MATH 725 Final Exam 12/15/21

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an follow-up oral exam if I suspect that you did not follow these rules.

The take-home exam is due on at 12:00 p.m. on 17 December 2021 (via email), and your submission should be a pdf file (typed or carefully scanned).

(1) Consider the vector space $\mathscr{P}_4(\mathbf{C})$ of polynomials of degree ≤ 4 and the linear function $D : \mathscr{P}_4(\mathbf{C}) \to \mathscr{P}_4(\mathbf{C})$ given by $D(p(x)) := p'(x)$.
  (a) Determine all generalized eigenspaces and the Jordan Normal Form of D.
  (b) Compute the minimal and characteristic polynomial of D.

(2) Let V be a complex vector space.
  (a) Determine all linear functions $g : V \to V$ for which each $v \in V \setminus \{0\}$ is an eigenvector. (*Hint:* consider distinct eigenvalues of g.)
  (b) Give an example of a complex vector space and a nonzero linear function $f : V \to V$ for which each $\lambda \in \mathbf{C}$ is an eigenvalue.

(3) Suppose $A = (a_{jk}) \in \mathbf{C}^{n\times n}$ and let

$$||A|| \ := \ \sqrt{\sum_{j,k=1}^{n} |a_{jk}|^2}.$$

Prove that

$$||A||^2 \ = \ s_1^2 + s_2^2 + \cdots + s_n^2,$$

where $s_1, s_2, \ldots, s_n$ are the singular values of A. (*Hint:* start by showing that $||A||^2$ equals the trace of A^*A.)

(4) (a) Suppose $x_1, x_2, \ldots, x_n \in \mathbf{C}$, and let $A : \mathbf{C}^n \to \mathbf{C}^n$ be given in matrix form (with respect to the standard basis of $\mathbf{C}^n$)

$$A \ := \ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Viewing $x_1, x_2, \ldots, x_n$ as variables, prove that $\det(A)$ is a polynomial in $x_1, x_2, \ldots, x_n$ of (total) degree at most $\frac{n(n-1)}{2}$.

(b) Show that $\det(A) = 0$ if $x_j = x_k$ for some $j \neq k$, and conclude that $x_k - x_j$ divides $\det(A)$.

(c) Prove that

$$\det(A) = \prod_{1 \leq j < k \leq n} \left(x_k - x_j\right).$$

(*Hint:* use (a) and (b) to show that $\det(A) = c \prod_{1 \leq j < k \leq n} \left(x_k - x_j\right)$ for some constant c, and then compute the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.)