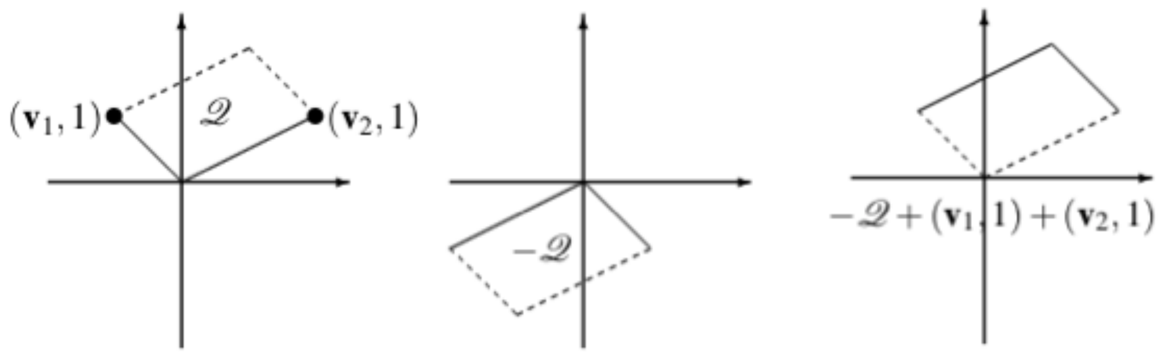
# Combinatorial Reciprocity Theorems





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Based on joint work with

Thomas Zaslavsky

Binghamton University (SUNY)

"In mathematics you don't understand things. You just get used to them."

John von Neumann (1903–1957)

# The Theme

Combinatorics

counting function depending on $k \in \mathbb{Z}_{>0}$

polynomial $p(k)$

What is $p(0)$?
$p(-1)$? $p(-2)$?

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$p(-1)$? $p(-2)$?

- Two-for-one charm of combinatorial reciprocity theorems

- “Big picture” motivation: understand/classify these polynomials

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$k$-coloring of $G$ — mapping $\boldsymbol{x} \in \{1, 2, \dots, k\}^V$

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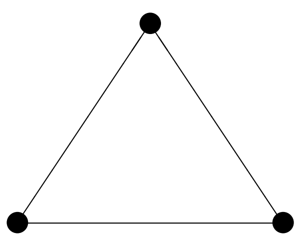
$\chi_G(k) := \#\,(\text{proper } k\text{-colorings of } G)$

Example:



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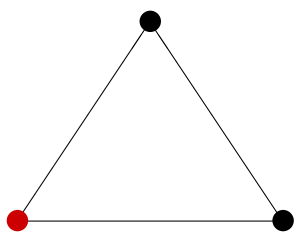

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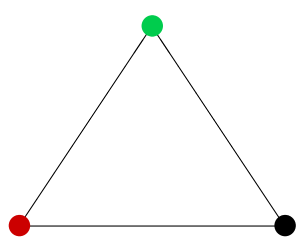

$$\chi_{K_3}(k) = k(k-1)\cdots$$

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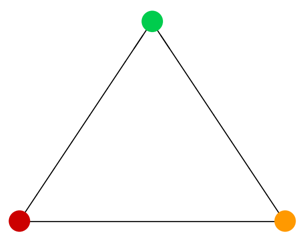

$\chi_{K_3}(k) = k(k-1)(k-2)$

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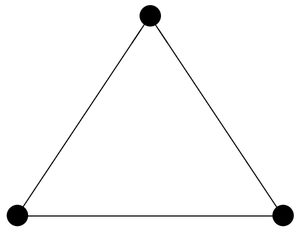

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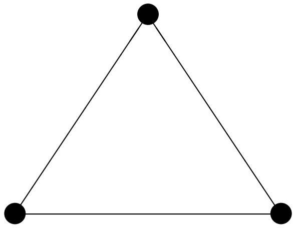

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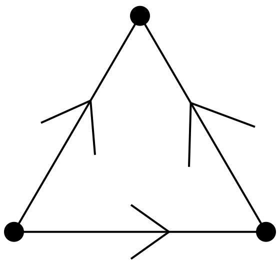

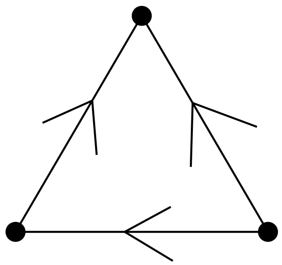

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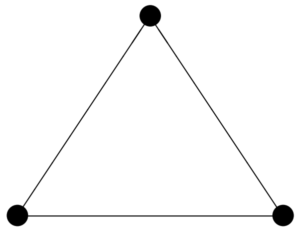

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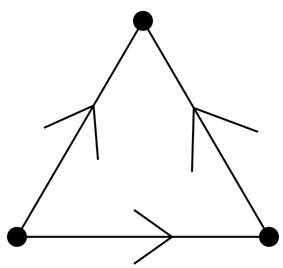

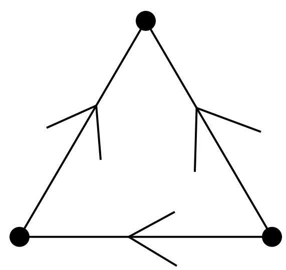

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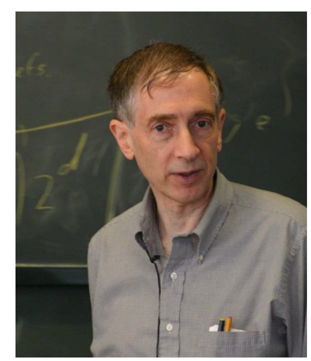
**Theorem** (Stanley 1973) $(-1)^{|V|}\chi_G(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation $\alpha$ of $G$ and a compatible $k$-coloring $\boldsymbol{x}$.
In particular, $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of $G$.



# If you get bored. . .

- Show that the coefficients of $\chi_G$ alternate in sign. [old news]

- Show that the absolute values of the coefficients form a unimodal sequence. [J. Huh, `arXiv:1008.4749`]

- Show that $\chi_G(4) > 0$ for any planar graph $G$. [impressive with or without a computer]

- Show that $\chi_G$ has no real root $\geq 4$. [open]

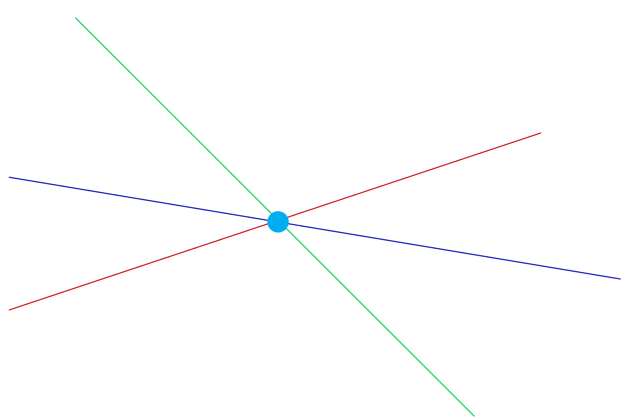
- Classify chromatic polynomials. [wide open]

