

REFINED UPPER BOUNDS FOR THE LINEAR DIOPHANTINE PROBLEM OF FROBENIUS

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ABSTRACT. We study the *Frobenius problem*: given relatively prime positive integers a_1, \dots, a_d , find the largest value of t (the *Frobenius number* $g(a_1, \dots, a_d)$) such that $\sum_{k=1}^d m_k a_k = t$ has no solution in nonnegative integers m_1, \dots, m_d . We introduce a method to compute upper bounds for $g(a_1, a_2, a_3)$, which seem to grow considerably slower than previously known bounds. Our computations are based on a formula for the restricted partition function, which involves Dedekind-Rademacher sums, and the reciprocity law for these sums.

1. INTRODUCTION

Given positive integers $a_1 < a_2 < \dots < a_d$ with $\gcd(a_1, \dots, a_d) = 1$, the *linear Diophantine problem of Frobenius* asks for the largest integer t for which we cannot find nonnegative integers m_1, \dots, m_d such that

$$t = m_1 a_1 + \dots + m_d a_d .$$

We call this largest integer the *Frobenius number* $g(a_1, \dots, a_d)$; its study was initiated in the 19th century. One fact which makes this problem attractive is that it can be easily described, for example, in terms of coins of denominations a_1, \dots, a_d ; the Frobenius number is the largest amount of money which cannot be formed using these coins. For $d = 2$, it is well known (most probably at least since Sylvester [12]) that

$$(1) \quad g(a_1, a_2) = a_1 a_2 - a_1 - a_2 .$$

For $d > 2$, all attempts to find explicit formulas have proved elusive. Two excellent survey papers on the Frobenius problem are [1] and [11].

Our goal is to establish upper bounds for $g(a_1, \dots, a_d)$. The literature on such bounds is vast; it includes results by Erdős and Graham [8]

$$(2) \quad g(a_1, \dots, a_d) \leq 2a_d \left\lfloor \frac{a_1}{d} \right\rfloor - a_1 ,$$

Selmer [11]

$$(3) \quad g(a_1, \dots, a_d) \leq 2a_{d-1} \left\lfloor \frac{a_d}{d} \right\rfloor - a_d ,$$

and Vitek [13]

$$(4) \quad g(a_1, \dots, a_d) \leq \left\lfloor \frac{1}{2}(a_2 - 1)(a_d - 2) \right\rfloor - 1 .$$

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Here $a_1 < a_2 < \cdots < a_d$, and $\lfloor x \rfloor$ denotes the greatest integer not exceeding x . Davison [6] established the lower bound

$$(5) \quad g(a_1, a_2, a_3) \geq \sqrt{3a_1a_2a_3} - a_1 - a_2 - a_3 .$$

Experimental data [4] shows that Davison's bound is sharp in the sense that it is very often very close to $g(a_1, a_2, a_3)$. On the other hand, the upper bounds given by (2), (3), and (4) seem to be quite large compared to the actual Frobenius numbers. In this paper, we derive a method of achieving sharper upper bounds for the Frobenius number. Our results are based on a formula for the restricted partition function (Section 2), which involves Dedekind-Rademacher sums, and the reciprocity law for these sums (Section 3). The main result is derived in Section 4; computations which illustrate our new bounds can be found in Section 5.

We focus on the first non-trivial case $d = 3$; any bound for this case yields a general bound, as one can easily see that $g(a_1, \dots, a_d) \leq g(a_1, a_2, a_3)$ if a_1, a_2 , and a_3 are relatively prime. If not then we can reduce by one variable at a time: Again by the definition of the Frobenius number, $g(a_1, \dots, a_d) \leq g(a_1, \dots, a_{d-1})$ if a_1, \dots, a_{d-1} are relatively prime. If not, we can use a formula of Brauer and Shockely [5]: If $n = \gcd(a_1, \dots, a_{d-1})$ then

$$(6) \quad g(a_1, \dots, a_d) = n \, g\left(\frac{a_1}{n}, \dots, \frac{a_{d-1}}{n}, a_d\right) + (n-1)a_d .$$

Hence

$$g(a_1, \dots, a_d) \leq n \, g\left(\frac{a_1}{n}, \dots, \frac{a_{d-1}}{n}\right) + (n-1)a_d .$$

2. THE RESTRICTED PARTITION FUNCTION

We approach the Frobenius problem through the study of the *restricted partition function*

$$p_{\{a_1, \dots, a_d\}}(n) = \# \left\{ (m_1, \dots, m_d) \in \mathbb{Z}_{\geq 0}^d : m_1a_1 + \cdots + m_da_d = n \right\} ,$$

the number of partitions of n using only a_1, \dots, a_d as parts. In view of this function, the Frobenius number $g(a_1, \dots, a_d)$ is the largest integer n such that $p_{\{a_1, \dots, a_d\}}(n) = 0$.

