MATH 725 Homework Set 4 due 9/19/11

(1) Suppose V and W are finite-dimensional vector spaces, $T \in L(V,W)$, and define $\operatorname{rank}(T) := \dim \operatorname{range}(T)$. Prove that if $\operatorname{rank}(T) = \dim W$ then T is surjective.

Proof. Suppose $\operatorname{rank}(T) = \dim W$. We know that $\operatorname{range}(T)$ is a subspace of W, so consider a basis of $\operatorname{range}(T)$, which we can then extend to a basis of W. But since $\operatorname{rank}(T) = \dim W$ these bases must have the same length, and so the basis of $\operatorname{rank}(T)$ is already a basis of W, whence $\operatorname{range}(T) = W$. □

(2) Suppose U, V, and W are finite-dimensional vector spaces, $S \in L(U,V)$, and $T \in L(V,W)$.

(a) Show that $\operatorname{rank}(TS) \le \min(\operatorname{rank}(S), \operatorname{rank}(T))$. Give examples to show that both equality and strict inequality are possible. Can two nonzero maps have composition equal to zero?

(b) If S has a right inverse, show that $\operatorname{rank}(TS) = \operatorname{rank}(T)$.

(c) If T has a left inverse, show that $\operatorname{rank}(TS) = \operatorname{rank}(S)$.

(d) Prove that the rank of a matrix (i.e., the rank of the underlying linear map) is not more than the number of rows or the number of columns of the matrix.

(e) A matrix is said to have *maximal rank* if the rank is equal to the the minimum of the number of rows and the number of columns. Show that a matrix has maximal rank if and only it is either injective or surjective.

Proof. (a) It suffices to show that $\operatorname{rank}(TS) \le \operatorname{rank}(S)$ and $\operatorname{rank}(TS) \le \operatorname{rank}(T)$. The latter follows directly from $\operatorname{range}(TS) \subseteq \operatorname{range}(T)$ (which we know by adapting the previous homework #4). To prove the former, note that

$$\dim U = \dim \operatorname{null}(TS) + \operatorname{rank}(TS) \qquad \text{and} \qquad \dim U = \dim \operatorname{null}(S) + \operatorname{rank}(S)\,.$$

Again by adaptation of the previous homework #4, we know that $\operatorname{null}(TS) \subseteq \operatorname{null}(S)$ and thus $\operatorname{rank}(S) \ge \operatorname{rank}(TS)$. An example that shows that $\operatorname{rank}(TS) = \min(\operatorname{rank}(S), \operatorname{rank}(T))$ is possible is given by $S = T =$ identity map on an arbitrary vector space $U = V = W$. An example that shows that $\operatorname{rank}(TS) < \min(\operatorname{rank}(S), \operatorname{rank}(T))$ is possible is given by $U = V = W = \mathscr{P}_2(\mathbf{R})$ and $S = T = \frac{d}{dx}$.

(b) Suppose SR is the identity map. By using part (a) twice,

$$\operatorname{rank}(T) \ge \operatorname{rank}(TS) \ge \operatorname{rank}(TSR) = \operatorname{rank}(T)\,,$$

and so we must have equality all around. (One can also argue via surjectivity of S.)

(c) Suppose LT is the identity map. By using part (a) twice,

$$\operatorname{rank}(S) \ge \operatorname{rank}(TS) \ge \operatorname{rank}(LTS) = \operatorname{rank}(S)\,.$$

(One can also argue via injectivity of T.)

(d) The rank of a linear map M is the dimension of $\operatorname{range}(M)$, which can be at most the dimension of the codomain of M, and that is the number of rows of the matrix corresponding to M. By our dimension–null space–rank theorem, the rank of M is also at most the dimension of the domain of M, which equals the number of columns of the matrix corresponding to M.

(e) Suppose the linear map underlying M is from V to W. Then

$$\begin{aligned}\operatorname{rank}(M) = \#\text{ rows of } M &\iff \dim \operatorname{range}(M) = \dim W \\ &\iff \operatorname{range}(M) = W\,,\end{aligned}$$

i.e., M is surjective, and

$$\begin{aligned}\operatorname{rank}(M) = \#\text{ columns of } M &\iff \dim \operatorname{range}(M) = \dim V \\ &\iff \dim \operatorname{null}(M)\,,\end{aligned}$$

i.e., M is injective. □

(3) Consider the linear operator $\frac{d}{dx} \in L(\mathscr{P}_n(\mathbf{F}))$ given by differentiation. Compute the matrix of $\frac{d}{dx}$ using the basis

(a) $1, x, x^2, \dots, x^n$;

(b) $\binom{x}{0}, \binom{x}{1}, \dots, \binom{x}{n}$.

