Inside-out polytopes

& a tale of seven polynomials

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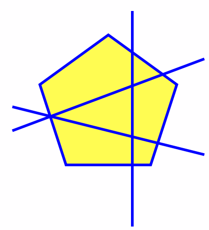

math.sfsu.edu/beck/

arXiv: math.CO/0309330 & math.CO/0309331 & . . .

Chromatic polynomials of graphs

$\Gamma = (V, E)$ – graph (without loops)

Proper k-coloring of Γ : mapping $x : V \to \{1, 2, \ldots, k\}$ such that $x_i \neq x_j$ if there is an edge ij

Theorem (Birkhoff 1912, Whitney 1932) $\chi_\Gamma(k) := \#$ (proper k-colorings of Γ) is a monic polynomial in k of degree $|V|$.

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Theorem (Stanley 1973) $(-1)^{|V|}\chi_\Gamma(-k)$ equals the number of pairs (α, x) consisting of an acyclic orientation α of Γ and a compatible k-coloring. In particular, $(-1)^{|V|}\chi_\Gamma(-1)$ equals the number of acyclic orientations of Γ.

(An orientation α of Γ and a k-coloring x are compatible if $x_j \geq x_i$ whenever there is an edge oriented from i to j. An orientation is acyclic if it has no directed cycles.)

Flow polynomials

Nowhere-zero A-flow on a graph $\Gamma = (V,E)$: mapping $x : E \to A \setminus \{0\}$ (A an abelian group) such that for every node $v \in V$

$$\sum_{h(e)=v} x(e) = \sum_{t(e)=v} x(e)$$

$$\left.\begin{array}{lll} h(e) & := & \text{head} \\ t(e) & := & \text{tail} \end{array}\right. \text{ of the edge } e \text{ in a (fixed) orientation of } \Gamma$$

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Theorem
(Tutte 1954) $\overline{\varphi}_\Gamma(|A|) := \#\,(\text{nowhere-zero } A\text{-flows})$ is a polynomial in $|A|$.
(Kochol 2002) $\varphi_\Gamma(k) := \#\,(\text{nowhere-zero } k\text{-flows})$ is a polynomial in k.

(Weak) semimagic squares

$H_n(t)$ – number of nonnegative integral $n \times n$-matrices in which every row and column sums to t

1	1	2
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and $H_n(-n-t) = (-1)^{n-1} H_n(t)$.

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What about "classical" magic squares?

Ehrhart (quasi-)polynomials

$\mathcal{P} \subset \mathbb{R}^d$ – convex rational polytope

For $t \in \mathbb{Z}_{>0}$ let $\mathrm{Ehr}_{\mathcal{P}}(t) := \#\left(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d\right)$

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Theorem
(Ehrhart 1962) $\mathrm{Ehr}_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree $\dim \mathcal{P}$ with leading term $\mathrm{vol}\, \mathcal{P}$ (normalized to $\mathrm{aff}\, \mathcal{P} \cap \mathbb{Z}^d$) and constant term $\mathrm{Ehr}_{\mathcal{P}}(0) = \chi(\mathcal{P}) = 1$.
(Macdonald 1971) $(-1)^{\dim \mathcal{P}}\, \mathrm{Ehr}_{\mathcal{P}}(-t)$ enumerates the interior lattice points in $t\mathcal{P}$.

(A quasipolynomial is an expression $c_d(t)\, t^d + \cdots + c_1(t)\, t + c_0(t)$ where $c_0, \dots, c_d$ are periodic functions in t.)

