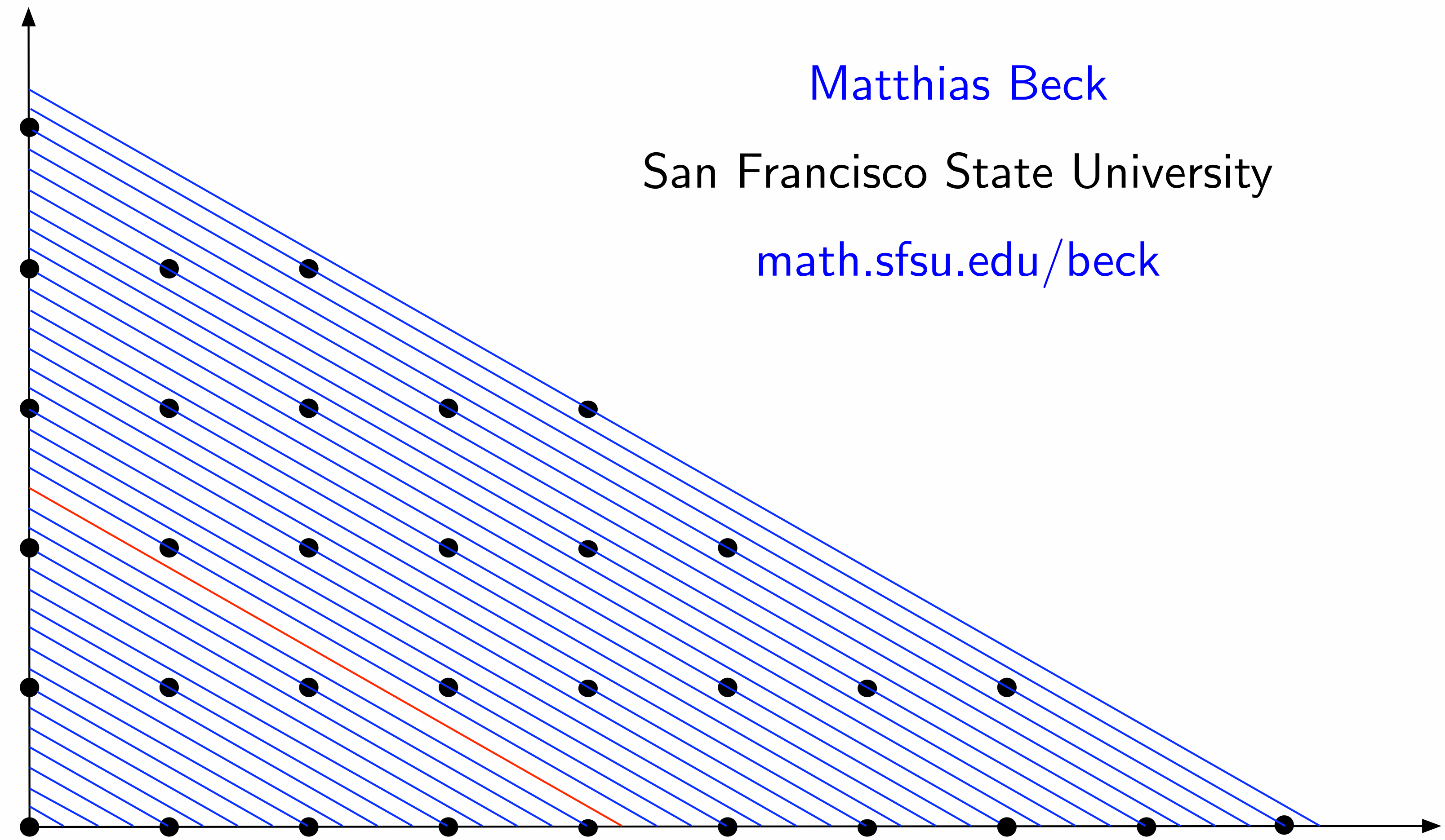
How to Change Coins, M&M's, or Chicken Nuggets: The Linear Diophantine Problem of Frobenius



Matthias Beck

San Francisco State University

math.sfsu.edu/beck

A warm-up problem

$$1, 11, 21, 1211, 111221, ?$$

The Frobenius Coin-Exchange Problem

Given coins of denominations 7, 10, and 50, what is the largest amount that cannot be changed?

The Frobenius Coin-Exchange Problem

Given coins of denominations 7, 10, and 50, what is the largest amount that cannot be changed?