Solution. (a) Since $\frac{d}{dx}x^k = kx^{k-1}$, the matrix of $\frac{d}{dx}$ with respect to the monomial basis has entries $a_{k-1,k} = k$ for $2 \le k \le n$ and $a_{jk} = 0$ otherwise.

(b) We claim that

$$\frac{d}{dx}\binom{x}{k} = \sum_{j \ge 1} \frac{(-1)^{j-1}}{j} \binom{x}{k-j}$$

(which is really a finite sum, as $\binom{x}{j} = 0$ when $j < 0$ by definition) and prove this by induction on k. The base case $k = 0$ just says $0 = 0$. The induction step follows with

$$\begin{aligned}
\frac{d}{dx}\binom{x}{k} &= \frac{d}{dx}\left(\binom{x}{k-1} + \binom{x-1}{k-1}\right) \\
&= \sum_{j \ge 1} \frac{(-1)^{j-1}}{j}\binom{x}{k-1-j} + \sum_{j \ge 1} \frac{(-1)^{j-1}}{j}\binom{x-1}{k-1-j} \\
&= \sum_{j \ge 1} \frac{(-1)^{j-1}}{j}\left(\binom{x}{k-1-j} + \binom{x-1}{k-1-j}\right) \\
&= \sum_{j \ge 1} \frac{(-1)^{j-1}}{j}\binom{x}{k-j}.
\end{aligned}$$

Thus the matrix of $\frac{d}{dx}$ with respect to the binomial-coefficient basis has entries

$$a_{jk} = \begin{cases} \frac{(-1)^{k-j-1}}{k-j} & \text{if } j < k, \\ 0 & \text{otherwise.} \end{cases}$$

□

(4) Suppose V is a finite-dimensional vector space and $S, T \in L(V)$. Prove that ST is invertible if and only if both S and T are invertible. Give an example that shows that this statement is not true for infinite-dimensional vector spaces.

Proof. Suppose ST is invertible and so, in particular, $\text{null}(ST) = \{\mathbf{0}\}$ and $\text{range}(ST) = V$. We have shown in a previous homework that $\text{null}(T) \subseteq \text{null}(ST) = \{\mathbf{0}\}$, and so T is injective and (by a theorem from class) invertible. We have also shown that $\text{range}(S) \supseteq \text{range}(ST) = V$, and so S is surjective and (by the same theorem) invertible.

An example of how this statement can fail if V is infinite dimensional is given by $V = \mathscr{P}(\mathbf{R})$, $S(p(x)) = p'(x)$, and $T(p(x)) = \int_0^x p(t)\,dt$: by the Fundamental Theorem of Calculus, ST is the identity map; however, S is not injective. □

(5) Suppose V and W are finite-dimensional vector spaces, and U is a subspace of V. Let $R : L(V,W) \to L(U,W)$ be the *restriction map* defined by $(R(T))(u) = T(u)$.

(a) Show that R is linear.
(b) Show that R is surjective.
(c) If U is a proper subspace of V, show that the restriction map is not injective.

Proof. (a) Given $S, T \in L(V,W)$ and $a \in \mathbf{F}$, we have for any $u \in U$

$$R(aS + T)(u) = aS(u) + T(u) = aR(S)(u) + R(T)(u).$$

(b) Let $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)$ be a basis of U, and extend it to a basis $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of V. Given $T \in L(U,W)$, define $S \in L(V,W)$ through

$$\mathbf{v} = \sum_{j=1}^{m} a_j \mathbf{u}_j + \sum_{j=1}^{n} b_j \mathbf{v}_j \qquad \mapsto \qquad S(\mathbf{v}) = \sum_{j=1}^{m} a_j T(\mathbf{u}_j).$$

This map S is by definition linear, and $R(S) = T$. Hence R is surjective.

(c) Let $m := \dim U$, $n := \dim V$, and $k := \dim W$. Then we know from a theorem in class that $\dim L(V,W) = nk$ and $\dim L(U,W) = mk$. If $U \subsetneq V$ then $m < n$, and so $\text{rank}(R)$ (which is at most $\dim L(U,W)$) cannot be equal $\dim L(V,W)$, i.e., R cannot be surjective. □