In the $d = 3$ case, we can additionally assume that $a = a_1$, $b = a_2$, and $c = a_3$ are *pairwise* relatively prime, a simplification due to Johnson's formula [9]: if $n = \gcd(a, b)$ then

$$g(a, b, c) = n \, g\left(\frac{a}{n}, \frac{b}{n}, c\right) + (n-1)c .$$

(This identity is a special case of (6).)

In the case that a, b, c are pairwise relatively prime, Beck, Diaz, and Robins derived the following result for the partition function $p_{\{a, b, c\}}$ [3, Theorem 3]:

$$(7) \quad \begin{aligned} p_{\{a, b, c\}}(n) = & \frac{n^2}{2abc} + \frac{n}{2} \left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) + \frac{1}{12} \left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \\ & + S_{-n}(b, c; a) + S_{-n}(c, a; b) + S_{-n}(a, b; c) . \end{aligned}$$

Here [3, Equation (14)]

$$(8) \quad S_t(a, b; c) = \sum_{m=0}^{c-1} \left(\left(\frac{-a^{-1}(bm+t)}{c} \right) \right) \left(\left(\frac{m}{c} \right) \right),$$

where $aa^{-1} \equiv 1 \pmod{c}$ and $((x)) = x - \lfloor x \rfloor - 1/2$, is a special case of a *Dedekind-Rademacher sum*; we will discuss these sums in the next section.

To bound the Frobenius number (from above), we need to bound $p_{\{a,b,c\}}$ (from below), whose only nontrivial ingredients are the Dedekind-Rademacher sums. A classical bound for the Dedekind-Rademacher sum yielded in [3] the inequality

$$g(a, b, c) \leq \frac{1}{2} \left(\sqrt{abc(a+b+c)} - a - b - c \right),$$

which is of comparable size to the other upper bounds given by (2), (3), and (4). However, we will show that one can obtain bounds of smaller magnitude.

3. DEDEKIND-RADEMACHER SUMS

The *Dedekind-Rademacher sum* [10] is defined for $a, b \in \mathbb{Z}$, $x, y \in \mathbb{R}$ as

$$R(a, b; x, y) = \sum_{k=0}^{b-1} \left(\left(\frac{a(k+y)}{b} + x \right) \right)^* \left(\left(\frac{k+y}{b} \right) \right)^*,$$

where

$$((x))^* = \begin{cases} ((x)) & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Rademacher's sum generalizes the classical Dedekind sum $R(a, b; 0, 0)$ [7]. An easy bound for the Dedekind-Rademacher sum $R(a, b; x, 0)$ can be obtained through the Cauchy-Schwartz inequality: if a and b are relatively prime then

$$(9) \quad \begin{aligned} |R(a, b; x, 0)| &= \left| \sum_{k=0}^{b-1} \left(\left(\frac{ak}{b} + x \right) \right)^* \left(\left(\frac{k}{b} \right) \right)^* \right| \\ &\leq \sqrt{\left(\sum_{k=0}^{b-1} \left(\left(\frac{ak}{b} + x \right) \right)^{*2} \right) \left(\sum_{k=0}^{b-1} \left(\left(\frac{k}{b} \right) \right)^{*2} \right)} \\ &= \sqrt{\left(\sum_{k=0}^{b-1} \left(\left(\frac{k}{b} + x \right) \right)^{*2} \right) \left(\sum_{k=1}^{b-1} \left(\frac{k}{b} - \frac{1}{2} \right)^2 \right)} \\ &\leq \sqrt{\left(\sum_{k=0}^{b-1} \left(\frac{k}{b} + \frac{1}{b} - \frac{1}{2} \right)^2 \right) \left(\frac{b}{12} - \frac{1}{4} + \frac{1}{6b} \right)} \\ &= \sqrt{\left(\frac{b}{12} + \frac{1}{6b} \right) \left(\frac{b}{12} - \frac{1}{4} + \frac{1}{6b} \right)} \end{aligned}$$

