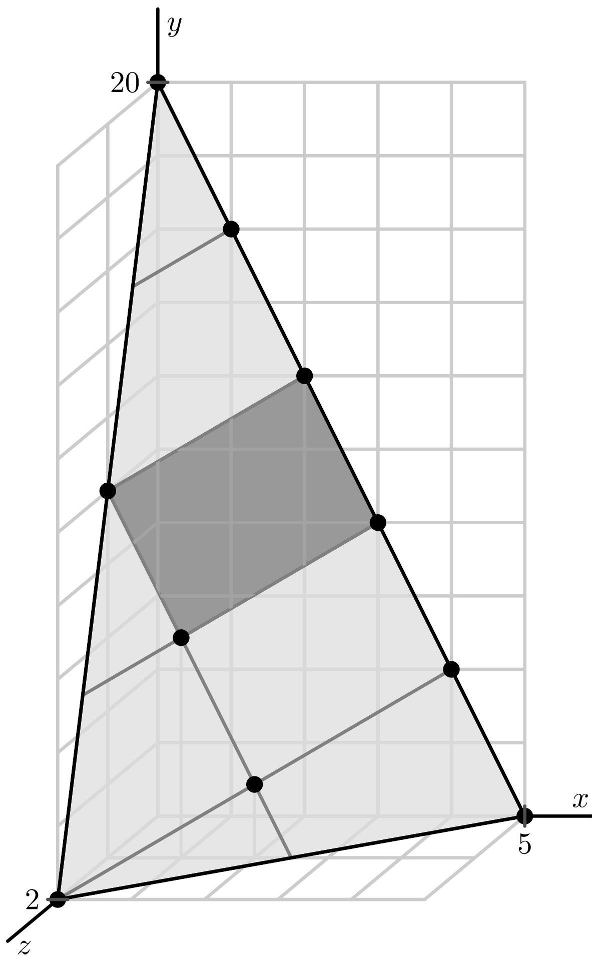
Digital

~~Discrete~~ Volume Computations for Polyhedra



Matthias Beck

San Francisco State University

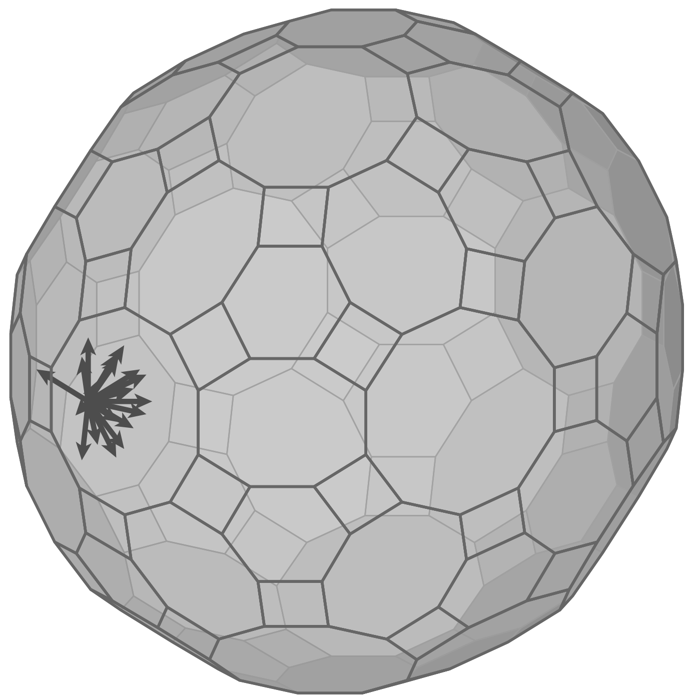
math.sfsu.edu/beck

DGCI 2016

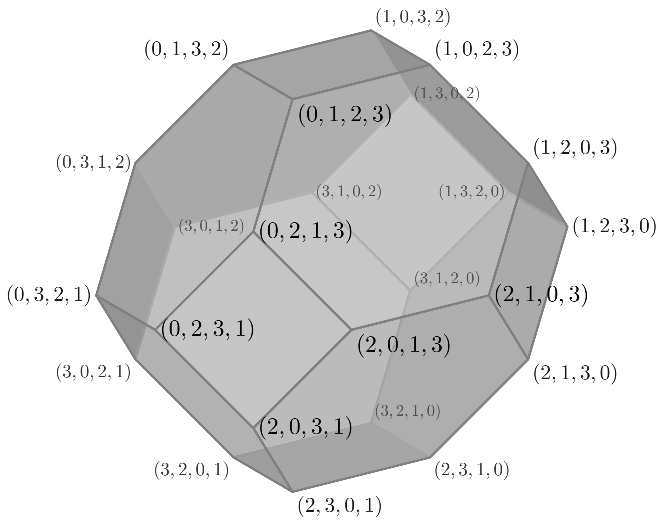
Institut de Recherche en Communications et Cybernétique de Nantes

"Science is what we understand well enough to explain to a computer, art is all the rest."

