AN ENUMERATION THEORY FOR MAGIC AND MAGILATIN LABELLINGS
(WORKING DRAFT August 19, 2003)

MATTHIAS BECK AND THOMAS ZASLAVSKY

Abstract. We present an enumerative theory for all kinds of magic and latin object such as squares, graphs, cubes, etc.

Note to publisher:
This paper does NOT have a "corresponding author" or "senior author".
All authors are EQUAL.
All authors are able to answer correspondence from readers.

For editorial purposes ONLY, contact the writer of the cover letter of submission.

Contents

1. It's all kind of magic 2
2. In which inside-out polytopes take the stage 4
3. A jumble of magic squares, magic graphs, the Birkhoff polytope, and general nonsense about equal line sums 6
3.1. Sorts of magic 6
3.2. Cubical magic 7
3.3. Affine magic 13
3.4. Limited and slack magic 17
4. In which latin squares, partially, join the fray 18
4.1. Cubical latinity 19
4.2. Affine latinity 20
5. Extensions and conjectures 21
5.1. Generalized exclusions 21
5.2. Periods 22
References 22

Date: Version of August 19, 2003.
2000 *Mathematics Subject Classification. Primary* 05B15, 05C78; *Secondary* 05A15, 05B35, 52B20, 52C35, 52C07.
Key words and phrases. Magic square, semimagic square, magic graph, latin square, magilatin square, lattice-point counting, rational convex polytope, arrangement of hyperplanes.
The research of the second author was partially supported by National Science Foundation grant DMS-0070729.

1. It's all kind of magic

We count magic squares and their innumerable relatives: semimagic and pandiagonal magic squares, magic cubes [1, 2] and hypercubes, magic graphs, and magical oddities like circles, spheres, and stars [1]. A *magic square* is an $n \times n$ array of distinct positive, or sometimes nonnegative, integers whose sum along any *line* (row, column, or main diagonal) is the same number, the *magic sum*. Magic squares date back to China in the first millenium B.C.E. [8]; underwent much further development in the Islamic world late in the first millenium C.E. and in the next millenium (or sooner; the data are lacking) in India [9]. From Islam they passed to Christendom in the later Middle Ages, probably initially through the Jewish community [9, Part II, pp. 290 ff.] and later possibly Byzantium [9, Part I, p. 198], and no later than the early eighteenth century (the data are buried in barely tapped archives) to sub-Saharan Africa [38, Chapter 12]. The contents of a magic square have varied with time and writer; usually they have been the first n^2 consecutive integers (*standard* squares), but often any arithmetic sequence or arbitrary numbers. The fixed ideas are that they are integers, positive, and distinct. (Even the mathematical treatises [1, 2, 6] take positivity and distinctness so much for granted as never to mention them.) The same holds for variants such as *pandiagonal magic squares*, where we include among the line sums the wrapped diagonals. In the past century, though, mathematicians have made some simplifications in the interest of obtaining results about the number of squares with a fixed magic sum. Thus we have *semimagic* squares, where the diagonals are omitted from the sum lines; sometimes a relaxation of positivity to nonnegativity; and, most significantly, the omission of the fundamental requirement of distinctness. Since the terminology is rather confused, we shall call a square *magic* or *strongly magic* if its entries are distinct and *weakly magic* if they need not be distinct.

The problem of counting magic squares (other than standard squares) seems to have occurred to anyone only in the twentieth century, no doubt because there was no way to approach the question previously. Recently there has grown up a literature on exact formulas for the numbers $M(t)$ and $M_0(t)$ of positive and nonnegative weak semimagic and magic squares with magic sum t, in part for the simple reason that they are amenable to analysis in terms of Ehrhart theory [11]. (See for example [12, 29] for semimagic squares; for magic squares see [3].) One treats a square as an integer vector $x \in t[0,1]^{n^2}$, confined to the subspace ts where $t = 1, 2, 3, \dots$ and

$$s := \{x \in \mathbb{R}^{n^2} : \text{ all line sums equal } 1\},$$

the *magic subspace*. (Exactly which subspace this is depends on whether we treat squares that are semimagic, magic, pandiagonal, or another type.) Thus a square x is an integer lattice point in the *magic polytope* $P := t\big([0,1]^{n^2} \cap s\big)$; moreover, $x \in P^\circ$, the relative interior of P, if and only if the square is positive. It follows from Ehrhart's fundamental results on integer-point enumeration in polytopes [11] that M and M_0 are quasipolynomials in t (polynomials in the semimagic case, because the matrix that defines s is totally unimodular so the vertices of P are all integral; however, in the magic case this is unfortunately not so and the period is not easy to calculate). A *quasipolynomial* is a function $Q(t) = \sum_0^d c_i t^i$ with coefficients c_i that are periodic functions of t (so that Q is a polynomial on each residue class modulo some integer, called the *period*; these polynomials are the *constituents* of Q). Ehrhart's *Reciprocity Theorem* [11] takes a remarkable form: adding 1 to every entry of a square with magic sum t creates a positive square whose magic sum is $t + n$; therefore

$M_0(t) = M(t+n)$, so by reciprocity, $M(t) = (-1)^{\dim s} M(n-t)$. This makes for a very nice, elegant theory applicable not only to magic squares but also similar objects, among them other kinds of squares, hypercubes, graphs, designs, and linear forms.

Still there were no exact (theoretical) formulas for strong squares (not even in the comprehensive tome [28]), with the exception of Stanley's [31, Exercise 4.10]. With the theory of *inside-out polytopes* [5] we can attack this and many related counting problems in a uniform, systematic way. We obtain not only a general result about magic counting functions but also a fascinating interpretation of reciprocity that leads to a new kind of question about permutations.

To apply inside-out theory, we supplement the polytope $P = [0,1]^{n^2} \cap s$ with the pair-equality hyperplane arrangement

$$\mathcal{H} := \mathcal{H}[K_{n^2}]^s = \{h_{ij} \cap s : i < j \leq n^2\},$$

where h_{ij} is the hyperplane $x_i = x_j$. ($\mathcal{H}[\Gamma] := \{h_{ij} : ij \in E\}$ is the hyperplane arrangement of the graph Γ with edge set E. K_n denotes the complete graph on n nodes.) The number of $n \times n$ squares corresponding to s with magic sum t is the number of integer points in $(t+1)\,(P^\circ \setminus \bigcup \mathcal{H})$. This is a quasipolynomial in t by the general inside-out polytopes theory. The inside-out reciprocity [5] gives the enumeration of weak nonnegative squares with multiplicity, reminiscent of Stanley's theorem on acylic orientations [29].

The basic and oldest kind of magic square is one of standard form, where the n^2 integers are the first n^2 positive integers; or one might say, the line sum is the smallest possible. Another famous kind of squares are latin squares and their relatives. Here each line has n different numbers. In a *latin square* these n numbers are the same in every line and are normally taken to be the first n positive integers. In any latin square in this broad meaning, every line has the same sum. Suppose we add this property to the definition of a latin square; call these *magilatin squares*. Then the only difference between a semimagic square and a magilatin square is in the inequations we assume between the entries. As with magic and semimagic squares, inside-out polytope theory yields theorems about the number of magilatin squares as a function either of the magic sum or of the largest allowed value of an entry in the square.

Still another topic to which we can apply inside-out theory are *latin rectangles*: $m \times n$ rectangular arrays filled by n symbols; allowing more symbols gives a *partial latin rectangle*. The asymptotic numbers of latin squares and rectangles of given dimensions have been the subject of many studies. Our geometric counting method, where we think of a partial latin rectangle as a point in $\mathbb{Z}^{mn}$ but not in $\bigcup \mathcal{H}[K_m \times K_n]$, leads in a different direction that, as far as we know, has not been studied. We look at the number of magilatin squares or partial latin rectangles of fixed dimensions as we vary the number of available symbols. Our treatment is similar to that of magic squares, but the distinctness requirement, and consequently the hyperplane arrangement, are fundamentally different. While in a magic square each entry must differ from every other, in a magilatin square it must differ only from those that are collinear with it, a *line* (as with semimagic squares) being a row or column.

The magic and latin properties generalize far beyond squares. Semimagic and pandiagonal magic squares suggest a general picture: that of a *covering cluster*, consisting of a finite set X of points together with a family $\mathcal{L}$ of subsets, called *lines* for no particular reason, of which none contains any other and none is empty, and whose union is X. We want to assign integers, possibly positive ones, to X so that all line sums are equal to a single number, the *magic sum*. Such a labelling is a *weakly* or *strongly magic* or *latin labelling* of the covering cluster, depending on the particular requirements. A *standard* labelling is strong and employs the label set $\{1, 2, \ldots, |X|\}$. A covering cluster with a (weakly) magic (or latin, etc.) labelling may itself be called *(weakly) magic*, etc. There are many interesting examples that have been the object of greatly varying degrees of attention. $(X, \mathcal{L})$ may be a finite affine or projective geometry, the "lines" being the subspaces of any fixed dimension; more generally it may be a block design. It may be an $n \times n \times \cdots \times n$ hypercubical array. It may be a *k-net*, where the lines fall into k parallel classes (with $k \geq 2$) so that each point belongs to a unique line in each parallel class. (A semimagic square is a 2-net and a pandiagonal square is a kind of 4-net.) All these examples have lines of equal size, a property that has certain advantages but is not necessary for the existence of the desired labellings. Consider for instance *magic graphs*: here the edges are assigned positive integers so that the sum of labels of all edges incident to any one node is the same, regardless of the node; these generate covering clusters in the obvious way. The type example is the complete bipartite graph $K_{n,n}$, which gives semimagic squares. There is a fairly extensive literature on such questions as which graphs have magic labellings; see [15, Section 5.1]. The closest approach we know to general covering clusters is the recent article [14] on "magic carpets", which are covering clusters with strongly magic standard labellings.

