Asymptotics of Ehrhart Series of Lattice Polytopes

Matthias Beck (San Francisco State)

math.sfsu.edu/beck

Joint with Alan Stapledon (Michigan)

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Warm-Up Trivia

- Let's say we add two random 100-digit integers. How often should we expect to carry a digit?

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The Eulerian polynomial $A_d(t)$ is defined through $\displaystyle\sum_{m \geq 0} m^d \, t^m = \frac{A_d(t)}{(1-t)^{d+1}}$

Persi Diaconis will tell you that the coefficients of $A_d(t)$ (the Eulerian numbers) play a role here. . .

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Theorem (Ehrhart 1962) $L_{\mathcal{P}}(m)$ is a polynomial in m of degree d . Equivalently,

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Write the Ehrhart h-vector of $\mathcal{P}$ as $h(t) = h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0$, then

$$L_{\mathcal{P}}(m) = \sum_{j=0}^{d} h_j \binom{m+d-j}{d}.$$

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Easier Problem Study $\mathrm{Ehr}_{n\mathcal{P}}(t) = 1 + \sum_{m \geq 1} L_{\mathcal{P}}(nm)\, t^m$ as n increases.

General Properties of Ehrhart h-Vectors

$$\mathrm{Ehr}_{\mathcal{P}}(t) = 1 + \sum_{m \ge 1} \# \left(m\mathcal{P} \cap \mathbb{Z}^d \right) t^m = \frac{h_d t^d + h_{d-1} t^{d-1} + \cdots + h_0}{(1-t)^{d+1}}$$

- (Ehrhart) $h_0 = 1$, $h_1 = \# \left(\mathcal{P} \cap \mathbb{Z}^d \right) - d - 1$, $h_d = \# \left(\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$
- (Ehrhart) $\mathrm{vol}\, \mathcal{P} = \frac{1}{d!} \left(h_d + h_{d-1} + \cdots + h_1 + 1 \right)$
- (Stanley 1980) $h_j \in \mathbb{Z}_{\ge 0}$
- (Stanley 1991) Whenever $h_s > 0$ but $h_{s+1} = \cdots = h_d = 0$, then $h_0 + h_1 + \cdots + h_j \le h_s + h_{s-1} + \cdots + h_{s-j}$ for all $0 \le j \le s$.
- (Hibi 1994) $h_0 + \cdots + h_{j+1} \ge h_d + \cdots + h_{d-j}$ for $0 \le j \le \left\lfloor \frac{d}{2} \right\rfloor - 1$.
- (Hibi 1994) If $h_d > 0$ then $h_1 \le h_j$ for $2 \le j < d$.

General Properties of Ehrhart h-Vectors

A triangulation τ of $\mathcal{P}$ is unimodular if for any simplex of τ with vertices $v_0, v_1, \ldots, v_d$, the vectors $v_1 - v_0, \ldots, v_d - v_0$ form a basis of $\mathbb{Z}^d$.

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- Recent papers of Reiner–Welker and Athanasiadis use this as a starting point to give conditions under which the Ehrhart h-vector is unimodal, i.e., $h_d \le h_{d-1} \le \cdots \le h_k \ge h_{k-1} \ge \cdots \ge h_0$ for some k.

The Main Question

Define $h_0(n), h_1(n), \ldots, h_d(n)$ through

$$\mathrm{Ehr}_{n\mathcal{P}}(t) = \frac{h_d(n)\, t^d + h_{d-1}(n)\, t^{d-1} + \cdots + h_0(n)}{(1-t)^{d+1}} \,.$$

What does the Ehrhart h-vector $(h_0(n), h_1(n), \ldots, h_d(n))$ look like as n increases?

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Let $h(t) = (1-t)^{d+1}\, \mathrm{Ehr}_{\mathcal{P}}(t)$. The operator U_n defined through

$$\mathrm{Ehr}_{n\mathcal{P}}(t) = 1 + \sum_{m \geq 1} L_{\mathcal{P}}(nm)\, t^m = \frac{\mathrm{U}_n\, h(t)}{(1-t)^{d+1}}$$

appears in Number Theory as a Hecke operator and in Commutative Algebra in Veronese subring constructions.

Motivation I: Unimodular Triangulations

Theorem (Kempf–Knudsen–Mumford–Saint-Donat–Waterman 1970's)
For every lattice polytope $\mathcal{P}$ there exists an integer m such that $m\mathcal{P}$ admits a regular unimodular triangulation.