Donald Knuth



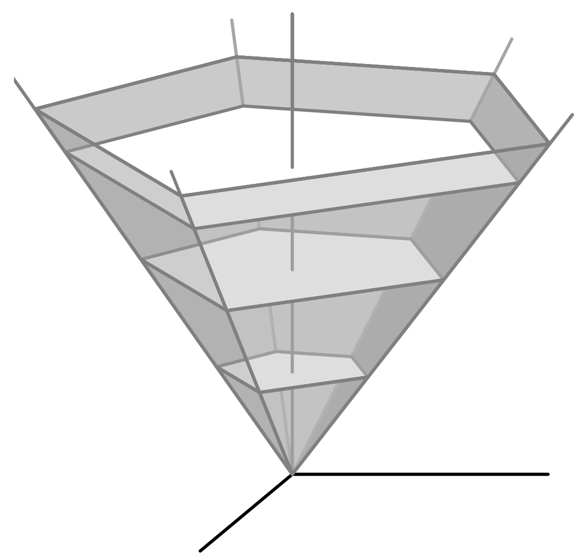
Themes



Discrete-geometric polynomials

Computation (complexity)

Generating functions



Combinatorial structures

Discrete Fourier analysis

A Sample Problem: Birkhoff–von Neumann Polytope

This site is supported by donations to The OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A037302 Normalized volume of Birkhoff polytope of n X n doubly-stochastic square matrices. If the volume is v(n), then a(n) = ((n-1)^2)! * v(n) / n^(n-1). 2

```
1, 1, 3, 352, 4718075, 14666561365176, 17832560768358341943028,
12816077964079346687829905128694016, 7658969897501574748537755050756794492337074203099,
5091038988117504946842559205930853037841762820367901333706255223000
```
(list; graph; refs; listen; history; text; internal format)

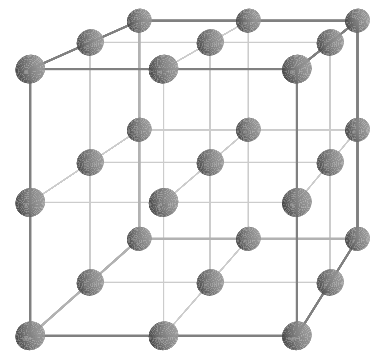
OFFSET 1,3

COMMENTS
```
The Birkhoff polytope is an (n-1)^2-dimensional polytope in n^2-dimensional
  space; its vertices are the n! permutation matrices.
Is a(n) divisible by n^2 for all n>=4? - Dean Hickerson, Nov 27 2002
```

$$B_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2}_{\geq 0} : \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients



Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .

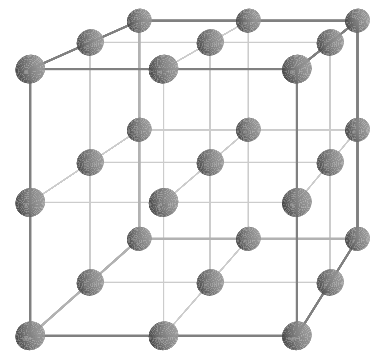
- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

- (count) $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$

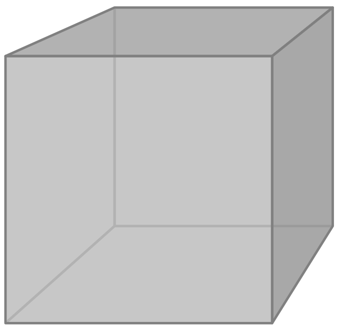
Discrete Volumes

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .



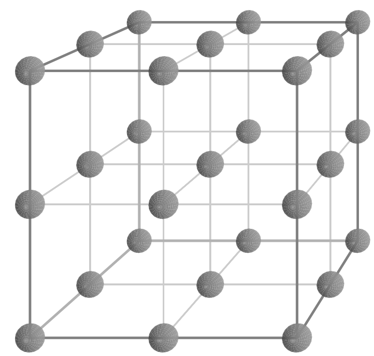
- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$
- (count) $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$
- (volume) $\displaystyle \mathrm{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \left| \mathcal{P} \cap \frac{1}{t} \mathbb{Z}^d \right|$



Discrete Volumes

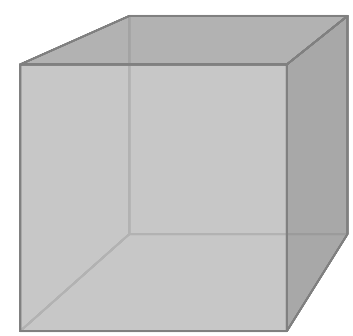
Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

Goal: understand $\mathcal{P} \cap \mathbb{Z}^d$ . . .



- (list) $\displaystyle\sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

- (count) $\left|\mathcal{P} \cap \mathbb{Z}^d\right|$



- (volume) $\operatorname{vol}(\mathcal{P}) = \displaystyle\lim_{t \to \infty} \frac{1}{t^d} \left|\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right|$

Ehrhart function $L_{\mathcal{P}}(t) := \left|\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right| = \left|t\mathcal{P} \cap \mathbb{Z}^d\right|$ for $t \in \mathbb{Z}_{>0}$

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").

- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").

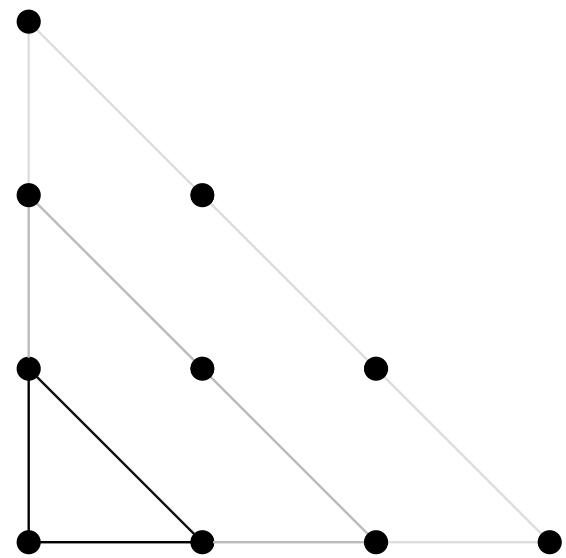
- Polytopes are basic geometric objects, yet even for these basic objects volume computation is hard and there remain many open problems.

