ENUMERATION OF ORTHOGONAL LATIN SQUARES

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CERTIFICATION OF APPROVAL

I certify that I have read *Enumeration of Orthogonal Latin Squares* by Andrew James Beyer, and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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ENUMERATION OF ORTHOGONAL LATIN SQUARES

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We present theory of two methods for counting the number of sets of mutually orthogonal magilatin arrays as functions of either a magic sum or a strict upper bound on the entries. The methods of enumeration are based on the theory of inside-out polytopes [1]. The theory of inside-out polytopes employs Ehrhart's theory of counting lattice-points in a convex polytope. In our case, the polytope is dissected by a hyperplane arrangement or an affine subspace arrangement, which act as extra boundaries in the interior of the polytope. We also demonstrate the method with some computational examples of sets of 2 mutually orthogonal 2×3 and 3×3 magilatin arrays.

I certify that the Abstract is a correct representation of the content of this thesis.

Chair, Thesis Committee   Date

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N. INTRODUCTION

This paper presents methods for counting k-sets of mutually orthogonal latin arrays. In Chapter I we begin by building up to the definition of these objects as an extension of classic latin squares and highlight some useful properties. We provide some motivation for the counting of such objects, offering theorems and conjectures, while focusing on the search for the existence of complete sets of mutually orthogonal latin squares, which may be used to obtain an optimal block code.

The methods of enumeration are based on the theory of inside-out polytopes [1]. Chapter II provides the mathematical background for this theory. We start with the language of polytopes in terms of the affine structure of $\mathbb{R}^d$. Generating functions and their use in Ehrhart's theory of lattice point enumeration follow. Next we give definitions from the world of posets and lattices, and subsequently the world of arrangements of hyperplanes and affine subspaces. The chapter concludes with the theory of counting lattice points contained within an inside-out polytope.

The main results of this paper appear in Chapter III and IV. We offer theory for two methods of counting k-sets of mutually orthogonal latin arrays in Chapter III. Both methods represent the k-sets as lattice points contained in a polytope that do not lie on any hyperplane or affine space of an arrangement. We then apply the theory of inside-out polytopes to enumerate these lattice points. In application, computer software can only implement one method. Chapter IV provides a discussion of the two methods with respect to this complication, and then implements the usable method in computer computation to give some numbers with respect to latin arrays and orthogonal mates.

I. GETTING FAMILIAR

1.1 Defining MOLS

Definition 1.1 [15, page 3]: A *classic latin square of order n* is an $n \times n$ array filled with n symbols such that no symbol is repeated in a row or a column.

$$\boxed{\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}} \quad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}} \quad \boxed{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{array}}$$

Figure 1.1

Often the n symbols of a classic latin square of order n are taken to be the first n positive integers. For example Figure 1.1 shows classic latin squares of order 2, 3, and 4 each filled with the first n positive integers. As a consequence of construction, the sum of the entries across each row and each column is the same, equaling the sum of the first n positive integers $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$.

Property 1.2: If a classic latin square of order n is filled with the first n positive integers, then the sum across a row or column is $\frac{n(n+1)}{2}$.

Proof: Consider only two rows of the square. Each row is filled with the first n positive integers. In the top row, list the integers from least to greatest. In the bottom row, list the integers from greatest to least. The sum down each column now equals $n+1$. Since there are n columns, the sum of all the integers in both rows is $n(n+1)$. Thus the sum across one row is $\frac{n(n+1)}{2}$. The proof for columns is identical once the columns are transposed to rows. ■

To create a larger class of squares, which include classic latin squares, we keep the property of the distinctness of entries and the equal row and column sums, but allow the entries to come from a larger set of positive integers.

Definition 1.3 [2, page 2]: A *magilatin square* is an $n \times n$ array filled with positive integers such that no integer is repeated in a row or a column, and such that the sum along each row and column is the same.

Magilatin squares of order 2, 3, and 4 are shown below in Figure 1.2.

2	5
5	2

1	2	7
3	5	2
6	3	1

4	5	7	9
12	9	3	1
3	10	4	8
6	1	11	7

Figure 1.2

Definition 1.4: A *classic latin rectangle* is an $m \times n$ array (with $m < n$) filled with n symbols such that no symbol is repeated in a row or a column.

1	2	3
3	1	2

1	2	3	4
4	3	2	1
2	1	4	3

1	2	3	4
4	3	2	1

Figure 1.3

Similar to the classic latin square, the classic latin rectangle is often filled with the first n positive integers. As a result, the sum along each row is the same, equaling

$\frac{n(n+1)}{2}$. However, the column sums may be different. In Figure 1.3 the left and center classic latin rectangles have equal row sums, but the symbols cannot be arranged to create equal column sums. Only some classic latin rectangles have equal row and column sums, such as the example on the right of Figure 1.3.

Property 1.5: If an $m \times n$ classic latin rectangle is filled with the first n positive integers, then the sum across a row is $\frac{n(n+1)}{2}$.

Proof: This is the same as the proof of Property 1.2, for the row sums only. ■

Preserving the distinctness of entries and the equal sum across the rows but not necessarily the columns, we create a larger class of rectangles that includes the classic latin rectangle.

Definition 1.6: A *magilatin rectangle* is an $m \times n$ array (with $m < n$) filled with positive integers such that no integer is repeated in a row or a column, and such that the sum along each row is the same.

6	2	7
4	8	3

1	2	3	5
5	3	2	1
2	1	5	3

1	4	5	8
8	5	4	1

Figure 1.4

Magilatin rectangles can be defined to include equal column sums. In this case, all row sums are equal and all column sums are equal, but the two sums are different. However, a definition that requires an equal column sum excludes many of the classic latin rectangles. Thus we have chosen to define magilatin rectangles without requiring equal column sums.

Definition 1.7: A *latin array* is a magilatin square or magilatin rectangle.

Our notation for an $m \times n$ latin array is the matrix notation $\mathbf{x} = \left(x_{ij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ where x_{ij} denotes the entry in the cell corresponding to row i and column j of the array.

Definition 1.8: When a latin array has a sum equal across rows, or equal across rows and columns, we call this integer value the *magic sum*.

We want the set of positive integers from which to choose entries for a latin array to be a finite set. Thus we create an integer parameter t to restrict the size of this set. One approach simply designates the parameter t to represent a strict upper bound on the entries $0 < x_{ij} < t$. A clever alternative designates the parameter t to represent the magic sum $\sum_j x_{ij} = t$. Both choices for the parameter t are developed in [2].

Let t be an integral parameter representing either the magic sum or a strict upper bound on the entries.

Defintition 1.9: We call t an *affine parameter* when it represents the magic sum.

Defintition 1.10: We call t a *cubical parameter* when it represents the strict upper bound on the entries.

Definition 1.11: Two latin arrays are *orthogonal* if when superimposed, the ordered pairs in the cells of the resulting array are all distinct.

When two latin arrays are orthogonal to each other, we call them *orthogonal mates*. Figure 1.5 gives two examples of orthogonal mates. The left pair are classic latin squares and the right pair are magilatin squares. Below each pair is the superimposed array of distinct ordered pairs.

$$\boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}} \text{ and } \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}} \qquad \boxed{\begin{array}{ccc} 3 & 10 & 2 \\ 5 & 2 & 8 \\ 7 & 3 & 5 \end{array}} \text{ and } \boxed{\begin{array}{ccc} 1 & 3 & 11 \\ 10 & 4 & 1 \\ 4 & 8 & 3 \end{array}}$$

$$\boxed{\begin{array}{ccc} (1,1) & (2,2) & (3,3) \\ (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,2) \end{array}} \qquad \boxed{\begin{array}{ccc} (3,1) & (10,3) & (2,11) \\ (5,10) & (2,4) & (8,1) \\ (7,4) & (3,8) & (5,3) \end{array}}$$

Figure 1.5

Definition 1.12: A *semimagic square* is an $n \times n$ array filled with distinct positive integers such that the sum along each row and column is equal.

A *magic square* is a semimagic square where the main diagonal sums equal the row and column sums. A *classic semimagic square* or a *classic magic square* takes the first n^2 positive integers as entries, and thus the row and column sums equal $\dfrac{n(n^2+1)}{2}$.

Property 1.13: A classic semimagic (magic) square filled with the first n^2 positive integers, has a sum across a row or column (or main diagonal) equaling $\frac{n(n^2+1)}{2}$.

Proof: Consider two sets of the n^2 positive integers. List the integers of one set from least to greatest in a row. From the other set, list the integers from greatest to least in a row beneath the first row. The sum down each column now equals n^2+1. Since there are n^2 columns, the sum of all the integers from the two sets is $n^2(n^2+1)$. Thus the sum of the integers from one set is $\frac{n^2(n^2+1)}{2}$. This sum must be divided equally to the n rows of the semimagic (magic) square. Thus the sum across a row of the semimagic (magic) square is $\frac{n^2(n^2+1)}{2n} = \frac{n(n^2+1)}{2}$. The proof for columns (main diagonals) is identical once the columns (main diagonals) are transposed to rows.

Definition 1.14: A *semimagic rectangle* is an $m \times n$ array (with $m < n$) filled with distinct positive integers such that the sum along each row is equal.

Definition 1.15: A *semimagic array* is a semimagic square or semimagic rectangle.

Definition 1.16 (Euler encoding): For a pair of orthogonal classic latin squares $\mathbf{x} = \left(x_{ij}\right)_{n\times n}$ and $\mathbf{y} = \left(y_{ij}\right)_{n\times n}$, the *Euler encoding* is defined by the linear combination $n(\mathbf{x}-\mathbf{1})+\mathbf{y} = n\left(x_{ij}-1\right)_{n\times n} + \left(y_{ij}\right)_{n\times n}$.

Property 1.17: For a pair of orthogonal classic latin squares $\mathbf{x} = \left(x_{ij}\right)_{n\times n}$ and $\mathbf{y} = \left(y_{ij}\right)_{n\times n}$, the Euler encoding produces a classic semimagic square $\mathbf{z} = \left(z_{ij}\right)_{n\times n} = n\left(x_{ij}-1\right)_{n\times n} + \left(y_{ij}\right)_{n\times n}$.

Proof: The Euler encoding creates an $n \times n$ array with distinct entries (x_{ij}, y_{ij}). Then $(x_{ij}-1, y_{ij}-1)$ is a distinct number from the first n^2-1 numbers in base n. Convert to the first n^2-1 nonnegative integers in base 10 by taking $n(x_{ij}-1)+(y_{ij}-1)$. Thus $n(x_{ij}-1)+(y_{ij}-1)+1 = n(x_{ij}-1)+y_{ij}$ gives the first n^2 positive integers. Since each line sum of the latin squares is $\frac{n(n+1)}{2}$, the Euler encoding creates a line sum of $n(\frac{n(n+1)}{2}-n)+\frac{n(n+1)}{2}=\frac{n(n^2+1)}{2}$. ■

$$\begin{array}{|ccc|}\hline (1,1) & (2,2) & (3,3) \\ (3,2) & (1,3) & (2,1) \\ (2,3) & (3,1) & (1,2) \\ \hline \end{array} \text{ becomes } \begin{array}{|ccc|}\hline 1 & 5 & 9 \\ 8 & 3 & 4 \\ 6 & 7 & 2 \\ \hline \end{array}$$

Figure 1.6

Figure 1.6 shows a pair of orthogonal classic latin squares on the left, and the Euler encoded classic semimagic square on the right. The Euler encoding can also be used for pairs of orthogonal magilatin squares with some limitation. When the magic sum or the strict upper bound on the entries gets large, the encoding may fail to produce a semimagic square. We offer an example of the encoding failing below in Figure 1.7. In Chapter III, we generalize the Euler encoding for use with pairs of orthogonal magilatin arrays. On the left of Figure 1.7 is a pair of orthogonal magilatin squares; on the right is the Euler encoded square, which has some repeated entries, and thus is not a semimagic square.

$$\begin{array}{|ccc|}\hline (3,1) & (10,3) & (2,11) \\ (5,10) & (2,4) & (8,1) \\ (7,4) & (3,8) & (5,3) \\ \hline \end{array} \text{ becomes } \begin{array}{|ccc|}\hline 7 & 30 & 14 \\ 22 & 7 & 22 \\ 22 & 14 & 15 \\ \hline \end{array}$$

Figure 1.7

Definition 1.18: A *set of mutually orthogonal latin arrays* is a collection of latin arrays that are pairwise orthogonal to each other.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

Figure 1.8

As a shorthand for a set of mutually orthogonal latin arrays, we borrow the abbreviation *set of MOLS* which comes from a set of mutually orthogonal (classic) latin squares. In this paper a *set of MOLS* refers to a set of mutually orthogonal latin arrays. In figure 1.8 we have 3 MOLS of order 4, each a classic latin square.

Definition 1.19: A *complete set of MOLS* is a set of $n-1$ mutually orthogonal classic latin squares of order n.

The latin squares in Figure 1.8 constitute a complete set of MOLS. For classic latin squares of order n, $n-1$ is the largest number of MOLS. See Theorem 1.30 for a proof of this statement.

Definition 1.20: A *line* of a latin array is any row or column of the array.

Definition 1.21: A *magic requirement* is an equal sum between two lines of a latin array.

Definition 1.22: An *entry requirement* is a lower bound or upper bound on the entries of a latin array.

Definition 1.23: A *latin exclusion* is a prohibited equality of two entries within a line of a latin array.

Definition 1.24: An *orthogonal exclusion* is a prohibited equality of two ordered pairs that result from pairwise orthogonality in a set of k MOLS.

Finally, we call a set with k elements a *k-set*. Thus a *k-set of MOLS* is a collection of k latin arrays that are pairwise orthogonal to each other. A k-set of $m \times n$ MOLS may be described as k $m \times n$ integral matrices subject to the following impositions: magic requirements, entry requirements, latin exclusions, orthogonal exclusions, and a parameter t that represents either a strict upper bound on the entries or the magic sum. Our notation for a k-set of MOLS is the matrix notation $\mathbf{x}_1 = \left(x_{1ij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$, $\mathbf{x}_2 = \left(x_{2ij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}, \ldots,$ $\mathbf{x}_k = \left(x_{kij}\right)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ where x_{lij} denotes the entry in the cell corresponding to row i and column j of latin array l. We consider both (unordered) k-sets and ordered k-sets of MOLS. For example, the equivalent sets $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and $\{\mathbf{x}_k, \mathbf{x}_{k-1}, \ldots, \mathbf{x}_1\}$ are not equivalent as ordered sets. Note that for a set of k elements, there are $k!$ unique ordered sets containing precisely all k members.

1.2 Motivation

Latin arrays inhabit a wide variety of mathematical topics. Much of the interest and application of latin arrays focuses on a set of MOLS comprised of classic latin squares. Applications range from the very pure to the most applied. On the pure side, orthogonal classic latin squares have a geometric life in the world of affine and projective planes. For a definition of affine and projective planes see [15]. In particular we wish to bring the following two theorems to attention.

Theorem 1.25 [15, Theorem 8.6]: A complete set of MOLS of order n exists if and only if an affine plane of order n exists.

Theorem 1.26 [15, Theorem 8.10]: A complete set of MOLS of order n exists if and only if a finite projective plane of order n exists.

Some of the noteworthy applications described in [15] include statistics, experimental design, error-correcting codes, cryptology, (t, m, s)-nets, conflict-free access to parallel memories, tournament design, network communications, electronic mail, log-on procedures, authentication schemes for both binary and non-binary systems, specialized codes, character recognition, hashing functions, compiler testing, and tomography. In fact, the motivation for this paper began as a synthesis of an applied use of complete sets of MOLS and an open question on the existence of such sets.

While investigating the rich subject of information theory and error-correcting codes, I happened upon the topic of constructing a particular optimal code (a code that attains the Singleton bound) using a complete set of MOLS. You can find a nice treatment of error-correcting codes and complete sets of MOLS in [15].

We start with a set of symbols that we call an *alphabet*. A string of symbols from the alphabet is called a *code word*. A code word replaces one piece of information during transmission of a message. The *length* of a code word is the number of symbols used to create it. The set of all code words is called a *code* and its size determines the diversity of messages we can transmit. When all code words of a code have the same length, then we call the code a *block code*. In the strings of two code words of a block code, the number of places the two words differ is called the *Hamming distance*. The *minimum distance* of a block code is the smallest Hamming distance between any pair of code words. During transmission some of the symbols in a codeword may be corrupted. A *c-error correcting code* is a code that can correct code words with up to *c* errors. A block code is a *c*-error correcting code when the minimum distance is $2c+1$.

We are interested in non-binary block codes on an alphabet of *q* symbols, called a *q-ary code*. When the code contains *w* code words with length *l* and minimum distance *d*, then we refer to the code as an (l,w,d) *q-ary code*. Given *q* symbols, length *l*, and minimum distance *d* the maximum number of code words *w* is given by the Singleton bound $w \le q^{l-d+1}$ [15].

