A

Vertex and Hyperplane Descriptions of Polytopes

Everything should be made as simple as possible, but not simpler.

Albert Einstein

In this appendix, we prove that every polytope has a vertex and a hyperplane description. This appendix owes everything to Günter Ziegler's beautiful exposition in [192]; in fact, these pages contain merely a few cherries picked from [192, Lecture 1].

As in Chapter 3, it is easier to move to the world of cones. To be as concrete as possible, let us call $\mathcal{K} \subseteq \mathbb{R}^d$ an **h-cone** if

$$\mathcal{K} = \left\{ \mathbf{x} \in \mathbb{R}^d : \, \mathbf{A}\,\mathbf{x} \le \mathbf{0} \right\}$$

for some $\mathbf{A} \in \mathbb{R}^{m \times d}$; in this case $\mathcal{K}$ is given as the intersection of m halfspaces determined by the rows of $\mathbf{A}$. We use the notation $\mathcal{K} = \operatorname{hcone}(\mathbf{A})$.

On the other hand, we call $\mathcal{K} \subseteq \mathbb{R}^d$ a **v-cone** if

$$\mathcal{K} = \{ \mathbf{B}\,\mathbf{y} : \, \mathbf{y} \ge \mathbf{0} \}$$

for some $\mathbf{B} \in \mathbb{R}^{d \times n}$, that is, $\mathcal{K}$ is a pointed cone with the column vectors of $\mathbf{B}$ as generators. In this case we use the notation $\mathcal{K} = \operatorname{vcone}(\mathbf{B})$.

Note that, according to our definitions, any h- or v-cone contains the origin in its apex. We will prove that every h-cone is a v-cone and vice versa. More precisely:

Theorem A.1. *For every* $\mathbf{A} \in \mathbb{R}^{m \times d}$ *there exists* $\mathbf{B} \in \mathbb{R}^{d \times n}$ *(for some n) such that* $\operatorname{hcone}(\mathbf{A}) = \operatorname{vcone}(\mathbf{B})$. *Conversely, for every* $\mathbf{B} \in \mathbb{R}^{d \times n}$ *there exists* $\mathbf{A} \in \mathbb{R}^{m \times d}$ *(for some m) such that* $\operatorname{vcone}(\mathbf{B}) = \operatorname{hcone}(\mathbf{A})$.

We will prove the two halves of Theorem A.1 in Sections A.1 and A.2. For now, let us record that Theorem A.1 implies our goal, that is, the equivalence of the vertex and halfspace description of a polytope:

Corollary A.2. *If $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$, then $\mathcal{P}$ is the intersection of finitely many half spaces in $\mathbb{R}^d$. Conversely, if $\mathcal{P}$ is given as the bounded intersection of finitely many half spaces in $\mathbb{R}^d$, then $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$.*

Proof. If $\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ for some $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^d$, then coning over $\mathcal{P}$ (as defined in Chapter 3) gives

$$\operatorname{cone}(\mathcal{P}) = \operatorname{vcone}\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ 1 & 1 & & 1 \end{pmatrix}.$$

By Theorem A.1 we can find a matrix $(\mathbf{A}, \mathbf{b}) \in \mathbb{R}^{m \times (d+1)}$ such that

$$\operatorname{cone}(\mathcal{P}) = \operatorname{hcone}(\mathbf{A}, \mathbf{b}) = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \, (\mathbf{A}, \mathbf{b})\, \mathbf{x} \le \mathbf{0} \right\}.$$

We recover the polytope $\mathcal{P}$ upon setting $x_{d+1} = 1$, that is,

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \, \mathbf{A}\mathbf{x} \le -\mathbf{b} \right\},$$

which is a hyperplane description of $\mathcal{P}$.

These steps can be reversed: Suppose the polytope $\mathcal{P}$ is given as

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \, \mathbf{A}\mathbf{x} \le -\mathbf{b} \right\}$$

for some $\mathbf{A} \in \mathbb{R}^{m \times d}$ and $\mathbf{b} \in \mathbb{R}^m$. Then $\mathcal{P}$ can be obtained from

$$\operatorname{hcone}(\mathbf{A}, \mathbf{b}) = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \, (\mathbf{A}, \mathbf{b})\, \mathbf{x} \le \mathbf{0} \right\}$$

by setting $x_{d+1} = 1$. By Theorem A.1 we can construct a matrix $\mathbf{B} \in \mathbb{R}^{(d+1) \times n}$ such that

$$\operatorname{hcone}(\mathbf{A}, \mathbf{b}) = \operatorname{vcone}(\mathbf{B}).$$