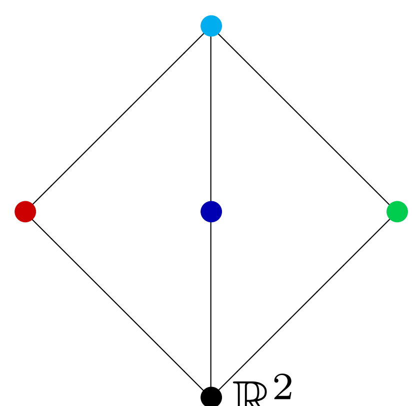
# Hyperplane Arrangements

$\mathcal{H} \subset \mathbb{R}^d$ — arrangement of affine hyperplanes
$\mathcal{L}(\mathcal{H})$ — all nonempty intersections of hyperplanes in $\mathcal{H}$

Möbius function $\mu(F) := \begin{cases} 1 & \text{if } F = \mathbb{R}^d \\ -\sum_{G \supsetneq F} \mu(G) & \text{otherwise} \end{cases}$

Characteristic polynomial $p_{\mathcal{H}}(k) := \sum_{F \in \mathcal{L}(\mathcal{H})} \mu(F)\, k^{\dim F}$





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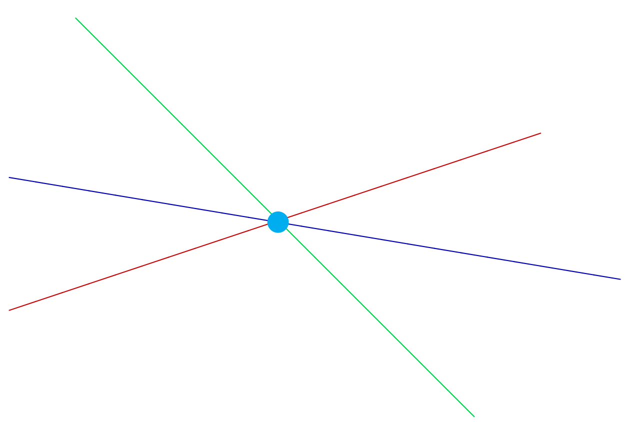

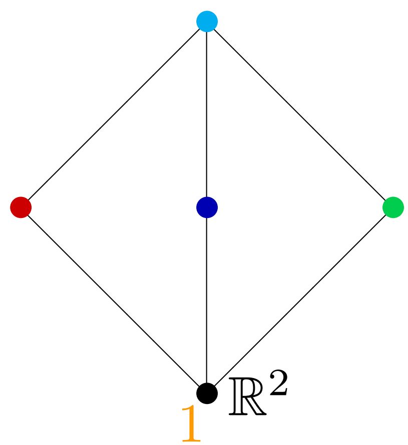

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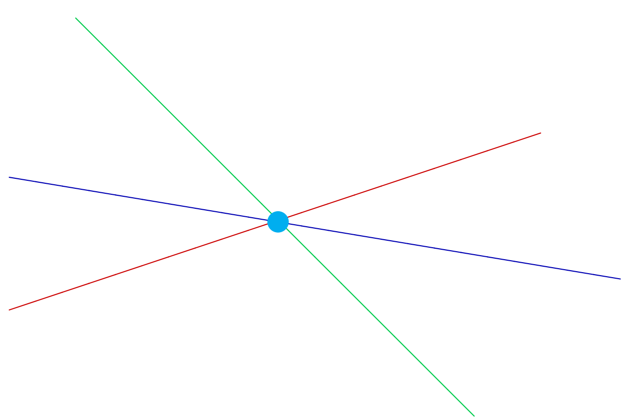

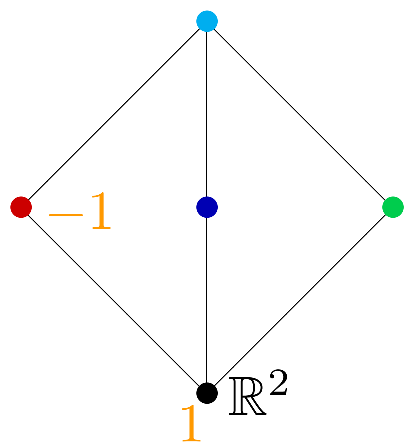

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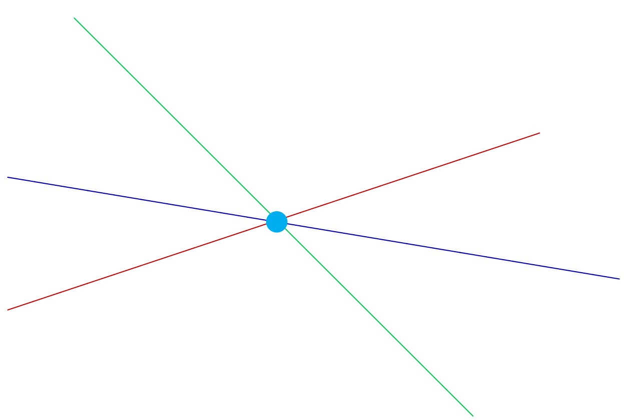

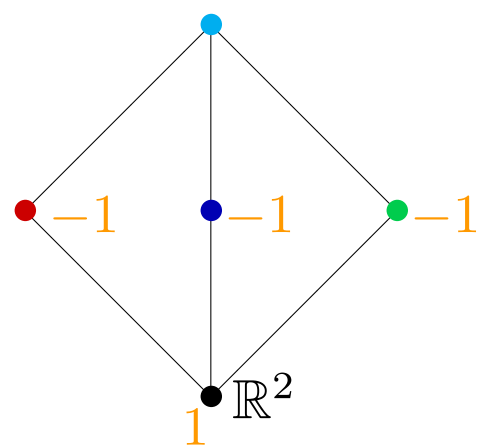

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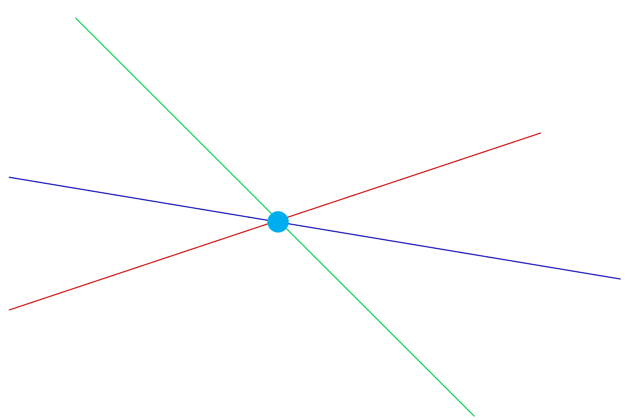