When equality is achieved in the Singleton bound, the code is called a *maximum distance separable code,* or simply an *MDS code*. In particular, MDS codes optimize the number of code words to $w = q^2$ when the code words have minimum distance *d* and length $l = d+1$.

Theorem 1.27 [15, Theorem 13.14]: There exists a *q*-ary $(d+1, q^2, d)$ code if and only if there exists $d-1$ MOLS of order *q*.

We note that when the order of classic latin squares is equivalent to the minimum distance of a code, then a complete set of MOLS exists if and only if a *q*-ary $(q+1, q^2, q)$ code exists.

Corollary 1.28 [15, Corollary 13.15]: When *q* is an order for which a complete set of MOLS exists and $d \le q$, then equality in the Singleton bound is achieved and thus there exists an MDS *q*-ary $(d+1, q^2, d)$ code.

To create a code word of an MDS code using a complete set of MOLS, we begin by recording a row and column position. Next we record the entry in that position from each square. From the MOLS in figure 1.8, row 2 and column 3 holds the entries 4, 1, and 2. This generates the code word 23412. Applying this process to all positions, we get 16 code words of length 5, listed in Figure 1.9. Any pair of code words differs in at least 4 digits, thus the minimum distance is 4. Thus this code achieves equality in the Singleton bound with $16 = 4^{5-4+1}$.

11111, 12222, 13333, 14444,

21234, 22143, 23412, 24321,

31342, 32431, 33124, 34213,

41423, 42314, 43241, 44132

Figure 1.9

Although complete sets of MOLS may be used to construct some optimal codes, the existence of complete sets of order *n* where *n* is a composite integer greater than 11 is unknown. There are conjectures on both the affirmative and negative sides of the existence of such sets. Before we present the conjectures on the existence of complete sets of MOLS, we begin with a history.

Near the end of the 18th century, Leonard Euler published two of the first papers that offer a mathematical analysis of a pair of orthogonal classic latin squares [13]. In [10] he names the new mathematical structure *Graeco-Latin square*, using Greek symbols to label one square and Latin symbols to label the other, and describes the encoding we present in Definition 1.16. In [11] Euler discusses the existence of Greaco-Latin squares for a given order. He offers the conjecture that Greaco-Latin squares do not exist for the order $4a+2$ when a is any nonnegative integer, which he calls *oddly even integers*.

The now famous context of Euler's conjecture is the 36 officers problem: Arrange 6 regiments each having 6 officers of 6 different ranks into a 6×6 square such that each row and each column holds a representative of each rank and each regiment. According to legend, Euler began his exploration of Greaco-Latin squares after the Empress Catherine the Great had posed this 36 officer question to him [13]. Although the 36 officer problem has no solution, it was not until 1960, nearly 200 years after Euler had made his conjecture, that it was proven false for all oddly even integers when $a \geq 2$ [6]. Once Euler's conjecture had been resolved, the existence of a pair of orthogonal classic latin squares of any order had been settled.

Theorem 1.29 [6]: There exists a pair of orthogonal classic latin squares for any order $n \neq 2,6$.

Let $\mathrm{Max}(n)$ denote the maximal size of a set of classic MOLS of order n. Restating Theorem 1.29, we obtain the lower bound $\mathrm{Max}(n) \geq 2$ for $n \neq 2,6$. By considering the permutations of the entries in the first row of an order n classic latin square, it is relatively simple to prove the following upper bound.

Theorem 1.30 [15, Theorem 2.1]: The size of a set of MOLS order n on n symbols can be at most $n-1$.

Proof: Without changing any of the pairwise orthogonal relations in a set of MOLS order n on n symbols, the entries of any one member of the set may be renamed. Thus for convenience, rename the entries in each member such that every first row is the list of the first $n-1$ non-negative integers. Let $g \neq h$ (distinct members), then by latin exclusion $x_{g21} \neq 0$ and $x_{h21} \neq 0$, and by orthogonal exclusion $x_{g21} \neq x_{h21}$. Thus there are at most $n-1$ symbols permitted in row 2, column 1, producing at most $n-1$ MOLS. ■

Note that with $n \times n$ magilatin squares the set of symbols is larger and there may be more than $n-1$ MOLS.

We have arrived at our original existence question: For what values of n is $\text{Max}(n) = n-1$? Some results are known. In Figure 1.5 and Figure 1.8 we have examples of complete sets of MOLS of order 3 and 4, respectively. Furthermore, Theorem 1.29 implies that no complete set of MOLS exists for order 6. In fact, it has been proven that a complete set of MOLS exists for any order that is a power of a prime [15]. However, it has been a long standing open question whether or not complete sets of MOLS exist for composite orders $n \geq 12$.

Much work has been done to improve the lower bound on the size of sets of classic MOLS of order n.

Theorem 1.31 [15, Theorem 2.10]: When an order n factors into distinct prime powers $q_1 \cdot q_2 \cdot q_3 \cdot \ldots \cdot q_r$ with $q_1 < q_2 < q_3 < \ldots < q_r$, then the size of a set of classic MOLS is at least $q_1 - 1$.

Conjecture 1.32 (McNeish) [15, Conjecture 2.11]: The number of classic MOLS of order n is precisely $q_1 - 1$, when q_1 is the smallest prime power in the factorization of n.

Conjecture 1.33 [15, Conjecture 2.25]: The MacNeish conjecture is false for all orders except 6 and prime powers.

If proven true, Conjecture 1.33 would bring the McNeish conjecture to a fate that resembles the history of the oddly even conjecture offered by Euler.

Conjecture 1.34 [15, Conjecture 2.26]: A complete set of MOLS exists if and only if the order is a prime power.

As a nod to the affirmative of Conjecture 1.34 Lam, Thiel, and Swiercz prove in [14] that for order 10 a complete set of MOLS does not exist.

Theorem 1.35 [15, Theorem 2.23]: For orders $n > 4$, if $n-3$ MOLS exist, then that set can be extended to a complete set of MOLS.

Note that with this theorem we know that the maximum number of classic MOLS of order 10 satisfies the inequality $2 \le \text{Max}(10) \le 6$. McNeish's conjecture predicts that $\text{Max}(10) = 2$. However, we are presently left to discover the exact value of $\text{Max}(10)$.

Much motivation for this paper comes from the open questions and conjectures posed above. Although this paper does not deal specifically with finding the exact value of integers such as $\text{Max}(10)$ or if $\text{Max}(12) = 11$, one could use our results to do so. We offer a theory for the enumeration of ordered *k*-sets of $m \times n$ MOLS (with $m \le n$) in terms of a parameter t, which is either a strict upper bound on the entries or the magic sum.

II. SUPPORT

2.1 Affine Structure

Recall that a *linear subspace of* $\mathbb{R}^d$ may be defined as a nonempty subset $V \subseteq \mathbb{R}^d$ such that $a\mathbf{u} + b\mathbf{v} \in V$ for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars $a, b \in \mathbb{R}$. A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is a vector $\mathbf{u} \in \mathbb{R}^d$ of the form $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$, where each $a_i \in \mathbb{R}$. The *linear span* of finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, denoted $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, is the linear subspace of $\mathbb{R}^d$ defined as the set of all linear combinations $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$.

Definition 2.1: An *affine subspace of* $\mathbb{R}^d$ is a subset $W \subseteq \mathbb{R}^d$ that is either the empty set or is a translation of a linear subspace of $\mathbb{R}^d$.

Thus in the latter case, for some linear subspace $V \subseteq \mathbb{R}^d$ and some vector $\mathbf{w} \in \mathbb{R}^d$ an affine subspace $W \subseteq \mathbb{R}^d$ may be written in the form $W = V + \mathbf{w} = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V\}$. The *dimension of an affine subspace* is the dimension of the corresponding linear subspace. An *affine space* is an affine subspace of some $\mathbb{R}^d$.

Definition 2.2: An *affine combination* is a linear combination such that the scalars sum to unity.

Thus an affine combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is a vector $\mathbf{u} \in \mathbb{R}^d$ of the form $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ such that $\sum_{i=1}^{n} a_i = 1$, where each $a_i \in \mathbb{R}$.

Definition 2.3: The *affine span* of finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is the set of all affine combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$.

Thus the affine span of a finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, denoted $\operatorname{aff}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, is the affine subspace of $\mathbb{R}^d$ defined as the set of all combinations $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ such that $\sum_{i=1}^{n} a_i = 1$, where each $a_i \in \mathbb{R}$. In particular, note that a subset $V \subseteq \mathbb{R}^d$ is an affine subspace if and only if $\operatorname{aff}(\mathbf{u}, \mathbf{v}) \subseteq V$ for all $\mathbf{u}, \mathbf{v} \in V$. Another name for affine span is *affine hull*. Next I define convex terms in analogy to affine terms. The analogy is less straight forward this iteration, and thus I use the definitions for line and segment as segue.

Definition 2.4: The *line* through $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ is the affine span of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$.

Thus we have that a line through $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ can be written in the form $\operatorname{aff}(\mathbf{u}, \mathbf{v}) = \{a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R},\ a + b = 1\} = \{a\mathbf{u} + (1-a)\mathbf{v} : a \in \mathbb{R}\}$.

Definition 2.5: The *closed segment* between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ is the set $\{a\mathbf{u} + b\mathbf{v} : a, b \ge 0,\ a + b = 1\} = \{a\mathbf{u} + (1-a)\mathbf{v} : a \in [0,1]\}$.

Definition 2.6: A *convex set* is a subset $X \subseteq \mathbb{R}^d$ that contains the closed segment between any $\mathbf{u}, \mathbf{v} \in X$.

Definition 2.7: A *convex combination* is an affine combination such that the scalars are nonnegative.

Thus a convex combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is a vector $\mathbf{u} \in \mathbb{R}^d$ of the form $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ such that $\sum_{i=1}^{n} a_i = 1$ and each $a_i \geq 0$.

Definition 2.8: The *convex span* of finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is the set of all convex combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$.

Thus the convex span of finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^d$, denoted $\operatorname{conv}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, is the convex subset of $\mathbb{R}^d$ defined as the set of all combinations $\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ such that $\sum_{i=1}^{n} a_i = 1$ and each $a_i \geq 0$. Another name for convex span is *convex hull*. To tie convex concepts back to the segue, note that the convex span of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ is the closed segment between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. In other words we have the following, $\operatorname{conv}(\mathbf{u}, \mathbf{v}) = \{a\mathbf{u} + b\mathbf{v} : a, b \geq 0,\ a + b = 1\} = \{a\mathbf{u} + (1-a)\mathbf{v} : a \in [0,1]\}$.

Definition 2.9: A *hyperplane in* $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$.

Thus a hyperplane is a set that may be described by the form $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle = b\}$ for $\mathbf{a} \in \mathbb{R}^{\mathbf{d}}$ and $b \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^d$. Intuitively, a hyperplane cuts the space $\mathbb{R}^d$ into two halves.

Definition 2.10: A *closed halfspace of* $\mathbb{R}^d$ is a set of the form $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle \leq b\}$ for $\mathbf{a} \in \mathbb{R}^{\mathbf{d}}$ and $b \in \mathbb{R}$.

An *open halfspace of* $\mathbb{R}^d$ is a set of the form $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle < b\}$ for $\mathbf{a} \in \mathbb{R}^{\mathbf{d}}$ and $b \in \mathbb{R}$. Thus intuitively, a halfspace consists of one of the two halves into which a hyperplane cuts the space, the closed halfspace includes the hyperplane and the open halfspace excludes the hyperplane. The hyperplane acts as a boundary for both halves.

Definition 2.11: A *polyhedron* is a subset $P \subseteq \mathbb{R}^d$ that is the intersection of finitely many halfspaces.

A *closed polyhedron* may be described by the form $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$ for $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$. An *open polyhedron* is the set of the form $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} < \mathbf{b}\}$ for $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$.

Definition 2.12: A *polytope* is a bounded polyhedron.

Formally, a polytope is *bounded* in the sense of the Euclidean metric, or simply that a finite maximal absolute value exists over every component of every point. Informally, the points of a polytope are bounded all around by hyperplanes. Explicitly a polytope can be described as the convex hull of a finite point set or the intersection of finitely many closed halfspaces [22, Theorem 1.1]. The former is called the *vertex description of a polytope* and has the form $\operatorname{conv}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ for some finite set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$. The latter is called the *halfspace description of a polytope* and has the form $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \le \mathbf{b}\}$ for $A \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$. We use the halfspace description to define the *relative interior of a polytope* P, denoted P°, to be $\{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} < \mathbf{b}\}$.

Definition 2.13: A polytope P of dimension *d* is called a *d-polytope*.

Note that a polytope $P \subset \mathbb{R}^d$ does not necessarily have dimension *d*.

Definition 2.14: A *supporting hyperplane* H of a polytope $P \subset \mathbb{R}^d$ is a hyperplane such that P is contained in one of the halfspaces defined by H.

Definition 2.15: A *face* of a polytope P is a set $P \cap H$ where H is a supporting hyperplane of P.

In particular, when the supporting hyperplane is the *degenerate hyperplane* $\mathbb{R}^d$, then $P \cap \mathbb{R}^d = P$ is a face of P. When H is a supporting hyperplane that does not meet P, then $P \cap H = \varnothing$ is a face of P. For a *d*-polytope, the $(d-1)$-dimensional faces are called *facets*, the 1-dimensional faces are called *edges*, and the 0-dimensional faces are the vertices of *P*. A polytope *P* is called *integral* if each vertex of *P* has integral coordinates, and *rational* when the coordinates are rational.

Definition 2.16: The *denominator* of a rational polytope P is the smallest positive integer *p* such that $p^{-1}\mathbb{Z}^d$ contains every vertex of P.

Definition 2.17: The *dilation of a polytope* P by a positive integer t is defined by $tP = \{t\mathbf{x} : \mathbf{x} \in P\}$.

2.2 Generating Functions

Definition 2.18: The *(ordinary) generating function* for a sequence of real numbers $(f(t))_{t=0}^{\infty}$ is a function $F(x)$ expressed as the formal power series

$$F(x) = \sum_{t \geq 0} f(t)x^t \ .$$

Theorem 2.19 [19, Corollary 4.3.1]: If

$$\sum_{t \geq 0} f(t)x^t = \frac{g(x)}{(1-x)^{d+1}} \, ,$$

then *f* is a polynomial of degree *d* if and only if *g* is a polynomial of degree at most *d* and $g(1) \neq 0$.

Definition 2.20: A *quasipolynomial of degree d* is a function $f : \mathbb{Z} \to \mathbb{R}$ defined by

$$f(t) = \sum_{i=0}^{d} \mathrm{c}_i(t)t^i$$

where each $\mathrm{c}_i(t)$ is a periodic function and $\mathrm{c}_d(t)$ is not the zero function.

The period of each $\mathrm{c}_i(t)$ is integral, and a period $N > 0$ common to all $\mathrm{c}_i(t)$ is called a *period of f.*

Theorem 2.21 [19, Proposition 4.4.1]: If

$$\sum_{t \geq 0} f(t)x^t = \frac{g(x)}{h(x)} \, ,$$

then f is a quasipolynomial of degree d with period dividing p if and only if g and h are polynomials such that $\deg g < \deg h$, all roots of h are p^{th} roots of unity of multiplicity at most $d+1$, and there is a root of multiplicity equal to $d+1$ (all of this assuming that $\frac{g}{h}$ has been reduced to lowest terms).

2.3 Ehrhart Theory

We wish to count the points of the discrete integral lattice $\mathbb{Z}^d$ contained in a rational convex polytope P. Furthermore, we count such points in terms of an integral parameter t that dilates the polytope tP or contracts the discrete lattice $t^{-1}\mathbb{Z}^d$.

Definition 2.22: The lattice point enumerator of a polytope $P \subset \mathbb{R}^d$ is the function $Num_P : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined by

$$Num_P(t) := \#(tP \cap \mathbb{Z}^d) = \#(P \cap t^{-1}\mathbb{Z}^d).$$

Since the function $Num_P(t)$ defines a sequence, we define the associated generating function.

Definition 2.23: The generating function for Num_P, called the *Ehrhart series of P*, is defined by

$$Ehr_P(z) := 1 + \sum_{t \geq 1} Num_P(t) z^t .$$

Lemma 2.24 [4, Proof of Theorem 3.8]: If P is a rational convex d-polytope with denominator p, then

$$Ehr_P(z) = \frac{g(z)}{(1-z^p)^{d+1}},$$

for some polynomial g of degree less than $p(d+1)$.

The form above is called the *Ehrhart rational generating function*.

