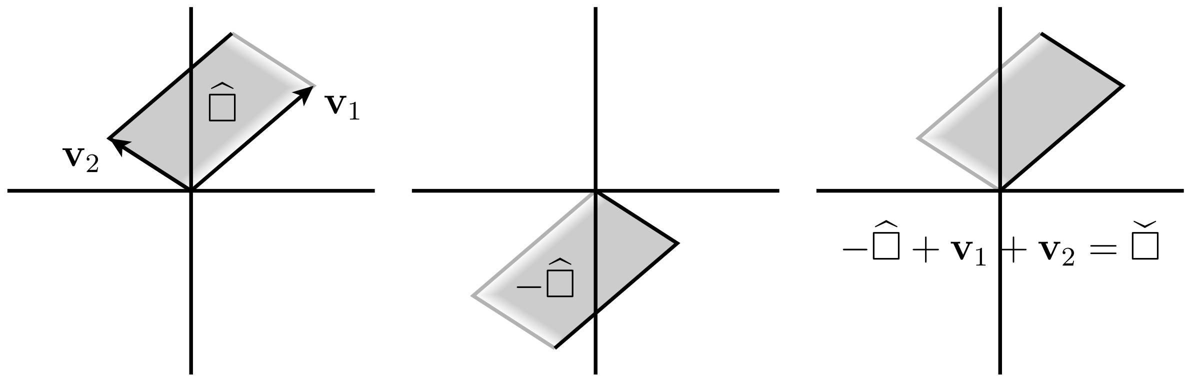
Combinatorial Reciprocity Theorems



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Combinatorial Reciprocity Theorems

or

What I've Learned From My Friends Raman Sanyal and Tom Zaslavsky

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The Theme

Combinatorics

counting function depending on $k \in \mathbb{Z}_{>0}$

polynomial $p(k)$

What is $p(0)$?
$p(-1)$? $p(-2)$?

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$p(-1)$? $p(-2)$?

- Two-for-one charm of combinatorial reciprocity theorems
- "Big picture" motivation: understand/classify these polynomials

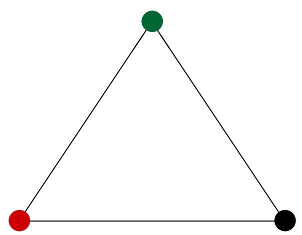
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper k-coloring of G — $\boldsymbol{x} \in [k]^V$ such that $x_i \neq x_j$ if $ij \in E$

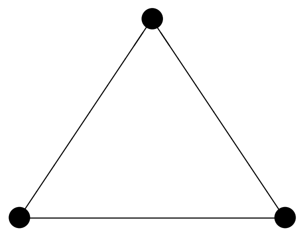
$\chi_G(k) := \#$ (proper k-colorings of G)

Example:



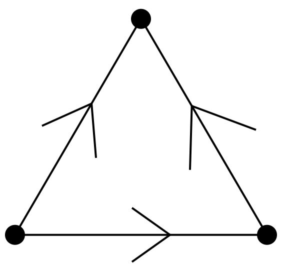
$$\chi_{K_3}(k) = k(k-1)(k-2)$$

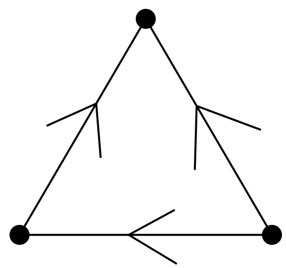
Chromatic Polynomials



$$\chi_{K_3}(k) = k(k-1)(k-2)$$

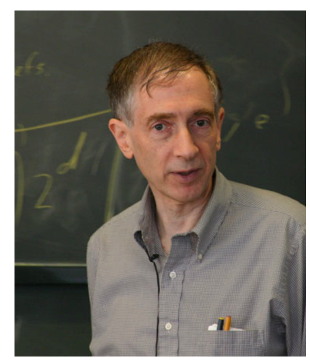
Theorem (Birkhoff 1912, Whitney 1932)
$\chi_G(k)$ is a polynomial in k.