More generally, given coins of denominations a, b, and c, what is the largest amount that cannot be changed?

The Frobenius Coin-Exchange Problem

Given coins of denominations 7, 10, and 50, what is the largest amount that cannot be changed?

More generally, given coins of denominations a, b, and c (with no common factor), what is the largest amount that cannot be changed?

The Frobenius Coin-Exchange Problem

Given coins of denominations 7, 10, and 50, what is the largest amount that cannot be changed?

More generally, given coins of denominations a, b, and c (with no common factor), what is the largest amount that cannot be changed?

Even more generally, given coins of denominations $a_1, a_2, \ldots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

Two Coins & the Euclidean Algorithm

Since 7 and 10 do not have any common factors (they are relatively prime), we can apply the Euclidean Algorithm to find integers m and n such that

$$1 = 7\,m + 10\,n\,.$$

(Note that one of m and n is negative.)

Two Coins & the Euclidean Algorithm

Since 7 and 10 do not have any common factors (they are relatively prime), we can apply the Euclidean Algorithm to find integers m and n such that

$$1 = 7\,m + 10\,n\,.$$

(Note that one of m and n is negative.)

But then any integer t can be written as an integral linear combination of 7 and 10:

$$t = 7\,m\,t + 10\,n\,t\,.$$

Two Coins & the Euclidean Algorithm

Since 7 and 10 do not have any common factors (they are relatively prime), we can apply the Euclidean Algorithm to find integers m and n such that

$$1 = 7\,m + 10\,n\,.$$

(Note that one of m and n is negative.)

But then any integer t can be written as an integral linear combination of 7 and 10:

$$t = 7\,m\,t + 10\,n\,t\,.$$

Claim: If t is sufficiently large then we can express it as a nonnegative integral linear combination of 7 and 10.

Two Coins & the Euclidean Algorithm

If a and b do not have any common factors (they are relatively prime), we can apply the Euclidean Algorithm to find integers m and n such that

$$1 = a\,m + b\,n\,.$$

(Note that one of m and n is negative.)

But then any integer t can be written as an integral linear combination of a and b:

$$t = a\,m\,t + b\,n\,t\,.$$

Claim: If t is sufficiently large then we can express it as a nonnegative integral linear combination of a and b.

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10) + 10(n - 7)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m-10) + 10(n+7)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 34) + 10(n - 7 \cdot 34)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 81) + 10(n - 7 \cdot 81)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m - 10 \cdot 39) + 10(n + 7 \cdot 39)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 92) + 10(n - 7 \cdot 92)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m - 10 \cdot 46) + 10(n + 7 \cdot 46)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 29) + 10(n - 7 \cdot 29)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m - 10 \cdot 74) + 10(n + 7 \cdot 74)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 57) + 10(n - 7 \cdot 57)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m - 10 \cdot 95) + 10(n + 7 \cdot 95)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 22) + 10(n - 7 \cdot 22)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m - 10 \cdot 63) + 10(n + 7 \cdot 63)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10 \cdot 84) + 10(n - 7 \cdot 84)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10\heartsuit) + 10(n - 7\heartsuit)\,.$$

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10\heartsuit) + 10(n - 7\heartsuit)\,.$$

There is a unique representation

$$t = 7\,m + 10\,n$$

for which $0 \leq m < 10$.

Two Coins & a well-defined problem

As we have just seen, we can write the (positive) integer t as an integral linear combination

$$t = 7\,m + 10\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = 7(m + 10\heartsuit) + 10(n - 7\heartsuit)\,.$$

There is a unique representation

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$. So if t is large enough (e.g., larger than $7 \cdot 10$) then we can find a nonnegative integral linear combination of 7 and 10.

Two Coins & a well-defined problem

Given two positive integers a and b with no common factor, we can write the (positive) integer t as an integral linear combination

$$t = a\,m + b\,n\,.$$

Once we have one such representation of t we can find many more:

$$t = a(m - b\heartsuit) + b(n + a\heartsuit)\,.$$

There is a unique representation

$$t = a\,m + b\,n$$

for which $0 \leq m < b$. So if t is large enough (e.g., larger than ab) then we can find a nonnegative integral linear combination of a and b.

What about more than two coins?

Homework Prove that the Frobenius problem for the coins $a_1, a_2, \ldots, a_d$ (with no common factor) is well defined for $d > 2$.

