Math 870 Lecture: Delaunay Triangulation (*Computational Geometry*, Ch.9)
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Motivation:

When talking about mapping the surface of the Earth, unless it is a country with minimal height changes such as the Netherlands, it is unrealistic to assume there is no relief. Thus, we must find a way to approximate a 3-dimensional space with limited sample points efficiently and which produces the most natural approximation.

Approach:

1) Model our points by first determining a *Triangulation* of *P*. (This is our focus!)
2) Lift each sample point to its correct height. (i.e. mapping every triangle in 2-d to its appropriate triangle in 3-d.)
3) We've created a polyhedral terrain to approximate the original terrain.

Questions:

1. How do we triangulate the set of sample points?
   There are many different possible triangulations available.
2. Which triangulation is the most appropriate to approximate a terrain?
   Unfortunately, there is not definitive answer to this question. With a given set of limited sample points, we want the one that is most natural. From this we can create a set of criterion that tells us which triangulation is best. The main criterion is maximizing the minimal angle of every triangle in our triangulation, a.k.a. a Delaunay Triangulation.

Glossary, Theorems, Lemmas, etc.:

Terrain: A 2-dimensional surface in a 3-dimensional space such that every vertical line intersects it in a point, if it intersects it at all. In other words, it is the graph of the function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ that assigns a height $f(p)$ to every point p in the domain, A, of the terrain.

Triangulation of P: A planar subdivision whose bounded faces are triangles and whose vertices are the points of *P*. (Recall from Chris's lecture that we know every bounded face (or polygon) can be triangulated.)

Polyhedral terrain: The graph of a continuous function that is piecewise linear.

P: Define $P \equiv \{p_1, p_2, \dots, p_n\}$ as the set of points in the plane

Maximal Planar Subdivision: A subdivision *S* such that no edge connecting two vertices can be added to *S* without destroying its planarity.

Triangulation of P: The maximal planar subdivision whose vertex set is *P*.

Angle-vector of T: $A(T) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{3m})$ is the angle vector of *T* where *T* is the triangulation of *P* and $\alpha_1, \alpha_2, \dots, \alpha_{3m}$ is the list of 3*m* angles in *T* sorted by increasing angle.

$A(T) > A(T')$: Let *T* and T' be triangulations of a point set P where $A(T) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{3m})$ and $A(T') \equiv (\alpha'_1, \alpha'_2, \dots, \alpha'_{3m})$ are the corresponding angle vectors. Then we say the angle-vector of *T* is larger than the angle-vector of T' if

$A(T)$ is lexicographically larger than $A(T')$. This means if there exists an index I with $1 \le i \le 3m$ such that

$$\alpha_j = \alpha'_j \text{ for all } j < i, \text{ and } \alpha_i > \alpha'_i$$

then $A(T) > A(T')$.

Angle-optimal: With the same triangulations as above, we say the triangulation T is *angle-optimal* if $A(T) \ge A(T')$.

Theorem. 9.2 (Thale's Theorem): Denote the smaller angle defined by three points p, q, r by $\measuredangle pqr$ Let C be a circle, l be a line intersecting C in points a and b, and p, q, r, and s points lying on the same side of l. Suppose that p and q lie on C, that r lies inside C, and that s lies outside C. Then

$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb.$$

Edge Flip: If an edge e is not an edge of the unbounded face of P, then it is incident to two triangles $p_i p_j p_k$ and $p_i p_j p_l$. If these two triangles form a convex quadrilateral, we can obtain a new triangulation T' by removing $\overline{p_i p_j}$ from T and inserting $\overline{p_k p_l}$ instead. This is called an *edge flip*.

Illegal Edge: We call the edge $e = \overline{p_i p_j}$ an *illegal edge* if

$$\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i.$$

Lemma 9.4: Let edge $\overline{p_i p_j}$ be incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$, and let C be the circle through p_i, p_j, and p_k. *The edge* $\overline{p_i p_j}$ *is illegal if and only if the point* p_l *lies in the interior of C.* Furthermore, if the points p_i, p_j, p_k, p_l form a convex quadrilateral and do not lie on a common circle, then exactly one of $\overline{p_i p_j}$ and $\overline{p_k p_l}$ is an illegal edge.

Legal Triangulation: A triangulation that does not contain any illegal edges.

Voronoi Diagram of P: The subdivision of the plane P into n regions, one for each site (point) in P, such that the region of a site $p \in P$ contains all points in the plane for which p is the closest site. We denote this as Vor(P).

Delaunay Graph of P: The strait line embedding of a graph G, which has a line connecting two nodes if their corresponding Voronoi cells share an edge, denoted as DG(P).

