ON STANLEY'S RECIPROCITY THEOREM FOR RATIONAL CONES

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Abstract. We give a short, self-contained proof of Stanley's reciprocity theorem for a rational cone $\mathcal{K} \subset \mathbb{R}^d$. Namely, let $\sigma_{\mathcal{K}}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$. Then $\sigma_{\mathcal{K}}(\mathbf{x})$ and $\sigma_{\mathcal{K}^\circ}(\mathbf{x})$ are rational functions which satisfy the identity $\sigma_{\mathcal{K}}(1/\mathbf{x}) = (-1)^d \sigma_{\mathcal{K}^\circ}(\mathbf{x})$. A corollary of Stanley's theorem is the Ehrhart-Macdonald reciprocity theorem for the lattice-point enumerator of rational polytopes. A distinguishing feature of our proof is that it uses neither the shelling of a polyhedron nor the concept of finite additive measures. The proof follows from elementary techniques in contour integration.

1. Introduction

Let $\mathcal{K}$ be a *rational cone*, that is, of the form

$$\mathcal{K} = \left\{ \mathbf{r} \in \mathbb{R}^d_{\geq 0} : \ \mathbf{A}\,\mathbf{r} \leq 0 \right\}$$

for some integral $d \times m$-matrix $\mathbf{A}$. Thus $\mathcal{K}$ is the positive real span of m vectors in $\mathbb{R}^d$. We assume that $\mathcal{K}$ is d-dimensional and pointed - i.e. $\mathcal{K}$ does not contain a line. In his study of nonnegative integral solutions to linear systems, Stanley was lead to consider the generating functions

$$\sigma_{\mathcal{K}}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$$

and its companion $\sigma_{\mathcal{K}^\circ}(\mathbf{x})$ for the interior $\mathcal{K}^\circ$ of $\mathcal{K}$. Here $\mathbf{x}^{\mathbf{m}}$ denotes the product $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$. The function $\sigma_{\mathcal{K}}$ (as well as $\sigma_{\mathcal{K}^\circ}$) is a rational function in the components of $\mathbf{x}$; this is proved, for example, by triangulating $\mathcal{K}$ into simplicial cones, for which one can explicitly form the rational functions representing their generating functions (see, for example, [5, Chapter 4]). The fundamental reciprocity theorem of Stanley [4] relates the two rational functions $\sigma_{\mathcal{K}}$ and $\sigma_{\mathcal{K}^\circ}$. We abbreviate the vector $(1/x_1, 1/x_2, \ldots, 1/x_d)$ by $1/\mathbf{x}$.

Theorem 1 (Stanley)**.** *As rational functions,* $\sigma_{\mathcal{K}}\left(1/\mathbf{x}\right) = (-1)^d \, \sigma_{\mathcal{K}^\circ}(\mathbf{x})$.

The proof of Stanley's theorem is not quite as simple as that of the rationality of $\sigma_{\mathcal{K}}$. All proofs of Theorem 1 that we are aware use either the concept of shelling of a polyhedron or the concept of finitely additive measures (also called valuations). Both concepts reduce the theorem to simplicial cones, for which Theorem 1 is not hard to prove. In this paper, we give a proof which does neither depend on shelling nor on valuations.

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2. Euler's generating function

We may assume that $\mathcal{K} \subset \mathbb{R}^d_{\geq 0}$, as we can unimodularly transform any rational cone into one in the nonnegative orthant (unimodular transformations preserve the integer lattice). Denote the columns of $\mathbf{A}$ by $\mathbf{a}_1, \dots, \mathbf{a}_d$. Our main tool for proving Theorem 1 is the following lemma, the idea of which goes back to at least Euler [2] and which is proved by simply expanding geometric series.

Lemma 2 (Euler). *$\sigma_{\mathcal{K}}(\mathbf{x})$ equals the constant $\mathbf{z}$-coefficient of the function*

$$\frac{1}{(1 - x_1 \mathbf{z}^{\mathbf{a}_1})(1 - x_2 \mathbf{z}^{\mathbf{a}_2}) \cdots (1 - x_d \mathbf{z}^{\mathbf{a}_d})(1 - z_1)(1 - z_2) \cdots (1 - z_m)}$$

expanded as a power series centered at $\mathbf{z} = 0$.