- Many discrete problems in various mathematical areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$



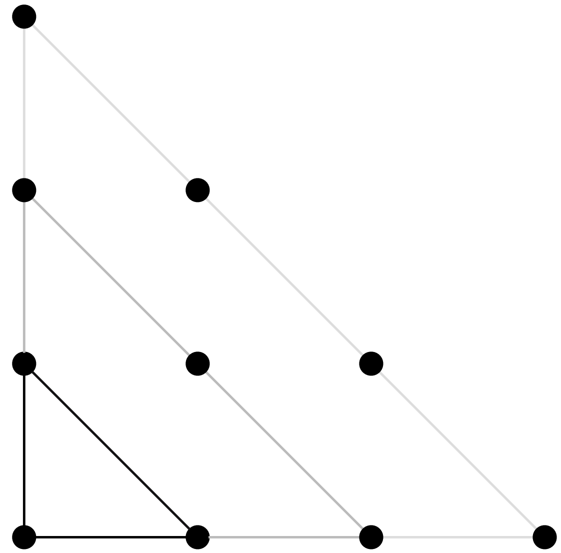
Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{(x,y) \in \mathbb{R}^2_{\geq 0} : x+y \leq 1\right\}$$

A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$



Example:

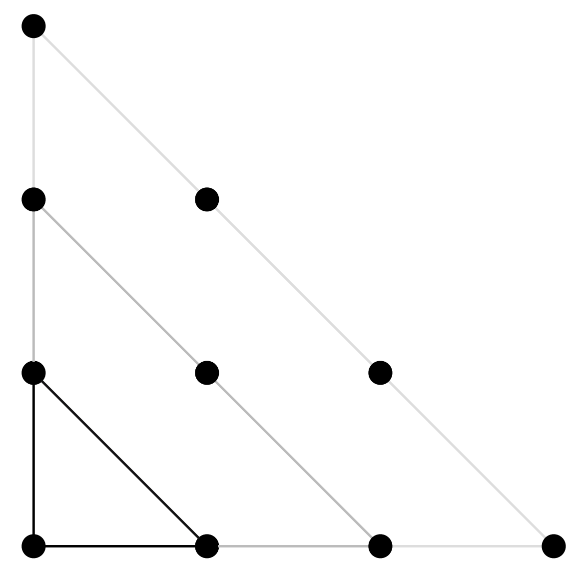
$$\begin{aligned}\Delta &= \operatorname{conv}\{(0,0),(1,0),(0,1)\} \\ &= \left\{(x,y) \in \mathbb{R}^2_{\geq 0} : x+y \leq 1\right\}\end{aligned}$$

$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$

A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\Delta = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$$
$$= \left\{(x,y) \in \mathbb{R}^2_{\geq 0} : x + y \leq 1\right\}$$

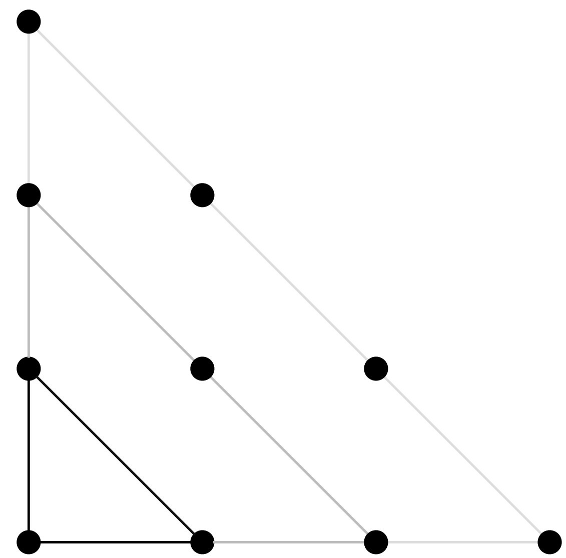
$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)\,,$$

a polynomial in t with leading coefficient $\operatorname{vol}(\Delta) = \dfrac{1}{2}$

A Warm-Up Ehrhart Function

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$



Example:

$$\begin{aligned} \Delta &= \operatorname{conv}\{(0,0),(1,0),(0,1)\} \\ &= \left\{ (x,y) \in \mathbb{R}^2_{\geq 0} : \, x+y \leq 1 \right\} \end{aligned}$$

$$L_\Delta(t) = \binom{t+2}{2} = \tfrac{1}{2}(t+1)(t+2)$$

comes with the friendly generating function $\displaystyle\sum_{t \geq 0} \binom{t+2}{2} z^t = \frac{1}{(1-z)^3}$

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t$ is rational:

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the Ehrhart h-vector $h(z)$ satisfies $h(0) = 1$ and $h(1) = (\dim \mathcal{P})! \operatorname{vol}(\mathcal{P})$.

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $\dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

Equivalently, $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t$ is rational:

$$\operatorname{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1-z)^{\dim \mathcal{P}+1}}$$

where the Ehrhart h-vector $h(z)$ satisfies $h(0) = 1$ and $h(1) = (\dim \mathcal{P})! \operatorname{vol}(\mathcal{P})$.

Seeming dichotomy: $\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^{\dim \mathcal{P}}} L_{\mathcal{P}}(t)$ can be computed discretely via a finite amount of data.

