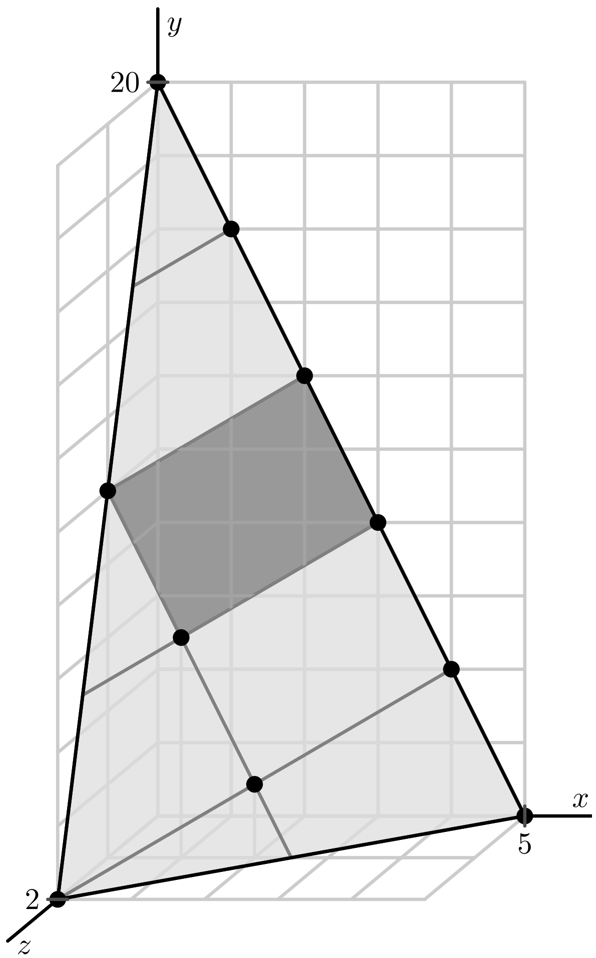
# Ehrhart Polynomials

## Day I: Appetizers




Matthias Beck

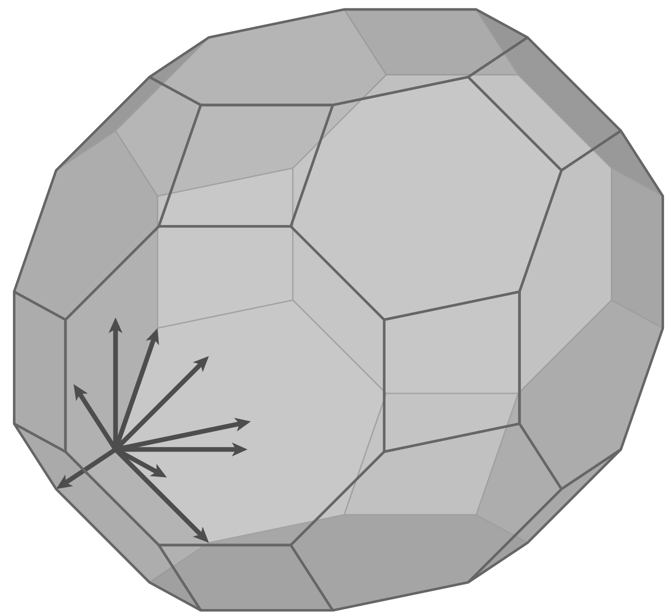
San Francisco State University

https://matthbeck.github.io/

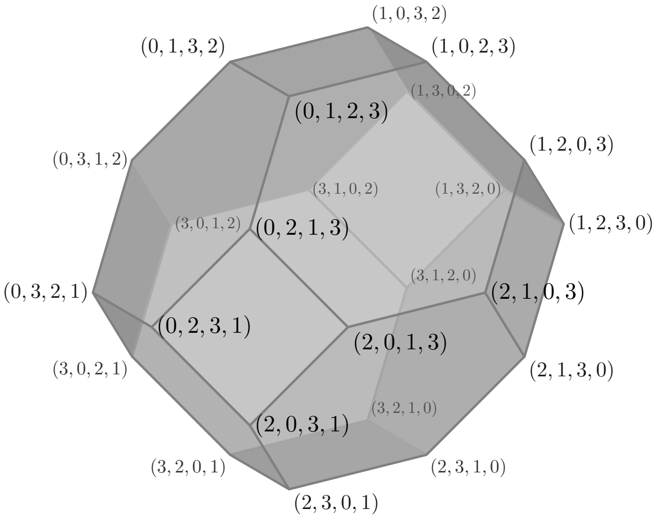
VIII Encuentro Colombiano
De Combinatoria

"Science is what we understand well enough to explain to a computer, art is all the rest."

Donald Knuth



# Themes



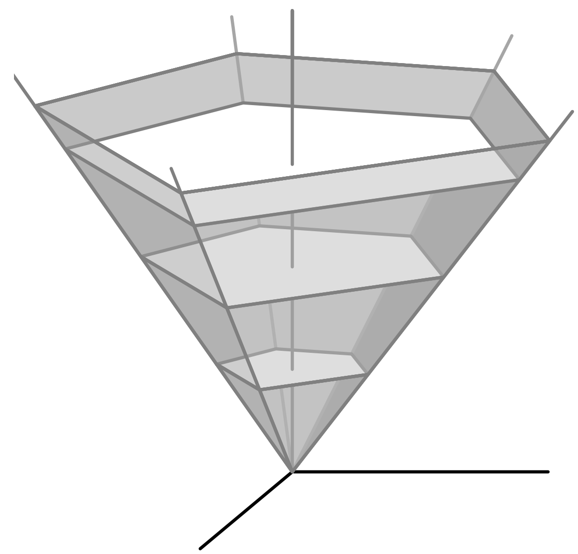

Discrete-geometric polynomials

Computation (complexity)

Generating functions

Combinatorial structures

Polyhedra



# A Sample Problem: Birkhoff–von Neumann Polytope

*This site is supported by donations to The OEIS Foundation.*

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A037302 Normalized volume of Birkhoff polytope of n X n doubly-stochastic square matrices. If the volume is v(n), then a(n) = ((n-1)^2)! * v(n) / n^(n-1). 2

1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028, 12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099, 5091038988117504946842559205930853037841762820367901333706255223000 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

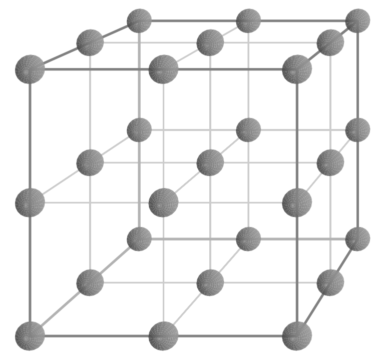
COMMENTS The Birkhoff polytope is an (n-1)^2-dimensional polytope in n^2-dimensional space; its vertices are the n! permutation matrices.
Is a(n) divisible by n^2 for all n>=4? - Dean Hickerson, Nov 27 2002

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geq 0} \ : \ \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

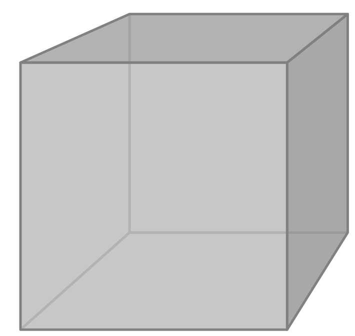
# Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ ...



- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$
- (count) $\left|\mathcal{P} \cap \mathbb{Z}^d\right|$
- (volume) $\displaystyle\mathrm{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left|\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right|$



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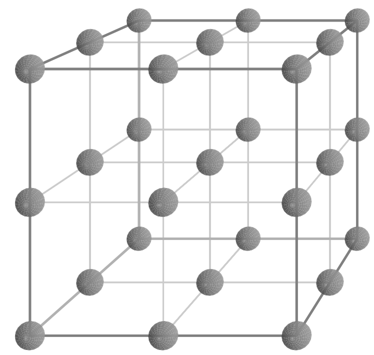

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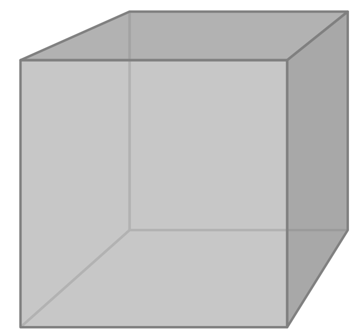

Ehrhart function $L_{\mathcal{P}}(t) := \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right| = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|$ for $t \in \mathbb{Z}_{>0}$

# Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

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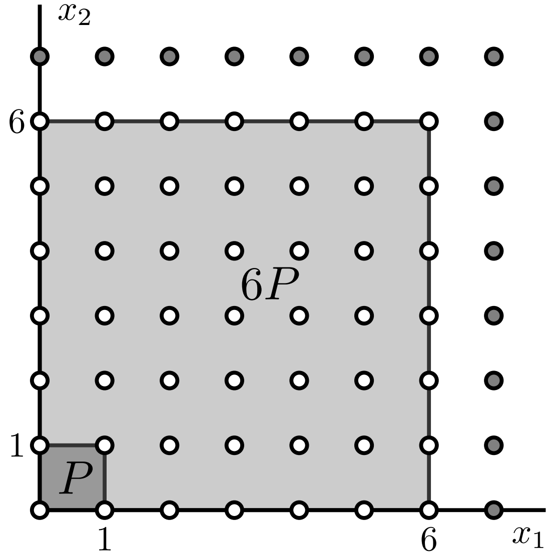
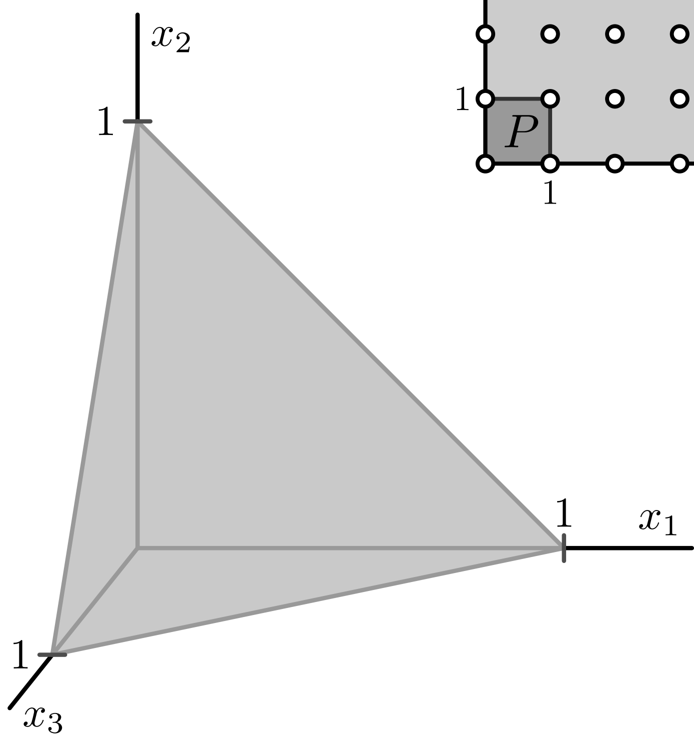
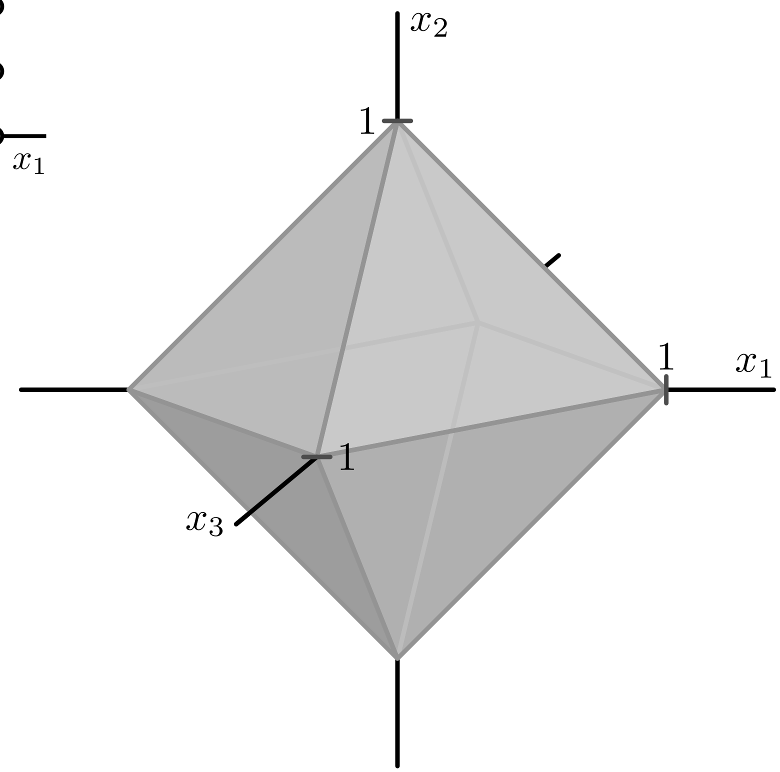
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Also, polytopes are cool.

# Today's Menu: Get Our Hands Dirty



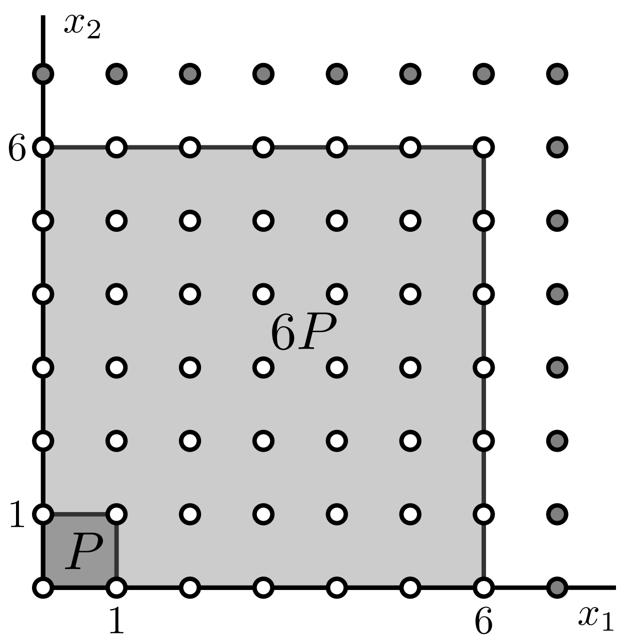



# The Unit Cube

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$

The unit cube in $\mathbb{R}^d$ is $\mathcal{P} = [0,1]^d = \left\{ \boldsymbol{x} \in \mathbb{R}^d : 0 \le x_j \le 1 \right\}$




$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$

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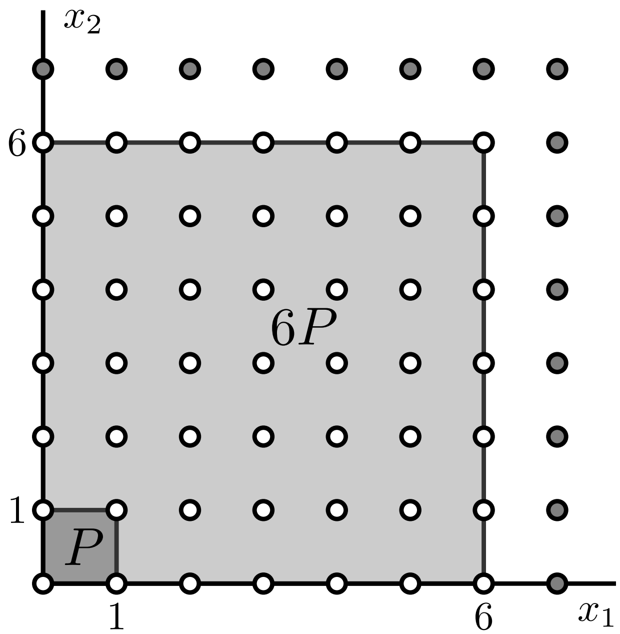


