Proposition 12.7. *Suppose $a, b, c \in \mathbb{R}$, where a and b are not both zero. Then the equation $ax^2 + bx + c = 0$ has a solution in $\mathbb{R}$ if and only if $b^2 - 4ac \geq 0$.*

Proof. We start with the case $a = 0$. Then $b^2 - 4ac = b^2 \geq 0$ is automatic, so that we have to show that the equation $ax^2 + bx + c = bx + c = 0$ *always* has a solution. But by assumption, $b \neq 0$, so $x = -c/b$ is a solution.

The second case is $a \neq 0$. Now we can rewrite the equation $ax^2 + bx + c = 0$ as follows:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \,. \end{aligned}$$

The statement of the proposition is hence equivalent to: the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{12.2}$$

has a solution $x \in \mathbb{R}$ if and only if $b^2 - 4ac \geq 0$.

Suppose first that (12.2) has a solution $x \in \mathbb{R}$. Since the left-hand side of (12.2) is a square, the right-hand side has to be ≥ 0. But $4a^2 \geq 0$, whence $b^2 - 4ac \geq 0$.

Conversely, suppose $b^2 - 4ac \geq 0$. Then $\frac{b^2-4ac}{4a^2} \geq 0$, so that we can take the square root of this number to obtain from (12.2)

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \,,$$

which is equivalent to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,,$$

which are both solutions to (12.2). $\square$