$|\chi_{K_3}(-1)| = 6$ counts the number of acyclic orientations of K_3.

Theorem (Stanley 1973) $(-1)^{|V|}\chi_G(-k)$ equals the number of pairs $(\alpha, \boldsymbol{x})$ consisting of an acyclic orientation α of G and a compatible k-coloring $\boldsymbol{x}$. In particular, $(-1)^{|V|}\chi_G(-1)$ equals the number of acyclic orientations of G.



Order Polynomials

$(\Pi, \preceq)$ — finite poset

$$\Omega_\Pi(k) \ := \ \#\left\{\phi \in [k]^\Pi : \ a \preceq b \ \Longrightarrow \ \phi(a) \le \phi(b)\right\}$$

$$\Omega^\circ_\Pi(k) \ := \ \#\left\{\phi \in [k]^\Pi : \ a \prec b \ \Longrightarrow \ \phi(a) < \phi(b)\right\}$$

Example: $\Pi = [d] \ \longrightarrow \ \Omega^\circ_\Pi(k) = \binom{k}{d}$ and $\binom{-k}{d} = (-1)^d \binom{k+d-1}{d}$

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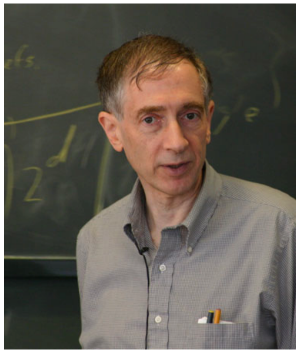
Theorem (Stanley 1970) $\Omega_\Pi(k)$ and $\Omega_\Pi^\circ(k)$ are polynomials related via $\Omega_\Pi^\circ(-k) = (-1)^{|\Pi|}\, \Omega_\Pi(k)$.



Order Polynomials

$(\Pi, \preceq)$ — finite poset

$$\sum_{k \ge 0} \Omega_\Pi^{(\circ)}(k)\, z^k = \frac{h_\Pi^{(\circ)}(z)}{(1-z)^{|\Pi|+1}}$$

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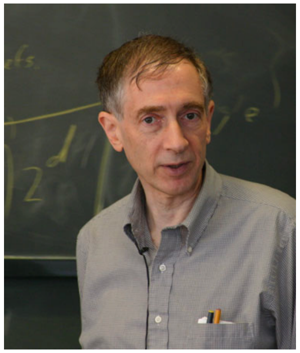


Theorem (Stanley 1970) $\Omega_\Pi(k)$ and $\Omega_\Pi^\circ(k)$ are polynomials related via $\Omega_\Pi^\circ(-k) = (-1)^{|\Pi|}\, \Omega_\Pi(k)$.

Equivalently, $z^{|\Pi|+1}\, h_\Pi^\circ(\frac{1}{z}) = h_\Pi(z)$.

Eulerian Simplicial Complexes

Γ — simplicial complex (collection of subsets of a finite set, closed under taking subsets)

Γ is Eulerian if it is pure and every interval has as many elements of even rank as of odd rank

$f_j \; := \; \#\ (j+1)$-subsets $\; = \; \#$ faces of dimension j

$$h(z) \; := \; \sum_{j=0}^{d+1} f_{j-1}\, z^j \, (1-z)^{d+1-j}$$

Theorem (Everyone 19xy) If Γ is Eulerian then $z^{d+1}\, h(\frac{1}{z}) \; = \; h(z)$.