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$.

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$.

If (and only if) we can find such a representation for which also $n \ge 0$ then t is representable.

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$.

If (and only if) we can find such a representation for which also $n \ge 0$ then t is representable. Hence the largest integer t that is not representable is

$$t = 7\ ? + 10\ ?$$

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$.

If (and only if) we can find such a representation for which also $n \ge 0$ then t is representable. Hence the largest integer t that is not representable is

$$t = 7(10-1) + 10 \; ?$$

A closer look for two coins

Let's say the integer t is representable (in terms of 7 and 10) if we can find nonnegative integers m and n such that

$$t = 7\,m + 10\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = 7\,m + 10\,n$$

for which $0 \le m < 10$.

If (and only if) we can find such a representation for which also $n \ge 0$ then t is representable. Hence the largest integer t that is not representable is

$$t = 7(10-1) + 10(-1) = 7 \cdot 10 - 7 - 10\,.$$

A closer look for two coins

Given two relatively prime integers a and b, we say the integer t is representable (in terms of a and b) if we can find nonnegative integers m and n such that

$$t = a\,m + b\,n\,.$$

We have seen already that we can always write t as an integral linear combination

$$t = a\,m + b\,n$$

for which $0 \leq m < b$.

If (and only if) we can find such a representation for which also $n \geq 0$ then t is representable. Hence the largest integer t that is not representable is

$$t = a(b-1) + b(-1) = ab - a - b\,.$$

a formula most likely known already to James J. Sylvester in the 1880's.

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

We define g_k as the largest k-representable integer. (So $g_0 = ab - a - b$ is the Frobenius number.)

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

We define g_k as the largest k-representable integer. (So $g_0 = ab - a - b$ is the Frobenius number.) Prove:

- g_k is well defined.

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\ge 0}$ to

$$t = ma + nb.$$

We define g_k as the largest k-representable integer. (So $g_0 = ab - a - b$ is the Frobenius number.) Prove:

- g_k is well defined.
- $g_k = (k+1)ab - a - b$

A homework with several representations

Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m,n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

We define g_k as the largest k-representable integer. (So $g_0 = ab - a - b$ is the Frobenius number.) Prove:

- g_k is well defined.
- $g_k = (k+1)ab - a - b$
- Given $k \geq 2$, the smallest k-representable integer is $ab(k-1)$.

What about more than two coins?

Frobenius Problem Given coins of denominations $a_1, a_2, \ldots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

What about more than two coins?

Frobenius Problem Given coins of denominations $a_1, a_2, \ldots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

Current state of affairs:

- $d = 2$ solved (probably by Sylvester in 1880's)

What about more than two coins?

Frobenius Problem Given coins of denominations $a_1, a_2, \ldots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

Current state of affairs:

- $d = 2$ solved (probably by Sylvester in 1880's)
- $d = 3$ solved algorithmically (Herzog 1970, Greenberg 1980, Davison 1994) and in not-quite-explicit form (Denham 2003, Ramirez-Alfonsin 2005)

What about more than two coins?

Frobenius Problem Given coins of denominations $a_1, a_2, \dots, a_d$ (with no common factor), what is the largest amount that cannot be changed?

Current state of affairs:

- $d = 2$ solved (probably by Sylvester in 1880's)
- $d = 3$ solved algorithmically (Herzog 1970, Greenberg 1980, Davison 1994) and in not-quite-explicit form (Denham 2003, Ramirez-Alfonsin 2005)
- $d \geq 4$ computationally feasible (Kannan 1992, Barvinok-Woods 2003), otherwise: completely open

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t \right\}$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t \right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0 \, .$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t \right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0\,.$$

Starting from this representation, we can obtain more...

$$t = (m+b)a + (n-a)b$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t \right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \; n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m + 2b)a + (n - 2a)b$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t \right\}$

If the positive integer t is representable then we can find m,n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m+3b)a + (n-3a)b$$

Let's start counting. . .

Let $c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t\right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \; n \geq 0 \,.$$

Starting from this representation, we can obtain more...

$$t = (m + 4b)a + (n - 4a)b$$

Let's start counting. . .

Let $c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t\right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \; n \geq 0\,.$$

Starting from this representation, we can obtain more...

$$t = (m + 5b)a + (n - 5a)b$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t \right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m + 6b)a + (n - 6a)b$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t \right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \; n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m + 7b)a + (n - 7a)b$$

Let's start counting. . .

