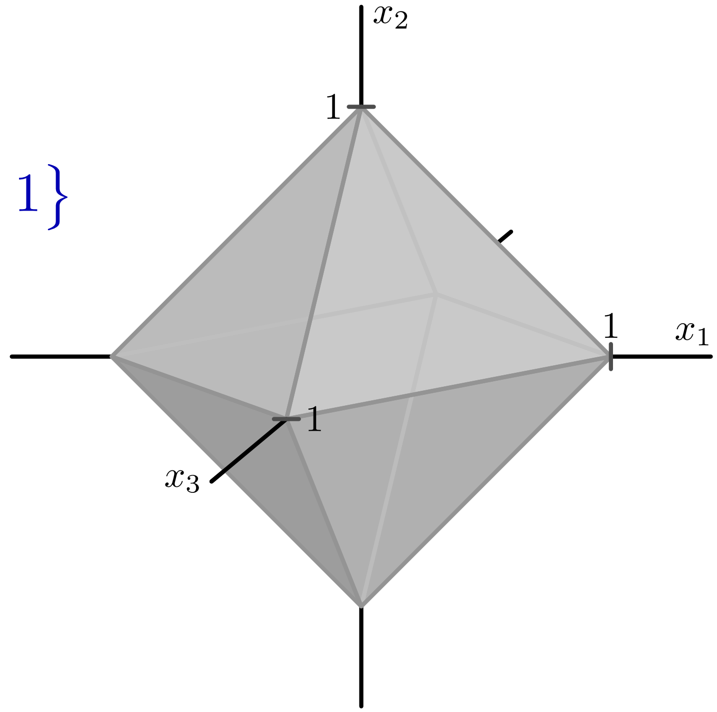

Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Interpolation

```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```

The Cross-Polytope

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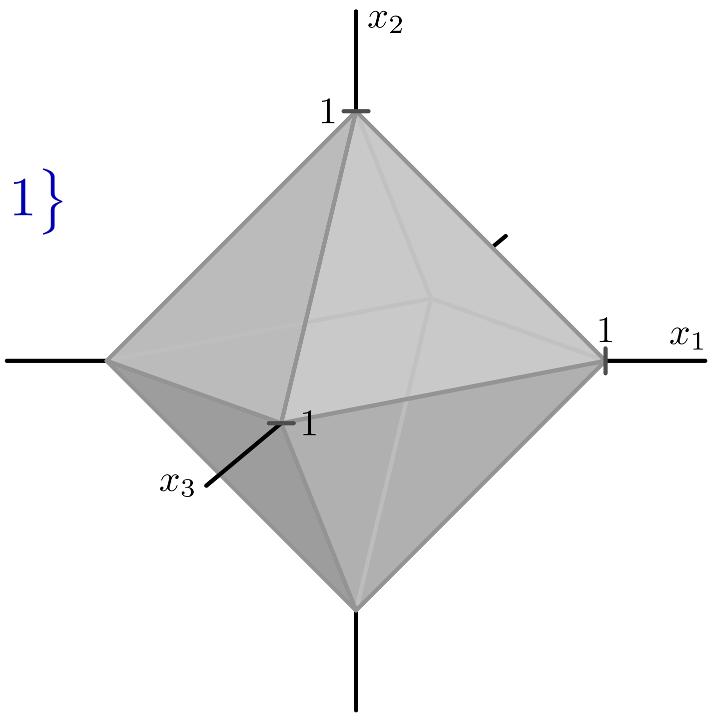


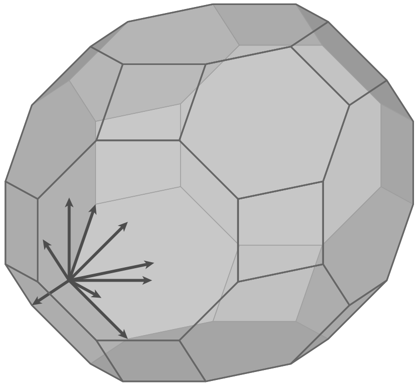
Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Generating function

$$\mathrm{Ehr}_\mathcal{P}(z) \;:=\; 1 + \sum_{t \ge 1} L_\mathcal{P}(t)\, z^t$$

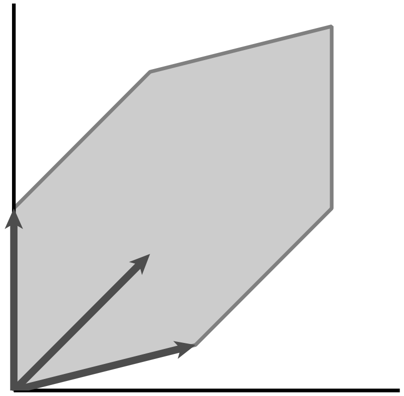
Exercise: $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{P})}(z) \;=\; \dfrac{1+z}{1-z}\,\mathrm{Ehr}_\mathcal{P}(z)$

Zonotopes



