Project 9.1. *Determine which of the following functions is injective, surjective, or bijective. Justify your assertions.*

(i) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = n^2$

(ii) $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0},\ f(n) = n^2$

(iii) $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0},\ f(n) = n^2$

(iv) $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 1$

(v) $f : \mathbb{R}_{\geq 0} \to \mathbb{R},\ f(x) = 3x + 1$

(vi) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 3x + 1$

(i) The function $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = n^2$ is neither injective, nor surjective, and hence certainly not bijective. f is not injective because $f(-1) = f(1)$, and f is not surjective because $f(n) \geq 0$ for all $n \in \mathbb{Z}$, and so there exists no $n \in \mathbb{Z}$ for which $f(n) = 3$.

(ii) The function $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0},\ f(n) = n^2$ is neither surjective nor injective, and hence not bijective, for the same reasons as in (i).

(iii) The function $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0},\ f(n) = n^2$ is injective but not surjective. It is not surjective for the same reasons as in (i). To prove that f is injective, suppose $f(m) = f(n)$, i.e., $m^2 = n^2$. This implies $m = \pm n$, and since $m, n \geq 0$, we can conclude that $m = n$.

(iv) The function $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 1$ is bijective. To prove that f is injective, suppose $f(m) = f(n)$, i.e., $3m + 1 = 3n + 1$. Then we can first subtract 1 from both sides of this equation and then cancel a 3 to arrive at $m = n$. To prove that f is surjective, suppose $m \in \mathbb{R}$ is given. Then we can choose $n = \frac{m-1}{3}$ and obtain $f(n) = 3 \cdot \frac{m-1}{3} + 1 = m$.

(v) The function $f : \mathbb{R}_{\geq 0} \to \mathbb{R},\ f(x) = 3x + 1$ is injective but not surjective, and hence not bijective. To show that f is injective, we can repeat the proof of injectivity in (iv). f is not surjective because $x \geq 0$ implies that $f(x) = 3x + 1 \geq 1$. Hence there is no $n \in \mathbb{R}_{\geq 0}$ for which $f(n) = 0$.

(vi) The function $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 3x + 1$ is also injective but not surjective, and hence not bijective. To show that f is injective, we can repeat the proof of injectivity in (iv). f is not surjective because $n \in \mathbb{Z}$ implies that $f(n) = 3n + 1 \equiv 1 \pmod 3$. Hence there is no $n \in \mathbb{Z}$ for which $f(n) = 2$.