These ideas generalize still further. Take rational linear forms $f_1, f_2, \ldots, f_m$. A magic labelling of $[d] := \{1, 2, \ldots, d\}$ with respect to $f_1, f_2, \ldots, f_m$ is an integer point $x \in \mathbb{R}^d$ such that all the values $f_i(x)$ are equal to the same number (the magic sum). The analog here of a covering cluster in which all lines have the same size is a system of forms for which all values $f_i(\mathbf{1})$ (the *weights*; $\mathbf{1}$ is the vector of all ones) are equal; such systems have nice properties.

Still other kinds of generalizations and examples will be found in the detailed treatments later in the paper. We mention especially the fact that there are two distinct interesting approaches to enumeration. Traditionally, magic and semimagic squares have been counted with the magic sum as the parameter; but another tack is to take as parameter the maximum allowed value of a label. The same two systems apply to latinity.

2. In which inside-out polytopes take the stage

The theory of inside-out polytopes [5] was motivated by the problem of counting those points of a discrete lattice D that are contained in a D-fractional convex polytope but not in an affine hyperplane arrangement. We call a polytope *D-integral* if its vertices all lie in D and *D-fractional* if the vertices lie in the contracted lattice $t^{-1}D$ for some positive integer t.

Suppose we are given a discrete lattice D, a D-fractional convex polytope P, and a D-fractional hyperplane arrangement $\mathcal{H}$, that is, each hyperplane in $\mathcal{H}$ is spanned by the D-fractional points it contains. Then $(P, \mathcal{H})$ is a *D-fractional inside-out polytope.*

A *region* of $\mathcal{H}$ is a connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$. A *region* of $(P, \mathcal{H})$ is the nonempty intersection of a region of $\mathcal{H}$ with P. A *vertex* of $(P, \mathcal{H})$ is a vertex of any such region. The *denominator* of $(P, \mathcal{H})$ is the smallest positive integer t for which $t^{-1}D$ contains every vertex of $(P, \mathcal{H})$.

The fundamental counting functions associated with $(P, \mathcal{H})$ are the (*closed*) *Ehrhart quasipolynomial* (we always assume P is closed),

$$E_{P,\mathcal{H}}(t) := \sum_{x \in t^{-1}D} m_{P,\mathcal{H}}(x),$$

where the *multiplicity* $m_{P,\mathcal{H}}(x)$ of $x \in \mathbb{R}^d$ with respect to $\mathcal{H}$ is defined through

$$m_{P,\mathcal{H}}(x) := \begin{cases} \text{the number of closed regions of } \mathcal{H} \text{ in } P \text{ that contain } x, & \text{if } x \in P, \\ 0, & \text{if } x \notin P, \end{cases}$$

and the *open Ehrhart quasipolynomial*,

$$E^{\circ}_{P,\mathcal{H}}(t) := \#\left(t^{-1}D \cap \left[P \setminus \bigcup \mathcal{H}\right]\right),$$

We denote by $\operatorname{vol}_D P$ the volume of P normalized with respect to D, that is, we take the volume of a fundamental domain of D to be 1 (in the case of the integer lattice $\mathbb{Z}^d$ this is the ordinary volume).

The names of our counting functions are justified by the fact that in the absence of $\mathcal{H}$ we recover Ehrhart's classical theory of lattice-point enumeration in polytopes [11], and by one of the main results in [5]:

Theorem 2.1 ([5, Theorem 4.1])**.** *If D is a full-dimensional discrete lattice and $(P, \mathcal{H})$ is a closed, full-dimensional, D-fractional inside-out polytope in $\mathbb{R}^d$ such that $\mathcal{H}$ does not contain the degenerate hyperplane, then $E_{P,\mathcal{H}}(t)$ and $E^{\circ}_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t, with period equal to a divisor of the denominator of $(P, \mathcal{H})$, with leading term $c_d t^d$ where $c_d = \operatorname{vol}_D P$, and with the constant term $E_{P,\mathcal{H}}(0)$ equal to the number of regions of $(P, \mathcal{H})$. Furthermore,*

$$E^{\circ}_{P^\circ,\mathcal{H}}(t) = (-1)^d E_{P,\mathcal{H}}(-t). \tag{2.1}$$

In particular, if $(P, \mathcal{H})$ is D-integral then $E_{P,\mathcal{H}}$ and $E^{\circ}_{P^\circ,\mathcal{H}}$ are polynomials.

The *Möbius function* of a finite partially ordered set S is the function $\mu : S \times S \to \mathbb{Z}$ defined recursively by

$$\mu(r, s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r, u) & \text{if } r < s. \end{cases}$$

Sources are, *inter alia*, [27] and [31, Section 3.7].

The *intersection semilattice* of $\mathcal{H}$ is defined as

$$\mathcal{L}(\mathcal{H}) := \left\{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H} \text{ and } \bigcap \mathcal{S} \neq \varnothing\right\},$$

ordered by reverse inclusion [39], and $\mu(s) = \mu(\hat{0}, s)$, the Möbius function of $\mathcal{L}(\mathcal{H})$ (with a minor exception). $\mathcal{L}$ is a *geometric semilattice* [26, 35] with $\hat{0} = \bigcap \varnothing = \mathbb{R}^d$; it is a *geometric lattice* (for which see [27, p. 357] or [31], etc.) if $\mathcal{H}$ has nonempty intersection, as when all the hyperplanes are homogeneous.

We will need the arrangement *induced* by $\mathcal{H}$ on s; this is

$$\mathcal{H}^s := \{h \cap s : h \in \mathcal{H}, h \not\supseteq s\}.$$

For the second theorem of [5] we will use here, we need the notion of transversality: $\mathcal{H}$ is *transverse* to P if every *flat* $u \in \mathcal{L}(\mathcal{H})$ that intersects P also intersects P°, and P does not lie in any of the hyperplanes of $\mathcal{H}$. Let $E_P(t) := \#\left(tP \cap \mathbb{Z}^d\right)$ the standard Ehrhart counting function (without any hyperplanes present).

Theorem 2.2 ([5, Theorem 4.2]). *If D, P, and $\mathcal{H}$ are as in Theorem 2.1 and furthermore P is closed, then*

$$E^{\circ}_{P,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E_{P \cap u}(t), \tag{2.2}$$

and if $\mathcal{H}$ is transverse to P,

$$E_{P,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{P \cap u}(t). \tag{2.3}$$

Often the polytope is not full-dimensional. Suppose, then, that D is a discrete lattice in $\mathbb{R}^d$ and s is any affine subspace. The *period* $p(s)$ of s with respect to D is the smallest positive integer p for which $p^{-1}D$ meets s. Then Theorem 2.1 implies

Corollary 2.3. *Let D be a discrete lattice in $\mathbb{R}^d$, P a D-fractional convex polytope, and $\mathcal{H}$ a hyperplane arrangement in $s := \operatorname{aff} P$ that does not contain the degenerate hyperplane. Then $E_{P,\mathcal{H}}(t)$ and $E^{\circ}_{P^{\circ},\mathcal{H}}(t)$ are quasipolynomials in t that satisfy the reciprocity law $E^{\circ}_{P^{\circ},\mathcal{H}}(t) = (-1)^{\dim s} E_{P,\mathcal{H}}(-t)$. Their period is a multiple of $p(s)$ and a divisor of the denominator of $(P, \mathcal{H})$. If $t \equiv 0 \bmod p(s)$, the leading term of $E_{P,\mathcal{H}}(t)$ is $(\operatorname{vol}_{p(s)^{-1}D} P) t^{\dim s}$ and its constant term is the number of regions of $(P, \mathcal{H})$; but if $t \not\equiv 0 \bmod p(s)$, then $E_{P,\mathcal{H}}(t) = E^{\circ}_{P^{\circ},\mathcal{H}}(t) = 0$.* ■

3. A jumble of magic squares, magic graphs, the Birkhoff polytope, and general nonsense about equal line sums

3.1. **Sorts of magic.** We pointed out in the introduction that the difference between weak and strong magic (or semimagic) squares lies in the fact that for the latter we require the entries to be distinct. We will therefore spend the beginning of this section studying the general setting of integer points in polytopes with distinct entries—but the beginning of the beginning in discussing additional diverse situations in which magic properties can arise.

The idea of magic labelling generalizes to nonzero, rational linear forms $f_1, f_2, \ldots, f_m$; that is, $f_i \in (\mathbb{R}^d)^*$. We do not wish to require positivity of the coefficients because of examples like *magic digraphs*, where the edge value is added at the head node and subtracted at the tail node, and the common generalization of these and magic graphs, *magic bidirected graphs*.

Let us explore the situation of nonequal coordinates in detail. Suppose a convex polytope $P \subseteq \mathbb{R}^d$, spanning an affine subspace s. The hyperplane arrangement that captures the distinctness of the entries of integer vectors is $\mathcal{H} := \mathcal{H}[K_d]^s$, the arrangement $\mathcal{H}[K_d]$ of the complete graph on d nodes intersected with s. Transversality means that, first of all, s is not a subspace of any hyperplane $\{x_j = x_k\}$ and, secondly, any flat of $\mathcal{H}[K_d]$ that meets P also meets P°. In many of the interesting special cases, though not in general, the latter condition is automatic. In the languate of inside-out counting, we recover one of our fundamental enumeration function as

$$E^{\circ}_{P^{\circ},\mathcal{H}}(t) = \text{ the number of strong positive squares with magic sum } t.$$

Suppose x is a point in $\mathbb{R}^d$ whose entries are all distinct. There is a unique permutation τ of $[d]$ such that $x_{\tau 1} < x_{\tau 2} < \cdots < x_{\tau d}$. We say x *realizes* τ. We call a permutation σ *realizable* in a subset $A \subseteq s$ if there is a vector $x \in A$ that realizes it. We are interested in realizability in P, but realizability in s is simpler. Fortunately, a permutation that is realizable in s is also realizable in P if P contains a positive multiple of $\mathbf{1} := (1, 1, \ldots, 1)$, since every closed region of $\mathcal{H}$ contains $\langle \mathbf{1} \rangle$. This is the case when every form has equal positive weight.

