INSIDE-OUT POLYTOPES
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Abstract. We present a common generalization of hyperplane dissections, lattice-point counting, graph coloring, and enumeration of nowhere-zero flows, magic squares and graphs, antimagic squares and graphs, and partial latin squares.

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1. In which we introduce polytopes, hyperplanes, and lattice points to the reader and each other

We study lattice-point counting in polytopes with boundary on the inside. To say this in a less mysterious way: we consider a convex polytope P together with an arrangement of hyperplanes that dissects the polytope, and we count points of a discrete lattice, such as the integer lattice $\mathbb{Z}^d$, that lie in P, with a multiplicity assigned to each point according to the number of regions of the arrangement that contain the point. It is an easy matter to show that standard Ehrhart theory [10] applies, but the consequences are quite interesting. Another interesting phenomenon is the relationship between the point count and the characteristic polynomial of the arrangement. Our first purpose, indeed, is to encompass colorings and acyclic orientations of graphs and signed graphs within the framework of counting lattice points in polytopes. Our second purpose is to apply the same framework to a multitude of counting problems in which there are forbidden values or relationships amongst the values of an integral function on a finite set, which may for instance be the edge set of a graph or the set of cells of an $n \times n$ square. Our main examples, aside from graph coloring, are nowhere-zero integral flows, magic, antimagic, and latin squares, magic and antimagic graphs, and generalizations involving rational linear forms. Our results are of three kinds: quasipolynomiality of counting functions, Möbius inversion formulas, and the appearance of quantities—the aforementioned multiplicities—that generalize the number of acyclic orientations of a graph but whose combinatorial interpretation is, in some examples at any rate, a mystery.

The origin of this work was in a geometrical interpretation of coloring of graphs and signed graphs. To keep things simple we begin with ordinary, unsigned graphs. A *coloring* in c colors of a graph Γ, with node set $V = [n] := \{1, 2, \ldots, n\}$, is a function $x : V \to [c]$. (By $[k]$ we mean the set $\{1, 2, \ldots, k\}$, the empty set if $k = 0$.) x is *proper* if, whenever there is an edge ij, $x_i \neq x_j$. It is a short step to regard x as a point in the real affine space $\mathbb{R}^n$ and call it proper if it lies in none of the hyperplanes $h_{ij} : x_i = x_j$ for $ij \in E$, the edge set of Γ. That is, if we write

$$\mathcal{H}[\Gamma] := \{h_{ij} : ij \in E\},$$

which is the hyperplane arrangement of the graph Γ, then counting proper colorings of Γ means counting integral points in $[c]^n \setminus \bigcup \mathcal{H}[\Gamma]$. From a general point of view, we are counting lattice points in a polytope but outside an arrangement of hyperplanes. The polytope is $(c+1)P^\circ$, a dilation of the interior of the unit hypercube $P := [0, 1]^n$. Write $E^\circ_{P^\circ, \mathcal{H}[\Gamma]}(c + 1)$ for the number of points we count; that is, the number of proper c-colorings of Γ. It is well known that this is a polynomial in c, called the *chromatic polynomial* of Γ.

An *arrangement* (that is, a finite set) $\mathcal{H}$ of real affine hyperplanes in $\mathbb{R}^d$ divides up the space into regions: an *open region* is a connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ and a *closed region* is the topological closure of an open region. The theory of arrangements began in 1826 with arrangements of lines, planes, circles, and spheres in the article of Steiner [31] ultimately inspired by the progressive educational reformer Pestalozzi. Steiner wished to count the number of regions of $\mathbb{R}^2$ or $\mathbb{R}^3$ dissected by an arrangement, a project that was not truly completed until 150 years later in [35] and then only for hyperplanes. There it was found that the number of regions into which a hyperplane arrangement $\mathcal{H}$ divides $\mathbb{R}^d$ is

$$r(\mathcal{H}) = (-1)^d p_{\mathcal{H}}(-1),$$

FIGURE 1. The lattice points in $(k+1)[0,1]^2$ that k-color the graph K_2, with $k=8$.

