Project 5.5 (Unions and intersections)**.** *Given sets $A_1, A_2, A_3, \ldots,$ develop recursive definitions for $\bigcup_{k=1}^{n} A_k$ and $\bigcap_{k=1}^{n} A_k$. Find and prove the extension of DeMorgan's laws (Theorem 4.5) for these unions and intersections.*

Here is a recursive definition for $\bigcup_{k=1}^{n} A_k$ for all $n \in \mathbb{N}$:

(i) Define $\bigcup_{k=1}^{1} A_k$ to be A_1.

(ii) Assuming $\bigcup_{k=1}^{n} A_k$ already defined, we define $\bigcup_{k=1}^{n+1} A_k$ to be $\bigcup_{k=1}^{n+1} A_k = (\bigcup_{k=1}^{n} A_k) \cup A_{n+1}$.

Analogously, here is a recursive definition for $\bigcap_{k=1}^{n} A_k$ for any $n \in \mathbb{N}$:

(i) Define $\bigcap_{k=1}^{1} A_k$ to be A_1.

(ii) Assuming $\bigcap_{k=1}^{n} A_k$ already defined, we define $\bigcap_{k=1}^{n+1} A_k$ to be $\bigcap_{k=1}^{n+1} A_k = (\bigcap_{k=1}^{n} A_k) \cap A_{n+1}$.

DeMorgan's laws for these arbitrary unions and intersections are, for given sets $A_1, A_2, \ldots, A_n \subseteq X$,

(1) $(\bigcup_{k=1}^{n} A_k)^c = \bigcap_{k=1}^{n} A_k^c$

(2) $(\bigcap_{k=1}^{n} A_k)^c = \bigcup_{k=1}^{n} A_k^c$.

We start by proving (1) by induction on n. For the base case $n = 1$, we have by definition $\left(\bigcup_{k=1}^{1} A_k\right)^c = A_1^c = \bigcap_{k=1}^{1} A_k^c$. For the induction step, assume that $(\bigcup_{k=1}^{n} A_k)^c = \bigcap_{k=1}^{n} A_k^c$. Then by definition,

$$\left(\bigcup_{k=1}^{n+1} A_k\right)^c = ((\bigcup_{k=1}^{n} A_k) \cup A_{n+1})^c .$$

By Theorem 4.5,

$$((\bigcup_{k=1}^{n} A_k) \cup A_{n+1})^c = (\bigcup_{k=1}^{n} A_k)^c \cap A_{n+1}^c .$$

By induction hypothesis,

$$(\bigcup_{k=1}^{n} A_k)^c \cap A_{n+1}^c = (\bigcap_{k=1}^{n} A_k^c) \cap A_{n+1}^c = \bigcap_{k=1}^{n+1} A_k^c ,$$

which proves $\left(\bigcup_{k=1}^{n+1} A_k\right)^c = \bigcap_{k=1}^{n+1} A_k^c$.

Analogously, we prove (2) by induction on n. For the base case $n = 1$, we have by definition $\left(\bigcap_{k=1}^{1} A_k\right)^c = A_1^c = \bigcup_{k=1}^{1} A_k^c$. For the induction step, assume that $(\bigcap_{k=1}^{n} A_k)^c = \bigcup_{k=1}^{n} A_k^c$. Then by definition,

$$\left(\bigcap_{k=1}^{n+1} A_k\right)^c = ((\bigcap_{k=1}^{n} A_k) \cap A_{n+1})^c .$$

By Theorem 4.5,

$$((\bigcap_{k=1}^{n} A_k) \cap A_{n+1})^c = (\bigcap_{k=1}^{n} A_k)^c \cup A_{n+1}^c .$$

By induction hypothesis,

$$(\bigcap_{k=1}^{n} A_k)^c \cup A_{n+1}^c = (\bigcup_{k=1}^{n} A_k^c) \cup A_{n+1}^c = \bigcup_{k=1}^{n+1} A_k^c ,$$

which proves $\left(\bigcap_{k=1}^{n+1} A_k\right)^c = \bigcup_{k=1}^{n+1} A_k^c$.