Chapter 9
Zonotopes

"And what is the use of a book," thought Alice, "without pictures or conversations?"

Lewis Carroll (*Alice in Wonderland*)

We have seen that the discrete volume of a general integral polytope may be quite difficult to compute. It is therefore useful to have an infinite class of integral polytopes such that their discrete volume is more tractable, and yet they are robust enough to be "closer" in complexity to generic integral polytopes. One initial class of more tractable polytopes are parallelepipeds, and as we will see in Lemma 9.2, the Ehrhart polynomial of a d-dimensional half-open integer parallelepiped $\mathcal{P}$ is equal to $\operatorname{vol}(\mathcal{P})\, t^d$. In this chapter, we generalize parallelepipeds to projections of cubes.

9.1 Definitions and Examples

In order to extend the notion of a parallelepiped, we begin by defining the **Minkowski sum** of the polytopes $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n \subset \mathbb{R}^d$ as

$$\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_n := \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n : \mathbf{x}_j \in \mathcal{P}_j\}.$$

For example, if $\mathcal{P}_1$ is the rectangle $[0,2] \times [0,1] \subset \mathbb{R}^2$ and $\mathcal{P}_2$ is the line segment $\left[\binom{0}{0}, \binom{2}{3}\right] \subset \mathbb{R}^2$, then their Minkowski sum $\mathcal{P}_1 + \mathcal{P}_2$ is the hexagon whose vertices are $\binom{0}{0}, \binom{2}{0}, \binom{0}{1}, \binom{4}{3}, \binom{2}{4}$, and $\binom{4}{4}$, as depicted in Figure 9.1. Parallelepipeds are special instances of Minkowski sums, namely those of line segments whose direction vectors are linearly independent, plus a point.

We will also make use of the following handy construct: Given a polynomial $p(z_1, z_2, \dots, z_d)$ in d variables, the **Newton polytope** $\mathcal{N}(p(z_1, z_2, \dots, z_d))$ of $p(z_1, z_2, \dots, z_d)$ is the convex hull of all exponent vectors appearing in the

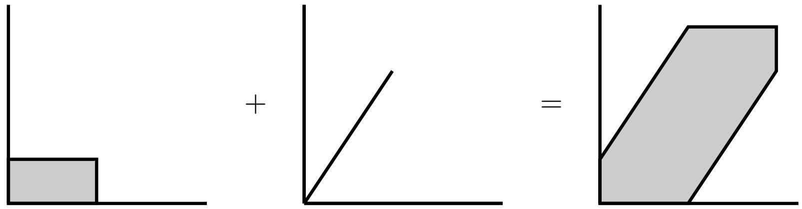

Fig. 9.1 The Minkowski sum $([0,2] \times [0,1]) + \left[\binom{0}{0}, \binom{2}{3}\right]$.

nonzero terms of $p(z_1, z_2, \ldots, z_d)$. For example, the hexagon in Figure 9.1 can be written as

$$\mathcal{N}\left(1 + 3z_1^2 - z_2 - 5z_1^4 z_2^3 + 34z_1^2 z_2^4 + z_1^4 z_2^4\right).$$

It turns out (Exercise 9.1) that the constructions of Newton polytopes and Minkowski sums are intimately related. Namely, if $p(z_1, z_2, \ldots, z_d)$ and $q(z_1, z_2, \ldots, z_d)$ are polynomials, then

$$\begin{aligned} &\mathcal{N}\big(p(z_1, z_2, \ldots, z_d)\, q(z_1, z_2, \ldots, z_d)\big) \\ &\qquad = \mathcal{N}\big(p(z_1, z_2, \ldots, z_d)\big) + \mathcal{N}\big(q(z_1, z_2, \ldots, z_d)\big). \end{aligned} \tag{9.1}$$

Suppose that we are now given n line segments in $\mathbb{R}^d$, such that each line segment has one endpoint at the origin and the other endpoint is located at the vector $\mathbf{u}_j \in \mathbb{R}^d$, for $j = 1, \ldots, n$. Then by definition, the Minkowski sum of these n segments is

$$\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) := \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n : \mathbf{x}_j = \lambda_j \mathbf{u}_j \text{ with } \lambda_j \in [0,1]\}.$$

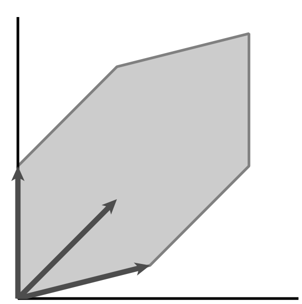

Fig. 9.2 The zonotope $\mathcal{Z}\left(\binom{0}{4}, \binom{3}{3}, \binom{4}{1}\right)$—a hexagon.

Let's rewrite the definition above in matrix form:

$$\begin{aligned}\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) &= \{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n : 0 \le \lambda_j \le 1\} \\ &= \left\{ (\mathbf{u}_1 \ \mathbf{u}_2 \ \cdots \ \mathbf{u}_n) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} : 0 \le \lambda_j \le 1 \right\} \\ &= \mathbf{A}\,[0,1]^n,\end{aligned}$$

where $\mathbf{A}$ is the $(d \times n)$-matrix whose j'th column is $\mathbf{u}_j$. Just a bit more generally, a **zonotope** is defined to be any translate of $\mathbf{A}\,[0,1]^n$, i.e.,

$$\mathbf{A}\,[0,1]^n + \mathbf{b}\,,$$

for some vector $\mathbf{b} \in \mathbb{R}^d$. So we now have two equivalent definitions of a zonotope—the first one is given by a Minkowski sum of line segments, while the second one is given by any projection of the unit cube $[0,1]^n$. Figures 9.2 and 9.3 show two example zonotopes.

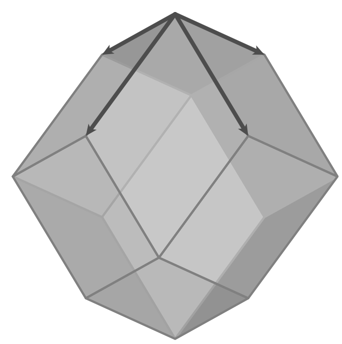

Fig. 9.3 The rhombic dodecahedron is a zonotope.

It is sometimes useful to translate a zonotope so that the origin becomes its new center of mass. To this end, we now dilate the matrix $\mathbf{A}$ by a factor of 2 and translate the resulting image so that its new center of mass is at the origin. Precisely, we have

$$\begin{aligned}&2\,\mathbf{A}\,[0,1]^n - (\mathbf{u}_1 + \cdots + \mathbf{u}_n) \\ &\quad = \{2\mathbf{u}_1\lambda_1 + \cdots + 2\mathbf{u}_n\lambda_n - (\mathbf{u}_1 + \cdots + \mathbf{u}_n) : 0 \le \lambda_j \le 1\} \\ &\quad = \{\mathbf{u}_1(2\lambda_1 - 1) + \cdots + \mathbf{u}_n(2\lambda_n - 1) : 0 \le \lambda_j \le 1\} \\ &\quad = \mathbf{A}\,[-1,1]^n, \qquad (9.2)\end{aligned}$$

where the last step holds because $-1 \le 2\lambda_j - 1 \le 1$ when $0 \le \lambda_j \le 1$. In other words, (9.2) is a linear image of the larger cube $[-1,1]^n$, centered at the origin. We thus define

$$\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n) := \mathbf{A}\,[-1,1]^n,$$

and we see that $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$ is indeed a zonotope, by definition. We say that a polytope $\mathcal{P}$ is **symmetric about the origin** when it has the property that $\mathbf{x} \in \mathcal{P}$ if and only if $-\mathbf{x} \in \mathcal{P}$. In general, a polytope $\mathcal{P}$ is called **centrally symmetric** if we can translate $\mathcal{P}$ by some vector $\mathbf{b}$ such that $\mathcal{P} + \mathbf{b}$ is symmetric about the origin. The zonotope $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$ is an example of a polytope that is symmetric about the origin. We go through the clean argument here, because it justifies our choice of a symmetric representation: Pick any $\mathbf{x} \in \mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$, so that $\mathbf{x} = \mathbf{A}\,\mathbf{y}$, with $\mathbf{y} \in [-1,1]^n$. Since $-\mathbf{y} \in [-1,1]^n$ as well, we have

$$-\mathbf{x} = \mathbf{A}\,(-\mathbf{y}) \in \mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)\,.$$

It is moreover true that each face of a zonotope is again a zonotope and that therefore every face of a zonotope is centrally symmetric (Exercise 9.2).

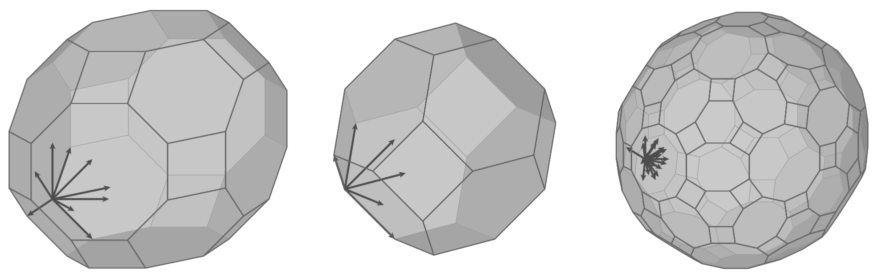

Fig. 9.4 A few more zonotopes.

9.2 Paving a Zonotope

Next we will show that any zonotope can be neatly decomposed into a disjoint union of half-open parallelepipeds. Suppose that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ are linearly independent, and let $\sigma_1, \sigma_2, \ldots, \sigma_m \in \{\pm 1\}$. Then we define

$$\Pi^{\sigma_1,\sigma_2,\ldots,\sigma_m}_{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m} := \left\{ \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_m\mathbf{w}_m : \begin{array}{l} 0 \le \lambda_j < 1 \text{ if } \sigma_j = -1 \\ 0 < \lambda_j \le 1 \text{ if } \sigma_j = 1 \end{array} \right\}.$$

In plain English, $\Pi^{\sigma_1,\sigma_2,\ldots,\sigma_m}_{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m}$ is a half-open parallelepiped generated by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, and the signs $\sigma_1, \sigma_2, \ldots, \sigma_m$ keep track of those facets of the parallelepiped that are included or excluded from the closure of the parallelepiped.