If $x \in \mathbb{R}^d$ satisfies $x_{\sigma 1} \leq x_{\sigma 2} \leq \cdots \leq x_{\sigma d}$, we say x and σ are *compatible.*

Theorem 3.1. *Suppose $P \subseteq \mathbb{R}^d$ is a closed, rational convex polytope transverse to $\mathcal{H}[K_d]$ and $s := \operatorname{aff} P$. The number $E^{\circ}_{P^{\circ},\mathcal{H}[K_d]^s}(t)$ of integer points in tP° with distinct entries is a quasipolynomial in positive integers t with leading term $(\operatorname{vol}_D P)t^{\dim s}$, where $D := s \cap \mathbb{Z}^d$, and with constant term equal to the number of permutations of $[d]$ that are realizable in P. Furthermore, $(-1)^{\dim s} E^{\circ}_{P^{\circ},\mathcal{H}[K_d]^s}(-t) = E_{P,\mathcal{H}[K_d]^s}(t) :=$ the number of pairs (x, σ) consisting of an integer point $x \in tP$ and a compatible P-realizable permutation σ of $[d]$.*

Proof. The first statement is a direct consequence of Corollary 2.3 along with the observation that, by transversality, a region that intersects P must also intersect P°. For the second, the multiplicity $m(x)$ of x equals the number of closed regions of $\mathcal{H}[K_d]^s$ that contain x. The regions of $\mathcal{H}[K_d]^s$ correspond to certain regions of $\mathcal{H}[K_d]$, which correspond to permutations of $[d]$. Clearly, a closed region contains x if and only if its permutation is compatible with x. Thus, $m(x)$ is the number of permutations that are both realizable in P and compatible with x. Now appeal to Corollary 2.3. ∎

Problem 3.2. The period and denominator present a puzzle. The denominator of the inside-out polytope $(P, \mathcal{H}[K_d])$ is obviously a multiple of the denominator of the standard polytope P. The first question is when the hyperplane arrangement $\mathcal{H}[K_d]$ changes this latter denominator, and in what way. As for the period, if in particular P has integral vertices then E_P is a polynomial. What conditions on P ensure that $E_{P,\mathcal{H}[K_d]}$ is also a polynomial? That it need not be is illustrated by the simple example of the line segment from $(0, 1)$ to $(1, 0)$ in $\mathbb{R}^2$ and the hyperplane $\{x_1 = x_2\}$.

Our desire to develop the ideas behind magic squares and graphs suggests two approaches to choosing P and s. Think of how they function in the problem. The subspace, s, represents the existence of a magic sum. The polytope, P, represents the constraints on the entries in the value vector. The magic sum constraints may be pure equalities:

(i) Set all linear forms equal to each other. (Homogeneous equations.)

Or, they may be set all equal to a controlled constant:

(ii) Set all linear forms equal to t. (Affine equations.)

(Sometimes one wants additional equations; see the discussion of centrally symmetric squares in Examples 3.13 and 3.26.) Similarly, the constraints on the components of x may be pure, independent of the magic sum conditions:

(I) All variables x_i satisfy $0 \leq x_i \leq t$. (Cubical constraints.)

Or, the constraints may be merely nonnegativity of the variables:

(II) All variables $x_i \geq 0$. (Nonnegativity.)

We think the natural combinations are (i) with (I) and (ii) with (II) and that is how we develop the theory.

3.2. **Cubical magic.** The cubical approach to magic squares counts them by the largest allowed value of the entry in a cell; if t is the parameter, the squares counted are those with entries $0 < y_{ij} < t$. Similarly, magic labellings of a bidirected graph are counted by the upper bound $t - 1$ on the edge labels.

In the general situation the magic subspace s is defined by homogeneous, rational linear equations

$$f_1(x) = f_2(x) = \cdots = f_m(x) = 0. \tag{3.1}$$

(These are obviously equivalent to (i), if the forms in 3.1 are the differences of the forms of (i).) The magic polytope is $P := [0,1]^d \cap s$, whence the name "cubical". The hyperplane arrangement is $\mathcal{H} := \mathcal{H}[K_d]^s$. In the example of magic squares, P is the set of all square matrices with real entries in $[0,1]$ such that all row, column, and diagonal sums are equal. For magic graphs P is the set of edge labellings by numbers in $[0,1]$ such that all node sums are equal.

One has to be careful that P spans s.

Lemma 3.3. *If P is not contained within a coordinate hyperplane, it affinely spans the magic subspace s.*

Proof. By the hypothesis P contains points x^i (not necessarily distinct) with $0 < (x^i)_i \le 1$. The barycenter of these points lies in $(0,1]^d \cap s$, hence in P. Therefore $P^\circ = s \cap (0,1)^d$, which clearly spans s. ∎

If P is contained in a coordinate hyperplane, one should reformulate the problem with fewer variables.

Now for the main theorem on cubical magic.

Theorem 3.4 (Magic enumeration by bounds)**.** *Suppose $s \subseteq \mathbb{R}^d$ is given by (3.1) and $P := [0,1]^d \cap s$ does not lie within a coordinate hyperplane. For $t = 1, 2, \ldots$ let*

$N^\circ(t) :=$ *the number of integer points $x \in s$ with distinct entries that satisfy $0 < x_i < t$,*

and let

$N(t) :=$ *the number of pairs (x, σ) consisting of an integer point $x \in s$ that satisfies $0 \le x_i \le t$ and a compatible s-realizable permutation σ of $[d]$.*

Then N° and N are quasipolynomials with leading term $(\mathrm{vol}_D P)t^{\dim s}$, where $D := s \cap \mathbb{Z}^d$, and with constant term $N(0)$ equal to the number of permutations of $[d]$ that are realizable in s. Furthermore, $(-1)^{\dim s} N^\circ(-t) = N(t)$.

Proof. Theorem 3.1 shows that $N = E_{P,\mathcal{H}}$ and $N^\circ = E^\circ_{P^\circ,\mathcal{H}}$ are reciprocal quasipolynomials. The constant term equals the number of permutations that are realizable in P. These are the same that are realizable in s, because a permutation realized by x is also realized by αx for any positive real α. By choosing small enough α we can put αx into P or $-P$. If the latter, then $\frac{1}{2}\mathbf{1} - \alpha x$ realizes the permutation and lies in P. ∎

Most interesting is the case in which all the forms f_i have weight zero. It is precisely then that $\mathcal{H}$ and P are transverse, as one can see by comparing the subspace $\bigcap \mathcal{H} = \langle \mathbf{1} \rangle \cap s$ with the definition of transversality; moreover, then P spans s.

Theorem 3.5. *With s defined by forms of weight zero, and assuming a magic labelling exists, we have*

$$N^\circ(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E^\circ_{[0,1]^d \cap u}(t) \ ,$$

$$N(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{[0,1]^d \cap u}(t) \ ,$$

where μ is the Möbius function of $\mathcal{L}(\mathcal{H})$.

Proof. Apply Theorem 2.2 in s. A magic labelling exists if and only if s does not lie in any hyperplane $\{x_j = x_k\}$. ■

A flat $u \in \mathcal{L}(\mathcal{H})$ has the form $v \cap s$ where v is given by a series of equations of coordinates: $x_{i_1} = x_{i_2} = x_{i_p}$, $x_{j_1} = x_{j_2} = x_{j_q}$, etc.; that is, v corresponds to a partition π of X. We can treat these equations as eliminating the variables $x_{i_2}, \ldots, x_{i_p}, x_{j_2}, \ldots, x_{j_q}, \ldots$ in favor of $x_{i_1}, x_{j_1}, \ldots$. With this substitution, $v = \mathbb{R}^{d'}$ for some $d' < d$ and $[0,1]^d \cap v = [0,1]^{d'}$. Then $[0,1]^d \cap u$ is essentially $[0,1]^{d'} \cap s'$, where s' is s after identifying variables.

Weight zero has another interesting consequence.

Corollary 3.6. *With s as in Theorem 3.5,*

$$E^{\circ}_{P,\mathcal{H}}(t) = (-1)^{\dim P} N(-t-2) \ .$$

Proof. By adding one to every coordinate of an integer point in $[0,t]^d \cap s$ with distinct entries, we get an integer point in $(0,t+2)^d \cap s$ with distinct entries. Hence

$$E^{\circ}_{P,\mathcal{H}}(t) = E^{\circ}_{P^{\circ},\mathcal{H}}(t+2)$$

and the result follows from reciprocity. ■

We describe some of the most interesting examples.

Example 3.7 (Lines of constant length). In a covering cluster $(X, \mathcal{L})$, suppose every line has the same number of points. Then the linear equations that express the existence of a magic sum take the form

$$\sum_{j \in L_1} x_j = \sum_{j \in L_2} x_j \text{ for all } L_1, L_2 \in \mathcal{L} \ .$$

Theorems 3.4 and 3.5 and Corollary 3.6 all apply: $N^{\circ}(t)$ and $N(t)$ are reciprocal quasipolynomials in the upper bound $t-1$ or t, respectively, and so on. Examples include magic, semimagic, and pandiagonal magic squares, affine and projective planes, k-nets, and magic hypercubes with or without diagonals of various kinds. One has to ask about the existence of magic labellings. There is no known general answer, but certainly there exist magic and semimagic squares of all orders $n \geq 3$ and pandiagonal magic squares of all orders $n \geq 4$, using the standard entries $\{1, 2, \ldots, n^2\}$ if $n \not\equiv 2 \bmod 4$ (see [2, pp. 203–211]).

Example 3.8 (Covering clusters and magic graphs). For a general covering cluster only Theorem 3.4 applies. Such covering clusters include magic edge labellings of graphs. For interesting work on characterizing graphs that have magic edge labellings see [21]. Another graphic example is magic *total labellings*, where both nodes and edges are labelled. One sums the numbers of each node and its incident edges (*node magic*) or an edge and its incident nodes (*edge magic*) or both (*total magic*) and desires all sums to be equal; see [15, Chapter 5] and [36].

