Proposition 10.4. *The rational number* $\frac{m}{n} \in \mathbb{Q}$ *is positive (i.e.,* $\frac{m}{n} \in \mathbb{R}_{>0}$*) if and only if either* $m > 0$ *and* $n > 0$*, or* $m < 0$ *and* $n < 0$*.*

Proof. We first prove a little lemma which has a philosophy similar to that of Proposition 8.4: $n \in \mathbb{Z}$ and $1/n$ have the same sign; they are either both positive or both negative. This follows by way of contradiction: if n and $1/n$ had opposite signs then $1 = n \cdot 1/n$ would be negative by Proposition 8.2, a contradiction to Proposition 8.3(i).

Now for the actual proof of the proposition. Suppose first that $m, n \in \mathbb{Z}$ are both positive or both negative. By the lemma, n and $1/n$ have the same sign, so m and $1/n$ have the same sign. If they are both positive then $m/n = m \cdot 1/n > 0$ by Axiom 8.6(ii). If m and $1/n$ are both negative then $m/n > 0$ by Proposition 8.2(iv). In both cases, m/n is positive.

Conversely, suppose that m and n have opposite signs. Again by the lemma, n and $1/n$ have the same sign, so m and $1/n$ have opposite signs. If $m > 0$ and $1/n < 0$ then $m/n < 0$, and if $m < 0$ and $1/n > 0$ then $m/n < 0$, both by Proposition 8.2(iv). In both cases, m/n is negative. $\square$

Proposition 10.5.

(i) *The sum of two positive rationals is a positive rational.*

(ii) *The product of two positive rationals is a positive rational.*

(iii) *For every* $\frac{m}{n} \in \mathbb{Q}$ *such that* $\frac{m}{n} \neq 0$*, either* $\frac{m}{n}$ *is positive or* $\frac{-m}{n}$ *is positive, and not both.*

Proof. (i) and (ii) follow immediately with Proposition 10.2, $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1 m_2}{n_1 n_2}$, Axiom 8.6, and the fact that $\mathbb{Q} \subseteq \mathbb{R}$. (iii) follows with Axiom 8.6 and the fact that $-\frac{m}{n} = \frac{-m}{n}$. $\square$