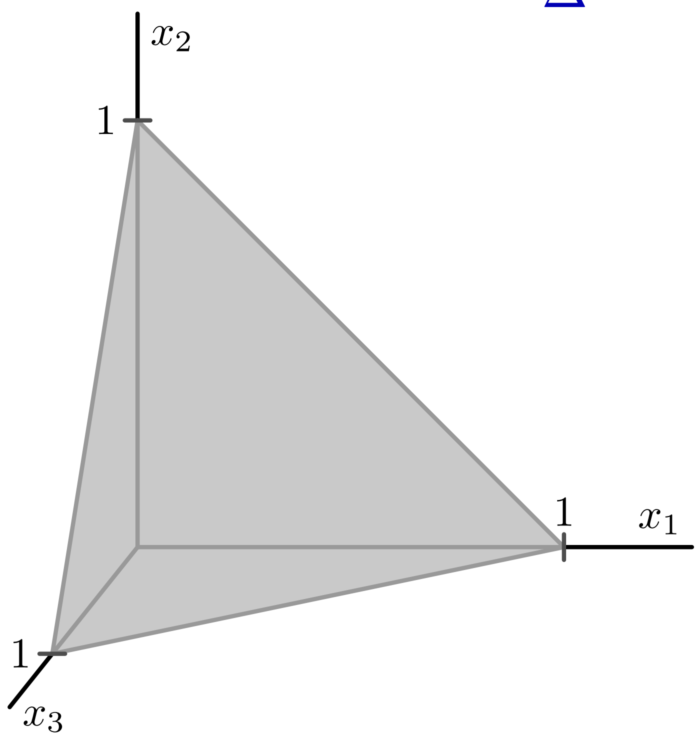

$$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$$

$$L_{\mathcal{P}^\circ}(t) = (t-1)^d$$

# The Standard Simplex

The standard simplex $\Delta \in \mathbb{R}^d$ is the convex hull of the unit vectors and the origin; alternatively,

$$\Delta = \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d \leq 1 \right\}$$



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$$\begin{aligned} L_\Delta(t) &= \# \left\{ (x_1, x_2 \ldots, x_d) \in \mathbb{Z}^d_{\ge 0} : \, x_1 + x_2 + \cdots + x_d \le t \right\} \\ &= \# \left\{ (x_1, x_2 \ldots, x_d, x_{d+1}) \in \mathbb{Z}^{d+1}_{\ge 0} : \, x_1 + x_2 + \cdots + x_{d+1} = t \right\} \\ &= \binom{d+t}{d} \end{aligned}$$

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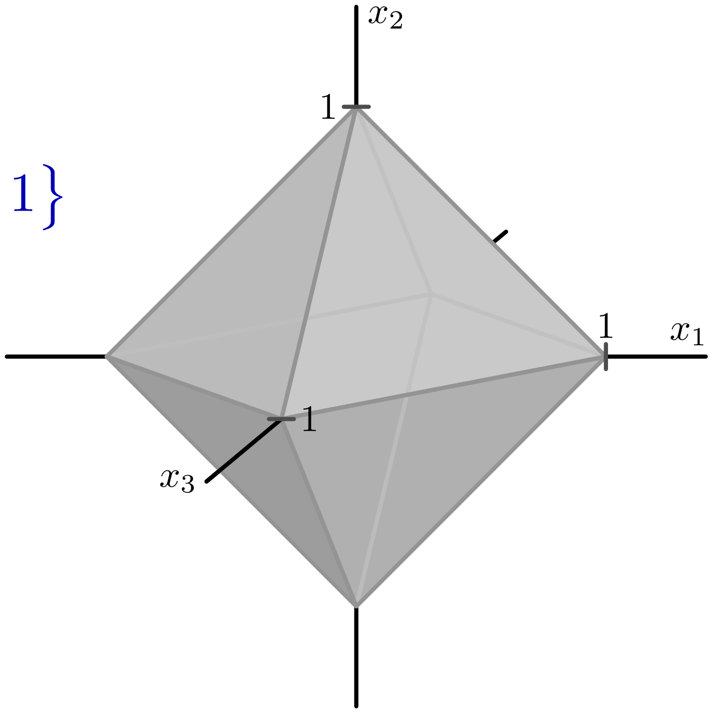
$$L_{\Delta^\circ}(t) \;=\; \binom{t-1}{d}$$

# The Cross-Polytope

The cross-polytope $\diamond \in \mathbb{R}^d$ is

$$\diamond \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^d : \, |x_1| + |x_2| + \cdots + |x_d| \le 1 \right\}$$



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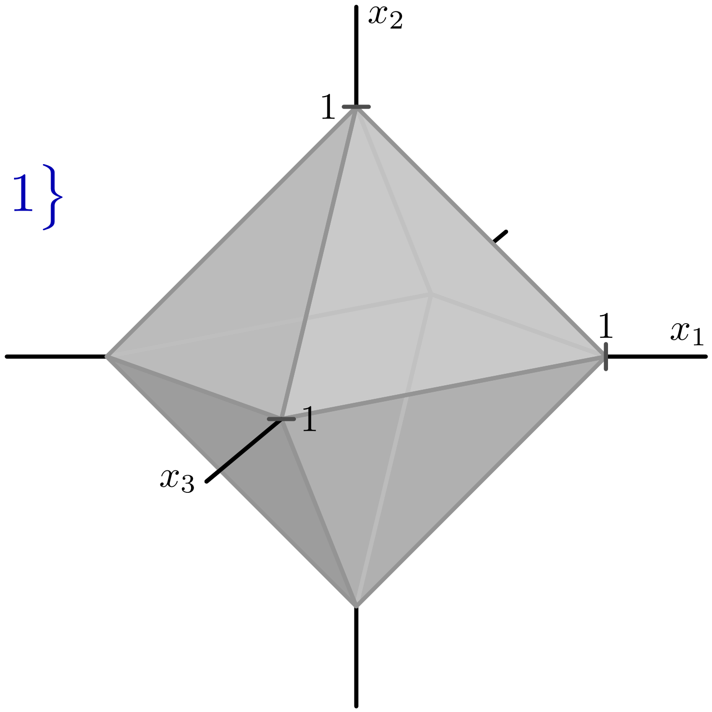


Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Triangulation
- Disjoint triangulation
- Interpolation
- Generating function

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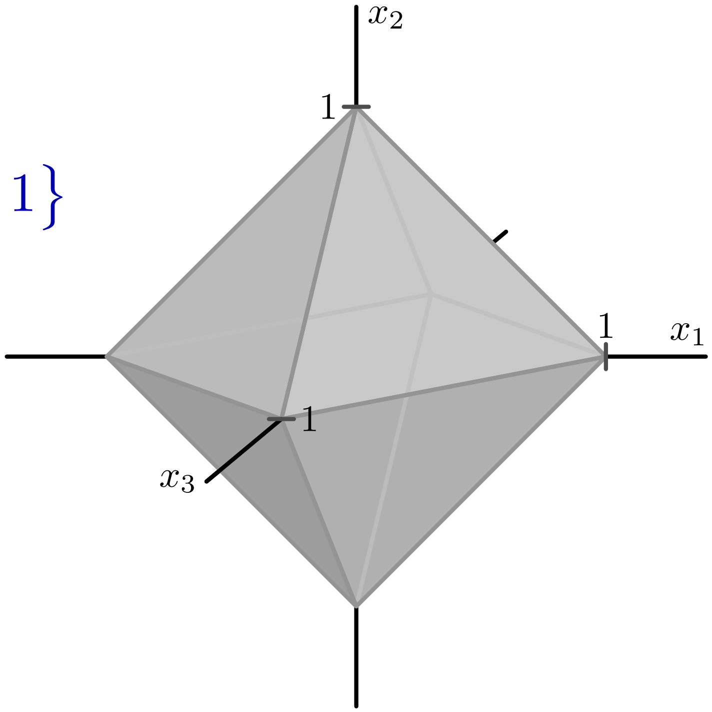


Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Triangulation

Dissect $\diamond$ into $8$ (standard) tetrahedra and use inclusion–exclusion to compute $L_\diamond(t)$

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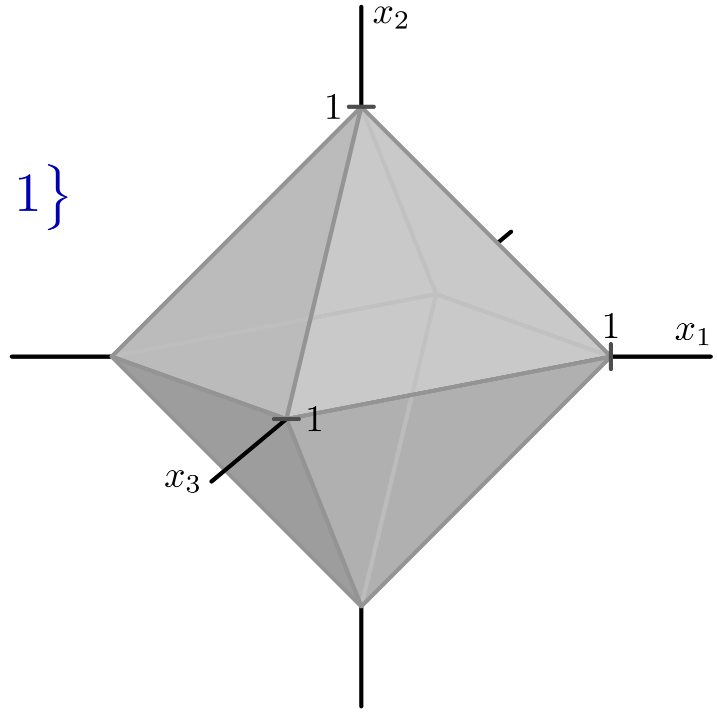


Let's compute $L_\diamond(t)$ for $d = 3$ . . .