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \ge 1} L_{\mathcal{P}}(t)\, z^t = \frac{h(z)}{(1-z)^{d+1}}$$

Equivalent descriptions of an Ehrhart polynomial:

- $L_{\mathcal{P}}(t) = c_d\, t^d + c_{d-1}\, t^{d-1} + \cdots + c_0$
- via roots of $L_{\mathcal{P}}(t)$
- $\mathrm{Ehr}_{\mathcal{P}}(z) \quad \longrightarrow \quad L_{\mathcal{P}}(t) = h_0 \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_d \binom{t}{d}$

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

$$\operatorname{Ehr}_{\mathcal{P}}(z) \ := \ 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t \ = \ \frac{h(z)}{(1-z)^{d+1}}$$

$$\longrightarrow \qquad L_{\mathcal{P}}(t) \ = \ h_0 \binom{t+d}{d} + h_1 \binom{t+d-1}{d} + \cdots + h_d \binom{t}{d}$$

Theorem (Macdonald 1971) $(-1)^d L_{\mathcal{P}}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$. Equivalently,

$$L_{\mathcal{P}^\circ}(t) \ = \ h_d \binom{t+d-1}{d} + h_{d-1} \binom{t+d-2}{d} + \cdots + h_0 \binom{t-1}{d}$$

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

$$\operatorname{Ehr}_{\mathcal{P}}(z) \ := \ 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t \ = \ \frac{h(z)}{(1-z)^{d+1}}$$

$$\longrightarrow \qquad L_{\mathcal{P}^\circ}(t) \ = \ h_d \binom{t+d-1}{d} + h_{d-1} \binom{t+d-2}{d} + \cdots + h_0 \binom{t-1}{d}$$

Theorem (Stanley 1980) $h_0, h_1, \dots, h_d$ are nonnegative integers.

Ehrhart Polynomials

Theorem (Ehrhart 1962) For any lattice polytope $\mathcal{P}$, $L_\mathcal{P}(t)$ is a polynomial in t of degree $d := \dim \mathcal{P}$ with leading coefficient $\operatorname{vol} \mathcal{P}$ and constant term 1.

$$\operatorname{Ehr}_\mathcal{P}(z) := 1 + \sum_{t \ge 1} L_\mathcal{P}(t)\, z^t = \frac{h(z)}{(1-z)^{d+1}}$$

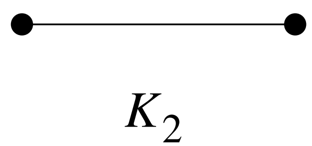
$$\longrightarrow \qquad L_{\mathcal{P}^\circ}(t) = h_d \binom{t+d-1}{d} + h_{d-1} \binom{t+d-2}{d} + \cdots + h_0 \binom{t-1}{d}$$

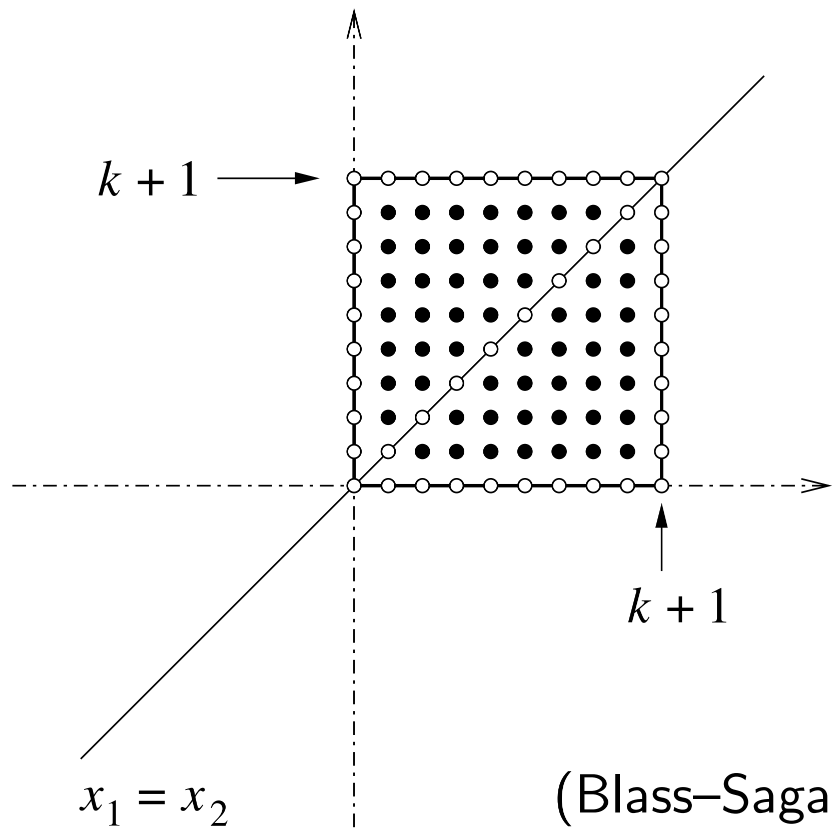
Theorem (Stanley 1980) $h_0, h_1, \ldots, h_d$ are nonnegative integers.

Corollary If $h_{d+1-k} > 0$ then $k\mathcal{P}^\circ$ contains an integer point.

