MATH 380 Final Exam 12/19/07 10:45–1:15 p.m.

Show complete work—that is, all the steps needed to completely justify your answer. You may refer to theorems proved in class and in the text.

(1) (10 points) Find all solutions to the equation $z^4 = -16$.

Solution: Let $z = r\, e^{i\phi}$. Then $z^4 = -16$ is equivalent to

$$\left(r\, e^{i\phi}\right)^4 = r^4\, e^{i4\phi} = 16\, e^{i\pi} = -16\ .$$

$r^4 = 16$ has as the only nonnegative real solution $r = 2$. From the argument comparison, $4\phi = \pi + 2\pi k$ or

$$\phi = \frac{\pi}{4} + \frac{\pi}{2}\, k\ ,$$

where k is any integer. This gives the four distinct angles $\phi = \pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. The four solutions are $2\, e^{i\pi/4} = \sqrt{2}\,(1+i)$, $2\, e^{i3\pi/4} = \sqrt{2}\,(-1+i)$, $2\, e^{i5\pi/4} = -\sqrt{2}\,(1+i)$, and $2\, e^{i7\pi/4} = \sqrt{2}\,(1-i)$.

(2) (10 points each) For each of the following functions, give the subset of $\mathbb{C}$ where the function is differentiable, respectively analytic, and find its derivative. (As usual, $z = x + iy$.)

(a) $f(z) = |z|^2 = x^2 + y^2$

(b) $f(z) = (z - 1 + 2i)^{3i}$

(c) $f(z) = (3i)^{z-1+2i}$

Solution:

(a) The real part of f is $u = x^2 + y^2$ and the imaginary part is $v = 0$. For the Cauchy-Riemann equations to hold, we need

$$u_x = 2x = 0 = v_y \qquad \text{and} \qquad u_y = 2y = 0 = -v_x\ ,$$

and these equations are only satisfied for $x = 0$ and $y = 0$. Hence the first part of the Cauchy-Riemann Theorem 2.4 says that f is not differentiable for all $z \in \mathbb{C}\setminus\{0\}$. Since u_x, u_y, v_x, v_y are continuous, the second part of the Cauchy-Riemann Theorem 2.4 implies that f is differentiable at 0, with derivative 0. Since a point contains no disk, f is nowhere analytic.

(b) By definition,

$$(z - 1 + 2i)^{3i} = \exp\left((3i)\, \mathrm{Log}(z - 1 + 2i)\right).$$

exp is an entire function, so $(z - 1 + 2i)^{3i}$ is analytic wherever $\mathrm{Log}(z - 1 + 2i)$ is. As we showed many times, Log is analytic everywhere but the nonpositive real axis ($z = x + iy$ with $x \le 0$ and $y = 0$), which implies that $\mathrm{Log}(z - 1 + 2i)$ is analytic everywhere but when $z-1+2i$ is real and nonpositive, that is, for $z = x+iy$ with $x \le$

1 and $y = -2$. Hence $(z-1+2i)^{3i}$ is analytic on $\mathbb{C} \setminus \{x+iy \in \mathbb{C} : \; x \leq 1, \; y = -2\}$. The derivative is

$$\exp\left((3i)\,\mathrm{Log}(z-1+2i)\right) \frac{3i}{z-1+2i} = 3i\,(z-1+2i)^{3i-1}.$$

(c) By definition,

$$(3i)^{z-1+2i} = \exp\left((z-1+2i)\,\mathrm{Log}(3i)\right).$$

This is the exponential function applied to a polynomial (note that $\mathrm{Log}(3i)$ is a constant). Both are entire functions, so $(3i)^{z-1+2i}$ is entire. The derivative is

$$\exp\left((z-1+2i)\,\mathrm{Log}(3i)\right)\mathrm{Log}(3i) = \mathrm{Log}(3i)\,(3i)^{z-1+2i}.$$

(3) (5 points each) Give the regions of convergence, absolute convergence, and uniform convergence for the following series:

(a) $\sum_{k\geq 0} \frac{1}{(2k+1)!}\, z^{2k+1}$

(b) $\sum_{k\geq 0} \left(\frac{1}{z-3}\right)^k$

(c) (*Bonus question*) What function is represented by the power series in (a)?

Solution:

(a) Since $\left|\sum_{k\geq 0} \frac{1}{(2k+1)!} z^{2k+1}\right| \leq \sum_{k\geq 0} \frac{1}{k!}|z|^k$, this series converges for all z for which the exponential series converges, and thus the radius of convergence is ∞. By Theorem 7.14, the series converges absolutely in $\mathbb{C}$ and uniformly for $|z| \leq r$ for any r.

(b) This is a geometric series:

$$\sum_{k\geq 0} \left(\frac{1}{z-3}\right)^k = \frac{1}{1-\frac{1}{z-3}}$$

and converges by Lemma 7.13 absolutely for $\left|\frac{1}{z-3}\right| < 1 \iff |z-3| > 1$ and uniformly for $\left|\frac{1}{z-3}\right| \leq r \iff |z-3| \geq \frac{1}{r}$ for any $r < 1$.

