

# Ehrhart Polynomials

## VIII Encuentro Colombiano De Combinatoria

### Day I: Appetizers

- (1) Given integers  $a, b, c, d$ , form the line segment  $[(a, b), (c, d)] \subset \mathbb{R}^2$  joining the points  $(a, b)$  and  $(c, d)$ . Show that the number of integer points on this line segment is  $\gcd(a - c, b - d) + 1$ .
- (2) Prove that a triangle with vertices on the integer lattice has no other interior/boundary lattice points if and only if it has area  $\frac{1}{2}$ . (*Hint*: You may begin by “doubling” the triangle to form a parallelogram.)
- (3) Pick four points in  $\mathbb{Z}^3$  and let  $\mathcal{P}$  be their convex hull (in  $\mathbb{R}^3$ ). Compute the Ehrhart polynomial of  $\mathcal{P}$ . (If you cannot think of a good example, consider the regular tetrahedron with vertices  $(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)$ .)

- (4) Katie likes  $\sum_{k=1}^{\infty} \frac{1}{n^{3k+1}} (x_2)$

Recall that the standard simplex  $\Delta \in \mathbb{R}^d$  is the convex hull of the unit vectors and the origin. Verify that

$$L_{\Delta}(t) = \binom{d+t}{d} \quad \text{and} \quad L_{\Delta^{\circ}}(t) = \binom{t-1}{d}.$$

(If you’d like to amuse your colleagues, we can also write  $L_{\Delta^{\circ}}(t) = (-1)^d \binom{d-t}{d}$ .)

- (5) Given a  $(d-1)$ -polytope  $\mathcal{Q}$  with vertices  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  such that the origin is in  $\mathcal{Q}$ , we define the bipyramid  $\text{BiPyr}(\mathcal{Q})$  over  $\mathcal{Q}$  as the convex hull of

$$(\mathbf{v}_1, 0), (\mathbf{v}_2, 0), \dots, (\mathbf{v}_m, 0), (0, \dots, 0, 1), \text{ and } (0, \dots, 0, -1).$$

Show that  $\text{Ehr}_{\text{BiPyr}(\mathcal{Q})}(z) = \frac{1+z}{1-z} \text{Ehr}_{\mathcal{Q}}(z)$ .

- (6) Compute the Ehrhart polynomial of the octahedron

$$\diamond = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$$

via the four different approaches outlined in the lecture:

- (a) triangulation into 8 standard tetrahedra & their faces (inclusion–exclusion);
- (b) disjoint triangulation into 8 standard tetrahedra;
- (c) [sage] interpolation;
- (d) [sage] generating function.

Generalize.

- (7) [sage] Plot the roots of the Ehrhart polynomials of cross polytopes in different dimensions. What’s going on here?

- (8) [research problem] Compute the Ehrhart polynomial of the Birkhoff—von Neumann polytope  $\mathcal{B}_{10}$  or the volume of  $\mathcal{B}_{11}$ .
- (9) Define the Eulerian number  $A(d, k)$  through<sup>1</sup>

$$\sum_{j \geq 0} j^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}.$$

Alternatively, we may think of the polynomial  $\sum_{k=0}^d A(d, k) z^k$  is the numerator of the rational function

$$\left(z \frac{d}{dz}\right)^d \left(\frac{1}{1-z}\right) = \underbrace{z \frac{d}{dz} \cdots z \frac{d}{dz}}_{d \text{ times}} \left(\frac{1}{1-z}\right).$$

Prove the following:

$$\begin{aligned} A(d, k) &= A(d, d+1-k), \\ A(d, k) &= (d-k+1) A(d-1, k-1) + k A(d-1, k), \\ \sum_{k=0}^d A(d, k) &= d!, \\ A(d, k) &= \sum_{j=0}^k (-1)^j \binom{d+1}{j} (k-j)^d. \end{aligned}$$

- (10) The permutahedron  $\mathcal{P}_d \in \mathbb{R}^d$  is defined as the convex hull of

$$\{(\pi(1)-1, \pi(2)-1, \dots, \pi(d)-1) : \pi \in S_d\},$$

where  $S_d$  is the set of all permutations of  $\{1, 2, \dots, d\}$ . Show that  $\mathcal{P}_d$  is a zonotope:

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d],$$

where  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$  are the standard unit vectors.