Let $c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; ma + nb = t \right\}$

If the positive integer t is representable then we can find m,n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \; n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m + kb)a + (n - ka)b$$

until $n - ka$ becomes negative.

Let's start counting. . .

Let $c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t\right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0 \, .$$

Starting from this representation, we can obtain more...

$$t = (m + kb)a + (n - ka)b$$

until $n - ka$ becomes negative. But then

$$\begin{aligned} t + ab &= (m + kb)a + (n - ka)b + ab \\ &= (m + kb)a + (n - (k-1)a)\, b \end{aligned}$$

has one more representation

Let's start counting. . .

Let $c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ ma + nb = t\right\}$

If the positive integer t is representable then we can find m, n such that

$$t = ma + nb \qquad \text{and} \qquad 0 \leq m \leq b-1, \ n \geq 0 \,.$$

Starting from this representation, we can obtain more...

$$t = (m + kb)a + (n - ka)b$$

until $n - ka$ becomes negative. But then

$$\begin{aligned} t + ab &= (m + kb)a + (n - ka)b + ab \\ &= (m + kb)a + (n - (k-1)a)\, b \end{aligned}$$

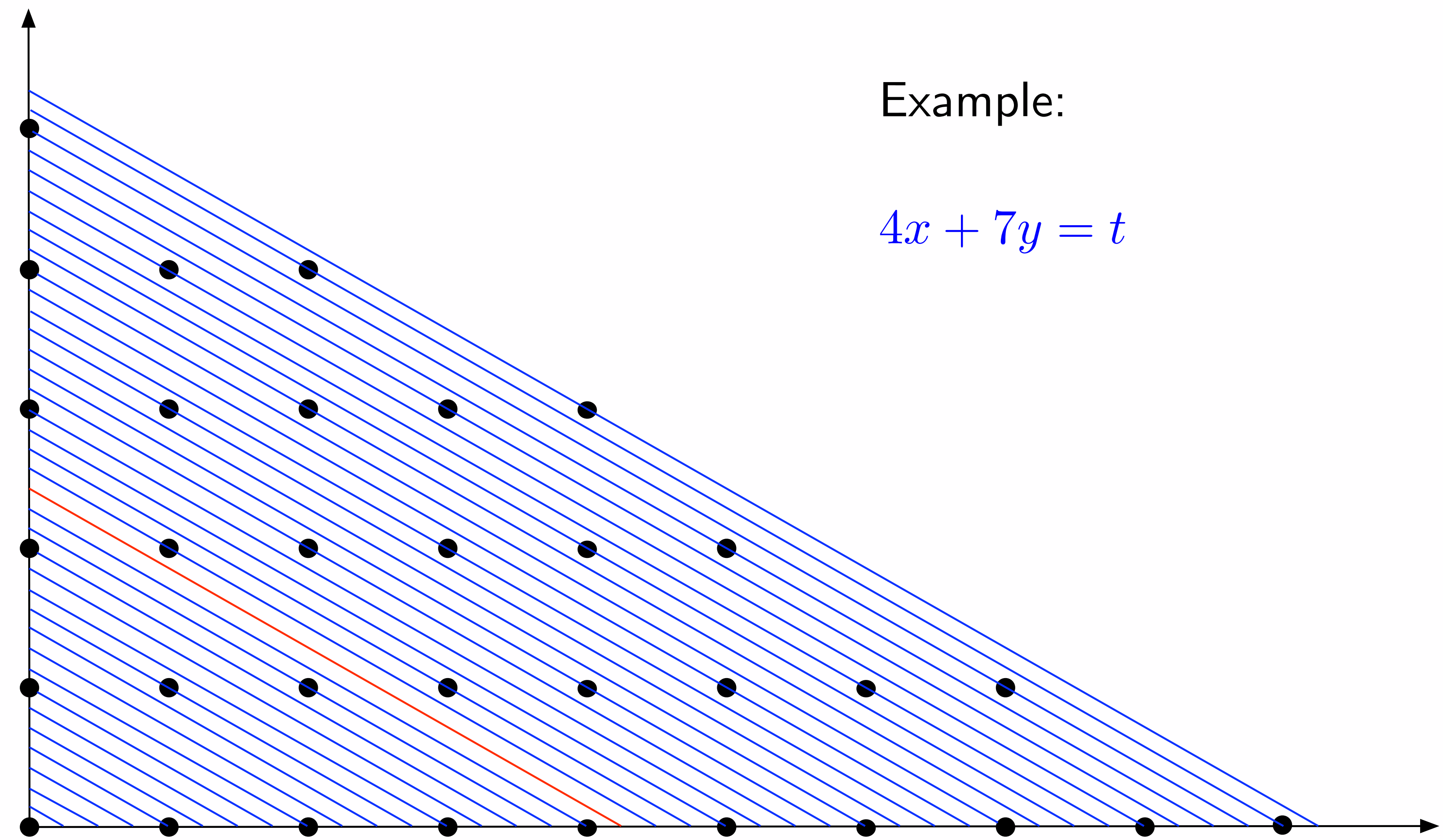
has one more representation, i.e., $c(t + ab) = c(t) + 1$.

