MATH 301 Midterm Exam 10/10/17 11:00–12:15 p.m.

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to propositions in the class notes (without stating the exact number of the proposition).

1. Suppose m is an integer.

   (a) Carefully define $-m$.

   (b) Prove that $-m = (-1)\,m$.

   Solution:

   (a) For any integer m, $-m$ is the integer that satisfies $m + (-m) = 0$.

   (b) $m+(-1)m = 1\cdot m+(-1)m = (1 + (-1))\,m = 0\cdot m = 0$ (using several axioms and a proposition). The equation $m + (-1)m = 0$ means that $(-1)m$ is the additive inverse of m, i.e., $-m = (-1)m$.

2. Suppose $m, n \in \mathbb{Z}$.

   (a) Carefully define the statement $m > n$.

   (b) Show that if $m > n$ and $p > 0$ then $mp > np$.

   Solution:

   (a) $m - n \in \mathbb{N}$.

   (b) Suppose $m > n$ and $p > 0$, that is, $m - n \in \mathbb{N}$ and $p - 0 = p \in \mathbb{N}$. Then

   $$mp - np \;=\; (m - n)p \;\in\; \mathbb{N}$$

   (here we have used the axiom that the product of two natural numbers is a natural number), and so $mp > np$.

3. Suppose $m, n \in \mathbb{Z}$.

   (a) Carefully define the statement m divides n.

   (b) Prove that for all $k \in \mathbb{N}$, 24 divides $5^{2k} - 1$.

   Solution:

   (a) $\exists\, k \in \mathbb{Z}$ such that $n = mk$.

   (b) We use induction on k. The base case $k = 1$ holds because 24 indeed divides $5^{2\cdot 1} - 1 = 24$.

   For the induction step, assume that 24 divides $5^{2n} - 1$ for some $n \in \mathbb{N}$, i.e., $5^{2n} - 1 = 24m$ for some $m \in \mathbb{Z}$. Then

   $$5^{2(n+1)} - 1 \;=\; 5^{2n}\,5^2 - 1 \;=\; (24m + 1)25 - 1 \;=\; 24(25m + 1)$$

   (here we have used properties of exponentials), and since $25m+1 \in \mathbb{Z}$, 24 divides $5^{2(n+1)} - 1$.