Line segment $[\boldsymbol{a}, \boldsymbol{b}] := \{(1-\lambda)\,\boldsymbol{a} + \lambda\,\boldsymbol{b} :\ 0 \le \lambda \le 1\}$

Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2 \ := \ \{\boldsymbol{p} + \boldsymbol{q} :\ \boldsymbol{p} \in \mathcal{K}_1,\ \boldsymbol{q} \in \mathcal{K}_2\}$



Zonotope $\mathcal{Z} := [\boldsymbol{a}_1, \boldsymbol{b}_1] + [\boldsymbol{a}_2, \boldsymbol{b}_2] + \cdots + [\boldsymbol{a}_m, \boldsymbol{b}_m]$

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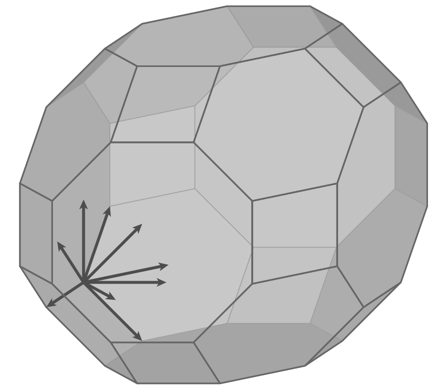

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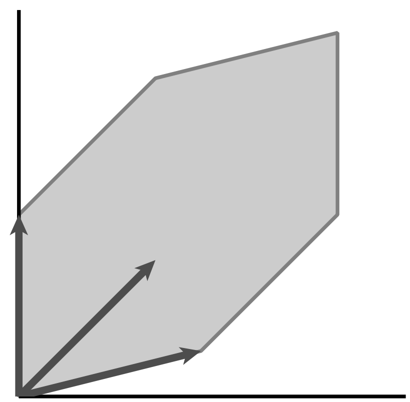