We will use an integral version of this lemma, for which we need additional variables to avoid integrating over singularities. Let

$$\theta_{\mathcal{K}}(\mathbf{x}, \mathbf{y}) = \theta_{\mathcal{K}}(x_1, \dots, x_d, y_1, \dots, y_m) = \mathrm{const}_{\mathbf{z}} \left(\frac{1}{\prod_{j=1}^d (1 - x_j \mathbf{z}^{\mathbf{a}_j}) \prod_{k=1}^m (1 - y_k z_k)} \right).$$

This is a rational function in the coordinates of $\mathbf{x}$ and $\mathbf{y}$, and $\sigma_{\mathcal{K}}(\mathbf{x}) = \theta_{\mathcal{K}}(\mathbf{x}, 1, 1, \dots, 1)$. We translate this constant-term identity into an integral identity. We abbreviate the measure $\frac{dz_1}{z_1} \frac{dz_2}{z_2} \cdots \frac{dz_m}{z_m}$ by $\frac{d\mathbf{z}}{\mathbf{z}}$, and we write $|\mathbf{x}| < 1$ to indicate that $|x_1|, |x_2|, \dots, |x_d| < 1$. Then for $|\mathbf{x}|, |\mathbf{y}| < 1$

$$\theta_{\mathcal{K}}(\mathbf{x}, \mathbf{y}) = \int \frac{1}{\prod_{j=1}^d (1 - x_j \mathbf{z}^{\mathbf{a}_j}) \prod_{k=1}^m (1 - y_k z_k)} \frac{d\mathbf{z}}{\mathbf{z}} ,$$

where the integral sign stands for $1/(2\pi i)^m$ times an m-fold integral, each one over the unit circle.

Analogously, we have for the open generating function

$$\sigma_{\mathcal{K}^\circ}(\mathbf{x}) = \mathrm{const} \left(\frac{x_1 \mathbf{z}^{\mathbf{a}_1}}{1 - x_1 \mathbf{z}^{\mathbf{a}_1}} \frac{x_2 \mathbf{z}^{\mathbf{a}_2}}{1 - x_2 \mathbf{z}^{\mathbf{a}_2}} \cdots \frac{x_d \mathbf{z}^{\mathbf{a}_d}}{1 - x_d \mathbf{z}^{\mathbf{a}_d}} \frac{z_1}{1 - z_1} \frac{z_2}{1 - z_2} \cdots \frac{z_m}{1 - z_m} \right)$$

and we define

$$\theta_{\mathcal{K}^\circ}(\mathbf{x}, \mathbf{y}) = \mathrm{const}_{\mathbf{z}} \left(\prod_{j=1}^d \frac{x_j \mathbf{z}^{\mathbf{a}_j}}{1 - x_j \mathbf{z}^{\mathbf{a}_j}} \prod_{k=1}^m \frac{y_k z_k}{1 - y_k z_k} \right).$$

Then $\sigma_{\mathcal{K}^\circ}(\mathbf{x}) = \theta_{\mathcal{K}^\circ}(\mathbf{x}, 1, 1, \dots, 1)$ and for $|\mathbf{x}|, |\mathbf{y}| < 1$

$$\theta_{\mathcal{K}^\circ}(\mathbf{x}, \mathbf{y}) = \int \prod_{j=1}^d \frac{x_j \mathbf{z}^{\mathbf{a}_j}}{1 - x_j \mathbf{z}^{\mathbf{a}_j}} \prod_{k=1}^m \frac{y_k z_k}{1 - y_k z_k} \frac{d\mathbf{z}}{\mathbf{z}} .$$