Theorem 2.25 [4, Theorem 3.23] (Ehrhart's Theorem for Rational Polytopes): If P is a rational convex d-polytope, then $Num_P(t)$ is a degree d quasipolynomial in t. The period of $Num_P(t)$ divides the least common multiple of the denominators of the coordinates of the vertices of P.

Definition 2.26: The generating function for Num_{P°, called the *open Ehrhart series*, is given as

$$Ehr_{P^\circ}(z) := \sum_{t \geq 1} Num_{P^\circ}(t) z^t .$$

Theorem 2.27 [4, Theorem 4.4] (Ehrhart Reciprocity): Suppose P is a convex rational polytope. The evaluation of the rational function Ehr_P for z^{-1} yields

$$Ehr_P\left(z^{-1}\right) = (-1)^{\dim P+1} Ehr_{P^\circ}(z).$$

These theorems lead to the Ehrhart-Macdonald theorem.

Theorem 2.28 [4, Theorem 4.1] (Ehrhart-Macdonald reciprocity): Suppose P is a convex rational polytope. Then evaluation of the quasipolynomial Num_P at negative integers yields

$$Num_P(-t) = (-1)^{\dim P} Num_{P^\circ}(t).$$

2.4 Posets and Lattices

Definition 2.29: A *partial order* is a relation $\leq$ on a nonempty set X such that

for all x in X, $x \leq x$ (reflexive);

if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric);

if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive).

Definition 2.30: A *partially ordered set* is a pair $(X, \leq)$ where X is a nonempty set and $\leq$ is a partial order on X.

A *poset* is a partially ordered set [19]. In general we will refer to a poset $(X, \leq)$ simply as the poset X.

Definition 2.31: An *upper bound* on elements x and y of a poset X is an element $z \in X$ such that $x \leq z$ and $y \leq z$. The *lower bound* is defined dually.

Definition 2.32: A *least upper bound* on elements x and y of a poset X is an upper bound z on x and y such that $z \leq w$ for every upper bound w on x and y. The *greatest lower bound* is defined dually.

When the least upper bound on elements x and y of a poset X exists, then it is unique and is denoted $x \vee y$. Similarly when the greatest lower bound on elements x and y of a poset X exists, then it is unique and is denoted $x \wedge y$.

Definition 2.33: There is a *minimal element of a poset* X, denoted $\hat{0}$, if there exists an element $\hat{0}$ of X such that $x \geq \hat{0}$ for all $x \in X$.

Definition 2.34: There is a *maximal element of a poset* X, denoted $\hat{1}$, if there exists an element $\hat{1}$ of X such that $x \leq \hat{1}$ for all $x \in X$.

Definition 2.35: A *lattice* is a poset L in which every pair of elements has both a least upper bound and a greatest lower bound.

For a lattice L the least upper bound $\vee$ and the greatest lower bound $\wedge$ are binary operations called the *join* and *meet* respectively. If every pair of elements of L has a join, but not necessarily a meet, then we call L a *join semilattice*. Similarly if every pair of elements of L has a meet, but not necessarily a join, then we call L a *meet semilattice*.

Definition 2.36: Elements x and y of a poset X are *comparable* if $x \leq y$ or $y \leq x$.

Definition 2.37: A *chain* is a poset in which any two elements are comparable.

Definition 2.38: The *length* of a finite chain is its number of elements minus one.

Definition 2.39: A poset X is *graded* if every maximal chain of X has the same length.

Definition 2.40: For elements x and y of a poset X, y *covers* x if $x < y$ and there does not exist a $z \in X$ such that $x < z < y$.

Definition 2.41: A finite *semimodular* lattice L is a graded lattice such that if x and y cover $x \wedge y$, then $x \vee y$ covers both x and y.

Definition 2.42: An *atom* of a finite lattice L is an element covering $\hat{0}$.

A finite lattice L is *atomic* if every element of L is the join of atoms.

Definition 2.43: A finite *geometric* lattice L is an atomic semimodular lattice.

Definition 2.44: The *Möbius function* of a finite poset X the function $\mu : X \times X \to \mathbb{Z}$ defined recursively by

$$\mu(r,s) := \begin{cases} 0 & \text{if } r \not\leq s \\ 1 & \text{if } r = s \\ -\sum_{r \leq u < s} \mu(r,u) & \text{if } r < s \end{cases}.$$

2.5 Hyperplane Arrangements

Definition 2.45: A *hyperplane arrangement* is a finite set H of hyperplanes in $\mathbb{R}^d$.

In the definition above, $\mathbb{R}^d$ is called the *ambient space* of the hyperplane arrangement. When the ambient space is $\mathbb{R}^2$, the hyperplanes are lines. When the ambient space is $\mathbb{R}^3$, the hyperplanes are planes. Sometimes it is useful to include the empty set and $\mathbb{R}^d$ in a hyperplane arrangement. In this case the empty set is called the *empty hyperplane*, and $\mathbb{R}^d$ is called the *degenerate hyperplane*. We call the hyperplane arrangement H *rational* when each hyperplane in H is spanned by the rational points that it contains [2]. One may imagine that a hyperplane arrangement slices up the ambient space into regions.

Definition 2.46: An *open region of* H is a connected component of $\mathbb{R}^d \setminus \bigcup H$.

Definition 2.47: A *closed region of* H is the topological closure of an open region.

Note that these regions are not necessarily bounded.

Definition 2.48: The *intersection semilattice of a hyperplane arrangement H* is the set of subsets defined by

$$L(H) := \left\{ \bigcap S : S \subseteq H, \bigcap S \neq \varnothing \right\},$$

partially ordered by reverse inclusion: $u \leq v$ iff $u \supseteq v$ for $u, v \in L(H)$.

Definition 2.49: The elements of $L(H)$ are called the *flats* of H.

For flats u and v in $L(H)$ such that $u \bigcap v \neq \varnothing$, the join operation is given by $u \vee v = u \bigcap v$ and the meet operation is given by $u \wedge v = \bigcap \left\{ h \in H : h \supseteq \left(u \bigcup v \right) \right\}$ [21]. The semilattice $L(H)$ is a meet-semilattice, and more specifically is a geometric semilattice with $\hat{0} = \mathbb{R}^d$ [1]. Furthermore, if the intersection of all hyperplanes is non-emtpy, then $L(H)$ is a geometric lattice with $\hat{1} = \bigcap H$ [1].

Definition 2.50: The *rank* of a flat $u \in L(H)$ is given by

$$\lambda(u) = \text{the length of the longest chain in } L(H) \text{ below it.}$$

2.6 Inside-Out Polytopes

Definition 2.51: An *inside-out polytope* is a polytope P together with a hyperplane arrangement H, and is denoted (P, H).

One may think of the hyperplanes as acting as boundaries inside the polytope, hence the name inside-out. We call (P, H) *closed* when P is closed.

Definition 2.52: We say (P, H) is a *rational inside-out polytope* when P is a rational polytope and H is a set of rational hyperplanes.

Definition 2.53: A *region of* (P, H) is the nonempty intersection of a region of H with P.

Definition 2.54: A *vertex of* (P,H) is a vertex of any region of (P,H).

Definition 2.55: The *denominator* of (P,H) is the smallest positive integer t such that $t^{-1}\mathbb{Z}^d$ contains every vertex of (P,H).

Definition 2.56: The *intersection poset of* (P°,H) is defined as

$$L(P^\circ,H) := \left\{P^\circ \cap \bigcap S : S \subseteq H\right\} \setminus \{\varnothing\},$$

ordered by reverse inclusion.

The poset $L(P^\circ,H)$ is a ranked poset, every interval is a geometric lattice, and it equals the intersection semilattice $L(H)$ if $\bigcap H$ meets P° [2].

Definition 2.57: The *arrangement induced by H* on a subset $s \subseteq \mathbb{R}^d$ is the set defined as $H^s := \left\{h \cap s : h \in H, h \not\supseteq s\right\}$.

Definition 2.58: A hyperplane arrangement H is *transverse* to a polytope P if every flat $u \in L(H)$ that intersects P also intersects the interior of the polytope P°, and P is not contained in $\bigcup H$.

2.7 Inside-Out Polytope Ehrhart Theory (Hyperplane Arrangements)

Definition 2.59: The characteristic polynomial of H is defined by

$$p_H(\lambda) = \begin{cases} 0 & \text{if } H \text{ contains the degenerate hyperplane} \\ \displaystyle\sum_{u \in L(H)} \mu\left(\hat{0},u\right)\lambda^{\dim u} & \text{otherwise} \end{cases}$$

where μ is the mobius function of $L(H)$.

For example, a hyperplane arrangement in $\mathbb{R}^2$ is shown in Figure 2.1.

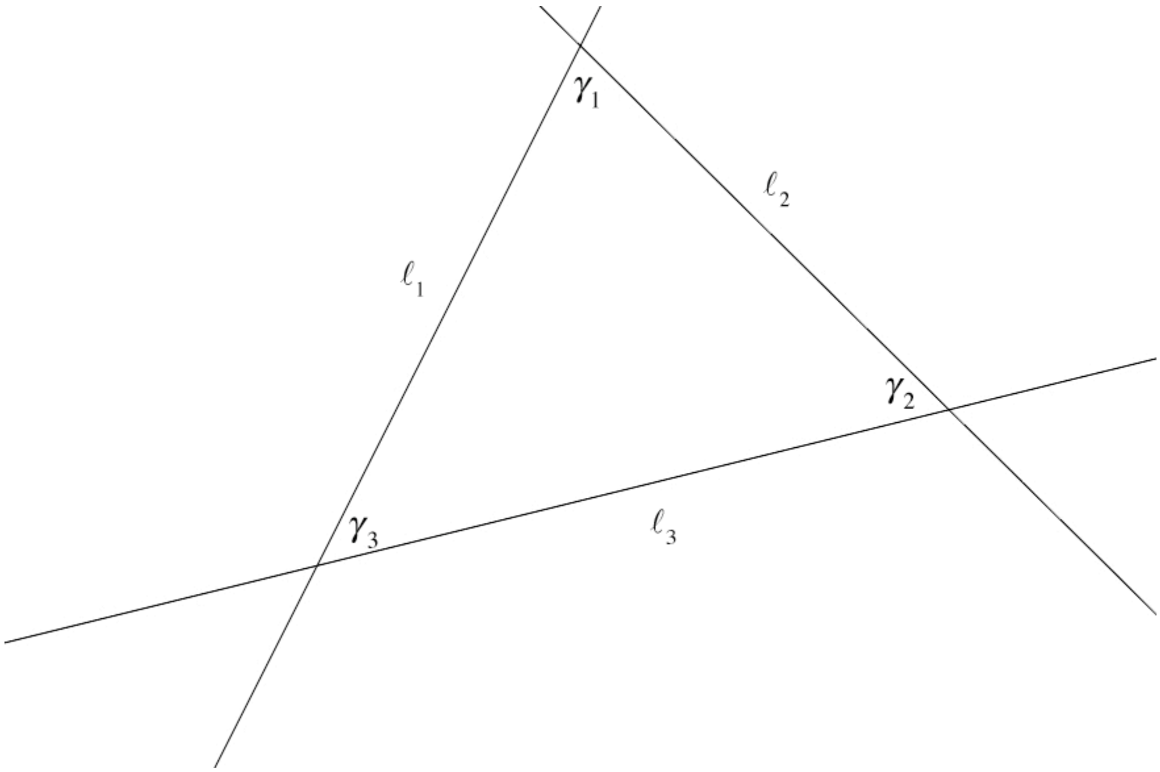

Figure 2.1

The arrangement is $H = \{\ell_1, \ell_2, \ell_3\}$. The hyperplanes are lines labeled ℓ_1, ℓ_2, ℓ_3. There are three intersections, each a point, given as $\gamma_1 = \ell_1 \cap \ell_2$, $\gamma_2 = \ell_2 \cap \ell_3$, $\gamma_3 = \ell_1 \cap \ell_3$.

The geometric lattice associated with H is shown in Figure 2.2.

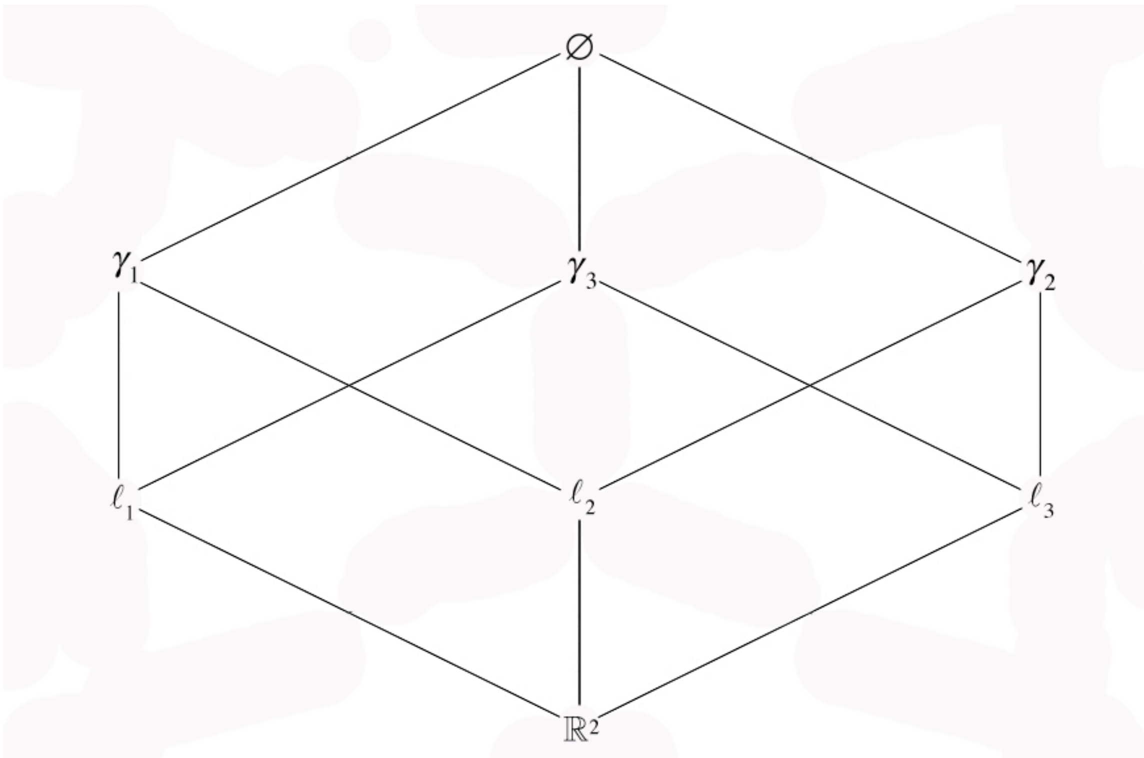

Figure 2.2

The intersection semilattice of the arrangement H is

$L(H) = \{\mathbb{R}^2, \{\ell_1\}, \{\ell_2\}, \{\ell_3\}, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}\}$.

Mobius function values are

$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$

$\mu(\mathbb{R}^2, \ell_1) = \mu(\mathbb{R}^2, \ell_2) = \mu(\mathbb{R}^2, \ell_3) = -\mu(\mathbb{R}^2, \mathbb{R}^2) = -1$

$\mu(\mathbb{R}^2, \gamma_1) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_1) + \mu(\mathbb{R}^2, \ell_2)\right) = 1$

$\mu(\mathbb{R}^2, \gamma_2) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_2) + \mu(\mathbb{R}^2, \ell_3)\right) = 1$

$\mu(\mathbb{R}^2, \gamma_3) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_1) + \mu(\mathbb{R}^2, \ell_3)\right) = 1$

$$
\begin{aligned}
&\mu(\mathbb{R}^2,\varnothing)\\
&=-\left(\mu(\mathbb{R}^2,\mathbb{R}^2)+\mu(\mathbb{R}^2,\ell_1)+\mu(\mathbb{R}^2,\ell_2)+\mu(\mathbb{R}^2,\ell_3)+\mu(\mathbb{R}^2,\gamma_1)+\mu(\mathbb{R}^2,\gamma_2)+\mu(\mathbb{R}^2,\gamma_3)\right)\\
&=-1
\end{aligned}
$$

Rank function values are

$\lambda(\mathbb{R}^2)=0$

$\lambda(\ell_1)=\lambda(\ell_2)=\lambda(\ell_3)=1$

$\lambda(\gamma_1)=\lambda(\gamma_2)=\lambda(\gamma_3)=2$

$\lambda(\varnothing)=3$

The characteristic polynomial of H is

$$
\begin{aligned}
&p_H(\lambda)\\
&=\mu(\mathbb{R}^2,\mathbb{R}^2)\lambda^2\\
&+\mu(\mathbb{R}^2,\ell_1)\lambda+\mu(\mathbb{R}^2,\ell_2)\lambda+\mu(\mathbb{R}^2,\ell_3)\lambda\\
&+\mu(\mathbb{R}^2,\gamma_1)\lambda^0+\mu(\mathbb{R}^2,\gamma_2)\lambda^0+\mu(\mathbb{R}^2,\gamma_3)\lambda^0\\
&=\lambda^2-3\lambda+3
\end{aligned}
$$

Definition 2.60: For a set or point X in $\mathbb{R}^d$,

$$H(X) \coloneqq \{h\in H : X\subseteq h\}.$$

Definition 2.61: For a set X of points in $\mathbb{R}^d$, the smallest flat of H that contains X is given by

$$sflat(X) \coloneqq \bigcap\{h\in H : X\subseteq h\}.$$

Definition 2.62: Let P be a full-dimensional convex polytope of $\mathbb{R}^d$ and H a transverse hyperplane arrangement. The multiplicity of $x\in\mathbb{R}^d$ with respect to P and H is given by

$$m_{P,H}(x) = \begin{cases} (-1)^{\operatorname{codim} sflat(x)} p_{H(x)}(-1) & \text{if } x \in P, \\ 0 & \text{if } x \notin P. \end{cases}$$

It may be noteworthy to the reader that when $x \notin P$, we have that $m_{P,H}(x)$ is the number of closed regions of (P,H) that contain x [1].