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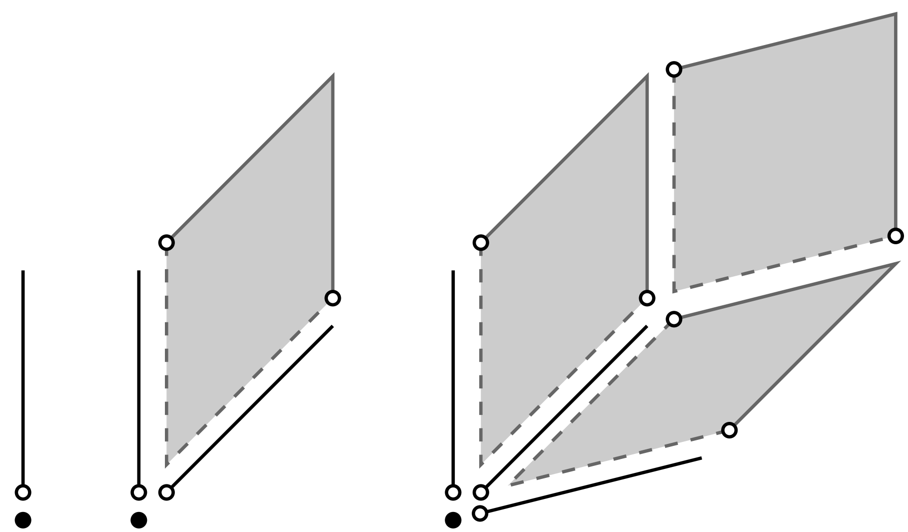
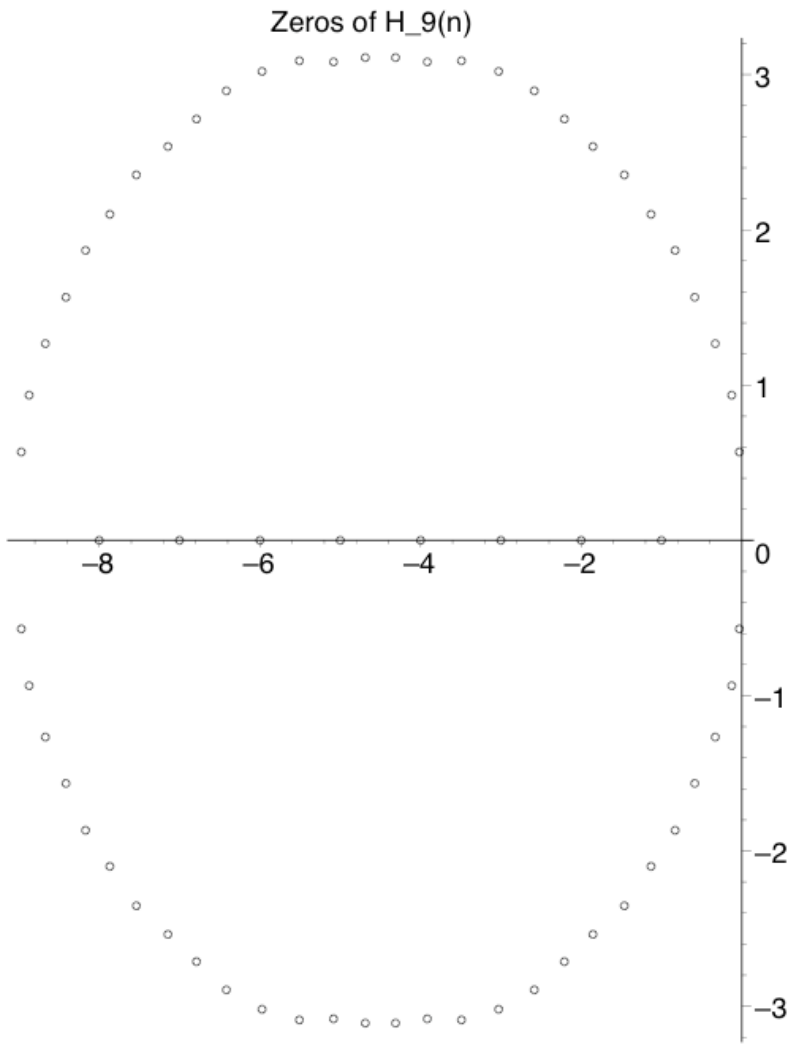
Every zonotope admits a tiling into parallelepipeds



$\mathcal{P}$ — half-open d-parallelepiped

$\longrightarrow \ L_{\mathcal{P}}(t) \ = \ t^d$

Birkhoff–von Neumann Revisited



For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).

Recap Day I

- Volume computations $\longrightarrow$ don't agonize, discretize
- Integer-point counting in dilated polytopes $\longrightarrow$ polynomials
- Interpolation
- Generating functions
- Dissections: triangulations, tilings
- Tomorrow: enough practice, how does this work in theory?