(c) *Bonus question*: This series contains the odd powers of the coefficients in the exponential series. We can retrieve those through

$$e^z - e^{-z} = \sum_{k\geq 0} \frac{1}{k!}\, z^k - \sum_{k\geq 0} \frac{1}{k!}\,(-z)^k = 2\sum_{k\geq 0} \frac{1}{(2k+1)!}\, z^{2k+1}.$$

Thus $\sum_{k\geq 0} \frac{1}{(2k+1)!}\, z^{2k+1} = \frac{1}{2}\left(e^z - e^{-z}\right) = \sinh z$.

(4) (5 points) Suppose f can be represented by the power series $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$. State a formula to compute the coefficients a_k.

Solution: See Corollary 8.3 or Theorem 8.5 in the lecture notes.

(5) (10 points each) Integrate each of the following functions over the circle $|z| = 2$, oriented counterclockwise.

(a) $\overline{z}$

(b) $\frac{1}{z^4}$

(c) $\frac{\sin z}{z-1}$

(d) $\frac{\sin z}{z-3}$

(e) $z^3 \cos\left(\frac{3}{z}\right)$

(f) $\frac{\exp z}{\sin z}$

Solution: A parametrization of the circle γ is $\gamma(t) = 2\,e^{it} = 2\cos t + 2i\sin t,\ 0 \le t \le 2\pi$. Note that $\gamma'(t) = 2i\,e^{it}$.

(a) $$\int_\gamma \overline{z}\,dz = \int_0^{2\pi} 2\,e^{-it}\,2i\,e^{it}\,dt = \int_0^{2\pi} 4i\,dt = 8\pi i\ .$$

(b) $\frac{1}{z^4}$ has the antiderivative $-\frac{1}{3z^3}$ on $\mathbb{C}\setminus\{0\}$, which contains γ, and thus the integral is zero.

(c) We can use Cauchy's Integral Formula (Theorem 4.7) with $G = \mathbb{C}$, $f(z) = \sin z$, and $w = 1$ (note that w is inside γ and that γ is G-contractible):

$$\int_\gamma \frac{\sin z}{z-1}\,dz = 2\pi i \sin 1\ .$$

(d) We can use Corollary 4.5 (to Cauchy's Theorem) with $G = \mathbb{C}\setminus\{3\}$, $f(z) = \sin z/(z-3)$: note that f is analytic in G and γ is closed and G-contractible, and so

$$\int_\gamma \frac{\sin z}{z-3}\,dz = 0\ .$$

(e) We expand the integrand into its Laurent series:

$$z^3\ \cos\left(\frac{3}{z}\right) = z^3 \sum_{k\ge0} \frac{(-1)^k}{(2k)!}\left(\frac{3}{z}\right)^{2k} = \sum_{k\ge0} \frac{(-1)^k 3^{2k}}{(2k)!} z^{-2k+3}\ .$$

The residue of this Laurent series is $\frac{3^4}{4!} = \frac{27}{8}$, hence

$$\int_\gamma z\cos\left(\tfrac{3}{z}\right)\,dz = 2\pi i\,\frac{27}{8} = \frac{27\pi i}{4}\ .$$

(f) The poles of the integrand are the zeros of the sine function, hence at $z = \pi k,\ k \in \mathbb{Z}$. Of those only $z = 0$ is inside γ, so the Residue Theorem gives

$$\int_\gamma \frac{\exp z}{\sin z}\,dz = 2\pi i\,\mathrm{Res}_{z=0}\left(\frac{\exp z}{\sin z}\right).$$

Since the sine has a simple zero at 0 and $\exp 0 = 1 \neq 0$, 0 is a simple pole of $\frac{\exp z}{\sin z}$ whose residue can be computed, for example, by Lemma 9.7 as

$$\operatorname{Res}_{z=0}\left(\frac{\exp z}{\sin z}\right) = \frac{\exp 0}{\cos 0} = 1 \ .$$

Hence

$$\int_\gamma \frac{\exp z}{\sin z}\, dz = 2\pi i \ .$$

(6) (10 points) Define the three types of isolated singularities and give an example for each one.

Solution: See pages 85 & 86 in the lecture notes.

(7) (10 points each) Let $f = \frac{1}{z(z-2)^2}$.

(a) Find a Laurent series for f centered at $z = 2$ and specify the region in which it converges.

(b) Compute $\int_\gamma f$ where γ is a positively oriented circle centered at 2 of radius 1.

Solution:

(a) For $|z-2| < 2$,

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{z-2}{2}\right)} = \frac{1}{2}\sum_{k\geq 0}\left(-\frac{z-2}{2}\right)^k = \sum_{k\geq 0}\frac{(-1)^k}{2^{k+1}}\,(z-2)^k \ .$$

Hence for $0 < |z-2| < 2$

$$\frac{1}{z(z-2)^2} = (z-2)^{-2}\sum_{k\geq 0}\frac{(-1)^k}{2^{k+1}}\,(z-2)^k = \sum_{k\geq 0}\frac{(-1)^k}{2^{k+1}}\,(z-2)^{k-2} = \sum_{k\geq -2}\frac{(-1)^k}{2^{k+3}}\,(z-2)^k \ .$$

(b) We can read off the residue of the integrand at $z = 2$ in the above Laurent series as $\frac{(-1)^{-1}}{2^{-1+3}} = -\frac{1}{4}$. Hence by the Residue Theorem

$$\int_\gamma \frac{1}{z(z-2)^2}\, dz = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2} \ .$$