If magic labellings exist on a covering cluster, the problems of existence and characterization of realizable permutations come to the fore. An example of nonexistence is the Fano plane. Examples of existence are magic squares of side $n \geq 3$. Regarding characterization we propose a conjecture. We may assume $X = [d]$. Given a covering cluster, a *magic permutation* (with respect to the covering cluster) is a permutation of $[d]$ that is realizable by a *positive* point $x \in P$. Obviously, x can be chosen to be rational if it exists at all. In the cubical situation we are discussing here, all points in P° are positive, so magic permutations are identical to P-realizable and therefore to s-realizable permutations. A permutation σ of $[d]$ defines a *reverse dominance order* on the power set $\mathcal{P}([d])$ by

> $L \preccurlyeq_\sigma L'$ if, when L and L' are written in decreasing order according to σ, say $L = \{\sigma j_1, \ldots, \sigma j_l\}$ where $j_1 > \cdots > j_l$ and $L' = \{\sigma j'_1, \ldots, \sigma j'_{l'}\}$ where $j'_1 > \cdots > j'_{l'}$, then $l \leq l'$ and $j_1 \leq j'_1, \ldots, j_l \leq j'_l$.

This is a partial order on $\mathcal{P}([d])$.

Conjecture 3.9 (Magic permutations). A permutation σ of $[d]$ is realizable by a positive point in the magic subspace s of the covering cluster $([d], \mathcal{L})$ if and only if $\mathcal{L}$ is an antichain in the reverse dominance order due to σ.

The archetype is magic squares. A magic permutation there is a permutation of the cells of the square that is obtained from some magic labelling by arranging the cells in increasing order. We have verified the conjecture for 3×3 magic and semimagic squares.

A permutation being a total order on $[d]$, one could generalize to total preorders (in which the antisymmetric law is not required); we propose the corresponding conjecture, replacing "permutation" by "total preorder".

The necessity for $\mathcal{L}$ to be an antichain is obvious. It is the same with the extension to linear forms. We define the *reverse dominance order due to* σ on the set $\left(\mathbb{R}^d\right)^*_+$ of positive linear forms on $\mathbb{R}^d$ by

> $f \preccurlyeq_\sigma f'$ if, writing $f = \sum_{k=1}^d a_k x_k$ and $f' = \sum_{k=1}^d a'_k x_k$, then
> $$\sum_{k=j}^{d} a_{\sigma k} \leq \sum_{k=j}^{d} a'_{\sigma k}$$
> for every $j = 1, 2, \ldots, d$.

Conjecture 3.10 (Magic permutations for linear forms). Let $f_1, \ldots, f_m$ be positive forms on $\mathbb{R}^d$ and s the subspace on which they are all zero. A permutation σ of $[d]$ is realizable by a positive point in s if and only if $\{f_1, \ldots, f_m\}$ is an antichain in the reverse dominance order of forms.

Example 3.11 (Magic bidirected graphs). A magic labelling of a bidirected graph is an injection $x : E \to \mathbb{Z}$ (usually requiring the values to be nonnegative, or positive) such that, if one calculates at each node v the sum of the labels of edges directed into v less the sum of the labels of the outwardly directed edges, the total is the same at every node. Such labellings were studied by Jeurissen [19, 20], excluding introverted edges (both ends directed away from the node).

Example 3.12 (Multiple covering clusters). A *multiple covering cluster* is a set X together with classes $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of nonempty subsets of X which we call *lines.* A *weakly magic labelling* of the multiple covering cluster is a function $x : X \to \mathbb{Z}_{\geq 0}$ such that the lines in each class have equal label sums; but the sums in different classes are independent of each other. For instance, X might be the cells of a rectangular matrix, $\mathcal{L}_1$ the set of rows, and $\mathcal{L}_2$ the set of columns. The number of weakly magic labellings that are positive with all values $x(e) < t$ equals the Ehrhart quasipolynomial $E_{P^\circ}(t)$ and the number that are nonnegative with all $x(e) \leq t$ is $E_P(t)$, where $P := [0,1]^X \cap s$ and s is the magic subspace defined by the line-sum equations. Weakly magic labellings with distinct values are *magic labellings.* The number of these is for which all values satisfy $0 < x(e) < t$ equals the inside-out Ehrhart quasipolynomial $N^\circ(t) = E^\circ_{P^\circ,\mathcal{H}}(t)$. $N(t) = E_{P,\mathcal{H}}(t)$ counts pairs (x, σ) with $0 \le x(e) \le t$ and σ a compatible s-realizable permutation of $[d]$. These quasipolynomials are reciprocal by Theorem 3.4.

If in addition all the lines in each class $\mathcal{L}_i$ have the same number of points c_i, then Theorem 3.5 and Corollary 3.6 apply.

An example of such a covering cluster is a *magic rectangle*: an $m \times n$ rectangle (with $m \le n$) filled with distinct integers so that each row has the same sum and all columns and diagonals have the same sum; but the two sums are not expected to be equal. By a "diagonal" we mean any geometrical diagonal line, of which there are $2(n-m+1)$. Omitting the diagonals yields a *semimagic rectangle*. Then there is a *pandiagonal magic rectangle*, in which the diagonals are supplemented by the wrapped diagonals as with pandiagonal magic squares. These examples generalize to higher dimensions. An *orthotope* is a d-dimensional $n_1 \times n_2 \times \cdots \times n_d$ array. One can define semimagic, magic, and pandiagonal magic orthotopes in all kinds of ways; the number k of covering clusters would naturally be d, but if some of the dimensions n_i are equal one could combine their covering clusters to reduce the multiplicity. As far as we know such objects have never been studied, but the enumerative theory will be ready when recreational mathematics takes an interest.

Example 3.13 (Cubically symmetric covering clusters). A *cubically symmetric* magic or semimagic square has the property that any two cells that lie opposite each other across the center have sum equal to t, the cell-value bound, and the center cell (if there is one) contains the value $\frac{1}{2}t$. This definition is generalized from that of *associated* square in [1] (*symmetrical* square in [2]), in which the entries are $1, 2, \ldots, n^2$ and a symmetrical pair sums to $n^2 + 1$. (Cubical symmetry contrasts with affine symmetry, which we shall treat shortly. The reader should not confuse magic-square symmetry, which is central, with matrix symmetry.) The novel feature is the additional linear restraints besides the magic sum conditions: these are $y_{ij} + y_{n+1-i,n+1-j} = t$ for all $i, j \in [m]$. Translated into the language of $\frac{1}{t}$-fractional vectors $x = \frac{1}{t}y \in \frac{1}{t}\mathbb{Z}^{n^2}$, we require

$$x_{ij} + x_{n+1-i,n+1-j} = 1. \tag{3.2}$$

The effect is to reduce the magic subspace s to a smaller subspace s', but Theorems 3.1 and 3.5 and Corollary 3.6 apply (with s' replacing s) if the line size is constant, due to the next lemma. In that lemma we generalize symmetry to fairly arbitrary covering clusters on the point set $[n]^2$, of which magic, semimagic, and pandiagonal squares are the most familiar examples. The definition is the same: we take the magic sum conditions and the symmetry equations (3.2). The magic subspace $s \subseteq \mathbb{R}^{n^2}$ is defined by equality of line sums; the *cubically symmetric magic subspace*, s', is the affine subspace of s in which (3.2) is valid; then $P = [0,1]^{n^2} \cap s'$ and $\mathcal{H} = \mathcal{H}[K_{n^2}]^{s'}$.

Lemma 3.14. *If $([n]^2, \mathcal{L})$ is a covering cluster of constant size such that s' is not contained in a hyperplane $\{x_{ij} = x_{kl}\}$ of $\mathcal{H}[K_{n^2}]$, and in particular if a cubically symmetric, strongly magic labelling exists, then P and $\mathcal{H}$ are transverse.*

Proof. Since $\frac{1}{2}\mathbf{1} \in P \cap \bigcap \mathcal{H}$, every flat of $\mathcal{H}$ intersects P°. ∎

The hypothesis on s' is satisfied in the case of cubically symmetric magic squares of side $n \ge 3$ because such squares are known to exist, using the values $1, 2, \ldots, n^2$ if $n \not\equiv 2 \bmod 4$; in fact, cubically symmetric pandiagonal squares exist if $n \ge 4$. See [2, pp. 204–211].

One can generalize the ideas of symmetry to hypercubes of various types (magic, semimagic, etc.) and, with multiple covering clusters, to orthotopes.

Example 3.15 (Magic squares of order 3: cubical count). The polytope P of weak magic squares of order 3 has denominator 12 and the inside-out polytope $(P, \mathcal{H})$ for enumerating strong magic squares has denominator 60. We computed these denominators and the quasipolynomials, and those in most of the similar examples in this and the following sections, by means of Maple. The strong quasipolynomial has period 12 ; it is

$$N^\circ(t) = \begin{cases} \frac{t^3-16t^2+76t-96}{6} = \frac{(t-2)(t-6)(t-8)}{6} & \text{if } t \equiv 0,2,6,8 \bmod 12, \\ \frac{t^3-16t^2+73t-58}{6} = \frac{(t-1)(t^2-15t+58)}{6} & \text{if } t \equiv 1 \bmod 12, \\ \frac{t^3-16t^2+73t-102}{6} = \frac{(t-3)(t^2-13t+34)}{6} & \text{if } t \equiv 3,11 \bmod 12, \\ \frac{t^3-16t^2+76t-112}{6} = \frac{(t-4)(t^2-12t+28)}{6} & \text{if } t \equiv 4,10 \bmod 12, \\ \frac{t^3-16t^2+73t-90}{6} = \frac{(t-2)(t-5)(t-9)}{6} & \text{if } t \equiv 5,9 \bmod 12, \\ \frac{t^3-16t^2+73t-70}{6} = \frac{(t-7)(t^2-9t+10)}{6} & \text{if } t \equiv 7 \bmod 12. \end{cases}$$

Compare this to the weak quasipolynomial, whose period is 2 :

$$\begin{cases} \frac{t^3-3t^2+5t-3}{6} = \frac{(t-1)(t^2-2t+3)}{6} & \text{if } t \text{ is odd}, \\ \frac{t^3-3t^2+8t-6}{6} = \frac{(t-1)(t^2-2t+6)}{6} & \text{if } t \text{ is even}. \end{cases}$$

The constant term of $N(t)$ is the number of magic permutations. This is 16. We found them all: they are the rotations and reflections of the patterns

$$\text{(a)} \quad \begin{array}{|c|c|c|} \hline 4 & 9 & 2 \\ \hline 3 & 5 & 7 \\ \hline 8 & 1 & 6 \\ \hline \end{array} \qquad\qquad \text{(b)} \quad \begin{array}{|c|c|c|} \hline 3 & 9 & 2 \\ \hline 4 & 5 & 6 \\ \hline 8 & 1 & 7 \\ \hline \end{array}$$

In these diagrams the numbers are not cell values but rather permutation positions: the largest value is in the cell marked 9, the next largest in that marked 8, and so on. To realize the permutations by magic squares, (a) can be left untouched (it is in fact the ancient Luo Shu square), but (b) needs numbers. The general form of a magic square is (up to the eight symmetries)

$T-\alpha$	$T+\beta$	$T+\alpha-\beta$
$T+2\alpha-\beta$	T	$T-2\alpha+\beta$
$T-\alpha+\beta$	$T-\beta$	$T+\alpha$

where $\beta > 2\alpha > 0$ and $b \neq 3\alpha$. We choose $T > \beta$ to make the square positive. If $\beta > 3\alpha$ we get the magic permutation (a); if $\beta < 3\alpha$ we get (b).