where $p_{\mathcal{H}}$ is the characteristic polynomial of $\mathcal{H}$ (to be defined later). We extend this formula to counting lattice points in a dissected convex polytope. For a point x let $m(x)$ be the number of closed regions of $\mathcal{H}$ in which x lies. In the graphic case, where the arrangement is $\mathcal{H}[\Gamma]$ in $\mathbb{R}^n$, each region corresponds (by [14] or [16, Section 7]) to an acyclic orientation of Γ that is compatible with each coloring x in the region, in the sense that each edge is oriented from a lower color to a color that is at least weakly higher. Thus $m(x)$, if x is a coloring point, equals the number of acyclic orientations of Γ that are compatible with the coloring. If we write $E_{P,\mathcal{H}[\Gamma]}(c)$ for the sum of $m(x)$ over all integral points $x \in cP$, then $E_{P,\mathcal{H}[\Gamma]}(c) = (-1)^n E^{\circ}_{P^\circ,\mathcal{H}[\Gamma]}(-c)$: this is known by graph theory [27]. We shall show (in Section 5) that it is also an instance of the general phenomenon of Ehrhart reciprocity [12, 29].

Ehrhart reciprocity applies to a closed, d-dimensional polytope P with rational vertices. It says that, if $E_P(t)$ is the number of points of the integer lattice $\mathbb{Z}^d$ that are contained in the dilation tP (t a positive integer) and $E_{P^\circ}(t)$ is the number in the interior tP°, then $E_{P^\circ}(t) = (-1)^d E_P(-t)$ [10, 20]. E_P is well defined at negative integers because of Ehrhart's fundamental theorem that E_P is a *quasipolynomial* in t: that is, $E_P(t) = \sum_0^d c_i t^i$ with coefficients c_i that, though not necessarily constant, at any rate are periodic functions of t (so that $E_P(t)$ is a polynomial on each residue class modulo some integer, called the *period*; these polynomials are the *constituents* of E_P). We generalize these results to inside-out polytopes.

Our principle motivating example, signed graph coloring, is slightly more complicated than ordinary graph coloring. A *signed graph* Σ is a graph in which each edge is signed, $+$ or $-$. (We allow graphs, especially signed graphs, to have loops and multiple edges.) A *c-coloring*

FIGURE 2. Left: The lattice points in $(k+1)[-1,1]^2$ that k-color the signed graph $\pm K_2$, with $k=8$. Right: The lattice points that k-color it without 0.

[37] of a signed graph with node set $V=[n]$ is a function

$$x : V \to \{-c, -(c-1), \ldots, 0, \ldots, c-1, c\};$$

we say x is *proper* if it has the property that, when there is an edge ij with sign ε, then $x_j \neq \varepsilon x_i$. Geometrically, $x \in \left(\{-c, -(c-1), \ldots, c\}\right)^n \setminus \bigcup \mathcal{H}[\Sigma]$ where

$$\mathcal{H}[\Sigma] := \{h_{ij}^{\varepsilon} : \Sigma \text{ has an edge } ij \text{ with sign } \varepsilon\}.$$

(h_{ij}^{ε} is the hyperplane $x_j = \varepsilon x_i$.) The function

$$\chi_\Sigma(2c+1) := \text{the number of proper } c\text{-colorings of } \Sigma$$

is a polynomial, called the *chromatic polynomial* of Σ [37]. We see that $\chi_\Sigma(2c+1) = E^\circ_{P^\circ,\mathcal{H}[\Sigma]}(c+1)$, the number of lattice points that lie in $(c+1)P^\circ$ (where now $P := [-1,1]^n$) but not in any of the hyperplanes of $\mathcal{H}[\Sigma]$. Furthermore, the regions of $\mathcal{H}[\Sigma]$ are known to correspond to the acyclic orientations of Σ [38] and the regions that contain a coloring x correspond to the acyclic orientations that are compatible with x [37]. Thus $E_{P,\mathcal{H}[\Sigma]}(c)$ is the number of pairs consisting of a coloring and a compatible acyclic orientation, which is known to equal $(-1)^n\chi_\Sigma(-(2c+1))$ [37]. Signed graphs have a second chromatic polynomial: the *zero-free chromatic polynomial*

$$\chi^*_\Sigma(2c) := \text{the number of proper } c\text{-colorings } x : V \to \pm[c],$$

that is, it counts colorings not taking the value 0. This is obviously also an inside-out Ehrhart polynomial, but it is not obvious that, as we prove in Section 5, χ_Σ and χ^*_Σ are the two constituent polynomials of a single Ehrhart quasipolynomial.

