Dedekind cotangent sums

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Define the sawtooth function $((x))$ by

$$((x)) := \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \, . \end{cases}$$

($\{x\} = x - [x] =$ fractional part of x .)

For $a, b \in \mathbb{N} := \{n \in \mathbb{Z} : n > 0\}$, we define the Dedekind sum as

$$\mathfrak{s}(a,b) := \sum_{k \bmod b} \left(\left(\frac{ka}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right)$$

$$= \frac{1}{4b} \sum_{k \bmod b} \cot \frac{\pi k a}{b} \, \cot \frac{\pi k}{b} \, .$$

Since their introduction by Dedekind in the 1880's, these sums and their generalizations have appeared in various areas such as analytic and algebraic number theory, topology, algebraic and combinatorial geometry, and algorithmic complexity.

The Bernoulli polynomials $B_k(x)$ are defined through

$$\frac{ze^{xz}}{e^z-1} = \sum_{k\geq 0} \frac{B_k(x)}{k!} z^k \ .$$

$(B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}$, etc.)

The Bernoulli numbers are $B_k := B_k(0)$. The Bernoulli functions $\overline{B}_k(x)$ are the periodized Bernoulli polynomials:

$$\overline{B}_k(x) := B_k(\{x\}) \ .$$

Apostol (1950's) introduced

$$\sum_{k \bmod b} \tfrac{k}{b} \, \overline{B}_n\left(\tfrac{ka}{b}\right) \ ,$$

generalized by Carlitz and Mikolás to

$$\mathfrak{s}_{m,n}(a;b,c) := \sum_{k \bmod b} \overline{B}_m\left(\tfrac{kb}{a}\right) \overline{B}_n\left(\tfrac{kc}{a}\right) \ .$$

For $a, b \in \mathbb{N}$, $x, y \in \mathbb{R}$, the Dedekind-Rademacher sum (1960's) is defined by

$$\mathfrak{s}(a, b; x, y) :=$$

$$\sum_{k \bmod b} \left(\left(a \, \frac{k+y}{b} - x \right) \right) \left(\left(\frac{k+y}{b} \right) \right) \; .$$

(A special version of this sum had been defined earlier by Meyer and Dieter.)

Takács (1970's) introduced

$$\sum_{k \bmod b} \overline{B}_1 \left(\frac{k+y}{b} \right) \overline{B}_n \left(a \, \frac{k+y}{b} - x \right) \; ,$$

further generalized by Halbritter (1980's) and Hall, Wilson, and Zagier (1990's) to

$$\mathfrak{s}_{m,n} \left(\begin{array}{c|cc} a & b & c \\ x & y & z \end{array} \right) :=$$

$$\sum_{k \bmod a} \overline{B}_m \left(b \, \frac{k+x}{a} - y \right) \overline{B}_n \left(c \, \frac{k+x}{a} - z \right) \; .$$

Around 1980, Meyer and Sczech, and Dieter independently introduced the cotangent sum, defined for $a, b \in \mathbb{N}$, $x, y \in \mathbb{R}$ by

$$\mathfrak{c}(a,b,c;x,y,z) :=$$

$$\frac{1}{c} \sum_{k \bmod c} \cot \pi \left(a \, \frac{k+z}{c} - x \right) \cot \pi \left(b \, \frac{k+z}{c} - y \right) \ .$$

These include as special cases various modified Dedekind sums introduced by Berndt, such as

$$\sum_{k=1}^{b} (-1)^{k+[ak/b]} \left(\left(\frac{k}{b} \right) \right) \ .$$

Finally, Zagier (1970's) introduced

$$\mathfrak{s}(a_0; a_1, \ldots, a_d) :=$$

$$\frac{(-1)^{d/2}}{a_0} \sum_{k=1}^{a_0 - 1} \cot \frac{\pi k a_1}{a_0} \cdots \cot \frac{\pi k a_d}{a_0} \ .$$

Definition. For $a_0, \ldots, a_d \in \mathbb{N}$, $m_0, \ldots, m_d \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $z_0, \ldots, z_d \in \mathbb{C}$, we define the Dedekind cotangent sum as

$$\mathfrak{c}\left(\begin{array}{c|ccc} a_0 & a_1 & \cdots & a_d \\ m_0 & m_1 & \cdots & m_d \\ z_0 & z_1 & \cdots & z_d \end{array}\right) :=$$

$$\frac{1}{a_0^{m_0+1}} \sum_{k \bmod a_0} \prod_{j=1}^{d} \cot^{(m_j)} \pi \left(a_j \, \frac{k+z_0}{a_0} - z_j \right) .$$

Reason for introducing cotangent derivatives:

- they appear in lattice point enumeration formulas for polyhedra (Diaz-Robins)
- they are essentially the discrete Fourier transforms of the Bernoulli functions: for $m \geq 2$,

$$\overline{B}_m\left(\frac{n}{p}\right) = \frac{B_m}{(-p)^m} +$$

$$m\left(\frac{i}{2p}\right)^m \sum_{k=1}^{p-1} \cot^{(m-1)}\left(\frac{\pi k}{p}\right) e^{2\pi kn/p} .$$

