MATH 725 Homework Set 1 due 8/29/11

(1) Find all solutions (in $\mathbf{C}$) to the following equations:
  (a) $z^6 = 1$
  (b) $z^4 = -16$
  (c) $z^6 = -9$
  (d) $z^6 - z^3 - 2 = 0$.

Solution. By the fundamental theorem of algebra, a polynomial equation of degree n has exactly n solutions (counting with multiplicities), in which case it thus suffices to give n distinct numbers that satisfy the equation.
  (a) $z = e^{\pi ik/3}$ for $1 \le k \le 6$ satisfy $z^6 = 1$.
  (b) $z = 2\,e^{\pi ik/4}$ for $k = 1,3,5,7$ satisfy $z^4 = -16$.
  (c) $z = \sqrt[3]{3}\,e^{\pi ik/3}$ for $k = 1,3,5$ satisfy $z^6 = -9$.
  (d) The quadratic equation $x^2 - x - 2 = 0$ has the solutions $x = -1, 2$, and so $z^6 - z^3 - 2 = 0$ has the solutions $z = e^{\pi ik/3}$ for $k = 1,3,5$ and $z = \sqrt[3]{2}\,e^{2\pi ik/3}$ for $k = 1,2,3$. $\square$

(2) Suppose $U_1, U_2, \dots, U_n$ are subspaces of V. Prove that $U_1 + U_2 + \cdots + U_n$ is a subspace of V.

Proof. We will check that $0 \in U_1 + U_2 + \cdots + U_n$ and that $U_1 + U_2 + \cdots + U_n$ is closed under addition and scalar multiplication.

The first assertion follows since $0 \in U_j$ for all j, and so $0 = 0 + o + \cdots + 0 \in U_1 + U_2 + \cdots + U_n$.

Now suppose $u_j, w_j \in U_j$ for each j; that is, we have two elements $u_1 + u_2 + \cdots + u_n \in U_1 + U_2 + \cdots + U_n$ and $w_1 + w_2 + \cdots + w_n \in U_1 + U_2 + \cdots + U_n$. Since $u_j + w_j \in U_j$ (because U_j is a subspace),

$$u_1 + u_2 + \dots u_n + w_1 + w_2 + \dots w_n = (u_1 + w_1) + (u_2 + w_2) + \cdots + (u_n + w_n) \in U_1 + U_2 + \cdots + U_n,$$

that is, $U_1 + U_2 + \cdots + U_n$ is closed under addition. Similarly, given $a \in \mathbf{F}$, we know that $au_j \in U_j$ (again because U_j is a subspace), and so

$$a\,(u_1 + u_2 + \cdots + u_n) = au_1 + au_2 + \cdots + au_n \in U_1 + U_2 + \cdots + U_n,$$

that is, $U_1 + U_2 + \cdots + U_n$ is closed under scalar multiplication. $\square$

(3) Carefully reason whether or not the following sets are subspaces of $\mathbf{R}^2$:
  (a) $\{(a,b) \in \mathbf{R}^2 : a, b \ge 0\}$
  (b) $\{(a,b) \in \mathbf{R}^2 : ab \ge 0\}$
  (c) $\{(a,b) \in \mathbf{R}^2 : a = b\}$
  (d) $\mathbf{Z}^2$

Solution. (a) $(1,1) \in \{(a,b) \in \mathbf{R}^2 : a,b \ge 0\}$ but $-(1,1) \notin \{(a,b) \in \mathbf{R}^2 : a,b \ge 0\}$, so this is not a subspace.
  (b) $(1,1)$ and $(-2,0)$ are both in $\{(a,b) \in \mathbf{R}^2 : ab \ge 0\}$ but $(1,1) + (-2,0) = (-1,1)$ is not, so this is not a subspace.
  (c) $\{(a,b) \in \mathbf{R}^2 : a = b\}$ is a subspace: $(0,0)$ is in it, it is closed under addition ($(x,x) + (y,y) = (x+y, x+y)$) and scalar multiplication ($a(x,x) = (ax, ax)$).
  (d) $(1,1) \in \mathbf{Z}^2$ but $\frac{1}{2}(1,1) \notin \mathbf{Z}^2$, so $\mathbf{Z}^2$ is not a subspace. $\square$

(4) Consider the subspace $U := \{p \in \mathscr{P}(\mathbf{F}) : \deg(p) \le 3\}$ of the vector space $\mathscr{P}(\mathbf{F})$ consisting of all polynomials with coefficients in $\mathbf{F}$. Construct a subspace W of $\mathscr{P}(\mathbf{F})$ such that $\mathscr{P}(\mathbf{F}) = U \oplus W$.

Solution. We claim that $W := \{x^4 p(x) : p \in \mathscr{P}(\mathbf{F})\}$ will do the trick (that is, W consists of the zero polynomial and all polynomials that do not have constant, linear, quadratic, or tertiary terms). W is a subspace because it contains 0 and is closed under addition and scalar multiplication. By construction we have $\mathscr{P}(\mathbf{F}) = U + W$ and $U \cap W = \{0\}$, so by Proposition 1.9, $\mathscr{P}(\mathbf{F}) = U \oplus W$. $\square$

(5) Suppose U and W are subspaces of V. Prove that $U \cap W$ is the largest subspace of V that is contained in both U and W; that is:
  (a) $U \cap W$ is a subspace of V, and

(b) any other subspace of V that is contained in both U and W is also contained in $U \cap W$.

Proof. Since U and W are subspaces, they both contain 0, and so $0 \in U \cap W$. Given $v_1, v_2 \in U \cap W$, they are both in U and W. As subspaces, U and W are closed under addition and scalar multiplication, and so $v_1 + v_2$ is in both U and V, that is, $v_1 + v_2 \in U \cap W$; similary, for $a \in \mathbf{F}$, av_1 is in both U and V, that is, $av_1 \in U \cap W$. Thus $U \cap W$ is also closed under addition and scalar multiplication, and this proves (a).

Now let S be a subspace of V that is contained in both U and W. Then (as a set) S is contained in $U \cap W$, and this proves (b). $\square$