Characteristic polynomials of hyperplane arrangements

$\mathcal{H} \subset \mathbb{R}^d$ – arrangement of affine hyperplanes

$\mathcal{L}(\mathcal{H}) := \left\{ \bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H} \text{ and } \bigcap \mathcal{S} \neq \emptyset \right\}$, ordered by reverse inclusion

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Möbius function $\mu(r,s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r,u) & \text{if } r < s. \end{cases}$

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$$p_{\mathcal{H}}(\lambda) := \sum_{s \in \mathcal{L}(\mathcal{H})} \mu\left(\mathbb{R}^d, s\right) \lambda^{\dim s}$$

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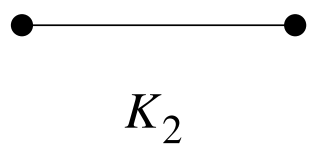
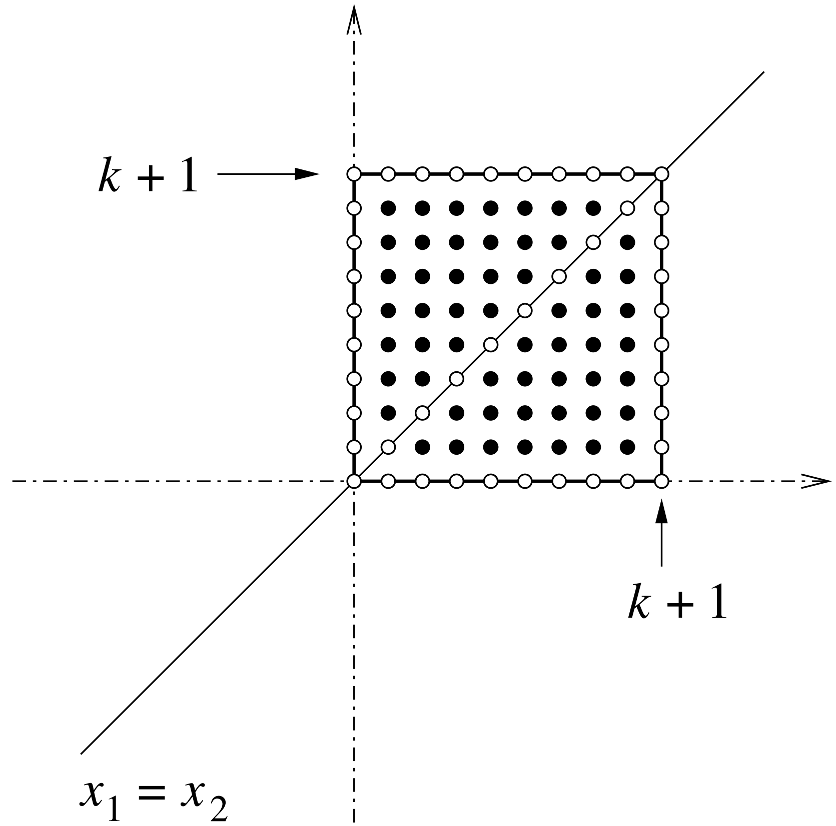
Theorem (Zaslavsky 1975) If $\mathbb{R}^d \notin \mathcal{H}$ then the number of regions into which a hyperplane arrangement $\mathcal{H}$ divides $\mathbb{R}^d$ is $(-1)^d p_{\mathcal{H}}(-1)$.

Graph coloring a la Ehrhart

$\chi_{K_2}(k) = k(k-1)$ ...





