MATH 725 Midterm Exam

Part II (take-home exam)

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book. As usual, **F** *stands for either* **R** *or* **C**.

You are welcome to use books and internet sources, but you are not allowed to discuss this exam with anyone (this includes live discussions, calls, chats, etc.). I reserve the right for an follow-up oral exam if I suspect that you did not follow these rules.

The take-home exam is due on at 9:30 a.m. on 22 October 2021 (via email), and your submission should be a pdf file (typed or carefully scanned).

(1) Let M be the vector space of all real $n \times n$ matrices, for some fixed $n \in \mathbf{Z}_{>0}$. For $A = \left(a_{jk}\right) \in M$, define the *trace* of A as

$$\operatorname{tr}(A) := \sum_{j=1}^{n} a_{jj}\,.$$

  (a) Show that $U := \{A \in M : \operatorname{tr}(A) = 0\}$ is a subspace of M.
  (b) Compute the dimension of U.

(2) As usual, let $\mathbf{R}[x]$ be the vector space of all polynomials with coefficients in $\mathbf{R}$.[1]
  (a) Show that $\frac{d}{dx}$ is a linear map $\mathbf{R}[x] \to \mathbf{R}[x]$. Is the map injective or surjective or both?
  (b) Fix $a \in \mathbf{R}$ and let $I_a : \mathbf{R}[x] \to \mathbf{R}[x]$ be defined by $I_a(f) := \int_a^x f(t)\,dt$. Show that I_a is linear. Is I_a injective or surjective or both?
  (c) Is it possible to choose a value of a so that I_a is the inverse of $\frac{d}{dx}$? Explain.

(3) Let V be vector space over $\mathbf{F}$, and let $f \in L(V)$. Suppose there exists $k \in \mathbf{Z}_{>0}$ such that $f^k = 0$.[2] Prove that 0 is the only eigenvalue of f.

(4) Consider the vector space $\mathscr{P}_2(\mathbf{C})$ of polynomials of degree ≤ 2, equipped with the inner product $\langle f, g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}\,dx$. Compute the orthogonal complement of $\mathscr{P}_1(\mathbf{C})$.

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[1] In this exercise, you may freely cite theorems from Calculus.

[2] Here 0 is the linear operator that returns $\mathbf{0}$ for every input vector.