Worksheet 6: Arithmetic Functions

1. Let p, q be two distinct primes. Compute $\phi(pq)$.

2. Let p be prime and $k \in \mathbb{Z}_{>0}$. Compute
   (a) $\phi(p^k)$
   (b) $\tau(p^k)$
   (c) $\sigma(p^k)$

3. Prove that τ and σ are *multiplicative*, that is, $\tau(mn) = \tau(m)\,\tau(n)$ and $\sigma(mn) = \sigma(m)\,\sigma(n)$ whenever $\gcd(m,n) = 1$. (*Hint:* start with the case $m = p^j$, $n = q^k$ for distinct primes p and q.)

4. Fix $m, n \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$. Consider the function $f : \mathbb{Z}_{mn}^* \to \mathbb{Z}_m^* \times \mathbb{Z}_n^*$ given by
$$f(k) := (k \bmod m,\ k \bmod n)\,.$$
   (a) Show that f is well defined.
   (b) Show that f is one-to-one.
   (c) Show that f is onto. (*Hint:* Chinese Remainder Theorem.)
   (d) Conclude that $\phi(mn) = \phi(m)\,\phi(n)$.

5. Derive formulas for $\phi(n)$, $\tau(n)$, and $\sigma(n)$ in terms of the prime factorization of n.

6. Fix $n \in \mathbb{Z}_{>0}$, and for $d|n$, let
$$S_d \ := \ \{m \in [n] \,:\, \gcd(m,n) = d\}\,.$$
   (a) Come up with a bijection $S_d \to \mathbb{Z}_{\frac{n}{d}}^*$.
   (b) Convince yourself that
$$[n] \ = \ \bigcup_{d|n} S_d$$
   as a disjoint union, and conclude that
$$\sum_{d|n} \phi(d) \ = \ n\,.$$

7. Andrews 6.1.1, 6.1.4, 6.2.2, 6.2.9.

8. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.