MATH 335 Final Exam 05/24/06 8:00–10:30 a.m.

Show complete work—that is, all the steps needed to completely justify your answer. You may refer to theorems proved in class (without stating the exact number of the theorem). Each problem is worth 20 points.

(1) Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3 \rangle \,/\, \langle 12 \rangle$ is isomorphic to $\mathbb{Z}_4$.

(2) Fix an integer $n \geq 2$, and consider the map $\phi : S_n \to \mathbb{Z}_2$ given by

$$\phi(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is even,} \\ 1 & \text{if } \alpha \text{ is odd.} \end{cases}$$

Show that ϕ is a homomorphism, determine its kernel, and conclude from this that A_n is a normal subgroup of S_n.

(3) Let H be a subgroup of the group G. Let

$$N(H) = \{x \in G : xHx^{-1} = H\}$$

(the *normalizer* of H) and

$$C(H) = \{x \in G : xhx^{-1} = h \text{ for all } h \in H\}$$

(the *centralizer* of H). Prove that $N(H)$ and $C(H)$ are subgroups of G.

(4) Prove that $\mathbb{Z}[i]/\langle 1 - i \rangle$ is a field. Conclude that $\langle 1 - i \rangle$ is a prime ideal in $\mathbb{Z}[i]$. (*Hint:* Start by showing that $2 \in \langle 1 - i \rangle$.)

(5) Show that $3x + 1$ is a unit in $\mathbb{Z}_9[x]$, but not a unit in $\mathbb{Z}[x]$. Determine $U(\mathbb{Z}[x])$.