Interlude: Graph Coloring a la Ehrhart

$$\chi_{K_2}(k) = 2\binom{k}{2} \ldots$$

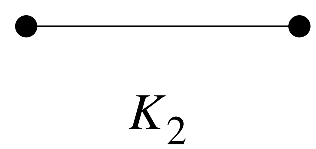


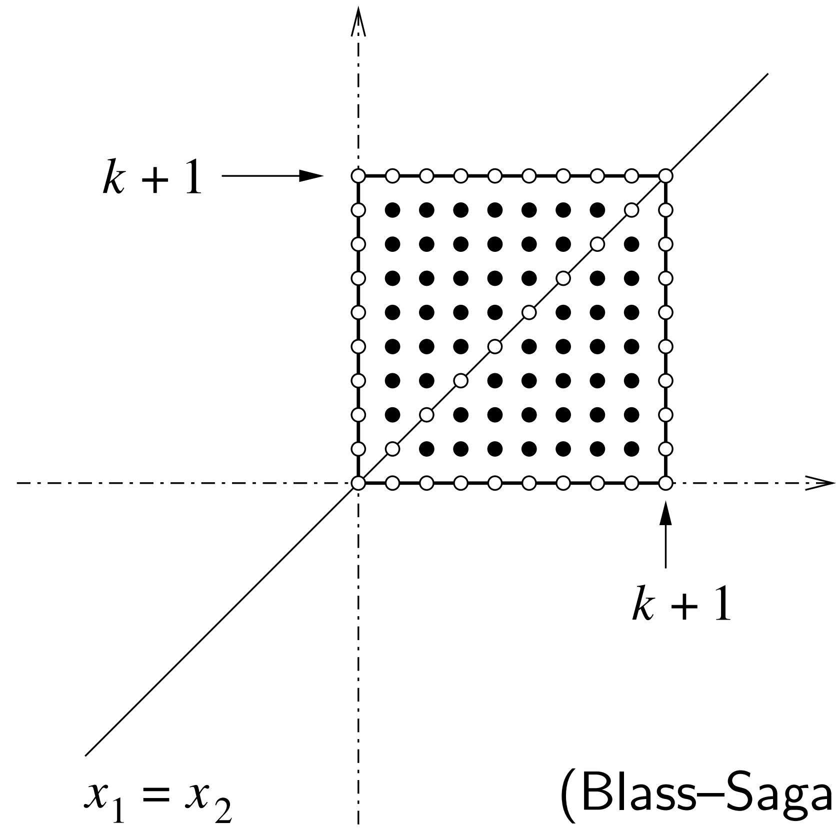


(Blass–Sagan)

Interlude: Graph Coloring a la Ehrhart

$$\chi_{K_2}(k) = 2\binom{k}{2} \cdots$$





(Blass–Sagan)

Similarly, for any given graph G on n nodes, we can write

$$\chi_G(k) = a_0\binom{k+n}{n} + a_1\binom{k+n-1}{n} + \cdots + a_n\binom{k}{n}$$

for some (meaningful) nonnegative integers $a_0, \ldots, a_n$

Computational Complexity of Integer-Point Transforms

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

$$\longrightarrow \sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$$ is a rational function in $z_1, z_2, \ldots, z_d$

Lenstra (1983) polynomial-time algorithm to decide whether $\sigma_{\mathcal{P}}(\mathbf{z}) = 0$

Barvinok (1994) polynomial-time algorithm to compute $\sigma_{\mathcal{P}}(\mathbf{z})$

Computational Complexity of Integer-Point Transforms

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

$\longrightarrow \ \sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$ is a rational function in $z_1, z_2, \dots, z_d$

Lenstra (1983) polynomial-time algorithm to decide whether $\sigma_{\mathcal{P}}(\mathbf{z}) = 0$

Barvinok (1994) polynomial-time algorithm to compute $\sigma_{\mathcal{P}}(\mathbf{z})$

Example $\mathcal{P} = [0, 1000]$

$$\sigma_{[0,1000]}(z) = 1 + z + \cdots + z^{1000} = \frac{1 - z^{1001}}{1 - z}$$

Computational Complexity of Integer-Point Transforms

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

$\longrightarrow \sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$ is a rational function in $z_1, z_2, \dots, z_d$

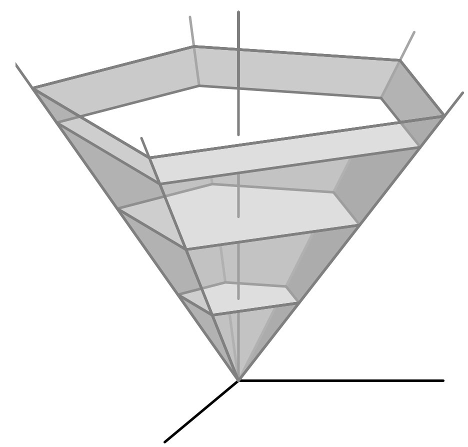
Lenstra (1983) polynomial-time algorithm to decide whether $\sigma_{\mathcal{P}}(\mathbf{z}) = 0$

Barvinok (1994) polynomial-time algorithm to compute $\sigma_{\mathcal{P}}(\mathbf{z})$

Given a polytope $\mathcal{P}$ we can compute

$\operatorname{Ehr}_{\mathcal{P}}(z) = \sigma_{\operatorname{cone}(\mathcal{P})}(1, 1, \dots, 1, z)$

where $\operatorname{cone}(\mathcal{P}) := \mathbb{R}_{\geq 0} (\mathcal{P} \times \{1\})$



Computational Complexity of Integer-Point Transforms

Rational polyhedron $\mathcal{P} \subset \mathbb{R}^d$ – solution set of a system of linear equalities & inequalities with integer coefficients

$\longrightarrow \sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$ is a rational function in $z_1, z_2, \ldots, z_d$

Lenstra (1983) polynomial-time algorithm to decide whether $\sigma_{\mathcal{P}}(\mathbf{z}) = 0$

Barvinok (1994) polynomial-time algorithm to compute $\sigma_{\mathcal{P}}(\mathbf{z})$

Implementations:

De Loera, Köppe et al `www.math.ucdavis.edu/~latte`

Verdoolaege `freshmeat.net/projects/barvinok`

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in t:

$$L_{\mathcal{P}}(t) \;=\; c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$$

where $c_0(t), \ldots, c_d(t)$ are periodic functions.