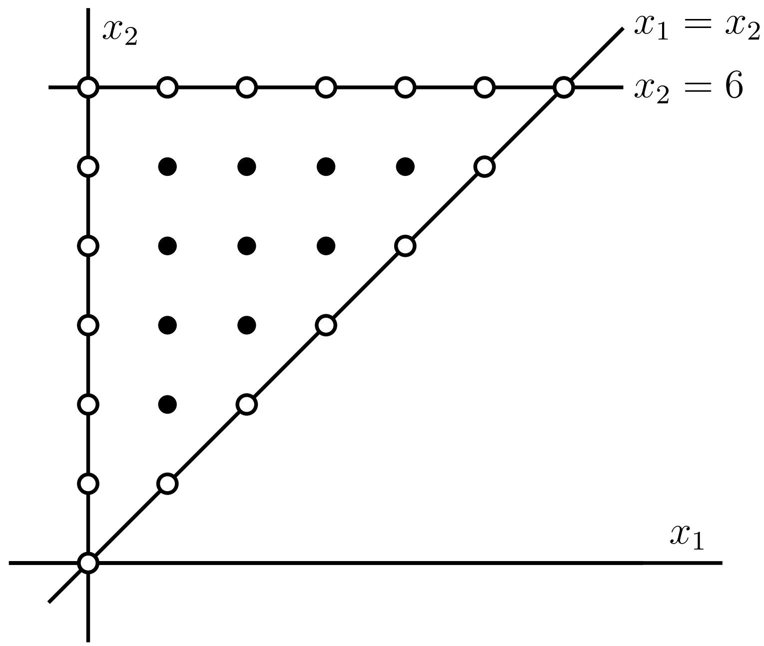
Key example (Dehn–Sommerville): $\Gamma =$ boundary complex of a simplicial polytope

Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $k \in \mathbb{Z}_{>0}$ let $\mathrm{ehr}_{\mathcal{P}}(k) := \# \left(k\mathcal{P} \cap \mathbb{Z}^d \right)$

Example:



$$\Delta = \mathrm{conv}\left\{(0,0),(0,1),(1,1)\right\}$$

$$= \left\{(x,y) \in \mathbb{R}^2 : \, 0 \le x_1 \le x_2 \le 1\right\}$$

$$\mathrm{ehr}_\Delta(k) = \binom{k+2}{2} = \frac{1}{2}(k+1)(k+2)$$

$$\mathrm{ehr}_\Delta(-k) = \binom{k-1}{2} = \mathrm{ehr}_{\Delta^\circ}(k)$$

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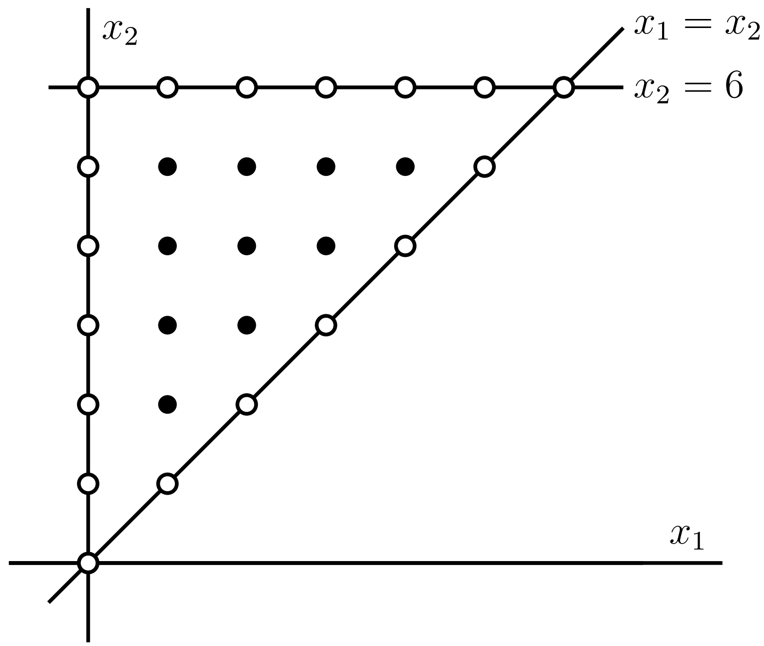

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For example, the evaluations $\mathrm{ehr}_\Delta(-1) = \mathrm{ehr}_\Delta(-2) = 0$ point to the fact that neither Δ nor 2Δ contain any interior lattice points.

Ehrhart Polynomials

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Theorem (Ehrhart 1962) $\mathrm{ehr}_{\mathcal{P}}(k)$ is a polynomial in k.

