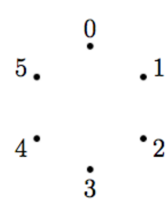
Worksheet 2: Primes

1. Let $a, b \in \mathbb{Z}_{>0}$. Show that, if $g = \gcd(a, b)$ then $\gcd(\frac{a}{g}, \frac{b}{g}) = 1$.
2. Give a careful definition of a *prime number*.
3. Let $a, b, c \in \mathbb{Z}_{>0}$.
   (a) Prove that, if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.
   (b) Conclude that if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.
   (c) Give a counterexample that shows the previous sentence is wrong if p is not prime.
4. Prove the *Fundamental Theorem of Arithmetic*: for every integer $n \geq 2$ there exist unique primes $p_1, p_2, \ldots, p_k$ and positive integers $a_1, a_2, \ldots, a_k$ such that
$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$
   (a) For existence, try induction on n.
   (b) For uniqueness, you may use 3(b).
5. Andrews 2.4.5 & 6.
6. Experiment with the `sage` commands `factor` and `is_prime`. Try them with a 100-digit number and a 150-digit number and compare the four running times (e.g., by using `%time` before the command). What's going on here?
7. *Preview: Clock Arithmetic.* The numbers on the 6-hour clock are the remainders we get when we divide by, in this case, 6. Adding 3 to 4 gets us to 1, which is also the remainder of dividing 3+4 by 6.
   (a) Explain why the number at the top of the clock is 0 rather than 6.
   (b) Complete the clock addition table and this clock multiplication table

+	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						



$\cdot$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

   (c) What patterns do you see in these two tables?
8. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.