Example $L_{[0,\frac{1}{2}]}(t) \;=\; \frac{1}{2}\, t + \frac{3 + (-1)^t}{4}$

Ehrhart Quasipolynomials

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

Theorem (Ehrhart 1962) $L_{\mathcal{P}}(t)$ is a quasipolynomial in t:

$$L_{\mathcal{P}}(t) \; = \; c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$$

where $c_0(t), \ldots, c_d(t)$ are periodic functions. Equivalently,

$$\mathrm{Ehr}_{\mathcal{P}}(z) \; := \; 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t \; = \; \frac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$$

for some (minimal) $p \in \mathbb{Z}_{>0}$ (the period of $L_{\mathcal{P}}(t)$).

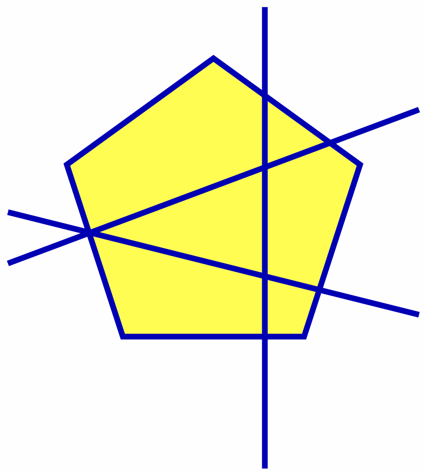
Example $\mathrm{Ehr}_{[0,\frac{1}{2}]}(z) \; = \; \sum_{t \geq 0} \left(\frac{1}{2} t + \frac{3 + (-1)^t}{4} \right) z^t \; = \; \frac{1+z}{(1-z^2)^2}$

Interlude: Magic Squares

joint with
Thomas Zaslavsky
Andrew Van Herick

$M_n(t)$ – number of $n \times n$ squares with distinct entries and row, column, and diagonal sums t

4	9	2
3	5	7
8	1	6



Similar to graph coloring, this is a polyhedral problem with forbidden hyperplanes:
inside-out polytopes

Interlude: Magic Squares

joint with
Thomas Zaslavsky
Andrew Van Herick

$M_n(t)$ – number of $n \times n$ squares with distinct entries and row, column, and diagonal sums t

4	9	2
3	5	7
8	1	6

Example

$$M_3(t) = \begin{cases} \frac{2t^2-32t+144}{9} = \frac{2}{9}(t^2-16t+72) & \text{if } t \equiv 0 \bmod 18 \\ \frac{2t^2-32t+78}{9} = \frac{2}{9}(t-3)(t-13) & \text{if } t \equiv 3 \bmod 18 \\ \frac{2t^2-32t+120}{9} = \frac{2}{9}(t-6)(t-10) & \text{if } t \equiv 6 \bmod 18 \\ \frac{2t^2-32t+126}{9} = \frac{2}{9}(t-7)(t-9) & \text{if } t \equiv 9 \bmod 18 \\ \frac{2t^2-32t+96}{9} = \frac{2}{9}(t-4)(t-12) & \text{if } t \equiv 12 \bmod 18 \\ \frac{2t^2-32t+102}{9} = \frac{2}{9}(t^2-16t+51) & \text{if } t \equiv 15 \bmod 18 \\ 0 & \text{if } t \not\equiv 0 \bmod 3 \end{cases}$$

$$\sum_{t \ge 0} M_3(t)\, z^t = \frac{8z^{15}\left(2z^3+1\right)}{\left(1-z^3\right)\left(1-z^6\right)\left(1-z^9\right)}$$

Fourier–Dedekind Sums

joint with
Sinai Robins

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

$$\longrightarrow \qquad \mathrm{Ehr}_{\mathcal{P}}(z) \; := \; 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t \; = \; \frac{h(z)}{\left(1 - z^p\right)^{\dim \mathcal{P} + 1}}$$

Natural ingredients of formulas for Ehrhart quasipolynomials:

$$s_n\left(c_1, \ldots, c_d; c\right) \; := \; \frac{1}{c} \sum_{k=1}^{c-1} \frac{e^{2\pi i\, n/c}}{\left(1 - e^{2\pi i\, c_1/c}\right)\left(1 - e^{2\pi i\, c_2/c}\right) \cdots \left(1 - e^{2\pi i\, c_d/c}\right)}$$

Examples $\quad s_n(c_1; c) \sim \left\lfloor \frac{n}{c} \right\rfloor$

Fourier–Dedekind Sums

joint with
Sinai Robins

Rational polytope $\mathcal{P} \subset \mathbb{R}^d$ – convex hull of finitely points in $\mathbb{Q}^d$

$$\longrightarrow \qquad \mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)\, z^t = \frac{h(z)}{(1 - z^p)^{\dim \mathcal{P} + 1}}$$

Natural ingredients of formulas for Ehrhart quasipolynomials:

$$s_n(c_1, \ldots, c_d; c) := \frac{1}{c} \sum_{k=1}^{c-1} \frac{e^{2\pi i\, n/c}}{\left(1 - e^{2\pi i\, c_1/c}\right)\left(1 - e^{2\pi i\, c_2/c}\right) \cdots \left(1 - e^{2\pi i\, c_d/c}\right)}$$