Problem 3.16. We do not know why the coefficients alternate in sign, nor what causes the small differences among the constituents. (Similar questions are raised by the data in [3] on the quasipolynomials that count small weakly magic squares.) Neither can we explain the symmetry of the constituents about residue 1, nor the striking fact that, for most but not all residues r, the r-constituent of N° has a factor $t-r$.

Some of the same questions, notably the alternating signs, arise in regard to our other examples in this and the next sections. Notice the factor $t-r$ in the next example.

Example 3.17 (Semimagic squares of order 3: cubical count). The polytope P of weak semimagic squares of order 3 has denominator 1 and the inside-out polytope $(P, \mathcal{H})$ for enumerating strong semimagic squares has denominator 60. The strong quasipolynomial, which has period 2, is :

$$N^\circ(t) = \begin{cases} \frac{72t^5-425t^4+1100t^3-1570t^2+1228t-405}{240} \\ \qquad = \frac{(t-1)(72t^4-353t^3+747t^2-823t+405)}{240} & \text{if } t \text{ is odd,} \\ \frac{72t^5-425t^4+1100t^3-1600t^2+1288t-480}{240} \\ \qquad = \frac{(t-2)(72t^4-281t^3+538t^2-524t+240)}{240} & \text{if } t \text{ is even.} \end{cases}$$

Compare this to the weak quasipolynomial, which is a polynomial :

$$\frac{3t^5-15t^4+35t^3-45t^2+32t-10}{10} = \frac{(t-1)(t^2-2t+2)(3t^2-6t+5)}{10}.$$

3.3. **Affine magic.** The affine approach counts magic squares, and magic labellings in general, by the magic sum. In the general situation the magic subspace s is defined by a rational, nonhomogeneous linear system

$$f_1(x) = f_2(x) = \cdots = f_m(x) = 1 \tag{3.3}$$

that we assume is consistent. The magic polytope P is the nonnegative part of s, that is,

$$P := s \cap O,$$

where $O := \mathbb{R}^d_{\geq 0}$, the nonnegative orthant, and the hyperplane arrangement is $\mathcal{H} := \mathcal{H}[K_d]^s$. As in the cubical treatment, one must make sure that P affinely spans s, or else change s in the theorems to aff P.

Lemma 3.18. *If P is not contained within a coordinate hyperplane, it spans s.*

Proof. As with Lemma 3.3. ∎

Affine magic is quite similar to cubical magic, but there is something new: one has to worry about boundedness of P. Obviously, P is bounded if the defining linear forms f_i in (3.3) are positive and every variable appears in at least one form. When P is bounded, Theorem 3.1 applies.

Theorem 3.19 (Magic enumeration by line sums). *Let $s \subseteq \mathbb{R}^d$ be the solution space of (3.3), where the f_i are rational linear forms, and let $P := s \cap O$. For $t = 1, 2, \ldots$, let*

$S^\circ(t) :=$ *the number of integer points $x \in ts$ with distinct positive entries,*

and let

$S(t) :=$ *the number of pairs (x, σ) consisting of a nonnegative integer point $x \in ts$ and a compatible s-realizable permutation σ of $[d]$.*

Suppose that P is bounded and does not lie within a coordinate hyperplane. Then S° and S are quasipolynomials with leading term $(\mathrm{vol}_D P)t^{\dim s}$, where $D := s \cap \mathbb{Z}^d$, and with constant term $S(0)$ equal to the number of permutations of $[d]$ that are realizable in P. Furthermore, $(-1)^{\dim s} S^\circ(-t) = S(t)$.

Proof. A straightforward application of Theorem 3.1. ■

We also want to know when P and $\mathcal{H}$ are transverse so that the Möbius-function formulas of Theorem 2.2 will apply.

Lemma 3.20. *P and $\mathcal{H}$ are transverse if s is not contained in any hyperplane $\{x_j = x_k\}$ and, for every $v \in \mathcal{L}(\mathcal{H}[K_d])$, $v \cap P$ is not contained in a coordinate hyperplane. Assuming all forms f_i have equal positive weight, the second condition can be omitted.*

Proof. We need to verify that for no $u \in \mathcal{L}(\mathcal{H})$ is $\varnothing \neq P \cap u \subseteq \partial P$. But $P \cap u = P \cap v$ for some $v \in \mathcal{L}(\mathcal{H}[K_d])$, and ∂P is contained in the union of the coordinate hyperplanes since they determine the facets of P.

If the forms have equal weight $c > 0$, then all $f_i(\mathbf{1}) = c$, so $c^{-1}\mathbf{1} \in \langle \mathbf{1} \rangle \cap s = s \cap \bigcap \mathcal{H}$ and $c^{-1}\mathbf{1} \in O$; this implies the second hypothesis. ■

Theorem 3.21. *With s defined by forms of constant positive weight, and assuming a magic labelling exists and $s \cap \mathbb{R}^d$ is bounded, we have*

$$S^\circ(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E^\circ_{u \cap O}(t) \ ,$$

$$S(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{u \cap O}(t),$$

where μ is the Möbius function of $\mathcal{L}(\mathcal{H})$.

Proof. Transversality holds by Lemma 3.20, since a magic labelling exists if and only if s does not lie in any hyperplane $\{x_j = x_k\}$. Apply Theorem 2.2 in s. ■

The analogy with cubical magic continues: there is a shifting formula when the forms have constant weight.

Corollary 3.22. *Suppose s is defined by forms of constant positive integral weight c. Then*

$$E^\circ_{P,\mathcal{H}}(t) = (-1)^{\dim P} E_{P,\mathcal{H}}(-t - c) \ .$$

Proof. $tP = O \cap ts$. Hence by adding one to every coordinate of an integer point in $O \cap ts$ with distinct entries, we get an integer point in $O^\circ \cap (t + c)s$. This adds c to the value of each $f_i(x)$. Hence,

$$E^\circ_{P,\mathcal{H}}(t) = E^\circ_{P^\circ,\mathcal{H}}(t + c) \ ,$$

and the result follows from Theorem 3.19. ■

Some interesting examples are the affine versions of the cubical examples we already mentioned.

Example 3.23 (Lines of constant length (cf. Example 3.7). The linear equations that express the existence of a magic sum t take the form

$$\sum_{j\in L} x_j = t \text{ for all } L \in \mathcal{L} \ .$$

Theorems 3.19 and 3.21 and Corollary 3.22 all apply as long as a magic labelling exists; thus, $S^\circ(t)$ and $S(t)$ are reciprocal quasipolynomials in the magic sum t, etc. Specific examples include all the varieties enumerated in Example 3.7 and an object of importance in its own right, the Birkhoff polytope.

Example 3.24 (Some bipartite magic graphs: the Birkhoff polytope). The *Birkhoff polytope* B_n, the set of all nonnegative $n \times n$ matrices with row and column sums equal to one, is the affine magic polytope of semimagic squares, or if you prefer of the graph $K_{n,n}$. It is an integral polytope by the Birkhoff–von Neumann theorem [7], which finds the exact vertices. Calculating its volume is a well-known open problem that has been solved, up to now, only for $n \leq 10$ [4]. $B_n = s \cap O$, the intersection polytope associated in the affine counting style with semimagic squares and, equivalently, magic labellings of $K_{n,n}$. That its volume is hard to compute suggests that explicitly finding just the leading coefficient of the enumerating quasipolynomial of the simpler magic examples is quite difficult.

Example 3.25 (Covering clusters; cf. Example 3.8). Theorem 3.19 applies, showing that $S(t)$ and $S^\circ(t)$ are reciprocal quasipolynomials in t—provided that magic labellings exist at all.

With covering clusters the affine magic subspace, call it s_1, defined by $\sum_{i\in L} x_i = 1$ for all $L \in \mathcal{L}$, is an affine subspace of the homogeneous magic subspace, call it s_0, of Example 3.8, and s_0 is the linear subspace generated by s_1. This means that the permutations realizable in s_0 and s_1 are the same, so all our comments on magic permutations with respect to a covering cluster, in the context of cubical counting, apply as well to affine enumeration, except that we do not in general know that all s-realizable permutations are P-realizable.