What we see in these examples is a relationship between the chromatic polynomial of a graph or signed graph and the Ehrhart quasipolynomial $E_{P,\mathcal{H}}$ of an associated inside-out polytope $(P, \mathcal{H})$, where P delimits the range of a coloring and $\mathcal{H}$ is the set of propriety constraints. We now come to the essential fact: the polynomial, regarded in the first instance as a polynomial in c and in the second as a polynomial in $2c+1$, is the characteristic polynomial of the arrangement $\mathcal{H}$, defined by

$$p_{\mathcal{H}}(\lambda) := \sum_{s \in \mathcal{L}(\mathcal{H})} \mu(s) \lambda^{\dim s},$$

where

$$\mathcal{L}(\mathcal{H}) := \left\{ \bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H} \text{ and } \bigcap \mathcal{S} \neq \varnothing \right\},$$

ordered by reverse inclusion [35], and $\mu(s) = \mu(\hat{0}, s)$, the Möbius function of $\mathcal{L}(\mathcal{H})$ (with a minor exception). It is one of our purposes to explain and generalize this coincidence. We do so through counting lattice points of an inside-out polytope.

We have two techniques for attacking the problem. In the first we dissect the polytope into its intersections with the regions of the hyperplane arrangement. The intersections are rational polytopes whose Ehrhart quasipolynomials sum to that of the inside-out polytope. Thus we get quasipolynomiality and reciprocity together with interpretations of the leading coefficient and constant term. The second technique is Möbius inversion over the lattice of flats of the arrangement, i.e., sophisticated inclusion-exclusion. The strongest results come in applications where the two methods meld, as most neatly in graph coloring. (Curiously, both techniques were anticipated to an extent by Stanley, as we recently learned [30]. Stanley used a method equivalent to dissection to give a second proof of his famous combinatorial interpretation of the chromatic polynomial at negative arguments; the proof is that via the order polynomial [27]. Much later, Kochol applied a dissection argument to nowhere-zero flows [18]. Then, in his textbook [29, Exercise 4.10] Stanley suggests Möbius inversion over the Boolean algebra or the partition lattice to find the number of nonnegative integral solutions x, with all coordinates distinct, of a rational linear system $Ax = 0$—such as the equations of a magic square.)

Beyond graph coloring are other examples: for one, nowhere-zero integral flows on graphs and signed graphs. A *nowhere-zero k-flow* on a graph Γ is a mapping $x : E \to [k-1]$, defined on some orientation of Γ, such that the total inflow equals the total outflow at each node. The existence of nowhere-zero integral flows of small width k is a major open question in graph theory, whose solution, if it is as conjectured, would imply the Four-Color Theorem. Our theorems do not answer the existence question but they do prove that the number of nowhere-zero k-flows is a polynomial function of k (a result recently discovered, *ad hoc*, by Kochol [18]) and they interpret the value of that function at negative arguments in terms of compatible pairs of k-flows and acyclic orientations. They also provide similar results for integral flows on signed graphs. This application is reported in a separate article [3].

In the same spirit as graphs and flows are applications to magic squares and their innumerable relatives: antimagic squares, semimagic and pandiagonal magic squares, magic cubes [1, 2] and hypercubes, magic graphs, and magical oddities like circles, spheres, and stars [1]. A *magic square* is an $n \times n$ array of distinct positive, or sometimes nonnegative, integers whose sum along any *line* (row, column, or main diagonal) is the same number, the *magic sum*. *Antimagic* is like magic but with difference: that is, the line sums all evaluate to different numbers rather than the same number. The rules for a magic square have varied with time and writer; but historically, in all events, the contents are integers, positive, and distinct. In the past century, though, mathematicians have made some simplifications in the interest of obtaining results about the number of squares with a fixed magic sum. Thus we have, most significantly, the omission of the fundamental requirement of distinctness. Since the terminology is rather confused, we shall call a square *magic* or *strongly magic* if its entries are distinct and *weakly magic* if they need not be distinct. The number of weakly magic squares is amenable to analysis in terms of Ehrhart theory, but there have been almost no exact (theoretical) formulas for strong magic. With inside-out polytopes we can attack this and many similar counting problems in a systematic way. We obtain not only a general result about their counting functions but also a fascinating interpretation of reciprocity that leads to a new kind of question about permutations.

