Cyclotomic Polytopes and Growth Series of Cyclotomic Lattices

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Math Research Letters

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M – subset that generates $\mathcal{L}$ as a monoid

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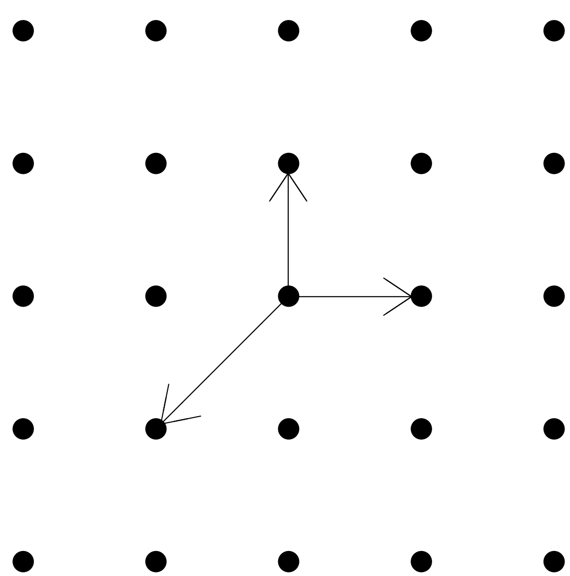

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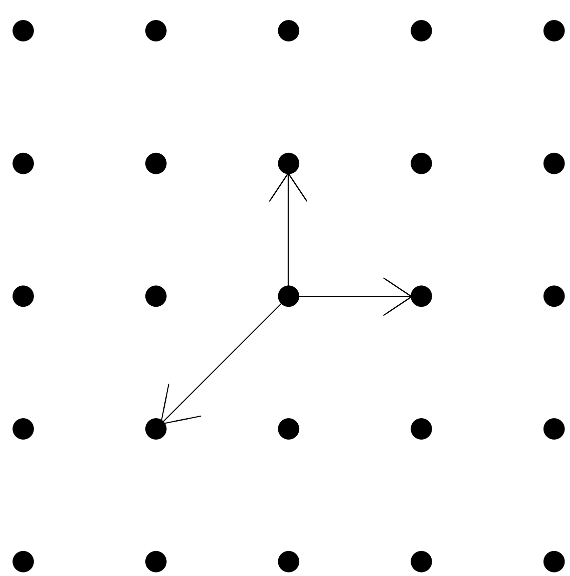

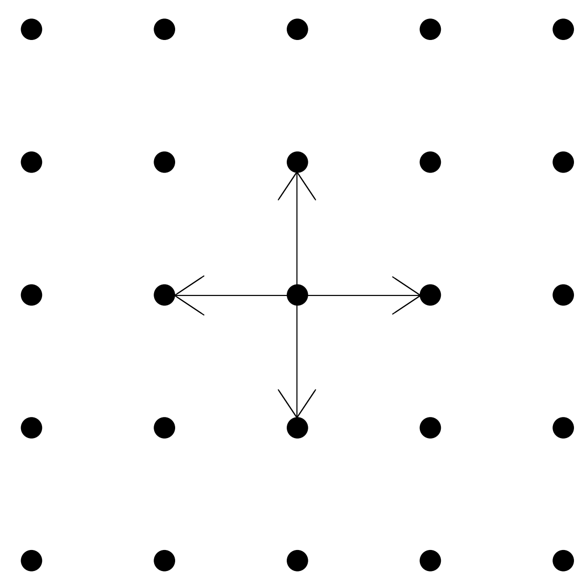

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Conjectures (Parker)

(1) $h_m(x) = g(x)^{\frac{m}{\sqrt{m}}}$ for a palindromic polynomial g of degree $\varphi(\sqrt{m})$.

(2) $h_{2p}(x) = \sum_{k=0}^{\frac{p-3}{2}} \left(x^k + x^{p-1-k} \right) \sum_{j=0}^{k} \binom{p}{j} + 2^{p-1} x^{\frac{p-1}{2}}$.

(3) $h_{15}(x) = \left(1 + x^8\right) + 7\left(x + x^7\right) + 28\left(x^2 + x^6\right) + 79\left(x^3 + x^5\right) + 130x^4$.

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Theorem (MB–Hoşten) Parker's Conjectures (2) & (3) are true.