Examples $\quad s_n(c_1; c) \sim \left\lfloor \frac{n}{c} \right\rfloor$

$$s_0(c_1, c_2; c) \sim \sum_{k=1}^{c-1} \left\lfloor \frac{k\, c_1}{c} \right\rfloor^2 \sim \text{ Dedekind sum}$$

Fourier–Dedekind Sums

joint with
Sinai Robins

$$s_n\left(c_1, \ldots, c_d; c\right) := \frac{1}{c} \sum_{k=1}^{c-1} \frac{e^{2\pi i\, n/c}}{\left(1 - e^{2\pi i\, c_1/c}\right)\left(1 - e^{2\pi i\, c_2/c}\right) \cdots \left(1 - e^{2\pi i\, c_d/c}\right)}$$

Fun Facts (Dedekind 1880s, Rademacher 1950s)

- $s_n(a, 1; b) \;=\; s_n(a \bmod b, 1; b)$
- $s_n(a, 1; b) + s_n(b, 1; a) \;=$ something simple [reciprocity]

Fourier–Dedekind Sums

joint with Sinai Robins

$$s_n\left(c_1, \ldots, c_d; c\right) := \frac{1}{c} \sum_{k=1}^{c-1} \frac{e^{2\pi i\, n/c}}{\left(1 - e^{2\pi i\, c_1/c}\right)\left(1 - e^{2\pi i\, c_2/c}\right) \cdots \left(1 - e^{2\pi i\, c_d/c}\right)}$$

Fun Facts (Dedekind 1880s, Rademacher 1950s)

- $s_n(a, 1; b) = s_n(a \bmod b, 1; b)$
- $s_n(a, 1; b) + s_n(b, 1; a) =$ something simple [reciprocity]

Corollary $s_n\left(c_1, c_2; c\right)$ can be computed in time $\log \max(c_1, c_2, c)$.

Fourier–Dedekind Sums

joint with Sinai Robins

$$s_n(c_1, \dots, c_d; c) := \frac{1}{c} \sum_{k=1}^{c-1} \frac{e^{2\pi i\, n/c}}{\left(1 - e^{2\pi i\, c_1/c}\right)\left(1 - e^{2\pi i\, c_2/c}\right) \cdots \left(1 - e^{2\pi i\, c_d/c}\right)}$$

Fun Facts (Dedekind 1880s, Rademacher 1950s)

- $s_n(a, 1; b) = s_n(a \bmod b, 1; b)$
- $s_n(a, 1; b) + s_n(b, 1; a) =$ something simple [reciprocity]

Corollary $s_n(c_1, c_2; c)$ can be computed in time $\log \max(c_1, c_2, c)$.

Corollary The Ehrhart quasipolynomial of any rational polygon $\mathcal{P}$ can be expressed in terms of Fourier–Dedekind sums and computed in linear time in the input size of $\mathcal{P}$.

Dedekind–Carlitz Polynomials

joint with
Christian Haase
Asia Matthews

Dedekind sum $s_0(a,1;b) \sim \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor^2 \sim \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor (k-1)$

Dedekind–Carlitz Polynomial $c\,(u,v;a,b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1}$

Dedekind–Carlitz Polynomials

joint with
Christian Haase
Asia Matthews

Dedekind sum $s_0(a,1;b) \sim \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor^2 \sim \sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor (k-1)$

Dedekind–Carlitz Polynomial $c\,(u,v;a,b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1}$

Theorem (Carlitz 1975) If a and b are relatively prime,

$$(v-1)\ c\,(u,v;a,b) + (u-1)\ c\,(v,u;b,a) \ = \ u^{a-1}v^{b-1} - 1$$

Applying $u\,\partial u$ twice and $v\,\partial v$ once gives Dedekind's reciprocity law.

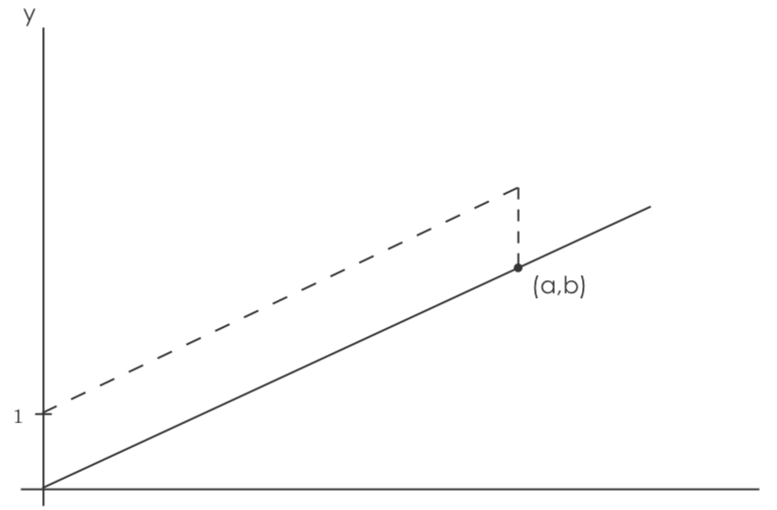
Dedekind–Carlitz Polynomials

joint with
Christian Haase
Asia Matthews

$$c(u,v;a,b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1}$$

Consider the 2-dimensional cone

$$\mathcal{K} := \{\lambda_1(0,1) + \lambda_2(a,b) : \lambda_1, \lambda_2 \geq 0\}$$