Lemma 9.1. *The zonotope $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_n)$ can be written as a disjoint union of translates of $\Pi^{\sigma_1, \sigma_2, \dots, \sigma_m}_{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m}$, where $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ ranges over all linearly independent subsets of $\{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_n\}$, each equipped with an appropriate choice of signs $\sigma_1, \sigma_2, \dots, \sigma_m$.*

Figure 9.5 illustrates the decomposition of a zonotope as suggested by Lemma 9.1.

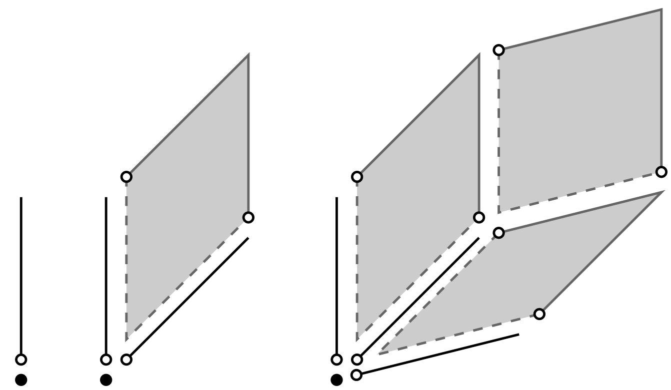

Fig. 9.5 A zonotopal decomposition of $\mathcal{Z}\left(\binom{0}{4}, \binom{3}{3}, \binom{4}{1}\right)$.

Proof of Lemma 9.1. We proceed by induction on n. If $n = 1$, $\mathcal{Z}(\mathbf{u}_1)$ is a line segment and $\mathbf{0} \cup (\mathbf{0}, \mathbf{u}_1]$ is a desired decomposition.

For general $n > 1$, we have by induction the decomposition

$$\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-1}) = \Pi_1 \cup \Pi_2 \cup \dots \cup \Pi_k$$

into half-open parallelepipeds of the form given in the statement of Lemma 9.1. Now we define the hyperplane $H := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{u}_n = 0\}$ and let $\pi : \mathbb{R}^d \to H$ denote the orthogonal projection onto H. Then $\pi(\mathbf{u}_1), \pi(\mathbf{u}_2), \dots, \pi(\mathbf{u}_{n-1})$ are line segments or points, and thus $\mathcal{Z}(\pi(\mathbf{u}_1), \pi(\mathbf{u}_2), \dots, \pi(\mathbf{u}_{n-1}))$ is a zonotope living in H. Once more by induction we can decompose

$$\mathcal{Z}(\pi(\mathbf{u}_1), \pi(\mathbf{u}_2), \dots, \pi(\mathbf{u}_{n-1})) = \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_m$$

into half-open parallelepipeds of the form given in Lemma 9.1. Each Φ_j is a half-open parallelepiped generated by some of the vectors $\pi(\mathbf{u}_1)$, $\pi(\mathbf{u}_2)$, …, $\pi(\mathbf{u}_{n-1})$; let $\widetilde{\Phi}_j$ denote the corresponding parallelepiped generated by their non-projected counterparts. Then (Exercise 9.5) the desired disjoint union of $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$ is given by $\Pi_1 \cup \Pi_2 \cup \dots \cup \Pi_k \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \dots \cup \mathcal{P}_m$ where

$$\mathcal{P}_j := \widetilde{\Phi}_j \times (\mathbf{0}, \mathbf{u}_n] \,. \qquad \square$$

This decomposition lemma is useful, e.g., to compute the Ehrhart polynomial of a zonotope. To this end, we now first work out the Ehrhart polynomial of a half-open parallelepiped, which in itself is a (particularly simple) zonotope.

Lemma 9.2. *Suppose $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_d \in \mathbb{Z}^d$ are linearly independent, and let*

$$\Pi := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_d \mathbf{w}_d : \, 0 \le \lambda_1, \lambda_2, \ldots, \lambda_d < 1\} \,.$$

Then

$$\# \left(\Pi \cap \mathbb{Z}^d\right) = \operatorname{vol} \Pi = \left|\det\left(\mathbf{w}_1, \ldots, \mathbf{w}_d\right)\right| ,$$

and for any positive integer t,

$$\# \left(t\, \Pi \cap \mathbb{Z}^d\right) = (\operatorname{vol} \Pi)\, t^d.$$

In other words, for the half-open parallelepiped Π, the discrete volume $\# \left(t\, \Pi \cap \mathbb{Z}^d\right)$ coincides with the continuous volume $(\operatorname{vol} \Pi)\, t^d$.

Proof. Because Π is half open, we can tile the t^{th} dilate $t\,\Pi$ by t^d translates of Π, and hence

$$L_\Pi(t) = \# \left(t\, \Pi \cap \mathbb{Z}^d\right) = \# \left(\Pi \cap \mathbb{Z}^d\right) t^d.$$

On the other hand, by the results of Chapter 3, $L_\Pi(t)$ is a polynomial with leading coefficient $\operatorname{vol} \Pi = \left|\det\left(\mathbf{w}_1, \ldots, \mathbf{w}_d\right)\right|$. Since we have equality of these polynomials for all positive integers t,

$$\# \left(\Pi \cap \mathbb{Z}^d\right) = \operatorname{vol} \Pi \,. \qquad \square$$

Our proof shows that Lemma 9.2 remains true if we switch the $\le$ and $<$ inequalities for some of the λ_j in the definition of Π, a fact that we will use freely, e.g., in the following tool to compute Ehrhart polynomials of zonotopes, which follows from a combination of Lemmas 9.1 and 9.2.

Corollary 9.3. *Decompose the zonotope $\mathcal{Z} \subset \mathbb{R}^d$ into half-open parallelepipeds according to Lemma 9.1. Then the coefficient c_k of the Ehrhart polynomial $L_\mathcal{Z}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$ equals the sum of the (relative) volumes of the k-dimensional parallelepipeds in the decomposition of $\mathcal{Z}$.*

9.3 The Permutahedron

To illustrate that Corollary 9.3 is useful for computations, we now study a famous polytope, the **permutahedron**

$$\mathcal{P}_d := \operatorname{conv}\left\{(\pi(1) - 1,\, \pi(2) - 1, \ldots,\, \pi(d) - 1) : \, \pi \in S_d\right\} ,$$

that is, the convex hull of $(0, 1, \ldots, d-1)$ and all points formed by permuting its entries. It is not hard to show (see Exercise 9.9) that $\mathcal{P}_d$ is $(d-1)$-dimensional and has $d!$ vertices. Figures 9.6 and 9.7 show P_3 and P_4 (as the latter is projected into $\mathbb{R}^3$).

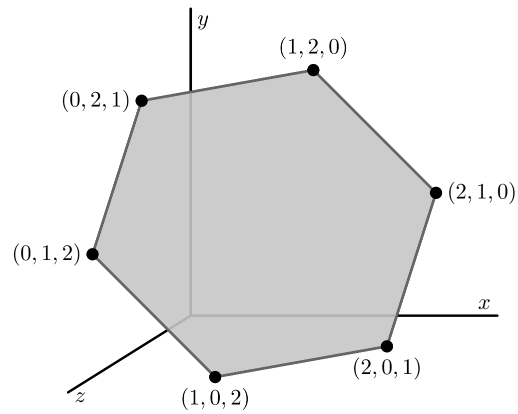

Fig. 9.6 The permutahedron P_3.

The reason that permutahedra make an appearance in this chapter is the following result.

Theorem 9.4.

$$\mathcal{P}_d = [\mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_1, \mathbf{e}_3] + \cdots + [\mathbf{e}_{d-1}, \mathbf{e}_d] \, ,$$

in words: the permutahedron $\mathcal{P}_d$ is the Minkowski sum of the line segments between each pair of unit vectors in $\mathbb{R}^d$.

Proof. We make use of a matrix that already made its debut in Section 3.7: namely, the permutahedron $\mathcal{P}_d$ is the Newton polytope of the polynomial

$$\det \begin{pmatrix} x_1^{d-1} & x_1^{d-2} & \cdots & x_1 & 1 \\ x_2^{d-1} & x_2^{d-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_d^{d-1} & x_d^{d-2} & \cdots & x_d & 1 \end{pmatrix}$$

—one can see the vertices of $\mathcal{P}_d$ appearing in the exponent vectors by computing this determinant by cofactor expansion. Now we use Exercise 3.20:

$$\det \begin{pmatrix} x_1^{d-1} & x_1^{d-2} & \cdots & x_1 & 1 \\ x_2^{d-1} & x_2^{d-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_d^{d-1} & x_d^{d-2} & \cdots & x_d & 1 \end{pmatrix} = \prod_{1 \le j < k \le d} (x_j - x_k) \, ,$$

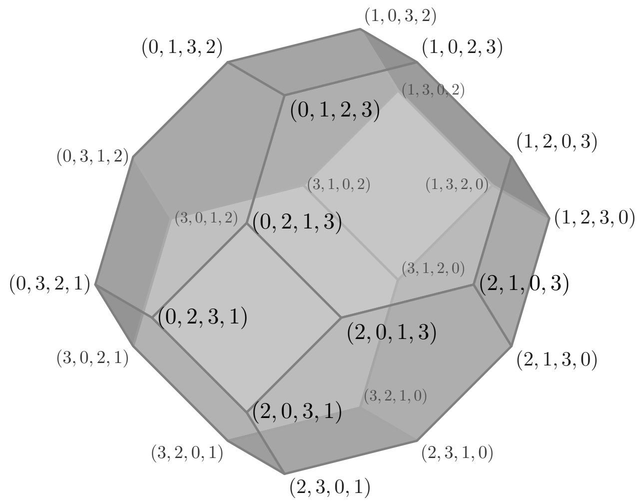

Fig. 9.7 The permutahedron P_4 (projected into $\mathbb{R}^3$).

and so by (9.1) we have

$$\mathcal{P}_d = \mathcal{N}(x_1 - x_2) + \mathcal{N}(x_1 - x_3) + \cdots + \mathcal{N}(x_{d-1} - x_d)\,.$$

The right-hand side is the Minkowski sum of the line segments $[\mathbf{e}_1, \mathbf{e}_2]$, $[\mathbf{e}_1, \mathbf{e}_3]$, $\dots$, $[\mathbf{e}_{d-1}, \mathbf{e}_d]$. $\square$

Now we will apply Corollary 9.3 to the special zonotope $\mathcal{P}_d$. A **forest** is a graph that does not contain any closed paths.

