Proposition 5.10. *For $x \neq 1$ and $n \in \mathbb{N} \cup \{0\}$,* $\displaystyle\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x}$.

Proof. For a fixed x, we will prove that $(1-x)\sum_{j=0}^{n} x^j = 1 - x^{n+1}$ by induction on $n \geq 0$.

The base case $n = 0$ says that $(1-x)\sum_{j=0}^{0} x^j = 1 - x$, which is true since $\sum_{j=0}^{0} x^j = x^0 = 1$.

For the induction step, assume that $(1-x)\sum_{j=0}^{n} x^j = 1 - x^{n+1}$. Then

$$\begin{aligned}(1-x)\sum_{j=0}^{n+1} x^j &= (1-x)\left(\sum_{j=0}^{n} x^j + x^{n+1}\right) = (1-x)\sum_{j=0}^{n} x^j + (1-x)x^{n+1} \\ &= 1 - x^{n+1} + x^{n+1} - x^{n+2} = 1 - x^{n+2}.\end{aligned}$$

In the penultimate step we used the induction hypothesis. $\square$