(In the third and fourth step we use the periodicity of $((x))^*$.) An important property of $R(a, b; x, y)$ is Rademacher's *reciprocity law* [10]: if a and b are relatively prime then

$$(10) \quad R(a, b; x, y) + R(b, a; y, x) = Q(a, b; x, y) .$$

Here

$$Q(a, b; x, y) = \begin{cases} -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right) & \text{if both } x, y \in \mathbb{Z}, \\ ((x))^* ((y))^* + \frac{1}{2} \left(\frac{a}{b} \psi_2(y) + \frac{1}{ab} \psi_2(ay + bx) + \frac{b}{a} \psi_2(x) \right) & \text{otherwise,} \end{cases}$$

where

$$\psi_2(x) = (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + 1/6$$

denotes the periodic second Bernoulli function. Among other things, this reciprocity law allows us to compute $R(a, b; x, y)$ in polynomial time, by means of a Euclidean-type algorithm using the first two variables: simply note that we can replace a in $R(a, b; x, y)$ by the least residue of a modulo b .

To express S in terms of R , we rewrite (8) as

$$S_t(a, b; c) = \sum_{m=0}^{c-1} \left(\left(\frac{-a^{-1}(bm+t)}{c} \right) \right)^* \left(\left(\frac{m}{c} \right) \right)^* + \begin{cases} \frac{1}{4} & \text{if } c|t, \\ -\frac{1}{2} \left(\left(-\frac{a^{-1}t}{c} \right) \right)^* - \frac{1}{2} \left(\left(-\frac{b^{-1}t}{c} \right) \right)^* & \text{otherwise.} \end{cases}$$

Accordingly,

$$S_t(a, b; c) = R \left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0 \right) + \begin{cases} \frac{1}{4} & \text{if } c|t, \\ -\frac{1}{2} \left(\left(-\frac{a^{-1}t}{c} \right) \right)^* - \frac{1}{2} \left(\left(-\frac{b^{-1}t}{c} \right) \right)^* & \text{otherwise.} \end{cases}$$

To ease our computations, we bound this as

$$(11) \quad S_t(a, b; c) \geq R \left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0 \right) - \frac{1}{2} .$$

4. UPPER BOUNDS FOR $g(a, b, c)$

To bound $S_t(a, b; c)$ from below (which yields an upper bound for $g(a, b, c)$), we use an interplay of (10) and (9) to obtain a bound for the Dedekind-Rademacher sum corresponding to S_t , according to (11). The idea is to reduce the arguments of the Dedekind-Rademacher sum after the application of (10), which means that the bound given by (9) will be more accurate. To illustrate this, let c_1 be the least nonnegative residue of $-a^{-1}b$ modulo c . Then

$$(12) \quad R \left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0 \right) = R \left(c_1, c; -\frac{a^{-1}t}{c}, 0 \right) = Q \left(c_1, c; -\frac{a^{-1}t}{c}, 0 \right) - R \left(c, c_1; 0, -\frac{a^{-1}t}{c} \right) .$$

If $c_1 = 1$ then the right-hand side can be simplified, as $R \left(c, 1; 0, -\frac{a^{-1}t}{c} \right) = 0$. If $c_1 \neq 1$ then the Dedekind-Rademacher sum on the right-hand side of (12) can be bounded (via (9)) sharper than the Dedekind-Rademacher sum on the left-hand side. In fact, by a repeated application of (10), we can achieve bounds which are even better. To keep the computations somewhat simple, we apply (10) once more and illustrate what this process yields in terms of lower bounds for S_t . Let c_2 be

the least nonnegative residue of c modulo c_1 . If $c_2 = 1$ then

$$\begin{aligned}
 R\left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0\right) &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - R\left(c, c_1; 0, -\frac{a^{-1}t}{c}\right) \\
 (13) \qquad \qquad \qquad &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - R\left(1, c_1; 0, -\frac{a^{-1}t}{c}\right) \\
 &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - Q\left(1, c_1; 0, -\frac{a^{-1}t}{c}\right),
 \end{aligned}$$

as $R\left(c_1, 1; -\frac{a^{-1}t}{c}, 0\right) = 0$. If $c_2 \neq 1$ then (12) can be refined as

$$\begin{aligned}
 R\left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0\right) &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - R\left(c_2, c_1; 0, -\frac{a^{-1}t}{c}\right) \\
 (14) \qquad \qquad \qquad &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - Q\left(c_2, c_1; 0, -\frac{a^{-1}t}{c}\right) + R\left(c_1, c_2; -\frac{a^{-1}t}{c}, 0\right).
 \end{aligned}$$

