Chromatic Polynomials, Symmetric Functions & Friends

Matthias Beck
San Francisco State University

Esme Bajo
San Diego Miramar College

Andrés R. Vindas-Meléndez
Harvey Mudd College

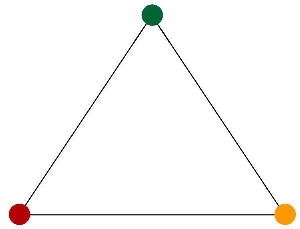
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example: $\chi_{K_3}(k) = k(k-1)(k-2)$



(Theorem due to Birkhoff 1912, Whitney 1932)

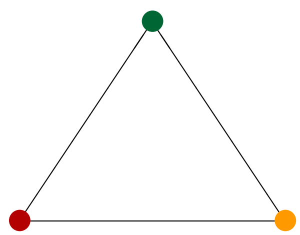
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \dots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#$ (proper n-colorings of G)

Example: $\chi_{K_3}(k) = k(k-1)(k-2)$



(Theorem due to Birkhoff 1912, Whitney 1932)

Class of chromatic polynomials $\longrightarrow$ two main research problems:

- Classification — which polynomials are chromatic?
- Detection — does a given polynomial determine the graph?

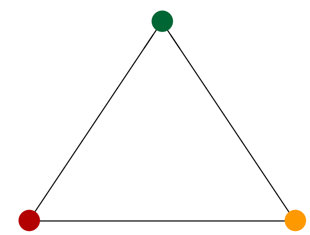
Chromatic Polynomials

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Example: $\chi_{K_3}(k) = k(k-1)(k-2)$



(Theorem due to Birkhoff 1912, Whitney 1932)

Polynomial classes in Combinatorics $\longrightarrow$ two main research problems:

- Classification — which polynomials are ...?
- Detection — does a given polynomial determine the ...?

Chromatic Polynomials

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

- Classification — which polynomials are chromatic?

... wide open, though we have structural results:

- $\chi_G(n)$ is monic, has constant term 0 and degree $|V|$.
- The coefficients of $\chi_G(n)$ alternate in sign.
- $|\chi_G(-1)|$ equals $\#$ acyclic orientations of G (Stanley 1973).
- The coefficients of $\chi_G(n)$ are unimodal (Huh 2012).

Chromatic Polynomials

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

- Detection — does a given polynomial determine the graph?

... fails spectacularly: If T is a tree with m edges then

$$\chi_T(n) \;=\; n(n-1)^m$$

Chromatic Symmetric Functions

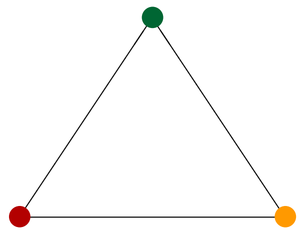
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \ldots) \ := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \ = \ 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



Chromatic Symmetric Functions

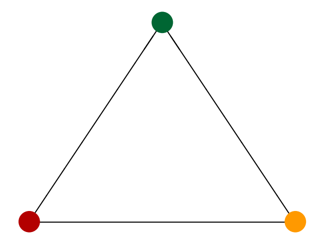
$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \; := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

Example: $X_{K_3}(k) \; = \; 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$



We recover $\chi_G(n) \; = \; X_G(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, 0, \dots)$

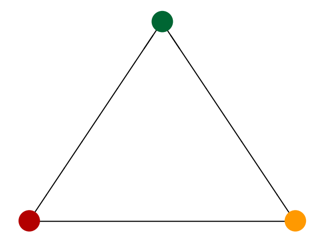
Chromatic Symmetric Functions

$G = (V, E)$ — graph (without loops)

Proper coloring — $\kappa : V \to \mathbb{Z}_{>0}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic symmetric function

$$X_G(x_1, x_2, \ldots) \; := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$



Example: $X_{K_3}(k) \; = \; 6\, x_1\, x_2\, x_3 + 6\, x_1\, x_2\, x_4 + \cdots$

We recover $\chi_G(n) \; = \; X_G(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

(Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$ distinguishes trees.

q-Chromatic Polynomials

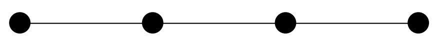
Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$

q-Chromatic Polynomials

Definition $\chi_G^{\lambda}(q,n) := \displaystyle\sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$

Example 

$$\begin{aligned}
\chi_{P_4}^{\mathbf{1}}(q,n) \quad = \quad & \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \\
& \Big(8q^{10}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}(1-q)^{n-3} \\
& +(4q^9+6q^8+4q^7)(1-q)^{n+1}(1-q)^n(1-q)^{n-1}(1-q)^{n-2} \\
& +2q^6(1-q)^{n+2}(1-q)^{n+1}(1-q)^n(1-q)^{n-1} \Big)
\end{aligned}$$

q-Chromatic Polynomial Structure

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1 + q + \cdots + q^{n-1}$$

q-Chromatic Polynomial Structure

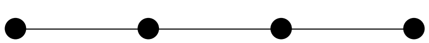
$$\chi_G^\lambda(q,n) \; := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \; = \; \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1 + q + \cdots + q^{n-1}$$