- Disjoint triangulation

Dissect $\diamond$ into $8$ half-open tetrahedra

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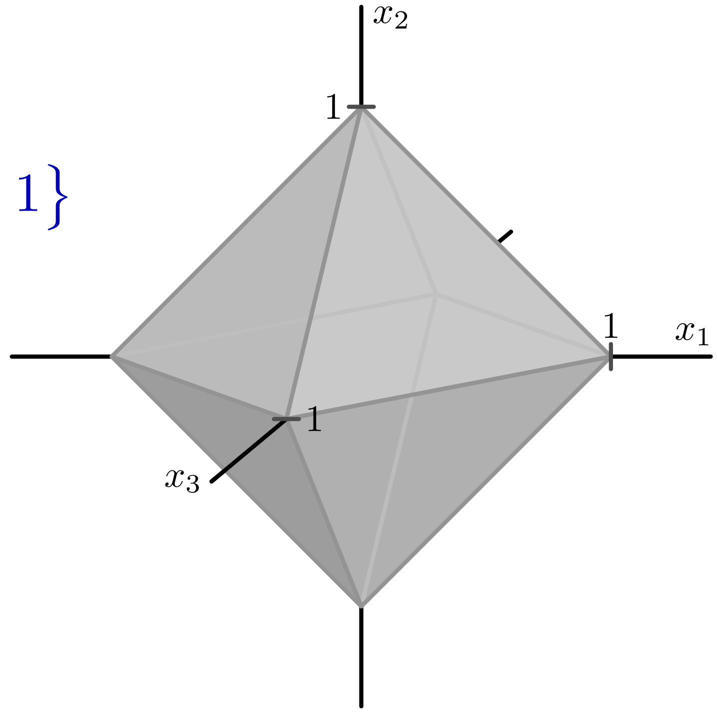


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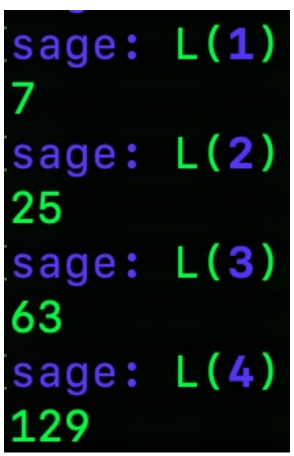
- Interpolation



```
sage: L(1)
7
sage: L(2)
25
sage: L(3)
63
sage: L(4)
129
```

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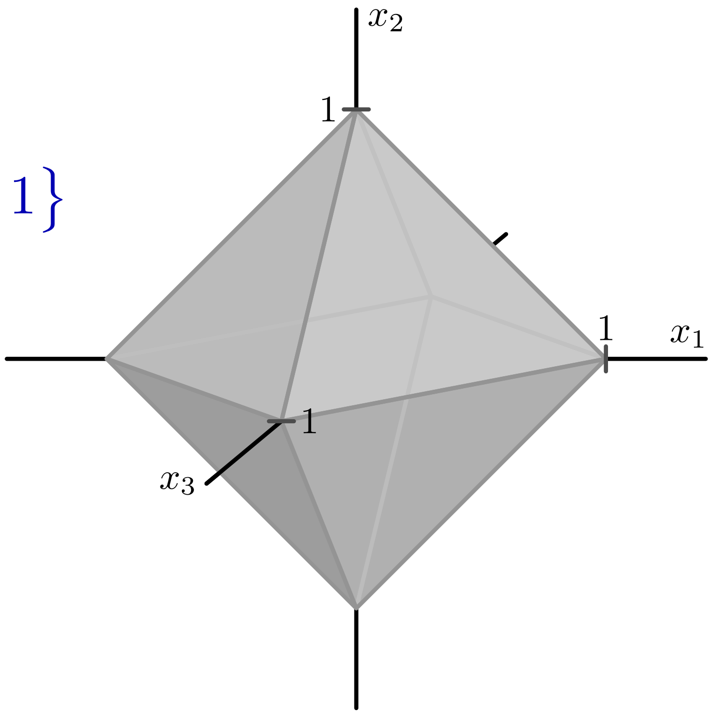



Let's compute $L_\diamond(t)$ for $d = 3$ . . .

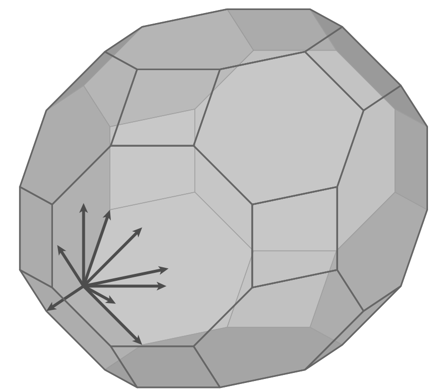
- Generating function

$$\mathrm{Ehr}_{\mathcal{P}}(z) \ := \ 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t)\, z^t$$

Exercise: $\mathrm{Ehr}_{\mathrm{BiPyr}(\mathcal{P})}(z) \ = \ \dfrac{1+z}{1-z}\,\mathrm{Ehr}_{\mathcal{P}}(z)$

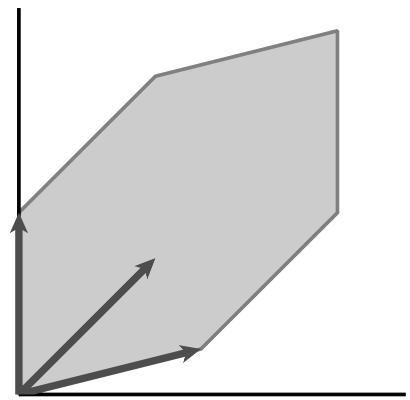
. . . for unit cubes $\longrightarrow$ Eulerian polynomials

# Zonotopes



Line segment $[\boldsymbol{a}, \boldsymbol{b}] := \{(1-\lambda)\,\boldsymbol{a} + \lambda\,\boldsymbol{b} : \ 0 \le \lambda \le 1\}$

Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2 \ := \ \{\boldsymbol{p} + \boldsymbol{q} : \ \boldsymbol{p} \in \mathcal{K}_1, \ \boldsymbol{q} \in \mathcal{K}_2\}$



Zonotope $\mathcal{Z} := [\boldsymbol{a}_1, \boldsymbol{b}_1] + [\boldsymbol{a}_2, \boldsymbol{b}_2] + \cdots + [\boldsymbol{a}_m, \boldsymbol{b}_m]$

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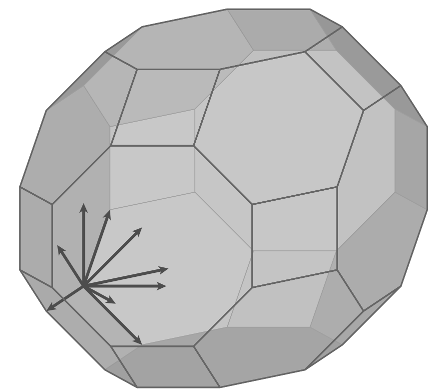