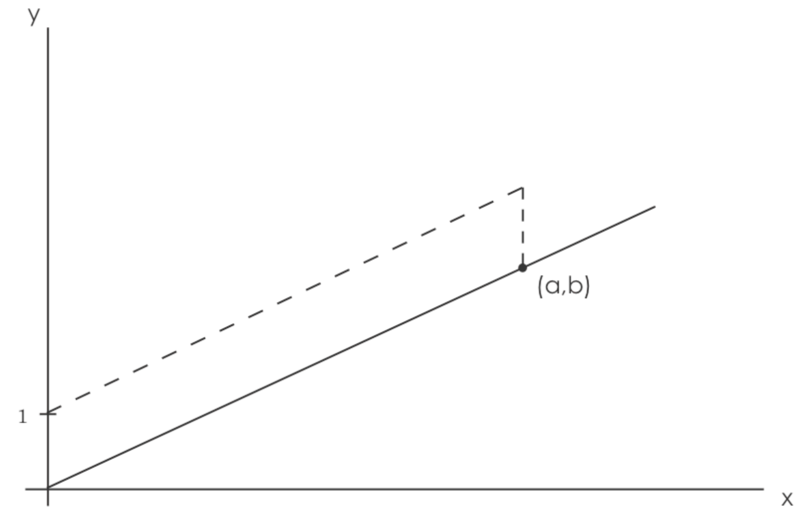


Fun Fact $\quad \sigma_{\mathcal{K}}(u,v) := \sum_{(m,n)\in\mathcal{K}\cap\mathbb{Z}^2} u^m v^n = \frac{1 + uv\,c(v,u;b,a)}{(1-v)(1-u^a v^b)}$

Dedekind–Carlitz Polynomials

joint with
Christian Haase
Asia Matthews

$$c(u,v;a,b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1}$$



Consider the 2-dimensional cone

$$\mathcal{K} := \{\lambda_1(0,1) + \lambda_2(a,b) : \lambda_1, \lambda_2 \ge 0\}$$

Fun Fact $\quad \sigma_{\mathcal{K}}(u,v) := \sum_{(m,n)\in\mathcal{K}\cap\mathbb{Z}^2} u^m v^n = \dfrac{1 + uv\, c(v,u;b,a)}{(1-v)(1-u^a v^b)}$

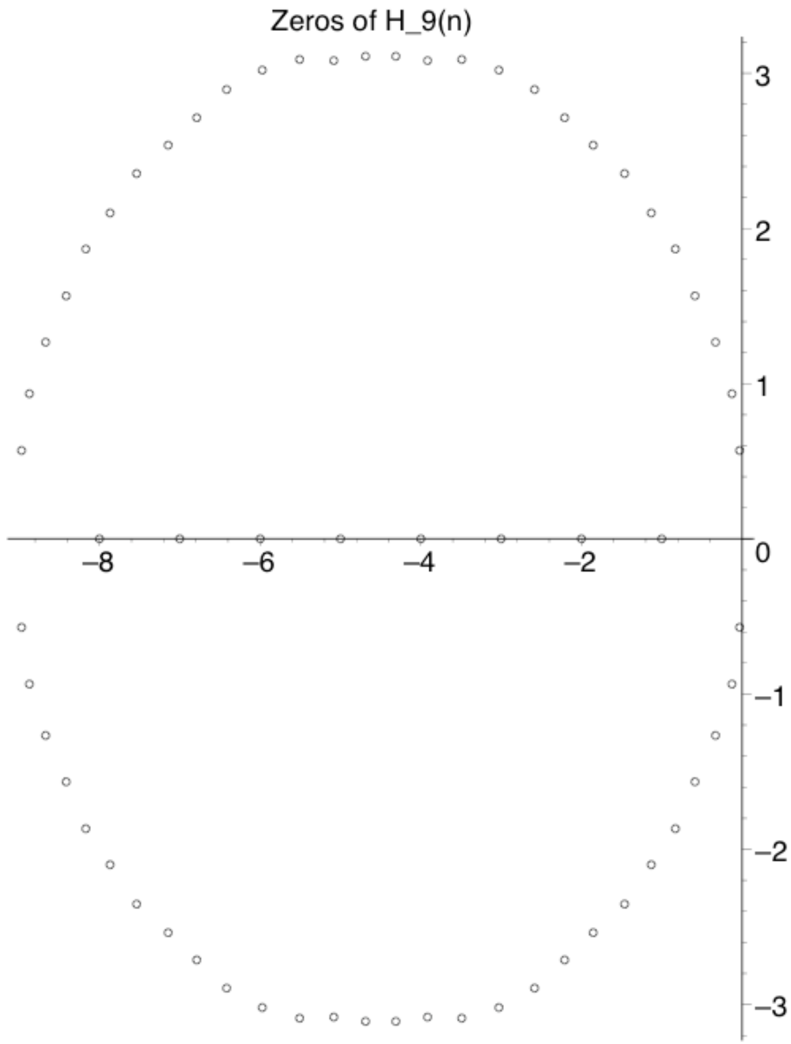
- Carlitz reciprocity $=$ two cones adding up to $\mathbb{R}^2_{\ge 0}$
- Complexity anyone?

A (Small) Bouquet of Open Problems

- Classify Ehrhart polynomials (or, alternatively, Ehrhart h-vectors)
- Find more inside-out polytopes
- Attack existence problems via (discrete-geometric) polynomials
- Study periods of Ehrhart quasipolynomials
- Study complexity of Fourier–Dedekind sums
- Compute $\mathrm{vol}(B_{11})$

Birkhoff–von Neumann Revisited

joint with
Dennis Pixton
Jesus De Loera
Mike Develin
Julian Pfeifle
Richard Stanley



For more about roots of (Ehrhart) polynomials, see Braun (2008) and Pfeifle (2010).