CSC/MATH 870 – Computational Discrete Geometry
Lecture 5 (10/3/05)

1 Polyhedra

Polyhedra are essential objects in mathematics, because they can be described by a linear system. Examples of polyhedra include line segments, half-lines, triangles, polygons, crystals, boxes, tetrahedra, (polyhedral) cones, and any convex object whose faces are all flat. Rigorously, a *(convex) polyhedron* is defined as an intersection of finitely many half-spaces in $\mathbb{R}^d$. (Recall that an *(affine) halfspace* is a set of the form $\left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} \le b \right\}$ for some $\mathbf{a} \in \mathbb{R}^d, b \in \mathbb{R}$.) For example, the intersection of two half-planes in $\mathbb{R}^2$ gives (usually) a cone, and if we intersect that with a third half-plane we obtain a triangle (if the half-planes are suitably oriented). A *polytope* is a bounded polyhedron. A 1-dimensional polytope is a line segment, and a 2-dimensional polytope is a *polygon*. Here are a few classes of polytopes that live in all dimensions:

- the *unit cube* $\square := \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \ 0 \le x_k \le 1 \text{ for all } k = 1, 2, \dots, d \right\}$;
- the *standard simplex* $\Delta := \left\{ (x_1, x_2 \dots, x_d) \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \le 1, \text{ and all } x_k \ge 0 \right\}$;
- the *cross polytope* $\diamond := \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \ |x_1| + |x_2| + \cdots + |x_d| \le 1 \right\}$.

Maybe the most familiar class of unbounded polyhedra is that containing

- the *nonnegative orthant* $O := \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : \ x_k \ge 0 \text{ for all } k = 1, 2, \dots, d \right\}$.

The *dimension* of a polyhedron P is the dimension of its *affine span*

$$\operatorname{span} P := \{ \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \ \mathbf{x}, \mathbf{y} \in P, \ \lambda \in \mathbb{R} \} .$$

If P has dimension d, we often call P a *d-polyhedron*. Note that $P \subseteq \mathbb{R}^d$ does not necessarily have dimension d. For example,

$$\left\{ (x, y, z) \in \mathbb{R}^3 : \ x, y, z \ge 0, \ x + y + z = 1 \right\}$$

is a 2-dimensional triangle living in $\mathbb{R}^3$.

The boundaries of the half-spaces that describe a polyhedron P are hyperplanes (that is, sets of the form $\left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} = b \right\}$ for some $\mathbf{a} \in \mathbb{R}^d, b \in \mathbb{R}$), the *defining hyperplanes* of P. A polyhedron whose defining hyperplanes meet in a point is called a *cone*. A d-cone is *simplicial* if it has exactly d defining hyperplanes. The nonnegative orthant O is an example of a simplicial cone. The analogous definition for polytopes is that of a *simplex*, which is a d-polytope with exactly $d + 1$ defining hyperplanes. For example, the standard simplex Δ is a simplex. Simplices have the minimal number of defining hyperplanes among all polytopes of a fixed dimension, and simplicial cones have the minimum number of defining hyperplanes among all polyhedra of a fixed dimension that do not contain lines.

Given a polyhedron $P \subseteq \mathbb{R}^d$, we say that the hyperplane

$$H = \{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} = b \}$$

is a *supporting hyperplane* of P if P lies entirely on one side of H, that is,

$$P \subseteq \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} \leq b \right\} \qquad \text{or} \qquad P \subseteq \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} \geq b \right\}.$$

A *face* of P is a set of the form $P \cap H$, where H is a supporting hyperplane of P. Note that both $\emptyset$ and P are faces of P, corresponding to the supporting hyperplanes $\emptyset$ and $\mathbb{R}^d$. The $(d-1)$-dimensional faces are called *facets*, the 1-dimensional faces *edges*, and the 0-dimensional faces *vertices* of P. Vertices are the "extreme points" of a polyhedron. We remark that each face of P is again polyhedron by definition. The faces of P are ordered by set inclusion, which gives the *face lattice* of P. In this lattice, the meet of two faces is their intersection and their join is the minimal face of P that contains both. One can show that the face lattice of a simplex is a Boolean lattice.

