MATH 301 Final Exam 12/20/06 Solutions

1. Suppose $m, n \in \mathbb{Z}$.

   (a) Carefully define the statement m divides n.

   (b) Prove that for all $n \in \mathbb{N}$, 24 divides $5^{2n} - 1$.

   (a) $m|n$ means that there exists $k \in \mathbb{Z}$ such that $n = km$.

   (b) *First solution:* For all $n \in \mathbb{N}$ (in fact, for all $n \in \mathbb{Z}$),

   $$5^{2n} - 1 = 25^n - 1 \equiv 1^n - 1 = 0 \pmod{24} .$$

   Second solution: We proceed by induction on n. For the base case $n = 1$, $5^2 - 1 = 24$ is divisible by 24, since $24 = 24 \cdot 1$. For the induction step, assume that $5^{2n} - 1$ is divisible by 24, i.e., there is $k \in \mathbb{Z}$ such that $5^{2n} - 1 = 24k$. Then

   $$5^{2(n+1)} - 1 = 5^{2n+2} - 1 = 5^{2n} 5^2 - 1 = (24k + 1)\, 25 - 1 .$$

   Here the last equation follows with the induction hypothesis. Hence

   $$5^{2(n+1)} - 1 = (24k + 1)\, 25 - 1 = 24 \cdot 25k + 24 = 24\, (25k + 1) .$$

   So we found an integer $j = 25k + 1$ such that $5^{2(n+1)} - 1 = 24j$, so by definition, 24 divides $5^{2(n+1)} - 1$, and our induction is complete.

2. Suppose $A, B \subseteq \mathbb{R}$ are sets.

   (a) Carefully define the statement $A = B$.

   (b) Recall that the *complement* of A (in $\mathbb{R}$) is defined as $A^c = \mathbb{R} - A$. Prove that $(A \cup B)^c = A^c \cap B^c$.

   (a) $A = B$ means that $A \subseteq B$ and $B \subseteq A$; in other words: $x \in A$ if and only if $x \in B$.

   (b) We need to show $(A \cup B)^c \subseteq A^c \cap B^c$ and $(A \cup B)^c \supseteq A^c \cap B^c$.
   Given $x \in (A \cup B)^c$, we know $x \in X$ but $x \notin A \cup B$; the last statement says that the statement "$x \in A$ or $x \in B$" does not hold, which means $x \notin A$ and $x \notin B$. Hence by definition of set intersection, $x \in A^c \cap B^c$. This proves $(A \cup B)^c \subseteq A^c \cap B^c$.
   These steps can be traversed backwards: $x \in A^c \cap B^c$ means $x \notin A$ and $x \notin B$, which is the negation of the statement "$x \in A$ or $x \in B$", i.e., $x \in (A \cup B)^c$. This proves $(A \cup B)^c \supseteq A^c \cap B^c$.

3. Suppose $m, n \in \mathbb{Z}$.

   (a) Carefully define the statement $m \equiv n \pmod{34}$.

   (b) Prove that the relation $\equiv \pmod{34}$ is an equivalence relation.

   (a) $m \equiv n \pmod{34}$ means that $34|(m - n)$.

(b) We have to prove that for all $k, m, n \in \mathbb{Z}$,

   (i) $k \equiv k \pmod{34}$,

   (ii) $k \equiv m \pmod{34}$ implies $m \equiv k \pmod{34}$,

   (iii) $k \equiv m \pmod{34}$ and $m \equiv n \pmod{34}$ implies $k \equiv n \pmod{34}$.

   (i) follows from the fact that $34|0$ (since $0 = 34 \cdot 0$), and so $34|(k-k)$.

   (ii) Suppose $k \equiv m \pmod{34}$, i.e., there exists $j \in \mathbb{Z}$ such that $k - m = 34j$. Then $m - k = 34(-j)$, and since $-j \in \mathbb{Z}$, we conclude that $m \equiv k$ (mod 34).

   (iii) Suppose $k \equiv m \pmod{34}$ and $m \equiv n \pmod{34}$, i.e., there exists $i, j \in \mathbb{Z}$ such that $k - m = 34i$ and $m - n = 34j$. Then

$$k - n = (k - m) + (m - n) = 34i + 34j = 34(i + j) \ .$$

   Since $i + j \in \mathbb{Z}$, we conclude that $k \equiv n \pmod{34}$.

4. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers.

   (a) Carefully define what it means for the sequence $(a_n)_{n=1}^{\infty}$ to converge.

   (b) Now let $a_n = \dfrac{2}{\sqrt{n}}$. Prove that $\displaystyle\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$.

   (a) $(a_n)_{n=1}^{\infty}$ converges means that there exists $L \in \mathbb{R}$ such that for any $\epsilon > 0$, there exists $N \in N$ such that for all $n \geq N$, $|a_n - L| < \epsilon$.

   (b) Suppose $\epsilon > 0$ is given. Then choose an integer $N > \left(\frac{\epsilon}{2}\right)^2$, and we have for $n \geq N$

$$\left|\frac{2}{\sqrt{n}} - 0\right| = \frac{2}{\sqrt{n}} \leq \frac{2}{\sqrt{N}} < \epsilon \ .$$

5. Suppose A is a set.

   (a) Carefully define what it means for A to be countable.

   (b) Prove that the set of all even integers is countable.

   (a) A is countable if either A is finite or A is countably infinite. (A is finite if either $A = \emptyset$ or for some $n \in \mathbb{N}$ there is a bijection from $[n]$ to A. A is countably infinite if there is a bijection from $\mathbb{N}$ to A.)

   (b) *First solution:* We proved in class that $\mathbb{Z}$ is countable, and that a subset of a countable set is countable. Since $2\mathbb{Z}$ is a subset of $\mathbb{Z}$, it is countable.

   Second solution: We proved in class that $\mathbb{Z}$ is countably infinite, so that there exists a bijection $\phi : \mathbb{N} \to \mathbb{Z}$. The function $\psi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\psi(n) = 2n$ is a bijection. Thus $\psi \circ \phi : \mathbb{N} \to 2\mathbb{Z}$ is a bijection, whence $2\mathbb{Z}$ is countably infinite.