The Dedekind-Rademacher sum on the right-hand side can be bounded according to (9) as

$$(15) \qquad R\left(c_1, c_2; -\frac{a^{-1}t}{c}, 0\right) \geq -\sqrt{\left(\frac{c_2}{12} + \frac{1}{6c_2}\right)\left(\frac{c_2}{12} - \frac{1}{4} + \frac{1}{6c_2}\right)}.$$

We still need to bound Q . ψ_2 has a minimum of $-1/12$ (at $x = 1/2$) and a maximum of $1/6$ (at $x = 0$). These extreme values yield for

$$Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) = \begin{cases} -\frac{1}{4} + \frac{1}{12}\left(\frac{c_1}{c} + \frac{1}{c_1c} + \frac{c}{c_1}\right) & \text{if } c|t, \\ \frac{1}{2}\left(\frac{c_1}{6c} + \frac{1}{6c_1c} + \frac{c}{c_1}\psi_2\left(-\frac{a^{-1}t}{c}\right)\right) & \text{otherwise,} \end{cases}$$

the lower bound

$$(16) \qquad Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) \geq -\frac{1}{4} + \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} = Q_{\text{low}}(c_1, c),$$

as well as the upper bound

$$(17) \qquad Q\left(c_2, c_1; 0, -\frac{a^{-1}t}{c}\right) \leq \frac{c_2}{12c_1} + \frac{1}{12c_2c_1} + \frac{c_1}{12c_2} = Q_{\text{up}}(c_2, c_1).$$

These inequalities yield the following.

Proposition. *Suppose a and b are relatively prime to c . Let c_1 be the least nonnegative residue of $-a^{-1}b$ modulo c , and let c_2 be the least nonnegative residue of c modulo c_1 .*

- (i) *If $c_1 = 1$ then $S_t(a, b; c) \geq -\frac{c}{24} + \frac{1}{6c} - \frac{3}{4}$.*
- (ii) *If $c_1 \neq 1$ and $c_2 = 1$ then $S_t(a, b; c) \geq \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} - \frac{1}{6c_1} - \frac{c_1}{12} - \frac{3}{4}$.*
- (iii) *If $c_1 \neq 1$ and $c_2 \neq 1$ then*

$$S_t(a, b; c) \geq \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} - \frac{c_2}{12c_1} - \frac{1}{12c_1c_2} - \frac{c_1}{12c_2} - \frac{3}{4} - \sqrt{\left(\frac{c_2}{12} + \frac{1}{6c_2}\right)\left(\frac{c_2}{12} - \frac{1}{4} + \frac{1}{6c_2}\right)}.$$

Proof. (i) Use (12) with $c_1 = 1$ in (11):

$$\begin{aligned} S_t(a, b; c) &\geq R\left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0\right) - \frac{1}{2} \\ &= Q\left(1, c; -\frac{a^{-1}t}{c}, 0\right) - \frac{1}{2} \\ &\geq -\frac{1}{4} + \frac{1}{6c} - \frac{c}{24} - \frac{1}{2}. \end{aligned}$$

Here the last inequality follows from (16).

(ii) Use (13) in (11) together with the bounds (16) and (17):

$$\begin{aligned} S_t(a, b; c) &\geq R\left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0\right) - \frac{1}{2} \\ &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - Q\left(1, c_1; 0, -\frac{a^{-1}t}{c}\right) - \frac{1}{2} \\ &\geq -\frac{1}{4} + \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} - \left(\frac{1}{6c_1} + \frac{c_1}{12}\right) - \frac{1}{2} \end{aligned}$$

(iii) Use (14) with the bounds given in (15), (16), and (17):

$$\begin{aligned} S_t(a, b; c) &\geq R\left(-a^{-1}b, c; -\frac{a^{-1}t}{c}, 0\right) - \frac{1}{2} \\ &= Q\left(c_1, c; -\frac{a^{-1}t}{c}, 0\right) - Q\left(c_2, c_1; 0, -\frac{a^{-1}t}{c}\right) + R\left(c_1, c_2; -\frac{a^{-1}t}{c}, 0\right) - \frac{1}{2} \\ &\geq -\frac{1}{4} + \frac{c_1}{12c} + \frac{1}{12c_1c} - \frac{c}{24c_1} - \left(\frac{c_2}{12c_1} + \frac{1}{12c_2c_1} + \frac{c_1}{12c_2}\right) \\ &\quad - \sqrt{\left(\frac{c_2}{12} + \frac{1}{6c_2}\right)\left(\frac{c_2}{12} - \frac{1}{4} + \frac{1}{6c_2}\right)} - \frac{1}{2}. \quad \square \end{aligned}$$

These lower bounds can be combined with (7) and the quadratic formula to give an upper bound on the Frobenius number.