A geometric interpretation

$c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \ge 0, \ a\,m + b\,n = t\right\}$ counts integer points in $\mathbb{R}^2_{\ge 0}$ on the line $a\,x + b\,y = t$.

A geometric interpretation

$c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ a\,m + b\,n = t\right\}$ counts integer points in $\mathbb{R}^2_{\geq 0}$ on the line $a\,x + b\,y = t$.

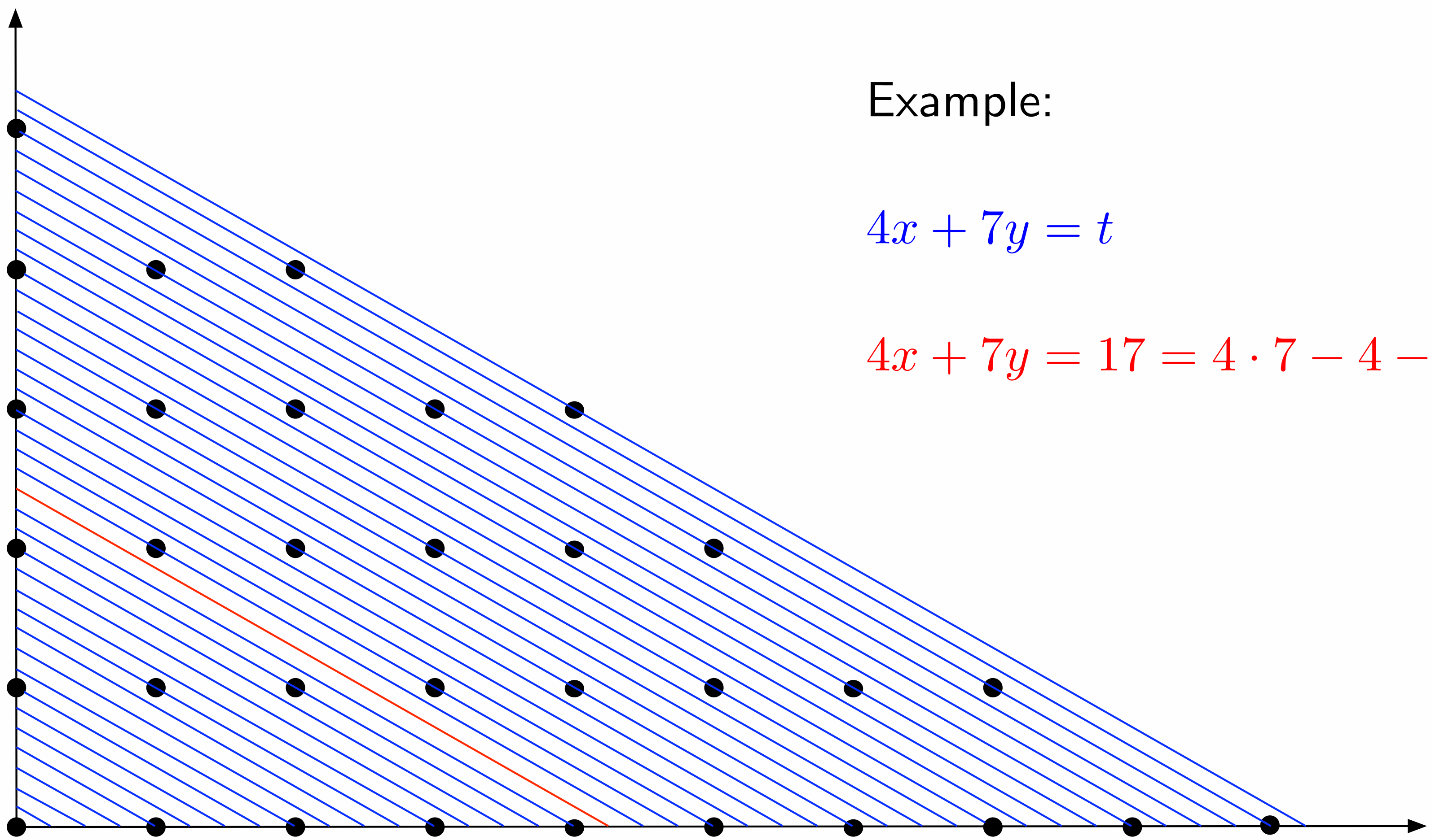


Example:

$4x + 7y = t$

A geometric interpretation

$c(t) = \#\left\{(m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ a\,m + b\,n = t\right\}$ counts integer points in $\mathbb{R}^2_{\geq 0}$ on the line $a\,x + b\,y = t$.



Example:

$4x + 7y = t$

$4x + 7y = 17 = 4 \cdot 7 - 4 - 7$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\, x + s_2\, x^2 + s_3\, x^3 + \cdots = \sum_{k \geq 0} s_k\, x^k .$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots = \sum_{k \ge 0} s_k\,x^k .$$

Example 0: if your favorite sequence looks like $(s_0, s_1, s_2, s_3, \dots, s_n, 0, 0, \dots)$ then its generating function is the polynomial

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots + s_n\,x^n .$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\, x + s_2\, x^2 + s_3\, x^3 + \cdots = \sum_{k \geq 0} s_k\, x^k .$$

Example 1: if your favorite sequence is $(1, 1, 1, \dots)$ then we obtain the geometric series $1 + x + x^2 + x^3 + \cdots = \sum_{k \geq 0} x^k$.

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\, x + s_2\, x^2 + s_3\, x^3 + \cdots = \sum_{k \ge 0} s_k\, x^k .$$

Example 1: if your favorite sequence is $(1, 1, 1, \dots)$ then we obtain the geometric series $1 + x + x^2 + x^3 + \cdots = \sum_{k \ge 0} x^k$. Since

$$\begin{aligned} (1 - x)\left(1 + x + x^2 + x^3 + \cdots\right) \quad &= \quad \left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad - x\left(1 + x + x^2 + x^3 + \cdots\right) \end{aligned}$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \ldots)$ we encode it into the generating function