Corollary. For $a, b, c \in \mathbb{N}$ pairwise relatively prime,

$$\mathfrak{s}_{m,n}(a;b,c) \stackrel{\text{def}}{=} \sum_{k \bmod a} \overline{B}_m\left(\frac{kb}{a}\right)\overline{B}_n\left(\frac{kc}{a}\right)$$

$$= \frac{B_m B_n}{a^{m+n-1}} + mn\,\frac{(-1)^{(m-n)/2}}{2^{m+n}a^{m+n-1}} \cdot$$

$$\cdot \sum_{k=1}^{a-1} \cot^{(m-1)}\left(\frac{\pi kc}{a}\right)\cot^{(n-1)}\left(\frac{\pi kb}{a}\right)$$

$$\stackrel{\text{def}}{=} mn\,\frac{(-1)^{(m-n)/2}}{2^{m+n}}\,\mathfrak{c}\left(\begin{array}{c|cc} a & b & c \\ m+n-2 & n-1 & m-1 \\ 0 & 0 & 0 \end{array}\right)$$

$$+\frac{B_m B_n}{a^{m+n-1}}\,.$$

Another note:

$$\sum_{k=1}^{a-1} \overline{B}_m\left(\frac{k}{a}\right)\left(\left(\frac{kb}{a}\right)\right)$$

appears naturally in the study of plane partition enumeration (Almkvist, 1990's).

Three themes for Dedekind sums:

1. Reciprocity laws
2. Petersson-Knopp identities
3. Computability properties

1.

"If you had done something twice, you are likely to do it again."

Brian Kernighan and Bob Pike
(*The Unix Programming Environment*, Prentice Hall, p. 97)

$$\mathfrak{s}(a,b) := \frac{1}{4b} \sum_{k \bmod b} \cot\frac{\pi k a}{b} \cot\frac{\pi k}{b}$$

Theorem (Dedekind). If $a, b \in \mathbb{N}$ are relatively prime then

$$\mathfrak{s}(a,b) + \mathfrak{s}(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right)$$

$$= \text{something simple} .$$

"Proof" (Rademacher? Carlitz?): Integrate the function

$$f(z) = \cot(\pi a z)\cot(\pi b z)\cot(\pi z)$$

along

$$\gamma = [x+iy, x-iy, x+1-iy, x+1+iy, x+iy] ,$$

for suitably chosen x and y .

$$\mathfrak{c}(a,b,c;x,y,z) :=$$

$$\frac{1}{c} \sum_{k \bmod c} \cot \pi \left(a\, \frac{k+z}{c} - x \right) \cot \pi \left(b\, \frac{k+z}{c} - y \right) \ .$$

Theorem (Dieter). Let $a, b, c \in \mathbb{N}$ be pairwise relatively prime. Then

$$\begin{aligned} &\mathfrak{c}\,(a,b,c;x,y,z) + \mathfrak{c}\,(b,c,a;y,z,x) \\ &\quad + \mathfrak{c}\,(c,a,b;z,x,y) \\ &= \text{something simple} \ . \end{aligned}$$

"Proof": Integrate

$$f(w) = \cot \pi(aw-x) \cot \pi(bw-y) \cot \pi(cw-z)$$

along γ .

$$\mathfrak{s}(a_0; a_1, \ldots, a_d) :=$$

$$\frac{(-1)^{d/2}}{a_0} \sum_{k=1}^{a_0-1} \cot \frac{\pi k a_1}{a_0} \cdots \cot \frac{\pi k a_d}{a_0} .$$

Theorem (Zagier). If $a_0, \ldots, a_d \in \mathbb{N}$ are pairwise relatively prime then

$$\sum_{n=0}^{d} \mathfrak{s}\left(a_n; a_0, \ldots, \widehat{a_n}, \ldots, a_d\right)$$

$$= \text{something simple} .$$

"Proof": Integrate

$$f(z) = \cot \pi a_0 z \ \cdots \ \cot \pi a_d z .$$

along γ .

$$\mathfrak{c}\left(\begin{array}{c|ccc} a_0 & a_1 & \cdots & a_d \\ m_0 & m_1 & \cdots & m_d \\ z_0 & z_1 & \cdots & z_d \end{array}\right) :=$$

$$\frac{1}{a_0^{m_0+1}} \sum_{k \bmod a_0} \prod_{j=1}^{d} \cot^{(m_j)} \pi \left(a_j \, \frac{k+z_0}{a_0} - z_j \right) \, .$$