Example

$$\widetilde{\chi}_{P_4}^{\mathbf{1}}(q,x) \; = \; \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \Big(\left(2q^8+4q^7+6q^6+4q^5+8q^4\right) x^4 - \left(6q^8+10q^7+18q^6+18q^5+20q^4\right) x^3 + \left(4q^8+10q^7+20q^6+22q^5+16q^4\right) x^2 - \left(4q^7+8q^6+8q^5+4q^4\right) x \Big)$$



Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^\lambda(q,n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q,n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q,x)$ distinguishes trees.

Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Remarks $\chi_G^{\mathbf{1}}(q, n)$ was previously studied by Loebl (2007).

$\chi_G^{\lambda}(q, n)$ is a special evaluation (with polynomial structure) of Crew–Spirkl's (2020) weighted chromatic symmetric function.

Why?

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

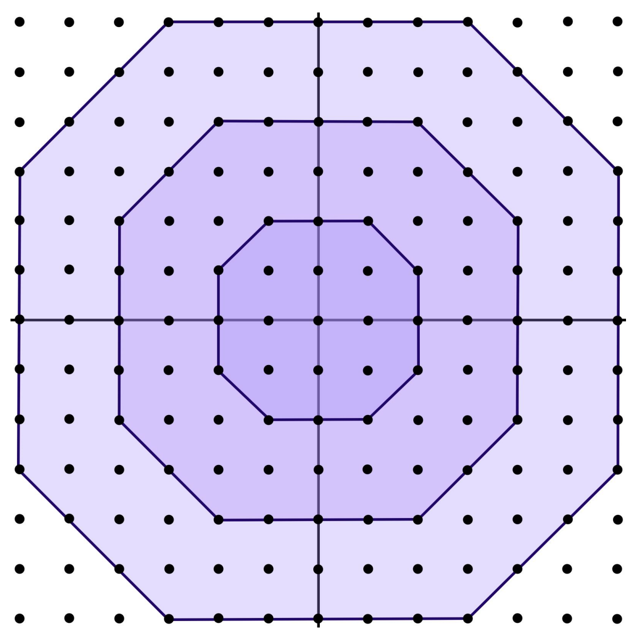
Actually... The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ is related to $X_G(q, q^2, q^3, \ldots)$ and so is $\widetilde{\chi}_G^{\mathbf{1}}(q, \frac{1}{1-q})$

Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $n \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(n) := \# \left(n\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in n. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left(n\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.



Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $n \in \mathbb{Z}_{>0}$ let $L_{\mathcal{P}}(n) := \# \left(n\mathcal{P} \cap \mathbb{Z}^d\right)$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in n. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left(n\mathcal{P}^\circ \cap \mathbb{Z}^d\right)$.

Example $(\Pi, \preceq)$ — (finite) partially ordered set $\longrightarrow$

$$\Omega_{\Pi}^{(\circ)}(n) := \# \text{ (strictly) order-preserving maps } \Pi \to [n]$$

Observation $\chi_G(n) = \sum_{\rho \in A(G)} \Omega^\circ_{\Pi_\rho}(n)$

where $A(G)$ is the set of acyclic orientations of G and Π_ρ is the poset corresponding to ρ

Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

$$L_{\mathcal{P}}(n) := \# \left(n\mathcal{P} \cap \mathbb{Z}^d \right) \qquad\qquad \chi_G(n) = \sum_{\rho \in A(G)} \Omega^{\circ}_{\Pi_\rho}(n)$$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in n. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left(n\mathcal{P}^{\circ} \cap \mathbb{Z}^d \right)$.

Now fix a linear form λ and let $L^{\lambda}_{\mathcal{P}}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}^{\lambda}_{\mathcal{P}}(q,x) \in \mathbb{Z}(q)[x]$ such that $L^{\lambda}_{\mathcal{P}}(q,n) = \widetilde{L}^{\lambda}_{\mathcal{P}}(q,[n]_q)$.

Where does all this come from?

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

$$L_{\mathcal{P}}(n) := \# \left(n\mathcal{P} \cap \mathbb{Z}^d \right) \qquad\qquad \chi_G(n) = \sum_{\rho \in A(G)} \Omega^{\circ}_{\Pi_\rho}(n)$$

Theorem (Ehrhart 1962, Macdonald 1971) $L_{\mathcal{P}}(n)$ is a polynomial in n. Furthermore, $L_{\mathcal{P}}(-n) = (-1)^{\dim \mathcal{P}} \# \left(n\mathcal{P}^{\circ} \cap \mathbb{Z}^d \right)$.