Example 3.26 (Affinely symmetric covering clusters; cf. Example 3.13). An *affinely symmetric* magic or semimagic square has the property that the average value of any two cells that lie opposite each other across the center equals the average cell value, t/n, and the center cell (if there is one) contains the value t/n. (Of course, one cannot expect such squares to exist unless $t \equiv 0 \pmod{n}$.) This definition is another generalization of that of associated square. The additional linear restraints are $y_{ij} + y_{n+1-i,n+1-j} = 2t/n$ for all $i, j \in [m]$. Translated into the language of $\frac{1}{t}$-fractional vectors $x = \frac{1}{t}y \in \frac{1}{t}\mathbb{Z}^{n^2}$, we require

$$x_{ij} + x_{n+1-i,n+1-j} = \frac{2}{n} \ . \tag{3.4}$$

The effect is to reduce the magic subspace s to a smaller subspace s'. With the extra hypothesis that all lines have n points, Theorems 3.1 and 3.21 and Corollary 3.22 (with $c = n$) apply, with s' replacing s, by the following lemma. In the lemma we generalize symmetry to covering clusters on $X = [n]^2$, with magic sum conditions and the symmetry equations (3.4). The magic subspace $s \subseteq \mathbb{R}^{n^2}$ is defined by equality of line sums; the *affinely symmetric magic subspace*, s', is the affine subspace of s in which (3.2) is valid; then $P = \mathbb{R}^{n^2}_{\geq 0} \cap s'$ and $\mathcal{H} = \mathcal{H}[K_{n^2}]^{s'}$.

Lemma 3.27. *If $([n]^2, \mathcal{L})$ is a covering cluster of size n such that s' is not contained in a hyperplane $\{x_{ij} = x_{kl}\}$ of $\mathcal{H}[K_{n^2}]$, and in particular if an affinely symmetric magic labelling exists, then P and $\mathcal{H}$ are transverse.*

Proof. Since $\frac{1}{n}\mathbf{1} \in P \cap \bigcap \mathcal{H}$, every flat of $\mathcal{H}$ intersects P°. ■

Problem 3.28 (Existence of magic labellings). As we remarked in Example 3.13, affinely symmetric magic labellings exist in the case of magic squares and even pandiagonal magic squares. Whether they exist also for more elaborate magical objects like hypercubes and orthotopes is an interesting question.

Example 3.29 (Magic squares of order 3: affine count). The affine polytope P of weak magic squares has denominator 3 [3] and the inside-out polytope, for enumerating strong magic squares, has denominator 18 (by a computer calculation with Maple). The strong quasipolynomial, with period 18 , is

$$S^\circ(t) = \begin{cases} \frac{2t^2-32t+144}{9} & \text{if } t \equiv 0 \bmod 18, \\[2ex] \frac{2t^2-32t+78}{9} = \frac{2(t-3)(t-13)}{9} & \text{if } t \equiv 3 \bmod 18, \\[2ex] \frac{2t^2-32t+120}{9} = \frac{2(t-6)(t-10)}{9} & \text{if } t \equiv 6 \bmod 18, \\[2ex] \frac{2t^2-32t+126}{9} = \frac{2(t-7)(t-9)}{9} & \text{if } t \equiv 9 \bmod 18, \\[2ex] \frac{2t^2-32t+96}{9} = \frac{2(t-4)(t-12)}{9} & \text{if } t \equiv 12 \bmod 18, \\[2ex] \frac{2t^2-32t+102}{9} & \text{if } t \equiv 15 \bmod 18, \\[2ex] 0 & \text{if } t \not\equiv 0 \bmod 3. \end{cases}$$

Compare to the weak quasipolynomial, whose period is 3 (since t must be a multiple of 3, because the center square contains $t/3$):

$$\begin{cases} \frac{2t^2-6t+9}{9} & \text{if } t \equiv 0 \bmod 3, \\ 0 & \text{if } t \not\equiv 0 \bmod 3. \end{cases}$$

This result is due to MacMahon [24, Vol. II, par. 409, p. 163].

The data here invite some of the same questions we raised in Problem 3.16.

For some small values of t we also calculated $S^\circ(t)$ by hand, which is feasible because the problem is 2-dimensional. The process of counting lattice points in a diagram drew our attention to some remarkable phenomena. Let $\delta := \dim s$. Since $t = 3T$ is necessary to get a nonzero constituent of the quasipolynomial, let us replace t by $3T$ and treat T as the variable. Finally, let c_k be the coefficient in the quasipolynomial N° and let c_k^w be that in the Ehrhart quasipolynomial of P. We observe that the variation in $c_{\delta-1}$ is exactly the same as that in $c_{\delta-1}^w$, i.e.,

$$c_{\delta-1}(T) - c_{\delta-1}(T-1) = c_{\delta-1}^w(T) - c_{\delta-1}^w(T-1),$$

but that is not so for c_0. Intuitively, the reason for this is that the hyperplane deduction—the number of lattice points excluded by the inequalities—is essentially linearly $\delta-1$-dimensional, but it is, on average, a fractional multiple of $t^{\delta-2}$ so that its effect has to show up in jumps which are felt in lower dimensions. The linear $\delta-1$-dimensional effect shows up in a uniform modification of $c_{\delta-1}$ compared with $c^w_{\delta-1}$, while the jumps modify $c^w_{\delta-2}$ irregularly, thereby increasing the period. Thus we propose some daring conjectures. Let p_k, p^w_k be the periods of c_k, c^w_k.

Conjecture 3.30. In an inside-out counting problem, normalized to eliminate zero constituents of the quasipolynomial, let $\delta := \dim P$.
(a) $p^w_k \mid p_k$ for $0 \leq k \leq \delta$, and $p_{\delta-1} = p^w_{\delta-1}$.
(b) If $p_j = p^w_j$ for all $j \geq k$, then the variation in c_k is the same as that in c^w_k.
(c) The period ratios increase by a multiplicative factor as k decreases:

$$\frac{p_k}{p^w_k} \;\Big|\; \frac{p_{k-1}}{p^w_{k-1}} \quad \text{for } 0 \leq k \leq \delta.$$

Example 3.31 (Semimagic squares of order 3: affine count). The polytope P of weak semimagic squares of order 3 (which is the polytope for weakly magic labellings of $K_{3,3}$) is integral (see Example 3.24) and the inside-out polytope $(P, \mathcal{H})$ for enumerating strong semimagic squares (which are identical to strongly magic labellings of $K_{3,3}$) has denominator $4620 = 2^2{\cdot}3{\cdot}5{\cdot}6{\cdot}11$ (found by computer calculation with Maple). The strong quasipolynomial's period is ???, however, far less than its denominator, according to our calculations; the quasipolynomial is

$$S^\circ(t) = \begin{cases} \quad = & \text{if } t \text{ is odd,} \\ \\ \\ \quad = & \text{if } t \text{ is even.} \end{cases}$$

Compare this to the weak quasipolynomial, or rather, polynomial. It was first computed by MacMahon [24, Vol. II, par. 407, p. 161]; it is

$$\frac{t^4 - 6t^3 + 15t^2 - 18t + 8}{8} = \frac{(t-1)(t-2)(t^2 - 3t + 4)}{8}.$$

3.4. **Limited and slack magic.** In the theorems and examples about magic we might replace the complete-graph arrangement by an arbitrary graphic hyperplane arrangement $\mathcal{H}[\Gamma]$ corresponding to a limited list of inequalities among coordinates of x. Then the regions correspond to acyclic orientations of Γ (as in [5, Section 5]) so the multiplicity of x equals the number of acylic orientations of Γ that are compatible with x and realizable in s. One such variation is especially interesting: latin squares and generalizations, treated in Section 4.

We conclude with a remark on general convex polytopes with rational vertices. Any such polytope living in, say, $\mathbb{R}^d$ can be described as the intersection of hyperplanes and halfspaces given by equalities and inequalities of the forms $a_1x_1 + \cdots + a_dx_d = b$ and $\leq b$. For the inequalities, which describe halfspaces, we may introduce nonnegative slack variables to convert an inequality to an equality. Furthermore, if we are only interested in counting integer points, we may translate our polytope by an integer vector into the positive orthant. In summary, we may assume that our polytope P is given as

$$P = \left\{ x \in \mathbb{R}^{d+s}_{\geq 0} : \; A\,x = b \right\}$$

where s is the number of slack variables we introduced (which we think of as the last s components of x), A is an integral matrix, and b is an integral vector. This description of P

follows closely that of affine magic polytopes, with one rather strong difference: distinctness applies only to the first d of the coordinates of an integer point, so we now have to use the hyperplane arrangement $\mathcal{H}[K_d \cup \bar{K}_s]$, which gives hyperplanes $x_j = x_k$ for $j < k \leq d$, affecting only the first d coordinates. This is a special case of the generalization to arbitrary inequality graphs; but the acyclic orientations are very simple, being just permutations of $[d]$.

4. In which latin squares, partially, join the fray

The general picture that encompasses latin squares is that of a covering cluster $(X, \mathfrak{L})$, as in Section 3, with an integer labelling $x : X \to \mathbb{Z}$ subject to the requirement that

$$x(e) \neq x(f) \qquad \text{if } e \text{ and } f \text{ lie in a line.}$$

This is a *latin labelling* of $(X, \mathfrak{L})$. The graph of forbidden equalities is therefore

$$\Gamma_{\mathfrak{L}} := \bigcup_{L \in \mathfrak{L}} K_L,$$

K_L being the complete graph with L as node set. Every graph is equal to $\Gamma_{\mathfrak{L}}$ for some choice of covering cluster.

In the case of a latin rectangle $\Gamma_{\mathfrak{L}} = K_m \times K_n$, the Cartesian product graph. A nice generalization is to a *partial latin orthotope*: a d-dimensional $n_1 \times n_2 \times \cdots \times n_d$ array in which all entries in any line are distinct; the associated graph is $K_{n_1} \times \cdots \times K_{n_d}$ (sometimes called a "Hamming graph"). Hypercubical latin orthotopes of side n with n symbols have been called "latin hypercubes" and "latin permutation hypercubes" [10].

A crucial decision is how to restrict the symbols of the latin square. One may simply specify the number of symbols allowed, say k, and an arbitrary symbol set, let us say $[k]$. Then the number of latin labellings equals the chromatic polynomial $\chi_{\Gamma_{\mathfrak{L}}}(k)$. This approach is trivial from our viewpoint, being merely an application of the graph coloring treated in [5, Section 5], but it seems worthwhile to record the result as it applies to latin labellings. Given an orientation α of $\Gamma_{\mathfrak{L}}$ and a node c-coloring $x : V \to [c]$, we call them *compatible* if $x_j \geq x_i$ whenever there is a $\Gamma_{\mathfrak{L}}$-edge oriented from i to j [30]. An orientation is *acyclic* if it has no directed cycles.