Theorem 9.5. *The coefficient c_k of the Ehrhart polynomial*

$$L_{\mathcal{P}_d}(t) = c_{d-1}\, t^{d-1} + c_{d-2}\, t^{d-2} + \cdots + c_0$$

of the permutahedron $\mathcal{P}_d$ equals the number of labeled forests on d nodes with k edges.

For example, we can compute $L_{\mathcal{P}_3}(t) = 3t^2 + 3t + 1$ by looking at the labeled forests in Figure 9.8.

An important ingredient to the proof of Theorem 9.5 is the following correspondence: to a subset $S \subseteq \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \dots, \mathbf{e}_{d-1} + \mathbf{e}_d\}$ we associate the graph G_S with node set $[d]$ and edge set

$$\{jk :\, \mathbf{e}_j + \mathbf{e}_k \in S\}\,.$$

We invite the reader to prove (Exercise 9.12):

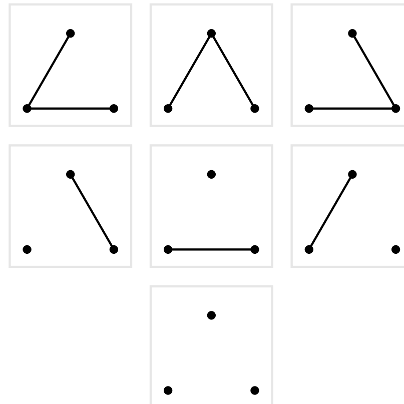

Fig. 9.8 The $3+3+1$ labeled forests on three nodes.

Lemma 9.6. *A subset $S \subseteq \{\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_3, \dots, \mathbf{e}_{d-1}+\mathbf{e}_d\}$ is linearly independent if and only if G_S is a forest.*

Proof of Theorem 9.5. Let I consist of all nonempty linearly independent subsets of

$$\{\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_3, \dots, \mathbf{e}_{d-1}+\mathbf{e}_d\}.$$

To each such subset S we will associate the half-open parallelepiped

$$\sum_{\mathbf{e}_j+\mathbf{e}_k \in S} (\mathbf{0}, \mathbf{e}_j+\mathbf{e}_k]$$

(written as a Minkowski sum). It has relative volume 1 (Exercise 9.13). These half-open parallelepipeds give rise to the disjoint union

$$\begin{aligned} \mathcal{P}_d &= \mathcal{Z}\left(\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_3, \dots, \mathbf{e}_{d-1}+\mathbf{e}_d\right) \\ &= \mathbf{0} \cup \bigcup_{S \in I} \left(\sum_{\mathbf{e}_j+\mathbf{e}_k \in S} (\mathbf{0}, \mathbf{e}_j+\mathbf{e}_k] \right), \end{aligned} \tag{9.3}$$

as the reader should verify in Exercise 9.14. Corollary 9.3 now says that the coefficient c_k of the Ehrhart polynomial

$$L_{\mathcal{P}_d}(t) = c_{d-1}\, t^{d-1} + c_{d-2}\, t^{d-2} + \cdots + c_0$$

equals the sum of the relative volumes of the k-dimensional parallelepipeds in our zonotopal decomposition of $\mathcal{P}_d$ given in (9.3). As mentioned above, each of these volumes is 1, and Lemma 9.6 now gives the statement of Theorem 9.5. $\square$

The leading coefficient of the Ehrhart polynomial of $\mathcal{P}_d$ is, of course, its relative volume, about which we can say slightly more. A **tree** is a connected forest.

Corollary 9.7. *The (relative) volume of the permutahedron $\mathcal{P}_d$ equals the number of labeled trees on d nodes.*

It turns out (Exercise 9.15) that there are precisely d^{d-2} labeled trees on d nodes, so the relative volume of $\mathcal{P}_d$ equals d^{d-2}.

Proof. By Theorem 9.5, the leading coefficient of $\mathcal{P}_d$ equals the number of labeled forests on d nodes with $d-1$ edges. But (as the reader should show in Exercise 9.16) any such forest is connected. □

9.4 The Ehrhart Polynomial of a Zonotope

Next we will generalize Lemma 9.2 to refine the formula of Corollary 9.3 for Ehrhart polynomials of zonotopes. Lemma 9.2 computed the (continuous and discrete) volume of a half-open parallelepiped Π spanned by d linearly independent vectors in $\mathbb{Z}^d$; this volume equals the number of integer points in Π. We need a version of this lemma in which Π is spanned by n linearly independent vectors in $\mathbb{Z}^d$ where $n < d$. It turns out that the (continuous and discrete) volume of Π is still given by the number of integer points in Π, but this number is not quite as easily computed as in Lemma 9.2.

Lemma 9.8. *Suppose $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in \mathbb{Z}^d$ are linearly independent, let*

$$\Pi := \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \cdots + \lambda_n \mathbf{w}_n : 0 \le \lambda_1, \lambda_2, \dots, \lambda_n < 1\},$$

and let V be the greatest common divisor of all $n \times n$ minors of the matrix formed by the column vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$. Then the relative volume of Π equals V. Furthermore,

$$\#\left(\Pi \cap \mathbb{Z}^d\right) = V$$

and for any positive integer t,

$$\#\left(t\,\Pi \cap \mathbb{Z}^d\right) = V\,t^n.$$

As with Lemma 9.2, the statement of Lemma 9.8 remains true if we switch the $\le$ and $<$ inequalities for some of the λ_j in the definition of Π.

Proof. We will make use of the *Smith normal form* of an integer matrix (or, more generally, a matrix with entries in an integral domain): more precisely, one proves in linear algebra that for any full-rank matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$, say where $m \ge n$, there exist invertible matrices $\mathbf{S} \in \mathbb{Z}^{m \times m}$ and $\mathbf{T} \in \mathbb{Z}^{n \times n}$ such that

$$\mathbf{S}\,\mathbf{A}\,\mathbf{T} = \begin{pmatrix} d_1 & 0 & \cdots & & 0 \\ 0 & d_2 & & & \\ \vdots & & \ddots & & \vdots \\ & & & d_{n-1} & 0 \\ 0 & & \cdots & 0 & d_n \\ 0 & & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

with diagonal integer entries whose product $d_1 d_2 \cdots d_n$ equals the gcd of all $n \times n$ minors of $\mathbf{A}$.

Geometrically, our matrices $\mathbf{S}$ and $\mathbf{T}$ transform Π into the half-open rectangular parallelepiped

$$\widetilde{\Pi} := [0, d_1) \times [0, d_2) \times \cdots \times [0, d_n) \subset \mathbb{R}^d$$

in such a way that the integer lattice $\mathbb{Z}^d$ is preserved; in particular, the relative volumes of Π and $\widetilde{\Pi}$ are equal, namely $d_1 d_2 \cdots d_n = V$. Furthermore,

$$\#\left(\Pi \cap \mathbb{Z}^d\right) = \#\left(\widetilde{\Pi} \cap \mathbb{Z}^d\right) = d_1 d_2 \cdots d_n = V.$$

The remainder of our proof follows that of Lemma 9.2: since Π is half open, we can tile the t^{th} dilate $t\,\Pi$ by t^n translates of Π, and so

$$\#\left(t\,\Pi \cap \mathbb{Z}^d\right) = \#\left(\Pi \cap \mathbb{Z}^d\right) t^n. \qquad \square$$

Lemma 9.8 is the crucial ingredient for the following refinement of Corollary 9.3.

Theorem 9.9. *Let $\mathcal{Z} := \mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be a zonotope generated by the integer vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Then the Ehrhart polynomial of $\mathcal{Z}$ is given by*

$$L_{\mathcal{Z}}(t) = \sum_S m(S)\, t^{|S|},$$

where S ranges over all linearly independent subsets of $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, and $m(S)$ is the greatest common divisor of all minors of size $|S|$ of the matrix whose columns are the elements of S.

For example, going to back to the zonotope $\mathcal{Z}\left(\binom{0}{4}, \binom{3}{3}, \binom{4}{1}\right)$ featured in Figure 9.5, we compute

$$\begin{aligned} L_{\mathcal{Z}\left(\binom{0}{4}, \binom{3}{3}, \binom{4}{1}\right)}(t) &= \left|\det \begin{pmatrix} 0 & 3 \\ 4 & 3 \end{pmatrix}\right| t^2 + \left|\det \begin{pmatrix} 0 & 4 \\ 4 & 1 \end{pmatrix}\right| t^2 + \left|\det \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}\right| t^2 \\ &\quad + \gcd(0,4)\, t + \gcd(3,3)\, t + \gcd(4,1)\, t + 1 \\ &= 37\, t^2 + 8\, t + 1\,. \end{aligned}$$

We remark that the numbers $m(S)$ appearing in Theorem 9.9 have geometric meaning: If $V(S)$ and $L(S)$ denote the vector space and lattice generated by S, respectively, then $m(S)$ equals the cardinality of the group $\left(V(S) \cap \mathbb{Z}^d\right)/L(S)$.