- (11) Prove that  $\mathcal{P}_d$  tiles the hyperplane spanned by it.
- (12) Show that a sequence  $f(n)$  is given by a polynomial of degree  $\leq d$  if and only if

$$\sum_{n \geq 0} f(n) z^n = \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial  $h(z)$  of degree  $\leq d$ . Furthermore,  $f(n)$  has degree  $d$  if and only if  $h(1) \neq 0$ .

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<sup>1</sup>There are two slightly conflicting definitions of *Eulerian numbers* in the literature: sometimes, they are defined through  $\sum_{j \geq 0} (j+1)^d z^j = \frac{\sum_{k=0}^d A(d, k) z^k}{(1-z)^{d+1}}$  instead.

# Ehrhart Polynomials

VIII Encuentro Colombiano De Combinatoria

## Day II: Generating Functions & Complexity

- (1) Compute the generating functions for

$$P_{\leq n} := \{\lambda \in \mathbb{Z}^n : 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\},$$

partitions into at most  $n$  parts. Adjust your computations for partitions into exactly  $n$  parts.

- (2) Compute the integer-point transform of the cone  $\mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}_{\geq 0} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ .

- (3) Let  $n \geq 3$  and

$$T_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} > \lambda_n\},$$

the set of all “ $n$ -gon partitions.”

- (a) Compute the generating function for  $T_3$ .  
(b) What makes your computation more complicated for  $n > 3$ ?  
(c) Compute the generating function for

$$\tilde{T}_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_n \geq \cdots \geq \lambda_1 \geq 1 \text{ and } \lambda_1 + \cdots + \lambda_{n-1} \leq \lambda_n\},$$

and conclude from it the generating function for  $T_n$ .

- (4) Recall the lecture-hall partitions

$$\text{LH}_n := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_n}{n} \right\}.$$

Compute the generators of the underlying cone, and verify the first few instances of the Lecture-Hall Theorem:

$$\sum_{\lambda \in \text{LH}_n} q^{\lambda_1 + \cdots + \lambda_n} = \frac{1}{(1-q)(1-q^3) \cdots (1-q^{2n-1})}.$$

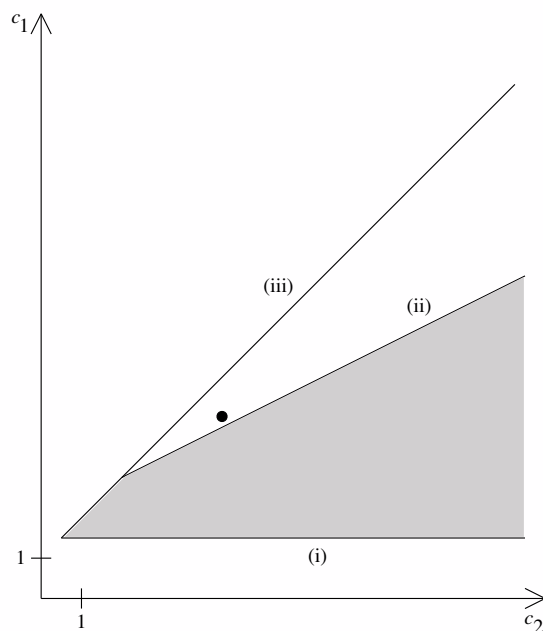
- (5) [sage] Pick five points in  $\mathbb{Z}^3$  and let  $\mathcal{P}$  be their convex hull (in  $\mathbb{R}^3$ ). Compute the Ehrhart polynomial of  $\mathcal{P}$ .  
(6) [research problem] Choose  $d+1$  of the  $2^d$  vertices of the unit  $d$ -cube, and let  $\Delta$  be the simplex defined by their convex hull.  
(a) Which choice of vertices maximizes  $\text{vol } \Delta$ ?  
(b) What is the maximum volume of such a  $\Delta$ ?

