Rational Ehrhart Theory

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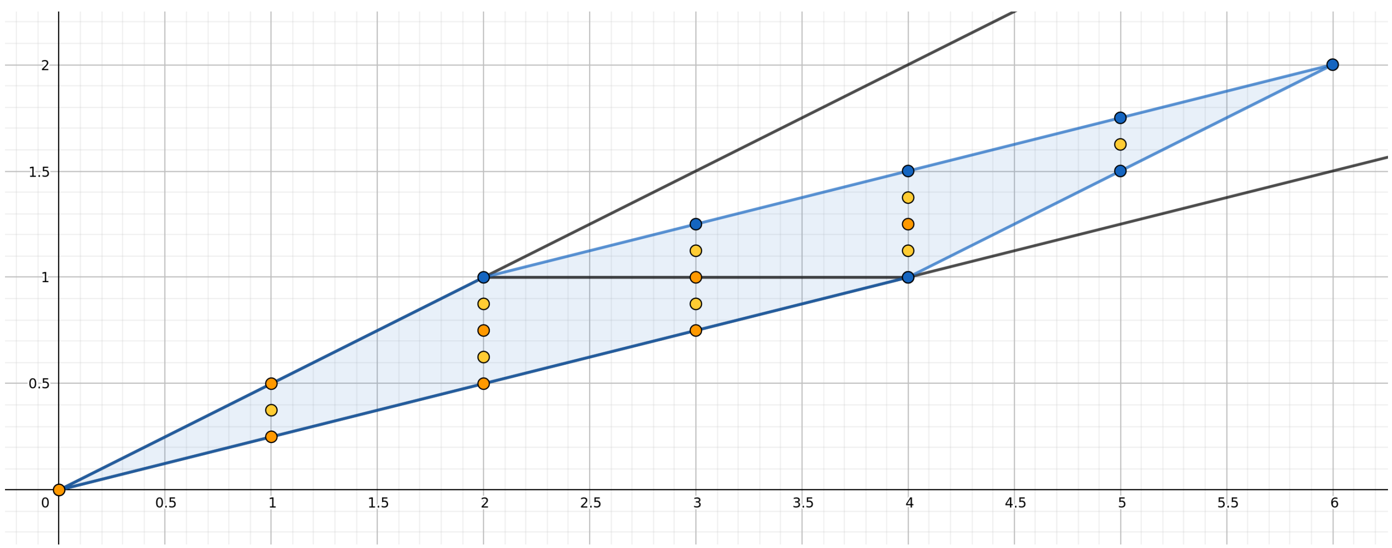


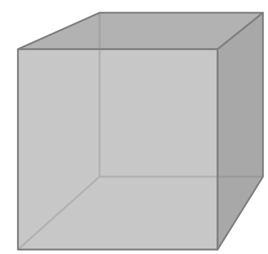
The Menu

- Ehrhart (quasi-)polynomials
- Motivation
- History
- Rational Ehrhart series
- Rational Gorenstein polytopes
- Symmetric h^*-decompositions
- Open problems

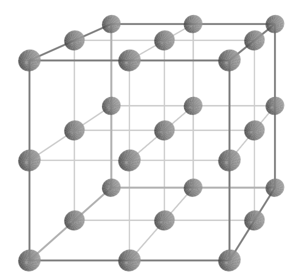
Measuring Polytopes

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$
— solution set of a system of linear (in-)equalities with integer coefficients

Goal: measuring...



- volume $\operatorname{vol}(\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n^d} \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right|$



- discrete volume $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$

Ehrhart function $\operatorname{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

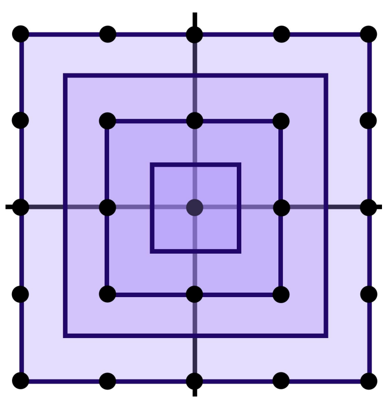
Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$

$q(n) = c_d(n)\, n^d + \cdots + c_0(n)$ is a quasipolynomial if $c_0(n), \dots, c_d(n)$ are periodic functions; the lcm of their periods is the period of $q(n)$.

Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\mathrm{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of P (the denominator of P).

Example $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]^2$



Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$

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Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\operatorname{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of P (the denominator q of P).

Equivalently, the Ehrhart series can be written as

$$\operatorname{Ehr}_{\mathbb{Z}}(\mathcal{P}; t) \;:=\; 1 + \sum_{n \geq 1} \operatorname{ehr}_{\mathbb{Z}}(\mathcal{P}; n)\, t^n \;=\; \frac{\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P}; t)}{(1 - t^q)^{\dim \mathcal{P} + 1}}$$

Why care about... Ehrhart (Quasi-)Polynomials

- Linear systems are everywhere, and so polyhedra are everywhere.
- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.
- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.
- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.

Motivation I: Let's be rational

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope

Ehrhart functions $\operatorname{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

$$\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) := \left| \lambda\mathcal{P} \cap \mathbb{Z}^d \right| \text{ for } \lambda \in \mathbb{Q}_{>0}$$

Fun Fact (Linke 2011, Baldoni–Berline–Köppe–Vergne 2013, Stapledon 2017). There is an Ehrhart theory for the quasipolynomial $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ in the rational (equivalently, real) variable λ.

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Examples $\operatorname{ehr}_{\mathbb{Q}}([1, 2]; \lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

$$\operatorname{ehr}_{\mathbb{Q}}([-1, \tfrac{2}{3}]; \lambda) = \tfrac{5}{3}\lambda + 1 - \{\tfrac{2}{3}\lambda\} - \{\lambda\}$$

Motivation II: Ehrhart Veronese

$\mathcal{P} \subset \mathbb{R}^d$ — lattice polytope

Ehrhart function $\mathrm{ehr}_{\mathbb{Z}}(\mathcal{P}; n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

Ehrhart series $\mathrm{Ehr}_{\mathbb{Z}}(\mathcal{P}; z) := 1 + \sum_{n \geq 1} \mathrm{ehr}_{\mathbb{Z}}(\mathcal{P}; n)\, z^n = \frac{\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P}; z)}{(1-z)^{\dim \mathcal{P} + 1}}$

Fun Fact (Brenti–Welker 2009, MB–Stapledon 2010, Jochemko 2018).
The Ehrhart series of $k\mathcal{P}$ becomes nicer as k increases.

A Bit of Real Ehrhart History

Theorem (Linke 2011) Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational polytope. Then $\mathrm{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda) = \left|\lambda\mathcal{P} \cap \mathbb{Z}^d\right|$ is a quasipolynomial in the rational variable $\lambda > 0$ whose period divides the smallest $q \in \mathbb{Q}_{>0}$ such that $q\mathcal{P}$ is a lattice polytope.

Linke also proved rational analogs of the Ehrhart–Macdonald reciprocity theorem and McMullen's (1978) theorem about the periods of the coefficient functions when writing

$$\mathrm{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda) \ = \ c_d(\lambda)\,\lambda^d + c_{d-1}(\lambda)\,\lambda^{d-1} + \cdots + c_0(\lambda)$$

She views these coefficient functions as piecewise polynomials and proved a differential equation for them.

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Baldoni–Berline–Köppe–Vergne (2013): algorithmic theory of intermediate sums on polyhedra, with $\mathrm{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ as a special case.

Royer (2017+): study of rational Gorenstein polytopes (and much more).

A Bit of Real Ehrhart History

Motivated by motivic integration, Stapledon (2008) introduced the weighted h^*-polynomial of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, which he later (2017) realizes via

$$1 + \sum_{\lambda \in \mathbb{Q}_{>0}} \left| \partial_{\neq 0}(\lambda \mathcal{P}) \cap \mathbb{Z}^d \right| t^\lambda \;=\; \frac{\widetilde{\mathrm{h}}(\mathcal{P};t)}{(1-t)^{\dim \mathcal{P}}}$$

and uses them to compute Ehrhart polynomials of free sums, generalizing work by Braun (2006) and MB–Jayawant–McAllister (2013).

He realizes that $\widetilde{\mathrm{h}}(\mathcal{P};t)$ is a polynomial in certain fractional powers of t with nonnegative coefficients. In the case that $\mathbf{0} \in \mathcal{P}^\circ$ he proves that $\widetilde{\mathrm{h}}(\mathcal{P};t)$ is symmetric.