Theorem 4.1. *Given a covering cluster $(X, \mathfrak{L})$. The number $L^\circ(t)$ of latin labellings with values in $(0, t)$ is a polynomial function of positive integers t with leading term $t^{|X|}$ and with constant term equal to the number of acyclic orientations of $\Gamma_{\mathfrak{L}}$. Furthermore, $(-1)^{|X|} L^\circ(-t) = L(t) :=$ the number of pairs consisting of an acyclic orientation of $\Gamma_{\mathfrak{L}}$ and a compatible latin labelling with values in $[0, t]$.*

Proof. By [5, Corollary 5.4], which is an inside-out reformulation of Stanley's Theorem on acyclic orientations [30]. ■

We focus on two other approaches, both taking a leaf from the book of magic squares (Section 3), which give rise to *magilatin squares*: we impose a summation condition on the lines, which may be either homogeneous:

(i) Set all line sums equal to each other.

or affine:

(ii) Set all line sums equal to t.

This tactic is not so strange as it may appear. A partial magilatin square with homogeneous line-sum requirements and symbols restricted to the interval $[1,n]$ is just a latin square with symbol set $[n]$. So is a partial latin square with affine line-sum restrictions, line sum $t = \binom{n+1}{2}$, and positive symbols. We are assuming that the affine constraint (ii) is supplemented by a positivity assumption and that the homogeneous constraint (i) is supplemented by the requirement that the symbols be drawn from the set $[k]$ for some k. To handle partial latin rectangles requires a generalization of (i) to multiple covering clusters; this is the approach we take in Section 4.1. In every case the hyperplane arrangement is $\mathcal{H} := \mathcal{H}[\Gamma_{\mathcal{L}}]^s$, where s is the subspace of $\mathbb{R}^X$ determined by the appropriate line sum conditions.

One could treat magilatin squares with prescribed symmetries, such as being equal to the transpose, by introducing additional equations. We omit the details, which are similar to those of Example 3.13 although simpler, since magic square symmetry is complicated by the fact that symmetric entries cannot be equal.

4.1. **Cubical latinity.** The cubical approach to latin labellings counts them by the largest allowed value; if t is the parameter, the labellings counted are those with $0 < x_i < t$. We assume a multiple covering cluster, $(X; \mathcal{L}_1, \ldots, \mathcal{L}_k)$, which in many applications will consist of just one covering cluster $\mathcal{L}_1$. To count the labellings we take the subspace s of $\mathbb{R}^d$ in which all line sums within each covering cluster $\mathcal{L}_i$ are equal. The polytope is $P := [0,1]^d \cap s$. This is as in Section 3.2, so Lemma 3.3 applies to assure that P spans s.

Theorem 4.2 (latin-sum labellings by bounds)**.** *Suppose $P := [0,1]^d \cap s$ does not lie within a coordinate hyperplane. For $t = 1, 2, \ldots$ let*

$L_0^\circ(t) :=$ *the number of latin labellings x with equal line sums within each covering cluster and with entries that satisfy $0 < x_i < t$,*

and let

$L_0(t) :=$ *the number of pairs consisting of a latin labelling x, with equal line sums within each covering cluster and with $0 \le x_i \le t$, and a compatible acyclic orientation of $\Gamma_{\mathcal{L}}$.*

Then L_0° and L_0 are quasipolynomials with leading term $(\operatorname{vol}_D P) t^{\dim s}$, where $D := s \cap \mathbb{Z}^d$, and with constant term $L_0(0)$ equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$ that are realizable in s. Furthermore, $(-1)^{\dim s} L_0^\circ(-t) = L_0(t)$.

Proof. Theorem 2.1 shows that $L_0 = E_{P,\mathcal{H}}$ and $L_0^\circ = E^\circ_{P^\circ,\mathcal{H}}$ are reciprocal quasipolynomials. The remainder of the proof is as in Theorem 3.4. ■

Theorem 4.3. *If all lines have the same size and a latin labelling exists, we have*

$$L_0^\circ(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E^\circ_{[0,1]^d \cap u}(t) \ ,$$

$$L_0(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{[0,1]^d \cap u}(t) \ ,$$

where μ is the Möbius function of $\mathcal{L}(\mathcal{H})$.

Proof. Apply Theorem 2.2. ■

The pertinent shifting formula is

Corollary 4.4. *If all lines have the same size and a latin labelling exists, then*

$$E^{\circ}_{P,\mathcal{H}}(t) = (-1)^{\dim P} L_0(-t-2) \ .$$

Proof. As with Corollary 3.6. ∎

Example 4.5 (Small latin shapes, counted cubically)*.* We have calculated two small magilatin examples: a square and a rectangle.

In the 2×2 magilatin square it must be that $x_{11} = x_{22} \neq x_{12} = x_{21}$. With cubical constraints, then,

$$L^{\circ}_0(t) = (t-1)(t-2).$$

For comparison, the number of squares without the magilatin distinctness requirement is t^2.

The 2×3 latin rectangle is much more complicated. Calculation with Maple shows that the denominator of P is 6 and that of $(P, \mathcal{H})$ is 12. This permits us to calculate the quasipolynomial, whose period turns out to be 4.

$$L^{\circ}_0(t) = \begin{cases} \frac{t^3-12t^2+41t-30}{4} = \frac{(t-1)(t-5)(t-6)}{4} & \text{if } t \equiv 1 \bmod 4, \\ \frac{t^3-12t^2+41t-42}{4} = \frac{(t-2)(t-3)(t-7)}{4} & \text{if } t \equiv 3 \bmod 4, \\ \frac{t^3-12t^2+44t-48}{4} = \frac{(t-2)(t-4)(t-6)}{4} & \text{if } t \text{ is even.} \end{cases}$$

For comparison, the number of rectangles without the distinctness requirement is

$$\begin{cases} \frac{t^3-3t^2+3t-1}{4} = \frac{(t-1)^3}{4} & \text{if } t \text{ is odd,} \\ \frac{t^3-3t^2+6t-4}{4} = \frac{(t-1)(t^2-2t+4)}{4} & \text{if } t \text{ is even.} \end{cases}$$

Problem 4.6. We do not know why L°_0 factors so nicely, why the coefficients alternate in sign, what causes the small differences among the constituents, nor what makes the even constituents have a smaller period than the odd constituents.

Another example is a 3×3, cubically counted magilatin square. TO BE ADDED LATER.

Still another example is a 3×3 cubical or affine magilatin square with diagonals. That is, all numbers in a row, column, or diagonal are distinct and have the same sum. It so happens that all such squares are magic, so the fomulas in Section 3 apply.

4.2. **Affine latinity.** For affine counting of latin labellings we take a covering cluster $(X, \mathcal{L})$. s is the subspace in which all line sums equal 1 and $P = s \cap O$, as in affine magic (Section 3.3). As there, we can apply Lemma 3.18 to conclude that P spans s in all interesting cases. One advantage over magic is that, because we only consider line sums and not general linear forms, P is certain to be bounded.

Theorem 4.7 (Latin enumeration by line sums)**.** *For $t = 1, 2, \ldots,$ let*

$L^{\circ}_1(t) :=$ *the number of latin labellings of $(X, \mathcal{L})$ with positive entries and all line sums equal to t,*

and let

$L_1(t) :=$ *the number of pairs consisting of a nonnegative latin labelling with all line sums equal to t and a compatible acyclic orientation of $\Gamma_{\mathcal{L}}$.*

Suppose that a positive latin labelling exists for some t. Then L°_1 and L_1 are quasipolynomials with leading term $(\mathrm{vol}_D P)t^{\dim s}$, where $D := s \cap \mathbb{Z}^d$, and with constant term $L_1(0)$ equal to the number of acyclic orientations of $\Gamma_{\mathcal{L}}$ that are realizable in P. Furthermore, $(-1)^{\dim s} L^{\circ}_1(-t) = L_1(t)$.

Proof. By Theorem 3.1, adapted to the latin inequalities. ■

If every line has the same size, then the acyclic orientations that are realizable in P are the same as those realizable in s, because then $\langle \mathbf{1} \rangle$ intersects P; see the discussion following Theorem 3.19.

Lemma 4.8. *P and $\mathcal{H}$ are transverse if a positive latin labelling exists.*

Proof. As with Lemma 3.20. The existence of a positive latin labelling implies that s is not contained in any hyperplane $\{x_j = x_k\}$ for which e_i and e_j are collinear and, for every $v \in \mathcal{L}(\mathcal{H}[\Gamma_{\mathcal{L}}])$, $v \cap P$ is not contained in a coordinate hyperplane. ■

Theorem 4.9. *Assuming a positive latin labelling exists, we have*

$$L_1^{\circ}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E^{\circ}_{u \cap O}(t) \ ,$$
$$L_1(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{u \cap O}(t),$$

where μ is the Möbius function of $\mathcal{L}(\mathcal{H})$.

Proof. Transversality holds by Lemma 4.8. Apply Theorem 2.2 in s. ■

The shifting formula for affine latinity is

Corollary 4.10. *If a positive latin labelling exists and all lines have the same size n, then*

$$E^{\circ}_{P,\mathcal{H}}(t) = (-1)^{\dim s} L_1(-t-n) \ .$$

■

Example 4.11 (Small latin shapes, counted affinely). It is easy to see that in a 2×2 magilatin square with affine constraints,

$$L_1^{\circ}(t) = \begin{cases} t-1 & \text{if } t \text{ is odd,} \\ t-2 & \text{if } t \text{ is even.} \end{cases}$$

The number of positive squares without the magilatin distinctness requirement is $t-1$.

The last example is 3×3 magilatin squares, counted affinely. The polytope is the same as for affine semimagic squares (Example 3.31).