We may normalize the generators of $\operatorname{vcone}(\mathbf{B})$, that is, the columns of $\mathbf{B}$, such that they all have $(d+1)$st variable equal to one:

$$\mathbf{B} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ 1 & 1 & & 1 \end{pmatrix}.$$

Since $\mathcal{P}$ can be recovered from $\operatorname{vcone}(\mathbf{B})$ by setting $x_{d+1} = 1$, we conclude that $\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. $\square$

A.1 Every h-cone is a v-cone

Suppose

$$\mathcal{K} = \operatorname{hcone}(\mathbf{A}) = \left\{ \mathbf{x} \in \mathbb{R}^d : \, \mathbf{A}\mathbf{x} \le \mathbf{0} \right\}$$

for some $\mathbf{A} \in \mathbb{R}^{m \times d}$. We introduce an auxiliary m-dimensional variable $\mathbf{y}$ and write

$$\mathcal{K} = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{d+m} : \mathbf{A}\,\mathbf{x} \le \mathbf{y} \right\} \cap \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{d+m} : \mathbf{y} = 0 \right\}. \tag{A.1}$$

(Strictly speaking, this is $\mathcal{K}$ lifted into a d-dimensional subspace of $\mathbb{R}^{d+m}$.) Our goal in this section is to prove the following two lemmas.

Lemma A.3. *The h-cone* $\left\{ \binom{\mathbf{x}}{\mathbf{y}} \in \mathbb{R}^{d+m} : \mathbf{A}\,\mathbf{x} \le \mathbf{y} \right\}$ *is a v-cone.*

Lemma A.4. *If* $\mathcal{K} \subseteq \mathbb{R}^d$ *is a v-cone, then so is* $\mathcal{K} \cap \left\{ \mathbf{x} \in \mathbb{R}^d : x_k = 0 \right\}$*, for any* k*.*

The first half of Theorem A.1 follows with these two lemmas, as we can start with (A.1) and intersect with one hyperplane $y_k = 0$ at a time.

Proof of Lemma A.3. We start by noting that

$$\begin{aligned}\mathcal{K} &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{d+m} : \mathbf{A}\,\mathbf{x} \le \mathbf{y} \right\} \\ &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{d+m} : (\mathbf{A}, -\mathbf{I}) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \le \mathbf{0} \right\}\end{aligned}$$

is an h-cone; here $\mathbf{I}$ represents an $m \times m$ identity matrix. Let us denote the k^{th} unit vector by $\mathbf{e}_k$. Then we can decompose

$$\begin{aligned}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} &= \sum_{j=1}^{d} x_j \begin{pmatrix} \mathbf{e}_j \\ \mathbf{A}\,\mathbf{e}_j \end{pmatrix} + \sum_{k=1}^{m} (y_k - (\mathbf{A}\,\mathbf{x})_k) \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_k \end{pmatrix} \\ &= \sum_{j=1}^{d} |x_j| \operatorname{sign}(x_j) \begin{pmatrix} \mathbf{e}_j \\ \mathbf{A}\,\mathbf{e}_j \end{pmatrix} + \sum_{k=1}^{m} (y_k - (\mathbf{A}\,\mathbf{x})_k) \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_k \end{pmatrix}.\end{aligned}$$

Note that if $\binom{\mathbf{x}}{\mathbf{y}} \in \mathcal{K}$ then $y_k - (\mathbf{A}\,\mathbf{x})_k \ge 0$ for all k, and so $\binom{\mathbf{x}}{\mathbf{y}}$ can be written as a nonnegative linear combination of the vectors $\operatorname{sign}(x_j) \binom{\mathbf{e}_j}{\mathbf{A}\,\mathbf{e}_j}$, $1 \le j \le d$, and $\binom{\mathbf{0}}{\mathbf{e}_k}$, $1 \le k \le m$. But this means that $\mathcal{K}$ is a v-cone. $\square$

Proof of Lemma A.4. Suppose $\mathcal{K} = \operatorname{vcone}(\mathbf{B})$, where $\mathbf{B}$ has the column vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in \mathbb{R}^d$; that is, $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ are the generators of $\mathcal{K}$. Fix $k \le d$ and construct a new matrix $\mathbf{B}_k$ whose column vectors are all $\mathbf{b}_j$ for which $b_{jk} = 0$, and the combinations $b_{ik}\mathbf{b}_j - b_{jk}\mathbf{b}_i$ whenever $b_{ik} > 0$ and $b_{jk} < 0$. We claim that

$$\mathcal{K} \cap \left\{ \mathbf{x} \in \mathbb{R}^d : x_k = 0 \right\} = \operatorname{vcone}(\mathbf{B}_k).$$