The Setup

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope $\longrightarrow$ $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

The codemoninator of $\mathcal{P}$ is $r := \operatorname{lcm}(\mathbf{b})$

Lemma (1) $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda) = \left|\lambda\mathcal{P} \cap \mathbb{Z}^d\right|$ is constant for $\lambda \in (\frac{n}{r}, \frac{n+1}{r}),\ n \in \mathbb{Z}_{\ge 0}$

(2) If $\mathbf{0} \in \mathcal{P}$ then $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)$ is monotone

Examples $\operatorname{ehr}_{\mathbb{Q}}([1,2];\lambda) = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

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Corollary If $\lambda \notin \frac{1}{r}\mathbb{Z}_{\ge 0}$ then $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda) = \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lfloor\lambda\rceil)$

where $\lfloor\lambda\rceil := \frac{2j+1}{2r}$ for $\left|\lambda - \frac{2j+1}{2r}\right| < \frac{1}{2r}$

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The Upshot $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda)$ is determined by $\left\{ \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P}; \lambda) : \lambda \in \frac{1}{2r}\mathbb{Z}_{\ge 0} \right\}$

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Rational Ehrhart Series

$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\, \mathbf{x} \le \mathbf{b} \right\}$ with codemoninator $r := \operatorname{lcm}(\mathbf{b})$

(Refined) rational Ehrhart series

$$\begin{aligned} \operatorname{Ehr}_{\mathbb{Q}}(\mathcal{P};t) &:= 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)\, t^{\lambda} \\ \operatorname{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P};t) &:= 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)\, t^{\lambda} \end{aligned}$$

Examples

$$\begin{aligned} \operatorname{Ehr}_{\mathbb{Q}}([-1, \tfrac{2}{3}];t) &= \frac{1 + t^{\frac{1}{2}} + t + t^{\frac{3}{2}} + t^2}{(1-t)(1-t^{\frac{3}{2}})} \\ \operatorname{Ehr}_{\mathbb{Q}}^{\text{ref}}([1,2];t) &= \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2} \end{aligned}$$

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\end{aligned}$$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\operatorname{Ehr}_{\mathbb{Q}}(\mathcal{P};t) = \frac{\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{h}^*_{\mathbb{Q}}$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

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where $\mathrm{h}^*_{\mathbb{Q}}(t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Proof Idea: $\operatorname{Ehr}_{\mathbb{Q}}(\mathcal{P}; t) = \operatorname{Ehr}_{\mathbb{Z}}\left(\frac{1}{r}\mathcal{P}; t^{\frac{1}{r}}\right)$

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where $\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Similar results hold for $\operatorname{Ehr}^{\text{ref}}_{\mathbb{Q}}(\mathcal{P};t)$, with r replaced by $2r$.

Corollary (Linke 2011) The period of the quasipolynomial $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)$ divides $\frac{m}{r}$. In particular, if $\mathcal{P}$ is a lattice polytope then $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)$ is a quasipolynomial of period 1.

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where $\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Corollary (Linke 2011) For a rational polytope, $(-1)^{\dim \mathcal{P}} \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};-\lambda)$ equals the number of interior lattice points in $\lambda\mathcal{P}$, for any $\lambda > 0$.

Remark If $\frac{m}{r} \in \mathbb{Z}$ then $\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P};t) = \operatorname{Int}\left[\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t)\right]$.

Gorenstein Musings

Theorem (Hibi 1991, Fiset–Kasprzyk 2008) Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. If the polar dual of $\mathcal{P}$ is a lattice polytope then $\mathrm{h}_{\mathbb{Z}}^*(\mathcal{P};t)$ is symmetric.

(This fits, more generally, with Stanley's theory of Gorenstein rings.)

Theorem Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{h}_{\mathbb{Q}}^*(\mathcal{P};t)$ is symmetric.

Proof Idea: $(1, \mathbf{0})$ is the Gorenstein point for $\mathrm{cone}\left(\{1\} \times \frac{1}{r}\mathcal{P}\right)$.

Remark This is a corollary of a more general, technical result, with similar musings for the h^*-polynomial of the refined Ehrhart series.