A polytope P is *simplicial* if all its faces, except possibly P itself, is a simplex. The cross polytope $\diamond$ is and example of a simplicial polytope. A d-polyhedron P is *simple* if each vertex of P is incident to exactly d edges. For example, every simplicial cone is simple. The unit cube $\square$ is an example of a simple polytope.

Given a polyhedron P, its *dual* is defined as

$$P^* := \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in P \right\}.$$

(The concept of duality is not confined to polyhedra but can be defined for any non-empty subset of $\mathbb{R}^d$. For more about polarity in the general setting, see for example [1, Chapter IV].) For example, the dual of the cross polytope $\diamond$ is the cube $[-1,1]^d$. For a polyhedron P that contains the origin, one can prove that $(P^*)^* = P$. The face lattice of P^* can be obtained by turning that of P upside down. The dual of a simplicial polytope is simple (and vice versa).

2 Convexity

A set $C \subset \mathbb{R}^d$ is *convex* if it contains the line segment $[\mathbf{x}, \mathbf{y}] := \{\lambda \mathbf{x} + \mu \mathbf{y} : \ \lambda, \mu \geq 0, \ \lambda + \mu = 1\}$ for any $\mathbf{x}, \mathbf{y} \in C$.

Theorem 1. *Given a family $\{C_j : \ j \in I\}$ of convex sets, their intersection $\bigcap_{j \in I} C_j$ is again convex.*

Proof. Given $\mathbf{x}, \mathbf{y} \in \bigcap_{j \in I} C_j$, both $\mathbf{x}$ and $\mathbf{y}$ are in each C_j and hence, since C_j is convex, so is $[\mathbf{x}, \mathbf{y}]$. $\square$

Theorem 2. *A half-space is convex.*

Proof. Given the half-space $H = \left\{ \mathbf{x} \in \mathbb{R}^d : \ \mathbf{a} \cdot \mathbf{x} \leq b \right\}$, let $\mathbf{x}, \mathbf{y} \in H$. That is, $\mathbf{a} \cdot \mathbf{x} \leq b$ and $\mathbf{a} \cdot \mathbf{y} \leq b$, and so for any $\lambda, \mu \geq 0$ that satisfy $\lambda + \mu = 1$,

$$\mathbf{a} \cdot (\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda \, (\mathbf{a} \cdot \mathbf{x}) + \mu \, (\mathbf{a} \cdot \mathbf{y}) \leq (\lambda + \mu) \, b = b \ ,$$

that is, $\lambda \mathbf{x} + \mu \mathbf{y} \in H$. $\square$

The combination of these two theorems immediately gives:

Corollary 3. *A polyhedron is convex.*

Theorem 1 implies much more. For example, it allows us to define the *convex hull* $\operatorname{conv} S$ of a set $S \subseteq \mathbb{R}^d$ as the intersection of all convex sets containing S; that is, $\operatorname{conv} S$ is the smallest convex set that contains S. Here is a result that is intuitively obvious, however, its proof takes a whole chapter of a textbook on polytopes:

Theorem 4. *The polytopes in $\mathbb{R}^d$ are precisely the convex hulls of finite sets of points in $\mathbb{R}^d$.*

Thus every polytope is the convex hull of its vertices. Here are the "vertex descriptions" of the three classes of polytopes on the first page:

$$\Box = \operatorname{conv}\{(x_1, x_2 \ldots, x_d) : \ x_k = 0 \text{ or } 1\} ;$$
$$\Delta = \operatorname{conv}\{(0,0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} ;$$
$$\Diamond = \operatorname{conv}\{(\pm1,0,\ldots,0),(0,\pm1,0,\ldots,0),\ldots,(0,\ldots,0,\pm1)\} .$$

Simplices are d-polytopes with precisely $d+1$ vertices. Theorem 4 is only one of many obvious sounding results for polytopes. We give two more as examples, the second being a consequence of the first:

Theorem 5. *Suppose P is a polytope with vertex set V, and F is one of the faces of P. Then*

$$F = \operatorname{conv}(V \cap F) .$$

Corollary 6. *The number of faces of a polytope is finite.*

The result analogous to Theorem 4 for cones is as follows:

Theorem 7. *The cones in $\mathbb{R}^d$ are precisely the sets of the form*

$$\{\mathbf{v} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_n\mathbf{v}_n : \ \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\}$$

for some $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$.