Definition 2.63: For $t = 1,2,3\ldots$ the Ehrhart quasipolynomials of (P,H) are the *closed Ehrhart quasipolynomial*

$$E_{P,H}(t) \coloneqq \sum_{\mathbf{x} \in t^{-1}\mathbb{Z}^d} m_{P,H}(\mathbf{x})$$

and the *open Ehrhart quasipolynomial*

$$E^{\circ}_{P^{\circ},H}(t) \coloneqq \#\left(t^{-1}\mathbb{Z}^d \cap \left[P^{\circ} \setminus \bigcup H\right]\right).$$

The theorem below justifies the use of the terminology in the above definition.

Theorem 2.64 [1, Theorem 4.1]: If the discrete lattice $\mathbb{Z}^d$ is full-dimensional and (P,H) is a closed, full-dimensional, rational inside-out polytope in $\mathbb{R}^d$ such that *H* does not contain the degenerate hyperplane, then $E_{P,H}$ and $E^{\circ}_{P^{\circ},H}$ are quasipolynomials in *t*, with period equal to a divisor of the denominator of (P,H), with leading term $\left(\operatorname{vol}_{\mathbb{Z}^d} P\right)t^d$, and with the constant term $E_{P,H}(0)$ equal to the number of regions of (P,H). Furthermore,

$$E^{\circ}_{P^{\circ},H}(t) = (-1)^d E_{P,H}(-t).$$

By $\operatorname{vol}_{\mathbb{Z}^d} P$ we denote the volume of *P* normalized with respect to $\mathbb{Z}^d \bigcap \operatorname{aff}(P)$, in other words we take the volume of the fundamental domain $\mathbb{Z}^d \bigcap \operatorname{aff}(P)$ to be 1. This is the ordinary volume when *P* is full dimensional.

Theorem 2.65 [1, Theorem 4.2]: If the discrete lattice $\mathbb{Z}^d$, the polytope *P*, and the hyperplane arrangement *H* are as in Theorem 2.1, except that *P* need not be closed, then

$$E^{\circ}_{P^{\circ},H}(t) = \sum_{u \in L(P^{\circ},H)} \mu\left(\hat{0},u\right) E_{P^{\circ} \bigcap u}(t)$$

and if *H* is transverse to *P*, then

$$E_{P,H}(t) = \sum_{u \in L(P^{\circ},H)} \left|\mu\left(\hat{0},u\right)\right| E_{P \bigcap u}(t).$$

2.8 Subspace Arrangements

Definition 2.66: An *affine subspace arrangement* is a finite set A of affine subspaces in a real vector space.

When the ambient space is $\mathbb{R}^2$, the affine subspaces may be lines or points. When the ambient space is $\mathbb{R}^3$, the affine subspaces may be planes, lines, or points. One may imagine that an affine subspace arrangement can slice up and poke holes in the ambient space.

Definition 2.67: The *intersection semilattice of an affine subspace arrangement A* is the set of subsets defined by

$$L(A) := \left\{ \bigcap S : S \subseteq A, \bigcap S \neq \varnothing \right\},$$

partially ordered by reverse inclusion: $u \leq v$ iff $u \supseteq v$ for $u, v \in L(A)$.

The semilattice $L(A)$ is not necessarily geometric or ranked, since it may not be semimodular.

Definition 2.68: The semilattice $L(A)$ is *extrinsically graded* by the rank function

$$\rho(u) = \operatorname{codim} u \, ,$$

and the total rank

$$\rho(L) = d \, ,$$

such that u has extrinsic corank

$$\rho(L) - \rho(u) = \dim u \, .$$

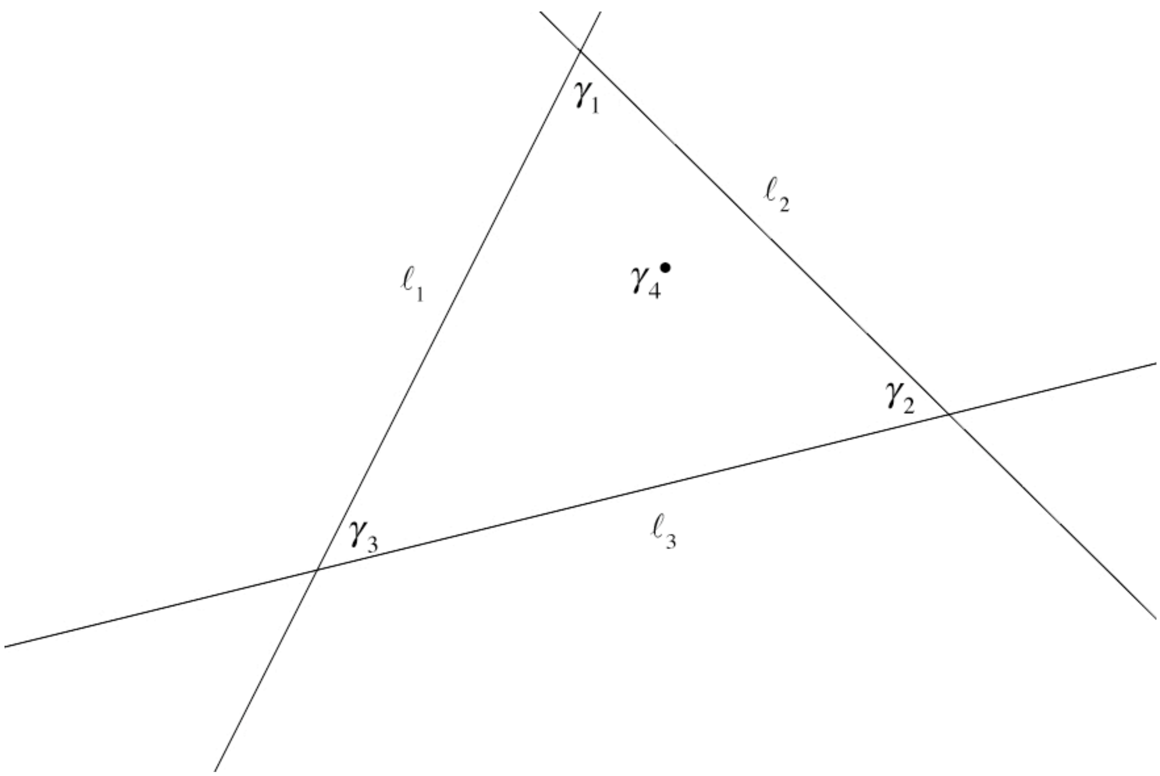

Figure 2.3

Take for example the arrangement presented in Figure 2.1. Simply include a point in the set to create an affine subspace arrangement in $\mathbb{R}^2$, as shown in Figure 2.3. The arrangement is $A = \{\ell_1, \ell_2, \ell_3, \gamma_4\}$. The affine subspaces are the lines ℓ_1, ℓ_2, ℓ_3, and the point γ_4. There are three intersections, each a point, given as $\gamma_1 = \ell_1 \cap \ell_2$, $\gamma_2 = \ell_2 \cap \ell_3$, $\gamma_3 = \ell_1 \cap \ell_3$.

The intersection semilattice of the arrangement A is

$L(A) = \left\{\mathbb{R}^2, \{\ell_1\}, \{\ell_2\}, \{\ell_3\}, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_4\}\right\}$.

Mobius function values are

$$\mu(\mathbb{R}^2, \mathbb{R}^2) = 1$$

$$\mu(\mathbb{R}^2, \ell_1) = \mu(\mathbb{R}^2, \ell_2) = \mu(\mathbb{R}^2, \ell_3) = -\mu(\mathbb{R}^2, \mathbb{R}^2) = -1$$

$$\mu(\mathbb{R}^2, \gamma_1) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_1) + \mu(\mathbb{R}^2, \ell_2)\right) = 1$$

$$\mu(\mathbb{R}^2, \gamma_2) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_2) + \mu(\mathbb{R}^2, \ell_3)\right) = 1$$

$$\mu(\mathbb{R}^2, \gamma_3) = -\left(\mu(\mathbb{R}^2, \mathbb{R}^2) + \mu(\mathbb{R}^2, \ell_1) + \mu(\mathbb{R}^2, \ell_3)\right) = 1$$

$$\mu(\mathbb{R}^2, \gamma_4) = -\mu(\mathbb{R}^2, \mathbb{R}^2) = -1$$

$$\begin{aligned}
&\mu(\mathbb{R}^2, \varnothing) \\
&= -(\mu(\mathbb{R}^2, \mathbb{R}^2) \\
&+\mu(\mathbb{R}^2, \ell_1) + \mu(\mathbb{R}^2, \ell_2) + \mu(\mathbb{R}^2, \ell_3) \\
&+\mu(\mathbb{R}^2, \gamma_1) + \mu(\mathbb{R}^2, \gamma_2) + \mu(\mathbb{R}^2, \gamma_3) \\
&+\mu(\mathbb{R}^2, \gamma_4)) \\
&= 0
\end{aligned}$$

Rank function values are

$\rho(\mathbb{R}^2) = 0$

$\rho(\ell_1) = \rho(\ell_2) = \rho(\ell_3) = 1$

$\rho(\gamma_1) = \rho(\gamma_2) = \rho(\gamma_3) = \rho(\gamma_4) = 2$

$\lambda(\varnothing) = 3$

The characteristic polynomial of A is

$$
\begin{aligned}
&p_H(\lambda) \\
&= \mu(\mathbb{R}^2,\mathbb{R}^2)\lambda^2 \\
&+\mu(\mathbb{R}^2,\ell_1)\lambda + \mu(\mathbb{R}^2,\ell_2)\lambda + \mu(\mathbb{R}^2,\ell_3)\lambda \\
&+\mu(\mathbb{R}^2,\gamma_1)\lambda^0 + \mu(\mathbb{R}^2,\gamma_2)\lambda^0 + \mu(\mathbb{R}^2,\gamma_3)\lambda^0 + \mu(\mathbb{R}^2,\gamma_4)\lambda^0 \\
&= \lambda^2 - 3\lambda + 2
\end{aligned}
$$

2.9 Inside-Out Polytope Ehrhart Theory (Subspace Arrangements)

Definition 2.69: For a set or a point X in $\mathbb{R}^d$,

$$A(X) \coloneqq \{a \in A : X \subseteq a\}.$$

Definition 2.70: Let P be a full-dimensional convex polytope of $\mathbb{R}^d$ and A a transverse affine subspace arrangement. The multiplicity of $x \in \mathbb{R}^d$ with respect to P and A is given by

$$m_{P,A}(x) = \begin{cases} (-1)^d p_{A(x)}(-1) = \sum_{u \in L(A): x \in u} \mu\left(\hat{0}, u\right)(-1)^{\rho(u)} & \text{if } x \in P, \\ 0 & \text{if } x \notin P. \end{cases}$$

Definition 2.71: For $t = 1,2,3\ldots$ the Ehrhart quasipolynomials of (P, A) are the *closed Ehrhart quasipolynomial*

$$E_{P,A}(t) \coloneqq \sum_{x \in t^{-1}\mathbb{Z}^d} m_{P,A}(x)$$

and the *open Ehrhart quasipolynomial*

$$E^{\circ}_{P^{\circ},A}(t) \coloneqq \#\left(t^{-1}\mathbb{Z}^d \cap \left[P^{\circ} \setminus \bigcup A\right]\right).$$

Theorem 2.72 [1, Theorem 8.2]: If P is a full-dimensional, rational convex polytope, and A is a rational affine subspace arrangement, then $E_{P,A}$ and $E^{\circ}_{P^{\circ},A}$ are quasipolynomials in *t*, with period equal to a divisor of the denominator of (P, A), with leading term $\left(\mathrm{vol}_{\mathbb{Z}^d}\ P\right)t^d$. Furthermore,

$$E^{\circ}_{P^{\circ},A}(t) = (-1)^d E_{\bar{P},A}(-t),$$

$$E^{\circ}_{P^{\circ},A}(t) = \sum_{u \in L(A)} \mu\left(\hat{0}, u\right) E_{P \cap u}(t),$$

and

$$E_{P,A}(t) = \sum_{u \in L(A)} \mu\left(\hat{0}, u\right) (-1)^{\operatorname{codim} u} E_{P \cap u}(t).$$

III. AN ENUMERATIVE GEOMETRY FOR *k*-SETS OF MUTUALLY ORTHOGONAL LATIN ARRAYS

3.1 Introducing the Two Methods

This paper presents two methods for the enumeration of MOLS. One method uses an encoding that relates each pair of k members from a k-set of MOLS to which we apply the general theory of inside-out polytopes where a polytope is supplemented with a hyperplane arrangement. We will call this the *MOLE method*, since the method uses a Mutually Orthogonal Latin Encoding. In the MOLE method, the encoding allows us to express all latin exclusions and all orthogonal exclusions as hyperplanes.

Without the encoding the orthogonal exclusions are affine subspaces defined by the intersection of a pair of hyperplanes. The second method considers such a setting to which we apply the theory of inside-out polytopes where a polytope is supplemented with an arrangement of affine subspaces. We will call this the *MOLS holes method*.

In both methods we count the number of MOLS as a function of a parameter t. We call the count *cubical* when t is the strict upper bound on the entries of the latin arrays. We call the count *affine* when t is the magic sum of the latin arrays. The following sections establish each count for each method. In all cases, we show that the theory of inside-out polytopes implies that the number of MOLS is a quasipolynomial in terms of the parameter t.

3.2 The MOLE Method

3.2.1 The MOLE Setting

One of the first mathematical papers to treat two orthogonal classic latin squares as a mathematical object was written by Euler in the eighteenth century. The paper [10] describes a method of encoding a pair of orthogonal classic latin squares of order *n* as a classic semimagic square of order *n*. We offer a proof of this in Chapter I.

A pair of latin arrays $\left(a_{ij}\right)_{\substack{1\le i\le m \\ 1\le j\le n}}$ and $\left(b_{ij}\right)_{\substack{1\le i\le m \\ 1\le j\le n}}$ are orthogonal when the ordered pairs $\left(a_{ij},b_{ij}\right)$ are distinct for $1\le i\le m$ and $1\le j\le n$. In this case an orthogonal exclusion is of the form $\left(a_{ij},b_{ij}\right)=\left(a_{pq},b_{pq}\right)$ where $i\ne p$ and $j\ne q$. Thus an orthogonal exclusion is two simultaneous equations $a_{ij}=a_{pq}$ and $b_{ij}=b_{pq}$. Each equation represents a hyperplane, and the two simultaneous equations represent the intersection of two hyperplanes. For the MOLE method we wish to extend the idea of the Euler encoding to latin arrays such that it captures an orthogonality exclusion in a single equation, and thus as a single hyperplane. The idea is to introduce a number d to encode the two simultaneous equations $\left(a_{ij},b_{ij}\right)=\left(a_{pq},b_{pq}\right)$ as the single equation $da_{ij}+b_{ij}=da_{pq}+b_{pq}$. Since our theorems require that d be rational, the encoding does not give us $\left(a_{ij},b_{ij}\right)=\left(a_{pq},b_{pq}\right)$ if and only if $da_{ij}+b_{ij}=da_{pq}+b_{pq}$, however it will suffice. We continue this discussion with respect to computation in section 4.1.

We begin by establishing a workable notation for k-sets of MOLS. For a k-set of $m \times n$ MOLS, let x_{lij} refer to the entry in row i and column j of latin array l. Then we denote k mutually orthogonal $m \times n$ latin arrays as $\left(x_{1ij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, $\left(x_{2ij}\right)_{\substack{1\le i\le m\\1\le j\le n}},\ldots,\left(x_{kij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$. We fix the integers $k \ge 2$, $m \ge 2$, and $n \ge 3$ with $m \le n$, and generalize the Euler encoding to a pair of orthogonal mates of a k-set of $m \times n$ MOLS.

