MATH 725 Homework Set 6 due 10/3/11

(1) Define the operator $T : \mathscr{P}_3(\mathbf{R}) \to \mathscr{P}_3(\mathbf{R})$ by $T(p)(x) = \frac{d^2}{dx^2}\left(\left(1+x+x^2\right)p(x)\right)$.
  (a) Show that T is linear.
  (b) Find all eigenvalues of T.
  (c) Is T invertible?

Proof. (a) Let $p(x), q(x) \in \mathscr{P}_3(\mathbf{R})$ and $a \in \mathbf{R}$. Then

$$\begin{aligned}\frac{d^2}{dx^2}\left(\left(1+x+x^2\right)\left(a\,p(x)+q(x)\right)\right) &= \frac{d^2}{dx^2}\left(a\left(1+x+x^2\right)p(x)+\left(1+x+x^2\right)q(x)\right)\\ &= a\frac{d^2}{dx^2}\left(\left(1+x+x^2\right)p(x)\right)+\frac{d^2}{dx^2}\left(\left(1+x+x^2\right)q(x)\right).\end{aligned}$$

(b) Fix the standard basis of $\mathscr{P}_3(\mathbf{R})$. The images of these basis vectors under T are

$$\begin{aligned}T(1) &= \frac{d^2}{dx^2}\left(1+x+x^2\right) = 2\\ T(x) &= \frac{d^2}{dx^2}\left(x+x^2+x^3\right) = 2+6x\\ T(x^2) &= \frac{d^2}{dx^2}\left(x^2+x^3+x^4\right) = 2+6x+12x^2\\ T(x^3) &= \frac{d^2}{dx^2}\left(x^3+x^4+x^5\right) = 6x+12x^2+20x^3\end{aligned}$$

and so the matrix representation of T with respect to this basis is upper triangular, with the eigenvalues 1, 6, 12, and 20 (which we can read off the diagonal).

(c) We found a matrix representation of T that is upper triangular with no zero on the diagonal, so by a theorem proved in class T is invertible. $\square$

(2) Suppose V is finite dimensional and $S, T \in L(V)$.
  (a) Prove that ST and TS have the same eigenvalues.
  (b) Show that (a) is false if V is infinite dimensional.

Proof. (a) If $\lambda \in \mathbf{F}$ is an eigenvalue of ST (with eigenvector $\mathbf{v}$) then $S(T(\mathbf{v})) = \lambda\,\mathbf{v}$. Thus

$$T(S(T(\mathbf{v}))) = \lambda\, T(\mathbf{v}),$$

and so λ is also an eigenvalue of TS (with eigenvector $T(\mathbf{v})$), unless $T(\mathbf{v}) = \mathbf{0}$.
So we still need to deal with the case $T(\mathbf{v}) = \mathbf{0}$, which means that $\operatorname{null}(T) \neq \{\mathbf{0}\}$. In this case, $\lambda = 0$ (because $\mathbf{0} = S(T(\mathbf{v})) = \lambda\,\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$). Since V is finite dimensional,

$$\dim V = \dim\operatorname{null}(T) + \operatorname{rank}(T) = \dim\operatorname{null}(TS) + \operatorname{rank}(TS).$$

Because $\operatorname{range}(TS) \subseteq \operatorname{range}(T)$, we conclude that $\dim\operatorname{null}(TS) \geq \dim\operatorname{null}(T) > 0$, so $\operatorname{null}(TS)$ is nontrivial, which means that TS has eigenvalue $\lambda = 0$.
Reversing the roles of S and T shows the other implication.

(b) Let $V = \mathbf{R}^\infty$, the vector space of all sequences in $\mathbf{R}$, and define the linear operators

$$S(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots) \qquad \text{and} \qquad T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Then ST is the identity map (which has 1 as its only eigenvalue) and

$$TS(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

which has 0 as an eigenvalue (with eigenspace $\{(x, 0, 0, \dots) : x \in \mathbf{R}\}$). $\square$

(3) Recall that a matrix M representing a linear operator in $L(V)$ is *diagonizable* if there exists a basis of V with respect to which M has nonzero entries only on its diagonal. Show that the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is diagonizable if and only if $a \neq d$ or $b = 0$.

