Proposition 8.2. *Let $a, b, c, d \in \mathbb{R}$.*

(i) *If $a < b$ and $b < c$ then $a < c$.*

(ii) *If $0 < a < b$ and $0 < c \leq d$ then $ac < bd$.*

(iii) *If $a \neq b$ then $a < b$ or $b < a$.*

(iv) *If $a < 0$ and $b < c$ then $ab > ac$.*

(v) *If $a \leq b \leq a$ then $a = b$.*

Proof. (i) The inequalities $a < b$ and $b < c$ mean that $b - a \in \mathbb{R}_{>0}$ and $c - b \in \mathbb{R}_{>0}$, from which we conclude with Axiom 8.6(i)

$$c - a = (c - b) + (b - a) \in \mathbb{R}_{>0} \ ,$$

which, in turn, means that $a < c$.

(ii) Suppose $0 < a < b$ and $0 < c \leq d$, that is, $a, b - a, c \in \mathbb{R}_{>0}$ and either $c = d$ or $d - c \in \mathbb{R}_{>0}$. In the first case $c = d$ we obtain

$$bd - ac = bc - ac = (b - a)\, c \in \mathbb{R}_{>0}$$

by Axiom 8.6(ii). In the second case $d - c \in \mathbb{R}_{>0}$,

$$bd - ac = (bd - ad) + (ad - ac) = (b - a)\, d + (d - c)\, a \in \mathbb{R}_{>0}$$

by a combination of Axioms 8.6(i) and 8.6(ii), after realizing that $d > 0$ by transitivity ($d > c > 0$).

(iii) Apply Axiom 8.6(iii) to the real number $a - b$: exactly one of $a - b > 0$, $-(a - b) < 0$, $a - b = 0$ is true. Suppose $a \neq b$, then this implies that either $a - b > 0$ or $-(a - b) < 0$ hold, which translate to $a > b$ or $a < b$.

(iv) Suppose $a < 0$ and $b < c$, that is, $-a \in \mathbb{R}_{>0}$ and $c - b \in \mathbb{R}_{>0}$. By Axiom 8.6(ii), $(-a)(c - b) \in \mathbb{R}_{>0}$, which simplifies to $ab - ac \in \mathbb{R}_{>0}$, which in turn means $ac < ab$.

(v) Suppose $a \leq b \leq a$. Assume, by means of contradiction, that $a \neq b$, which means that we have the inequalities $a < b < a$. By part (i), $a < a$, which implies that $0 = a - a \in \mathbb{R}_{>0}$, a contradiction to Axiom 8.6(iii). $\square$