Proposition 3.19. *The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.*

Proof. Assume (by means of contradiction) there there is an $x \in \mathbb{Z}$ such that $x^2 = -1$. We distinguish two cases:

1. case: $x = 0$. Then $x^2 = 0 \cdot 0 = 0$ (by Proposition 2.5), and so $0 = -1$. Adding 1 to both sides yields $1 = 0$, which contradicts Axiom 2.3.

2. case: $x \neq 0$. Then Proposition 3.18 applies, and we can conclude that $x^2 \in \mathbb{N}$, and so $-1 \in \mathbb{N}$. This implies by Axiom 3.1(ii) that $-1 + 1 = 0 \in \mathbb{N}$, which contradicts Axiom 3.1(iv).

We obtain a contradiction in both cases, so our original assumption must be false, i.e., there is no $x \in \mathbb{Z}$ such that $x^2 = -1$. □