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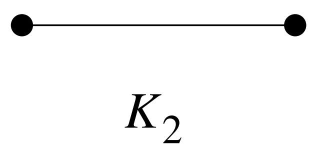
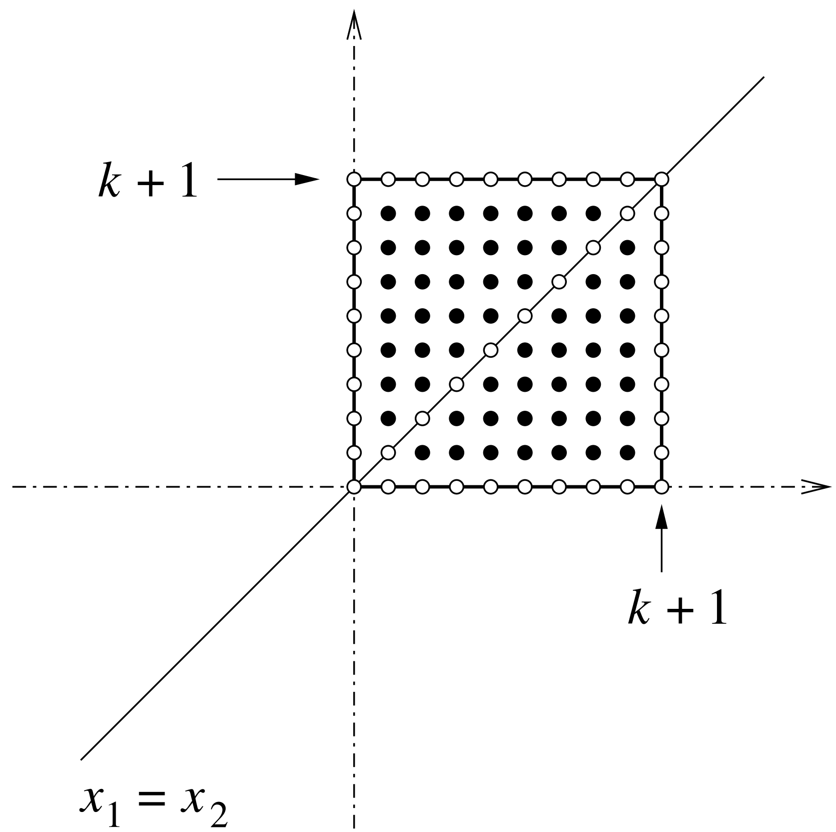
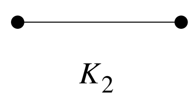
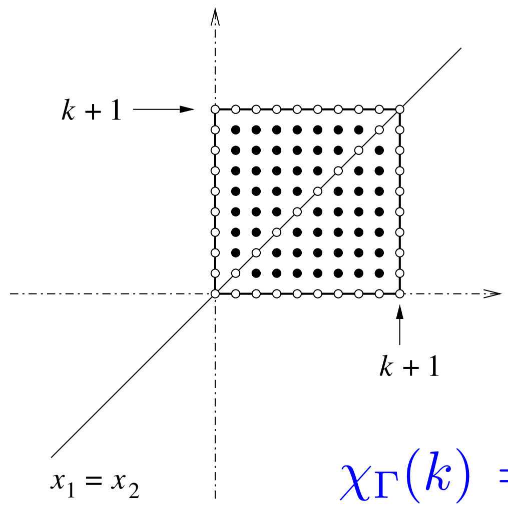




$$\chi_\Gamma(k) = \#\left(\left((0,1)^V \setminus \bigcup \mathcal{H}(\Gamma)\right) \cap \frac{1}{k+1}\mathbb{Z}^V\right)$$

Stanley's Theorem a la Ehrhart





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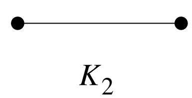

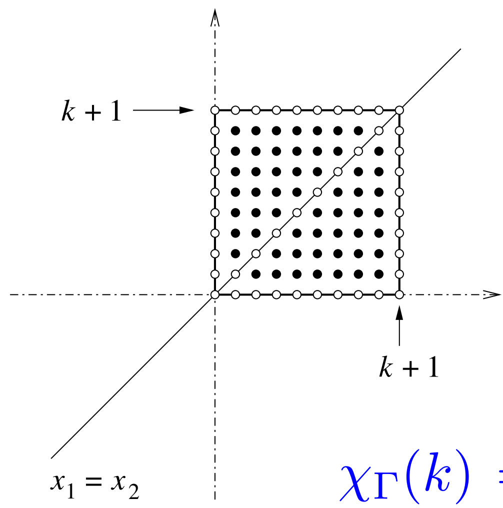

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Greene's observation

region of $\mathcal{H}(\Gamma)$ $\iff$ acyclic orientation of Γ

$x_i < x_j \iff i \longrightarrow j$

Chromatic polynomials of signed graphs

Σ – signed graph (without loops): each edge is labelled $+$ or $-$

Proper k-coloring of Σ : mapping $x : V \to \{-k, -k+1, \dots, k\}$ such that, if edge ij has sign ϵ then $x_i \neq \epsilon x_j$

