MATH 725 Homework Set 10 due 11/7/11

(1) Define a linear map $T \in L(\mathbf{R}^3)$ through $T(x_1,x_2,x_3) = (x_3, 3x_1, 2x_2)$. Compute an isometry S such that $T = S\sqrt{T^*T}$.

Solution. Fix the standard basis of $\mathbf{R}^3$; then T has the matrix form

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \text{and} \qquad T^* = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus

$$T^*T = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sqrt{T^*T} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so

$$S = T\left(\sqrt{T^*T}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

i.e., $S(x_1,x_2,x_3) = (x_3,x_1,x_2)$ will do the trick. □

(2) Prove that $T \in L(V)$ is invertible if and only if there exists a unique isometry S such that $T = S\sqrt{T^*T}$. (*Hint* for the "if" direction: look at our proof of the polar decomposition theorem.)

Proof. It will be useful (also for later) to know that for $T \in L(V)$,

$$T \text{ is invertible} \qquad \Longleftrightarrow \qquad \sqrt{T^*T} \text{ is invertible,} \qquad (\star)$$

so we'll prove that first. Suppose that $T \in L(V)$ is invertible. Then T^* is also invertible (if $ST = TS = \mathrm{I}$ then $T^*S^* = S^*T^* = \mathrm{I}$), and thus so is T^*T. But then

$$\sqrt{T^*T}\left(\sqrt{T^*T}\,(T^*T)^{-1}\right) = \left((T^*T)^{-1}\sqrt{T^*T}\right)\sqrt{T^*T} = \mathrm{I},$$

i.e., $\sqrt{T^*T}$ is invertible. Conversely, if $\sqrt{T^*T}$ is invertible, then

$$\mathrm{I} = \left(\left(\sqrt{T^*T}\right)^{-1}\right)^2 T^*T\,.$$

So if $\mathbf{v} \in \operatorname{null}(T)$ then

$$\mathbf{v} = \left(\left(\sqrt{T^*T}\right)^{-1}\right)^2 T^*T(\mathbf{v}) = \left(\left(\sqrt{T^*T}\right)^{-1}\right)^2 T^*(\mathbf{0}) = \mathbf{0},$$

i.e., $\operatorname{null}(T) = \{\mathbf{0}\}$, and so T is invertible.

Now for the actual problem: Suppose that T is invertible. Then (as we have just proved) $\sqrt{T^*T}$ is also invertible, and so the isometry S (which is guaranteed by the polar decomposition theorem) equals $T\left(\sqrt{T^*T}\right)^{-1}$ and is hence unique.

Conversely, suppose there exists a unique isometry S such that $T = S\sqrt{T^*T}$. This means that the isometry $S(\mathbf{u}+\mathbf{w}) = F(\mathbf{u}) + G(\mathbf{w})$ that we constructed in our proof is unique, and thus the isometries

$$F : \operatorname{range}\left(\sqrt{T^*T}\right) \to \operatorname{range}(T) \qquad \text{and} \qquad G : \left(\operatorname{range}\left(\sqrt{T^*T}\right)\right)^{\perp} \to (\operatorname{range}(T))^{\perp}$$

are unique. However, we could have easily replaced G by $-G$, which is still an isometry. Thus G has to be the zero map, and so $\left(\operatorname{range}\left(\sqrt{T^*T}\right)\right)^{\perp} = \{0\} = (\operatorname{range}(T))^{\perp}$. But this means $\operatorname{range}(T) = V$, i.e., T is surjective and thus invertible. □

(3) Suppose n is an odd positive integer and $T \in L(\mathbf{R}^n)$ is an isometry. Prove that there exists a nonzero vector $\mathbf{v} \in \mathbf{R}^n$ such that $T^2(\mathbf{v}) = \mathbf{v}$.

Proof. We know that there exists an orthonormal basis with respect to which T has block-diagonal form, with 1×1 blocks (of the form ± 1) and 2×2 blocks. Since n is odd, there must be a 1×1 block, and so there must be an eigenvector $\mathbf{v}$ with eigenvalue $\lambda = \pm 1$. But then

$$T^2(\mathbf{v}) = \lambda^2 \mathbf{v} = \mathbf{v}.$$

□

(4) Prove that $T \in L(V)$ is
(a) invertible if and only if 0 is not a singular value of T;
(b) an isometry if and only if all the singular values of T are 1.

Proof. (a)

$$\begin{aligned} \text{0 is not a singular value of } T \quad &\iff \quad \text{0 is not an eigenvalue of } \sqrt{T^*T} \\ &\iff \quad \text{null}\,\sqrt{T^*T} = \{\mathbf{0}\} \\ &\iff \quad \sqrt{T^*T} \text{ is invertible} \\ &\overset{(\star)}{\iff} \quad T \text{ is invertible.} \end{aligned}$$

(b) We proved in class that T is an isometry if and only if $T^*T = \mathrm{I}$. This, in turn, holds if and only if $\sqrt{T^*T} = \mathrm{I}$, which holds if and only if all eigenvalues of $\sqrt{T^*T}$ are 1, which holds if and only if all singular values of T are 1. □

(5) Suppose $T \in L(V)$ has smallest singular value s and largest singular value l. Show that for all $\mathbf{v} \in V$

$$s\,||\mathbf{v}|| \leq ||T(\mathbf{v})|| \leq l\,||\mathbf{v}||.$$

Proof. Denote the singular values of T by $s = s_1, s_2, \ldots, s_n = l$. Suppose $\mathbf{v} \in V$. By the singular-value decomposition theorem, there exist orthonormal bases $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ and $(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n)$ such that

$$T(\mathbf{v}) = \sum_{j=1}^{n} s_j \left\langle \mathbf{v}, \mathbf{e}_j \right\rangle \mathbf{f}_j$$

Since $(\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n)$ is orthonormal,

$$||T(\mathbf{v})||^2 = \sum_{j=1}^{n} \left| s_j \left\langle \mathbf{v}, \mathbf{e}_j \right\rangle \right|^2 = \sum_{j=1}^{n} s_j^2 \left| \left\langle \mathbf{v}, \mathbf{e}_j \right\rangle \right|^2$$

and so

$$s^2\,||\mathbf{v}||^2 = s^2 \sum_{j=1}^{n} \left| \left\langle \mathbf{v}, \mathbf{e}_j \right\rangle \right|^2 \leq ||T(\mathbf{v})||^2 \leq l^2 \sum_{j=1}^{n} \left| \left\langle \mathbf{v}, \mathbf{e}_j \right\rangle \right|^2 = l^2\,||\mathbf{v}||^2.$$

(Here we used the fact that $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is orthonormal.) □