Related to magic squares are something we call magilatin squares. A *latin square* is an $n \times n$ array in which each cell is filled by one of n symbols so that no row or column contains the same symbol twice. Allowing more symbols, but retaining the property of equal line sums, we have a *magilatin square*. We look at the number of magilatin squares of fixed dimensions as we vary either the number of available symbols or the magic sum. Our treatment is similar to that of magic squares, but the distinctness requirement, and consequently the hyperplane arrangement, are different. While in a magic square each entry must differ from every other, in a magilatin square it must differ only from those that are collinear with it, a *line* here being a row or column. We treat magic and latin objects in the separate paper [4].

Counting antimagic squares, whether strong or weak ones, cannot be done with standard methods, because it is intrinsic to the problem that one has *in*equalities between the line sums. That is just what our approach is well suited to handle. We treat the process of line summation as a linear transformation into a vector space of sums, in which there is a hyperplane arrangement guaranteeing unequal sum values. We treat antimagic in the separate paper [5].

We conclude this paper with two short sections on supplemental topics: subspace arrangements and general valuations. These are intended to clarify the phenomena by indicating the essential requirements for a theory of our type. A lattice-point count is one kind of valuation; another example is the *combinatorial Euler characteristic*, which is the alternating sum $a_0 - a_1 + \cdots$ of the number a_i of open cells of each dimension i into which a geometrical object can be decomposed. In Section 7 we show that the Möbius inversion formulas (as will be no surprise) apply to any valuation.

2. In which more characters take the stage

The *Möbius function* of a finite partially ordered set (a *poset*) S is the function $\mu : S \times S \to \mathbb{Z}$ defined recursively by

$$\mu(r,s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r,u) & \text{if } r < s. \end{cases}$$

Sources are, *inter alia*, [24] and [29, Section 3.7]. S may be the class of closed sets of a closure operator; in that case if $\varnothing$ is not closed we define $\mu(\varnothing, s) := 0$ for $s \in S$.

In a poset P, $\hat{0}$ denotes the minimum element and $\hat{1}$ the maximum element, if they exist. An *algebraic lattice* (commonly called simply a "lattice" but we must differentiate it from a discrete lattice) is a poset in which any two elements have a meet and a join: a greatest lower bound and a least upper bound. A *meet semilattice* is a poset in which meets exist but not necessarily joins.

One kind of poset is the *intersection semilattice* of an affine arrangement $\mathcal{H}$ of hyperplanes, that is, $\mathcal{L}(\mathcal{H})$ defined in the introduction. The elements of $\mathcal{L}(\mathcal{H})$ are called the *flats* of $\mathcal{H}$. $\mathcal{L}$ is a *geometric semilattice* (of which the theory is developed in [33, 23]) with $\hat{0} = \bigcap \varnothing = \mathbb{R}^d$; it is a *geometric lattice* (for which see [24, p. 357] or [29], etc.) if $\mathcal{H}$ has nonempty intersection, as when all the hyperplanes are homogeneous. A hyperplane arrangement decomposes the ambient space into relatively open cells called *open faces* of $\mathcal{H}$, whose topological closures are the *closed faces*. For a more precise definition we need the arrangement *induced* by $\mathcal{H}$ on a flat s; this is

$$\mathcal{H}^s := \{h \cap s : h \in \mathcal{H}, h \not\supseteq s\}.$$

A face of $\mathcal{H}$ is then a region of any $\mathcal{H}^s$ for $s \in \mathcal{L}(\mathcal{H})$. One face is $\hat{1} = \bigcap \mathcal{H}$ itself, if nonempty. An oddity about hyperplane arrangements is that, for technical reasons, one wants to treat the whole space as a hyperplane (called the *degenerate hyperplane*) that may or may not belong to $\mathcal{H}$. If $\mathcal{H}$ contains the degenerate hyperplane, it has no regions, because $\mathbb{R}^d \setminus \bigcup \mathcal{H} = \varnothing$. However, $\mathcal{H}$ does have faces; e.g., its d-dimensional faces are the regions of $\mathcal{H}^{\hat{0}}$, the arrangement induced in $\hat{0} = \mathbb{R}^d$.