Proof of Theorem 9.9. Decompose $\mathcal{Z}$ as described in Lemma 9.1. Corollary 9.3 says that the coefficient c_k of the Ehrhart polynomial

$$L_{\mathcal{Z}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$$

equals the sum of the relative volumes of the k-dimensional parallelepipeds in this decomposition. Lemma 9.8 implies that these relative volumes are the gcd's of the minors of the matrices formed by the generators of these parallelepipeds. $\square$

Notes

1. The word "zonotope" appears to have originated from the fact that for each line segment $\mathbf{u}_j$ that comes into the Minkowski-sum definition of a zonotope $\mathcal{Z}$, there corresponds a "zone," composed of all of the facets of $\mathcal{Z}$ that contain parallel translates of $\mathbf{u}_j$. This zone separates the zonotope into two isometric pieces, a "northern" hemisphere and a "southern" hemisphere, a property that is sometimes useful in their study.

2. The combinatorial study of zonotopes was brought to the forefront (if not initiated) by Peter McMullen [164]; see also his paper [167] with Geoffrey Shephard. Lemma 9.1 first appeared explicitly in [164].

3. The permutahedron seems to have been first studied by Pieter Hendrik Schoute [206] in 1911. It tends to play a central role whenever one tries to 'geometrize' a situation involving the symmetric group. The permutahedron is a *simple* zonotope and as such quite a rare animal. It is not clear from the literature who first realized Theorem 9.5, but we suspect it was Richard Stanley [224, Exercises 4.63 and 4.64]. There has been a recent flurry of research activities on *generalized permutahedra* (initiated by Alexander Postnikov [190]), which share many of the fascinating geometric and arithmetic properties of permutahedra.

4. Corollary 9.3 and Theorem 9.9 are due to Stanley [222] (see also [220, Example 3.1] and [224, Exercise 4.31]), and we follow his proof closely in this chapter. The formula for the leading coefficient of the Ehrhart polynomial in Theorem 9.9, i.e., the volume of the zonotope, goes back to McMullen [213, Equation (57)].

5. Zonotopes have played a central role in the recently established theory of *arithmetic matroids* and *arithmetic Tutte polynomials*, and Corollary 9.3 was restated in this language by Michele D'Adderio and Luca Moci [88, Theorem 3.2]. Ehrhart polynomials of zonotopes and their arithmetic Tutte siblings are in general harder to compute than what this chapter might convey; some examples for computable formulas (for arithmetic Tutte polynomials for the classical root systems) were given by Federico Ardila, Federico Castillo, and Michael Henley [10].

Exercises

9.1. ♣ Prove (9.1): if $p(z_1, z_2, \dots, z_d)$ and $q(z_1, z_2, \dots, z_d)$ are polynomials,

$$\begin{aligned}&\mathcal{N}\big(p(z_1, z_2, \dots, z_d)\, q(z_1, z_2, \dots, z_d)\big)\\ &\qquad = \mathcal{N}\big(p(z_1, z_2, \dots, z_d)\big) + \mathcal{N}\big(q(z_1, z_2, \dots, z_d)\big)\,.\end{aligned}$$

9.2. Prove that every face of a zonotope is a zonotope, and conclude that every face of a zonotope is centrally symmetric.

9.3. Let $\mathcal{P}$ be a d-dimensional integral parallelepiped in $\mathbb{R}^n$. Prove that if there exists only one integer point $\mathbf{x}$ in the interior of $\mathcal{P}$, then $\mathbf{x}$ must be the center of mass of $\mathcal{P}$.

9.4. Let $\mathcal{P}$ be a d-dimensional integral parallelepiped in $\mathbb{R}^n$. Prove that the convex hull of all the integer points in the interior of $\mathcal{P}$ is a centrally symmetric polytope.

9.5. ♣ Complete the proof of Lemma 9.1 by showing (using the notation from the proof) that $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$ equals the disjoint union of $\Pi_1, \Pi_2, \dots, \Pi_k$ and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m$.

9.6. (a) Show that for any real vectors $\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w} \in \mathbb{R}^d$,

$$\mathcal{Z}(\mathbf{u}_1, \dots, \mathbf{u}_n, -\mathbf{w}) = \mathcal{Z}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w}) - \mathbf{w},$$

where the latter difference is defined as a translation of the zonotope $\mathcal{Z}(\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{w})$ by the vector $-\mathbf{w}$.

(b) Show that given any zonotope $\mathcal{Z}$, we may always pick all of the generators of a translate of $\mathcal{Z}$ in some halfspace containing the origin on its boundary.

9.7. Prove that the following statements are equivalent for a polytope $\mathcal{P}$:

(a) $\mathcal{P}$ is a zonotope;
(b) every 2-dimensional face of $\mathcal{P}$ is a zonotope;
(c) every 2-dimensional face of $\mathcal{P}$ is centrally symmetric.

9.8. Given a zonotope $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n) \subset \mathbb{R}^d$, consider the hyperplane arrangement $\mathcal{H}$ consisting of the n hyperplanes in $\mathbb{R}^d$ through the origin with normal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Show that there is a one-to-one correspondence between the vertices of $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$ and the regions of $\mathcal{H}$ (i.e., the maximal connected components of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$). Can you give an analogous correspondence for the other faces of $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$?

9.9. ♣ Show that the dimension of the permutahedron

$$\mathcal{P}_d = \operatorname{conv}\{(\pi(1)-1, \pi(2)-1, \ldots, \pi(d)-1) : \pi \in S_d\}$$

is $d-1$ and that P_d has $d!$ vertices.

9.10. Show that the permutahedron $\mathcal{P}_d$ is the image of the Birkhoff–von Neumann polytope $\mathcal{B}_d$ from Chapter 6 under a suitable linear map.

9.11. According to Exercise 9.9, the permutahedron $\mathcal{P}_d$ lies in a hyperplane $H \subset \mathbb{R}^d$. Show that $\mathcal{P}_d$ tiles H. (*Hint:* start by drawing the case $d = 3$. For the general case, the viewpoint of Exercise 9.8 might be useful.)

9.12. Prove Lemma 9.6: A subset $S \subseteq \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \ldots, \mathbf{e}_{d-1} + \mathbf{e}_d\}$ is linearly independent if and only if G_S is a forest.

9.13. Let S be a linearly independent subset of

$$\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \ldots, \mathbf{e}_{d-1} + \mathbf{e}_d\}.$$

Show that the half-open parallelepiped

$$\sum_{\mathbf{e}_j + \mathbf{e}_k \in S} (\mathbf{0}, \mathbf{e}_j + \mathbf{e}_k]$$

(written as a Minkowski sum) has relative volume 1.

9.14. Prove (9.3): namely,

$$\mathcal{P}_d = \mathbf{0} \cup \bigcup_{S \in I} \left(\sum_{\mathbf{e}_j + \mathbf{e}_k \in S} (\mathbf{0}, \mathbf{e}_j + \mathbf{e}_k] \right)$$

as a disjoint union.

9.15. Prove that there are precisely d^{d-2} labeled trees on d nodes.

9.16. Show that any forest on d nodes with $d-1$ edges is connected, i.e., a tree.

9.17. Given a graph G with node set $[d]$, we define the *graphical zonotope* $\mathcal{Z}_G$ as the Minkowski sum of the line segments $[\mathbf{e}_j, \mathbf{e}_k]$ for all edges jk of G. (Thus the permutahedron $\mathcal{P}_d$ is the graphical zonotope of the complete graph on d nodes.) Prove that the volume of $\mathcal{Z}_G$ equals the number of spanning trees of G. Give an interpretation of the Ehrhart coefficients of $\mathcal{Z}_G$, in analogy with Theorem 9.5.

9.18. Let G be a graph with node set $[d]$, and let $\deg(j)$ denote the number of edges incident to j (the **degree** of the node j). The vector

$$\deg(G) := (\deg(1), \deg(2), \ldots, \deg(d))$$

is the **labeled degree sequence** of G. Let $\mathcal{Z}_d$ be the zonotope generated by the vectors $\mathbf{e}_j + \mathbf{e}_k$ for $1 \le j < k \le d$.

(a) Show that every labeled degree sequence of a graph with d nodes is an integer point in $\mathcal{Z}_d$.
(b) If you know the *Erdős–Gallai Theorem* (and if you don't, look it up), prove that every integer point in $\mathcal{Z}_d$ whose coordinate sum is even is a labeled degree sequence of a graph with d nodes.

9.19. Let K be a compact, convex set in $\mathbb{R}^d$, symmetric about the origin, whose volume is greater than 2^d. Prove that K must contain a non-zero integer point. (This result is known as *Minkowski's fundamental theorem* and lies at the core of the geometry of numbers, as well as algebraic number theory. Its applications have shown how interesting and useful symmetric bodies can be.)

Open Problems

9.20. Classify the polynomials in $\mathbb{Z}[t]$ that are Ehrhart polynomials of integral zonotopes. (This is hard: if the matrix defining a zonotope is unimodular, then its Ehrhart coefficients are the entries of the f-vector of the underlying matroid—see Note 5; the classification of f-vectors of matroids is an old and difficult problem.)

9.21. The **unlabeled degree sequence** (often simply called **degree sequence**) of a graph G is the vector $\deg(G)$ (defined in Exercise 9.18) redefined such that its entries are in decreasing order. How many distinct degree sequences are there for graphs on n nodes? (For known values for the first few n, see [1, Sequence A004251]. The analogous question for *labeled* degree sequences was answered in [222]; that proof starts with Exercise 9.18.)

9.22. Let $K \subset \mathbb{R}^d$ be a d-dimensional convex body with the origin as its barycenter. If K contains only the origin as an interior lattice point, then $\operatorname{vol}(K) \le \frac{(d+1)d}{d!}$, where equality holds if and only if K is unimodularly equivalent to $(d+1)\Delta$, where Δ is the d-dimensional standard simplex from Section 2.3. (This is a conjecture of Ehrhart from 1964. Ehrhart himself proved this upper bound for all d-dimensional simplices, and also completely in dimension 2, but it remains open in general. The interested reader may consult [179] about the current state of Ehrhart's conjecture and its relation to the Ricci curvature of Fano manifolds.)