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} \, \mathrm{ehr}_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

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$$\mathrm{Ehr}_{\mathcal{P}}(z) \ := \ 1 + \sum_{k>0} \mathrm{ehr}_{\mathcal{P}}(k)\, z^k \ = \ \frac{h^*_{\mathcal{P}}(z)}{(1-z)^{\dim(\mathcal{P})+1}}$$

Theorem (Macdonald 1971) $(-1)^{\dim \mathcal{P}} \, \mathrm{ehr}_{\mathcal{P}}(-k)$ enumerates the interior lattice points in $k\mathcal{P}$.

$$z^{\dim(\mathcal{P})} \, h^*_{\mathcal{P}}(\tfrac{1}{z}) \ = \ h^*_{\mathcal{P}^\circ}(z)$$

Combinatorial Reciprocity

Common theme: a combinatorial function, which is a priori defined on the positive integers,

(1) can be algebraically extended beyond the positive integers (e.g., because it is a polynomial), and

(2) has (possibly quite different) meaning when evaluated at negative integers.

Generating-function version: evaluate at reciprocals.

Ehrhart $\longrightarrow$ Order Polynomials

$(\Pi, \preceq)$ — finite poset

Order polytope $\mathcal{O}_\Pi \ := \ \left\{ \phi \in [0,1]^\Pi : \ a \preceq b \ \Longrightarrow \ \phi(a) \le \phi(b) \right\}$

$$\Omega_\Pi(k) \ = \ \# \left\{ \phi \in [k]^\Pi : \ a \preceq b \ \Longrightarrow \ \phi(a) \le \phi(b) \right\} \ = \ \mathrm{ehr}_{\mathcal{O}_\Pi}(k-1)$$

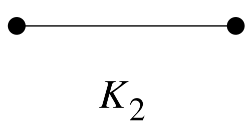
$$\Omega_\Pi^\circ(k) \ = \ \# \left\{ \phi \in [k]^\Pi : \ a \prec b \ \Longrightarrow \ \phi(a) < \phi(b) \right\} \ = \ \mathrm{ehr}_{\mathcal{O}_\Pi^\circ}(k+1)$$

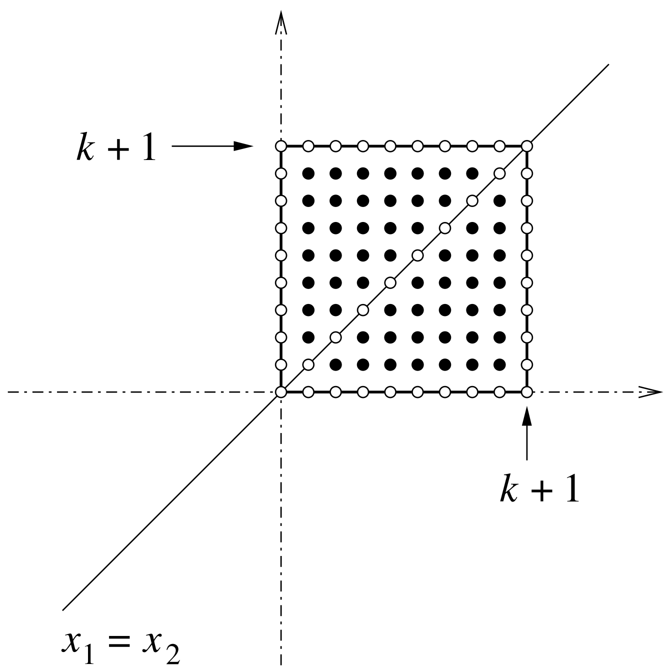
and so $\mathrm{ehr}_\mathcal{P}^\circ(-k) = (-1)^{\dim \mathcal{P}} \, \mathrm{ehr}_\mathcal{P}(k)$ implies $\Omega_\Pi^\circ(-k) = (-1)^{|\Pi|} \, \Omega_\Pi(k)$