To give some insight into the relationship between Theorems 4 and 7, we introduce a process called *coning over a polytope*. Given a convex polytope $P \subset \mathbb{R}^d$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we lift these vertices into $\mathbb{R}^{d+1}$, by adding a 1 as their last coordinate. So let

$$\mathbf{w}_1 = (\mathbf{v}_1, 1), \mathbf{w}_2 = (\mathbf{v}_2, 1), \ldots, \mathbf{w}_n = (\mathbf{v}_n, 1) .$$

Now we define the *cone over P* as

$$\operatorname{cone} P = \{\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{w}_2 + \cdots + \lambda_n\mathbf{w}_n : \ \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \subset \mathbb{R}^{d+1}.$$

This cone has the origin as vertex, and we can recover our original polytope P (strictly speaking, the set $\{(\mathbf{x}, 1) : \ \mathbf{x} \in P\}$) by cutting $\operatorname{cone} P$ with the hyperplane $x_{d+1} = 1$. This construction can be used to show the equivalence of Theorems 4 and 7. Not only are the proofs of any of Theorems 4 and 7 hard, it is also an algorithmically nontrivial problem to obtain one of the hyperplane or vertex description of a polytope from the other. The most powerful implementation I'm aware of is `cdd`[1], which is also incorporated in the useful tool `polymake`[2].

We finish this section with a common generalization of Theorems 4 and 7. For two sets $S, T \subseteq \mathbb{R}^d$, we define their *Minkowski sum*

$$S + T := \{s + t : \ s \in S, \, t \in T\} .$$

[1] http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html
[2] http://www.math.tu-berlin.de/polymake/

Theorem 8. *The polyhedra in* $\mathbb{R}^d$ *are precisely the sets of the form*

$$\operatorname{conv} V + \{\lambda_1 \mathbf{w}_1 + \lambda_2 \mathbf{w}_2 + \lambda_n \mathbf{w}_n : \ \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\}$$

for some finite set $V \subset \mathbb{R}^d$ *and some* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in \mathbb{R}^d$.

Polyhedra are beautiful objects with a rich history and interesting theory, which we have only glimpsed here. For a good introduction to convexity and polyhedra, I recommend the monographs cited at the end of these lectures.

3 Face Numbers

We already remarked that much of the discrete structure of a polyhedron is encoded in its face lattice. In reality, trying to classify them—even just polytopes, on which we will concentrate from now on—by their face lattices turns out to be an ambitious goal. Slightly easier (and less general) is the study of the *face numbers* f_k of a polytope P; f_k records how many faces of P have dimension k. The f*-vector* of the d-polytope P is the d-tuple $(f_0, f_1, \ldots, f_{d-1})$. A major (for $d \geq 4$ open) question is which d-tuples are f-vectors of polytopes. For example, for $d = 2$, the only possible f-vectors are (n, n) for some integer $n \geq 3$.

We start with the *Euler Relation* (which Euler only proved for $d = 3$; the first general proof is due to Schläfli, although his, and numerous later proofs, assumed another "obvious" fact, namely that every polytope is *shellable*, a concept that was only developed and understood four decades ago).

Theorem 9. *If* P *is a* d*-polytope, then* $\sum_{j=0}^{d} (-1)^j f_j = 1$.