Theorem 9.5: The Delaunay graph of a planar point set is a plane graph.

General Position: We say that a set of points is in *general position* if it contains no four points on a circle. Therefore, P in *general position* means every vertex of the Voronoi diagram has degree three, and consequently all bounded faces of DG(P) are triangles.

Theorem 9.6: Let P be a set of points in the plane.

(i) Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the Delaunay graph of P if and only if the circle through p_i, p_j, p_r contains no point of P in its interior.

(ii) Two points $p_i, p_j \in P$ form an edge of the Delaunay graph of *P* if and only if there is a closed disc *C* that contains p_i and p_j on its boundary and does not contain any other point of *P*.

Theorem 9.7: Let *P* be a set of points in the plane, and let *T* be a triangulation of *P*. Then *T* is a Delaunay triangulation of *P* if and only if the circumcircle of any triangle of *T* does not contain a point of *P* in its interior.

Theorem 9.8: Let *P* be a set of points in the plane. A triangulation *T* of *P* is legal if and only if *T* is a Delaunay triangulation of *P*.

Theorem 9.9: Let *P* be a set of points in the plane. Any angle-optimal triangulation of *P* is a Delaunay triangulation of *P*. Furthermore, any Delaunay triangulation of *P* maximizes the minimum angle over all triangulations of *P*.

Randomized Incremental: Input points in random order.

Lemma 9.10: Every new edge created in DELAUNAYTRIANGULATION or in LEGALIZEEDGE during the insertion of p_r is an edge f the Delaunay graph of $\Omega \cup \{p_1, ..., p_r\}$, where $\Omega = \{p_{-1}, p_{-2}, p_{-3}\}$.

Theorem 9.12: The Delaunay triangulation of a set *P* of *n* points in the plane can be computed in $O(n \log n)$ expected time, using $O(n)$ expected storage.

<u>*Algorithms*:</u>

Algorithm DELAUNAYTRIANGULATION (*P*)

Input. A set *P* of *n* points in the plane

Output. A Delaunay triangulation of *P*.

Let p_{-1}, p_{-2}, and p_{-3} be a suitable set of three points such that *P* is contained in the triangle $p_{-1}p_{-2}p_{-3}$.
Initialize *T* as the triangulation consisting of the single triangle $p_{-1}p_{-2}p_{-3}$.
Compute a random permutation of $p_1, p_2, ..., p_n$ of *P*.
for $r \leftarrow 1$ **to** *n*
    do(*Insert p_r into *T*:*)
        Find a triangle $p_i p_j p_k \in T$ containing p_r.
        if p_r lies in the interior of the triangle $p_i p_j p_k$
            then Add edges from p_r to the three vertices of $p_i p_j p_k$, thereby splitting $p_i p_j p_k$ into three triangles.
                LEGALIZEEDGE$\left(p_r, \overline{p_i p_j}, T\right)$
                LEGALIZEEDGE$\left(p_r, \overline{p_j p_k}, T\right)$
                LEGALIZEEDGE$\left(p_r, \overline{p_k p_i}, T\right)$
            else (* p_r lies on an edge of $p_i p_j p_k$, say $\overline{p_i p_j}$ *)
                Add edges from p_r to p_k and to the third vertex p_l of the other triangle that is incident to $\overline{p_i p_j}$, thereby splitting the two

triangles incident to $\overline{p_i p_j}$ into four triangles.
LEGALIZEEDGE$\left(p_r, \overline{p_i p_l}, T\right)$
LEGALIZEEDGE$\left(p_r, \overline{p_l p_j}, T\right)$
LEGALIZEEDGE$\left(p_r, \overline{p_j p_k}, T\right)$
LEGALIZEEDGE$\left(p_r, \overline{p_k p_i}, T\right)$
Discard $p_{-1}, p_{-2},$ and p_{-3} with all their incident edges from *T*.
return *T*

LEGALIZEEDGE$\left(p_r, \overline{p_i p_j}, T\right)$

(*The point being inserted is p_r, and $\overline{p_i p_j}$ is the edge of *T* that may need to be flipped.*)
if $\overline{p_i p_j}$ is illegal
then let $p_i p_j p_k$ be the triangle adjacent to $p_r p_i p_j$ along $\overline{p_i p_j}$.
(*Flip $\overline{p_i p_j}$:*) Replace $\overline{p_i p_j}$ with $\overline{p_r p_k}$.
LEGALIZEEDGE$\left(p_r, \overline{p_i p_k}, T\right)$
LEGALIZEEDGE$\left(p_r, \overline{p_k p_j}, T\right)$