MATH 725 Final Exam 12/16/11 10:45 a.m. – 1:15 p.m.

Show complete work—that is, all the steps needed to completely justify your answer. You may refer to theorems in the book (without referencing theorem numbers etc.).

(1) Let V be a vector space.
  (a) Carefully define *eigenvalues*, *eigenvectors* and *generalized eigenvectors* of $T \in L(V)$.
  (b) Now suppose $T \in L(V)$ is invertible. Show that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.
  (c) Prove that T and T^{-1} have the same generalized eigenspaces.

Proof. (b) First note that, since T is invertible, $\text{null}(T) = \{\mathbf{0}\}$, and so 0 cannot be an eigenvalue of T. Now

$$\begin{aligned} \lambda \text{ is an eigenvalue of } T &\iff T(\mathbf{v}) = \lambda\,\mathbf{v} \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\iff \mathbf{v} = T^{-1}(\lambda\,\mathbf{v}) \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\iff \tfrac{1}{\lambda}\,\mathbf{v} = T^{-1}(\mathbf{v}) \text{ for some } \mathbf{v} \neq \mathbf{0} \\ &\iff \tfrac{1}{\lambda} \text{ is an eigenvalue of } T^{-1}. \end{aligned}$$

(c) Let $n = \dim V$, and consider an eigenvalue λ of T. Then

$$\text{null}(T - \lambda \mathrm{I})^n = \text{null}\left(\lambda T\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)\right)^n = \text{null}\left((\lambda T)^n\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n\right)$$

for some $j \in \mathbf{Z}_{>0}$. (Here the last equality holds because T commutes with both I and T^{-1}.) But since λT is invertible,

$$\text{null}\left((\lambda T)^n\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n\right) = \text{null}\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n,$$

and so we have

$$\text{null}(T - \lambda \mathrm{I})^n = \text{null}\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n,$$

in words: the generalized eigenspace of T corresponding to λ equals the generalized eigenspace of T^{-1} corresponding to $\frac{1}{\lambda}$. □

(2) Let V be a complex inner-product space.
  (a) Define what it means for $T \in L(V)$ to be *normal* and what it means for T to be *self adjoint*.
  (b) Prove that a normal operator in $L(V)$ is self adjoint if and only if all its eigenvalues are real. (*Hint:* you may use the spectral theorem.)

Proof of (b). Suppose T is self adjoint, and λ is an eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then

$$\lambda\,||\mathbf{v}||^2 = \langle \lambda\,\mathbf{v}, \mathbf{v} \rangle = \langle T(\mathbf{v}), \mathbf{v} \rangle = \langle \mathbf{v}, T(\mathbf{v}) \rangle = \langle \mathbf{v}, \lambda \mathbf{v} \rangle = \overline{\lambda}\,||\mathbf{v}||^2$$

and so, since $||\mathbf{v}|| \neq 0$, $\lambda = \overline{\lambda}$, i.e., $\lambda \in \mathbf{R}$.

Conversely, suppose all eigenvalues of $T \in L(V)$ are real. By the (complex version of the) spectral theorem, there exists an orthonormal basis of eigenvectors of T, and with respect to this basis, T has a diagonal matrix whose diagonal entries are the eigenvalues. But since these entries are real, $T^* = T$ (neither conjugation nor transposing changes the matrix), i.e., T is self adjoint. □

(3) Let V be an inner-product space.
   (a) Define what it means for $T \in L(V)$ to be an *isometry*.
   (b) Suppose n is an odd positive integer and $T \in L(\mathbf{R}^n)$ is an isometry. Prove that T has eigenvalue 1 or -1. (*Hint:* you may use the existence of a certain block-diagonal form of a matrix of T.)

Proof of (b). We proved in class that there exists an orthonormal basis with respect to which T has block-diagonal form, with 1×1 blocks (of the form ± 1) and 2×2 blocks. Since n is odd, there must be a 1×1 block, and so there must be an eigenvalue ± 1. □

(4) (a) Carefully define the *characteristic* and *minimal polynomials* of an operator $T \in L(\mathbf{C}^n)$.
   (b) Describe what a *Jordan normal form* for T is.
   (c) If $T \in L(V)$ has minimal polynomial $(x-28)^3(x-34)$ and characteristic polynomial $(x-28)^6(x-34)^2$, what are the possible different Jordan normal forms for T?

Solution for (c). Because the minimal polynomial of T is $(x-28)^3(x-34)$, all the Jordan forms must have a 3×3 Jordan block with eigenvalue 28 (of the form $\begin{bmatrix} 28 & 1 & 0 \\ 0 & 28 & 1 \\ 0 & 0 & 28 \end{bmatrix}$) and a 1×1 blocks with eigenvalue 34. Since the characteristic polynomial is $(x-28)^6(x-34)^2$, the possible variations are
   (i) two 3×3 Jordan blocks with eigenvalue 28 and two 1×1 blocks with eigenvalue 34,
   (ii) one 3×3 Jordan block with eigenvalue 28, one 2×2 Jordan block with eigenvalue 28, one 1×1 Jordan block with eigenvalue 28, and two 1×1 blocks with eigenvalue 34, and
   (iii) one 3×3 Jordan block with eigenvalue 28, three 1×1 Jordan blocks with eigenvalue 28, and two 1×1 blocks with eigenvalue 34. □

(5) (a) Define the *determinant* of $T \in L(\mathbf{C}^n)$.
   (b) Suppose $x_1, x_2, \ldots, x_n \in \mathbf{C}$, and let $A \in L(\mathbf{C}^n)$ be given in matrix form (with respect to the standard basis of $\mathbf{C}^n$)

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

   Viewing $x_1, x_2, \ldots, x_n$ as variables, prove that $\det(A)$ is a polynomial in $x_1, x_2, \ldots, x_n$ of (total) degree at most $\frac{n(n-1)}{2}$.
   (c) Show that $\det(A) = 0$ if $x_j = x_k$ for some $j \neq k$, and conclude that $x_k - x_j$ divides $\det(A)$.
   (d) Prove that

$$\det(A) = \prod_{1 \le j < k \le n} (x_k - x_j).$$

   (*Hint:* use (b) and (c) to show that $\det(A) = c \prod_{1 \le j < k \le n} (x_k - x_j)$ for some constant c, and then compute the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ on both sides.)

Proof. (b) The determinant formula for a matrix we proved in class gives

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{j=1}^{n} x_{\pi(j)}^{j-1}$$

which is a polynomial of degree at most $\sum_{j=1}^{n}(j-1) = \frac{n(n-1)}{2}$.

(c) If $x_j = x_k$ for some $j \neq k$ then two rows of A are equal, in which case we know that $\det(A) = 0$. Viewing $\det(A)$ as a polynomial in x_k, this means that x_j is a root, and so $x_k - x_j$ divides $\det(A)$.

(d) From part (c) we know that $\prod_{1 \le j < k \le n} \left(x_k - x_j\right)$, which is a polynomial of degree $\frac{n(n-1)}{2}$, divides $\det(A)$. Part (b) then implies that the degree of $\det(A)$ must equal $\frac{n(n-1)}{2}$, and so

$$\det(A) = c \prod_{1 \le j < k \le n} \left(x_k - x_j\right)$$

for some constant c. The coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in $\det(A)$ is $\operatorname{sign}(\mathrm{I}) = 1$, as is the coefficient of $x_1^0 x_2^1 \cdots x_n^{n-1}$ in $\prod_{1 \le j < k \le n} \left(x_k - x_j\right)$, and so $c = 1$. $\square$

Remark: We have just computed the famous *Vandermonte determinant*.