Definition 3.1 (generalized Euler encoding): For k mutually orthogonal $m \times n$ latin arrays $\left(x_{1ij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, $\left(x_{2ij}\right)_{\substack{1\le i\le m\\1\le j\le n}},\ldots,\left(x_{kij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ we introduce a number d and encode a pair $\left(x_{gij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ and $\left(x_{hij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, with $g \ne h$, as an array $\mathbf{y} = \left(y_{ghij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ defined by the linear combination

$$\left(y_{ghij}\right)_{\substack{1\le i\le m\\1\le j\le n}} = d\left(x_{gij}\right)_{\substack{1\le i\le m\\1\le j\le n}} + \left(x_{hij}\right)_{\substack{1\le i\le m\\1\le j\le n}}.$$

Definition 3.2: A *k-set of mutually orthogonal latin encoded $m \times n$ arrays*, or simply a *k-MOLE*, is a k-set of $m \times n$ MOLS with the generalized Euler encoding.

We have defined a k-MOLE in terms of k latin arrays with certain requirements and exclusions, and a parameter t (either the strict upper bound on the entries of the latin arrays or the magic sum or the latin arrays). The latin arrays of a k-MOLE inherit the entry requirements, magic requirements, parameter t, and the latin exclusions from the latin arrays. The orthogonality exclusions are encoded as linear combinations using the generalized Euler encoding. Taken together, we will use the entry requirements, the magic requirements, and the parameter t to define a polytope. Secondly, we will use the latin exclusions and orthogonality exclusions to define hyperplanes of a hyperplane arrangement.

We treat a k-MOLE as an integer vector $\mathbf{z} \in \mathbb{Z}_{>0}^{kmn}$. We use the magic requirements to determine a subspace s, called the *magic subspace*, in which the integer vector lives. The magic subspace depends on whether the latin arrays are square or rectangular. We define both below.

Definition 3.3: For $m = n$ the k latin arrays are square and the magic subspace is

$$s_{k,n,n} := \left\{ \mathbf{z} \in \mathbb{R}^{kn^2} : \sum_j x_{1ij} = \sum_i x_{1ij} = \sum_j x_{2ij} = \sum_i x_{2ij} = \cdots = \sum_j x_{lij} = \sum_i x_{lij} \right\}.$$

Definition 3.4: For $m < n$ the k latin arrays are rectangular and the magic subspace is

$$s_{k,m,n} := \left\{ \mathbf{z} \in \mathbb{R}^{kmn} : \sum_j x_{1ij} = \sum_j x_{2ij} = \cdots = \sum_j x_{lij} \right\}.$$

When the latin arrays are square, then all line sums are equal to each other. When the latin arrays are rectangular, only the sums across each row are set equal to each other. The difference between two sides of any equality from $s_{k,n,n}$ or $s_{k,m,n}$ is equal to zero, and thus produces a homogeneous, linear equation with rational coefficients. Furthermore in both $s_{k,n,n}$ and $s_{k,m,n}$ we take sums across each row of each latin array. Thus each entry of all latin arrays appears in at least one of the homogeneous, linear equations with rational coefficients. Let s denote the magic subspace defined by $s_{k,n,n}$ or $s_{k,m,n}$.

The arrays of a k-MOLE also have entry requirements, specifically that the entries be positive. Thus we wish to restrict the vector $\mathbf{z}$ to $\mathbb{R}_{>0}^{kmn}$. However, we wish to use the coordinate hyperplanes in the definition of a polytope. To achieve both of these goals, we first confine the vector $\mathbf{z}$ to $\mathbb{R}_{\geq 0}^{kmn}$. Secondly we do not count vectors on coordinate hyperplanes, thus eliminating any vectors with 0 entries.

In addition to the magic requirements and the entry requirements, there is the requirement determined by the parameter t, the magic sum (affine) or entry strict upper bound (cubical). Taken together the magic requirements, the entry requirements, and the parameter t are used to define a *magic polytope* P. Depending whether the parameter t is affine or cubical, the integer vector $\mathbf{z}$ is confined to one of two magic polytopes. We define both below.

Definition 3.5: If the (affine) parameter t represents the magic sum of the latin arrays, then the magic polytope is

$$P_a := s \bigcap \left\{ \mathbf{z} \in \mathbb{R}_{\geq 0}^{kmn} : \sum_j x_{lij} = 1 \right\},$$

called the *affine magic polytope*.

Consequently when the latin arrays are square, all line sums equal 1 in the affine magic polytope P_a. When the latin arrays are rectangular, all row sums equal 1 in P_a. Also notice that in P_a the latin array entries are bounded below $x_{lij} \geq 0$. Thus all the points of P_a are nonnegative. Furthermore, all the points in the interior of P_a, denoted P_a°, do not lie on coordinate hyperplanes, and thus are positive. Since every entry variable x_{lij} appears in at least one of the equations of magic from the magic subspace, P_a is bounded.

Definition 3.6: If the (cubical) parameter t represents a strict upper bound on the entries of the latin arrays, then the magic polytope is

$$P_c := s \bigcap [0,1]^{kmn},$$

called the *cubical magic polytope.*

In P_c the latin array entries are bounded $0 \le x_{lij} \le 1$, and thus P_c is bounded by the *kmn*-dimensional hypercube. Note that the points in P_c are nonnegative, and the points in the interior of P_c, denoted P_c°, are positive since they do not lie on coordinate hyperplanes.

We let P denote the magic polytope P_a or P_c. The interior of the magic polytope is denoted P°. Finally we use tP, tP°, tP_a, tP_a°, tP_c, and tP_c° to denote the *t*-dilate of the respective magic polytope.

To apply our theorems, we offer the following lemma to verify that the magic polytope *P* affinely spans the magic subspace *s*.

Lemma 3.7: If the magic polytope P is not contained within a coordinate hyperplane, then *P* affinely spans the magic subspace *s*.

Proof: By hypothesis, P contains points z^i (not necessarily distinct) with $0 < (z^i)_i \le 1$. The centroid of these points lies in $s \bigcap (0,1]^{kmn}$, and thus in P. Hence $P^\circ = s \bigcap (0,1]^{kmn}$, which spans s.

■

Thus far we have created a setting to count k-MOLEs as integer vectors $\mathbf{z} \in \mathbb{Z}_{>0}^{kmn}$ that are interior to a magic polytope P. However, we do not want to count the vectors whose entries satisfy the latin or orthogonal exclusions. We will use these exclusions to define the hyperplanes of a hyperplane arrangement.

Thus we supplement the magic polytope with the hyperplane arrangement. In the MOLE method, each hyperplane captures either a latin exclusion or an orthogonality exclusion. The latin exclusions guarantee that the entries in each latin array be distinct within a row or a column. The orthogonality exclusions guarantee that the generalized Euler encoded linear combinations are distinct with respect to each pair of arrays from the k-MOLE.

For the following definitions, we recall some of the notation used in the definition of the generalized Euler encoding. The generalized Euler encoding expresses an orthogonal exclusion as the linear combination $y_{ghij} = dx_{gij} + x_{hij}$, where x_{lij} denotes the entry in row i and column j of latin array l. For the following definitions and theorems we require d to be rational.

Definition 3.8: Each *latin exclusion of a k-MOLE* is a rational equation of the form

$$x_{lgh} = x_{lgj},$$

where $1 \le l \le k$, $1 \le g \le m$, and $1 \le h < j \le n$,

or the form

$$x_{lgh} = x_{lih},$$

where $1 \le l \le k$, $1 \le h \le n$, and $1 \le g < i \le m$.

Thus the set of latin exclusions is

$$H_{latin} = \left\{ s \cap (x_{111} = x_{112}), s \cap (x_{111} = x_{121}), \cdots, s \cap (x_{k(m-1)n} = x_{kmn}) \right\},$$

where s is the magic subspace.

For the next definition, we obtain y_{ghij} and y_{ghpq} using the generalized Euler encoding from Definition 3.1.

Definition 3.9: Each *orthogonal exclusion of a k-MOLE* is a linear equation with rational coefficients and has the form

$$y_{ghij} = y_{ghpq},$$

where y represent generalized Euler encodings in Definition 3.1, and $g \ne h$ (pair of distinct MOLS), and $i \ne p$ and $j \ne q$.

Thus the set of orthogonal exclusions is

$$H_{orthog} = \begin{Bmatrix} s \cap (y_{1211} = y_{1222}), s \cap (y_{1211} = y_{1223}), \cdots, s \cap (y_{12(m-1)(n-1)} = y_{12mn}), \\ s \cap (y_{1311} = y_{1322}), s \cap (y_{1311} = y_{1323}), \cdots, s \cap (y_{13(m-1)(n-1)} = y_{13mn}), \\ ,\ldots, \\ s \cap (y_{(k-1)k11} = y_{(k-1)k22}), s \cap (y_{(k-1)k11} = y_{1223}), \cdots, s \cap (y_{(k-1)k(m-1)(n-1)} = y_{(k-1)kmn}) \end{Bmatrix},$$

where s is the magic subspace.

Note that H_{orthog} does not include any orthogonal exclusion $y_{ghij} = y_{ghpq}$ that has $i = p$ or $j = q$. These exclusions are already captured in H_{latin}. The difference between the two sides of either the latin exclusions or the orthogonal exclusions is equal to zero, and thus produces homogeneous, linear equations with rational coefficients. Thus we have the rational linear hyperplane arrangement

$$H = H_{latin} \bigcup H_{orthog}.$$

For both the affine and cubical parameter t, (P, H) is a closed rational inside-out polytope in $\mathbb{R}^{\dim s}$. To verify that P and H are transverse, we offer the following lemma.

Lemma 3.10: P and H are transverse if a k-MOLE with positive entries exists.

Proof: This proof applies when P is either P_a or P_c. Since $\frac{1}{n}\mathbf{1} \in \bigcap H$ and $\frac{1}{n}\mathbf{1} \in P^\circ$, we have that $P^\circ \cap \bigcap H \neq \varnothing$. A k-MOLE with positive entries exists if and only if the magic subspace s does not lie in $\bigcup H$. When a k-MOLE with positive entries exist, then $P \not\subseteq \bigcup H$. Hence H is transverse to P, when a k-MOLE with positive entries exists. ■

In this setting, a k-MOLE is an integer lattice point $\mathbf{z}$ in the t-dilate of the magic polytope tP. Furthermore, if a k-MOLE has positive entries, then the integer lattice point $\mathbf{z}$ lies in the t-dilate of the relative interior of the magic polytope tP°. We count the number of such integer points $\mathbf{z}$ that do not lie in any of the hyperplanes of H. Thus $\#\left(\mathbb{Z}^{kmn} \cap \left[tP^\circ \setminus \bigcup H\right]\right) = \#\left(\frac{1}{t}\mathbb{Z}^{kmn} \cap \left[P^\circ \setminus \bigcup H\right]\right)$ gives the number of k-MOLEs with positive entries in terms of a parameter t, which is either the latin array magic sum or the strict upper bound on entries of the latin arrays. A k-MOLE represents an ordered set of k MOLS (although not in 1-1 correspondence, see section 4.1 for the details). Ordered sets are equivalent if and only if the sets have all the same members listed in the same order.

Thus, if we want the number of (unordered) sets, we simply divide out the number of permutations $k!$. We detail these results for the cubical and affine parameter in the theorems of the following sections.

3.2.2 Cubical MOLE

With the cubical approach, we count the number of *k*-MOLEs with entries that satisfy the strict bounds $0 < x_{lij} < t$. Thus these sets of arrays are the lattice points of $\mathbb{Z}^{kmn}$ that are contained in the interior of the dilated inside-out polytope $tP_c^\circ \setminus \bigcup H$.

Definition 3.11: Fix integers $k, m \geq 2$ and $n \geq 3$ with $m \leq n$. For $t = 1, 2, 3\ldots$ let

$MOLE_c^\circ(t) :=$ the number of *k*-MOLEs with latin array entries that satisfy $0 < x_{lij} < t$.

Theorem 3.12 (MOLE enumeration by upper bound): $MOLE_c^\circ(t)$ is a quasipolynomial in *t* with the leading term $t^{\dim s}(\operatorname{vol} P_c)$.

Proof: Since $\mathbf{1} \in P_c$, then P_c is not contained in a coordinate hyperplane, and thus by Lemma 3.7 we have that P_c affinely spans *s*. Thus (P_c, H) is a closed, rational inside-out polytope in $\mathbb{R}^{\dim s}$. The results follow by Theorem 2.64 applied in the magic subspace *s* with $MOLE_c^\circ(t) = E_{P_c^\circ, H}^\circ(t)$. ■

Theorem 3.13: Suppose (P_c, H) is as in Theorem 3.12, then

$$MOLE_c^\circ(t) = \sum_{u \in L(P_c^\circ, H)} \mu(\hat{0}, u) E_{P_c^\circ \cap u}(t)$$

where μ is the Mobius function of $L(P_c^\circ, H)$.

Proof: We have transversality by Lemma 3.10. The result follows by Theorem 2.65 applied in the magic subspace s. ■

3.2.3 Affine MOLE

The affine approach counts the number of k-MOLEs with positive entries $x_{lij} > 0$ and magic sum t. These sets of arrays are the lattice points of $\mathbb{Z}^{kmn}$ that are contained in the interior of the dilated inside-out polytope $tP_a^\circ \setminus \bigcup H$.

Definition 3.14: Fix integers $k, m \geq 2$ and $n \geq 3$ with $m \leq n$. For $t = 1, 2, 3\ldots$ let

$MOLE_a^\circ(t) :=$ the number of k-MOLEs with latin array magic sum t and positive entries.

Theorem 3.15 (MOLE enumeration by magic sum): $MOLE_a^\circ(t)$ is a quasipolynomial in t with leading term $t^{\dim s}(\mathrm{vol}\, P_a)$.

Proof: Since $\frac{1}{n}\mathbf{1} \in P_a$, then P_a is not contained in a coordinate hyperplane, and thus we have that P_a affinely spans s by Lemma 3.7. Thus (P_a, H) is a closed, rational inside-out polytope in $\mathbb{R}^{\dim s}$. The result follows by Theorem 2.64 applied in the magic subspace s with $MOLE_a^\circ(t) = E^\circ_{P_a^\circ, H}(t)$. ■

Theorem 3.16: Suppose (P_a, H) is as in Theorem 3.15, then

$$MOLE_a^\circ(t) = \sum_{u \in L(P_a^\circ, H)} \mu(\hat{0}, u) E_{P_a^\circ \cap u}(t)$$

where μ is the Mobius function of $L(P_a^\circ, H)$.

Proof: P_a and H are transverse by Lemma 3.10. The results follow by Theorem 2.65 applied in the magic subspace s. ■

3.3 The MOLS Holes Method

3.3.1 The MOLS Holes Setting

We treat a k-set of $m\times n$ MOLS as an integer vector $\mathbf{z}\in\mathbb{Z}_{>0}^{kmn}$. Since the magic requirements, the entry requirements, and the parameter t come from the latin arrays, they play the same role in this method as they do in the MOLE method. Thus the magic subspaces $s_{k,n,n}$, $s_{k,m,n}$, and s as well as the magic polytopes P_c, P_a, and P are the same.

In contrast, the MOLS holes method does not use the generalized Euler encoding, and consequently not all the exclusions are hyperplanes. The latin exclusions remain as hyperplanes, defined the same as in the MOLE method, and constitute the set H_{latin}. However the orthogonality exclusions are no longer hyperplanes, but are instead each the intersection of two hyperplanes, yielding affine subspaces. We think of these affine subspaces like holes inside the polytope, hence we call the method *MOLS holes*.

For k mutually orthogonal $m\times n$ latin arrays $\left(x_{1ij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, $\left(x_{2ij}\right)_{\substack{1\le i\le m\\1\le j\le n}},\ldots,\left(x_{kij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ a pair $\left(x_{gij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ and $\left(x_{hij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$, with $g\ne h$, are orthogonal mates when the ordered pairs $\left(x_{gij},x_{hij}\right)_{\substack{1\le i\le m\\1\le j\le n}}$ are distinct. Thus we prohibit $\left(x_{gab},x_{hcd}\right)=\left(x_{gab},x_{hcd}\right)$ where $a\ne c$ and

$b \neq d$. As a result each prohibition creates an affine space in terms of an intersection of two hyperplanes, since $\left(x_{gab}, x_{hcd}\right) = \left(x_{gab}, x_{hcd}\right)$ if and only if $x_{gab} = x_{\mathrm{gcd}}$ and $x_{hab} = x_{hcd}$.