Theorem. *Suppose a , b , and c are pairwise relatively prime. Denote the lower bounds for $S_t(b, c; a)$, $S_t(c, a; b)$, and $S_t(a, b; c)$ according to the previous proposition by α , β , and γ , respectively. Then*

$$g(a, b, c) \leq \sqrt{\frac{1}{4}(a+b+c)^2 - \frac{1}{6}(a^2+b^2+c^2) - 2abc(\alpha+\beta+\gamma)} - \frac{1}{2}(a+b+c).$$

One should note that $\alpha + \beta + \gamma$ is negative. We can see that the growth behavior of this upper bound is dominated by $-2abc(\alpha + \beta + \gamma)$ under the square root. This means that if we can make $-(\alpha + \beta + \gamma)$ somewhat smaller than $\min(a, b, c)$ then we get a bound which grows considerably less than the bounds given by (2), (3), and (4). In fact, we can see this difference in example computations already when we use the bounds α, β, γ as given by our proposition. What is more important, however, is the fact that we can easily obtain even better bounds by improving our proposition through additional applications of Rademacher's reciprocity law (10). We illustrate this with the following algorithm, whose result is a bound on $S_t(a, b; c)$, which can be used in the above theorem (instead of the bounds coming from the proposition).

Algorithm. *Input:* a, b, c (pairwise relatively prime) and N (number of iterations). *Output:* lower bound S for $S_t(a, b; c)$.

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c_1 := - a^{-1} b modulo c (least nonnegative residue)
S := 0
n := 1

REPEAT {

    c_2 := c modulo c_1 (least nonnegative residue)
    S_1 := S + Q_low (c_1, c)
    S_2 := S_1 - Q_up (c_2, c_1)

    IF c_1 = 1 THEN S := S_1
    Else S := S_2

    IF c_1 = 1 OR c_2 = 1 OR n = N THEN BREAK

    c := c_2
    c_1 := c_1 modulo c_2 (least nonnegative residue)
    n := n + 1

}

IF c_1 > 1 AND c_2 > 2 THEN S := S - sqrt((c_2/12 + 1/(6 c_2) - 1/4) (c_2/12 + 1/(6 c_2)))

S := S - 1/2

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The algorithm repeats the steps described in the proposition N times, at each step bounding Q coming from Rademacher reciprocity according to (16) and (17). It stops prematurely if one of the variables is 1, in which case the remaining Dedekind-Rademacher sum is zero.

5. COMPUTATIONS

In the present section we illustrate the newly proposed upper bound for $g(a, b, c)$ numerically. In order to compare the results also with the lower bound given by Davison (5) we present here the values

$$f(a, b, c) = g(a, b, c) + a + b + c .$$

For these Frobenius numbers, Davison's lower bound is

$$f(a, b, c) \geq \sqrt{3} z ,$$

where $z = \sqrt{abc}$. In [4] we presented together with David Einstein an algorithm for the exact computation of $f(a, b, c)$. Einstein computed 20000 “admissible” (see [4]) values of $f(a, b, c)$ for relatively prime arguments chosen at random from the set $\{3, \dots, 750\}$. In [4] we arrived at the empirical conjecture that $f(a, b, c) \leq \sqrt{abc}^{5/4}$. The objectives of our current presentations are:

- (i) to compare our new upper bound with the known upper bound, which is, according to (2), (3), and (4),

$$\min \left(2c \left\lfloor \frac{a}{3} \right\rfloor - a, 2b \left\lfloor \frac{c}{3} \right\rfloor - c, \left\lfloor \frac{1}{2}(b-1)(c-2) \right\rfloor - 1 \right) + a + b + c ;$$

- (ii) to compare our new upper bound with the conjectured upper bound $z^{5/4}$;
 (iii) to compare the new upper bound to the true value of $f(a, b, c)$.