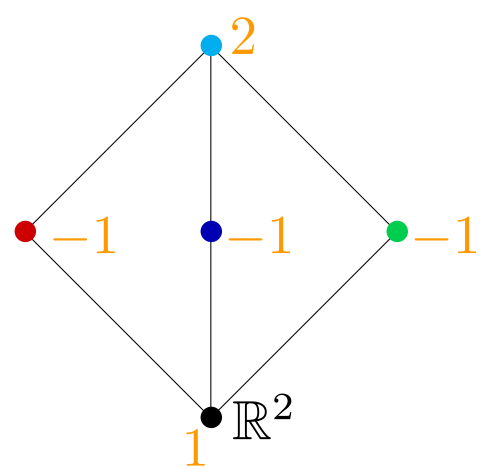

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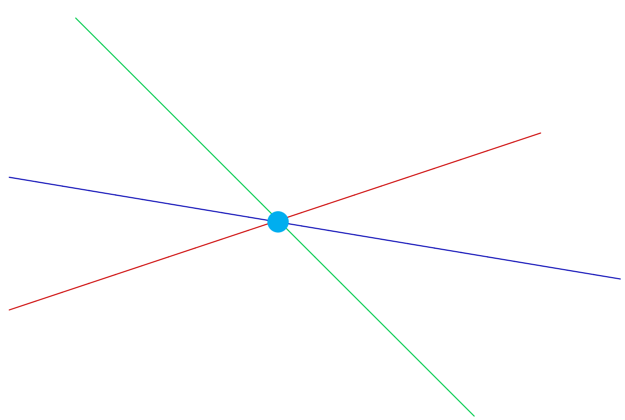

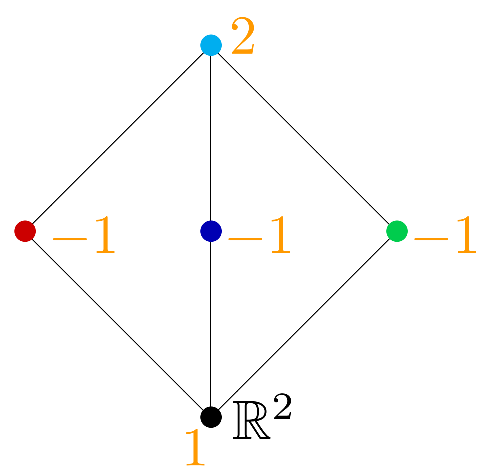

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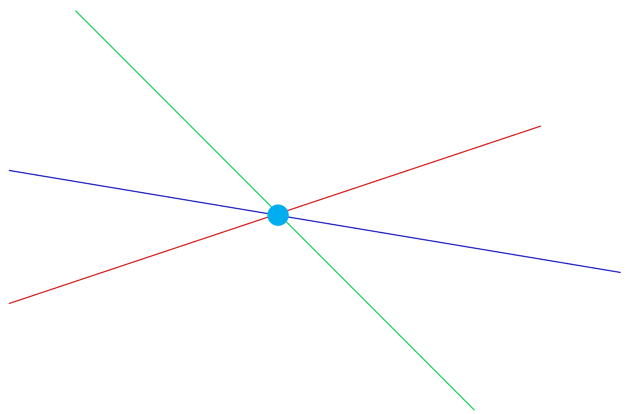

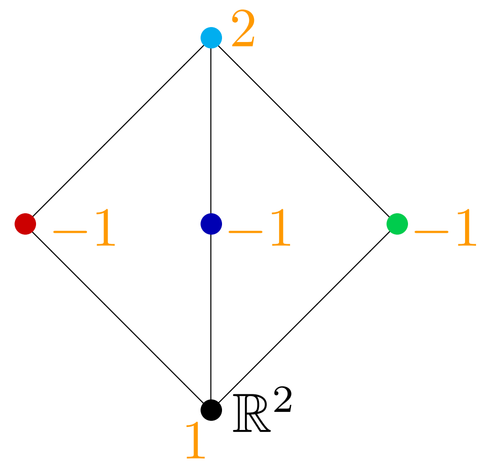


$$p_{\mathcal{H}}(k) = \sum_{F \in \mathcal{L}(\mathcal{H})} \mu(F)\, k^{\dim F} = k^2 - 3k + 2$$

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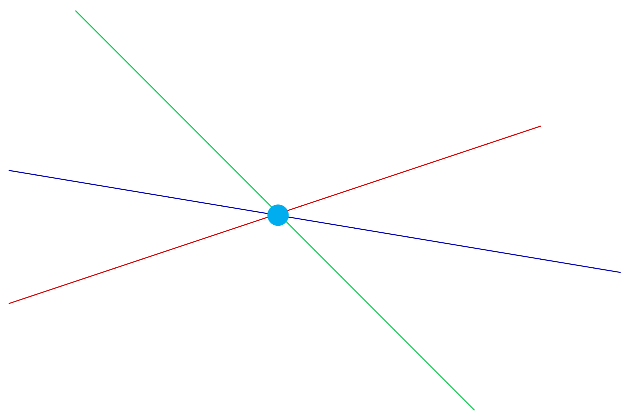

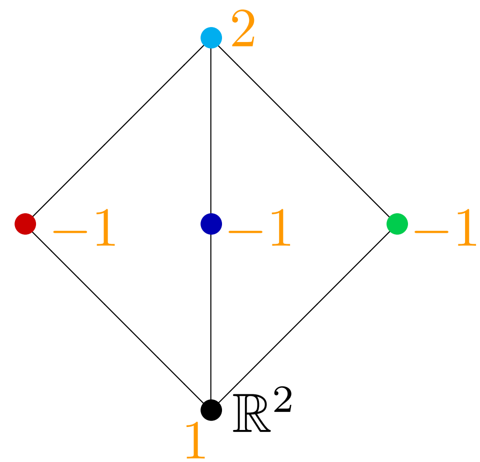


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Theorem (Zaslavsky 1975) $(-1)^d\, p_{\mathcal{H}}(-1)$ equals the number of regions into which a hyperplane arrangement $\mathcal{H}$ divides $\mathbb{R}^d$.