Chapter 10
h-Polynomials and h^*-Polynomials

Life is the twofold internal movement of composition and decomposition at once general and continuous.

Henri de Blainville (1777–1850)

In Chapters 2 and 3, we developed the Ehrhart polynomial and Ehrhart series of an integral polytope $\mathcal{P}$ and realized that the arithmetic information encoded in an Ehrhart polynomial is equivalent to the information encoded in its Ehrhart series. More precisely, when the Ehrhart series is written as a rational function, we introduced the name *h^*-polynomial* for its numerator:

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{h^*(z)}{(1-z)^{\dim(\mathcal{P})+1}} \, .$$

Our goal in this chapter is to prove several decomposition formulas for $h^*(z)$ based on triangulations of $\mathcal{P}$. As we will see, these decompositions will involve both arithmetic data from the simplices of the triangulation and combinatorial data from the face structure of the triangulation.

10.1 Simplicial Polytopes and (Unimodular) Triangulations

In Chapter 5—more precisely, in Theorem 5.1—we derived relations among the face numbers f_k of a simple polytope. The twin sisters of these *Dehn–Sommerville relations* for **simplicial** polytopes—all of whose nontrivial faces are simplices—were the subject of Exercise 5.9. It turns out they can be stated in a compact way by introducing the **h-polynomial** of the d-polytope $\mathcal{P}$:

$$h_{\mathcal{P}}(z) := \sum_{k=-1}^{d-1} f_k \, z^{k+1} \, (1-z)^{d-1-k} \, ,$$

where we set $f_{-1} := 1$. The following is a nifty restatement of Exercise 5.9, as the reader should check (Exercise 10.1).

Theorem 10.1 (Dehn–Sommerville relations for simplicial polytopes). *If $\mathcal{P}$ is a simplicial d-polytope, then $h_{\mathcal{P}}$ is a palindromic polynomial.*

We need this simplicial version of the Dehn–Sommerville relations because it naturally connects to triangulations. The h-polynomial of $\mathcal{P}$ encodes combinatorial information about the faces in the boundary of $\mathcal{P}$, and so the natural setting in the world of triangulations will consist of a triangulation of the *boundary* of a polytope. By analogy with the face numbers of a polytope, given a triangulation T of the boundary $\partial\mathcal{P}$ of a given d-dimensional polytope $\mathcal{P}$, we define f_k to be the number of k-simplices in T; they will be encoded, as above, in the h-polynomial

$$h_T(z) := \sum_{k=-1}^{d-1} f_k \, z^{k+1} \, (1-z)^{d-1-k} \, ,$$

where again we set $f_{-1} := 1$. The typical scenario we will encounter is that we are given a triangulation T of the polytope $\mathcal{P}$ and then consider the **induced** triangulation

$$\{\Delta \in T : \, \Delta \subset \partial\mathcal{P}\}$$

of the boundary of $\mathcal{P}$.

Theorem 10.2 (Dehn–Sommerville relations for boundary triangulations). *Given a regular triangulation of the polytope $\mathcal{P}$, the h-polynomial of the induced triangulation of $\partial\mathcal{P}$ is palindromic.*

Proof. Given a regular triangulation T of $\mathcal{P} \subseteq \mathbb{R}^d$, let $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ be the corresponding lifted polytope as in (3.1). Choose a point $\mathbf{v} \in \mathcal{P}^\circ$ and lift it to $(\mathbf{v}, h+2) \in \mathbb{R}^{d+1}$, where h is the maximal height among the vertices of $\mathcal{Q}$. Let $\mathcal{R}$ be the convex hull of $(\mathbf{v}, h+2)$ and the vertices of the lower hull of $\mathcal{Q}$, and let

$$\mathcal{S} := \mathcal{R} \cap \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \, x_{d+1} = h+1 \right\} .$$

Figure 10.1 shows an instance of this setup. (The polytope $\mathcal{S}$ is a *vertex figure* of $\mathcal{Q}$ at $\mathbf{v}$.) Exercise 10.2 says that $\mathcal{S}$ is a simplicial polytope whose face numbers f_k equal the face numbers of the triangulation of $\partial\mathcal{P}$ induced by T. Thus the h-polynomial of this triangulation of $\partial\mathcal{P}$ equals $h_{\mathcal{S}}(z)$, which is palindromic by Theorem 10.1. □

When we are given a triangulation T of a polytope (rather than of its boundary), we adjust our definition of the accompanying h-polynomial to

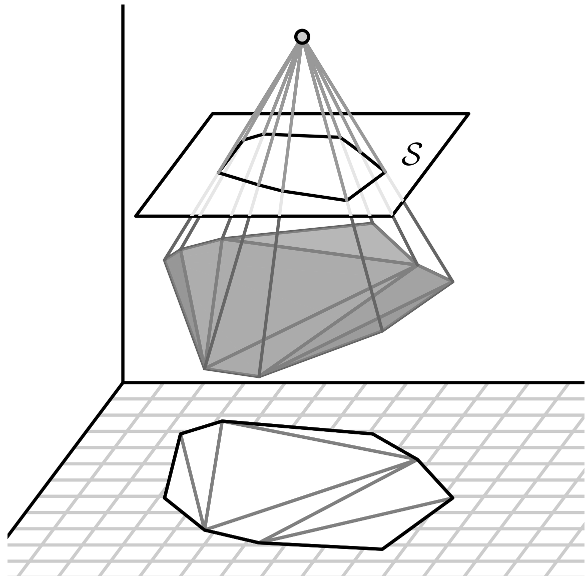

Fig. 10.1 The geometry of our proof of Theorem 10.2.

$$h_T(z) := \sum_{k=-1}^{d} f_k \, z^{k+1} \, (1-z)^{d-k} \, .$$

In general, there is no analogue of Theorem 10.2 for this h-polynomial, though an (important) exception is given in Exercise 10.3.

There is a second reason for us to introduce the h-polynomial of a triangulation. We call a triangulation T **unimodular** if each simplex

$$\Delta = \operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\} \in T$$

has the property that the vectors $\mathbf{v}_1 - \mathbf{v}_0$, $\mathbf{v}_2 - \mathbf{v}_0$, $\ldots$, $\mathbf{v}_k - \mathbf{v}_0$ form a lattice basis of $\operatorname{span}(\Delta) \cap \mathbb{Z}^d$. In this case, we also call the simplex Δ **unimodular**. One example of a unimodular simplex is the standard k-simplex Δ of Section 2.3. We recall that in this case

$$\operatorname{Ehr}_\Delta(z) = \frac{1}{(1-z)^{k+1}} \tag{10.1}$$

and invite the reader to prove that the same Ehrhart series comes with *any* unimodular k-simplex (Exercise 10.4).

Here is the main result of this section.

Theorem 10.3. *If $\mathcal{P}$ is an integral d-polytope that admits a unimodular triangulation T then*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_T(z)}{(1-z)^{d+1}} \, .$$

In words, the h^*-polynomial of $\mathcal{P}$ is given by the h-polynomial of the triangulation T.

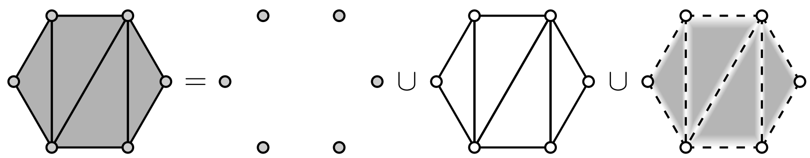

Fig. 10.2 A triangulation of a hexagon and the corresponding decomposition (10.2).

Proof. We start by writing $\mathcal{P}$ as the union of all open simplices in T, pictured in an example in Figure 10.2:

$$\mathcal{P} = \bigcup_{\Delta \in T} \Delta^\circ \tag{10.2}$$

(here we are using Exercise 5.4). Since this union is disjoint, we can compute

$$\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{\Delta \in T} \mathrm{Ehr}_{\Delta^\circ}(z) = 1 + \sum_{\Delta \in T} \left(\frac{z}{1-z} \right)^{\dim \Delta + 1} .$$

Here the last equality follows from the Ehrhart–Macdonald Reciprocity Theorem 4.4 applied to the Ehrhart series (10.1) of a unimodular simplex. Since the above summands depend only on the dimension of each simplex, we can rewrite

$$\begin{aligned} \mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{k=0}^{d} f_k \left(\frac{z}{1-z} \right)^{k+1} &= \sum_{k=-1}^{d} f_k \left(\frac{z}{1-z} \right)^{k+1} \\ = \frac{\sum_{k=-1}^{d} f_k \, z^{k+1} (1-z)^{d-k}}{(1-z)^{d+1}} &= \frac{h_T(z)}{(1-z)^{d+1}} \, . \end{aligned}$$

□

Theorem 10.3 says something remarkable: namely, that the arithmetic of a polytope (its discrete volume) is completely determined by the combinatorics of the unimodular triangulation (its face structure). Unfortunately, not all integral polytopes admit unimodular triangulations—in fact, most do not (see Exercise 10.5). Our next goal is to find an analogue of Theorem 10.3 that holds for *any* integral polytope.

10.2 Fundamental Parallelepipeds Open Up, With an h-Twist

Our philosophy in constructing a generalization of Theorem 10.3 rests in revisiting the cone over a given polytope from Chapter 3, using a decomposition of $\mathcal{P}$ into open simplices.

Given an integral d-polytope $\mathcal{P}$, fix a (not necessarily unimodular) triangulation T. As in (10.2), we can write $\mathcal{P}$ as the disjoint union of the open simplices in T; from this point of view, the following definition should look natural. Given a simplex $\Delta \in T$, let $\Pi(\Delta)$ be the fundamental parallelepiped of $\operatorname{cone}(\Delta)$ (as defined in Section 3.3), and let

$$B_\Delta(z) := \sigma_{\Pi(\Delta)^\circ}(1, 1, \dots, 1, z) .$$

Thus $B_\Delta(z)$ is an "open" variant of $h^*_\Delta(z) = \sigma_{\Pi(\Delta)}(1, 1, \dots, 1, z)$, an equation we used several times in Chapters 3 and 4.