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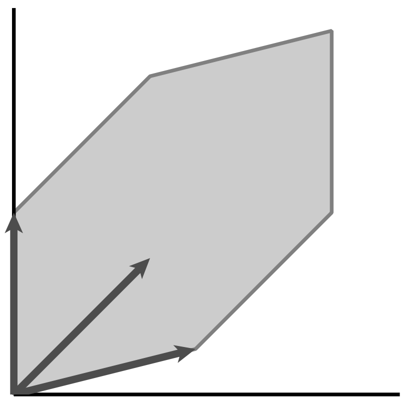
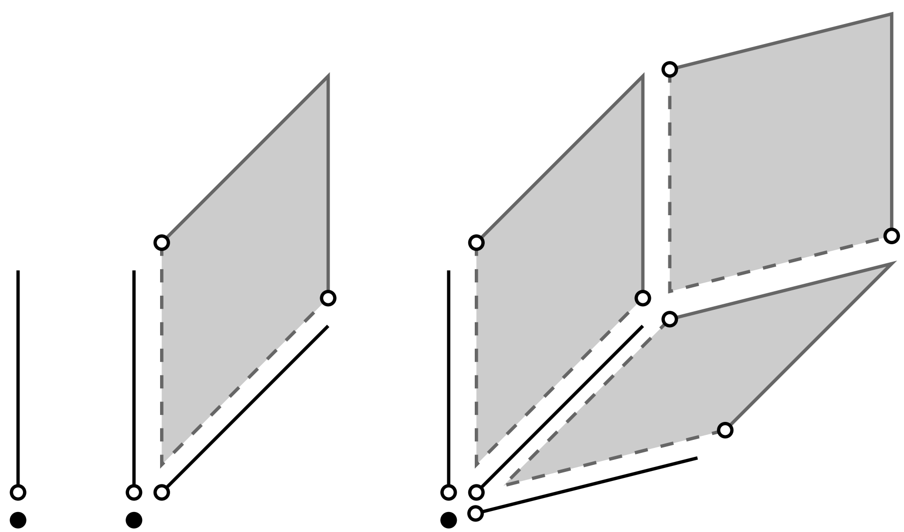


Zonotope $\mathcal{Z} := [\boldsymbol{a}_1, \boldsymbol{b}_1] + [\boldsymbol{a}_2, \boldsymbol{b}_2] + \cdots + [\boldsymbol{a}_m, \boldsymbol{b}_m]$

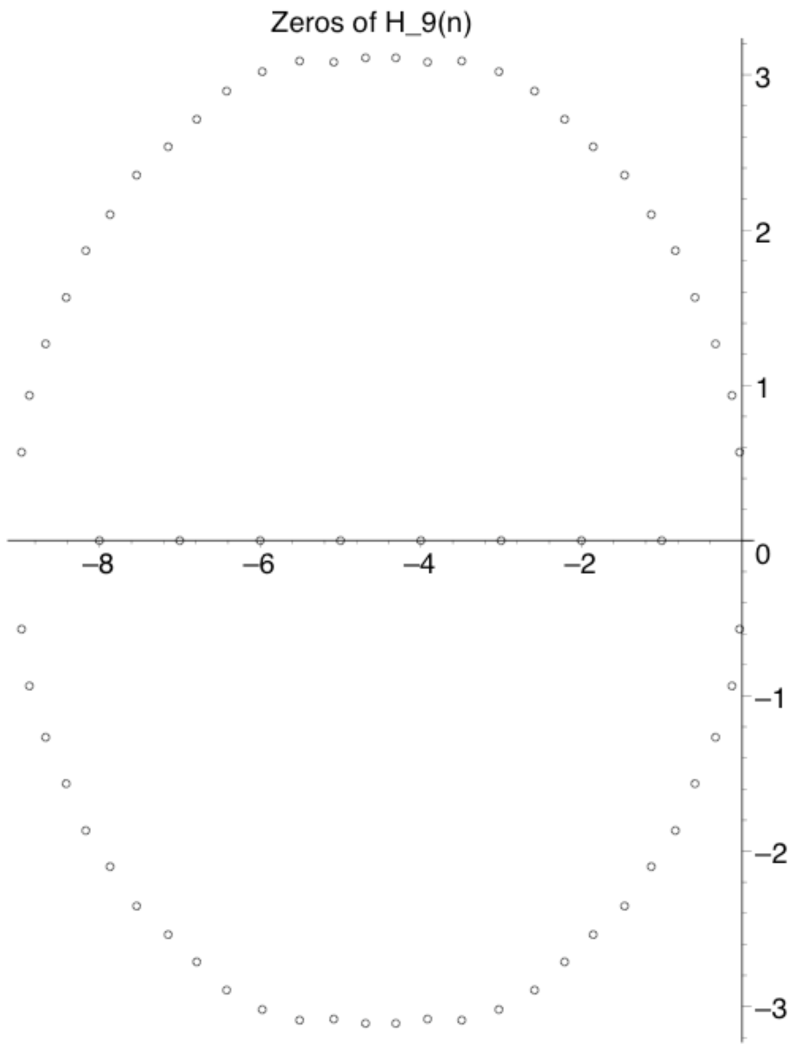
Every zonotope admits a tiling into parallelepipeds



$\mathcal{P}$ — half-open $d$-parallelepiped

$\longrightarrow \; L_{\mathcal{P}}(t) \; = \; t^d$

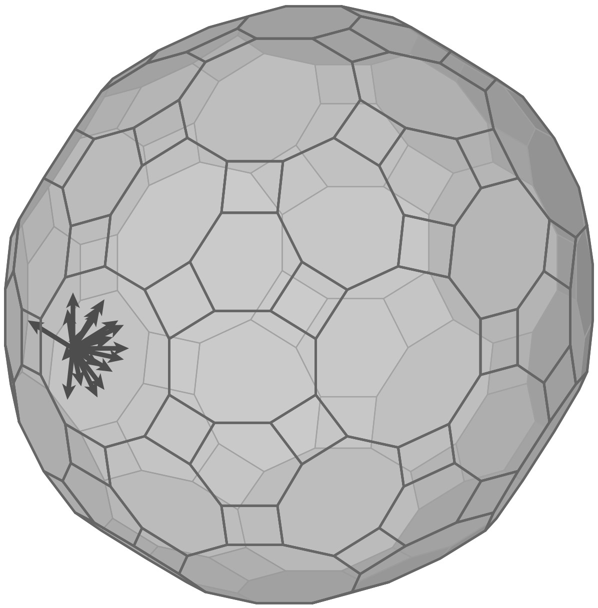
# Birkhoff–von Neumann Revisited




For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).

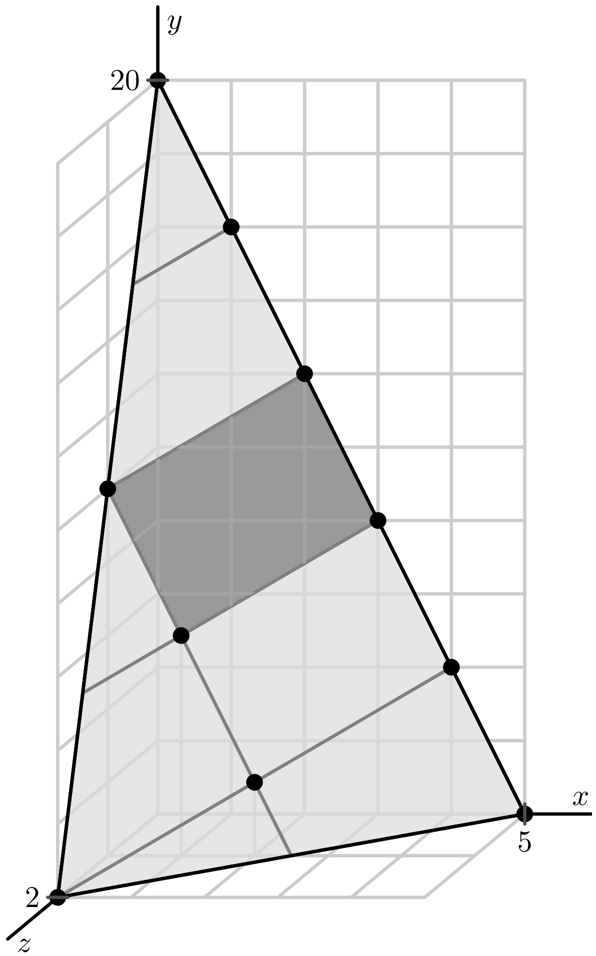
# Recap Day I

- Volume computations $\longrightarrow$ don't agonize, discretize
- Integer-point counting in dilated polytopes $\longrightarrow$ polynomials
- Interpolation
- Generating functions
- Dissections: triangulations, tilings
- Tomorrow: enough practice, how does this work in theory?