# If you get bored...

- Compute $p_{\mathcal{H}}(k)$ for [old news]
  - the Boolean arrangement $\mathcal{H} = \{x_j = 0 : 1 \le j \le d\}$
  - the braid arrangement $\mathcal{H} = \{x_j = x_k : 1 \le j < k \le d\}$
  - an arrangement $\mathcal{H}$ in $\mathbb{R}^d$ of $n$ hyperplanes in general position.
- Show that the coefficients of $p_{\mathcal{H}}(k)$ alternate in sign. [old news]
- Show that the absolute values of the coefficients form a unimodal sequence. [J. Huh, `arXiv:1008.4749`]
- Classify characteristic polynomials. [wide open]

# Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $k \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(k) := \# \left( k\mathcal{P} \cap \mathbb{Z}^d \right)$

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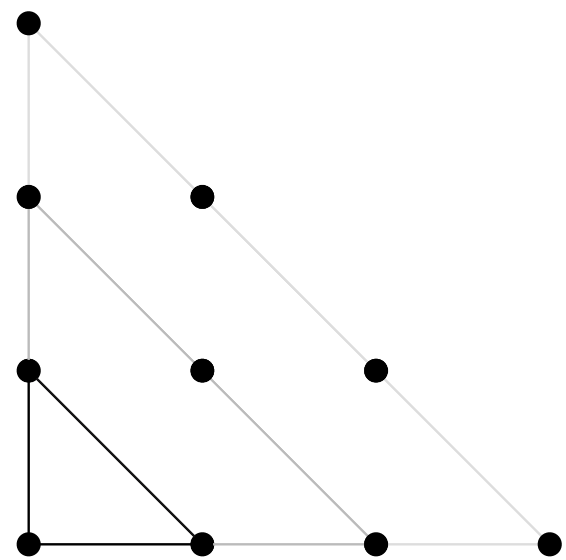
Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \, x,y \geq 0, \, x+y \leq 1 \right\}$$

$$L_\Delta(k) = \dots$$



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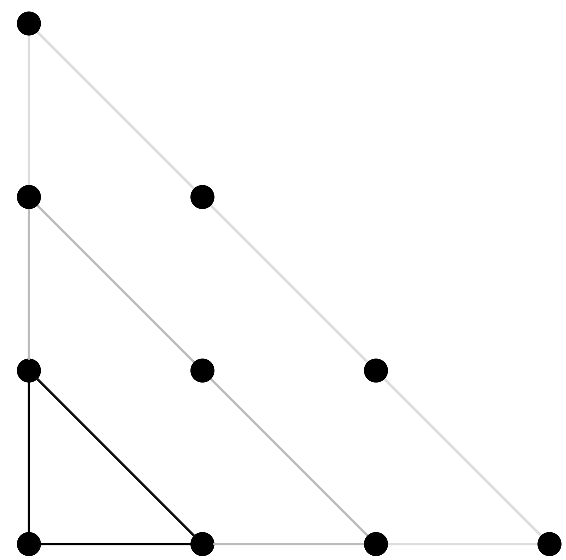

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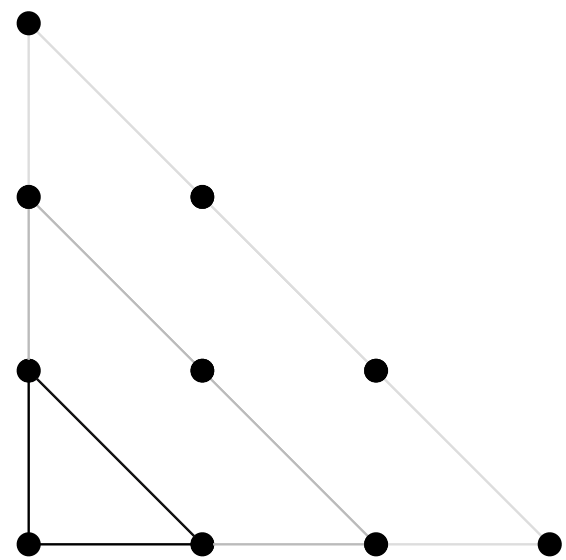

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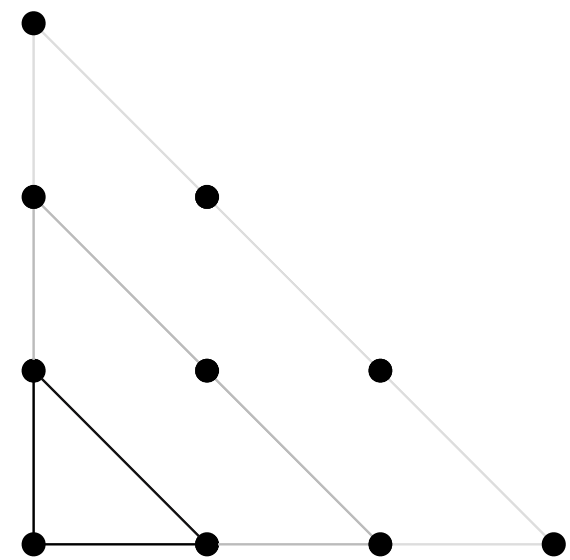

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$$L_\Delta(-k) = \binom{k-1}{2} = L_{\Delta^\circ}(k)$$

For example, the evaluations $L_\Delta(-1) = L_\Delta(-2) = 0$ point to the fact that neither $\Delta$ nor $2\Delta$ contain any interior lattice points.

# Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $k \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(k) := \#\left(k\mathcal{P} \cap \mathbb{Z}^d\right)$


Theorem (Ehrhart 1962) $L_{\mathcal{P}}(k)$ is a polynomial in $k$.

# Ehrhart Polynomials

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Theorem (Ehrhart 1962) $L_{\mathcal{P}}(k)$ is a polynomial in $k$.

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} L_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

# If you get bored. . .

- Show how the previous page for $d = 2$ follows from Pick's Theorem.

- Compute the Ehrhart polynomial of your favorite lattice polytope. Here are two of my favorites:
  - the cross polytope, the convex hull of the unit vectors in $\mathbb{R}^d$ and their negatives [old news]
  - the Birkhoff–von Neumann polytope of all doubly-stochastic $n \times n$ matrices. [open for $n \geq 10$]

- Classify Ehrhart polynomials of lattice polygons. [Scott 1976]

- Classify Ehrhart polynomials of lattice $3$-polytopes. [open]

# Combinatorial Reciprocity

Common theme: a combinatorial function, which is a priori defined on the positive integers,

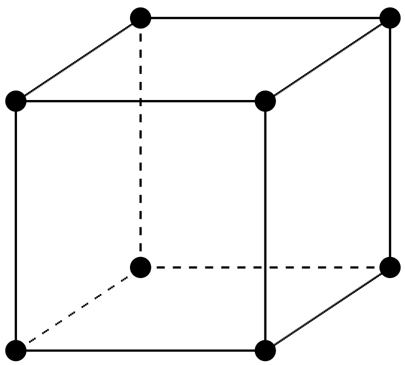
(1) can be algebraically extended beyond the positive integers (e.g., because it is a polynomial), and

(2) has (possibly quite different) meaning when evaluated at negative integers.

# The Mother of All Combinatorial Reciprocity Theorems

Polyhedron $\mathcal{P}$ – intersection of finitely many halfspaces

$$f_{\mathcal{P}}(k) := \sum_{\mathcal{F} \text{ face of } \mathcal{P}} k^{\dim \mathcal{F}} = k^3 + 6k^2 + 12k + 8$$



Note that $f_{\mathcal{P}}(-1) = 1$ . . .