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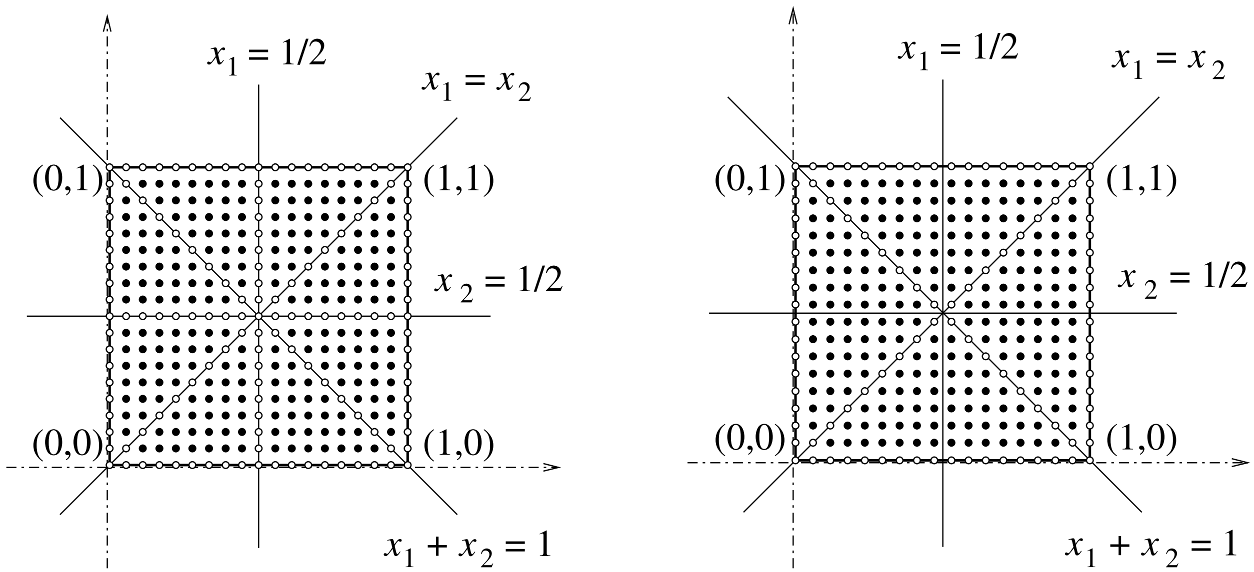
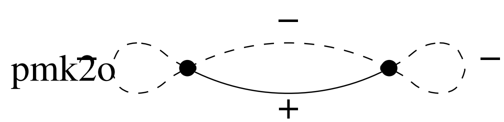
Theorem (Zaslavsky 1982) $\chi_\Sigma(2k+1) := \#\,(\text{proper } k\text{-colorings of } \Sigma)$ and $\chi^*_\Sigma(2k) := \#\,(\text{proper zero-free } k\text{-colorings of } \Sigma)$ are monic polynomials of degree $|V|$. The number of compatible pairs (α, x) consisting of an acyclic orientation α and a k-coloring x of Σ is equal to $(-1)^{|V|}\chi_\Sigma(-(2k+1))$. The number in which x is zero-free equals $(-1)^{|V|}\chi^*_\Sigma(-2k)$. In particular, $(-1)^{|V|}\chi_\Sigma(-1)$ equals the number of acyclic orientations of Σ.

Signed-graph coloring a la Ehrhart





Theorem $\chi_\Sigma(2k+1)$ and $\chi^*_\Sigma(2k)$ are two halves of one inside-out quasipolynomial.

