Proposition 6.6. *The integer m is odd if and only if there exists $k \in \mathbb{Z}$ such that $m = 2k + 1$.*

Proof. Suppose $m = 2k+1$. If m was even then there exists $j \in \mathbb{Z}$ such that $m = 2k+1 = 2j$, and hence $1 = 2(j-k)$ would be even, which is a contradiction. Hence m is odd.

Conversely, suppose m is odd. By Proposition 6.5, we can find $k \in \mathbb{Z}$ and $j = 0$ or 1 such that $m = 2k + j$. However, $j = 0$ would imply that m is even, and so $j = 1$, that is, $m = 2k + 1$. □

Proposition 6.7. *The integer m is even if and only if m^2 is even.*

Proof. Suppose that m is even, that is, there exists $k \in \mathbb{Z}$ such that $m = 2k$. Then

$$m^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$$

is also even.

Conversely, suppose that m is odd. By Proposition 6.6, there exists $k \in \mathbb{Z}$ such that $m = 2k+1$. Then

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$$

is also odd, again by Proposition 6.6. □