Proposition 3.5.

(i) *For any integer* $m \neq 0$*,* m *is not divisible by* 0*.*

(ii) *For any* $n \in \mathbb{N}$*,* $n^3 + 2n$ *is divisible by* 3*.*[4]

(iii) *For any* $n \in \mathbb{N}$*,* $n^3 + 5n$ *is divisible by* 6*.*

Proof of (i) and (iii). (i) Suppose, by way of contradiction, that the integer $m \neq 0$ is divisible by 0, i.e., there exists $k \in \mathbb{Z}$ such that $m = k \cdot 0$. But then Proposition 2.5 implies $m = 0$, which is a contradiction to $m \neq 0$. So our assumption must be false, that is, m is not divisible by 0.

(iii) We use induction on n. For $n = 1$, $1^3 + 5 \cdot 1 = 6 = 6 \cdot 1$ is divisible by 6. For the induction step, assume that $n^3 + 5n$ is divisible by 6, i.e., there exists $k \in \mathbb{Z}$ such that $n^3 + 5n = 6k$. Below we prove a lemma that says that 2 divides $n^2 + n$, i.e., there exists $m \in \mathbb{Z}$ such that $n^2 + n = 2m$. Hence

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 8n + 6 = 6k + 3n^2 + 3n + 6 = 6k + 6m + 6 = 6\,(k + m + 1)\,.$$

So we found an integer, namely $k+m+1$, such that $(n+1)^3+5(n+1)$ equals 6 times that integer, so by definition, 6 divides $(n+1)^3 + 5(n+1)$, and our induction is complete. □

Lemma. *For any* $n \in \mathbb{N}$*,* 2 *divides* $n^2 + n$*.*

Proof. We proceed by induction on n. For $n = 1$, $1^2 + 1 = 2 = 2 \cdot 1$. For the induction step, assume that 2 divides $n^2 + n$, i.e., there exists $m \in \mathbb{Z}$ such that $n^2 + n = 2m$. Then

$$(n+1)^2 + n + 1 = n^2 + 3n + 2 = 2m + 2n + 2 = 2(m + n + 1)\ .$$

So we found an integer, namely $m + n + 1$, such that $(n+1)^2 + n + 1$ equals 2 times that integer, so by definition, 2 divides $(n+1)^2 + n + 1$, and our induction is complete. □

[4] By n^3 we mean $n \cdot n \cdot n$.