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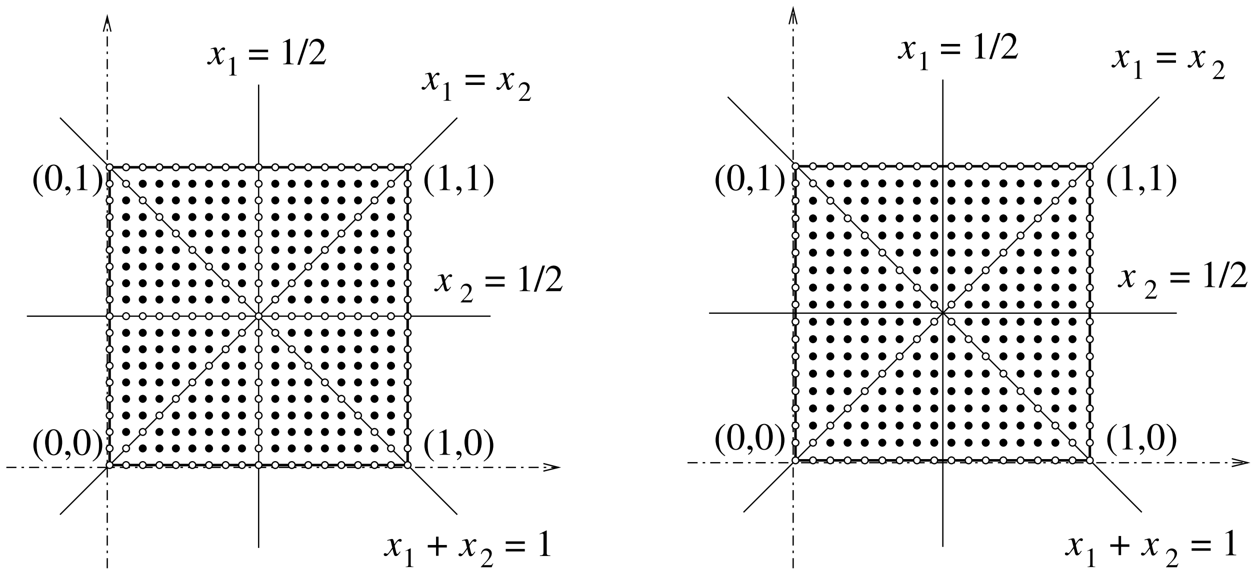

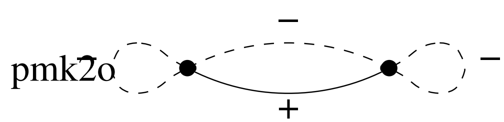

Theorem $\chi_\Sigma(2k+1)$ and $\chi_\Sigma^*(2k)$ are two halves of one inside-out quasipolynomial.

Open problem Is there a combinatorial interpretation of $\chi_\Sigma^*(-1)$?

Flow polynomials revisited

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\begin{aligned}
\varphi_\Gamma(k) &:= \#\,(\text{nowhere-zero } k\text{-flows}) \\
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\end{aligned}
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Theorem $(-1)^{|E|-|V|+c(\Gamma)}\varphi_\Gamma(-k)$ equals the number of pairs (τ, x) consisting of a totally cyclic orientation τ and a compatible $(k+1)$-flow x. In particular, the constant term $\varphi_\Gamma(0)$ equals the number of totally cyclic orientations of Γ.

(An orientation of Γ is totally cyclic if every edge lies in a coherent circle, that is, where the edges are oriented in a consistent direction around the circle. A totally cyclic orientation τ and a flow x are compatible if $x \geq 0$ when it is expressed in terms of τ.)

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Corollary $\varphi_\Gamma(0) = (-1)^{|E|-|V|+c(\Gamma)}\overline{\varphi}_\Gamma(-1)$

$\exists$ analogous theorems for signed graphs

Open problems

Find a formula for, or a combinatorial interpretation of, the leading coefficient of φ_Γ.

