MATH 420/720 Homework Quiz 1 (5 February 2025)

(a) Define a permutation of $[n]$.

(b) The *Lucas numbers* are defined by $L_0 = 2$, $L_1 = 1$, and

$$L_n = L_{n-1} + L_{n-2} \qquad \text{for } n \geq 2\,.$$

Let C_n be the set of tilings of n boxes arranged in a circle with dominos and monominos. Show that $\#C_n = L_n$ for $n \geq 1$.

MATH 420/720 Homework Quiz 2 (12 February 2025)

(a) Define the Stirling numbers $S(n,k)$ of the second kind.

(b) Show that

(i) $S(n,n) = 1$

(ii) $S(n,n-1) = \binom{n}{2}$

(iii) $S(n,n-2) = \binom{n}{3} + 3\binom{n}{4}\,.$

MATH 420/720 Homework Quiz 3 (19 February 2025)

(a) Define a partition λ of n.

(b) Denote by $p_e(n,k)$ the number of partitions of n having exactly k parts. Prove that

$$p_e(n,k) = p(n-k,k)\,.$$

MATH 420 Homework Quiz 4 (5 March 2025)

(a) State the Principle of Inclusion–Exclusion.

(b) Let $A(n)$ be the number of partitions of $[n]$ such that i and $i+1$ never occur in the same block. Show that

$$A(n) \;=\; \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} B(n-i)$$

where $B(n)$ is the nth Bell number.

MATH 720 Homework Quiz 4 (5 March 2025)

(a) State the Principle of Inclusion–Exclusion.

(b) A graph G is *planar* if it can be drawn in the plane $\mathbb{R}^2$ without edge crossings. In this case the *regions* of G are the topologically connected components of the set-theoretic differences $\mathbb{R}^2 - G$. Let R be the set of regions of G. If $r \in R$, then let $\deg r$ be the number of edges on the boundary of r. Show that

$$\sum_{r \in R} \deg r \;\leq\; 2|E|\,.$$

MATH 420/720 Homework Quiz 5 (12 March 2025)

(a) Define what it means for a sequence (a_k) to be unimodal.

(b) Suppose $0 \leq k < n$. Prove that

$$\binom{n}{k}^2 \;\geq\; \binom{n-1}{k}\binom{n+1}{k}.$$

MATH 420/720 Homework Quiz 6 (19 March 2025)

(a) Let $A(x) := \sum_{k \geq 0} a_k \, x^k$ and $B(x) := \sum_{k \geq 0} b_k \, x^k$. Define the coefficient c_k for the generating function $A(x) \cdot B(x) = \sum_{k \geq 0} c_k \, x^k$.

(b) Prove that

$$\sum_{k=0}^{n} c(n,k) \, x^k \; = \; x(x+1)(x+2) \cdots (x+n-1)$$

where $c(n,k)$ are the Stirling numbers of the first kind.

MATH 420/720 Homework Quiz 7 (9 April 2025)

(a) Define the exponential generating function of the sequence a_n.

(b) Given $m \geq 2$, use generating functions to show that the number of partitions of n where each part is repeated fewer than m times equals the number of partitions of n into parts not divisible by m.

MATH 420 Homework Quiz 8 (16 April 2025)

(a) Define the Eulerian polynomial $A_n(x)$.

(b) Use the recursion

$$D(n) \; = \; n \, D(n-1) + (-1)^n$$

to (re-)derive the exponential generating function for the derangement numbers.

MATH 720 Homework Quiz 8 (16 April 2025)

(a) Define the Eulerian polynomial $A_n(x)$.

(b) Recall that we proved

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{m\geq 0}(m+1)^n\, x^m.$$

Show that

$$(m+1)^n = \sum_{k\geq 0} A(n,k)\binom{m+n-k}{n}.$$