Looking back at our proof of Theorem 10.3, it makes moral sense to include $\varnothing$ in the collection of faces of a triangulation of a given polytope, with the convention $\dim(\varnothing) := -1$. (This explains in hindsight our convention $f_{-1} := 1$.) We will assume for the rest of this chapter that any triangulation T includes the "empty simplex" $\varnothing$. Along the same lines, we define $B_\varnothing(z) := 1$.

We need one more concept to be able to state our generalization of Theorem 10.3. Given a simplex $\Delta \in T$, let

$$\operatorname{link}_T(\Delta) := \{\Omega \in T : \Omega \cap \Delta = \varnothing, \ \Omega \subseteq \Phi \text{ for some } \Phi \in T \text{ with } \Delta \subseteq \Phi\} ,$$

the **link** of Δ. In words, every simplex in $\operatorname{link}_T(\Delta)$ is disjoint from Δ, yet it is the face of a simplex in T that also contains Δ as a face; see Figure 10.3 for two examples. When the triangulation T is clear from the context,

Fig. 10.3 The links of a vertex and an edge in a triangulation of the 3-cube.

we suppress the subscript and simply write $\operatorname{link}(\Delta)$. We remark that, by definition, $\operatorname{link}(\varnothing) = T$. In general, $\operatorname{link}(\Delta)$ consists of a collection of simplices in T, the largest dimension of which is $d - \dim(\Delta) - 1$ (Exercise 10.8), and so it is reasonable to define

$$h_{\operatorname{link}(\Delta)}(z) := \sum_{k=-1}^{d-\dim(\Delta)-1} f_k \, z^{k+1} \, (1-z)^{d-\dim(\Delta)-1-k} \,,$$

where f_k denotes the number of k-simplices in $\operatorname{link}(\Delta)$. The h-polynomial of $\operatorname{link}(\Delta)$ encodes combinatorial data coming from the simplices in T that contain Δ. This statement can be made more precise, as we invite the reader to prove in Exercise 10.9:

$$h_{\operatorname{link}(\Delta)}(z) = (1-z)^{d-\dim(\Delta)} \sum_{\Phi \supseteq \Delta} \left(\frac{z}{1-z} \right)^{\dim(\Phi)-\dim(\Delta)} , \qquad (10.3)$$

where the sum is over all simplices $\Phi \in T$ containing Δ.

Theorem 10.4 (Betke–McMullen decomposition of h^*). *Fix a triangulation T of the integer d-polytope $\mathcal{P}$. Then*

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{\sum_{\Delta \in T} h_{\operatorname{link}(\Delta)}(z) \, B_\Delta(z)}{(1-z)^{d+1}} \,.$$

Before proving Theorem 10.4, we explain its relation to Theorem 10.3. If a simplex $\Delta \in T$ is unimodular then the corresponding fundamental parallelepiped $\Pi(\Delta)$ of $\operatorname{cone}(\Delta)$ contains the origin as its sole integer lattice point, and thus $B_\Delta(z) = 0$, unless $\Delta = \varnothing$. Thus, if the triangulation T is unimodular then the sum giving the h^*-polynomial in Theorem 10.4 collapses to $h_{\operatorname{link}(\varnothing)}(z) \, B_\varnothing(z) = h_T(z)$, and so Theorem 10.3 follows as a special case of Theorem 10.4.

Proof of Theorem 10.4. We start, as in our proof of Theorem 10.3, by writing $\mathcal{P}$ as the disjoint union of all open nonempty simplices in T as in (10.2), and thus, using Ehrhart–Macdonald Reciprocity (Theorem 4.4),

$$\begin{aligned}\operatorname{Ehr}_{\mathcal{P}}(z) &= 1 + \sum_{\Delta \in T \setminus \{\varnothing\}} \operatorname{Ehr}_{\Delta^\circ}(z) = 1 + \sum_{\Delta \in T \setminus \{\varnothing\}} (-1)^{\dim(\Delta)+1} \frac{h^*_\Delta\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)^{\dim \Delta + 1}} \\ &= 1 + \frac{\sum_{\Delta \in T \setminus \{\varnothing\}} z^{\dim(\Delta)+1} (1-z)^{d-\dim(\Delta)} h^*_\Delta(\frac{1}{z})}{(1-z)^{d+1}} \,. \qquad (10.4)\end{aligned}$$

Here $h^*_\Delta(z)$ denotes the h^*-polynomial of the simplex Δ. Now we use Exercise 10.10:

$$\sigma_{\Pi(\Delta)}(\mathbf{z}) = 1 + \sum_{\substack{\Omega \subseteq \Delta \\ \Omega \neq \varnothing}} \sigma_{\Pi(\Omega)^\circ}(\mathbf{z}) \,,$$

where the sum is over all nonempty faces of Δ. This identity specializes, choosing $\mathbf{z} = (1, 1, \dots, 1, z)$, to

$$h^*_\Delta(z) = \sum_{\Omega \subseteq \Delta} B_\Omega(z)\,,$$

where now the sum is over all faces of Δ, including $\varnothing$ (recall that $B_\varnothing(z) = 1$). Substituting this back into (10.4) yields

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{P}}(z) &= 1 + \frac{\sum_{\Delta \in T \setminus \{\varnothing\}} z^{\dim(\Delta)+1}(1-z)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_\Omega(\frac{1}{z})}{(1-z)^{d+1}} \\ &= \frac{\sum_{\Delta \in T} z^{\dim(\Delta)+1}(1-z)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_\Omega(\frac{1}{z})}{(1-z)^{d+1}}\,. \end{aligned} \tag{10.5}$$

Recall that the polynomial $B_\Delta(z)$ encodes data about the lattice points in the interior of the fundamental parallelepiped of $\mathrm{cone}(\Delta)$. The symmetry of this parallelepiped gets translated into the palindromy of the associated polynomial (Exercise 10.7):

$$B_\Delta(z) = z^{\dim(\Delta)+1} B_\Delta\left(\tfrac{1}{z}\right). \tag{10.6}$$

This allows us to rewrite (10.5) further:

$$\begin{aligned} h^*_{\mathcal{P}}(z) &= \sum_{\Delta \in T} z^{\dim(\Delta)+1}(1-z)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} B_\Omega\left(\frac{1}{z}\right) \\ &= \sum_{\Delta \in T} z^{\dim(\Delta)+1}(1-z)^{d-\dim(\Delta)} \sum_{\Omega \subseteq \Delta} z^{-\dim(\Omega)-1} B_\Omega(z) \\ &= \sum_{\Omega \in T} \sum_{\Delta \supseteq \Omega} z^{\dim(\Delta)-\dim(\Omega)}(1-z)^{d-\dim(\Delta)} B_\Omega(z) \\ &= \sum_{\Omega \in T} (1-z)^{d-\dim(\Omega)} B_\Omega(z) \sum_{\Delta \supseteq \Omega} \left(\frac{z}{1-z}\right)^{\dim(\Delta)-\dim(\Omega)}. \end{aligned}$$

Theorem 10.4 follows now with (10.3). $\square$

10.3 Palindromic Decompositions of h^*-Polynomials

Our next goal is to refine Theorem 10.4 in the case that T comes from a *boundary* triangulation of $\mathcal{P}$, for which we can exploit Theorem 10.2—rather, its analogue for links, which is the subject of Exercise 10.12. To keep our exposition accessible, we will only discuss the case that $\mathcal{P}$ contains an interior lattice point; the identities in this and the next section have analogues without this condition, which we will address in the Notes at the end of the chapter.

Theorem 10.5. *Suppose $\mathcal{P}$ is an integral d-polytope that contains an interior lattice point. Then there exist unique polynomials $a(z)$ and $b(z)$ such that*

$$h^*_{\mathcal{P}}(z) = a(z) + z\, b(z)\,,$$

$a(z) = z^d\, a(\frac{1}{z})$, *and* $b(z) = z^{d-1}\, b(\frac{1}{z})$.

The identities for $a(z)$ and $b(z)$ say that $a(z)$ and $b(z)$ are palindromic polynomials; the degree of $a(z)$ is necessarily d (because the constant coefficient of $a(z)$ is $h^*_{\mathcal{P}}(0) = 1$), while the degree of $b(z)$ is $d-1$ or smaller. In fact, $b(z)$ can be the zero polynomial—this happens if and only if $\mathcal{P}$ is the translate of a reflexive polytope, by Theorem 4.6.

Proof. After a harmless lattice translation, we may assume that $\mathbf{0} \in \mathcal{P}^\circ$. Fix a regular boundary triangulation T of $\mathcal{P}$ and let

$$T_0 := T \cup \{\operatorname{conv}(\Delta, \mathbf{0}) :\ \Delta \in T\}\,.$$

Thus T_0 is a triangulation of $\mathcal{P}$ whose simplices come in two flavors, those on the boundary of $\mathcal{P}$ and those of the form $\operatorname{conv}(\Delta, \mathbf{0})$ for some $\Delta \in T$. (We say that T_0 comes from **coning over** the boundary triangulation T.) Figure 10.4 shows an example. Maybe not surprisingly, the two kinds of simplices in

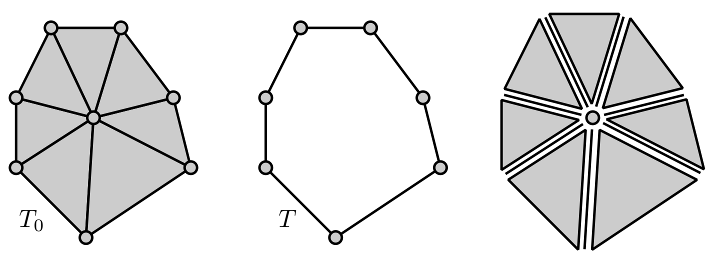

Fig. 10.4 A triangulation T_0 and its two types of simplices; the middle picture shows the boundary triangulation T.

