**Proposition 5.8.** (i) $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ .

(ii) $\displaystyle\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ .

*(In particular,* $n(n+1)$ *is divisible by* 2 *and* $n(n+1)(2n+1)$ *is divisible by* 6*.)*

*Proof.* (i) We proceed by induction on $n$. For $n = 1$, we have $\sum_{i=1}^{1} i = 1 = \frac{1\cdot 2}{2}$.

For the induction step, assume that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Then, by the recursive definition of sums and the induction hypothesis,

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2} .$$

(ii) Again, we use induction on $n$. For $n = 1$, we have $\sum_{i=1}^{1} i^2 = 1 = \frac{1\cdot 2\cdot 3}{6}$.

For the induction step, assume that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{2n^3 + 3n^2 + n + 6n^2 + 12n + 6}{6} \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6} .\end{aligned}$$

□

**Project 5.9.** *Find (and prove) a formula for* $\sum_{j=1}^{n} j^3$*.*

One can guess a solution from the evaluations of this sum for the first couple of integers $n$, and then proceed with a proof by induction. Here we give a direct proof that depends on the formulas of Proposition 5.8. Namely, we evaluate the expression $\sum_{j=1}^{n}(j+1)^4 - \sum_{j=1}^{n} j^4$ in two different ways.

First, by Proposition 5.7, $\sum_{j=1}^{n}(j+1)^4 = \sum_{j=2}^{n+1} j^4$, and so, again by Proposition 5.7,

$$\sum_{j=1}^{n}(j+1)^4 - \sum_{j=1}^{n} j^4 = \sum_{j=2}^{n+1} j^4 - \sum_{j=1}^{n} j^4 = (n+1)^4 - 1 .$$

A second evaluation of $\sum_{j=1}^{n}(j+1)^4 - \sum_{j=1}^{n} j^4$ is obtained by expanding $(j+1)^4$:

$$\begin{aligned}\sum_{j=1}^{n} \left((j+1)^4 - j^4\right) &= \sum_{j=1}^{n} \left(j^4 + 4j^3 + 6j^2 + 4j + 1 - j^4\right) \\ &= 4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 \\ &= 4\sum_{j=1}^{n} j^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n .\end{aligned}$$

In the last step, we used Proposition 5.8. Equating the two different expressions we obtained for $\sum_{j=1}^{n}(j+1)^4 - \sum_{j=1}^{n} j^4$ gives

$$(n+1)^4 - 1 = 4\sum_{j=1}^{n} j^3 + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + n \ ,$$

or, after some simplification,

$$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4} \ .$$