Math 310 – Generating Functions

(1) Compute the generating function for the sequence

$$a_k = \begin{cases} 1 & \text{if } k \text{ is a multiple of } 7, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Compute the sequence (a_k) that gives rise to the generating function $\sum_{k \ge 0} a_k \, x^k = \left(\frac{1}{1-x} \right)^2$, by looking at the product $\left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$. (If you look at the result, can you think of a different way to compute (a_k)?)

(3) Define a recursive sequence by setting $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for $n \ge 0$.

(a) Conjecture a formula for a_k by experimenting.

(b) Now put the sequence (a_k) into a generating function $g(x)$ and find a formula for $g(x)$ by utilizing the recursive definition of a_k.

(c) Expand your formula for $g(x)$ into partial fractions, and use the result to prove your conjectured formula for a_k.

(4) We define a second recursive sequence by setting $a_0 = 1$ and $a_{n+1} = 2a_n + n$ for $n \ge 0$. Find a formula for a_k.

Matthias Beck math.sfsu.edu/beck