Every $\mathbf{x} \in \operatorname{vcone}(\mathbf{B}_k)$ satisfies $x_k = 0$ by construction of $\mathbf{B}_k$, and so $\operatorname{vcone}(\mathbf{B}_k) \subseteq \mathcal{K} \cap \left\{ \mathbf{x} \in \mathbb{R}^d : x_k = 0 \right\}$ follows immediately. We need to do some more work to prove the reverse containment.

Suppose $\mathbf{x} \in \mathcal{K} \cap \left\{ \mathbf{x} \in \mathbb{R}^d : \, x_k = 0 \right\}$, that is, $\mathbf{x} = \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \cdots + \lambda_n \mathbf{b}_n$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$ and $x_k = \lambda_1 b_{1k} + \lambda_2 b_{2k} + \cdots + \lambda_n b_{nk} = 0$. This allows us to define

$$\Lambda = \sum_{i:\, b_{ik} > 0} \lambda_i b_{ik} = - \sum_{j:\, b_{jk} < 0} \lambda_j b_{jk} \, .$$

Note that $\Lambda \ge 0$. Now consider the decomposition

$$\mathbf{x} = \sum_{j:\, b_{jk} = 0} \lambda_j \mathbf{b}_j + \sum_{i:\, b_{ik} > 0} \lambda_i \mathbf{b}_i + \sum_{j:\, b_{jk} < 0} \lambda_j \mathbf{b}_j \, . \tag{A.2}$$

If $\Lambda = 0$ then $\lambda_i b_{ik} = 0$ for all i such that $b_{ik} > 0$, and so $\lambda_i = 0$ for these i. Similarly, $\lambda_j = 0$ for all j such that $b_{jk} < 0$. Thus we conclude from $\Lambda = 0$ that

$$\mathbf{x} = \sum_{j:\, b_{jk} = 0} \lambda_j \mathbf{b}_j \, \in \, \operatorname{vcone}(\mathbf{B}_k) \, .$$

Now assume $\Lambda > 0$. Then we can expand the decomposition (A.2) into

$$\begin{aligned} \mathbf{x} = {} & \sum_{j:\, b_{jk} = 0} \lambda_j \mathbf{b}_j + \frac{1}{\Lambda} \left(- \sum_{j:\, b_{jk} < 0} \lambda_j b_{jk} \right) \left(\sum_{i:\, b_{ik} > 0} \lambda_i \mathbf{b}_i \right) \\ & \qquad + \frac{1}{\Lambda} \left(\sum_{i:\, b_{ik} > 0} \lambda_i b_{ik} \right) \left(\sum_{j:\, b_{jk} < 0} \lambda_j \mathbf{b}_j \right) \\ = {} & \sum_{j:\, b_{jk} = 0} \lambda_j \mathbf{b}_j + \frac{1}{\Lambda} \sum_{\substack{i:\, b_{ik} > 0 \\ j:\, b_{jk} < 0}} \lambda_i \lambda_j \left(b_{ik} \mathbf{b}_j - b_{jk} \mathbf{b}_i \right) , \end{aligned}$$

which is by construction in $\operatorname{vcone}(\mathbf{B}_k)$. □

A.2 Every v-cone is an h-cone

Suppose

$$\mathcal{K} = \operatorname{vcone}(\mathbf{B}) = \{ \mathbf{B}\,\mathbf{y} : \, \mathbf{y} \ge \mathbf{0} \}$$

for some $\mathbf{B} \in \mathbb{R}^{d \times n}$. Then $\mathcal{K}$ is the projection of

$$\left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{d+n} : \, \mathbf{y} \ge \mathbf{0}, \, \mathbf{x} = \mathbf{B}\,\mathbf{y} \right\} \tag{A.3}$$

to the subspace $\left\{ \binom{\mathbf{x}}{\mathbf{y}} \in \mathbb{R}^{d+n} : \, \mathbf{y} = \mathbf{0} \right\}$. The constraints for (A.3) can be written as

$$\mathbf{y} \ge \mathbf{0} \qquad \text{and} \qquad (\mathbf{I}, -\mathbf{B}) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \mathbf{0} \, .$$

Thus the set (A.3) is an h-cone, for which we can project one component of $\mathbf{y}$ at a time to obtain $\mathcal{K}$. This means that it suffices to prove the following lemma to finish the second half of Theorem A.1.

