PLANE PARTITION DIAMONDS

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In *Partition Analysis VIII*, Andrews, Paule and Riese study partitions λ satisfying the "diamond" conditions

$$\lambda_1 \geq \begin{matrix}\lambda_2 \\ \lambda_3\end{matrix} \geq \lambda_4 \geq \begin{matrix}\lambda_5 \\ \lambda_6\end{matrix} \geq \lambda_7 \geq \cdots \geq \begin{matrix}\lambda_{3n-1} \\ \lambda_{3n}\end{matrix} \geq \lambda_{3n+1} \, .$$

We compute the generating function $F_n(q) := \sum_\lambda q^{\lambda_1 + \cdots + \lambda_{3n+1}}$ from first principles, giving a new proof of the following result.

Theorem 1 (Andrews–Paule–Riese)**.** *The generating function for the plane partition diamonds is*

$$F_n(q) = \frac{(1+q^2)(1+q^5)(1+q^8)\cdots(1+q^{3n-1})}{(1-q)(1-q^2)\cdots(1-q^{3n+1})} \, .$$

Proof. Define the auxiliary function

$$f(q, \lambda_4) := \sum_{\lambda_3 \geq \lambda_4} \sum_{\lambda_2 \geq \lambda_4} \sum_{\lambda_1 \geq \max(\lambda_2, \lambda_3)} q^{\lambda_1 + \lambda_2 + \lambda_3} ,$$

which we can compute:

$$\begin{aligned}
f(q, \lambda_4) &= \sum_{\lambda_3 \geq \lambda_4} q^{\lambda_3} \sum_{\lambda_2 \geq \lambda_4} q^{\lambda_2} \sum_{\lambda_1 \geq \max(\lambda_2, \lambda_3)} q^{\lambda_1} = \sum_{\lambda_3 \geq \lambda_4} q^{\lambda_3} \sum_{\lambda_2 \geq \lambda_4} q^{\lambda_2} \frac{q^{\max(\lambda_2, \lambda_3)}}{1-q} \\
&= \frac{1}{1-q} \sum_{\lambda_3 \geq \lambda_4} q^{\lambda_3} \left(q^{\lambda_3} \sum_{\lambda_2 = \lambda_4}^{\lambda_3 - 1} q^{\lambda_2} + \sum_{\lambda_2 \geq \lambda_3} q^{2\lambda_2} \right) \\
&= \frac{1}{1-q} \sum_{\lambda_3 \geq \lambda_4} q^{\lambda_3} \left(q^{\lambda_3} \frac{q^{\lambda_4} - q^{\lambda_3}}{1-q} + \frac{q^{2\lambda_3}}{1-q^2} \right) \\
&= \frac{q^{3\lambda_4}}{1-q} \left(\frac{1}{(1-q)(1-q^2)} - \frac{1}{(1-q)(1-q^3)} + \frac{1}{(1-q^2)(1-q^3)} \right) \\
&= q^{3\lambda_4} \, \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)} \, .
\end{aligned}$$

Thus we can recursively compute

$$F_1(q) = \sum_{\lambda_4 \geq 0} q^{\lambda_4} f(q, \lambda_4) = \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)} \frac{1}{1-q^4} \, ,$$

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$$\begin{aligned}
F_2(q) &= \sum_{\lambda_7 \geq 0} \sum_{\lambda_6 \geq \lambda_7} \sum_{\lambda_5 \geq \lambda_7} \sum_{\lambda_4 \geq \max(\lambda_5, \lambda_6)} q^{\lambda_4} q^{\lambda_5} q^{\lambda_6} q^{\lambda_7} f(q, \lambda_4) \\
&= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)} \sum_{\lambda_7 \geq 0} \sum_{\lambda_6 \geq \lambda_7} \sum_{\lambda_5 \geq \lambda_7} \sum_{\lambda_4 \geq \max(\lambda_5, \lambda_6)} q^{4\lambda_4} q^{\lambda_5} q^{\lambda_6} q^{\lambda_7} \\
&= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)} \sum_{\lambda_7 \geq 0} f(q^4, \lambda_7) q^{\lambda_7} \\
&= \frac{1+q^2}{(1-q)(1-q^2)(1-q^3)} \frac{1+q^5}{(1-q^4)(1-q^5)(1-q^6)} \frac{1}{1-q^7}
\end{aligned}$$

and so on. □

Generalize this theorem by replacing each 2×2 diamond by an $n \times m$ diamond (a general plane partition). Read up on plane partitions starting with Pak's expository article on partitions.