Now fix a linear form λ and let $L^{\lambda}_{\mathcal{P}}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}^{\lambda}_{\mathcal{P}}(q,x) \in \mathbb{Z}(q)[x]$ such that $L^{\lambda}_{\mathcal{P}}(q,n) = \widetilde{L}^{\lambda}_{\mathcal{P}}(q,[n]_q)$. Furthermore,

$$\widetilde{L}^{\lambda}_{\mathcal{P}}\left(\tfrac{1}{q}, [-n]_{\frac{1}{q}}\right) = (-1)^{\dim \mathcal{P}} \sum_{\mathbf{m} \in n\mathcal{P}^{\circ}} q^{\lambda(\mathbf{m})}$$

Where does all this come from?

Fix a linear form λ and let $L_{\mathcal{P}}^{\lambda}(q,n) := \sum_{\mathbf{m} \in n\mathcal{P}} q^{\lambda(\mathbf{m})}$

Theorem (Chapoton 2015) Under some mild assumptions, there exists a polynomial $\widetilde{L}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $L_{\mathcal{P}}^{\lambda}(q,n) = \widetilde{L}_{\mathcal{P}}^{\lambda}(q,[n]_q)$. Furthermore,

$$\widetilde{L}_{\mathcal{P}}^{\lambda}\left(\frac{1}{q},[-n]_{\frac{1}{q}}\right) = (-1)^{\dim \mathcal{P}} \sum_{\mathbf{m} \in n\mathcal{P}^\circ} q^{\lambda(\mathbf{m})}$$

Extensions (MB–Kunze 2025+)

- Explicit formulas in terms of the vertex cones of $\mathcal{P}$
- Bounds on the poles of the cofficients
- Behavior as $n \to \infty$ via $x = \frac{1}{1-q}$
- Quasipolynomials for rational polytopes

q-Chromatic Polynomial Formulas

$$\chi_G^\lambda(q,n) \;:=\; \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \;=\; \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+)

$$\widetilde{\chi}_G^\lambda(q,x) \;=\; q^{\Lambda_V} \sum_{\text{flats } S\subseteq E} \mu(\varnothing,S) \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

where $P(S)$ denotes the collection of vertex sets of the connected components induced by S and $\Lambda_W := \sum_{v\in W}\lambda_v$. In particular, for a tree

$$\widetilde{\chi}_T^\lambda(q,x) \;=\; q^{\Lambda_V} \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

$\longrightarrow$ highly-structured formulas for paths, stars, . . .

The Leading Coefficient for Trees

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)} \ = \ \widetilde{\chi}_G^\lambda(q, [n]_q)$$

Corollary Given a tree T, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$\begin{aligned} c_T^{\mathbf{1}}(q) &= (q-q^2)^d \sum_{S \subseteq E} \prod_{C \in P(S)} \frac{1}{1-q^{\Lambda_C}} \\ &= \frac{1}{[d]_q!} \sum_{(\rho,\sigma)} q^{d + \operatorname{maj} \sigma} \qquad\qquad d := |V| \end{aligned}$$

where the sum ranges over all pairs of acyclic orientations ρ of T and linear extensions σ of the poset induced by ρ

Corollary $c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, X_T\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \dots\right)$

G-Partitions

Given a poset $P = ([d], \preceq)$, a strict P-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_j < m_k \ \text{ whenever } \ j \prec k$$

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function

$$P_G(q) \ := \ \sum_{n>0} p_G(n)\, q^n \ = \ X_G(q, q^2, q^3, \dots)$$

G-Partitions

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \dots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \;\; \text{whenever} \;\; vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Theorem

$$P_G(q) \;=\; \frac{q^{\binom{d+1}{2}} \sum_{(\rho,\sigma)} q^{-\operatorname{maj}\sigma}}{(1-q)(1-q^2)\cdots(1-q^d)}$$

where the sum ranges over all pairs of acyclic orientations ρ of G and linear extensions σ of the poset induced by ρ

G-Partitions

Given a (simple) graph $G = ([d], E)$, a G-partition of $n \in \mathbb{Z}_{>0}$ is a tuple $(m_1, \ldots, m_d) \in \mathbb{Z}_{>0}^d$ such that

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_v \neq m_w \ \text{ whenever } \ vw \in E$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$

Collorary Given a tree T on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q, n)$ equals

$$c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, P_T\left(\frac{1}{q}\right)$$

Conjecture The G-partition function $p_G(n)$ distinguishes trees.

Stanley's Tree Conjecture Revisited

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Theorem (MB–Braun–Cornejo 2025+) Fix $k \geq d$ and $\lambda_j := k^j$. Then $\widetilde{\chi}_G^{\lambda}(q, x)$ distinguishes graphs on d nodes.