Proposition 3.13. *If $m, n \in \mathbb{Z}$ satisfy $m \leq n \leq m$ then $m = n$.*

Proof. Suppose that $m \leq n$ and $n \leq m$. Assume (by means of contradiction) that $m \neq n$. But then $m \leq n$ implies that $m < n$, and, similarly, $n \leq m$ implies that $n < m$. By Proposition 3.9, we obtain $m < m$, i.e., $0 = m - m \in \mathbb{N}$, a contradiction. Therefore, $m \neq n$ cannot hold, i.e., $m = n$. $\square$

Proposition 3.14. *For all $m, n, p \in \mathbb{Z}$:*

(i) *If $m < n$ then $m + p < n + p$.*

(ii) *If $m < n$ and $0 < p$ then $mp < np$.*

(iii) *If $m < n$ and $p < 0$ then $np < mp$.*

Proof. (i) Suppose $m < n$, i.e., $n - m \in \mathbb{N}$. Then (Axioms 2.2, 2.4, and associativity)

$$n - m = n - m + 0 = n - m + p - p = (n + p) - (m + p) \in \mathbb{N} ,$$

i.e., $m + p < n + p$.

(ii) Suppose $m < n$ and $0 < p$, i.e., $n - m \in \mathbb{N}$ and (using the definition of subtraction, Proposition 2.11, and Axiom 2.2) $p - 0 = p \in \mathbb{N}$. By Proposition 3.7, $(n - m)p \in \mathbb{N}$, which we can distribute: $np - mp \in \mathbb{N}$. But this just means $mp < np$.

(iii) Suppose $m < n$ and $p < 0$, i.e., $n - m \in \mathbb{N}$ and (using the definition of subtraction and Axiom 2.2) $0 - p = -p \in \mathbb{N}$. By Proposition 3.7, $(n - m)(-p) \in \mathbb{N}$, which we can distribute: $n(-p) - m(-p) \in \mathbb{N}$. The last expression equals, by the definition of subtraction and Proposition 2.15, $mp - np \in \mathbb{N}$, i.e., $np < mp$. $\square$