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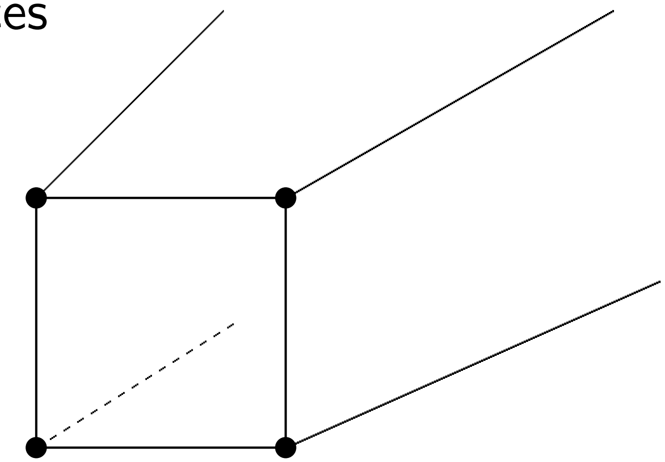

Note that $f_{\mathcal{P}}(-1) = 0$ . . .

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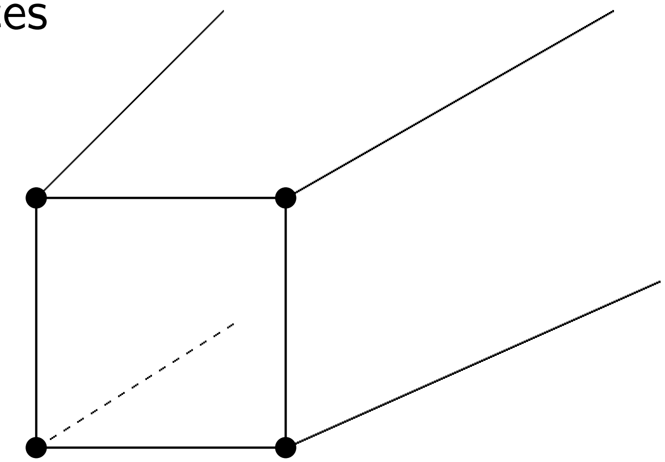

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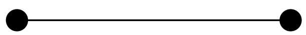
Note that $f_{\mathcal{P}}(-1) = 0$ . . .

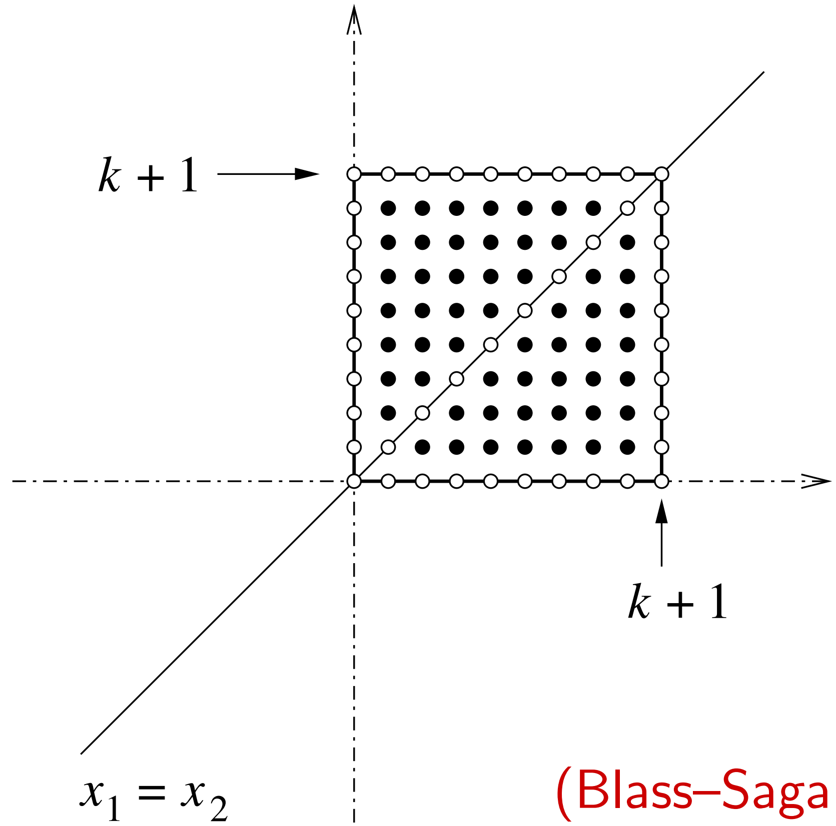
Theorem (Euler–Poincaré) For any polyhedron, $f_{\mathcal{P}}(-1) = 0$ or $\pm 1$.

# Graph Coloring a la Ehrhart

$\chi_{K_2}(k) = k(k-1)$ ...



$K_2$




(Blass–Sagan)

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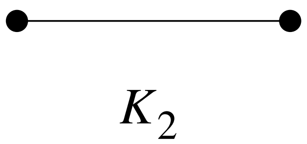


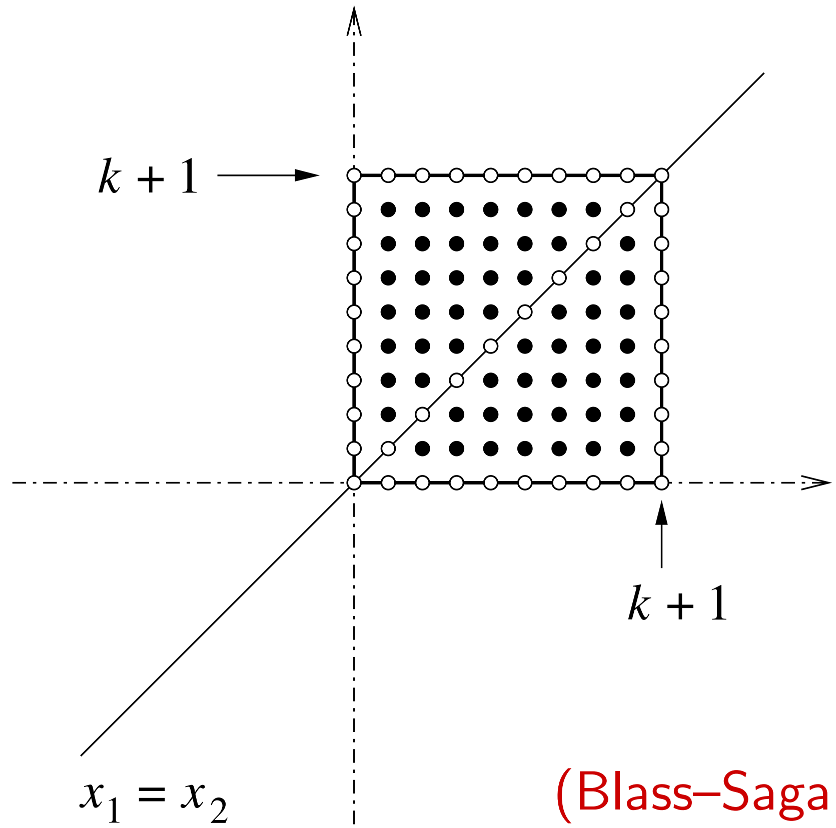


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$$\chi_G(k) \quad = \quad \#\Big(\big(\{1,2,\ldots,k\}^V \setminus \mathcal{H}\big) \cap \mathbb{Z}^V\Big)$$