5. Extensions and conjectures

5.1. **Generalized exclusions.** We concentrated our treatment on exclusions of magic and latin type: that is, where all values, or all collinear values, are unequal. These seem to us the most interesting special cases, but one can choose any graph of unequal values, independent of the covering cluster or forms involved in the defining equations of the magic subspace.

Other kinds of exclusion are also possible. We want to mention the very natural *complementation* restrictions. In cubical enumeration we call x_i and x_j *complementary* if $x_i + x_j = k$. If we forbid certain pairs of values to be complementary, we pass from graphs to signed graphs, since the rule $x_i + x_j \neq k$ corresponds to a negative edge $-ij$. ($x_i \neq x_j$ corresponds to a positive edge, $+ij$; thus ordinary edges are positive. The exact application of signed graphs involves translation and halving of the centrally symmetric polytope $[-1, 1]^n$ to $[0, 1]^n$, along with corresponding translation of the signed-graphic hyperplane arrangement, as explained in [5, Section 5].) This signed-graphic interpretation allows one to apply formulas for chromatic polynomials, and thereby Möbius functions, of various kinds of signed graphs as in [**?**]. Because signed-graphic hyperplanes, as translated, give half-integral vertices (see [5, Section 5]), we expect a counting quasipolynomial with inequalities and noncomplementarities given by a signed graph to have, generally speaking, twice the period of a counting quasipolynomial pertaining to a similar unsigned graph of inequalties. This is necessarily vague; we intend only a suggestion for research that we invite readers to explore.

5.2. **Periods.** We observed in the examples some remarkable phenomena concerning periods.

We suggest that the real phenomenon is that each new edge in the graph of inequalities has the same effect. Fixing the polytope P, let p denote periods with reference to an inequality graph Γ, and for an edge ij in Γ let p' denote those with reference to $\Gamma' := \Gamma \setminus \{ij\}$. Then we propose the relationships (i)–(iii) betweeen the periods of the quasipolynomials $E' := E^{\circ}_{P^{\circ},\mathcal{H}[\Gamma']}$ and $E := E^{\circ}_{P^{\circ},\mathcal{H}[\Gamma]}$.

References

[1] W. S. Andrews and others, *Magic Squares and Cubes.* 2nd ed. Open Court, La Salle, Ill., 1917. Repr. by Dover, New York. 1960. MR 22 #5582.

[2] W. W. Rouse Ball, *Mathematical Recreations and Essays.* 11th ed., revised by H. S. M. Coxeter. Macmillan, London, 1939. MR 8, 440b. Zbl. 22, 1a (e: 022.00101).

[3] Matthias Beck, Moshe Cohen, Jessica Cuomo, and Paul Gribelyuk, The number of "magic" squares and hypercubes. *Amer. Math. Monthly*, to appear.

[4] Matthias Beck and Dennis Pixton, The Ehrhart polynomial of the Birkhoff polytope. *Discrete Comput. Geom.*, to appear.

[5] Matthias Beck and Thomas Zaslavsky, Inside-out polytopes. In preparation.

[6] William H. Benson and Oswald Jacoby, *New Recreations with Magic Squares.* Dover, New York, 1976.

[7] Garrett Birkhoff,

[8] Schuyler Cammann, Old Chinese magic squares. *Sinologica* **7** (1962), 14–53.

[9] ——, Islamic and Indian magic squares, Parts I and II. *History of Religions* **8** (1969), 181–209; 271–299.

[10] J. Dénes and A. D. Keedwell, *Latin Squares and their Applications.* Academic Press, New York, 1974. MR 50 #4338. Zbl. 283.05014.

[11] Eugène Ehrhart, Sur un problème de géometrie diophantienne linéaire. I: Polyèdres et réseaux. II: Systèmes diophantiens linéaires. *J. reine angew. Math.* **226** (1967), 1–29; **227** (1967), 25–49. Correction. *ibid.* **231** (1968), 220. MR 35 #4184, 36 #105. Zbl. 155.37503, 164.05304.

[12] ——, Sur les carrés magiques. C. R. Acad. Sci. Paris Sér. A–B **277** (1973), A651–A654. MR 48 #10859. Zbl. 267.05014.

[13] ——, *Polynômes arithmétiques et Méthode des Polyèdres en Combinatoire.* Internat. Ser. Numerical Math., Vol. 35. Birkhäuser, Basel, 1977. MR 55 #5544. Zbl. 337.10019.

[14] Erich Friedman and Mike Keith, Magic carpets. *J. Integer Seq.* **3** (2000), no. 2, Article 00.2.5 (HTML, electronic). MR 2001e:05026. Zbl. 953.05011.

[15] Joseph A. Gallian, A dynamic survey of graph labeling. *Electronic J. Combin.* **5** (1998), Dynamic Survey # 8, 43 pp. (electronic). Revised ed. (2002), 106 pp. (electronic). MR 99m:05141. Zbl. 953.05067.

[16] Curtis Greene and Thomas Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.* **280** (1983), 97–126. MR 84k:05032. Zbl. 539.05024.

[17] Nora Hartsfield and Gerhard Ringel, *Pearls in Graph Theory.* Academic Press, Boston, 1990. MR 91f:05001. Zbl. 703.05001.

[18] I. Heller and C.B. Tompkins, An extension of a theorem of Dantzig's. In: H. W. Kuhn and A. W. Tucker, eds., *Linear Inequalities and Related Systems*, pp. 247–252. Annals of Math. Studies, No. 38. Princeton Univ. Press, Princeton, N.J., 1956. MR 18, 459. Zbl. 72, 378 (e: 072.37804).

[19] R.H. Jeurissen, Disconnected graphs with magic labellings. *Discrete Math.* **43** (1983), 47–53. MR 84c:05064. Zbl. 499.05053.

[20] ——, Pseudo-magic graphs. *Discrete Math.* **43** (1983), 207–214. MR 84g:05122. Zbl. 514.05054.

[21] ——, Magic graphs, a characterization. *European J. Combin.* **9** (1988), 363–368. MR 89f:05138. Zbl. 657.05065.

[22] Dénes König, *Theorie der endlichen und unendlichen Graphen*. Mathematik und ihre Anwendungen, Band 16. Akademische Verlagsgesellschaft, Leipzig, 1936. Repr. Chelsea, New York, 1950. MR 12, 195. Zbl. 13, 228 (e: 013.22803). Enlish trans. *Theory of Finite and Infinite Graphs*. Birkhäuser, Boston, 1990. MR 91f:01026. Zbl. 695.05015.

[23] I. G. Macdonald, Polynomials associated with finite cell-complexes. *J. London Math. Soc.* (2) **4** (1971), 181–192. MR 45 #7594. Zbl. 216, 452 (e: 216.45205).

[24] Percy A. MacMahon, *Combinatory Analysis*, Vols. I and II. Cambridge University Press, Cambridge, 1915–1916. Reprinted in one volume by Chelsea, New York, 1960. MR 25 #5003. Zbl. 101, 251b (e: 101.25102).

[25] P. McMullen, Lattice invariant valuations on rational polytopes. *Arch. Math. (Basel)* **31** (1978/79), 509–516. MR 80d:52011. Zbl. 387.52007.

[26] Jennifer McNulty, Generalized affine matroids. Proc. Twenty-fifth Southeastern Internat. Conf. Combinatorics, Graph Theory and Computing (Boca Raton, Fla., 1994). *Congr. Numer.* **101** (1994), 243–254. MR 96k:05034. Zbl. 842.05019.

[27] Gian-Carlo Rota, On the foundations of combinatorial theory: I. Theory of Möbius functions. *Z. Wahrsch. verw. Gebiete* **2** (1964), 340–368. MR 30 #4688. Zbl. 121, 24f (e: 121.02406). Many times reprinted.

[28] Society of Go Philosophers of the North River, Magic in mathematics: A methodological and metalogical examination. *Rep. S.G.P.N.R.* **96** (1992), No. 2.

[29] Richard P. Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs. *Duke Math. J.* **40** (1973), 607–632. MR 47 #6519. Zbl. 269.05109.

[30] ——, Acyclic orientations of graphs. *Discrete Math.* **5** (1973), 171–178. MR 47 #6537. Zbl. 258.05113.

[31] ——, *Enumerative Combinatorics*, Vol. I. Wadsworth & Brooks/Cole, Monterey, Calif., 1986. MR 87j:05003. Corrected reprint: Cambridge Stud. Adv. Math., Vol. 49. Cambridge University Press, Cambridge, Eng., 1997. MR 98a:05001. Zbl. 889.05001, 945.05006.

[32] ——, personal communication, June 16, 2003.

[33] Frank Swetz, *Legacy of the Luoshu: The 4,000 Year Search for the Meaning of the Magic Square of Order Three*. Open Court, Chicago, 2002.

[34] W.T. Tutte, *Graph theory*. Encyc. Math. Appl., 21. Addison-Wesley, Reading, Mass., 1984. MR 87c:05001. Zbl. 554.05001. Reprinted by Cambridge University Press, Cambridge, 2001. MR 2001j:05002. Zbl. 964.05001.

[35] Michelle L. Wachs and James W. Walker, On geometric semilattices. *Order* **2** (1986), 367–385. MR 87f:06004. Zbl. 589.06005.

[36] W.D. Wallis, *Magic graphs*. Birkhäuser, Boston, 2001. MR 2003h:05165. Zbl. 979.05001.

[37] H. Whitney, A logical expansion in mathematics. *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[38] Claudia Zaslavsky, *Africa Counts: Number and Pattern in African Culture*. Prindle, Weber & Schmidt, Boston, 1973. MR 58 #20993.

[39] Thomas Zaslavsky, *Facing Up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes*. Mem. Amer. Math. Soc., No. 154. American Mathematical Society, Providence, R.I., 1975. MR 50 #9603. Zbl. 296.50010.

[40] ——, Geometric lattices of structured partitions. I: Gain-graphic matroids and group valued partitions. Manuscript, 1985.

[41] Günther Ziegler, *Lectures on Polytopes*. Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995. MR 96a:52011. Zbl. 823.52002.

Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA 94720-5070, U.S.A.

Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902-6000, U.S.A.