Proof. If $a \neq d$ then (since M is upper triangular) there are two linearly independent eigenvectors, which thus form a basis of V; writing the linear operator with respect to this basis yields a diagonal matrix. If $b = 0$ then M is diagonal.

Conversely, suppose $a = d$ and $b \neq 0$. Again M is upper triangular, and so by theorem in class, a is the only eigenvalue. The corresponding eigenspace

$$\operatorname{null}(M - a\mathrm{I}) = \operatorname{null}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

is one-dimensional, and thus there does not exist a basis of V consisting of eigenvectors; consequently, M is not diagonalizable. □

(4) Let M and N be $n \times n$ matrices, and let D be a diagonal $n \times n$ matrix, all with entries in $\mathbf{C}$. Prove:
  (a) $MN = ND$ if and only if the diagonal elements of D are eigenvalues of M and the columns of N are the corresponding eigenvectors.
  (b) If N is invertible and $M = NDN^{-1}$ then M is diagonalizable.

Proof. (a) For a matrix A, we use the notation $A[k]$ to denote the k'th column vector of A and $A[j,k]$ to denote the (j,k)-entry of A. Note that $(AB)[k] = A(B[k])$, and if D is diagonal, $(AD)[k] = D[k,k]A[k]$. If $MN = ND$, then

$$M(N[k]) = (MN)[k] = (ND)[k] = D[k,k]\,N[k]\,,$$

in other words, $N[k]$ is an eigenvector of M with eigenvalue $D[k,k]$.
Converseley, if the $D[k,k]$'s are the eigenvalues of M, each coming with the eigenvector $N[k]$, then

$$(MN)[k] = M(N[k]) = D[k,k]\,N[k] = (ND)[k]\,,$$

that is, the columns of the matrices MN and ND are identical. But that means $MN = ND$.

(b) Suppose N is invertible and $M = NDN^{-1}$. The latter implies (by multiplying N on the right) $MN = ND$, and so by (a), the columns of N are the eigenvectors of M. Since N is invertible, these eigenvectors form a basis of $\mathbf{F}^n$, and so (by a theorem proved in class) M is diagonizable. □

(5) Let $U := \left\{(x_1,x_2,x_3,x_4) \in \mathbf{R}^4 : x_1 + x_2 + x_3 + x_4 = 0 = x_1 + 2x_2 + 3x_3 + 4x_4\right\}$. Explicitly construct two different projections P and Q from $\mathbf{R}^4$ onto U by giving matrix representations of P and Q with respect to the standard basis.

Proof. Given any basis $(\mathbf{u}_1, \mathbf{u}_2)$ of the subspace U, the matrices

$$P = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_1\ \mathbf{u}_2] \qquad \text{and} \qquad P = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_2\ \mathbf{u}_1]$$

(written in terms of their columns) will be two such projections. One such basis is formed by $\mathbf{u}_1 = (1,-2,1,0)$ and $\mathbf{u}_2 = (2,-3,0,1)$.

(Here's a generic way to obtain such a basis: Let $u = (x_1,x_2,x_3,x_4) \in U$; then $x_1 + x_2 + x_3 + x_4 = 0 = x_1 + 2x_2 + 3x_3 + 4x_4$. Thus $x_2 + 2x_3 + 3x_4 = 0$, and so we can express x_1 and x_2 in terms of x_3 and x_4:

$$\begin{aligned} x_2 &= -2x_3 - 3x_4 \\ x_1 &= 2x_3 + 3x_4 - x_3 - x_4 = x_3 + 2x_4\,. \end{aligned}$$

So we can write

$$U = \left\{(a+2b, -2a-3b, a, b) \in \mathbf{R}^4 : a, b \in \mathbf{R}\right\}$$

and our above basis comes from choosing (a,b) to be $(1,0)$ and $(0,1)$. □