Order $\longrightarrow$ Chromatic Polynomials

$$\chi_G(k) \;=\; \#\,(\text{proper } k\text{-colorings of } G) \;=\; \sum_{\Pi \text{ acyclic}} \Omega^\circ_\Pi(k)$$

$$(-1)^{|V|}\,\chi_G(-k) \;=\; \sum_{\Pi \text{ acyclic}} (-1)^{|\Pi|}\,\Omega^\circ_\Pi(-k) \;=\; \sum_{\Pi \text{ acyclic}} \Omega_\Pi(k)$$





Chain Partitions

$(\Pi, \preceq)$ — finite graded poset with $\hat{0}$ and $\hat{1}$

$\phi : \Pi \setminus \{\hat{0}, \hat{1}\} \to \mathbb{Z}_{>0}$ order preserving

(Π, ϕ)-chain partition of $n \in \mathbb{Z}_{>0}$: $n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
for some multichain $\hat{1} \succ c_m \succeq c_{m-1} \succeq \cdots \succeq c_1 \succ \hat{0}$

$\mathrm{cp}_{\Pi,\phi}(k) :=$ #(chain partitions of k) $\qquad \mathrm{CP}_{\Pi,\phi}(z) := 1 + \sum_{k>0} \mathrm{cp}_{\Pi,\phi}(k)\, z^k$

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Example: $A = \{a_1 < a_2 < \cdots < a_d\} \subset \mathbb{Z}_{>0} \qquad \Pi = [d] \qquad \phi(j) := a_j$

$\longrightarrow$ $\mathrm{cp}_{\Pi,\phi}(k)$ is the restricted partition function with parts in A

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ϕ is ranked if $\mathrm{rank}(a) = \mathrm{rank}(b) \Longrightarrow \phi(a) = \phi(b)$

$\phi_j := \phi(a)$ for $\mathrm{rank}(a) = j$

Theorem $\quad (-1)^{\mathrm{rank}(\Pi)}\, \mathrm{CP}_{\Pi,\phi}(\frac{1}{z}) \;=\; z^{\phi_1 + \cdots + \phi_d}\, \mathrm{CP}_{\Pi,\phi}(z)$

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$\operatorname{cp}_{\Pi,\phi}(k) := \#(\text{chain partitions of } k)$ $\qquad$ $\operatorname{CP}_{\Pi,\phi}(z) := 1 + \sum_{k>0} \operatorname{cp}_{\Pi,\phi}(k)\, z^k$

For $S = \{s_1 < s_2 < \cdots < s_m\}$ let

$$\alpha_\Pi(S) := \#\left\{\hat{0} \prec c_1 \prec c_2 \prec \cdots \prec c_m \prec \hat{1} : \operatorname{rank}(c_j) = s_j\right\}$$

flag f-vector of Π : $\quad (\alpha_\Pi(S) : S \subseteq [d])$

Observation $\quad \operatorname{CP}_{\Pi,\phi}(z) \;=\; \dfrac{\sum_{S \subseteq [d]} \alpha(S) \prod_{s \in S} z^{\phi_s} \prod_{s \notin S} (1 - z^{\phi_s})}{(1 - z^{\phi_1})(1 - z^{\phi_2}) \cdots (1 - z^{\phi_d})}$

Chain Partitions for Simplicial Complexes

$\Pi = \Gamma \cup \{\hat{1}\}$ for a d-simplicial complex Γ with ground set V

$\phi(\sigma) = \mathrm{rank}(\sigma) = |\sigma|$

(Π, ϕ)-chain partition of $n \in \mathbb{Z}_{>0}$: $\quad n = \phi(c_m) + \phi(c_{m-1}) + \cdots + \phi(c_1)$
for some multichain $\hat{1} \succ c_m \succeq c_{m-1} \succeq \cdots \succeq c_1 \succ \hat{0}$

$$\mathrm{cp}_{\Pi,\phi}(k) \ := \ \#(\text{chain partitions of } k) \ = \ \sum_{j=0}^{d+1} f_{j-1} \binom{k}{j}$$

Canonical geometric realization of Γ in $\mathbb{R}^V$:

$$\mathcal{R}[\Gamma] \ := \ \left\{ \mathrm{conv}\{\mathbf{e}_v : \, v \in \sigma\} : \, \sigma \in \Gamma \right\}$$