The integral representations of $\theta_{\mathcal{K}}$ and $\theta_{\mathcal{K}^\circ}$ now suggest how to prove Theorem 1—make a change of variables $\mathbf{z} \to 1/\mathbf{z}$, say in $\theta_{\mathcal{K}^\circ}$:

$$\theta_{\mathcal{K}^\circ}(\mathbf{x}, \mathbf{y}) = \int \prod_{j=1}^d \frac{x_j \mathbf{z}^{-\mathbf{a}_j}}{1 - x_j \mathbf{z}^{-\mathbf{a}_j}} \prod_{k=1}^m \frac{y_k z_k^{-1}}{1 - y_k z_k^{-1}} \frac{d\mathbf{z}}{\mathbf{z}} .$$

Hence the rational function $\theta_{\mathcal{K}^\circ}(1/\mathbf{x}, 1/\mathbf{y})$ has the integral representation

$$\begin{aligned}\theta_{\mathcal{K}^\circ}(1/\mathbf{x}, 1/\mathbf{y}) &= \int \prod_{j=1}^{d} \frac{x_j^{-1}\mathbf{z}^{-\mathbf{a}_j}}{1 - x_j^{-1}\mathbf{z}^{-\mathbf{a}_j}} \prod_{k=1}^{m} \frac{y_k^{-1}z_k^{-1}}{1 - y_k^{-1}z_k^{-1}} \frac{d\mathbf{z}}{\mathbf{z}} \\ &= \int \prod_{j=1}^{d} \frac{1}{x_j\mathbf{z}^{\mathbf{a}_j} - 1} \prod_{k=1}^{m} \frac{1}{y_k z_k - 1} \frac{d\mathbf{z}}{\mathbf{z}} \\ &= (-1)^{d+m} \int \prod_{j=1}^{d} \frac{1}{1 - x_j\mathbf{z}^{\mathbf{a}_j}} \prod_{k=1}^{m} \frac{1}{1 - y_k z_k} \frac{d\mathbf{z}}{\mathbf{z}} \ ,\end{aligned}$$

valid for $|\mathbf{x}|, |\mathbf{y}| > 1$. It remains to prove that the rational function given by the integral

$$\int \frac{1}{\prod_{j=1}^{d}\left(1 - x_j\mathbf{z}^{\mathbf{a}_j}\right)\prod_{k=1}^{m}\left(1 - y_k z_k\right)} \frac{d\mathbf{z}}{\mathbf{z}} \tag{1}$$

with $|\mathbf{x}|, |\mathbf{y}| < 1$ equals the rational function given by the integral

$$(-1)^m \int \frac{1}{\prod_{j=1}^{d}\left(1 - x_j\mathbf{z}^{\mathbf{a}_j}\right)\prod_{k=1}^{m}\left(1 - y_k z_k\right)} \frac{d\mathbf{z}}{\mathbf{z}} \tag{2}$$

with $|\mathbf{x}|, |\mathbf{y}| > 1$.

Let us consider as the innermost integral the one with respect to z_1. Almost all of the poles of the integrand $f(z_1)$ are at the solutions z_1 of the equations $1 - x_j\mathbf{z}^{\mathbf{a}_j} = 0$ (for $j = 1, \dots, d$) and $1 - y_1 z_1 = 0$. Since $|z_2|, |z_3|, \dots, |z_m| = 1$, each z_1-pole is inside or outside the unit circle, depending on the exponent of z_1 in $\mathbf{z}^{\mathbf{a}_j}$ and on whether a given x_j or y_1 has magnitude smaller or larger than 1. But this means that the z_1-integrals in (1) and (2) pick up the residues of complementary singularities.

The only other potential poles are at zero and infinity, induced by the extra factor of $\frac{1}{z_1}$. We claim that there are no residues at these poles, if they exist. Indeed, as z_1 approaches zero, the residue $z_1 f(z_1)$ is the product of the factors $\frac{1}{1 - x_j\mathbf{z}^{\mathbf{a}_j}}$ and $\frac{1}{1 - y_k z_k}$. Since K is in the nonnegative orthant, in each inequality at least one coefficient must be negative; therefore, in at least one of the former factors, the exponent of z_1 must be negative. This factor then goes to zero as z_1 does, while the norms of all of the other factors either go to zero (if the exponent of z_1 is negative), one (if the exponent is positive), or a constant (if z_1 does not appear at all.) Therefore, this residue is equal to zero.

