Grid Graphs, Gorenstein Polytopes, and Domino Stackings

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“The hardest thing being with a mathematician is that they always have problems.”

Tendai Chitewere

Ehrhart Polynomials

$\mathcal{P} \subset \mathbb{R}^n$ – lattice polytope of dimension d, i.e., the vertices of $\mathcal{P}$ are in $\mathbb{Z}^n$

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$$\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{h(z)}{(1-z)^{d+1}} \,,$$

where $h(z)$ is a polynomial, the Ehrhart h-vector of $\mathcal{P}$.

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(Easier) Open Problem Construct and study special classes of lattice polytopes.

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Some sample problems

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- Study Ehrhart h-vectors of special classes, e.g., simplicial polytopes.

Gorenstein Polytopes

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- the Birkhoff polytope

$$\left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geq 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

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Goal Construct classes of Gorenstein polytopes.

Suggested Tools

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- barvinok (http://freshmeat.net/projects/barvinok/)
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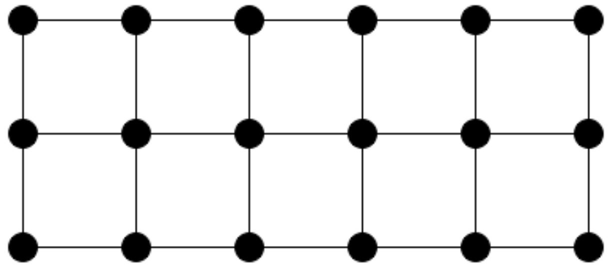
The perfect matching polytope associated to a graph G is the convex hull in $\mathbb{R}^{E(G)}$ of the incidence vectors of all perfect matchings of G.

Perfect Matchings of Grid Graphs

The $m \times n$ grid graph $\mathcal{G}(m,n)$ has vertex set $\{(i,j) \in \mathbb{Z}^2 : \, 0 \le i < n, \, 0 \le j < m\}$ and (i,j) and (i',j') are adjacent if $|i-i'| + |j-j'| = 1$.



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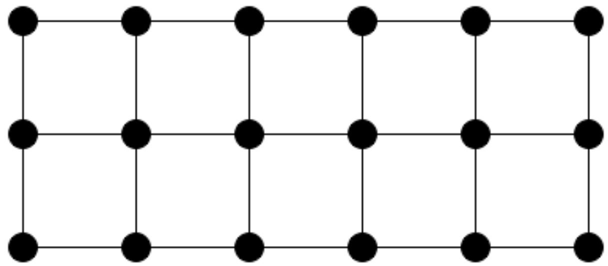

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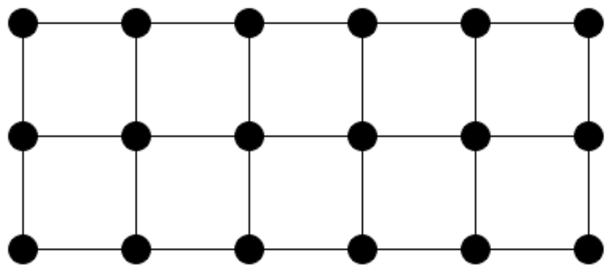

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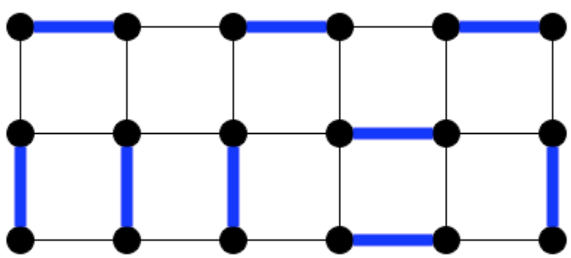
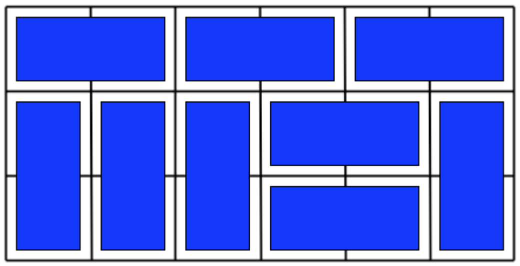
The number $T(m,n,1)$ of perfect matchings of $\mathcal{G}(m,n)$ can be interpreted as the number of domino tilings of an $m \times n$ board.





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Theorem (BHS) Assume $m \leq n$. The perfect matching polytope $\mathcal{P}(m,n)$ is Gorenstein (of index k) if and only if one of the following holds:
(1) $m = 1$ and n is even (in which case $\mathcal{P}$ is a point)
(2) $m = 2$ (in which case $k = 2$ if $n = 2$, and $k = 3$ for $n > 2$)
(3) $m = 3$ and n is even (in which case $k = 5$)
(4) $m = n = 4$ (in which case $k = 4$).

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Theorem (BHS) If $\mathcal{P}(m,n)$ is Gorenstein then $\mathcal{P}(m,n)$ has a unimodal Ehrhart h-vector.

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Note that $T(2,n,1)$ is a shift of the Fibonacci sequence.

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Propp (2001) This recurrence relation for $T(m,n,1)$ satisfies the reciprocity relation

$$T(m,n,1) = \begin{cases} (-1)^n \, T(m,-n-2,1) & \text{if } m \equiv 2 \bmod 4, \\ T(m,-n-2,1) & \text{otherwise.} \end{cases}$$

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A domino stacking (of height t) of an $m \times n$ rectangular board is a collection of t domino tilings piled on top of one another.

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Proposition (BHS) Every magic labelling of sum t of $\mathcal{G}(m,n)$ can be realized as a domino stacking of height t of an $m \times n$ rectangular board.

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Propp's reciprocity relation naturally extends to $(T(m,n,1)^t)_{n\ge 0}$.

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- Is there a recurrence relation for the number $T(m,n,t)$ of magic labellings of $\mathcal{G}(m,n)$ with sum t when m and n are both allowed to vary?