Proposition 3.1. *Let A, B, C be nonempty*[1] *sets.*

(i) $A \subseteq A$

(ii) *If* $A \subseteq B$ *and* $B \subseteq C$ *then* $A \subseteq C$.

Proof. (i) $A \subseteq A$ means "if $x \in A$ then $x \in A$", which is apparently a true statement.

(ii) Suppose $A \subseteq B$ and $B \subseteq C$. We need to show that if $x \in A$ then $x \in C$. Given $x \in A$, $A \subseteq B$ implies that $x \in B$. This, in turn, implies with $B \subseteq C$ that $x \in C$. □

Proposition 3.2. *Let A, B, C be nonempty sets.*

(i) $A = A$

(ii) *If* $A = B$ *then* $B = A$.

(iii) *If* $A = B$ *and* $B = C$ *then* $A = C$.

Proof. (i) follows from Proposition 3.1(i): $A \subseteq A$, so by definition, $A = A$.

(ii) Suppose $A = B$, i.e., $A \subseteq B$ and $B \subseteq A$. But this also means, by definition, that $B = A$.

(iii) Suppose $A = B$, i.e., $A \subseteq B$ and $B \subseteq A$, and $B = C$, i.e., $B \subseteq C$ and $C \subseteq B$. Then by Proposition 3.1(ii), $A \subseteq C$ (since $A \subseteq B$ and $B \subseteq C$) and $C \subseteq A$ (since $C \subseteq B$ and $B \subseteq A$), i.e., $A = C$. □

---

[1] A set is *nonempty* if it contains at least one element. The *empty set*, which contains no element, will be defined in Chapter 4; then you can convince yourself that this and the next proposition also hold if one of A, B, C are empty.