A similar argument eliminates the residue at infinity. After a change of variables $z_1 \to \frac{1}{z_1}$, this residue at infinity is (up to sign) the limit of the product of these same factors. However, the factor $\frac{1}{1 - y_1 z_1}$ goes to zero, while all other factors again go to zero, one, or a constant, depending on whether the exponent of z_1 is positive, negative, or zero respectively. This completes the argument that the z_1-integrals in (1) and (2) differ by a minus sign.

We can use the same argument for the next variable if we know that the z_1-integral results in a rational function with a similar-looking denominator as the integrands in (1) and (2). But this follows from Euler's generating function: The z_1-integral in (1) gives the generating function of the cone described by $\mathbf{r} \geq 0$ and the first row inequality in $\mathbf{A}\,\mathbf{r} \leq 0$, in the variables

$x_1\left(z_2,\ldots,z_m\right)^{\mathbf{a}'_1},\ldots,x_d\left(z_2,\ldots,z_m\right)^{\mathbf{a}'_d}$ and $y_2,\ldots,y_m$, where $\mathbf{a}'_1,\ldots,\mathbf{a}'_d$ are the column vectors of $\mathbf{A}$ after we removed the first row. It is not hard to show (from the simplicial case) that the generating function of any cone has as denominator a product with terms of the form 1 minus a monomial of the variables, so the z_2-integrand has the desired form.

This process can be repeated for the integrals in all the other variables. Each time we consider a new integral, we obtain the generating function of the cone described by one more row constraint of $\mathbf{A}\,\mathbf{r}\le 0$ than in the previous step. Since the integrals with respect to each variable in (1) and (2) differ by a minus sign, we get at the end a factor $(-1)^m$. This completes the proof.

Well... almost. Can anyone tell me what's going on with the possible poles at 0 and ∞? Mike convinced me that there are no such poles in the first variable, but I'm not sure what happens with the next and so on.

3. Extensions and applications

An interesting extension of Theorem 1, already realized by Stanley [4], is the following: Let

$$\mathcal{K}_1=\left\{\mathbf{r}\in\mathbb{R}^d_{\ge 0}:\ \mathbf{A}\,\mathbf{r}\le 0 \text{ and } \mathbf{B}\,\mathbf{r}<0\right\}$$

and

$$\mathcal{K}_2=\left\{\mathbf{r}\in\mathbb{R}^d_{>0}:\ \mathbf{A}\,\mathbf{r}<0 \text{ and } \mathbf{B}\,\mathbf{r}\le 0\right\},$$

that is, $\mathcal{K}_1$ and $\mathcal{K}_2$ are *half-open* cones whose constraints are defined by the same matrix, however, those facets (codimension-1 faces) which are contained in $\mathcal{K}_1$ are not in $\mathcal{K}_2$ and vice versa. (The condition that $\mathcal{K}_1\subseteq\mathbb{R}^d_{\ge 0}$ while $\mathcal{K}_2\subseteq\mathbb{R}^d_{>0}$ is there to keep the exposition simple; one could also include *some* of the coordinate hyperplanes in $\mathcal{K}_1$ and the others in $\mathcal{K}_2$.) The extension of Theorem 1 is as follows.

Theorem 3 (Stanley)**.** *As rational functions, $\sigma_{\mathcal{K}_1}(1/\mathbf{x})=(-1)^d\,\sigma_{\mathcal{K}_2}(\mathbf{x})$.*