$$s_0 + s_1\, x + s_2\, x^2 + s_3\, x^3 + \cdots = \sum_{k \ge 0} s_k\, x^k .$$

Example 1: if your favorite sequence is $(1, 1, 1, \ldots)$ then we obtain the geometric series $1 + x + x^2 + x^3 + \cdots = \sum_{k \ge 0} x^k$. Since

$$\begin{aligned}(1-x)\left(1 + x + x^2 + x^3 + \cdots\right) &= \left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad - \left(x + x^2 + x^3 + x^4 \cdots\right)\end{aligned}$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots = \sum_{k \geq 0} s_k\,x^k .$$

Example 1: if your favorite sequence is $(1, 1, 1, \dots)$ then we obtain the geometric series $1 + x + x^2 + x^3 + \cdots = \sum_{k \geq 0} x^k$. Since

$$\begin{aligned} (1-x)\left(1 + x + x^2 + x^3 + \cdots\right) &= \left(1 + x + x^2 + x^3 + \cdots\right) \\ &\quad - \left(x + x^2 + x^3 + x^4 \cdots\right) \\ &= 1 \end{aligned}$$

we conclude that $\sum_{k \geq 0} x^k = \frac{1}{1-x}$.

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots = \sum_{k \geq 0} s_k\,x^k.$$

Example 2: if your favorite sequence is $(1, 0, 0, 1, 0, 0, \dots)$ its generating function is

$$1 + x^3 + x^6 + x^9 + \cdots$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots = \sum_{k\ge0} s_k\,x^k .$$

Example 2: if your favorite sequence is $(1, 0, 0, 1, 0, 0, \dots)$ its generating function is

$$1 + x^3 + x^6 + x^9 + \cdots = \sum_{k\ge0} x^{3k}$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \ldots)$ we encode it into the generating function

$$s_0 + s_1\, x + s_2\, x^2 + s_3\, x^3 + \cdots = \sum_{k \ge 0} s_k\, x^k .$$

Example 2: if your favorite sequence is $(1, 0, 0, 1, 0, 0, \ldots)$ its generating function is

$$1 + x^3 + x^6 + x^9 + \cdots = \sum_{k \ge 0} x^{3k} = \sum_{k \ge 0} \left(x^3\right)^k$$