Cyclotomic Polytopes

We choose a specific basis for $\mathbb{Z}[e^{2\pi i/m}]$ consisting of certain powers of $e^{2\pi i/m}$ which we then identify with the unit vectors in $\mathbb{R}^{\varphi(m)}$. The other powers of $e^{2\pi i/m}$ are integer linear combinations of this basis; hence they are lattice vectors in $\mathbb{Z}[e^{2\pi i/m}] \subset \mathbb{R}^{\varphi(m)}$. The m^{th} cyclotomic polytope $\mathcal{C}_m$ is the convex hull of all of these m lattice points in $\mathbb{R}^{\varphi(m)}$, which correspond to the m^{th} roots of unity.

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When $m = p$ is a prime number, let $\zeta = e^{2\pi i/p}$ and fix the $\mathbb{Z}$-basis $1, \zeta, \zeta^2, \dots, \zeta^{p-2}$ of the lattice $\mathbb{Z}[\zeta]$. Together with $\zeta^{p-1} = -\sum_{j=0}^{p-2} \zeta^j$, these p elements form a monoid basis for $\mathbb{Z}[\zeta]$. We identify them with $e_0, e_1, \dots, e_{p-2}, -\sum_{j=0}^{p-2} e_j$ in $\mathbb{R}^{p-1}$ and define the cyclotomic polytope $\mathcal{C}_p \subset \mathbb{R}^{p-1}$ as the simplex

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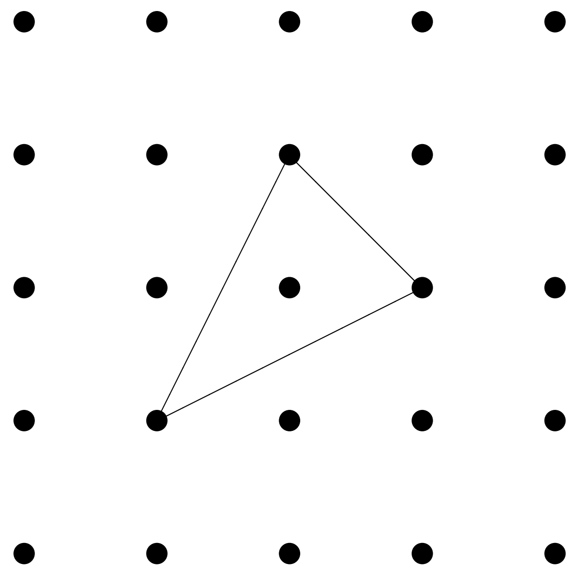

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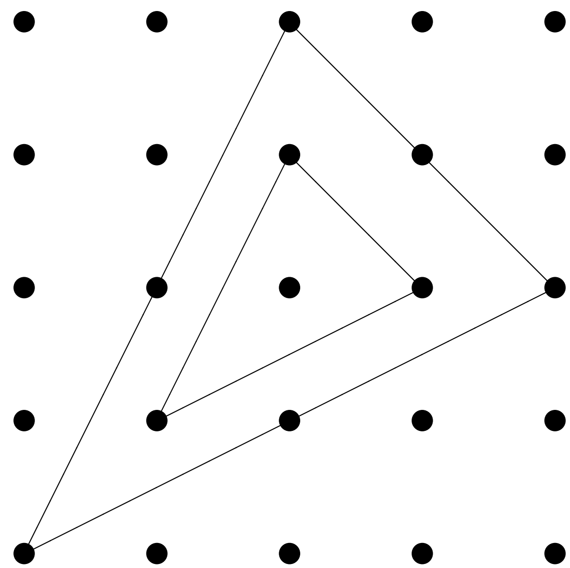

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For two polytopes $P \subset \mathbb{R}^{d_1}$ and $Q \subset \mathbb{R}^{d_2}$, each containing the origin in its interior, we define the direct sum $P \circ Q := \operatorname{conv}\left(P \times \mathbf{0}_{d_2}, \mathbf{0}_{d_1} \times Q\right)$. For a prime p, we define the cyclotomic polytope

