Proposition 5.3. *For any $n \in \mathbb{N}$:*

(i) $5^{2n} - 1$ *is divisible by* 24.

(ii) $2^{2n+1} + 1$ *is divisible by* 3.

(iii) $10^n + 3 \cdot 4^{n+2} + 5$ *is divisible by* 9.

Proof. (i) We proceed by induction on n. For the base case $n = 1$, $5^2 - 1 = 24$ is divisible by 24, since $24 = 24 \cdot 1$. For the induction step, assume that $5^{2n} - 1$ is divisible by 24, i.e., there is $k \in \mathbb{Z}$ such that $5^{2n} - 1 = 24k$. Then by Proposition 5.2,

$$5^{2(n+1)} - 1 = 5^{2n+2} - 1 = 5^{2n}5^2 - 1 = (24k + 1)\, 25 - 1\ .$$

Here the last equation follows with the induction hypothesis. Hence

$$5^{2(n+1)} - 1 = (24k + 1)\, 25 - 1 = 24 \cdot 25k + 24 = 24\,(25k + 1)\,.$$

So we found an integer $j = 25k + 1$ such that $5^{2(n+1)} - 1 = 24j$, so by definition, 24 divides $5^{2(n+1)} - 1$, and our induction is complete.

(ii) We proceed by induction on n. For $n = 1$, $2^3 + 1 = 9$ is divisible by 3, since $9 = 3 \cdot 3$. For the induction step, assume that $2^{2n+1} + 1$ is divisible by 3, i.e., there is $k \in \mathbb{Z}$ such that $2^{2n+1} + 1 = 3k$. Then

$$2^{2(n+1)+1} + 1 = 2^{2n+3} + 1 = 4 \cdot 2^{2n+1} + 1 = 4\,(3k - 1) + 1 = 12k - 3 = 3\,(4k - 1)\,.$$

So we found an integer, namely $4k - 1$, such that $2^{2(n+1)+1} + 1$ equals 3 times that integer, so by definition, 3 divides $2^{2(n+1)+1} + 1$, and our induction is complete.

(iii) We first give a lemma, namely that for all $n \in \mathbb{N}$, $10^n - 1$ is divisible by 9.

We prove this lemma by induction on n. For $n = 1$, $10^1 - 1 = 9$ is certainly divisible by 9. For the induction step, assume that $10^n - 1$ is divisible by 9, that is, there is $j \in \mathbb{Z}$ such that $10^n - 1 = 9j$. Then by Proposition 5.2,

$$10^{n+1} - 1 = 10 \cdot 10^n - 1 = 10\,(9j + 1) - 1 = 90j + 9 = 9\,(10j + 1)\,.$$

So we found an integer $m = 10j + 1$ such that $10^{n+1} - 1 = 9m$, so by definition, 9 divides $10^{n+1} - 1$, which finishes the proof of our lemma.

To prove (iii), we proceed by induction on n. For $n = 1$, $10^1 + 3 \cdot 4^3 + 5 = 207$ is divisible by 9, since $207 = 9 \cdot 23$. For the induction step, assume that $10^n + 3 \cdot (4^{n+2}) + 5$ is divisible by 9, i.e., there is $k \in \mathbb{Z}$ such that $10^n + 3 \cdot 4^{n+2} + 5 = 9k$. Now by the above lemma, there exists $j \in \mathbb{Z}$ such that $10^n = 9j + 1$. This together with Proposition 5.2 and our induction hypothesis gives

$$\begin{aligned} 10^{n+1} + 3 \cdot 4^{n+3} + 5 &= 10 \cdot 10^n + 3 \cdot 4^{n+2} \cdot 4 + 5 = 10\,(9j + 1) + 4\,(9k - 10^n - 5) + 5 \\ &= 90j + 36k - 4 \cdot 10^n - 5 = 90j + 36k - 4\,(9j + 1) - 5 \\ &= 54j + 36k - 9 = 9\,(6j + 4k - 1)\,. \end{aligned}$$

So we found an integer $m = 6j + 4k - 1$ such that $10^{n+1} + 3 \cdot 4^{n+3} + 5 = 9m$, that is, 9 divides $10^{n+1} + 3 \cdot 4^{n+3} + 5$, and our induction is complete. $\square$

Project 5.4. *Determine for which natural numbers $n^2 < 2^n$ and prove your answer.*

For $n = 1$ and $n \geq 5$, we have $n^2 < 2^n$.

Proof. For $n = 1$, $n^2 = 1 < 2 = 2^n$.

Now we will prove that $n^2 < 2^n$ for all $n \geq 5$ by induction on n. The base case $n = 5$ follows with $5^2 = 25 < 32 = 2^n$.

For the induction step, suppose that we know that $n^2 < 2^n$ for some $n \geq 5$. We wish to prove that $(n+1)^2 < 2^{n+1}$. $(n+1)^2 = n^2 + 2n + 1 < 2^n + 2n + 1$, by our induction hypothesis. If we can prove that $2n+1 < 2^n$, then we're done, since it then follows that $2^n+2n+1 < 2^n+2^n = 2 \cdot 2^n = 2^{n+1}$ (by the recursive definition of exponentials).

So now we'll prove that $2n + 1 < 2^n$ for all $n \geq 5$, which we will do again by induction on n. The base case $n = 5$ follows with $2 \cdot 5 + 1 = 11 < 32 = 2^5$. For the induction step, suppose that we know that $2n + 1 < 2^n$ for some $n \geq 5$. Then $2(n + 1) + 1 = 2n + 3 = (2n + 1) + 2 < 2^n + 2$. If we can prove that $2 < 2^n$, then it follows that $2^n + 2 < 2^n + 2^n = 2^{n+1}$, which would finish our second induction step.

So it remains to prove that $2 < 2^n$ for all $n \geq 5$, which we'll do with an easy induction proof on n. The base case $n = 5$ follows with $2 < 32 = 2^5$. For the induction step, assume that $2 < 2^n$. Then $2 < 2^n + 2^n = 2^{n+1}$, which finishes our third induction. $\square$