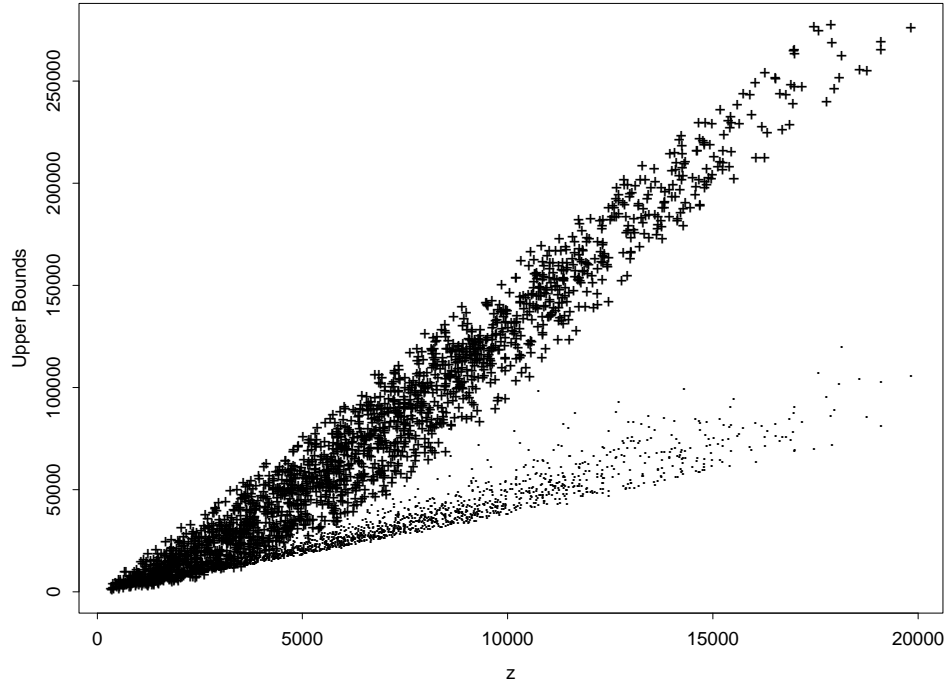


FIGURE 1. The new and old upper bounds for the Frobenius number

For these objectives we computed the new upper bound and the known upper bound for two thousand values of (a, b, c) , randomly chosen from Einstein's data. In all computations we used the minimum of the lower bounds given by the proposition and the algorithm for $N = 2$ in the theorem to obtain an upper bound for $f(a, b, c)$. In Figure 1 we plot the new upper bound (\cdot) and the known upper bound ($+$) as functions of z .

Among the 2000 points only less than 100 have known upper bounds smaller than the new upper bound. In 50% of the cases the ratio of the known upper bound to the new upper bound is greater than 2.44. In Figure 2 we plot the known upper bound as a function of the new upper bound. This figure complements the inferences from Figure 1.

In Figure 3 we plot the new upper bound as a function of z , and compare the points with Davison's lower bound (5) and with the conjectured upper bound $z^{5/4}$. We see that most values of the new

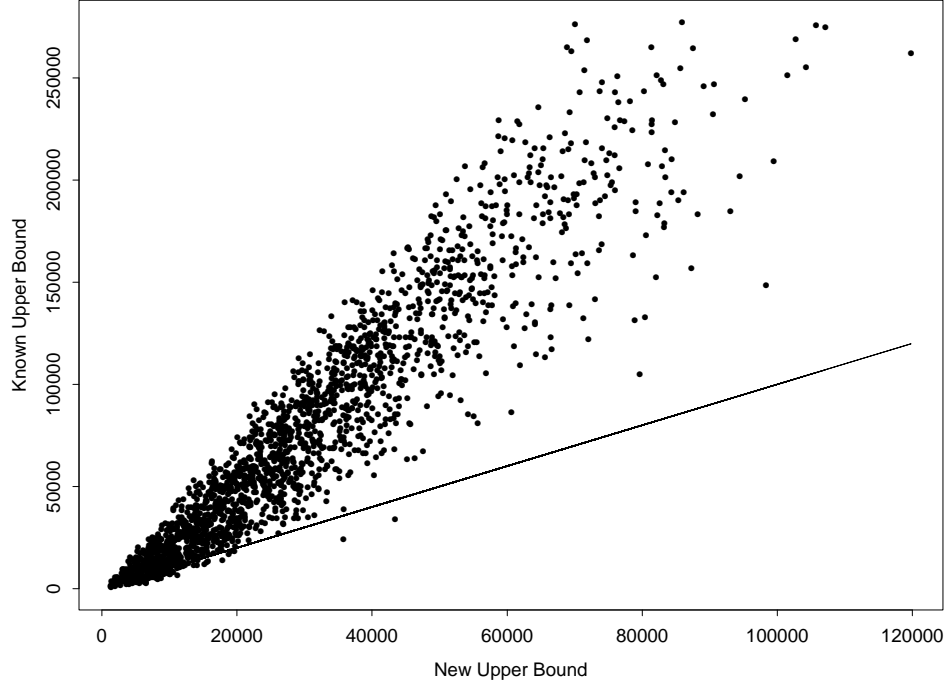


FIGURE 2. The new and old upper bounds compared

upper bound are smaller than $z^{5/4}$. This gives additional credence to the empirical conjecture in [4].

