Theorem 4.5 (DeMorgan's laws)**.** *Given two subsets $A, B \subseteq X$,*

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

Proof. For the first equality, we need to show $(A \cup B)^c \subseteq A^c \cap B^c$ and $(A \cup B)^c \supseteq A^c \cap B^c$.

Given $x \in (A \cup B)^c$, we know $x \in X$ but $x \notin A \cup B$; the last statement says that the statement "$x \in A$ or $x \in B$" does not hold, which means $x \notin A$ and $x \notin B$. Hence by definition of set intersection, $x \in A^c \cap B^c$. This proves $(A \cup B)^c \subseteq A^c \cap B^c$.

These steps can be traversed backwards: $x \in A^c \cap B^c$ means $x \notin A$ and $x \notin B$, which is the negation of the statement "$x \in A$ or $x \in B$", i.e., $x \in (A \cup B)^c$. This proves $(A \cup B)^c \supseteq A^c \cap B^c$.

To prove the second equality, we show directly that $x \in (A \cap B)^c$ is equivalent to $A^c \cup B^c$. Given $x \in (A \cap B)^c$, i.e., the statement "$x \in A$ and $x \in B$" does not hold. This negation is equivalent to saying $x \notin A$ or $x \notin B$, i.e., $x \in A^c \cup B^c$. □

Project 4.6. *Determine which of the following set identities are true; prove your assertions.*

$$A - (B \cup C) = (A - B) \cup (A - C)$$
$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

The first identity is wrong. As a counterexample, consider $A = B = \{0, 1\}, C = \{1\}$. Then $A - (B \cup C) = \{0, 1\} - \{0, 1\} = \emptyset$, but $(A - B) \cup (A - C) = \emptyset \cup \{0\} = \{0\}$.

The second identity is true; we prove it in the usual two steps of showing both set inclusions.

To prove $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$, suppose $x \in A \cap (B - C)$, that is, $x \in A$ and $x \in B - C$. This, in turn means that $x \in A$ and $x \in B$ but $x \notin C$. Then $x \in A \cap B$ (by the "and" statement) and x is not in $A \cap C$ (since $A \cap C \subseteq C$ and $x \notin C$). That is, $x \in (A \cap B) - (A \cap C)$.

Conversely, to prove $A \cap (B - C) \supseteq (A \cap B) - (A \cap C)$, suppose $x \in (A \cap B) - (A \cap C)$, that is, $x \in A \cap B$ but $x \notin A \cap C$. The first statement says that $x \in A$ and $x \in B$, and the last statement translates into $x \notin A$ or $x \notin C$. If both statements are true, then $x \notin A$ cannot hold (since the first statement says that $x \in A$), and so the second statement implies that $x \notin C$ has to hold. In summary, we have $x \in A$ and $x \in B$ and $x \notin C$, that is, $x \in A \cap (B - C)$.