Our proof of Theorem 1 is easily adjusted to this more general setting. Suppose $\mathbf{A}\in\mathbb{Z}^{d\times m}$ has columns $\mathbf{a}_1,\ldots,\mathbf{a}_d$, and $\mathbf{B}\in\mathbb{Z}^{d\times n}$ has columns $\mathbf{b}_1,\ldots,\mathbf{b}_d$, then Euler's Lemma 2 gives

$$\sigma_{\mathcal{K}_1}(\mathbf{x})=\operatorname{const}_{\mathbf{z},\mathbf{w}}\left(\frac{1}{\left(1-x_1\mathbf{z}^{\mathbf{a}_1}\mathbf{w}^{\mathbf{b}_1}\right)\cdots\left(1-x_d\mathbf{z}^{\mathbf{a}_d}\mathbf{w}^{\mathbf{b}_d}\right)}\frac{1}{(1-z_1)\cdots(1-z_m)}\frac{w_1}{1-w_1}\cdots\frac{w_n}{1-w_n}\right)$$

and

$$\sigma_{\mathcal{K}_2}(\mathbf{x})=\operatorname{const}_{\mathbf{z},\mathbf{w}}\left(\frac{x_1\mathbf{z}^{\mathbf{a}_1}}{1-x_1\mathbf{z}^{\mathbf{a}_1}\mathbf{w}^{\mathbf{b}_1}}\cdots\frac{x_d\mathbf{z}^{\mathbf{a}_d}\mathbf{w}^{\mathbf{b}_d}}{1-x_d\mathbf{z}^{\mathbf{a}_d}}\frac{z_1}{1-z_1}\cdots\frac{z_m}{1-z_m}\frac{1}{(1-w_1)\cdots(1-w_m)}\right).$$

The proof that

$$\theta_{\mathcal{K}_1}(\mathbf{x},\mathbf{y})=\int\prod_{j=1}^{d}\frac{1}{1-x_j\mathbf{z}^{\mathbf{a}_j}\mathbf{w}^{\mathbf{b}_j}}\prod_{k=1}^{m}\frac{1}{1-y_kz_k}\prod_{i=1}^{n}\frac{y_{m+i}w_i}{1-y_{m+i}w_i}\frac{d\mathbf{z}}{\mathbf{z}}\frac{d\mathbf{w}}{\mathbf{w}}$$

and

$$\theta_{\mathcal{K}_2}(\mathbf{x},\mathbf{y})=\int\prod_{j=1}^{d}\frac{x_j\mathbf{z}^{\mathbf{a}_j}\mathbf{w}^{\mathbf{b}_j}}{1-x_j\mathbf{z}^{\mathbf{a}_j}\mathbf{w}^{\mathbf{b}_j}}\prod_{k=1}^{m}\frac{y_kz_k}{1-y_kz_k}\prod_{i=1}^{n}\frac{1}{1-y_{m+i}w_i}\frac{d\mathbf{z}}{\mathbf{z}}\frac{d\mathbf{w}}{\mathbf{w}},$$

both defined for $|\mathbf{x}|, |\mathbf{y}| < 1$, satisfy a Stanley-type reciprocity identity proceeds along the exact same lines as our proof of Theorem 1.

A particular nice application of Theorem 1 concerns the counting function $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$ for a rational convex polytope $\mathcal{P}$, that is, the convex hull of finitely many points in $\mathbb{Q}^d$. Ehrhart proved in [1] the fundamental structural result about $L_{\mathcal{P}}(t)$, namely that it is a quasi-polynomial in t (for a definition and nice discussion of quasi-polynomials see [5, Chapter 4]). Ehrhart conjectured and partially proved the following reciprocity theorem, which was proved by Macdonald [3].

Theorem 4 (Ehrhart-Macdonald)**.** *The quasi-polynomials $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ satisfy*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t) \ .$$

As with Stanley's theorem, we are not aware of any proof of Ehrhart-Macdonald reciprocity which does not rely on shellings of a polytope or finite additive measures.

It is a fun exercise to deduce Theorem 4 from Theorem 1, for example by considering the generating function of the $(d+1)$-cone generated by $(\mathbf{v}_1, 1), \dots, (\mathbf{v}_n, 1)$, where $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the vertices of $\mathcal{P}$, applying Stanley reciprocity, and then specializing the rational generating functions by setting the first d variables to 1.

Finally, there exists an extension of Theorem 4 corresponding to Theorem 3: one includes some of the facets of the polytope on one side, and the complementary set of facets on the other side.

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