Proposition 7.3. *The n^{th} Fibonacci number is given directly by the formula*

$$f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Proof. Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. We prove $P(n) : f(n) = \frac{1}{\sqrt{5}}(a^n - b^n)$ by (strong) induction on $n \in \mathbb{N}$. For starters, we check $P(1)$ and $P(2)$, for which the formula gives $f(1) = 1 = f(2)$. For the induction step, assume that $P(k)$ is true for $1 \le k \le n$. Then, by definition of the Fibonacci sequence and induction assumption,

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) \\ &= \frac{1}{\sqrt{5}}(a^n - b^n) + \frac{1}{\sqrt{5}}\left(a^{n-1} - b^{n-1}\right) \\ &= \frac{1}{\sqrt{5}}\left(a^n + a^{n-1} - b^n - b^{n-1}\right) \\ &= \frac{1}{\sqrt{5}}\left(a^{n-1}(a+1) - b^{n-1}(b+1)\right) \\ &= \frac{1}{\sqrt{5}}\left(a^{n-1}a^2 - b^{n-1}b^2\right) \\ &= \frac{1}{\sqrt{5}}\left(a^{n+1} - b^{n+1}\right). \end{aligned}$$

Here the penultimate step follows with

$$a+1 = \frac{3+\sqrt{5}}{2} = \frac{1+2\sqrt{5}+5}{4} = a^2 \qquad \text{and} \qquad b+1 = \frac{3-\sqrt{5}}{2} = \frac{1-2\sqrt{5}+5}{4} = b^2. \qquad \square$$

Proposition 7.4. $f(m+n) = f(m-1)f(n) + f(m)f(n+1)$.

Proof. Fix an arbitrary $m \in \mathbb{N}$. We will prove the statement

$$P(n): \; f(m+n) = f(m-1)f(n) + f(m)f(n+1)$$

by induction on $n \in \mathbb{N}$.

We need two base cases, namely $n = 1$ and 2. The right-hand side of $P(1)$ is

$$f(m-1)f(1) + f(m)f(2) = f(m-1) + f(m) = f(m+1),$$

by the definition of the Fibonacci sequence; this proves $P(1)$. The right-hand side of $P(2)$ is

$$f(m-1)f(2)+f(m)f(3) = f(m-1)+2f(m) = f(m-1)+f(m)+f(m) = f(m+1)+f(m) = f(m+2);$$

again we used the definition of the Fibonacci sequence (to compute $f(3)$, and we used the recurrence relation twice). This proves $P(2)$.

Now for the induction step, assume that $P(k)$ holds for all $1 \le k \le n$; we will prove that then $P(n+1)$ also holds. Using the recurrence relation for the Fibonacci sequence, we obtain

$$f(m+n+1) = f(m+n)+f(m+n-1) = f(m-1)f(n)+f(m)f(n+1)+f(m-1)f(n-1)+f(m)f(n),$$

by using the induction hypothesis for $P(n)$ and $P(n-1)$. This can be simplified to

$$\begin{aligned} f(m+n+1) &= f(m-1)\,(f(n)+f(n-1)) + f(m)\,(f(n+1)+f(n)) \\ &= f(m-1)f(n+1) + f(m)f(n+2)\ , \end{aligned}$$

once more by the recurrence relation for the Fibonacci sequence. But the identity $f(m+n+1) = f(m-1)f(n+1)+f(m)f(n+2)$ is precisely $P(n+1)$, and our induction step is complete. $\square$