Generating functions

Given your favorite infintite sequence $(s_0, s_1, s_2, s_3, \dots)$ we encode it into the generating function

$$s_0 + s_1\,x + s_2\,x^2 + s_3\,x^3 + \cdots = \sum_{k \ge 0} s_k\,x^k .$$

Example 2: if your favorite sequence is $(1, 0, 0, 1, 0, 0, \dots)$ its generating function is

$$1 + x^3 + x^6 + x^9 + \cdots = \sum_{k \ge 0} x^{3k} = \sum_{k \ge 0} \left(x^3\right)^k = \frac{1}{1 - x^3} .$$

Frobenius generatingfunctionology

For our two integers 7 and 10, recall that

$$c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \; m,n \geq 0, \; 7m + 10n = t \right\} .$$

Frobenius generatingfunctionology

For our two integers 7 and 10, recall that

$$c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ 7m + 10n = t \right\} .$$

Consider the product of the geometric series

$$\frac{1}{\left(1 - x^7\right)\left(1 - x^{10}\right)} = \left(1 + x^7 + x^{2 \cdot 7} + x^{3 \cdot 7} + \cdots\right)\left(1 + x^{10} + x^{2 \cdot 10} + x^{3 \cdot 10} + \cdots\right)$$

Frobenius generatingfunctionology

For our two integers 7 and 10, recall that

$$c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ 7m + 10n = t \right\}.$$

Consider the product of the geometric series

$$\frac{1}{(1-x^7)(1-x^{10})} = \left(1 + x^7 + x^{2\cdot 7} + x^{3\cdot 7} + \cdots\right)\left(1 + x^{10} + x^{2\cdot 10} + x^{3\cdot 10} + \cdots\right)$$

A typical term looks like x^{7m+10n} for some $m,n \geq 0$, and so

$$\frac{1}{(1-x^7)(1-x^{10})} = \sum_{t \geq 0} c(t)\, x^t$$

is the generating function associated to the counting function $c(t)$.

Frobenius generatingfunctionology

For two integers a and b, recall that

$$c(t) = \# \left\{ (m,n) \in \mathbb{Z}^2 : \ m,n \geq 0, \ am + bn = t \right\} .$$

Consider the product of the geometric series

$$\frac{1}{(1-x^a)\,(1-x^b)} = \left(1 + x^a + x^{2a} + x^{3a} + \cdots \right) \left(1 + x^b + x^{2b} + x^{3b} + \cdots \right)$$

A typical term looks like x^{am+bn} for some $m,n \geq 0$, and so

$$\frac{1}{(1-x^a)\,(1-x^b)} = \sum_{t \geq 0} c(t)\, x^t$$

is the generating function associated to the counting function $c(t)$.

A Frobenius generatingfunctionological homework

Recall that $c(t+ab) = c(t) + 1$. Use this to prove

$$\sum_{t \text{ representable}} x^t = \frac{1 - x^{ab}}{\left(1 - x^a\right)\left(1 - x^b\right)}$$

A Frobenius generatingfunctionological homework

Recall that $c(t+ab) = c(t) + 1$. Use this to prove

$$\sum_{t \text{ representable}} x^t = \frac{1 - x^{ab}}{(1-x^a)(1-x^b)}$$

This is the generating function $\sum_{t \geq 0} s_t \, x^t$ of the sequence

$$s_t = \begin{cases} 1 & \text{if } t \text{ is representable,} \\ 0 & \text{if } t \text{ is not representable.} \end{cases}$$

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t \;=\; \frac{1}{1-x} - \frac{1-x^{ab}}{\left(1-x^a\right)\left(1-x^b\right)}$$

Why this was a cool homework

$$
\begin{aligned}
\sum_{t \text{ not representable}} x^t &= \frac{1}{1-x} - \frac{1-x^{ab}}{\left(1-x^a\right)\left(1-x^b\right)} \\
&= \frac{1-x^a-x^b+x^{a+b}-\left(1-x-x^{ab}+x^{ab+1}\right)}{\left(1-x\right)\left(1-x^a\right)\left(1-x^b\right)}
\end{aligned}
$$

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}$$

$$= \frac{1-x^a-x^b+x^{a+b}-\left(1-x-x^{ab}+x^{ab+1}\right)}{(1-x)(1-x^a)(1-x^b)}$$

$$= \frac{x-x^a-x^b+x^{a+b}+x^{ab}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}$$

$$= \frac{1-x^a-x^b+x^{a+b}-\left(1-x-x^{ab}+x^{ab+1}\right)}{(1-x)(1-x^a)(1-x^b)}$$

$$= \frac{x-x^a-x^b+x^{a+b}+x^{ab}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

This polynomial has degree. . .