Definition 3.17: Each *orthogonal exclusion of a k-set of MOLS* is two simultaneous linear equations with rational coefficients and has the form

$$x_{gab} = x_{\mathrm{gcd}} \text{ and } x_{hab} = x_{hcd},$$

where $a \neq c$ and $b \neq d$.

Thus the set of orthogonal exclusions is

$$A_{orthog} = \left\{ \begin{array}{l} s \bigcap \left(x_{111} = x_{122}\right) \bigcap \left(x_{211} = x_{222}\right), \ldots, s \bigcap \left(x_{111} = x_{122}\right) \bigcap \left(x_{k11} = x_{k22}\right), \\ s \bigcap \left(x_{111} = x_{123}\right) \bigcap \left(x_{211} = x_{223}\right), \ldots, s \bigcap \left(x_{111} = x_{123}\right) \bigcap \left(x_{k11} = x_{k23}\right), \\ \ldots \\ s \bigcap \left(x_{1(m-1)(n-1)} = x_{1mn}\right) \bigcap \left(x_{2(m-1)(n-1)} = x_{2mn}\right), \ldots, \\ s \bigcap \left(x_{1(m-1)(n-1)} = x_{1mn}\right) \bigcap \left(x_{k(m-1)(n-1)} = x_{kmn}\right) \end{array} \right\}.$$

Note that when $a = c$ or $b = d$, the above definition becomes the intersection of latin exclusions. Thus they are not needed in the above definition. The difference between the sides of each equality is equal to zero, and thus an orthogonal exclusion produces two simultaneous homogeneous, linear equations with rational coefficients.

The MOLS holes method supplements a magic polytope P with the rational affine subspace arrangement

$$A = H_{latin} \bigcup A_{orthog},$$

comprised of the latin and orthogonality exclusions. Thus (P, A) is a closed, full-dimensional, rational inside-out polytope with respect to affine arrangements. For our results, we will need P and A to be transverse.

Lemma 3.18: P and A are transverse if a *k*-set of $m \times n$ MOLS with positive entries exists.

Proof: This proof applies when P is either P_a or P_c. Since $\frac{1}{n}\mathbf{1} \in \bigcap A$ and $\frac{1}{n}\mathbf{1} \in P^\circ$, we have that $P^\circ \cap \bigcap A \neq \varnothing$. A *k*-set of MOLS with positive entries exists if and only if the magic subspace *s* does not lie in $\bigcup A$. When a *k*-set of MOLS with positive entries exist, then $P \not\subseteq \bigcup A$. Hence A is transverse to P, when a *k*-set of MOLS with positive entries exists. ■

A *k*-set of $m \times n$ MOLS with positive entries is an integer point $\mathbf{z}$ in the *t*-dilate of the magic polytope relative interior tP° that does not lie in any of the affine subspaces of A. Thus $\#\left(\mathbb{Z}^{kmn} \cap \left[tP^\circ \setminus \bigcup A\right]\right) = \#\left(\frac{1}{t}\mathbb{Z}^{kmn} \cap \left[P^\circ \setminus \bigcup A\right]\right)$ is the number of ordered *k*-sets $m \times n$ MOLS with positive entries and parameter *t*, the latin array magic sum or strict upper bound on entries of the latin arrays. If we want the count of (unordered) sets, we simply divide out the number of permutations $k!$. These results are detailed in the theorems of the following sections.

3.3.2 Cubical MOLS Holes

The cubical approach to counting *k*-sets of MOLS by inside-out polytopes with holes establishes the parameter *t* to be the largest allowed value for an entry in any of the latin arrays. The setting is the same as the MOLE method where we replace the hyperplane arrangement H with the affine subspace arrangement A. Thus we count the number of ordered *k*-sets of $m \times n$ MOLS with entries that satisfy $0 < x_{lij} < t$, which are the lattice points of $\mathbb{Z}^{kmn}$ that are contained in the dilated inside-out polytope $tP_c^\circ \setminus \bigcup A$.

Definition 3.19: Fix integers $k, m \geq 2$ and $n \geq 3$ with $m \leq n$. For $t = 1, 2, 3 \ldots$ let

$MOLS_c^\circ(t) :=$ the number of ordered *k*-sets of $m \times n$ MOLS with entries that satisfy

$$0 < x_{lij} < t.$$

Theorem 3.20 (MOLS enumeration by upper bound): $MOLS_c^\circ(t)$ is a quasipolynomial in *t* with leading term $t^{\dim s}(\operatorname{vol} P_c)$. Furthermore,

$$MOLS_c^\circ(t) = \sum_{u \in L(A)} \mu(\hat{0}, u) E_{P_c^\circ \cap u}(t)$$

where μ is the Mobius function of $L(A)$.

Proof: P_c and *s* have not changed from Theorem 3.12, thus P_c affinely spans *s*. Thus (P_c, A) is a closed, rational inside-out polytope in $\mathbb{R}^{\dim s}$. P_c and A are transverse by Lemma 3.18. The results follow by Theorem 2.72 applied in the magic subspace *s* with $MOLS_c^\circ(t) = E^\circ_{P_c^\circ, A}(t)$. ■

3.3.3 Affine MOLS Holes

The affine approach to counting k-sets of $m \times n$ MOLS by inside-out polytopes with holes establishes the parameter t to be the magic sum of the latin arrays. The magic polytope P_a is the same as the affine MOLE method and the affine subspace arrangement A is the same as the cubic MOLS holes method. Thus we count the number of ordered k-sets of $m \times n$ MOLS with positive entries $x_{lij} > 0$ and magic sum t, which are the lattice points of $\mathbb{Z}^{kmn}$ that are contained in the interior of the dilated inside-out polytope $tP_a^\circ \setminus \bigcup A$.

Definition 3.21: Fix integers $k, m \geq 2$ and $n \geq 3$ with $m \leq n$. For $t = 1, 2, 3\ldots$ let

$MOLS_a^\circ(t) :=$ the number of ordered k-sets of $m \times n$ MOLS with positive entries $x_{lij} > 0$ and magic sum t.

Theorem 3.22 (MOLS enumeration by magic sum): $MOLS_a^\circ(t)$ is a quasipolynomial in t with leading term $t^{\dim s}(\mathrm{vol}\, P_a)$. Furthermore,

$$MOLS_a^\circ(t) = \sum_{u \in L(A)} \mu(\hat{0}, u) E_{P_a^\circ \cap u}(t)$$

where μ is the Mobius function of $L(A)$.

Proof: P_a and s have not changed from Theorem 3.15, thus P_a affinely spans s. Thus (P_a, A) is a closed, rational inside-out polytope in $\mathbb{R}^{\dim s}$. P_a and A are transverse by

Lemma 3.18. The results follow by Theorem 2.72 applied in the magic subspace s with $MOLS_a^{\circ}(t) = E^{\circ}_{P_a^{\circ},A}(t)$. ■

IV. COMPUTATIONS

4.1 Methods in Computation

We offer some experimental numbers with respect to latin arrays and orthogonal mates using the generalized Euler encoding of the MOLE method developed in Chapter III. The software used to produce these numbers was developed by Andrew van Herick, and is presented in [20]. A special thank you goes to the SFSU Center for Computing for Life Sciences (CCLS) for allowing me to run these computations on their cluster. I suspect I may have been responsible for causing the cluster to crash, trying to obtain the generating function for the 3×3 orthogonal mates. However, this challenge lead me to consider rectangles, as opposed to squares, and gather the counts for the 2×3 rectangles and orthogonal mates.

As stated in definitions 3.19 and 3.21, the functions $MOLS_c^{\circ}(t)$ and $MOLS_a^{\circ}(t)$ count the number of ordered k-sets of $m \times n$ MOLS. These functions are quasipolynomials as per theorems 3.20 and 3.22. However, the MOLS holes method does not use the generalized Euler encoding, and consequently an orthogonality exclusion is the intersection of two hyperplanes, which may yield an affine space. When the union of these affine spaces is intersected with a polytope, the resulting regions may not necessarily be polytopes themselves. The lattice points in these objects cannot be counted by computational software such as that developed in [20], since such programs count lattice points inside polytopes. Thus we use the MOLE method for computation.

In the MOLE method, we must fix a number d for the generalized Euler encoding on ordered pairs $\left(a_{ij},b_{ij}\right)$. Since $a_{ij},b_{ij},a_{pq},b_{pq}\in\mathbb{Z}_{>0}$, choosing d to be irrational would make da_{ij} and da_{pq} irrational numbers, and then the equation $da_{ij}+b_{ij}=da_{pq}+b_{pq}$ holds precisely when $\left(a_{ij},b_{ij}\right)=\left(a_{pq},b_{pq}\right)$ holds. However, the computational algorithms require rational coefficients, and thus we are required to work with a rational number d.

For computation using the MOLE method applied to k mutually orthogonal $m\times n$ latin arrays $\left(x_{1ij}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}$, $\left(x_{2ij}\right)_{\substack{1\le i\le m\\ 1\le j\le n}},\ldots,\left(x_{kij}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}$, we wish to choose a rational number d.

From the entry requirements we have $1\le x_{hij},x_{hpq}\le t-1$ and $1\le x_{gij},x_{gpq}\le t-1$.

We would like to choose d to guarantee that $(x_{hij}-x_{hpq})+d(x_{gij}-x_{gpq})=0$ holds, only when $x_{hij}-x_{hpq}=0$ and $x_{gij}-x_{gpq}=0$ holds.

In the cubical count, if $x_{hij}-x_{hpq}\ne 0$ and $x_{gij}-x_{gpq}\ne 0$,

then $1\le\left|x_{hij}-x_{hpq}\right|\le t-2$ and $d\le d\left|x_{gij}-x_{gpq}\right|\le d(t-2)$.

Thus we wish to fix d such that $d>t-2$.

In the affine count, if $x_{hij}-x_{hpq}\ne 0$ and $x_{gij}-x_{gpq}\ne 0$,

then $1\le\left|x_{hij}-x_{hpq}\right|\le t-1-\sum_{r=1}^{n-1}r$ and $d\le d\left|x_{gij}-x_{gpq}\right|\le d\left(t-1-\sum_{r=1}^{n-1}r\right)$.

Thus we wish to fix d such that $d>t-1-\sum_{r=1}^{n-1}r$.

In either case, the choice of d for computation depends on the parameter t. Thus as the dilates t of the inside-out polytope get large relative to the choice of d, it is possible that some of the desired lattice points will be contained in the hyperplane arrangement, and thus not counted. In other words the counting functions $MOLE_c^{\circ}(t)$ and $MOLE_a^{\circ}(t)$ may miss some of the k-sets of MOLS as t gets large relative to d. We give an example of such an occurrence in Chapter I, Figure 1.7, where the Euler encoding of a pair of orthogonal mates does not yield a semimagic square.

Although we have not proven that a rational d will cause the counting functions $MOLE_c^{\circ}(t)$ and $MOLE_a^{\circ}(t)$ to accurately count k-sets of MOLS for all t, we may use the functions to extract the correct quasipolynomials by experimentation. As shown above we may choose d such that the encoding is accurate up to a specified choice of t. Assume the quasipolynomial we are after has period p and degree q. Then such a quasipolynomial may be thought of as a collection of p polynomials. Thus we need to determine p polynomials of degree q. Each of these polynomials is determined by $q+1$ of its values. Each of these polynomials evaluated at t (depending on t mod p), is the quasipolynomial evaluated at t. Thus we need the computation correct for $t = 1, 2, \ldots,$ $p(q+1)$. Therefore by choosing d to be larger than $p(q+1)$, the encoding is sufficiently accurate to collect the necessary correct values from which we may extract the quasipolynomial.

In practice we may have to experiment with the magnitude of d, since the period likely is unknown. Once a choice of d_0 is made, then a generating function can be computed, and subsequently the period p_0. If $d_0 > p_0(q+1)$, then the computation is complete. Otherwise fix $d_1 > p_0(q+1)$, compute the generating function and the period p_1. Continue the process until $d_i > p_i(q+1)$. We conjecture that the process will terminate.

We offer some numbers for 2×3 rectangles in section 4.2. These counts are for the rectangles only, not sets of mutually orthogonal rectangles, thus the generalized Euler encoding was not used. These counts are the actual counts.

In section 4.3 we offer some experimental counts for 2-sets of 2×3 rectangles. We fixed $d = 99$ and obtained the corresponding generating functions. For section 4.4, we fixed $d = 999$ and attempted to compute the generating function for 2-sets of 3×3 squares. Although the computation of the generating function proved intractable, we do provide the experimental count of regions.

4.2 2×3 Rectangle Counts

1. Cubic count for the number of 2×3 latin rectangles that have rows with equal line sums and columns with equal line sums.

Ambient dimension is 6.

Inside-out polytope dimension is 3.

Number of regions is 12.

Rational generating function is

$12/(z^8 - 2z^7 + 2z^5 - 2z^4 + 2z^3 - 2z + 1)$.

2. Affine count for the number of 2×3 latin rectangles that have row sums equal to t and column sums equal to $\frac{2}{3}t$.

Ambient dimension is 6.

Inside-out polytope dimension is 2.

Number of regions is 12.

Rational generating function is

$12/(-z^{12} + 2z^9 - 2z^3 + 1)$.

3. Cubic count for the number of 2×3 latin rectangles that have rows with equal line sums (no column sum).

Ambient dimension is 6.

Inside-out polytope dimension is 5.

Number of regions is 132.

Rational generating function is

$(-12z^6 - 12z^5 - 132z^4 - 216z^3 - 204z^2 - 84z - 132)/(-z^{10} + 3z^9 - z^8 - 4z^7 + 2z^6 + 2z^5 + 2z^4 - 4z^3 - z^2 + 3z - 1)$.

4. Affine count for the number of 2×3 latin rectangles that have row sums equal to t (no column sum).

Ambient dimension is 6.

Inside-out polytope dimension is 4.

Number of regions is 132.

Rational generating function is

$(24z^{12} + 24z^{11} + 120z^{10} + 288z^{9} + 456z^{8} + 504z^{7} + 864z^{6} + 696z^{5} + 744z^{4} + 576z^{3} + 408z^{2} + 216z + 264)/(-2z^{18} + 2z^{16} + 4z^{15} + 2z^{14} - 4z^{13} - 2z^{12} - 4z^{11} + 4z^{7} + 2z^{6} + 4z^{5} - 2z^{4} - 4z^{3} - 2z^{2} + 2)$.

5. Count for the number of 2×3 rectangles that have no line sums.

Ambient dimension is 6.

Inside-out polytope dimension is 6.

Number of regions is 204.

Rational generating function is

$(12z^{3} + 180z^{2} + 324z + 204)/(-z^{7} + 7z^{6} - 21z^{5} + 35z^{4} - 35z^{3} + 21z^{2} - 7z + 1)$.

4.3 2×3 Orthogonal Mate Counts

1. Cubic count for the number of pairs of orthogonal mates of 2×3 latin rectangles that have rows with equal line sums and columns with equal line sums.

Ambient dimension is 12.

Inside-out polytope dimension is 5.

Number of regions is 384.