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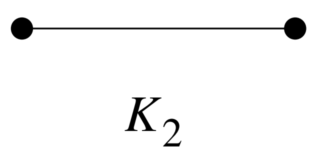
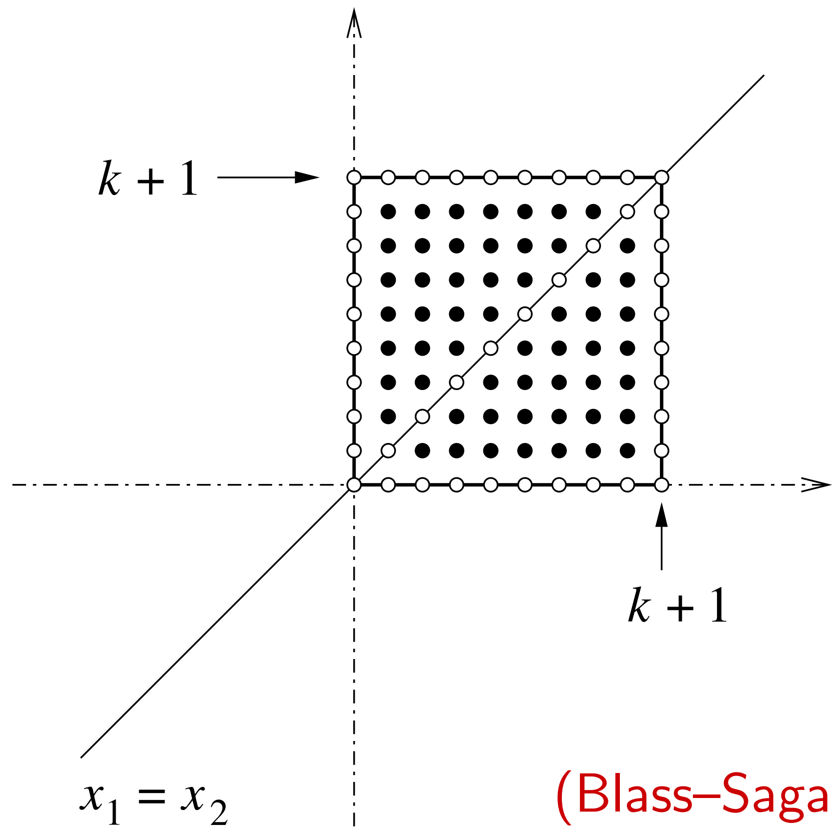



(Blass–Sagan)

$$\begin{aligned} \chi_G(k) &= \#\Big(\left(\{1,2,\dots,k\}^V \setminus \mathcal{H}\right) \cap \mathbb{Z}^V\Big) \\ &= \#\left((k+1)\left(\Box^\circ \setminus \mathcal{H}\right) \cap \mathbb{Z}^V\right) \end{aligned}$$

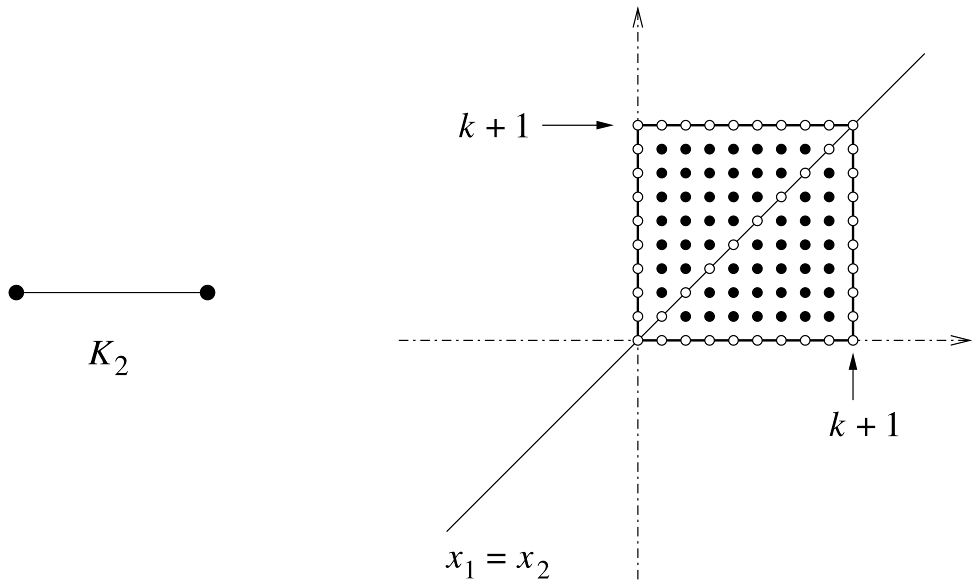
where $\Box$ is the unit cube in $\mathbb{R}^V$.

# Stanley's Theorem a la Ehrhart

$$\chi_G(k) = \#\left( (k+1)\left(\Box^\circ \setminus \mathcal{H}\right) \cap \mathbb{Z}^V \right)$$

Write $\Box^\circ \setminus \mathcal{H} = \bigcup_j \mathcal{P}_j^\circ$, then by Ehrhart–Macdonald reciprocity

$$(-1)^{|V|} \chi_G(-k) = \sum_j L_{\overline{P_j}}(k-1)$$




So $(-1)^{|V|} \chi_G(-k)$ counts lattice points in $k\,\Box$ with multiplicity #regions.

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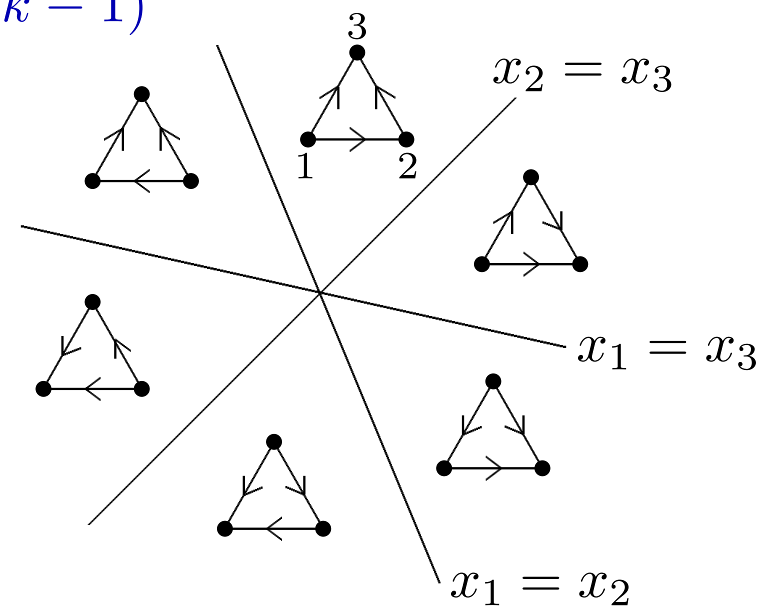


Greene's observation

region of $\mathcal{H}(G)$ $\iff$ acyclic orientation of $G$

$x_i < x_j \iff i \longrightarrow j$

Stanley's Theorem $(-1)^{|V|}\chi_G(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation $\alpha$ of $G$ and a compatible $k$-coloring $\boldsymbol{x}$.