Finally, in Figure 4 we plot the points of the new upper bound versus the true $f(a, b, c)$ values. We see that even for large values of $f(a, b, c)$ there are cases where the new upper bound yields close values. In 50% of the cases the ratio of the new upper bound to the true $f(a, b, c)$ is greater than 2.

6. FINAL REMARKS

As stated in the last section, we used “only” two iterations ($N = 2$) in our algorithm to compute bounds for $S_t(a, b, c)$, which in turn lead to bounds on the Frobenius number $f(a, b, c)$. It is interesting to compare these values with the ones we get when just using the proposition, that is, one iteration ($N = 1$). Figure 5 illustrates this comparison.

It is reasonable to expect even better results when one uses more than two iterations in the algorithm. However, we found that—at least for our range of variables—in the vast majority of cases the algorithm terminates prematurely after one or two iterations; accordingly, there is not much gain from increasing the number N of iterations.

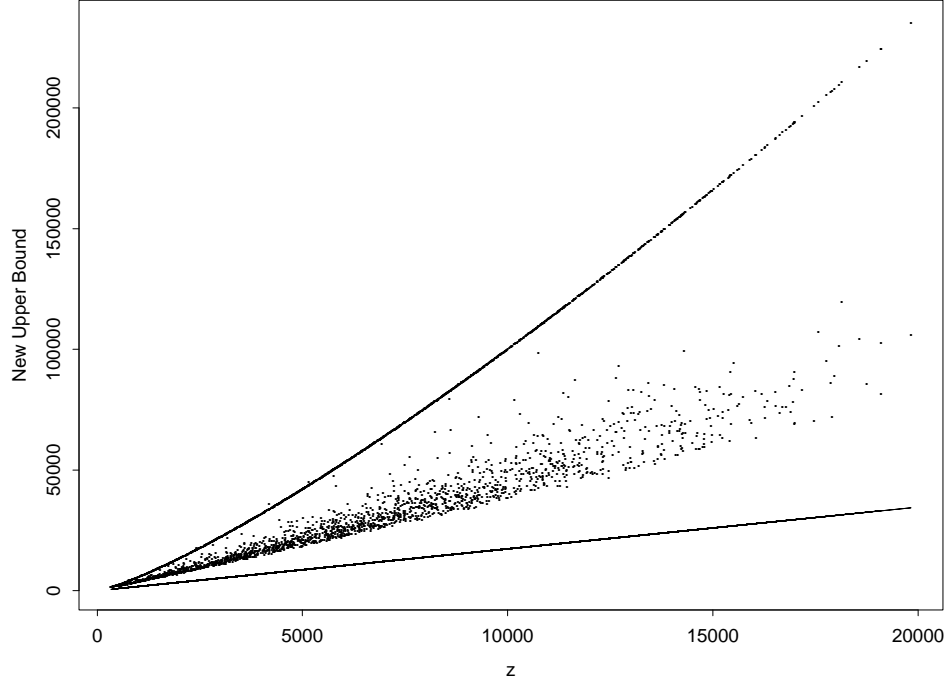


FIGURE 3. The new upper bounds and the conjecture

The question which remains, even with a higher number of iterations, is how our bound can possibly be improved. The quality of the upper bound for $f(a, b, c)$ clearly depends only on the quality of the lower bound for $S_t(a, b, c)$, and this bound is computed in our algorithm. There are three steps in the algorithm where we use bounds, namely, when adding[subtracting] $Q_{\text{low}}[Q_{\text{up}}]$, respectively, in the second to last step where we use the Cauchy-Schwartz inequality, and in the last step where we use (11). Let us assume that we use enough iterations so that we only leave the REPEAT loop when $c_1 = 1$ or $c_2 = 1$; in this case we don't use the Cauchy-Schwartz inequality. The very last step does not entail a crude inequality, as we might be off maximally by one. This leaves the bounds Q_{low} and Q_{up} given by (16) and (17), respectively, and this is where we lose quality: here we might be off by a fractional factor of c/c_1 , which then gets multiplied in the theorem, with possibly grave consequences.

We close with a remark on the general Frobenius problem, that is, with an arbitrary number of arguments a_1, \dots, a_d . There are formulas analogous to (7) for $d > 3$ [3]; they involve higher-dimensional analogs of Dedekind-Rademacher sums. However, it is not clear how to compute them efficiently. According to [2], it is possible to compute these counting functions quickly; we just don't know an actual way to do so. It is our hope that the ideas in this paper can be extended to this general setting.