Rational generating function is

$(288z^{1384} + 864z^{1382} + 3168z^{1380} + 4896z^{1378} + 9504z^{1376} + 12384z^{1374} + 19296z^{1372} + 23328z^{1370} + 32544z^{1368} + 37728z^{1366} + 49248z^{1364} + 55584z^{1362} + 69408z^{1360} + 76896z^{1358} + 93024z^{1356} + 101664z^{1354} + 120096z^{1352} + 129888z^{1350} + 150624z^{1348} + 161568z^{1346} + 184608z^{1344} + 196704z^{1342} + 222048z^{1340} + 235296z^{1338} + 262944z^{1336} + 277344z^{1334} + 307296z^{1332} + 322848z^{1330} + 355104z^{1328} + 371808z^{1326} + 406368z^{1324} + 424224z^{1322} + 461088z^{1320} + 480096z^{1318} + 519264z^{1316} + 539424z^{1314} + 580896z^{1312} + 602208z^{1310} + 645984z^{1308} + 668448z^{1306} + 714528z^{1304} + 738144z^{1302} + 786528z^{1300} +$

$$811296z^{1298} + 861984z^{1296} + 887904z^{1294} + 940896z^{1292} + 967968z^{1290} + 1023264z^{1288} + 1051488z^{1286} + 1109088z^{1284} + 1138464z^{1282} + 1198368z^{1280} + 1228896z^{1278} + 1291104z^{1276} + 1322784z^{1274} + 1387296z^{1272} + 1420128z^{1270} + 1486944z^{1268} + 1520928z^{1266} + 1590048z^{1264} + 1625184z^{1262} + 1696608z^{1260} + 1732896z^{1258} + 1806624z^{1256} + 1844064z^{1254} + 1920096z^{1252} + 1958688z^{1250} + 2037024z^{1248} + 2076768z^{1246} + 2157408z^{1244} + 2198304z^{1242} + 2281248z^{1240} + 2323296z^{1238} + 2408544z^{1236} + 2451744z^{1234} + 2539296z^{1232} + 2583648z^{1230} + 2673504z^{1228} + 2719008z^{1226} + 2811168z^{1224} + 2857824z^{1222} + 2952288z^{1220} + 3000096z^{1218} + 3096864z^{1216} + 3145824z^{1214} + 3244896z^{1212} + 3295008z^{1210} + 3396384z^{1208} + 3447648z^{1206} + 3551328z^{1204} + 3603744z^{1202} + 3709728z^{1200} + 3763296z^{1198} + 3871584z^{1196} + 3926304z^{1194} + 3808752z^{1192} + 4085712z^{1190} + 4426656z^{1188} + 4262208z^{1186} + 4376832z^{1184} + 4432704z^{1182} + 4548480z^{1180} + 4603200z^{1178} + 4720128z^{1176} + 4773696z^{1174} + 4891776z^{1172} + 4944192z^{1170} + 5063424z^{1168} + 5114688z^{1166} + 5235072z^{1164} + 5285184z^{1162} + 5406720z^{1160} + 5455680z^{1158} + 5578368z^{1156} + 5626176z^{1154} + 5750016z^{1152} + 5796672z^{1150} + 5921664z^{1148} + 5967168z^{1146} + 6093312z^{1144} + 6137664z^{1142} + 6264960z^{1140} + 6308160z^{1138} + 6436608z^{1136} + 6478656z^{1134} + 6608256z^{1132} + 6649152z^{1130} + 6779904z^{1128} + 6819648z^{1126} + 6951552z^{1124} + 6990144z^{1122} + 7123200z^{1120} + 7160640z^{1118} + 7294848z^{1116} + 7331136z^{1114} + 7466496z^{1112} + 7501632z^{1110} + 7638144z^{1108} + 7672128z^{1106} + 7809792z^{1104} + 7842624z^{1102} + 7981440z^{1100} + 8013120z^{1098} + 8153088z^{1096} + 8183616z^{1094} + 8324736z^{1092} + 8354112z^{1090} + 8496384z^{1088} + 8524608z^{1086} + 8668032z^{1084} + 8695104z^{1082} + 8839680z^{1080} + 8865600z^{1078} + 9011328z^{1076} + 9036096z^{1074} + 9182976z^{1072} + 9206592z^{1070} + 9354624z^{1068} + 9377088z^{1066} + 9526272z^{1064} + 9547584z^{1062} + 9697920z^{1060} + 9718080z^{1058} + 9869568z^{1056} + 9888576z^{1054} + 10041216z^{1052} + 10059072z^{1050} + 10212864z^{1048} + 10229568z^{1046} + 10384512z^{1044} + 10400064z^{1042} + 10556160z^{1040} + 10570560z^{1038} + 10727808z^{1036} + 10741056z^{1034} + 10899456z^{1032} + 10911552z^{1030} + 11071104z^{1028} + 11082048z^{1026} + 11242752z^{1024} + 11252544z^{1022} + 11414400z^{1020} + 11423040z^{1018} + 11586048z^{1016} + 11593536z^{1014} + 11757696z^{1012} + 11764032z^{1010} + 11929344z^{1008} + 11934528z^{1006} + 12100992z^{1004} + 12105024z^{1002} + 12272640z^{1000} + 12275520z^{998} + 12444288z^{996} + 11766288z^{994} + 12608928z^{992} + 13289232z^{990} + 12787296z^{988} + 12786144z^{986} + 12956064z^{984} + 12952608z^{982} + 13121376z^{980} + 13115616z^{978} + 13283232z^{976} + 13275168z^{974} + 13441632z^{972} + 13431264z^{970} + 13596576z^{968} + 13583904z^{966} + 13748064z^{964} + 13733088z^{962} + 13896096z^{960} + 13878816z^{958} + 14040672z^{956} + 14021088z^{954} + 14181792z^{952} + 14159904z^{950} + 14319456z^{948} + 14295264z^{946} + 14453664z^{944} + 14427168z^{942} + 14584416z^{940} + 14555616z^{938} + 14711712z^{936} + 14680608z^{934} + 14835552z^{932} + 14802144z^{930} + 14955936z^{928} + 14920224z^{926} + 15072864z^{924} + 15034848z^{922} + 15186336z^{920} + 15146016z^{918} + 15296352z^{916} + 15253728z^{914} + 15402912z^{912} + 15357984z^{910} + 15506016z^{908} + 15458784z^{906} + 15605664z^{904} + 15556128z^{902} + 15701856z^{900} + 15650016z^{898} + 15794592z^{896} + 15740448z^{894} + 15883872z^{892} + 15827424z^{890} + 15969696z^{888} +$$

$15910944z^{886} + 16052064z^{884} + 15991008z^{882} + 16130976z^{880} + 16067616z^{878} +$
$16206432z^{876} + 16140768z^{874} + 16278432z^{872} + 16210464z^{870} + 16346976z^{868} +$
$16276704z^{866} + 16412064z^{864} + 16339488z^{862} + 16473696z^{860} + 16398816z^{858} +$
$16531872z^{856} + 16454688z^{854} + 16586592z^{852} + 16507104z^{850} + 16637856z^{848} +$
$16556064z^{846} + 16685664z^{844} + 16601568z^{842} + 16730016z^{840} + 16643616z^{838} +$
$16770912z^{836} + 16682208z^{834} + 16808352z^{832} + 16717344z^{830} + 16842336z^{828} +$
$16749024z^{826} + 16872864z^{824} + 16777248z^{822} + 16899936z^{820} + 16802016z^{818} +$
$16923552z^{816} + 16823328z^{814} + 16943712z^{812} + 16841184z^{810} + 16960416z^{808} +$
$16855584z^{806} + 16973664z^{804} + 16866528z^{802} + 16983456z^{800} + 16874016z^{798} +$
$16037280z^{796} + 16877904z^{794} + 17945136z^{792} + 16879104z^{790} + 16993152z^{788} +$
$16879104z^{786} + 16993152z^{784} + 16879104z^{782} + 16993152z^{780} + 16879104z^{778} +$
$16993152z^{776} + 16879104z^{774} + 16993152z^{772} + 16879104z^{770} + 16993152z^{768} +$
$16879104z^{766} + 16993152z^{764} + 16879104z^{762} + 16993152z^{760} + 16879104z^{758} +$
$16993152z^{756} + 16879104z^{754} + 16993152z^{752} + 16879104z^{750} + 16993152z^{748} +$
$16879104z^{746} + 16993152z^{744} + 16879104z^{742} + 16993152z^{740} + 16879104z^{738} +$
$16993152z^{736} + 16879104z^{734} + 16993152z^{732} + 16879104z^{730} + 16993152z^{728} +$
$16879104z^{726} + 16993152z^{724} + 16879104z^{722} + 16993152z^{720} + 16879104z^{718} +$
$16993152z^{716} + 16879104z^{714} + 16993152z^{712} + 16879104z^{710} + 16993152z^{708} +$
$16879104z^{706} + 16993152z^{704} + 16879104z^{702} + 16993152z^{700} + 16879104z^{698} +$
$16993152z^{696} + 16879104z^{694} + 16993152z^{692} + 16879104z^{690} + 16993152z^{688} +$
$16879104z^{686} + 16993152z^{684} + 16879104z^{682} + 16993152z^{680} + 16879104z^{678} +$
$16993152z^{676} + 16879104z^{674} + 16993152z^{672} + 16879104z^{670} + 16993152z^{668} +$
$16879104z^{666} + 16993152z^{664} + 16879104z^{662} + 16993152z^{660} + 16879104z^{658} +$
$16993152z^{656} + 16879104z^{654} + 16993152z^{652} + 16879104z^{650} + 16993152z^{648} +$
$16879104z^{646} + 16993152z^{644} + 16879104z^{642} + 16993152z^{640} + 16879104z^{638} +$
$16993152z^{636} + 16879104z^{634} + 16993152z^{632} + 16879104z^{630} + 16993152z^{628} +$
$16879104z^{626} + 16993152z^{624} + 16879104z^{622} + 16993152z^{620} + 16879104z^{618} +$
$16993152z^{616} + 16879104z^{614} + 16993152z^{612} + 16879104z^{610} + 16993152z^{608} +$
$16879104z^{606} + 16993152z^{604} + 16879104z^{602} + 16993152z^{600} + 15959520z^{598} +$
$16993152z^{596} + 17798688z^{594} + 16992864z^{592} + 16878240z^{590} + 16989984z^{588} +$
$16874208z^{586} + 16983648z^{584} + 16866720z^{582} + 16973856z^{580} + 16855776z^{578} +$
$16960608z^{576} + 16841376z^{574} + 16943904z^{572} + 16823520z^{570} + 16923744z^{568} +$
$16802208z^{566} + 16900128z^{564} + 16777440z^{562} + 16873056z^{560} + 16749216z^{558} +$
$16842528z^{556} + 16717536z^{554} + 16808544z^{552} + 16682400z^{550} + 16771104z^{548} +$
$16643808z^{546} + 16730208z^{544} + 16601760z^{542} + 16685856z^{540} + 16556256z^{538} +$
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$16399008z^{526} + 16473888z^{524} + 16339680z^{522} + 16412256z^{520} + 16276896z^{518} +$
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$768)/(2z^{1392} - 4z^{1391} + 2z^{1390} - 2z^{1388} + 4z^{1387} - 2z^{1386} - 2z^{1194} + 4z^{1193} - 2z^{1192} + 2z^{1190} -$
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$2);$
Ehr_int $= (384z^{1392} + 600z^{1390} + 1560z^{1388} + 2544z^{1386} + 4848z^{1384} + 6288z^{1382} +$
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$2131104z^{206} + 2213328z^{204} + 2042856z^{202} + 1904376z^{200} + 1963152z^{198} + 1935792z^{196} +$
$1881648z^{194} + 1854864z^{192} + 1801872z^{190} + 1775664z^{188} + 1723824z^{186} + 1698192z^{184} +$
$1647504z^{182} + 1622448z^{180} + 1572912z^{178} + 1548432z^{176} + 1500048z^{174} + 1476144z^{172} +$
$1428912z^{170} + 1405584z^{168} + 1359504z^{166} + 1336752z^{164} + 1291824z^{162} + 1269648z^{160} +$
$1225872z^{158} + 1204272z^{156} + 1161648z^{154} + 1140624z^{152} + 1099152z^{150} + 1078704z^{148} +$
$1038384z^{146} + 1018512z^{144} + 979344z^{142} + 960048z^{140} + 922032z^{138} + 903312z^{136} +$
$866448z^{134} + 848304z^{132} + 812592z^{130} + 795024z^{128} + 760464z^{126} + 743472z^{124} +$
$710064z^{122} + 693648z^{120} + 661392z^{118} + 645552z^{116} + 614448z^{114} + 599184z^{112} +$
$569232z^{110} + 554544z^{108} + 525744z^{106} + 511632z^{104} + 483984z^{102} + 470448z^{100} +$
$443952z^{98} + 430992z^{96} + 405648z^{94} + 393264z^{92} + 369072z^{90} + 357264z^{88} + 334224z^{86} +$
$322992z^{84} + 301104z^{82} + 290448z^{80} + 269712z^{78} + 259632z^{76} + 240048z^{74} + 230544z^{72} +$
$212112z^{70} + 203184z^{68} + 185904z^{66} + 177552z^{64} + 161424z^{62} + 153648z^{60} + 138672z^{58} +$
$131472z^{56} + 117648z^{54} + 111024z^{52} + 98352z^{50} + 92304z^{48} + 80784z^{46} + 75312z^{44} +$
$64944z^{42} + 60048z^{40} + 50832z^{38} + 46512z^{36} + 38448z^{34} + 34704z^{32} + 27792z^{30} +$
$24624z^{28} + 18864z^{26} + 16272z^{24} + 11664z^{22} + 9648z^{20} + 6192z^{18} + 4752z^{16} + 2448z^{14} +$

$1584z^{12} + 432z^{10} + 144z^{8})/(z^{1392} - 2z^{1391} + z^{1390} - z^{1388} + 2z^{1387} - z^{1386} - z^{1194} + 2z^{1193} -$
$z^{1192} + z^{1190} - 2z^{1189} + z^{1188} - z^{996} + 2z^{995} - z^{994} + z^{992} - 2z^{991} + z^{990} + z^{798} - 2z^{797} + z^{796} -$
$z^{794} + 2z^{793} - z^{792} - z^{600} + 2z^{599} - z^{598} + z^{596} - 2z^{595} + z^{594} + z^{402} - 2z^{401} + z^{400} - z^{398} +$
$2z^{397} - z^{396} + z^{204} - 2z^{203} + z^{202} - z^{200} + 2z^{199} - z^{198} - z^{6} + 2z^{5} - z^{4} + z^{2} - 2z + 1).$

2. Affine count for the number of pairs of orthogonal mates of 2×3 latin rectangles that have row sums equal to t and column sums equal to $\frac{2}{3}t$.

Ambient dimension is 12.

Inside-out polytope dimension is 4.

Number of regions is 384.

Rational generating function is

$(144z^{2076} + 432z^{2073} + 1584z^{2070} + 2448z^{2067} + 4752z^{2064} + 6192z^{2061} + 9648z^{2058} +$
$11664z^{2055} + 16272z^{2052} + 18864z^{2049} + 24624z^{2046} + 27792z^{2043} + 34704z^{2040} +$
$38448z^{2037} + 46512z^{2034} + 50832z^{2031} + 60048z^{2028} + 64944z^{2025} + 75312z^{2022} +$
$80784z^{2019} + 92304z^{2016} + 98352z^{2013} + 111024z^{2010} + 117648z^{2007} + 131472z^{2004} +$
$138672z^{2001} + 153648z^{1998} + 161424z^{1995} + 177552z^{1992} + 185904z^{1989} + 203184z^{1986} +$
$212112z^{1983} + 230544z^{1980} + 240048z^{1977} + 259632z^{1974} + 269712z^{1971} + 290448z^{1968} +$
$301104z^{1965} + 322992z^{1962} + 334224z^{1959} + 357264z^{1956} + 369072z^{1953} + 393264z^{1950} +$
$405648z^{1947} + 430992z^{1944} + 443952z^{1941} + 470448z^{1938} + 483984z^{1935} + 511632z^{1932} +$
$525744z^{1929} + 554544z^{1926} + 569232z^{1923} + 599184z^{1920} + 614448z^{1917} + 645552z^{1914} +$
$661392z^{1911} + 693648z^{1908} + 710064z^{1905} + 743472z^{1902} + 760464z^{1899} + 795024z^{1896} +$
$812592z^{1893} + 848304z^{1890} + 866448z^{1887} + 903312z^{1884} + 922032z^{1881} + 960048z^{1878} +$
$979344z^{1875} + 1018512z^{1872} + 1038384z^{1869} + 1078704z^{1866} + 1099152z^{1863} +$
$1140624z^{1860} + 1161648z^{1857} + 1204272z^{1854} + 1225872z^{1851} + 1269648z^{1848} +$
$1291824z^{1845} + 1336752z^{1842} + 1359504z^{1839} + 1405584z^{1836} + 1428912z^{1833} +$
$1476144z^{1830} + 1500048z^{1827} + 1548432z^{1824} + 1572912z^{1821} + 1622448z^{1818} +$
$1647504z^{1815} + 1698192z^{1812} + 1723824z^{1809} + 1775664z^{1806} + 1801872z^{1803} +$
$1854864z^{1800} + 1881648z^{1797} + 1935792z^{1794} + 1963152z^{1791} + 1904376z^{1788} +$
$2042856z^{1785} + 2213328z^{1782} + 2131104z^{1779} + 2188416z^{1776} + 2216352z^{1773} +$
$2274240z^{1770} + 2301600z^{1767} + 2360064z^{1764} + 2386848z^{1761} + 2445888z^{1758} +$
$2472096z^{1755} + 2531712z^{1752} + 2557344z^{1749} + 2617536z^{1746} + 2642592z^{1743} +$
$2703360z^{1740} + 2727840z^{1737} + 2789184z^{1734} + 2813088z^{1731} + 2875008z^{1728} +$
$2898336z^{1725} + 2960832z^{1722} + 2983584z^{1719} + 3046656z^{1716} + 3068832z^{1713} +$
$3132480z^{1710} + 3154080z^{1707} + 3218304z^{1704} + 3239328z^{1701} + 3304128z^{1698} +$
$3324576z^{1695} + 3389952z^{1692} + 3409824z^{1689} + 3475776z^{1686} + 3495072z^{1683} +$