# Ehrhart Polynomials

## Day II: Generating Functions & Complexity





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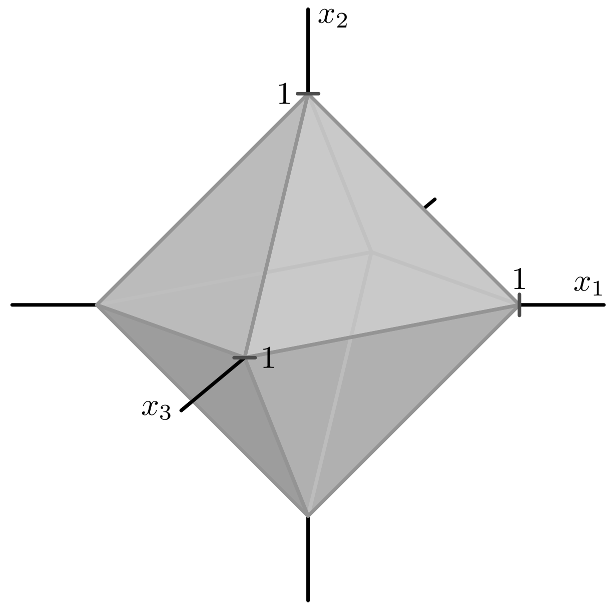
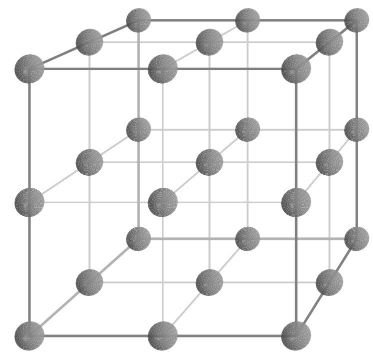
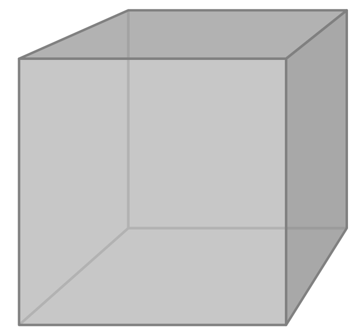
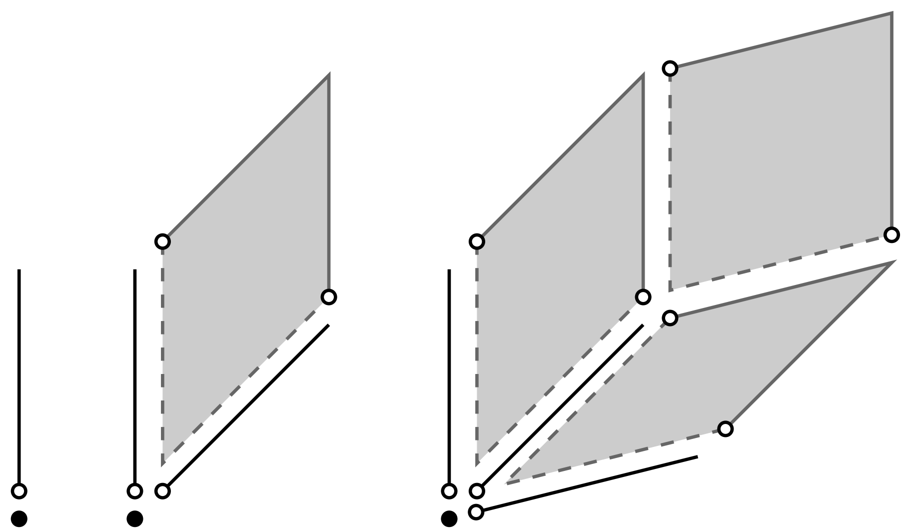
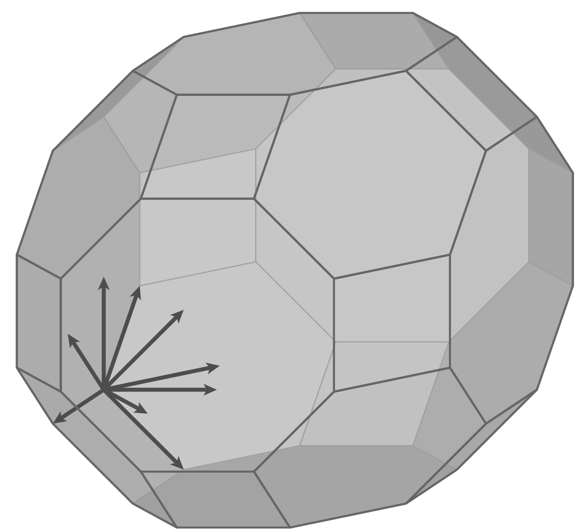
# Any questions about yesterday?









# Warm-Up: Partition Generating Functions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of an integer $k \geq 0$ satisfies

$$k = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \text{and} \qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Goal Compute $\sum_{\lambda} q^{\lambda_1 + \cdots + \lambda_n}$ over your favorite partition family

Example $P_{\leq 3}$ — family of partitions into at most $3$ parts

$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

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$$\sum_{\lambda \in P_{\leq 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} = \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

Idea $P_{\leq 3} = \left\{ \lambda \in \mathbb{Z}^3 : \, 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \right\} = \mathcal{K} \cap \mathbb{Z}^3$

$$\mathcal{K} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \, 0 \leq x_1 \leq x_2 \leq x_3 \right\} \longleftarrow \text{ polyhedral cone } \heartsuit$$

# Warm-Up: Partition Generating Functions

$$\mathcal{K} \;=\; \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \, 0 \le x_1 \le x_2 \le x_3 \right\} \;=\; \mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\ge 0} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a rational, simplicial, unimodular cone

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -1$$

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Integer-point transform

$$\begin{aligned} \sigma_{\mathcal{K}}(z_1, z_2, z_3) &= \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^3} z_1^{m_1} z_2^{m_2} z_3^{m_3} \\ &= \frac{1}{(1 - z_3)(1 - z_2 z_3)(1 - z_1 z_2 z_3)} \end{aligned}$$

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$$\sum_{\lambda \in P_{\le 3}} q^{\lambda_1 + \lambda_2 + \lambda_3} \;=\; \sigma_{\mathcal{K}}(q, q, q) \;=\; \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

# Variations on a Theme

$P_3$ — family of partitions into exactly $3$ parts

$$P_3 \;=\; \left\{ \lambda \in \mathbb{Z}^3 : \, 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \right\} \;=\; \widetilde{\mathcal{K}} \cap \mathbb{Z}^3$$

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# Integer-point Complexity of a Simplicial Cone

What if $\mathcal{K}$ is (still simplicial and rational but) not unimodular?
Say $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{Z}^3$ are linearly independent, $\det[\mathbf{w}_1 \, \mathbf{w}_2 \, \mathbf{w}_3] = D > 1$

$$\mathcal{K} = \mathbb{R}_{\geq 0} \, \mathbf{w}_1 + \mathbb{R}_{\geq 0} \, \mathbf{w}_2 + \mathbb{R}_{\geq 0} \, \mathbf{w}_3$$

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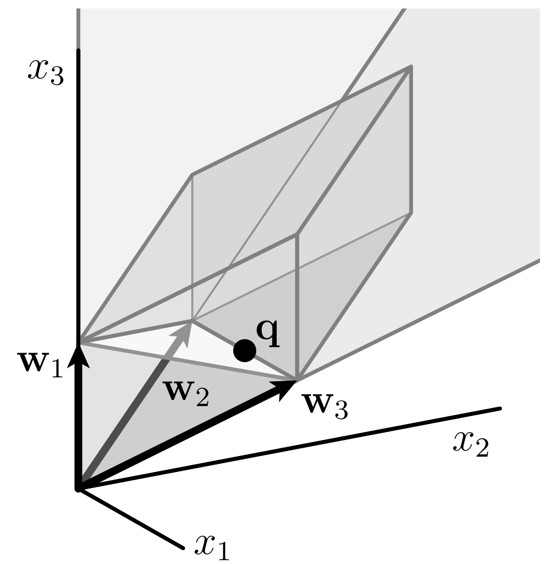
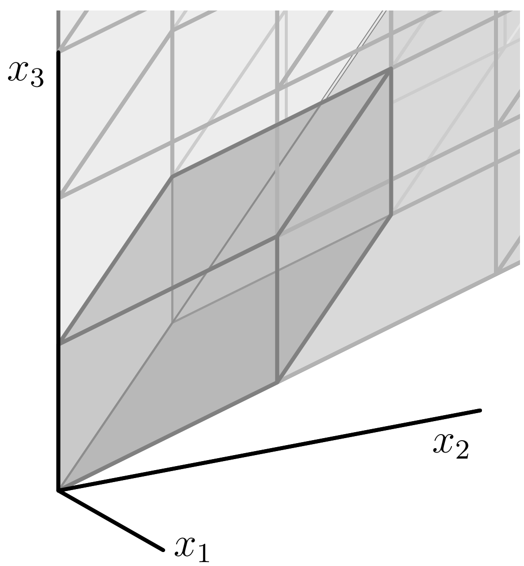



