

Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day II: Generating Functions & Complexity

- (1) Compute the generating functions for

$$P_{\leq n} := \{\lambda \in \mathbb{Z}^n : 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\},$$

partitions into at most n parts. Adjust your computations for partitions into exactly n parts.

- (2) Let $n \geq 3$ and

$$T_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\},$$

the set of all “ n -gon partitions.”

- (a) Compute the generating function for T_3 .
- (b) What makes your computation more complicated for $n > 3$?
- (c) Compute the generating function for

$$\tilde{T}_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \leq \lambda_n\},$$

and conclude from it the generating function for T_n .

- (3) Compute the integer-point transform of the cone $\mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.
- (4) Give an explicit bijection between the faces (including \emptyset) of a given polytope \mathcal{P} and the faces (excluding \emptyset) of its homogenization cone(\mathcal{P}).
- (5) Pick five points in \mathbb{Z}^3 and let \mathcal{P} be their convex hull (in \mathbb{R}^3). Compute the Ehrhart polynomial of \mathcal{P} .
- (6) Give an example of a polynomial $f(n)$ with (some) negative coefficients whose corresponding generating function numerator polynomial $h(z)$ has only positive coefficients.
- (7) For a lattice polytope \mathcal{P} , the numerator of the generating function is the h^* -polynomial of \mathcal{P} . Give a non-unimodal² example of an h^* -polynomial.
- (8) [research problem] Now let $\mathcal{P} = \{\mathbf{x} \in [0, 1]^d : x_1 + x_2 + \cdots + x_d = k\}$, for your favorite integers $2 \leq k \leq d - 2$. (This is the (d, k) -hypersimplex.) Prove that the h^* -polynomial of \mathcal{P} is unimodal.

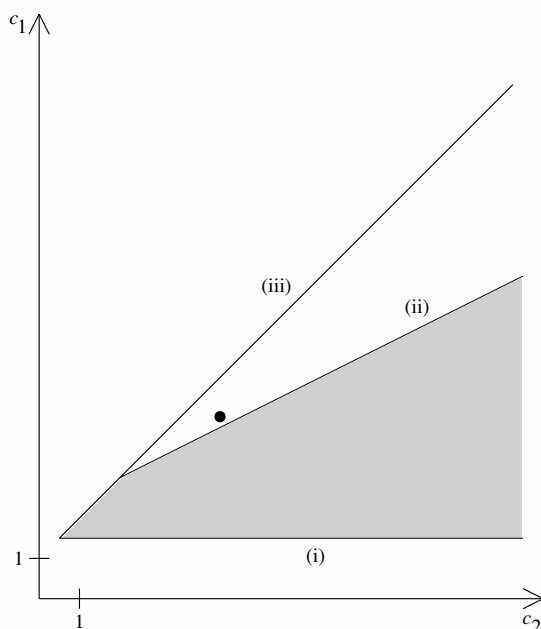
²A polynomial is *unimodal* if its coefficients increase up to some point and then decrease.

- (9) Suppose $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$ are lattice polytopes. Prove that the *convolution* of their Ehrhart polynomials,

$$L(t) := \sum_{s=0}^t L_{\mathcal{P}}(s) L_{\mathcal{Q}}(t-s)$$

equals the Ehrhart quasipolynomial of the polytope given by the convex hull of $\mathcal{P} \times \{\mathbf{0}_n\} \times \{0\}$ and $\{\mathbf{0}_m\} \times \mathcal{Q} \times \{1\}$. Here $\mathbf{0}_d$ denotes the origin in \mathbb{R}^d .

- (10) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials $c_2 t^2 + c_1 t + 1$: every half-integral point in the figure below corresponds to an Ehrhart polynomial.



- (11) [research problem] Give the corresponding classification picture of degree-3 Ehrhart polynomials.
- (12) This exercise constructs triangulations. Given a polytope $\mathcal{P} \subseteq \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, randomly choose $h_1, h_2, \dots, h_n \in \mathbb{R}$, and define the new polytope $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ as the convex hull of $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_n, h_n)$. The *lower hull* of \mathcal{Q} consists of all points that are *visible from below*: all points $(x_1, x_2, \dots, x_{d+1}) \in \mathcal{Q}$ for which there is no $\epsilon > 0$ such that $(x_1, x_2, \dots, x_{d+1} - \epsilon) \in \mathcal{Q}$. A *lower face* of \mathcal{Q} is a face of \mathcal{Q} that is in the lower hull. Let $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ be the projection that forgets the last coordinate. Show that all lower faces of \mathcal{Q} are simplices, and that their projections under π form a triangulation of \mathcal{P} .