# Inside-Out Polytopes

Underlying setup of our proof of Stanley's theorem:

$\mathcal{P}$ — (rational) polytope in $\mathbb{R}^d$

$\mathcal{H}$ — (rational) hyperplane arrangement

$$\mathcal{P}^\circ \setminus \mathcal{H} = \bigcup_j \mathcal{P}_j^\circ$$

Say we're interested in the counting function

$$f(k) := \# \left( k \left( \mathcal{P}^\circ \setminus \mathcal{H} \right) \cap \mathbb{Z}^d \right) = \sum_j L_{\mathcal{P}_j^\circ}(k) \, .$$

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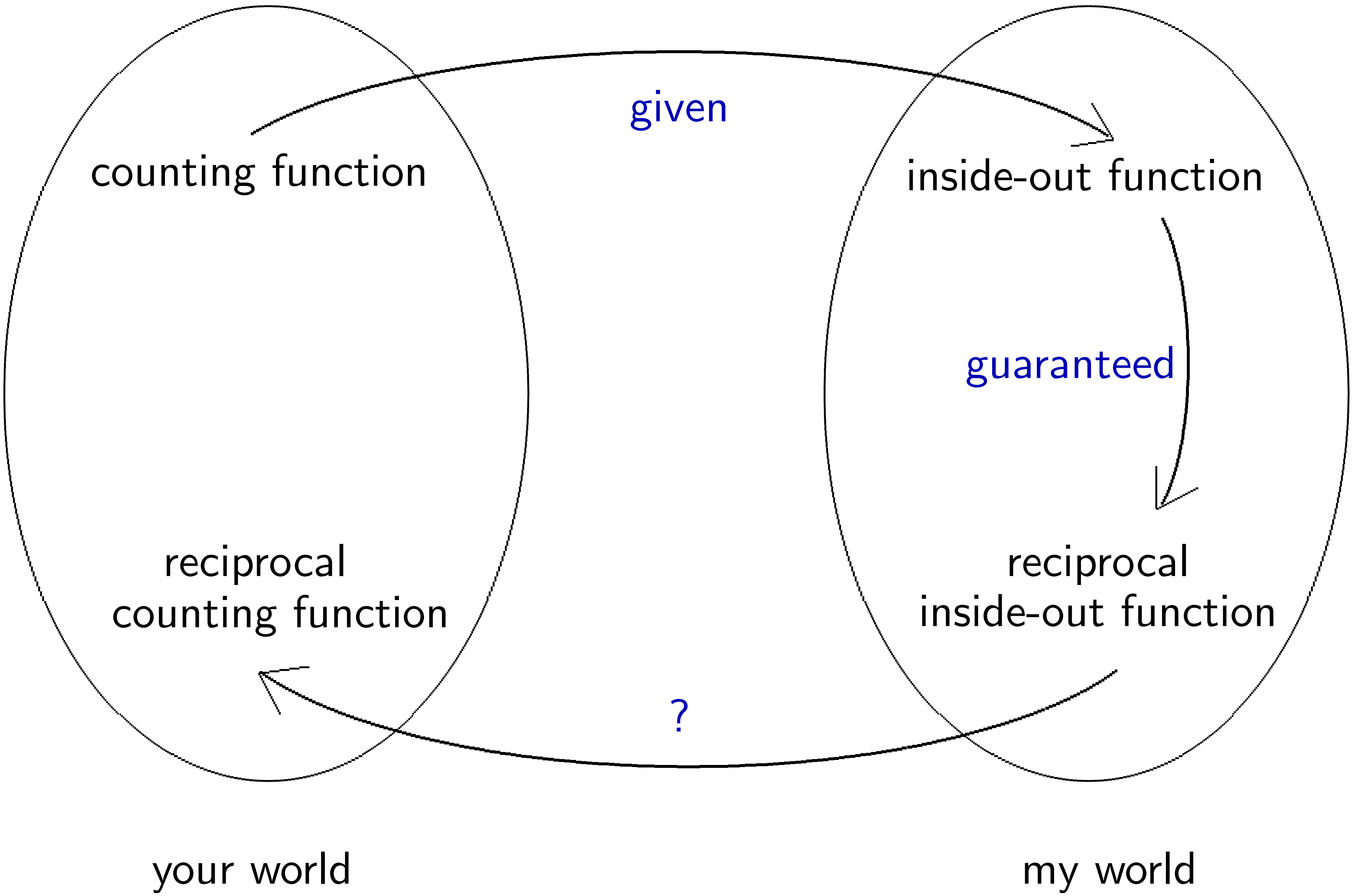
Ehrhart says that this is a (quasi-)polynomial, and by Ehrhart–Macdonald reciprocity,

$$f(-k) = (-1)^d \sum_j L_{\overline{P_j}}(k)$$

i.e., $(-1)^d f(-k)$ counts lattice points in $k\mathcal{P}$ with multiplicity $\#$regions.

# Make-Your-Own Combinatorial Reciprocity Theorem



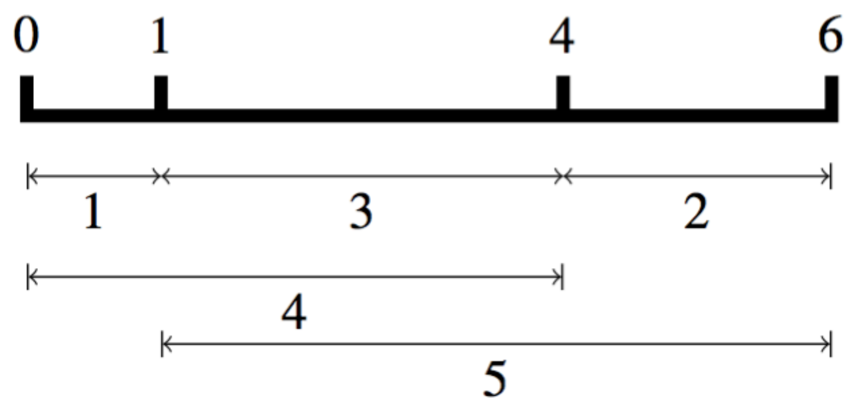
# Applications

- Nowhere-zero flow polynomials (MB–Zaslavsky 2006, Breuer–Dall 2011, Breuer–Sanyal 2011)
- Magic & Latin squares (MB–Zaslavsky 2006, MB–Van Herick 2011)
- Antimagic graphs (MB–Zaslavsky 2006, MB–Jackanich 201?)
- Nowhere-harmonic & bivariable graph colorings (MB–Braun 2011, MB–Chavin–Hardin 201?)
- Golomb rulers (MB–Bogart–Pham 201?)