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(b) For every d there exists an integer m_d such that, if $\mathcal{P}$ is a d-dimensional lattice polytope, then $m_d\mathcal{P}$ admits a regular unimodular triangulation.

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Motivation II: Unimodal Ehrhart h-Vectors

Theorem (Athanasiadis–Hibi–Stanley 2004) If the d-dimensional lattice polytope $\mathcal{P}$ admits a regular unimodular triangulation, then the Ehrhart h-vector of $\mathcal{P}$ satisfies

(a) $h_{j+1} \geq h_{d-j}$ for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor - 1$,

(b) $h_{\lfloor \frac{d+1}{2} \rfloor} \geq h_{\lfloor \frac{d+1}{2} \rfloor + 1} \geq \cdots \geq h_{d-1} \geq h_d$,

(c) $h_j \leq \binom{h_1 + j - 1}{j}$ for $0 \leq j \leq d$.

In particular, if the Ehrhart h-vector of $\mathcal{P}$ is symmetric and $\mathcal{P}$ admits a regular unimodular triangulation, then the Ehrhart h-vector is unimodal.

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There are (many) lattice polytopes for which (some of these) inequalities fail and one may hope to use this theorem to construct a counter-example to the Knudsen–Mumford–Waterman Conjectures.

Veronese Polynomials Are Eventually Unimodal

Theorem (Brenti–Welker 2008) For any $d \in \mathbb{Z}_{>0}$, there exists real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{d-1} < \alpha_d = 0$, such that, if $h(t)$ is a polynomial of degree at most d with nonnegative integer coefficients and constant term 1, then for n sufficiently large, $\mathrm{U}_n \, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ and $\lim_{n \to \infty} \beta_j(n) = \alpha_j$.

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If the polynomial $p(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0$ has negative roots, then its coefficients are (strictly) log concave ($a_j^2 > a_{j-1} a_{j+1}$) which, in turn, implies that the coefficients are (strictly) unimodal ($a_d < a_{d-1} < \cdots < a_k > a_{k-1} > \cdots > a_0$ for some k).

A General Theorem

The Eulerian polynomial $A_d(t)$ is defined through $\displaystyle\sum_{m \ge 0} m^d \, t^m = \frac{A_d(t)}{(1-t)^{d+1}}$.

Theorem (MB–Stapledon) Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of $A_d(t)$. There exist M, N depending only on d such that, if $h(t)$ is a polynomial of degree at most d with nonnegative integer coefficients and constant term 1, then for $n \ge N$, $\mathrm{U}_n \, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\displaystyle\lim_{n \to \infty} \beta_j(n) = \rho_j$, and the coefficients of $\mathrm{U}_n \, h(t)$ satisfy $h_j(n) < M \, h_d(n)$.

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In particular, the coefficients of $\mathrm{U}_n\, h(t)$ are unimodal for $n \geq N$.

Furthermore, if $h_0 + \cdots + h_{j+1} \geq h_d + \cdots + h_{d-j}$ for $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1$, with at least one strict inequality, then we may choose N such that, for $n \geq N$,

$$h_0 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_j(n) < h_{d-j}(n) < h_{j+1}(n)$$
$$< \cdots < h_{\lfloor \frac{d+1}{2} \rfloor}(n) < M\, h_d(n)\,.$$

An Ehrhartian Corollary

Corollary (MB–Stapledon) Fix a positive integer d and let $\rho_1 < \rho_2 < \cdots < \rho_d = 0$ denote the roots of the Eulerian polynomial $A_d(t)$. There exist M, N depending only on d such that, if P is a d-dimensional lattice polytope with Ehrhart series numerator $h(t)$, then for $n \geq N$, $\mathrm{U}_n\, h(t)$ has negative real roots $\beta_1(n) < \beta_2(n) < \cdots < \beta_{d-1}(n) < \beta_d(n) < 0$ with $\lim_{n \to \infty} \beta_j(n) = \rho_j$.

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In particular, the coefficients of $\mathrm{U}_n \, h(t)$ are unimodal for $n \geq N$.

Furthermore, they satisfy

$$1 = h_0(n) < h_d(n) < h_1(n) < \cdots < h_j(n) < h_{d-j}(n) < h_{j+1}(n) \\ < \cdots < h_{\lfloor \frac{d+1}{2} \rfloor}(n) < M \, h_d(n) \, .$$

Open Problems

Find optimal choices for M and N in any of our theorems.

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Conjecture For Ehrhart series of d-dimensional polytopes, $N = d$.

(Open for $d \geq 3$)

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Homework Figure out what all of this has to do with carrying digits when summing 100-digit numbers.