$3561600z^{1680} + 3580320z^{1677} + 3647424z^{1674} + 3665568z^{1671} + 3733248z^{1668} + 3750816z^{1665} + 3819072z^{1662} + 3836064z^{1659} + 3904896z^{1656} + 3921312z^{1653} + 3990720z^{1650} + 4006560z^{1647} + 4076544z^{1644} + 4091808z^{1641} + 4162368z^{1638} + 4177056z^{1635} + 4248192z^{1632} + 4262304z^{1629} + 4334016z^{1626} + 4347552z^{1623} + 4419840z^{1620} + 4432800z^{1617} + 4505664z^{1614} + 4518048z^{1611} + 4591488z^{1608} + 4603296z^{1605} + 4677312z^{1602} + 4688544z^{1599} + 4763136z^{1596} + 4773792z^{1593} + 4848960z^{1590} + 4859040z^{1587} + 4934784z^{1584} + 4944288z^{1581} + 5020608z^{1578} + 5029536z^{1575} + 5106432z^{1572} + 5114784z^{1569} + 5192256z^{1566} + 5200032z^{1563} + 5278080z^{1560} + 5285280z^{1557} + 5363904z^{1554} + 5370528z^{1551} + 5449728z^{1548} + 5455776z^{1545} + 5535552z^{1542} + 5541024z^{1539} + 5621376z^{1536} + 5626272z^{1533} + 5707200z^{1530} + 5711520z^{1527} + 5793024z^{1524} + 5796768z^{1521} + 5878848z^{1518} + 5882016z^{1515} + 5964672z^{1512} + 5967264z^{1509} + 6050496z^{1506} + 6052512z^{1503} + 6136320z^{1500} + 6137760z^{1497} + 6222144z^{1494} + 5883144z^{1491} + 6304464z^{1488} + 6644616z^{1485} + 6393648z^{1482} + 6393072z^{1479} + 6478032z^{1476} + 6476304z^{1473} + 6560688z^{1470} + 6557808z^{1467} + 6641616z^{1464} + 6637584z^{1461} + 6720816z^{1458} + 6715632z^{1455} + 6798288z^{1452} + 6791952z^{1449} + 6874032z^{1446} + 6866544z^{1443} + 6948048z^{1440} + 6939408z^{1437} + 7020336z^{1434} + 7010544z^{1431} + 7090896z^{1428} + 7079952z^{1425} + 7159728z^{1422} + 7147632z^{1419} + 7226832z^{1416} + 7213584z^{1413} + 7292208z^{1410} + 7277808z^{1407} + 7355856z^{1404} + 7340304z^{1401} + 7417776z^{1398} + 7401072z^{1395} + 7477968z^{1392} + 7460112z^{1389} + 7536432z^{1386} + 7517424z^{1383} + 7593168z^{1380} + 7573008z^{1377} + 7648176z^{1374} + 7626864z^{1371} + 7701456z^{1368} + 7678992z^{1365} + 7753008z^{1362} + 7729392z^{1359} + 7802832z^{1356} + 7778064z^{1353} + 7850928z^{1350} + 7825008z^{1347} + 7897296z^{1344} + 7870224z^{1341} + 7941936z^{1338} + 7913712z^{1335} + 7984848z^{1332} + 7955472z^{1329} + 8026032z^{1326} + 7995504z^{1323} + 8065488z^{1320} + 8033808z^{1317} + 8103216z^{1314} + 8070384z^{1311} + 8139216z^{1308} + 8105232z^{1305} + 8173488z^{1302} + 8138352z^{1299} + 8206032z^{1296} + 8169744z^{1293} + 8236848z^{1290} + 8199408z^{1287} + 8265936z^{1284} + 8227344z^{1281} + 8293296z^{1278} + 8253552z^{1275} + 8318928z^{1272} + 8278032z^{1269} + 8342832z^{1266} + 8300784z^{1263} + 8365008z^{1260} + 8321808z^{1257} + 8385456z^{1254} + 8341104z^{1251} + 8404176z^{1248} + 8358672z^{1245} + 8421168z^{1242} + 8374512z^{1239} + 8436432z^{1236} + 8388624z^{1233} + 8449968z^{1230} + 8401008z^{1227} + 8461776z^{1224} + 8411664z^{1221} + 8471856z^{1218} + 8420592z^{1215} + 8480208z^{1212} + 8427792z^{1209} + 8486832z^{1206} + 8433264z^{1203} + 8491728z^{1200} + 8437008z^{1197} + 8018640z^{1194} + 8438952z^{1191} + 8972568z^{1188} + 8439552z^{1185} + 8496576z^{1182} + 8439552z^{1179} + 8496576z^{1176} + 8439552z^{1173} + 8496576z^{1170} + 8439552z^{1167} + 8496576z^{1164} + 8439552z^{1161} + 8496576z^{1158} + 8439552z^{1155} + 8496576z^{1152} + 8439552z^{1149} + 8496576z^{1146} + 8439552z^{1143} + 8496576z^{1140} + 8439552z^{1137} + 8496576z^{1134} + 8439552z^{1131} + 8496576z^{1128} + 8439552z^{1125} + 8496576z^{1122} + 8439552z^{1119} + 8496576z^{1116} + 8439552z^{1113} + 8496576z^{1110} + 8439552z^{1107} + 8496576z^{1104} + 8439552z^{1101} + 8496576z^{1098} + 8439552z^{1095} + 8496576z^{1092} + 8439552z^{1089} + 8496576z^{1086} + 8439552z^{1083} + 8496576z^{1080} + 8439552z^{1077} + 8496576z^{1074} + 8439552z^{1071} + 8496576z^{1068} +$

$8439552z^{1065} + 8496576z^{1062} + 8439552z^{1059} + 8496576z^{1056} + 8439552z^{1053} +$
$8496576z^{1050} + 8439552z^{1047} + 8496576z^{1044} + 8439552z^{1041} + 8496576z^{1038} +$
$8439552z^{1035} + 8496576z^{1032} + 8439552z^{1029} + 8496576z^{1026} + 8439552z^{1023} +$
$8496576z^{1020} + 8439552z^{1017} + 8496576z^{1014} + 8439552z^{1011} + 8496576z^{1008} +$
$8439552z^{1005} + 8496576z^{1002} + 8439552z^{999} + 8496576z^{996} + 8439552z^{993} + 8496576z^{990}$
$+ 8439552z^{987} + 8496576z^{984} + 8439552z^{981} + 8496576z^{978} + 8439552z^{975} + 8496576z^{972}$
$+ 8439552z^{969} + 8496576z^{966} + 8439552z^{963} + 8496576z^{960} + 8439552z^{957} + 8496576z^{954}$
$+ 8439552z^{951} + 8496576z^{948} + 8439552z^{945} + 8496576z^{942} + 8439552z^{939} + 8496576z^{936}$
$+ 8439552z^{933} + 8496576z^{930} + 8439552z^{927} + 8496576z^{924} + 8439552z^{921} + 8496576z^{918}$
$+ 8439552z^{915} + 8496576z^{912} + 8439552z^{909} + 8496576z^{906} + 8439552z^{903} + 8496576z^{900}$
$+ 7979760z^{897} + 8496576z^{894} + 8899344z^{891} + 8496432z^{888} + 8439120z^{885} + 8494992z^{882}$
$+ 8437104z^{879} + 8491824z^{876} + 8433360z^{873} + 8486928z^{870} + 8427888z^{867} + 8480304z^{864}$
$+ 8420688z^{861} + 8471952z^{858} + 8411760z^{855} + 8461872z^{852} + 8401104z^{849} + 8450064z^{846}$
$+ 8388720z^{843} + 8436528z^{840} + 8374608z^{837} + 8421264z^{834} + 8358768z^{831} + 8404272z^{828}$
$+ 8341200z^{825} + 8385552z^{822} + 8321904z^{819} + 8365104z^{816} + 8300880z^{813} + 8342928z^{810}$
$+ 8278128z^{807} + 8319024z^{804} + 8253648z^{801} + 8293392z^{798} + 8227440z^{795} + 8266032z^{792}$
$+ 8199504z^{789} + 8236944z^{786} + 8169840z^{783} + 8206128z^{780} + 8138448z^{777} + 8173584z^{774}$
$+ 8105328z^{771} + 8139312z^{768} + 8070480z^{765} + 8103312z^{762} + 8033904z^{759} + 8065584z^{756}$
$+ 7995600z^{753} + 8026128z^{750} + 7955568z^{747} + 7984944z^{744} + 7913808z^{741} + 7942032z^{738}$
$+ 7870320z^{735} + 7897392z^{732} + 7825104z^{729} + 7851024z^{726} + 7778160z^{723} + 7802928z^{720}$
$+ 7729488z^{717} + 7753104z^{714} + 7679088z^{711} + 7701552z^{708} + 7626960z^{705} + 7648272z^{702}$
$+ 7573104z^{699} + 7593264z^{696} + 7517520z^{693} + 7536528z^{690} + 7460208z^{687} + 7478064z^{684}$
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$+ 7213680z^{663} + 7226928z^{660} + 7147728z^{657} + 7159824z^{654} + 7080048z^{651} + 7090992z^{648}$
$+ 7010640z^{645} + 7020432z^{642} + 6939504z^{639} + 6948144z^{636} + 6866640z^{633} + 6874128z^{630}$
$+ 6792048z^{627} + 6798384z^{624} + 6715728z^{621} + 6720912z^{618} + 6637680z^{615} + 6641712z^{612}$
$+ 6557904z^{609} + 6560784z^{606} + 6476400z^{603} + 6115848z^{600} + 6396696z^{597} + 6759600z^{594}$
$+ 6308448z^{591} + 6308160z^{588} + 6223200z^{585} + 6222336z^{582} + 6137952z^{579} + 6136512z^{576}$
$+ 6052704z^{573} + 6050688z^{570} + 5967456z^{567} + 5964864z^{564} + 5882208z^{561} + 5879040z^{558}$
$+ 5796960z^{555} + 5793216z^{552} + 5711712z^{549} + 5707392z^{546} + 5626464z^{543} + 5621568z^{540}$
$+ 5541216z^{537} + 5535744z^{534} + 5455968z^{531} + 5449920z^{528} + 5370720z^{525} + 5364096z^{522}$
$+ 5285472z^{519} + 5278272z^{516} + 5200224z^{513} + 5192448z^{510} + 5114976z^{507} + 5106624z^{504}$
$+ 5029728z^{501} + 5020800z^{498} + 4944480z^{495} + 4934976z^{492} + 4859232z^{489} + 4849152z^{486}$
$+ 4773984z^{483} + 4763328z^{480} + 4688736z^{477} + 4677504z^{474} + 4603488z^{471} + 4591680z^{468}$
$+ 4518240z^{465} + 4505856z^{462} + 4432992z^{459} + 4420032z^{456} + 4347744z^{453} + 4334208z^{450}$
$+ 4262496z^{447} + 4248384z^{444} + 4177248z^{441} + 4162560z^{438} + 4092000z^{435} + 4076736z^{432}$
$+ 4006752z^{429} + 3990912z^{426} + 3921504z^{423} + 3905088z^{420} + 3836256z^{417} + 3819264z^{414}$
$+ 3751008z^{411} + 3733440z^{408} + 3665760z^{405} + 3647616z^{402} + 3580512z^{399} + 3561792z^{396}$
$+ 3495264z^{393} + 3475968z^{390} + 3410016z^{387} + 3390144z^{384} + 3324768z^{381} + 3304320z^{378}$
$+ 3239520z^{375} + 3218496z^{372} + 3154272z^{369} + 3132672z^{366} + 3069024z^{363} + 3046848z^{360}$
$+ 2983776z^{357} + 2961024z^{354} + 2898528z^{351} + 2875200z^{348} + 2813280z^{345} + 2789376z^{342}$

$+ 2728032z^{339} + 2703552z^{336} + 2642784z^{333} + 2617728z^{330} + 2557536z^{327} + 2531904z^{324}$
$+ 2472288z^{321} + 2446080z^{318} + 2387040z^{315} + 2360256z^{312} + 2301792z^{309} + 2274432z^{306}$
$+ 2096640z^{303} + 2192112z^{300} + 2254704z^{297} + 2102928z^{294} + 2046480z^{291} + 2018544z^{288}$
$+ 1963248z^{285} + 1935888z^{282} + 1881744z^{279} + 1854960z^{276} + 1801968z^{273} + 1775760z^{270}$
$+ 1723920z^{267} + 1698288z^{264} + 1647600z^{261} + 1622544z^{258} + 1573008z^{255} + 1548528z^{252}$
$+ 1500144z^{249} + 1476240z^{246} + 1429008z^{243} + 1405680z^{240} + 1359600z^{237} + 1336848z^{234}$
$+ 1291920z^{231} + 1269744z^{228} + 1225968z^{225} + 1204368z^{222} + 1161744z^{219} + 1140720z^{216}$
$+ 1099248z^{213} + 1078800z^{210} + 1038480z^{207} + 1018608z^{204} + 979440z^{201} + 960144z^{198} +$
$922128z^{195} + 903408z^{192} + 866544z^{189} + 848400z^{186} + 812688z^{183} + 795120z^{180} +$
$760560z^{177} + 743568z^{174} + 710160z^{171} + 693744z^{168} + 661488z^{165} + 645648z^{162} +$
$614544z^{159} + 599280z^{156} + 569328z^{153} + 554640z^{150} + 525840z^{147} + 511728z^{144} +$
$484080z^{141} + 470544z^{138} + 444048z^{135} + 431088z^{132} + 405744z^{129} + 393360z^{126} +$
$369168z^{123} + 357360z^{120} + 334320z^{117} + 323088z^{114} + 301200z^{111} + 290544z^{108} +$
$269808z^{105} + 259728z^{102} + 240144z^{99} + 230640z^{96} + 212208z^{93} + 203280z^{90} + 186000z^{87}$
$+ 177648z^{84} + 161520z^{81} + 153744z^{78} + 138768z^{75} + 131568z^{72} + 117744z^{69} + 111120z^{66}$
$+ 98448z^{63} + 92400z^{60} + 80880z^{57} + 75408z^{54} + 65040z^{51} + 60144z^{48} + 50928z^{45} +$
$46608z^{42} + 38544z^{39} + 34800z^{36} + 27888z^{33} + 24720z^{30} + 18960z^{27} + 16368z^{24} +$
$11760z^{21} + 9744z^{18} + 6288z^{15} + 4848z^{12} + 2544z^{9} + 1560z^{6} + 600z^{3} + 384)/(-z^{2088} + z^{2085}$
$+ z^{2082} - z^{2079} + z^{1791} - z^{1788} - z^{1785} + z^{1782} + z^{1494} - z^{1491} - z^{1488} + z^{1485} - z^{1197} + z^{1194} +$
$z^{1191} - z^{1188} + z^{900} - z^{897} - z^{894} + z^{891} - z^{603} + z^{600} + z^{597} - z^{594} - z^{306} + z^{303} + z^{300} - z^{297} +$
$z^{9} - z^{6} - z^{3} + 1);$
Ehr_int $= (-384z^{2088} - 600z^{2085} - 1560z^{2082} - 2544z^{2079} - 4848z^{2076} - 6288z^{2073} -$
$9744z^{2070} - 11760z^{2067} - 16368z^{2064} - 18960z^{2061} - 24720z^{2058} - 27888z^{2055} - 34800z^{2052}$
$- 38544z^{2049} - 46608z^{2046} - 50928z^{2043} - 60144z^{2040} - 65040z^{2037} - 75408z^{2034} -$
$80880z^{2031} - 92400z^{2028} - 98448z^{2025} - 111120z^{2022} - 117744z^{2019} - 131568z^{2016} -$
$138768z^{2013} - 153744z^{2010} - 161520z^{2007} - 177648z^{2004} - 186000z^{2001} - 203280z^{1998} -$
$212208z^{1995} - 230640z^{1992} - 240144z^{1989} - 259728z^{1986} - 269808z^{1983} - 290544z^{1980} -$
$301200z^{1977} - 323088z^{1974} - 334320z^{1971} - 357360z^{1968} - 369168z^{1965} - 393360z^{1962} -$
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3. Cubic count for the number of pairs of orthogonal mates of 2×3 latin rectangles that have rows with equal line sums (no column sum).

Ambient dimension is 12.

Inside-out polytope dimension is 9.

Number of regions is 186,780.

4. Affine count for the number of pairs of orthogonal mates of 2×3 latin rectangles that have row sums equal to t (no column sum).

Ambient dimension is 12.

Inside-out polytope dimension is 8.

Number of regions is 186,780.

5. Count for the number of pairs of orthogonal mates of 2×3 rectangles that have no line sums.

Ambient dimension is 12.

Inside-out polytope dimension is 12.

Number of regions is 687,852.

4.4 3×3 Orthogonal Mate Counts

1. Affine count for the number of pairs of orthogonal mates of 3×3 latin squares that have row sums and column sums equal to t.

Ambient dimension is 18.

Inside-out polytope dimension is 8.

Number of regions is 5,862,888.

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