Idea  Tile $\mathcal{K}$ with the half-open parallelepiped
$\Pi = [0,1)\, \mathbf{w}_1 + [0,1)\, \mathbf{w}_2 + [0,1)\, \mathbf{w}_3$



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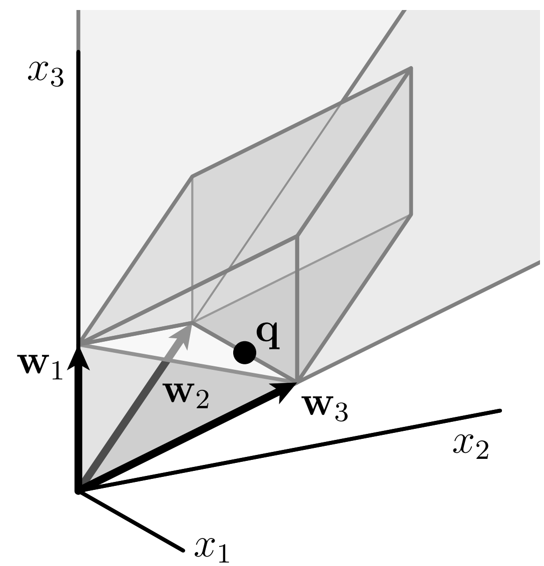


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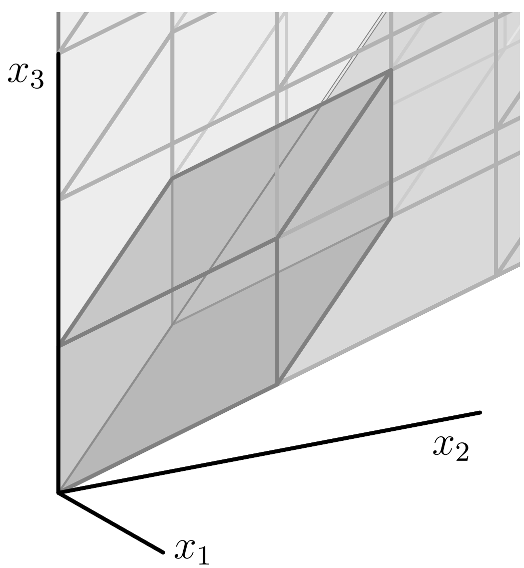


$\downarrow$

$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) = \frac{\sigma_{\Pi}(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

where $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} z_3^{m_3}$

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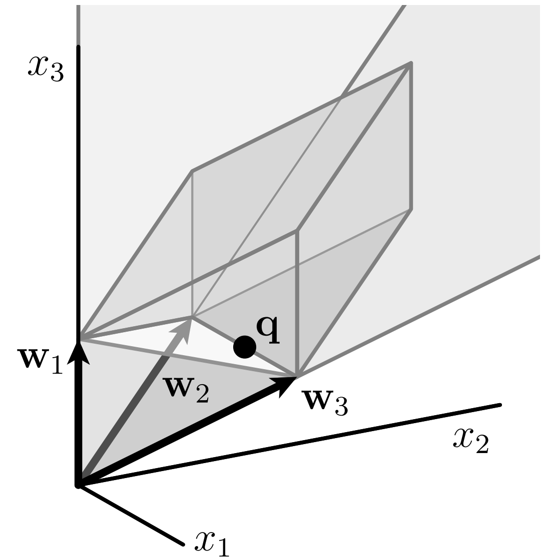


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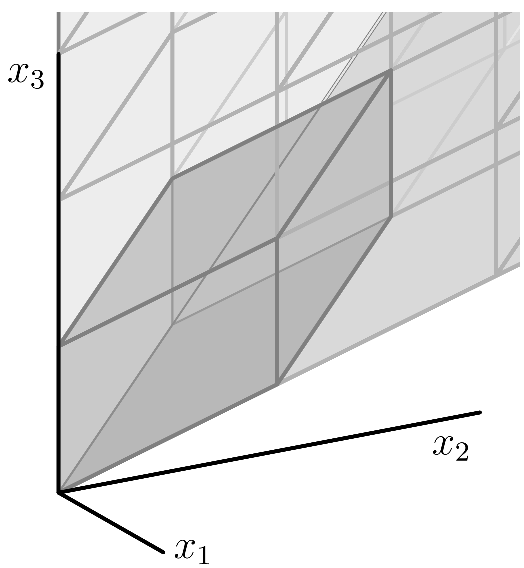


$\downarrow$

$$\sigma_{\mathcal{K}}(z_1, z_2, z_3) =$$

$$\frac{\sigma_\Pi(z_1, z_2, z_3)}{(1 - \mathbf{z}^{\mathbf{w}_1})(1 - \mathbf{z}^{\mathbf{w}_2})(1 - \mathbf{z}^{\mathbf{w}_3})}$$

Complexity: $\sigma_\Pi(z_1, z_2, z_3)$ has $D$ terms

# Ehrhart Polynomials


Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $\dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term $1$.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t$ is rational:

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the $h^*$-polynomial $h^*_{\mathcal{P}}(z)$ satisfies $h^*_{\mathcal{P}}(0) = 1$ and $h^*_{\mathcal{P}}(1) = (\dim \mathcal{P})! \operatorname{vol}(\mathcal{P})$.

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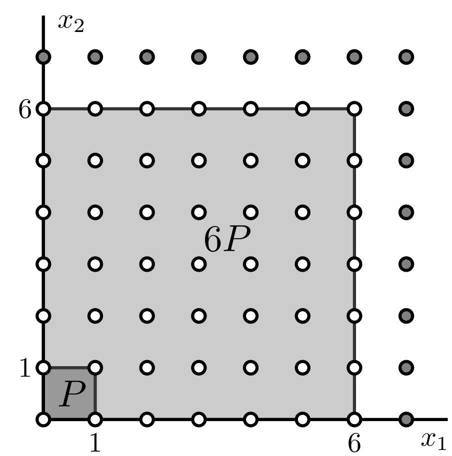
We saw instances yesterday: $\mathcal{P} = [0,1]^d$

$\longrightarrow \quad L_{\mathcal{P}}(t) = (t+1)^d$

$h^*_{\mathcal{P}}(z)$ — Eulerian polynomial



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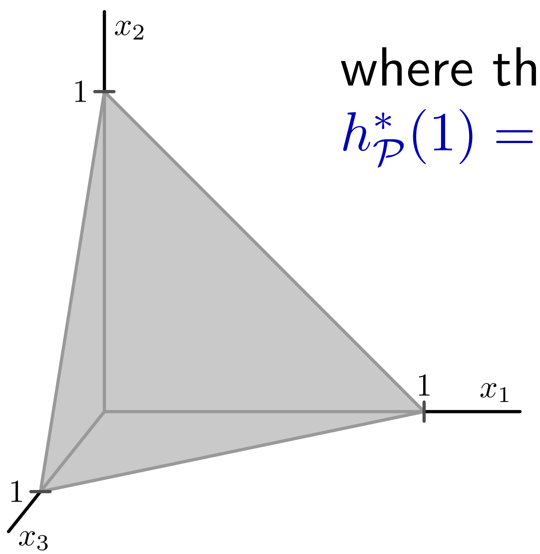


$$\Delta = \left\{ \boldsymbol{x} \in \mathbb{R}^d_{\ge 0} : x_1 + x_2 + \cdots + x_d \le 1 \right\}$$

$$L_\Delta(t) = \binom{d+t}{d} \qquad h^*_\mathcal{P}(z) = 1$$

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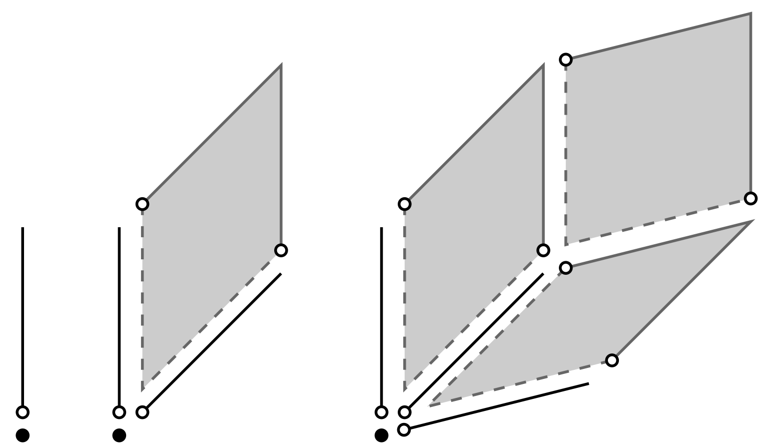
$\mathcal{P}$ — half-open $d$-parallelepiped

$\longrightarrow \quad L_{\mathcal{P}}(t) = t^d$



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Seeming dichotomy: $\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

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**Theorem** (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term $1$.