T_0 are related: Exercise 10.11 says that for any nonempty simplex $\Delta \in T$,

$$h_{\operatorname{link}_{T_0}(\Delta)}(z) = h_{\operatorname{link}_{T_0}(\operatorname{conv}(\Delta,\mathbf{0}))}(z) = h_{\operatorname{link}_T(\Delta)}(z)\,. \tag{10.7}$$

Thus Theorem 10.4 gives in this case

$$\begin{aligned} h^*_{\mathcal{P}}(z) &= \sum_{\Delta \in T_0} h_{\operatorname{link}_{T_0}(\Delta)}(z)\, B_\Delta(z) \\ &= \sum_{\Delta \in T} h_{\operatorname{link}_T(\Delta)}(z) \left(B_\Delta(z) + B_{\operatorname{conv}(\Delta,\mathbf{0})}(z)\right). \end{aligned}$$

Now let

$$a(z) := \sum_{\Delta \in T} h_{\mathrm{link}_T(\Delta)}(z)\, B_\Delta(z) \tag{10.8}$$

$$b(z) := \frac{1}{z} \sum_{\Delta \in T} h_{\mathrm{link}_T(\Delta)}(z)\, B_{\mathrm{conv}(\Delta,\mathbf{0})}(z)\,. \tag{10.9}$$

The fact that each $B_{\mathrm{conv}(\Delta,\mathbf{0})}(z)$ has constant term zero ensures that $b(z)$ is a polynomial (of degree at most $d-1$). Furthermore, by Exercise 10.12 and (10.6),

$$\begin{aligned} z^d\, a\left(\frac{1}{z}\right) &= \sum_{\Delta \in T} z^{d-\dim(\Delta)-1} h_{\mathrm{link}_T(\Delta)}\left(\frac{1}{z}\right) z^{\dim(\Delta)+1} B_\Delta\left(\frac{1}{z}\right) \\ &= \sum_{\Delta \in T} h_{\mathrm{link}_T(\Delta)}(z)\, B_\Delta(z) = a(z)\,, \end{aligned}$$

and $b(z) = z^{d-1}\, b(\frac{1}{z})$ follows analogously. Thus our setup gives rise to the decomposition

$$h^*_{\mathcal{P}}(z) = a(z) + z\, b(z)$$

with palindromic polynomials $a(z)$ and $b(z)$, and Exercise 10.13 says that this decomposition is unique. (Note that this means, in particular, that *any* boundary triangulation yields the same decomposition.) $\square$

10.4 Inequalities for h^*-Polynomials

The proof of Theorem 10.5 has a powerful consequence due to the following fact, whose proof would lead us too far astray (though we outline a self-contained proof in Exercises 10.16–10.18).

Theorem 10.6. *The h-polynomial of a link in a regular triangulation has nonnegative coefficients.*

Looking back how the polynomials $a(z)$ and $b(z)$ were constructed (see (10.8) and (10.9)), we can thus deduce:

Corollary 10.7. *The polynomials $a(z)$ and $b(z)$ appearing in Theorem 10.5 have nonnegative coefficients.*

Naturally, the fact that the coefficients of $a(z)$ and $b(z)$ have nonnegative coefficients can be translated into inequalities among the coefficients of the accompanying h^*-polynomials: the following corollary of Corollary 10.7 is easily proved (Exercise 10.14).

Corollary 10.8. *Suppose $\mathcal{P}$ is an integral d-polytope that contains an interior lattice point. Then its h^*-polynomial $h^*_{\mathcal{P}}(x) = h^*_d x^d + h^*_{d-1} x^{d-1} + \cdots + h^*_0$ satisfies*

$$h_0^*+h_1^*+\cdots+h_{j+1}^* \geq h_d^*+h_{d-1}^*+\cdots+h_{d-j} \qquad \textit{for} \qquad 0 \leq j \leq \lfloor \tfrac{d}{2} \rfloor -1 \quad (10.10)$$

and

$$h_0^* + h_1^* + \cdots + h_j^* \leq h_d^* + h_{d-1}^* + \cdots + h_{d-j}^* \qquad \textit{for} \qquad 0 \leq j \leq d. \qquad (10.11)$$

Notes

1. Theorem 10.3 is only one indication how important (and special) unimodular triangulations are; see [97, Chapter 9] for more. If an integral polytope $\mathcal{P}$ admits a unimodular triangulation, then it is **integrally closed**: for every positive integer k and every integer point $\mathbf{x} \in k\mathcal{P}$ there exist integer points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k \in \mathcal{P}$ such that $\mathbf{y}_1+\mathbf{y}_2+\cdots+\mathbf{y}_k = \mathbf{x}$.[1] (See Exercise 10.6.) There are integrally closed polytopes that do not admit a unimodular triangulation. These two notions are part of an interesting hierarchy of integral polytopes, with connections to commutative algebra and algebraic geometry; see, e.g., [73].

2. Theorem 10.4 is due to Ulrich Betke and Peter McMullen, as is Theorem 10.5, published in an influential paper [54] in 1985. Corollary 10.7 *should* have maybe been in [54] (the nonnegativity of the h-polynomials in Betke–McMullen's formulas was established in the 1970's), but it only appeared—in the guise of Corollary 10.8—in the 1990's, in papers by Takayuki Hibi [128] and Richard Stanley [221]. Neither paper gives the impression that the inequalities of Corollary 10.8 follow immediately from Betke–McMullen's work, though both hold without the condition of the existence of an interior lattice point (here d has to be replaced by the degree of $h_{\mathcal{P}}^*(z)$ in (10.11)) and in situations more general than the realm of Ehrhart series.

3. Theorem 10.5 and Corollary 10.7 are not the end of the story. Building on work of Sam Payne [182] (which gave a multivariate version of Theorem 10.4), Alan Stapledon [228] generalized Theorem 10.5 and Corollary 10.7 to arbitrary integral polytopes. His theorem says that, if $h_{\mathcal{P}}^*(z)$ has degree s (recall from Chapter 4 that in this case we say $\mathcal{P}$ has degree s), then there exist unique polynomials $a(z)$ and $b(z)$ with nonnegative coefficients such that

$$\left(1 + z + \cdots + z^{d-s}\right) h_{\mathcal{P}}^*(z) = a(z) + z^{d+1-s}\, b(z)\,, \qquad (10.12)$$

$a(z) = z^d\, a(\frac{1}{z})$, $b(z) = z^{d-l}\, b(\frac{1}{z})$, and, writing $a(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$,

$$1 = a_0 \leq a_1 \leq a_j \qquad \text{for} \qquad 2 \leq j \leq d-1. \qquad (10.13)$$

[1] There is a related notion for integral polytopes, namely that of *normality*. For full-dimensional polytopes, the adjectives *integrally closed* and *normality* are equivalent, but this is not the case when the subgroup $\sum_{\mathbf{x},\,\mathbf{y} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbb{Z}(\mathbf{x}-\mathbf{y})$ of $\mathbb{Z}^d$ is not a direct summand of $\mathbb{Z}^d$. For more about this subtlety (and much more), see [73, 87].

These inequalities imply Corollary 10.8 without the condition that $\mathcal{P}$ contains an interior lattice point (again here d has to be replaced by s in (10.11)); see Exercise 10.15. Stapledon has recently improved this theorem further, giving infinitely many classes of linear inequalities among the h^*-coefficients [226]. This exciting new line of research involves additional techniques from additive number theory. He also introduced a weighted variant of the h^*-polynomial which is *always* palindromic, motivated by motivic integration and the cohomology of certain toric varieties [227]. One can easily recover $h^*_{\mathcal{P}}(x)$ from this weighted h^*-polynomial, but one can also deduce the palindromy of both $a(x)$ and $b(x)$ as coming from the same source (and this perspective has some serious geometric applications).

Exercises

10.1. ♣ Show that the f-vector and the h-vector of a simplicial d-polytope are related via

$$h_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{d-k} f_{j-1} \qquad \text{and} \qquad f_{k-1} = \sum_{j=0}^{k} \binom{d-j}{k-j} h_j \,,$$

and conclude that the Dehn–Sommerville relations in Exercise 5.9 are equivalent to Theorem 10.1.

10.2. ♣ As in the setup of our proof of Theorem 10.2, let $\mathcal{Q} \subseteq \mathbb{R}^{d+1}$ be the lifted polytope giving rise to a regular triangulation T of the polytope $\mathcal{P} \subseteq \mathbb{R}^d$, choose $\mathbf{v} \in \mathcal{P}^\circ$ and lift it to $(\mathbf{v}, h+2) \in \mathbb{R}^{d+1}$, where h is the maximal height among the vertices of $\mathcal{Q}$. Let $\mathcal{R}$ be the convex hull of $(\mathbf{v}, h+2)$ and the vertices of the lower hull of $\mathcal{Q}$, and let

$$\mathcal{S} := \mathcal{R} \cap \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \, x_{d+1} = h+1 \right\} .$$

Prove that $\mathcal{S}$ is a simplicial polytope whose face numbers f_k equal the face numbers of the triangulation of $\partial\mathcal{P}$ induced by T.

10.3. Given a triangulation T of the boundary of a d-polytope $\mathcal{P}$ and a point $\mathbf{v} \in \mathcal{P}^\circ$, construct a triangulation K of $\mathcal{P}$ consisting of T appended by the simplices $\operatorname{conv}(\Delta, \mathbf{v})$ for all $\Delta \in T$; i.e., the new triangulation K comes from coning over T. Prove that

$$h_K(z) = h_T(z) \,.$$

10.4. ♣ Show that if Δ is a unimodular k-simplex then

$$\operatorname{Ehr}_\Delta(z) = \frac{1}{(1-z)^{k+1}} \,.$$

10.5. Show that any integral polygon admits a unimodular triangulation. Give an example of an integral d-polytope that does not admit a unimodular triangulation, for any $d \geq 3$.