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Remark This is a corollary of a more general, technical result, with similar musings for the h^*-polynomial of the refined Ehrhart series.

Corollary Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P};t)$ is the Veronese of a symmetric polynomial.

Example $\mathrm{Ehr}_{\mathbb{Q}}([-1,\frac{2}{3}];t) \ =$

$$\frac{1+t^{\frac{1}{2}}+2t+3t^{\frac{3}{2}}+4t^2+4t^{\frac{5}{2}}+4t^3+4t^{\frac{7}{2}}+3t^4+2t^{\frac{9}{2}}+t^5+t^{\frac{11}{2}}}{(1-t^3)^2}$$

Symmetric Decompositions

Theorem Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t)$ is symmetric.

Corollary (Betke–McMullen 1985, MB–Braun–Vindas-Meléndez 2022) Let $\mathcal{P}$ be a rational polytope with denominator k and $\mathbf{0} \in \mathcal{P}^\circ$. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that

$$\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P};t) = a(t) + t\, b(t)\,, \quad t^{k(d+1)-1}\, a(\tfrac{1}{t}) = a(t)\,, \quad t^{k(d+1)-2}\, b(\tfrac{1}{t}) = b(t)\,.$$

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Proof Idea ($k=1$):

$$\mathrm{Ehr}_{\mathbb{Q}}(\mathcal{P};t) \ = \ \frac{\mathrm{h}^*_{\mathbb{Z}}\left(\partial(\frac{1}{r}\mathcal{P});t^{\frac{1}{r}}\right)}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}$$

Note that $\mathrm{h}^*_{\mathbb{Z}}(\partial(\frac{1}{r}\mathcal{P});t)$ is symmetric and nonnegative, and

$$\begin{aligned}
\mathrm{h}^*_{\mathbb{Q}}(\mathcal{P};t) \ &= \ \mathrm{Int}\left[\left(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\mathrm{h}^*_{\mathbb{Z}}\left(\partial(\tfrac{1}{r}\mathcal{P});t^{\frac{1}{r}}\right)\right] \\
&= \ \mathrm{Int}\left[\mathrm{h}^*_{\mathbb{Z}}\left(\partial(\tfrac{1}{r}\mathcal{P});t^{\frac{1}{r}}\right)\right] + \mathrm{Int}\left[\left(t^{\frac{1}{r}}+t^{\frac{2}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\mathrm{h}^*_{\mathbb{Z}}\left(\partial(\tfrac{1}{r}\mathcal{P});t^{\frac{1}{r}}\right)\right]
\end{aligned}$$

A Remark on Complexity

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope with codemoninator r

To capture $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)$ (or $\operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)$), we need to compute...

$$\mathbf{0} \notin \mathcal{P} \quad \longrightarrow \quad \operatorname{Ehr}_{\mathbb{Q}}^{\text{ref}}(\mathcal{P};t) \;=\; 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)\, t^{\lambda} \;=\; \frac{\mathrm{h}_{\mathbb{Q}}^{*\text{ref}}(\mathcal{P};t)}{(1-t^q)^{d+1}}$$

$$\mathbf{0} \in \partial\mathcal{P} \quad \longrightarrow \quad \operatorname{Ehr}_{\mathbb{Q}}(\mathcal{P};t) \;=\; 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{ehr}_{\mathbb{Q}}(\mathcal{P};\lambda)\, t^{\lambda} \;=\; \frac{\mathrm{h}_{\mathbb{Q}}^{*}(\mathcal{P};t)}{(1-t^q)^{d+1}}$$

$\mathbf{0} \in \mathcal{P}^{\circ} \quad \longrightarrow \quad \mathrm{h}_{\mathbb{Q}}^{*}(\mathcal{P};t)$ symmetric

A Few Open Problems

- Periods of rational Ehrhart quasipolynomials
- Classification questions a la Kasprzyk–Nill ("reflexive polytopes of higher index" 2012)
- Stapledon's (2009) symmetric decomposition of $\mathrm{h}^*_{\mathbb{Z}}(\mathcal{P};t)$
- Connections to the Fine (1983) interior of a lattice polytope (Batyrev 2017, Batyrev–Kasprzyk–Schaller 2022)
- Symmetric polytopes, equivariant Ehrhart theory
- Further applications