A real short proof of Pick's theorem

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We study the number of integer points in polytopes, which live in some real space $\mathbb{R}^n$. Since the integer points $\mathbb{Z}^n$ form a lattice in $\mathbb{R}^n$, we frequently call them **lattice points**. The first interesting case is dimension $n = 2$. Consider a simple, closed polygon whose vertices have integer coordinates. Denote the number of integer points inside the polygon by I, and the number of integer points on the polygon by B. In 1899, Pick ([Pi]) discovered the astonishing fact that the area A inside the polygon can be computed simply by counting lattice points:

Theorem 1 (Pick)

$$A = I + \frac{1}{2}B - 1 \ .$$

We give an elementary proof of Pick's theorem, essentially due to Varberg ([Va]).

Proof. We start by proving that Pick's identity has an additive character: suppose our polygon has more than 3 vertices. Then we can write the 2-dimensional polytope $\mathcal{P}$ bounded by our polygon as the union of two 2-dimensional polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, such that the interiors of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not meet. Both have fewer vertices than $\mathcal{P}$. We claim that the validity of Pick's identity for $\mathcal{P}$ is equivalent to the validity of Pick's identity for $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote the area, number of interior lattice points, and number of boundary lattice points of $\mathcal{P}_k$ by A_k, I_k, and B_k, respectively, for $k = 1, 2$. Clearly,

$$A = A_1 + A_2 \ .$$

Furthermore, if we denote the number of lattice points on the edges common to $\mathcal{P}_1$ and $\mathcal{P}_2$ by L, then

$$I = I_1 + I_2 + L - 2 \qquad \text{and} \qquad B = B_1 + B_2 - 2L + 2 \ .$$

Hence

$$\begin{aligned} I + \frac{1}{2}B - 1 &= I_1 + I_2 + L - 2 + \frac{1}{2}B_1 + \frac{1}{2}B_2 - L + 1 - 1 \\ &= I_1 + \frac{1}{2}B_1 - 1 + I_2 + \frac{1}{2}B_2 - 1 \ . \end{aligned}$$

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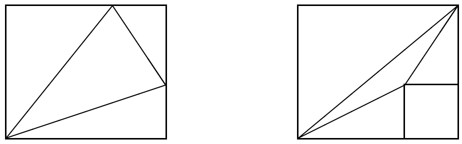
This proves the claim. Therefore, we can triangulate $\mathcal{P}$, and it suffices to prove Pick's theorem for triangles. Moreover, by further triangulations, we may assume that there are no lattice points on the boundary of the triangle other than the vertices. To prove Pick's theorem for such triangles, embed them into rectangles:



Again by additivity, we conclude that it suffices to prove Pick's theorem for rectangles and rectangular triangles, which have no lattice points on the hypothenuse, and whose other two sides are parallel to a coordinate axis. If these two sides have lengths a and b, respectively, we have

$$A = \frac{1}{2}ab \qquad \text{and} \qquad B = a + b + 1 \ .$$

Furthermore, by thinking of the triangle as 'half' of a rectangle, we obtain

$$I = \frac{1}{2}(a-1)(b-1) \ .$$

(Here it is crucial that there are no points on the hypothenuse.) Pick's identity is now a straightforward consequence for these triangles. Finally, for a rectangle whose sides have length a and b, it is easy to see that

$$A = ab, \ I = (a-1)(b-1), \ B = 2a + 2b,$$

and Pick's theorem follows for rectangles, which finishes our proof. $\square$

References

[Pi] G. Pick, Geometrisches zur Zahlenlehre, *Sitzungsber. Lotos (Prague)* **19** (1899), 311-319.

[Va] D. E. Varberg, Pick's theorem revisited, *Amer. Math. Monthly* **92** (1985), 584-587.