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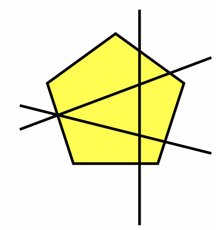
For some graphs, both φ_Γ and $\overline{\varphi}_\Gamma$ have integral coefficients and φ_Γ is a multiple of $\overline{\varphi}_\Gamma$. Is there a general reason for these facts?

Inside-out counting functions



Inside-out polytope : $(\mathcal{P}, \mathcal{H})$

Multiplicity of $x \in \mathbb{R}^d$:

$$m_{\mathcal{P},\mathcal{H}}(x) := \begin{cases} \# \text{ closed regions of } \mathcal{H} \text{ in } \mathcal{P} \text{ that contain } x & \text{if } x \in \mathcal{P}, \\ 0 & \text{if } x \notin \mathcal{P} \end{cases}$$

Closed Ehrhart quasipolynomial $E_{P,\mathcal{H}}(t) := \sum_{x \in \frac{1}{t}\mathbb{Z}^d} m_{\mathcal{P},\mathcal{H}}(x)$

Open Ehrhart quasipolynomial $E^{\circ}_{\mathcal{P},\mathcal{H}}(t) := \# \left(\frac{1}{t}\mathbb{Z}^d \cap [\mathcal{P} \setminus \bigcup \mathcal{H}] \right)$

Basic inside-out results

Theorem If $(\mathcal{P}, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope, then $E_{\mathcal{P},\mathcal{H}}(t)$ and $E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}}(t)$ are quasipolynomials in t of degree $\dim \mathcal{P}$ with leading term $\mathrm{vol}\, P$, and with constant term $E_{\mathcal{P},\mathcal{H}}(0)$ equal to the number of regions of $(\mathcal{P}, \mathcal{H})$. Furthermore,

$$E^{\circ}_{\mathcal{P}^{\circ},\mathcal{H}}(t) = (-1)^d E_{\mathcal{P},\mathcal{H}}(-t).$$

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Theorem $(\mathcal{P}, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope, then

$$E^\circ_{\mathcal{P},\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\mathbb{R}^d, u) \operatorname{Ehr}_{\mathcal{P}\cap u}(t),$$

and if $\mathcal{H}$ is transverse to $\mathcal{P}$

$$E_{\mathcal{P},\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\mathbb{R}^d, u)| \operatorname{Ehr}_{\mathcal{P}\cap u}(t).$$

($\mathcal{H}$ is transverse to $\mathcal{P}$ if every flat $u \in \mathcal{L}(\mathcal{H})$ that intersects $\mathcal{P}$ also intersects P°, and $\mathcal{P}$ does not lie in any of the hyperplanes of $\mathcal{H}$.)

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Corollary $\mathrm{Mag}_n(t)$ is a quasipolynomial in t of degree $n - 2n - 1$.

Open problem Can anything be said about the period of Mag_n? Even in the weak case, do we ever get a polynomial?

Enumeration of integer points with distinct entries

$\mathcal{P} \subset \mathbb{R}^d$ – rational convex polytope, transverse to

$\mathcal{H} := \mathcal{H}[K_d]^{\operatorname{aff} \mathcal{P}}$ – arrangement corresponding to K_d, induced on $\operatorname{aff} \mathcal{P}$

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Theorem The number $E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(t)$ of integer points in $t\mathcal{P}^\circ$ with distinct entries is a quasipolynomial with constant term equal to the number of permutations of $[d]$ that are realizable in $\mathcal{P}$. Furthermore, $(-1)^{\dim s} E^{\circ}_{\mathcal{P}^\circ,\mathcal{H}}(-t) = E_{\mathcal{P},\mathcal{H}}(t) :=$ the number of pairs (x, σ) consisting of an integer point $x \in t\mathcal{P}$ and a compatible $\mathcal{P}$-realizable permutation σ of $[d]$.