An Ehrhartian Interlude for Polytopal Complexes

$\mathcal{C}$ — d-dimensional complex of lattice polytopes
with Euler characteristic $1-(-1)^{d+1}$

Ehrhart polynomial $\operatorname{ehr}_{\mathcal{C}}(k) := \#\left(k\,|\mathcal{C}| \cap \mathbb{Z}^d\right)$

We call $\mathcal{C}$ self reciprocal if $\operatorname{ehr}_{\mathcal{C}}(-k) \;=\; (-1)^d \operatorname{ehr}_{\mathcal{C}}(k)$. Equivalently,

$\operatorname{Ehr}_{\mathcal{C}}(z) \;:=\; 1+\sum_{k>0} \operatorname{ehr}_{\mathcal{C}}(k)\, z^k \;=\; \dfrac{h^*_{\mathcal{C}}(z)}{(1-z)^{d+1}}$ satisfies $z^{d+1} h^*_{\mathcal{C}}(\frac{1}{z}) = h^*_{\mathcal{C}}(z)$

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Key examples:

$\mathcal{C} =$ boundary complex of a lattice polytope

$\mathcal{C} =$ Eulerian complex of lattice polytopes

Back to Chain Partitions for Simplicial Complexes

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Observation 1 $\;\mathrm{cp}_{\Pi,\phi}(k) \; := \; \#(\text{chain partitions of } k) \; = \; \mathrm{ehr}_{\mathcal{R}[\Gamma]}(k)$

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Observation 1 $\; \mathrm{cp}_{\Pi,\phi}(k) \; := \; \#(\text{chain partitions of } k) \; = \; \mathrm{ehr}_{\mathcal{R}[\Gamma]}(k)$

Observation 2 $\; \mathrm{CP}_{\Pi,\phi}(k) \; = \; \mathrm{Ehr}_{\mathcal{R}[\Gamma]}(z) \; = \; \dfrac{h^*_{\mathcal{R}[\Gamma]}(z)}{(1-z)^{d+1}} \; = \; \dfrac{h_\Gamma(z)}{(1-z)^{d+1}}$

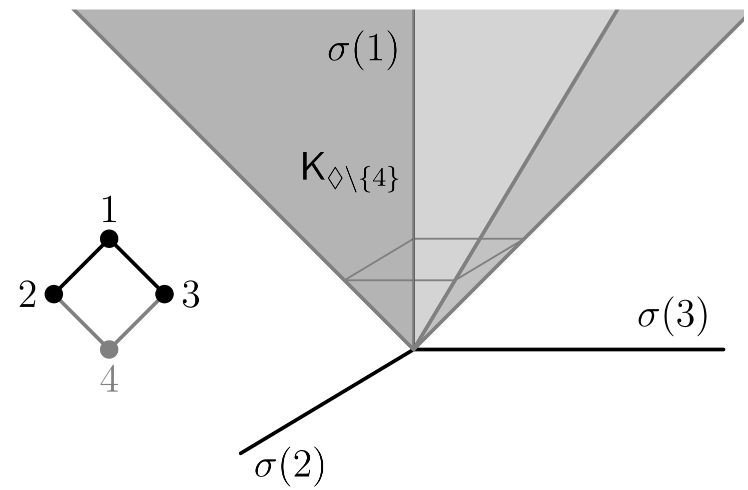
Corollary If Γ is Eulerian then $\; z^{d+1} \, h_\Gamma(\frac{1}{z}) \; = \; h_\Gamma(z)$.

The Answer To Most Of Life's Questions

$(\Pi, \preceq)$ — finite poset

Order cone $\mathcal{K}_\Pi := \left\{ \phi \in \mathbb{R}^\Pi_{\geq 0} : \, a \preceq b \implies \phi(a) \leq \phi(b) \right\}$

- Interesting geometry
- Linear extensions $\longrightarrow$ triangulations
- Order polynomials
- P-partitions
- Euler–Mahonian statistics



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