. . . with lots of open questions.

# Golomb Rulers

Sequences of $n$ distinct integers whose pairwise differences are distinct



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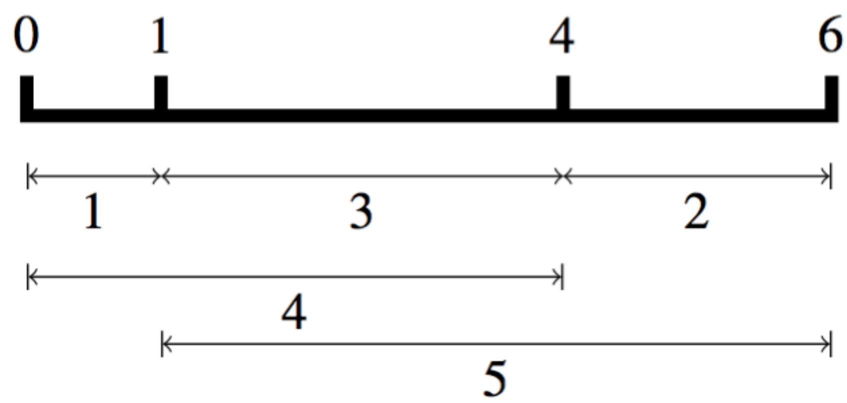


- Natural applications to error-correcting codes and phased array radio antennas
- Classical studies in additive number theory, more recent studies on existence problems (e.g., optimal Golomb rulers)

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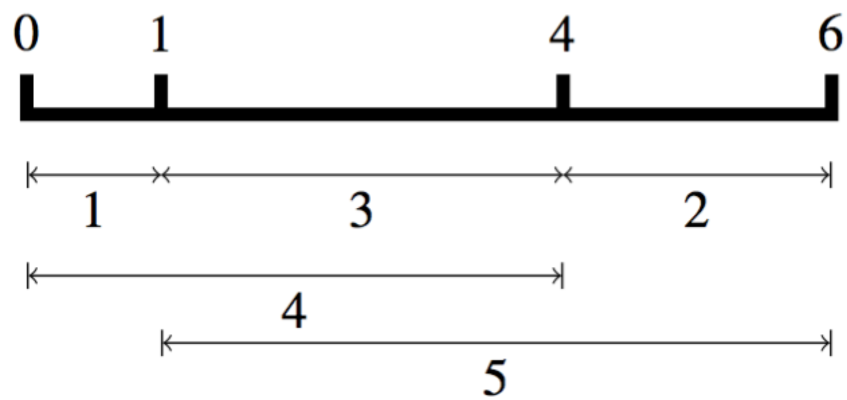


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$$g_m(t) := \# \left\{ \boldsymbol{x} \in \mathbb{Z}^{m+1} : \begin{array}{l} 0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = t \\ \text{all } x_j - x_k \text{ distinct} \end{array} \right\}$$

# Enumeration of Golomb Rulers

Goal Study/compute

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$$= \quad \# \left\{ \boldsymbol{z} \in \mathbb{Z}^m_{>0} : \begin{array}{l} z_1 + z_2 + \cdots + z_m = t \\ \sum_{j \in U} z_j \neq \sum_{j \in V} z_j \text{ for all dpcs } U, V \subset [m] \end{array} \right\}$$

where dpcs means disjoint proper consecutive subset.

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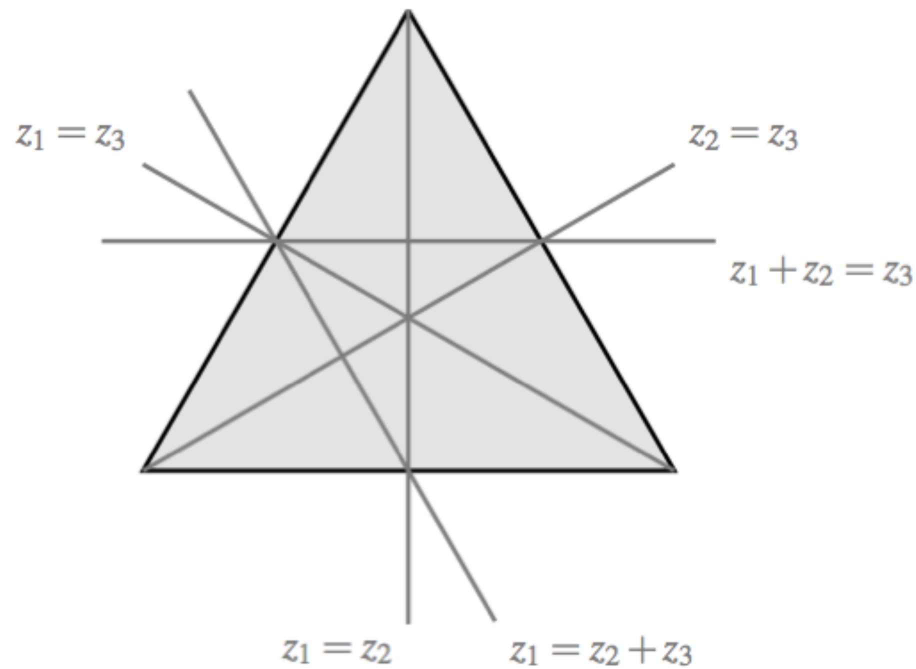

# Golomb Ruler Reciprocity

Real Golomb ruler — $\boldsymbol{z} \in \mathbb{R}^m_{\geq 0}$ satisfying $z_1 + z_2 + \cdots + z_m = t$ and

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$\boldsymbol{z}, \mathbf{w} \in \mathbb{R}_{\geq 0}^m$ are combinatorially equivalent if for any dpcs $U, V \subset [m]$

$$\sum_{j \in U} z_j < \sum_{j \in V} z_j \qquad \Longleftrightarrow \qquad \sum_{j \in U} w_j < \sum_{j \in V} w_j$$

Golomb multiplicty of $\boldsymbol{z} \in \mathbb{Z}_{\geq 0}^m$ — number of combinatorially different real Golomb rulers in an $\epsilon$-neighborhood of $\boldsymbol{z}$

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Golomb multiplicty of $\boldsymbol{z} \in \mathbb{Z}^m_{\geq 0}$ — number of combinatorially different real Golomb rulers in an $\epsilon$-neighborhood of $\boldsymbol{z}$

Theorem $g_m(t)$ is a quasipolynomial in $t$ whose evaluation $(-1)^m g_m(-t)$ equals the number of rulers in $\mathbb{Z}^m_{\geq 0}$ of length $t$, each counted with its Golomb multiplicity. Furthermore, $(-1)^m g_m(0)$ equals the number of combinatorially different Golomb rulers.

# More Golomb Counting

- Natural correspondence to certain mixed graphs
- Regions of the Golomb inside-out polytope correspond to acyclic orientations
- General reciprocity theorem for mixed graphs

# For much more. . .

math.sfsu.edu/beck/crt.html