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}$$

$$= \frac{1 - x^a - x^b + x^{a+b} - \left(1 - x - x^{ab} + x^{ab+1}\right)}{(1-x)(1-x^a)(1-x^b)}$$

$$= \frac{x - x^a - x^b + x^{a+b} + x^{ab} - x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

This polynomial has degree $ab + 1 - (1 + a + b) = ab - a - b$.

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}$$

$$= \frac{1-x^a-x^b+x^{a+b}-\left(1-x-x^{ab}+x^{ab+1}\right)}{(1-x)(1-x^a)(1-x^b)}$$

$$= \frac{x-x^a-x^b+x^{a+b}+x^{ab}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

This polynomial has degree $ab+1-(1+a+b) = ab-a-b$.

The number of non-representable positive integers is

$$\lim_{x\to 1} \frac{x-x^a-x^b+x^{ab}+x^{a+b}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

Why this was a cool homework

$$\sum_{t \text{ not representable}} x^t = \frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)}$$

$$= \frac{1-x^a-x^b+x^{a+b}-\left(1-x-x^{ab}+x^{ab+1}\right)}{(1-x)(1-x^a)(1-x^b)}$$

$$= \frac{x-x^a-x^b+x^{a+b}+x^{ab}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

This polynomial has degree $ab+1-(1+a+b) = ab-a-b$.

The number of non-representable positive integers is

$$\lim_{x \to 1} \frac{x-x^a-x^b+x^{ab}+x^{a+b}-x^{ab+1}}{(1-x)(1-x^a)(1-x^b)} = \frac{(a-1)(b-1)}{2}.$$

A higher-dimensional homework

We say that t is representable by the positive integers $a_1, a_2, \ldots, a_d$ if there is a solution $(m_1, m_2, \ldots, m_d)$ in nonnegative integers to

$$m_1 a_1 + m_2 a_2 + \cdots + m_d a_d = t$$

A higher-dimensional homework

We say that t is representable by the positive integers $a_1, a_2, \ldots, a_d$ if there is a solution $(m_1, m_2, \ldots, m_d)$ in nonnegative integers to

$$m_1 a_1 + m_2 a_2 + \cdots + m_d a_d = t$$

Prove

$$\sum_{t \text{ representable}} x^t = \frac{p(x)}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_d})}$$

for some polynomial p.

Another homework with several representations

Recall: Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m, n) \in \mathbb{Z}_{\geq 0}^2$ to

$$t = ma + nb.$$

Another homework with several representations

Recall: Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m, n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

Prove:

- There are exactly $ab - 1$ integers that are uniquely representable.

Another homework with several representations

Recall: Given two positive integers a and b with no common factor, we say the integer t is k-representable if there are exactly k solutions $(m, n) \in \mathbb{Z}^2_{\geq 0}$ to

$$t = ma + nb.$$

Prove:

- There are exactly $ab - 1$ integers that are uniquely representable.
- Given $k \geq 2$, there are exactly ab k-representable integers.

What about more than two coins?

Given integers $a_1, a_2, \ldots, a_d$ with no common factor, let

$$F(x) = \sum_{t \text{ representable}} x^t = \frac{p(x)}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_d})}.$$

- (Denham 2003) For $d = 3$, the polynomial $p(x)$ has either 4 or 6 terms, given in semi-explicit form.
- (Bresinsky 1975) For $d \geq 4$, there is no absolute bound for the number of terms in $p(x)$.
- (Barvinok–Woods 2003) For fixed d, the rational generating function $F(x)$ can be written as a "short" sum of rational functions.

What else is known

- Frobenius number and number of non-representable integers in special cases: arithmetic progressions and variations, extension cases
- Upper and lower bounds for the Frobenius number
- Algorithms
- Generalizations: vector version, k-representable Frobenius number