Lemma A.5. *If $\mathcal{K}$ is an h-cone, then the projection $\{\mathbf{x} - x_k \mathbf{e}_k : \mathbf{x} \in \mathcal{K}\}$ is also an h-cone, for any k.*

Proof. Suppose $\mathcal{K} = \operatorname{hcone}(\mathbf{A})$ for some $\mathbf{A} \in \mathbb{R}^{m \times d}$. Fix k and consider

$$\mathcal{P}_k = \{\mathbf{x} + \lambda \mathbf{e}_k : \mathbf{x} \in \mathcal{K}, \lambda \in \mathbb{R}\} .$$

The projection we're after can be constructed from this set as

$$\{\mathbf{x} - x_k \mathbf{e}_k : \mathbf{x} \in \mathcal{K}\} = \mathcal{P}_k \cap \left\{\mathbf{x} \in \mathbb{R}^d : x_k = 0\right\} ,$$

so that it suffices to prove that $\mathcal{P}_k$ is an h-cone.

Suppose $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are the row vectors of $\mathbf{A}$. We construct a new matrix $\mathbf{A}_k$ whose row vectors are all $\mathbf{a}_j$ for which $a_{jk} = 0$, and the combinations $a_{ik}\mathbf{a}_j - a_{jk}\mathbf{a}_i$ whenever $a_{ik} > 0$ and $a_{jk} < 0$. We claim that $\mathcal{P}_k = \operatorname{hcone}(\mathbf{A}_k)$.

If $\mathbf{x} \in \mathcal{K}$ then $\mathbf{A}\,\mathbf{x} \le \mathbf{0}$, which implies $\mathbf{A}_k\,\mathbf{x} \le \mathbf{0}$ because each row of $\mathbf{A}_k$ is a nonnegative linear combination of rows of $\mathbf{A}$; that is, $\mathcal{K} \subseteq \operatorname{hcone}(\mathbf{A}_k)$. However, the k^{th} component of $\mathbf{A}_k$ is zero by construction, and so $\mathcal{K} \subseteq \operatorname{hcone}(\mathbf{A}_k)$ implies $\mathcal{P}_k \subseteq \operatorname{hcone}(\mathbf{A}_k)$.

Conversely, suppose $\mathbf{x} \in \operatorname{hcone}(\mathbf{A}_k)$. We need to find a $\lambda \in \mathbb{R}$ such that $\mathbf{A}(\mathbf{x} - \lambda \mathbf{e}_k) \le \mathbf{0}$, that is,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1k}(x_k - \lambda) + \cdots + a_{1d}x_d &\le 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mk}(x_k - \lambda) + \cdots + a_{md}x_d &\le 0 . \end{aligned}$$

The j^{th} constraint is $\mathbf{a}_j \cdot \mathbf{x} - a_{jk}\lambda \le 0$, that is, $\mathbf{a}_j \cdot \mathbf{x} \le a_{jk}\lambda$. This gives the following conditions on λ:

$$\begin{aligned} \lambda &\ge \frac{\mathbf{a}_i \cdot \mathbf{x}}{a_{ik}} && \text{if } a_{ik} > 0 , \\ \lambda &\le \frac{\mathbf{a}_j \cdot \mathbf{x}}{a_{jk}} && \text{if } a_{jk} < 0 . \end{aligned}$$

Such a λ exists because if $a_{ik} > 0$ and $a_{jk} < 0$ then (since $\mathbf{x} \in \operatorname{hcone}(\mathbf{A}_k)$)

$$(a_{ik}\mathbf{a}_j - a_{jk}\mathbf{a}_i) \cdot \mathbf{x} \le 0 ,$$

which is equivalent to

$$\frac{\mathbf{a}_i \cdot \mathbf{x}}{a_{ik}} \le \frac{\mathbf{a}_j \cdot \mathbf{x}}{a_{jk}} .$$

Thus we can find a λ that satisfies

$$\frac{\mathbf{a}_i \cdot \mathbf{x}}{a_{ik}} \le \lambda \le \frac{\mathbf{a}_j \cdot \mathbf{x}}{a_{jk}} ,$$

which proves $\operatorname{hcone}(\mathbf{A}_k) \subseteq \mathcal{P}_k$. □