Theorem. Let $a_0, \ldots, a_d \in \mathbb{N}$, $m_0, \ldots m_d \in \mathbb{N}_0$, $z_0, \ldots, z_d \in \mathbb{C}$. Then

$$\sum_{n=0}^{d} (-1)^{m_n} m_n! \sum_{\substack{l_0, \ldots, \widehat{l_n}, \ldots, l_d \geq 0 \\ l_0 + \cdots + \widehat{l_n} + \cdots + l_d = m_n}} \frac{a_0^{l_0} \cdots \widehat{a_n^{l_n}} \cdots a_d^{l_d}}{l_0! \cdots \widehat{l_n!} \cdots l_d!} \cdot$$

$$\cdot \, \mathfrak{c}\left(\begin{array}{c|ccccc} a_n & a_0 & \cdots & \widehat{a_n} & \cdots & a_d \\ m_n & m_0+l_0 & \cdots & \widehat{m_n+l_n} & \cdots & m_d+l_d \\ z_n & z_0 & \cdots & \widehat{z_n} & \cdots & z_d \end{array}\right)$$

$$= \begin{cases} (-1)^{d/2} & \text{if all } m_k = 0 \text{ and } d \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

if for all distinct $i, j \in \{0, \ldots, d\}$ and all $m, n \in \mathbb{Z}$,

$$\frac{m+z_i}{a_i} - \frac{n+z_j}{a_j} \notin \mathbb{Z} \,.$$

"Proof": Integrate

$$f(z) = \prod_{j=0}^{d} \cot^{(m_j)} \pi \left(a_j z - z_j \right)$$

along γ .

2.

"Mathematics is a collection of cheap tricks and dirty jokes."

Lipman Bers

Theorem (Petersson-Knopp). If $a, b \in \mathbb{N}$ are relatively prime then

$$\sum_{d|n} \sum_{k \bmod d} \mathfrak{s}\left(\frac{n}{d}b + ka, ad\right) = \sigma(n)\,\mathfrak{s}(b, a)\,.$$

This identity was stated by Petersson in the 1970's (with additional congruence restrictions on a and b) and proved by Knopp in 1980. For n prime, the identity was already known to Dedekind.

It was generalized by Parson and Rosen to Apostol's generalized Dedekind sums, by Apostel and Vu to their 'sums of the Dedekind type' (both 1980's), and, most broadly, by Zheng (1990's) to what we will call sums of Dedekind type with weight (m_1, m_2) .

Definition. Let $a, a_1, \ldots, a_d \in \mathbb{N}$.

$$S\,(a; a_1, \ldots, a_d) := \sum_{k \bmod a} f_1\left(\frac{ka_1}{a}\right) \cdots f_d\left(\frac{ka_d}{a}\right)$$

is called of Dedekind type with weight $(m_1, \ldots, m_d)$ if for all $j = 1, \ldots, d$,

$$f_j(x+1) = f_j(x)$$

and for all $a \in \mathbb{N}$,

$$\sum_{k \bmod a} f_j\left(x + \frac{k}{a}\right) = a^{m_j}\, f_j(ax)\ .$$

Note that the Bernoulli functions $\overline{B}_m(x)$ satisfy this identity (with 'weight' $-m+1$), as do the functions $\cot^{(m)}(\pi x)$ (with 'weight' $m+1$). Zheng's theorem is the 'two-dimensional' case of the following

Theorem. Let $n, a, a_1, \ldots, a_d \in \mathbb{N}$. If

$$S\left(a; a_1, \ldots, a_d\right) := \sum_{k \bmod a} f_1\left(\frac{ka_1}{a}\right) \cdots f_d\left(\frac{ka_d}{a}\right)$$

is of Dedekind type with weight $(m_1, \ldots, m_d)$ then

$$\sum_{b|n} b^{-m_1-\cdots-m_d} \sum_{r_1,\ldots,r_d \bmod b} S\left(ab; \tfrac{n}{b}a_1 + r_1 a \ , \ \ldots \ , \ \tfrac{n}{b}a_d + r_d a\right)$$

$$= \ n \ \sigma_{d-1-m_1-\cdots-m_d}(n) \ S\left(a; a_1, \ldots, a_d\right) \ .$$

Here $\sigma_m(n) := \sum_{d|n} d^m$.

Our proof is along the exact same lines as Zheng's proof for $d = 2$, a slick application of the Möbius μ-function.

Corollary. For $n, a_0, \ldots, a_d \in \mathbb{N}, m_0, \ldots, m_d \in \mathbb{N}_0$,

$$\sum_{b|n} b^{m_0+1-m_1-\cdots-m_d-d} \sum_{r_1,\ldots,r_d \bmod b} \mathfrak{c}\left(\begin{array}{c|ccc} a_0 b & \frac{n}{b}a_1+r_1a_0 & \cdots & \frac{n}{b}a_d+r_da_0 \\ m_0 & m_1 & \cdots & m_d \\ 0 & 0 & \cdots & 0 \end{array}\right)$$

$$= \; n \; \sigma_{-m_1-\cdots-m_d-1}(n) \; \mathfrak{c}\left(\begin{array}{c|ccc} a_0 & a_1 & \cdots & a_d \\ m_0 & m_1 & \cdots & m_d \\ 0 & 0 & \cdots & 0 \end{array}\right) \; .$$

Corollary. For $n, a_0, \ldots, a_d \in \mathbb{N}$,

$$\sum_{b|n} b^{1-d} \sum_{r_1,\ldots,r_d \bmod b} \mathfrak{s}(a_0 b; \tfrac{n}{b}a_1 + r_1 a_0, \ldots, \tfrac{n}{b}a_d + r_d a_0)$$

$$= \sigma(n) \; \mathfrak{s}(a_0; a_1, \ldots, a_d) \; .$$