(The point $x \in \mathbb{R}^d$ and the permutation τ are compatible if $x_{\tau 1} < x_{\tau 2} < \cdots < x_{\tau d}$. τ is realizable in X if there exists a compatible $x \in X$.)

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Applications (strong) magic squares, rectangles, cubes, graphs, ...

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Compute $\mathrm{Mag}_4, \mathrm{Mag}_5, \ldots$ (possibly using `LattE` and the Möbius function of the intersection lattice of $\mathcal{H}[K_d]$).

Latin squares and beyond

Covering cluster $(X, \mathcal{L})$ – a finite set X of points together with a family $\mathcal{L} \subseteq P(X)$ of lines

Latin labelling of $(X, \mathcal{L})$ – assignment of integers to X such that all entries in a line are distinct.

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To make counting fun, we restrict the entries to the set $(0, t)$. This corresponds to the inside-out polytope $\left([0,1]^X, \mathcal{H}[\Gamma_{\mathcal{L}}]\right)$, where $\Gamma_{\mathcal{L}} = \bigcup_{L \in \mathcal{L}} K_L$. (Every graph is isomorphic to one of those.)

Latin squares and beyond

Covering cluster $(X, \mathcal{L})$ – a finite set X of points together with a family $\mathcal{L} \subseteq P(X)$ of lines

Latin labelling of $(X, \mathcal{L})$ – assignment of integers to X such that all entries in a line are distinct.

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Example : latin rectangle – lines are rows & columns, $\Gamma_\mathcal{L} = K_m \times K_n$. Slightly more general are (partial) latin orthotopes with $\Gamma_\mathcal{L} = K_{m_1} \times \cdots \times K_{m_j}$ (a "Hamming graph").

Stanley's Theorem and latinity

Theorem The number $L^{\circ}(t)$ of latin labellings of $(X, \mathcal{L})$ with values in $(0, t)$ is a monic polynomial of degree $|X|$ with constant term equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$. Furthermore, $(-1)^{|X|} L^{\circ}(-t)$ enumerates pairs consisting of an acyclic orientation of $\Gamma_{\mathcal{L}}$ and a compatible latin labelling with values in $[0, t]$.

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Example : latin squares, with $t = \binom{n+1}{2}$

Note that the hyperplane arrangement gets more complicated, namely $\mathcal{H}[\Gamma_{\mathcal{L}}]^{s}$, where s is the subspace of $\mathbb{R}^{X}$ determined by the line sum conditions.

Open problem

The magic subspace of the covering cluster $([d], \mathcal{L})$ is defined by all line sums given by $\mathcal{L}$ being equal.

A permutation σ of $[d]$ defines a reverse dominance order on the power set $P([d])$ by $L \preccurlyeq_\sigma L'$ if, when L and L' are written in decreasing order according to σ, say $L = \{\sigma j_1, \dots, \sigma j_l\}$ where $j_1 > \dots > j_l$ and $L' = \{\sigma j'_1, \dots, \sigma j'_{l'}\}$ where $j'_1 > \dots > j'_{l'}$, then $l \le l'$ and $j_1 \le j'_1, \dots, j_l \le j'_l$.

Conjecture A permutation σ of $[d]$ is realizable by a positive point in the magic subspace of the covering cluster $([d], \mathcal{L})$ if and only if $\mathcal{L}$ is an antichain in the reverse dominance order due to σ.

Antimagic

$f_1, \dots, f_m \in (\mathbb{R}^d)^*$ – linear forms

$A^\circ(t) :=$ # integer points $x \in (0,t)^d$ such that

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Pullback $\mathcal{H}[K_m]^\sharp \subseteq \mathbb{R}^d$ obtained from $f^{-1}(h)$ for all $h \in \mathcal{H}[K_m]$

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Examples : antimagic graphs and relatives (bidirected antimagic graphs, node antimagic, total graphical antimagic), antimagic squares, cubes, etc.

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If that's too hard, try trees.