- (7) Give an explicit bijection between the faces (including  $\emptyset$ ) of a given polytope  $\mathcal{P}$  and the faces (excluding  $\emptyset$ ) of its homogenization cone( $\mathcal{P}$ ).
- (8) Suppose  $\mathcal{P} \subset \mathbb{R}^m$  and  $\mathcal{Q} \subset \mathbb{R}^n$  are lattice polytopes. Prove that the *convolution* of their Ehrhart polynomials,

$$L(t) := \sum_{s=0}^t L_{\mathcal{P}}(s) L_{\mathcal{Q}}(t-s)$$

equals the Ehrhart quasipolynomial of the polytope given by the convex hull of  $\mathcal{P} \times \{\mathbf{0}_n\} \times \{0\}$  and  $\{\mathbf{0}_m\} \times \mathcal{Q} \times \{1\}$ . Here  $\mathbf{0}_d$  denotes the origin in  $\mathbb{R}^d$ .

- (9) Verify (parts of) the classification picture of degree-2 Ehrhart polynomials  $c_2 t^2 + c_1 t + 1$ : every half-integral point in the figure below corresponds to an Ehrhart polynomial.



- (10) [research problem] Give the corresponding classification picture of degree-3 Ehrhart polynomials.
- (11) This exercise constructs triangulations. Given a polytope  $\mathcal{P} \subseteq \mathbb{R}^d$  with vertices  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ , randomly choose  $h_1, h_2, \dots, h_n \in \mathbb{R}$ , and define the new polytope  $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$  as the convex hull of  $(\mathbf{v}_1, h_1), (\mathbf{v}_2, h_2), \dots, (\mathbf{v}_n, h_n)$ . The *lower hull* of  $\mathcal{Q}$  consists of all points that are *visible from below*: all points  $(x_1, x_2, \dots, x_{d+1}) \in \mathcal{Q}$  for which there is no  $\epsilon > 0$  such that  $(x_1, x_2, \dots, x_{d+1} - \epsilon) \in \mathcal{Q}$ . A *lower face* of  $\mathcal{Q}$  is a face of  $\mathcal{Q}$  that is in the lower hull. Let  $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  be the projection that forgets the last coordinate. Show that all lower faces of  $\mathcal{Q}$  are simplices, and that their projections under  $\pi$  form a triangulation of  $\mathcal{P}$ .

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## Day III: Positivity, Reciprocity & Friends

- (1) Let  $\mathcal{P} \subset \mathbb{R}^d$  be a full-dimensional polyhedron with dissection  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$ . If  $\mathbf{q} \in \mathbb{R}^d$  is generic relative to each  $\mathcal{P}_i$ , then

$$\mathbb{H}_{\mathbf{q}}\mathcal{P} = \mathbb{H}_{\mathbf{q}}\mathcal{P}_1 \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_2 \uplus \cdots \uplus \mathbb{H}_{\mathbf{q}}\mathcal{P}_m.$$

- (2) Prove Ehrhart's theorem for half-open lattice polytopes.
- (3) [sage] Give an example of a polynomial  $f(n)$  with (some) negative coefficients whose corresponding generating function numerator polynomial  $h(z)$  has only positive coefficients.
- (4) [sage] For a lattice polytope  $\mathcal{P}$ , the numerator of the generating function is the  $h^*$ -polynomial of  $\mathcal{P}$ . Give a non-unimodal<sup>2</sup> example of an  $h^*$ -polynomial.
- (5) [research problem] Now let  $\mathcal{P} = \{\mathbf{x} \in [0, 1]^d : x_1 + x_2 + \cdots + x_d = k\}$ , for your favorite integers  $2 \leq k \leq d - 2$ . (This is the  $(d, k)$ -hypersimplex.) Prove that the  $h^*$ -polynomial of  $\mathcal{P}$  is unimodal.
- (6) For any polynomial  $h(z)$  of degree  $d$ , show there exist unique polynomials  $a(z)$  and  $b(z)$  such that

$$h(z) = a(z) + z b(z) \quad \text{where} \quad a(z) = z^d a\left(\frac{1}{z}\right) \quad \text{and} \quad b(z) = z^{d-1} b\left(\frac{1}{z}\right).$$

(There are many variations of this; e.g., we could leave out the  $z$  factor in front of  $b(z)$ .)

- (7) Derive inequalities for the coefficients of  $h(z)$  if we know that both  $a(z)$  and  $b(z)$  have only nonnegative coefficients.

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<sup>2</sup>A polynomial is *unimodal* if its coefficients increase up to some point and then decrease.