10.6. Recall from the Notes that an integral polytope $\mathcal{P}$ is **integrally closed** if for every positive integer k and every integer point $\mathbf{x} \in k\mathcal{P}$ there exist integer points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k \in \mathcal{P}$ such that $\mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_k = \mathbf{x}$. Prove that any integral polytope that admits a unimodular triangulation is integrally closed.

10.7. ♣ Prove (10.6): $B_\Delta(z) = z^{\dim(\Delta)+1} B_\Delta\left(\frac{1}{z}\right)$.

10.8. ♣ Let T be a triangulation of the d-polytope $\mathcal{P}$. Given $\Delta \in T$, show that the largest dimension occurring among the simplices in $\operatorname{link}(\Delta)$ is $d - \dim(\Delta) - 1$.

10.9. ♣ Prove (10.3): given a triangulation T of a d-dimensional polytope, then for any simplex $\Delta \in T$,

$$h_{\operatorname{link}(\Delta)}(z) = (1-z)^{d-\dim(\Delta)} \sum_{\Phi \supseteq \Delta} \left(\frac{z}{1-z}\right)^{\dim(\Phi)-\dim(\Delta)},$$

where the sum is over all simplices $\Phi \in T$ that contain Δ.

10.10. ♣ For an integral simplex Δ, let $\Pi(\Delta)$ denote the fundamental parallelepiped of $\operatorname{cone}(\Delta)$. Show that

$$\sigma_{\Pi(\Delta)}(\mathbf{z}) = 1 + \sum_{\Omega \subseteq \Delta} \sigma_{\Pi(\Omega)^\circ}(\mathbf{z}),$$

where the sum is over all nonempty faces of Δ.

10.11. ♣ Prove (10.7): Fix a regular triangulation T_0 of a polytope $\mathcal{P}$ with $\mathbf{0} \in \mathcal{P}^\circ$, whose vertices are $\mathbf{0}$ and the vertices of $\mathcal{P}$, and let $T := \{\Delta \in T_0 : \Delta \subseteq \partial\mathcal{P}\}$. Then for any nonempty $\Delta \in T$,

$$h_{\operatorname{link}(\Delta)}(z) = h_{\operatorname{link}(\operatorname{conv}(\Delta,\mathbf{0}))}(z).$$

10.12. ♣ Let Δ be a simplex in a regular boundary triangulation of a d-dimensional polytope. Prove that $h_{\operatorname{link}(\Delta)}(z)$ is palindromic. (*Hint:* establish a one-to-one correspondence between $\operatorname{link}(\Delta)$ and the boundary faces of a polytope of dimension $d - \dim(\Delta) - 1$, respecting the face relations.)

10.13. ♣ Let $p(z)$ be a polynomial of degree d. Show that there are unique polynomials $a(z)$ and $b(z)$ such that

$$p(z) = a(z) + z\, b(z),$$

$a(z) = z^d\, a(\frac{1}{z})$, and $b(z) = z^{d-1}\, b(\frac{1}{z})$.

10.14. ♣ Consider the polynomial decomposition $h^*_{\mathcal{P}}(z) = a(z) + z\,b(z)$ of an integral d-polytope that contains an interior lattice point, given in Theorem 10.5. Prove that the fact that $a(z)$ has nonnegative coefficients implies (10.10):

$$h_0^* + h_1^* + \cdots + h_{j+1}^* \geq h_d^* + h_{d-1}^* + \cdots + h_{d-j} \quad \text{for} \quad 0 \leq j \leq \lfloor \tfrac{d}{2} \rfloor - 1,$$

and the fact that $b(z)$ has nonnegative coefficients implies (10.11):

$$h_0^* + h_1^* + \cdots + h_j^* \leq h_d^* + h_{d-1}^* + \cdots + h_{d-j}^* \quad \text{for} \quad 0 \leq j \leq d.$$

10.15. Derive the analogue of Corollary 10.8 without the condition that $\mathcal{P}$ contains an interior lattice point, by utilizing (10.12) and (10.13).

10.16. A **shelling** of the d-polyhedron $\mathcal{P}$ is a linear ordering of the facets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ of $\mathcal{P}$ such that the following recursive conditions hold:

(1) $\mathcal{F}_1$ has a shelling.
(2) For any $j > 1$, $(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{j-1}) \cap \mathcal{F}_j$ is of dimension $d-2$ and

$$(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{j-1}) \cap \mathcal{F}_j = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_k\,,$$

where $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_k \cup \cdots \cup \mathcal{G}_n$ is a shelling of $\mathcal{F}_j$.[2]

Given a d-polytope $\mathcal{P} \subset \mathbb{R}^d$, choose a generic vector $\mathbf{v} \in \mathbb{R}^d$ (e.g., you can pick a vector at random). If we think of $\mathcal{P}$ as a planet and we place a hot air balloon on one of the facets of $\mathcal{P}$, we will slowly see more and more facets, one at a time, as the balloon rises in the direction of $\mathbf{v}$. Here "seeing" means the following: we say that a facet $\mathcal{F}$ is **visible** from the point $\mathbf{x} \notin \mathcal{P}$ if the line segment between $\mathbf{x}$ and any point $\mathbf{y} \in \mathcal{F}$ intersects $\mathcal{P}$ only in $\mathbf{y}$.

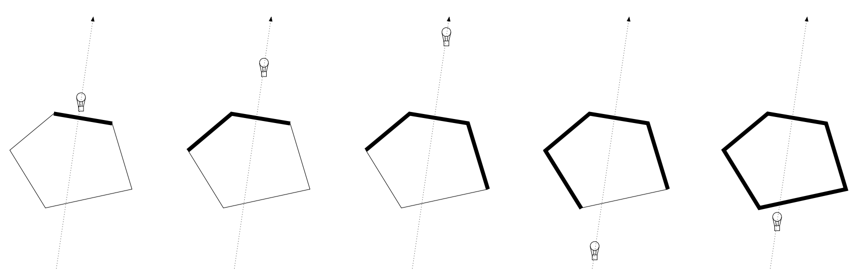

Fig. 10.5 Constructing a line shelling of a pentagon.

Our balloon ride gives rise (no pun intended) to an ordering $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ of the facets of $\mathcal{P}$. Namely, $\mathcal{F}_1$ is the facet we're starting our journey on, $\mathcal{F}_2$ is the next visible facet, etc., until we're high enough so that no more visible

[2] Note that this condition implies that $(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{j-1}) \cap \mathcal{F}_j$ is connected for $d \geq 3$.

facets can be added. At this point we "pass through infinity" and let the balloon approach the polytope planet from the opposite side. The next facet in our list is the first one that will disappear as we're moving towards the polytope, then the next facet that will disappear, etc., until we're landing back on the polytope. (That is, the second half of our list of facets is the reversed list of what we would get had we started our ride at $\mathcal{F}_m$.)

Prove that this ordering $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ of the facets of $\mathcal{P}$ is a shelling (called a **line shelling**).

10.17. Suppose $\mathcal{P}$ is a simplicial d-polytope with facets $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$, ordered by some line shelling. We collect the vertices of $\mathcal{F}_j$ in the set F_j. Let $\widetilde{F}_j$ be the set of all vertices $\mathbf{v} \in F_j$ such that $F_j \setminus \{\mathbf{v}\}$ is contained in one of $F_1, F_2, \dots, F_{j-1}$ (which define the facets coming earlier in the shelling order).

(a) Prove that the new faces appearing in the j^{th} step of the construction of the line shelling, i.e., the faces in $\mathcal{F}_j \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_{j-1})$, are precisely those sets $\operatorname{conv}(G)$ for some $\widetilde{F}_j \subseteq G \subseteq F_j$.

(b) Show that

$$f_{k-1} = \sum_{m=0}^{k} \binom{d-m}{k-m} \widetilde{h}_m \, ,$$

where $\widetilde{h}_m$ denotes the number of sets $\widetilde{F}_j$ of cardinality m.

(c) Conclude that $\widetilde{h}_m = h_m$ and thus that $h_m \geq 0$.

10.18. Given a regular triangulation of the polytope $\mathcal{P}$, prove that the h-polynomial of the induced triangulation of $\partial\mathcal{P}$ has nonnegative coefficients, as does the h-polynomial of the link of any simplex in this triangulation.

10.19. Show that the construction in Exercise 10.17 gives the following characterization of the h-polynomial of a *simple* polytope $\mathcal{P}$: fix a generic direction vector $\mathbf{v}$ and orient the graph of $\mathcal{P}$ so that each oriented edge (viewed as a vector) forms an acute angle with $\mathbf{v}$. Then h_k equals the number of vertices of this oriented graph with in-degree k.

Open Problems

10.20. Which integral polytopes admit a unimodular triangulation? (The property of admitting unimodular triangulations is part of an interesting hierarchy of integral polytopes; see [73].)

10.21. Find a complete set of inequalities for the coefficients of any h^*-polynomials of degree ≤ 3, i.e., piecewise linear regions in $\mathbb{R}^3$ all of whose integer points (h_1^*, h_2^*, h_3^*) come from an h^*-polynomial of a 3-dimensional integral polytope. (See also Open Problem 3.42.)

10.22. A linear inequality $a_0x_0 + a_1x_1 + \dots a_dx_d \geq 0$ is **balanced** if $a_0 + a_1 + \cdots + a_d = 0$. Find a complete set of balanced inequalities for the coefficients of any h^*-polynomial of degree ≤ 6. (For degree ≤ 5, see [226].)

10.23. Prove that every integrally closed reflexive polytope has a unimodal h^*-polynomial. More generally, prove that every integrally closed polytope has a unimodal h^*-polynomial. Even more generally, classify integral polytopes with a unimodal h^*-polynomial. (See [12, 63, 74, 204] for possible starting points.)