

Worksheet 11: Continued Fractions

1. Find continued fraction expansions for

(a) $\frac{100}{37}$

(c) $\frac{21}{13}$

(e) $\frac{13}{35}$

(b) $\frac{1001}{45}$

(d) $\frac{1000}{301}$

(f) $\frac{\sqrt{5}-1}{2}$

2. Compute

(a) $[2, 3, 2, 3, \dots]$

(b) $[1, 2, 1, 2, \dots]$

(c) $[1, 2, 2, 2, \dots]$

3. Compute the start of a continued fraction expansion for π and compare the accuracy of $[a_0, a_1, \dots, a_n]$ with that of the decimal expansion up to n digits, for $n = 1, 2, 3, 4$.

4. Compute the continued fraction expansions of e , $\sqrt{19}$, and $\tan(1)$ with sage.

5. Prove that $[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$ where

$p_0 = a_0$

$p_1 = a_1 p_0 + 1$

$p_n = a_n p_{n-1} + p_{n-2}$

for $n \geq 2$

$q_0 = 1$

$q_1 = a_1$

$q_n = a_n q_{n-1} + q_{n-2}$

6. Keeping the notation from 5., show that

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}.$$

(Hint: an easy way to proceed is to extend the definition of the p_j 's and q_j 's by setting $p_{-2} = q_{-1} = 0$ and $q_{-2} = p_{-1} = 1$.) Conclude that $\gcd(p_n, q_n) = 1$, i.e., the fraction $[a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$ is written in lowest terms.

7. Prove that the sequence $(\frac{p_n}{q_n})_{n \geq 1}$ converges. (Hint: first show that $(q_n)_{n \geq 1}$ is strictly increasing, and then prove that $(\frac{p_n}{q_n})_{n \geq 1}$ is a Cauchy sequence.)

8. Show that

$$p_n q_{n-2} - q_n p_{n-2} = (-1)^n a_n$$

and conclude that $(\frac{p_{2n}}{q_{2n}})_{n \geq 1}$ increases and $(\frac{p_{2n+1}}{q_{2n+1}})_{n \geq 1}$ decreases.

9. Suppose $a \in \mathbb{R} \setminus \mathbb{Q}$ has an eventually periodic continued fraction expansion, i.e.,

$$a = [a_0, a_1, \dots, a_n, a_{n+1}, \dots, a_{n+k}, a_{n+1}, \dots, a_{n+k}, \dots]$$

for some positive integers n and k . Prove that $[a_{n+1}, \dots, a_{n+k}, a_{n+1}, \dots, a_{n+k}, \dots]$ satisfies a quadratic equation and conclude that $a = b + cx$ where $b, c \in \mathbb{Q}$ and $x \in \mathbb{R}$ satisfies a quadratic equation, i.e., a is a *quadratic irrational*.

10. Stein 5.1–4.

11. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.