Project 9.7. *Suppose $f : A \to B$ and $g : B \to C$. Decide if each of the following is true or false; support your answers.*

(i) *If f is injective and g is surjective then $g \circ f$ is surjective.*

(ii) *If $g \circ f$ is bijective then g is surjective and f is injective.*

(i) is false. As a counterexample you could take your favorite injective function $f : \mathbb{Z} \to \mathbb{R}$, e.g., the one defined by $f(x) = x$, and choose $g = \mathrm{id}_{\mathbb{R}}$ (so g is certainly surjective). Then $g \circ f = f$, however, this function cannot be surjective, e.g., for $f(x) = x$ there is no $x \in \mathbb{Z}$ such that $f(x) = 1/2$.

(ii) is true: Suppose $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f : A \to C$ is bijective.

We first prove that g is surjective. Given $c \in C$, we need to construct $b \in B$ such that $g(b) = c$. Given such a c, we use the fact that $g \circ f$ is surjective, so there exists $a \in A$ such that $g(f(a)) = c$. Choose $b = f(a)$; then $g(b) = g(f(a)) = c$, as desired.

Next we prove that f is injective. Given $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, we need to conclude that $a_1 = a_2$. We can apply the function g to the number $f(a_1) = f(a_2)$ to obtain $g(f(a_1)) = g(f(a_2))$. But now, since g is injective, we can deduce that $a_1 = a_2$.