MATH 725 Homework Set 5 due 9/26/11

(1) Consider the operator $T \in L(\mathbf{C}^n)$ whose matrix (with respect to the standard basis of $\mathbf{C}^n$) consists of all 1's. Find all eigenvalues and eigenvectors of T.

Solution. If $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbf{C}^n$ is a nonzero eigenvector of T, then $T(\mathbf{v}) = \lambda \mathbf{v}$ for some $\lambda \in \mathbf{C}$. Since $T(\mathbf{v})$ is a vector all of whose entries are $v_1 + v_2 + \cdots + v_n$, this gives the linear system

$$v_1 + v_2 + \cdots + v_n = \lambda v_j, \qquad 1 \le j \le n.$$

But this means, in particular, that $\lambda v_1 = \lambda v_2 = \cdots = \lambda v_n$, and so either $\lambda = 0$ or $v_1 = v_2 = \cdots = v_n$. The latter case forces $\lambda = n$, which is an eigenvalue with eigenvector $(1, 1, \ldots, 1)$ (and its scalar multiples). The eigenvalue $\lambda = 0$ comes with the space of eigenvalues $\{(v_1, v_2, \ldots, v_n) \in \mathbf{C}^n : v_1 + v_2 + \cdots + v_n = 0\}$ (which is of dimension $n - 1$). □

(2) Suppose $T \in L(V)$ is invertible.
 (a) Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. (In particular, such an eigenvalue cannot be zero.)
 (b) If λ is an eigenvalue of T with eigenvector $\mathbf{v}$, show that λ^k is an eigenvalue of T^k with eigenvector $\mathbf{v}$. (Note that $k \in \mathbf{Z}$, so you need to consider both positive and negative powers of T.)

Proof. (a) λ is an eigenvalue of $T \iff T(\mathbf{v}) = \lambda \mathbf{v}$ for some $\mathbf{v} \in V \iff \mathbf{v} = T^{-1}(\lambda \mathbf{v})$ for some $\mathbf{v} \in V$ $\iff \frac{1}{\lambda}\mathbf{v} = T^{-1}(\mathbf{v})$ for some $\mathbf{v} \in V \iff \frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
 (b) We prove the result for $k \ge 0$ by induction. The base case $k = 0$ follows because $T^0(\mathbf{v}) = \mathbf{v}$ and so $\lambda^0 = 1$ is indeed an eigenvalue. For the induction step, assume that $T^k(\mathbf{v}) = \lambda^k \mathbf{v}$ for some $k \ge 0$. Then

$$T^{k+1}(\mathbf{v}) = T\left(T^k(\mathbf{v})\right) = T\left(\lambda^k \mathbf{v}\right) = \lambda^k T(\mathbf{v}) = \lambda^k \cdot \lambda \mathbf{v} = \lambda^{k+1} \mathbf{v}.$$

The case $k < 0$ follows with the above case $k > 0$ and part (a).

□

(3) Let U_1 and U_2 be vector spaces, and let $V := U_1 \oplus U_2$. Define $T : V \to V$ by $T(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{u}_1$ (note the improved notation...). Find the eigenvalues and eigenspaces (i.e., subspaces of eigenvectors corresponding to each eigenvalue) of T.

Solution. An eigenvector $\mathbf{u}_1 + \mathbf{u}_2 \in V$ of T with eigenvalue λ satisfies $\mathbf{u}_1 = \lambda(\mathbf{u}_1 + \mathbf{u}_2)$, i.e.,

$$\mathbf{u}_1 = \lambda \mathbf{u}_1 \qquad \text{and} \qquad \mathbf{0} = \lambda \mathbf{u}_2.$$

The second equation implies that either $\lambda = 0$ (which forces $\mathbf{u}_1 = \mathbf{0}$) or $\mathbf{u}_2 = \mathbf{0}$ (which forces $\lambda = 1$ by the first equation, because then $\mathbf{u}_1 \neq \mathbf{0}$). Thus we have the eigenvalues

$$\lambda = 0 \ \text{ with eigenspace } U_2$$
$$\lambda = 1 \ \text{ with eigenspace } U_1.$$ □

(4) Give an example of a vector space V, a basis B of V, and a linear operator $T \in L(V)$ whose matrix (with respect to B) contains
 (a) only 0's on the diagonal, yet T is invertible;
 (b) only nonzero numbers on the diagonal, yet T is not invertible.

Solution. (a) $V = \mathbf{R}^2$ with the standard basis, $T(x, y) = (y, x)$. Then $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 0)$, so the matrix of T contains only 0's on the diagonal. Since T^2 is the identity, $T^{-1} = T$, and so T is invertible.
 (b) $V = \mathbf{R}^2$ with the standard basis, $T(x, y) = (x + y, x + y)$. Then $T(1, 0) = (1, 1) = T(0, 1)$, so all of the entries (in particular, on the diagonal) of the matrix of T are 1. However, $\operatorname{range}(T) = \left\{(x, y) \in \mathbf{R}^2 : x = y\right\}$ is one-dimensional, so T is not invertible. □

(5) (a) Suppose V is a vector space over $\mathbf{C}$, $T \in L(V)$, $p \in \mathscr{P}(\mathbf{C})$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of $p(T)$ if and only if $\lambda = p(\mu)$ for some eigenvalue μ of T.
 (b) Show that (a) does not hold if $\mathbf{C}$ is replaced by $\mathbf{R}$.

Proof. (a) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathscr{P}(\mathbf{C})$.
Suppose λ is an eigenvalue of $p(T)$, i.e., $\operatorname{null}(p(T) - \lambda \mathrm{I}) \neq \{\mathbf{0}\}$, where I denotes the identity map. By the fundamental theorem of algebra, the polynomial $p(x) - \lambda$ has n roots, say $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{C}$. Furthermore, by the same reasoning that we gave in class, this means that at least one of the operators

$$T - \alpha_1 \mathrm{I}, T - \alpha_2 \mathrm{I}, \ldots, T - \alpha_n \mathrm{I}$$

has a nontrivial null space, say, $\operatorname{null}(T - \alpha_k \mathrm{I}) \neq \{\mathbf{0}\}$. This means that α_k is an eigenvalue of T; note that $p(\alpha_k) - \lambda = 0$, so $\mu = \alpha_k$ will do the trick.
Conversely, suppose μ is an eigenvalue of T with eigenvector $\mathbf{v}$, and let $\lambda := p(\mu)$. Then with Exercise (2),

$$\begin{aligned}
p(T)(\mathbf{v}) &= a_n T^n(\mathbf{v}) + a_{n-1}T^{n-1}(\mathbf{v}) + \cdots + a_1 T(\mathbf{v}) + a_0 \mathbf{v} \\
&= a_n \mu^n \mathbf{v} + a_{n-1}\mu^{n-1}\mathbf{v} + \cdots + a_1 \mu \mathbf{v} + a_0 \mathbf{v} \\
&= p(\mu)\mathbf{v} = \lambda \mathbf{v}.
\end{aligned}$$

□

(b) Let $T \in L(\mathbf{R}^2)$ be given by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and let $p(x) = x^2$. The operator T has no real eigenvalues (it has the complex eigenvalues $\pm i$). However, $p(T) = T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ which has the eigenvalue -1.