Worksheet 7: The Möbius Function

1. Show that $\mu(n)$ is multiplicative.

2. Prove that
$$\sum_{d|n} \mu(d) \;=\; \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$
Hint: for $n > 1$, try induction on the number of prime factors of n.

3. Suppose that $g(n)$ is multiplicative, and let
$$f(n) \;:=\; \sum_{d|n} g(d)\,.$$
Prove that $f(n)$ is also multiplicative.

4. Prove the *Möbius Inversion Formula*:
$$f(n) \;=\; \sum_{d|n} g(d) \qquad \text{if and only if} \qquad g(n) \;=\; \sum_{d|n} \mu(d)\, f(\tfrac{n}{d})\,.$$
Hint: write sums like the one on the right-hand side as
$$\sum_{d|n} \mu(d)\, f(\tfrac{n}{d}) \;=\; \sum_{de=n} \mu(d)\, f(e)\,.$$

5. Andrews 6.4.1, 6.4.3, 6.4.7, 6.4.8, 6.4.11.

6. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.