Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

Day I: Appetizers

(1) Given integers a, b, c, d, form the line segment $[(a, b), (c, d)] \subset \mathbb{R}^2$ joining the points (a, b) and (c, d). Show that the number of integer points on this line segment is $\gcd(a-c, b-d)+1$.

(2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area $\frac{1}{2}$. (*Hint:* You may begin by "doubling" the triangle to form a parallelogram.)

(3) Pick four points in $\mathbb{Z}^3$ and let $\mathcal{P}$ be their convex hull (in $\mathbb{R}^3$). Compute the Ehrhart polynomial of $\mathcal{P}$. (If you cannot think of a good example, consider the regular tetrahedron with vertices $(0,0,0), (1,1,0), (1,0,1), (0,1,1)$.)

(4) Recall that the standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin. Verify that

$$L_\Delta(t) = \binom{d+t}{d} \qquad \text{and} \qquad L_{\Delta^\circ}(t) = \binom{t-1}{d}.$$

(If you'd like to amuse your colleagues, we can also write $L_{\Delta^\circ}(t) = (-1)^d \binom{d-t}{d}$.)

(5) Given a $(d-1)$-polytope $\mathcal{Q}$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ such that the origin is in $\mathcal{Q}$, we define the bipyramid $\mathrm{BiPyr}(\mathcal{Q})$ over $\mathcal{Q}$ as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \ldots, (\mathbf{v}_m, 0), (0, \ldots, 0, 1), \text{ and } (0, \ldots, 0, -1).$$

Show that $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{Q})}(z) = \dfrac{1+z}{1-z} \mathrm{Ehr}_{\mathcal{Q}}(z)$.

(6) Compute the Ehrhart polynomial of the octahedron

$$\Diamond = \left\{ \mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1 \right\}$$

via the four different approaches outlined in the lecture:

(a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);
(b) disjoint triangulation into 8 standard tetrahedra;
(c) [`sage`] interpolation;
(d) [`sage`] generating function.

Generalize.

(7) [`sage`] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What's going on here?

(8) [`research problem`] Compute the Ehrhart polynomial of the Birkhoff—von Neumann polytope $\mathcal{B}_{10}$ or the volume of $\mathcal{B}_{11}$.

(9) Define the Eulerian number $A(d,k)$ through[1]

$$\sum_{j\geq 0} j^d\, z^j \;=\; \frac{\sum_{k=0}^d A(d,k)\, z^k}{(1-z)^{d+1}}\,.$$

Alternatively, we may think of the polynomial $\sum_{k=0}^d A(d,k)\, z^k$ is the numerator of the rational function

$$\left(z\frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) \;=\; \underbrace{z\frac{d}{dz}\cdots z\frac{d}{dz}}_{d \text{ times}} \left(\frac{1}{1-z}\right).$$

Prove the following:

$$\begin{aligned} A(d,k) &= A(d,d+1-k)\,,\\ A(d,k) &= (d-k+1)\, A(d-1,k-1) + k\, A(d-1,k)\,,\\ \sum_{k=0}^d A(d,k) &= d!\,,\\ A(d,k) &= \sum_{j=0}^k (-1)^j \binom{d+1}{j} (k-j)^d. \end{aligned}$$

(10) The permutahedron $\mathcal{P}_d \in \mathbb{R}^d$ is defined as the convex hull of

$$\{(\pi(1)-1,\, \pi(2)-1, \ldots,\, \pi(d)-1) :\; \pi \in S_d\}\,,$$

where S_d is the set of all permutations of $\{1,2,\ldots,d\}$. Show that P_d is a zonotope:

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d]\,,$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ are the standard unit vectors.

(11) Prove that $\mathcal{P}_d$ tiles the hyperplane spanned by it.

(12) Show that a sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n\geq 0} f(n)\, z^n \;=\; \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$. Furthermore, $f(n)$ has degree d if and only if $h(1) \neq 0$.

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[1]There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through $\sum_{j\geq 0}(j+1)^d\, z^j = \frac{\sum_{k=0}^d A(d,k) z^k}{(1-z)^{d+1}}$ instead.