Since $f_d = 1$, we can reformulate Euler's identity as $\sum_{j=0}^{d-1} (-1)^j f_j = 1 - (-1)^d$. Like Theorem 4, the proof of the general Euler Relation is beyond the scope of this course. However, we can give an easy proof for dimension three, which draws connections to planar graphs and shows that the Euler Relation also holds in that setting:

Proof of Theorem 9 for $d = 3$. Project the 3-polytope P in such a way that the facets get projected to the (polygonal) faces of a planar graph. We will sum up the angles in all bounded faces in two different ways. First, using that the the sum of the angles in a k-gon is $(k-2)\pi$, each face of the planar graph contributes $(e-2)\pi$ to the angle sum, where e is the number of edges of the face. Denote the number of interior edges by I, the number of boundary edges by B, and the number of faces by F. Since each interior edge contributes to two faces, we obtain a total angle sum of

$$(2I + B - 2(F-1))\,\pi \ .$$

On the other hand, we can compute the angles directly at each vertex. An interior vertex will contribute 2π to the total sum, whereas the angles at the boundary vertices sum up to $(B-2)\pi$. Hence if we denote number of vertices by V, the total angle sum equals (note that there are B boundary vertices)

$$(V - B)2\pi + (B-2)\pi \ .$$

Comparing this with the previous computation gives

$$(2I + B - 2(F-1))\,\pi = (V-B)2\pi + (B-2)\pi \ ,$$

and this simplifies to

$$V - I - B + F = 2 \ ,$$

which is the Euler Relation, as $V = f_0$, $I + B = f_1$, and $F = f_2$. $\square$

We can get more information about the face numbers when P is simple (or simplicial, by duality), namely, in form of the *Dehn–Sommerville Relations.*

Theorem 10. *If P is a simple d-polytope and $k \le d$ then*

$$f_k = \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} f_j \ .$$

Thus, for $d = 3$ the Dehn–Sommerville Relations determine the face numbers once we know one of f_0, f_1, or f_2. For example, every simple 3-polytope with 6 facets will have 12 edges and 8 vertices, just like the unit 3-cube, because the Dehn–Sommerville Relations give for $d = 3$

$$f_0 = 2f_2 - 4 \qquad \text{and} \qquad f_1 = 3f_2 - 6 \ .$$

Proof of Theorem 10, assuming Theorem 9. Suppose F is a k-face of P. Then the Euler Relation (Theorem 9) applied to F gives

$$1 = \sum_{G \subseteq F} (-1)^{\dim G},$$

where the sum is over all nonempty faces G of F. Now sum both left– and right–hand sides over all k-faces and rearrange the sum on the right–hand side:

$$\begin{aligned}
f_k &= \sum_{\substack{F \subseteq P \\ \dim F = k}} \sum_{G \subseteq F} (-1)^{\dim G} \\
&= \sum_{\substack{F \subseteq P \\ \dim F = k}} \sum_{j=0}^{k} (-1)^j \sum_{\substack{G \subseteq F \\ \dim G = j}} 1 \\
&= \sum_{j=0}^{k} (-1)^j \sum_{\substack{F \subseteq P \\ \dim F = k}} \sum_{\substack{G \subseteq F \\ \dim G = j}} 1 \\
&= \sum_{j=0}^{k} (-1)^j \sum_{\substack{G \subseteq P \\ \dim G = j}} f_k(P/G) 1 \\
&= \sum_{j=0}^{k} (-1)^j \sum_{\substack{G \subseteq P \\ \dim G = j}} \binom{d-j}{d-k} 1 \\
&= \sum_{j=0}^{k} (-1)^j \binom{d-j}{d-k} F_j(-t) \ .
\end{aligned}$$

Here $f_k(P/G)$ denotes the number of k-faces of P containing a given j-face G of P. Since P is simple, this number equals $\binom{d-j}{d-k}$. $\square$

References

[1] Alexander Barvinok, *A course in convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, RI, 2002. MR 2003j:52001

[2] Arne Brøndsted, *An introduction to convex polytopes*, Graduate Texts in Mathematics, vol. 90, Springer-Verlag, New York, 1983. MR MR683612 (84d:52009)

[3] Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. MR 2004b:52001

[4] Günter M. Ziegler, *Lectures on polytopes*, Springer-Verlag, New York, 1995, Revised edition, 1998; "Updates, corrections, and more" at `www.math.tu-berlin.de/~ziegler`. MR 96a:52011