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Equivalent descriptions of an Ehrhart polynomial:

- $L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$
- via roots of $L_{\mathcal{P}}(t)$
- $\operatorname{Ehr}_{\mathcal{P}}(z) \quad \longrightarrow \quad L_{\mathcal{P}}(t) = h^*_0 \binom{t+d}{d} + h^*_1 \binom{t+d-1}{d} + \cdots + h^*_d \binom{t}{d}$

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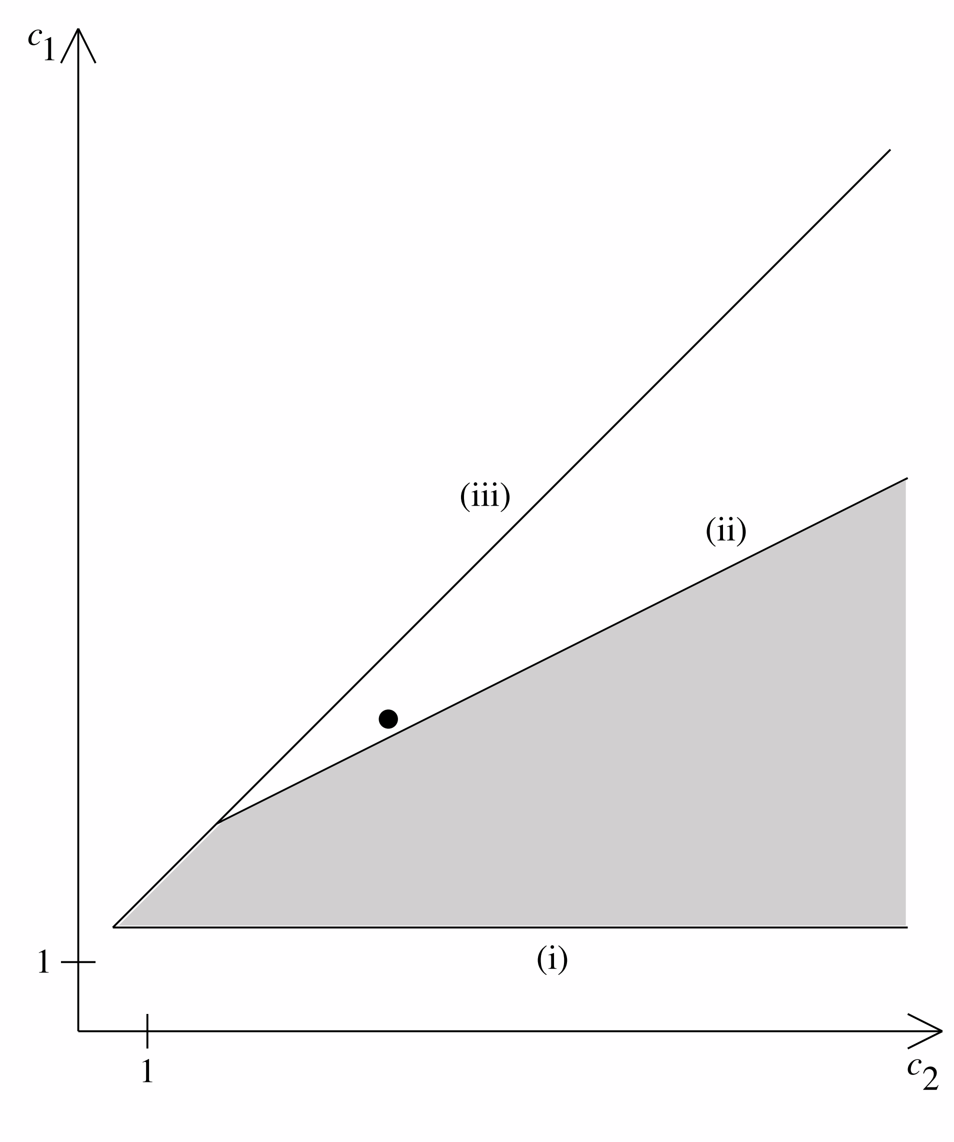
Open Problem Classify Ehrhart polynomials.

# Ehrhart Polynomials in Dimension 2




$\mathcal{P}$ — lattice polygon

$\longrightarrow \quad L_{\mathcal{P}}(t) = c_2\, t^2 + c_1\, t + 1$

# Ehrhart Polynomials


Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term $1$.

$$\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t)\, z^t = \frac{h(z)}{(1-z)^{d+1}}$$

$$\longrightarrow \qquad L_{\mathcal{P}}(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \cdots + h_d^* \binom{t}{d}$$

Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) = h_d^* \binom{t+d-1}{d} + h_{d-1}^* \binom{t+d-2}{d} + \cdots + h_0^* \binom{t-1}{d}$$

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Theorem (Stanley 1980) $h_0^*, h_1^*, \dots, h_d^*$ are nonnegative integers.

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Theorem (Stanley 1980) $h_0^*, h_1^*, \ldots, h_d^*$ are nonnegative integers.

Corollary If $h_{d+1-k}^* > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

# Positivity Among Ehrhart Polynomials


Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in $t$ of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term $1$.

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Theorem (Betke–McMullen 1985, Stapledon 2009) If $h_d^* > 0$ then

$$h(z) = a(z) + z\, b(z)$$

where $a(z) = z^d\, a(\frac{1}{z})$ and $b(z) = z^{d-1}\, b(\frac{1}{z})$ with nonnegative coefficients.

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Open Problem Try to prove the analogous theorem for your favorite combinatorial polynomial with nonnegative coefficients.

# Computational Complexity of Integer-Point Transforms

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

$$\longrightarrow \sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$$ is a rational function in $z_1, z_2, \dots, z_d$

Lenstra (1983) polynomial-time algorithm to decide whether $\sigma_{\mathcal{P}}(\mathbf{z}) = 0$

Barvinok (1994) polynomial-time algorithm to compute $\sigma_{\mathcal{P}}(\mathbf{z})$

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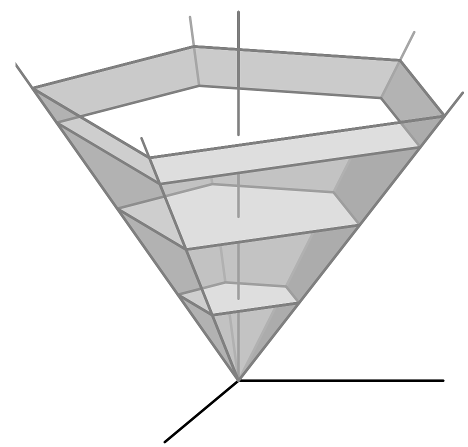
Given a polytope $\mathcal{P}$ we can compute



$\mathrm{Ehr}_{\mathcal{P}}(z) = \sigma_{\mathrm{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$

where $\mathrm{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\})$

## Computational Complexity of Integer-Point Transforms

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Implementations:

De Loera, Köppe et al `www.math.ucdavis.edu/~latte`

Verdoolaege `freshmeat.net/projects/barvinok`

# Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in $t$:

$$L_{\mathcal{P}}(t) \;=\; c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$$

where $c_0(t), \ldots, c_d(t)$ are periodic functions.

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for some (minimal) $p \in \mathbb{Z}_{>0}$ (the period of $L_{\mathcal{P}}(t)$).

Open Problem Study periods of Ehrhart quasipolynomials.