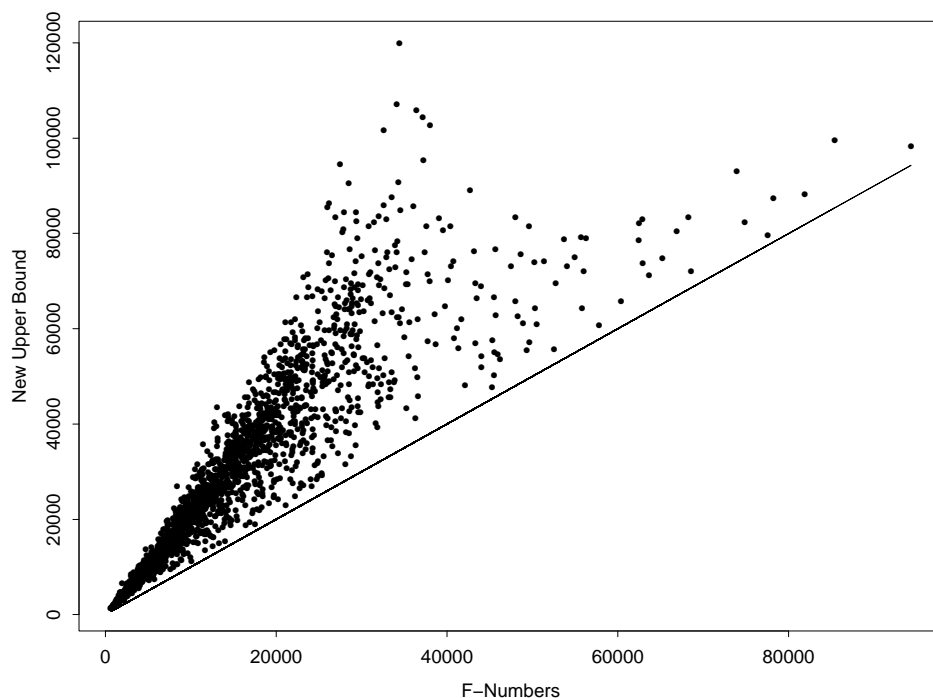


FIGURE 4. The new upper bounds compared to the Frobenius numbers

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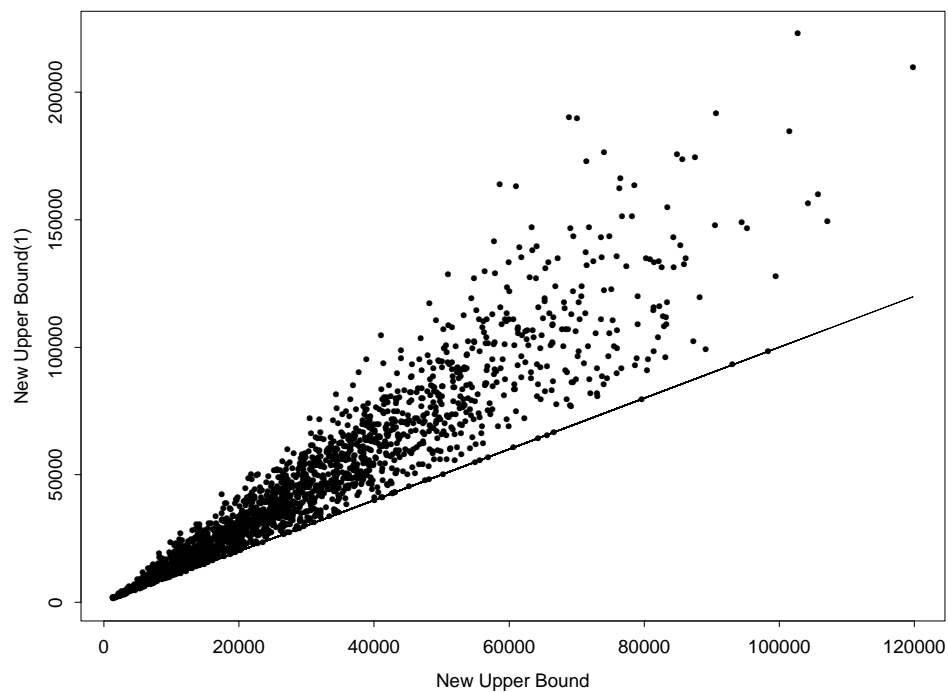


FIGURE 5. The new upper bounds compared (1 iteration vs. 2 iterations)

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