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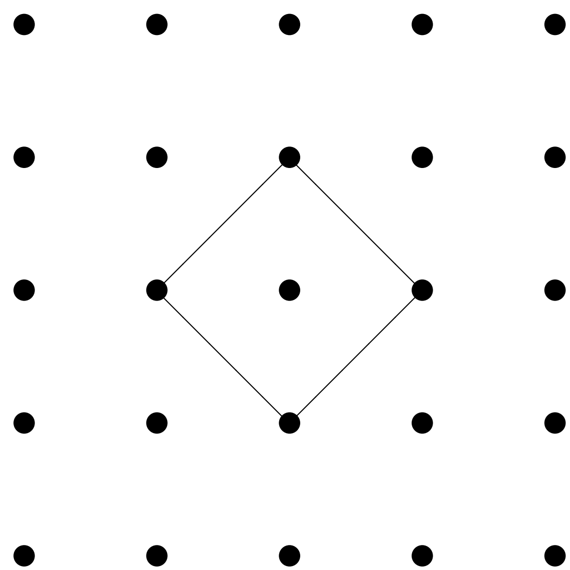

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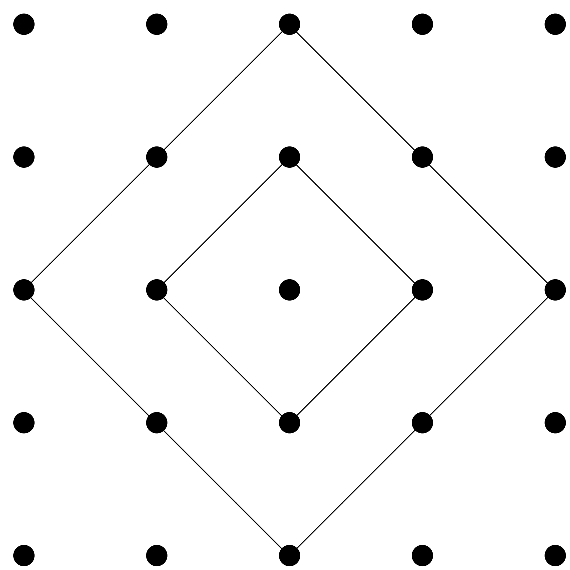

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For general m,

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a $0/\pm 1$ polytope with the origin as the sole interior lattice point.

Hilbert Series

$\mathcal{L} \cong \mathbb{Z}^d$ a lattice, M a minimal set of monoid generators, K a field

The vectors in $M' = \{(u, 1) : u \in M \cup \{0\}\}$ encoded as monomials give rise to the monoid algebra $K[M']$, in which each monomial corresponds to (v, k) where $v = \sum_{u_i \in M \cup \{0\}} n_i u_i$ with nonnegative integer coefficients n_i such that $\sum n_i = k$.

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In the conditions of our theorem, the Hilbert series of $\mathcal{C}_m \circ \mathcal{C}_m$ equals $(1-x)$ times the square of the Hilbert series of $\mathcal{C}_m$, whence $h_m(x) = h_{\sqrt{m}}(x)^{\frac{m}{\sqrt{m}}}$.

Total Unimodularity and Normality

A polytope $\mathcal{P}$ is totally unimodular if every submatrix of the matrix consisting of the vertices of $\mathcal{P}$ has determinant $0, \pm 1$.

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Remark Total unimodularity breaks down already for $\mathcal{C}_{3pq}$ for distinct primes $p, q > 3$. This is an indication that Parker's Conjecture (1) might not be true in general.

Total Unimodularity and Normality

Theorem (MB–Hoşten) Suppose m is divisible by at most two odd primes.
(2) $h_{\sqrt{m}}(x)$ is the h-polynomial of a simplicial polytope.

. . . follows now because $\mathcal{C}_{\sqrt{m}}$ has a unimodular triangulation, which induces a unimodular triangulation of the boundary of $\mathcal{C}_{\sqrt{m}}$. This boundary equals the boundary of a simplicial polytope $\mathcal{Q}$ (Stanley), and $h_{\sqrt{m}}$ is the h-polynomial of $\mathcal{Q}$ (which is palindromic, unimodal, and nonnegative).

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Remark If $\mathcal{C}_m$ is a simplicial polytope then the coordinator polynomial h_m equals the h-polynomial of $\mathcal{C}_m$. The polytope $\mathcal{C}_m$ is simplicial, e.g., for m a prime power or the product of two primes (the latter was proved by R. Chapman and follows from the fact that $\mathcal{C}_{pq}$ is dual to a transportation polytope with margins p and q).

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- S. Sullivant computed that the dual of $\mathcal{C}_{105}$ is not a lattice polytope, i.e., $\mathcal{C}_{105}$ is not reflexive. If we knew that $\mathcal{C}_{105}$ is normal, a theorem of Hibi would imply that the coordinator polynomial h_{105} is not palindromic, and hence that Parker's Conjecture (1) is not true in general.