The *characteristic polynomial* of $\mathcal{H}$ is defined in terms of the Möbius function of $\mathcal{L}(\mathcal{H})$ by

$$p_{\mathcal{H}}(\lambda) := \begin{cases} 0 & \text{if } \mathcal{H} \text{ contains the degenerate hyperplane, and} \\ \sum_{s \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, s) \lambda^{\dim s} & \text{otherwise.} \end{cases}$$

And three more definitions: for a set or point X in $\mathbb{R}^d$,

$$\mathcal{H}(X) := \{h \in \mathcal{H} : X \subseteq h\},$$

$$s(X) := \bigcap \mathcal{H}(X) = \bigcap \{h \in \mathcal{H} : X \subseteq h\},$$

the smallest flat of $\mathcal{H}$ that contains X, and

$$F(X) := \text{the unique open face of } \mathcal{H} \text{ that contains } X$$

provided that X is contained in an open face, as for instance when it is a point.

A *convex polytope* P (for which see, e.g., [40]) is a bounded, nonempty set that is the intersection of a finite number of open and closed half spaces in $\mathbb{R}^d$; P may be closed, relatively open, or neither. A closed convex polytope is also the convex hull of a finite set of points in $\mathbb{R}^d$. Another kind of poset is the face lattice of P. A *closed face* of P is either $\bar{P}$, the topological closure of P, or the intersection with $\bar{P}$ of any hyperplane h such that $P \setminus h$ is connected. (Then h is a *supporting hyperplane* of P.) An *open face* is the relative interior F° of a closed face F. (The relative interior of a point is the point.) The null set is a face; it and $\bar{P}$ (or P°) are the *improper* faces. The *face lattice* $\mathcal{F}(P)$ is the set of open faces, partially ordered by inclusion of the closures. A *vertex* is a 0-dimensional face. A *facet* is a face whose dimension is $\dim P - 1$; a *facet hyperplane* is the affine span of a facet. If $P \subseteq \mathbb{R}^d$ is not full-dimensional, then a facet hyperplane is a relative hyperplane of the affine flat spanned by P.

A *dilation* of a set $X \subseteq \mathbb{R}^d$ (also called a *contraction* if $t < 1$) is any set $tX = \{tx : x \in X\}$ for a real number $t > 0$.

3. In which we encounter facially weighted enumerations

Our first main result expresses the Ehrhart quasipolynomials of an inside-out polytope $(P, \mathcal{H})$ in terms of the combinatorics of $\mathcal{H}$; but its natural domain is far more general. We may take any discrete set D in $\mathbb{R}^d$, any bounded convex set C, and any hyperplane arrangement $\mathcal{H}$ that is *transverse* to C: every flat $u \in \mathcal{L}(\mathcal{H})$ that intersects the topological closure $\bar{C}$ also intersects C°, the relative interior of C, and C does not lie in any of the hyperplanes of $\mathcal{H}$. (The definition of transversality does not require C to be bounded or convex. A convenient sufficient condition for transversality is that $C^\circ \cap \bigcap \mathcal{H} \neq \varnothing$ and $C \not\subseteq \bigcup \mathcal{H}$.) We think of $(C, \mathcal{H})$ as having both an *external boundary* ∂C, the topological boundary of C, and an *internal boundary* $C \cap (\cup \mathcal{H})$, together constituting the *complete boundary* $\partial(C, \mathcal{H})$. The *relative interior* $(C, \mathcal{H})^\circ$ is $C^\circ \setminus \bigcup \mathcal{H}$. An *open face of* $\mathcal{H}$ *in* C (or, an *open face of* $(C, \mathcal{H})$) is a nonvoid intersection $F \cap C$ of C with an open face F of $\mathcal{H}$ such that $F(F \cap C) = F$ (for nontriviality). A *closed face of* $\mathcal{H}$ *in* C is the closure in C of an open face of $\mathcal{H}$ in C. In particular, a *region of* $\mathcal{H}$ *in* C is one of the components of $C \setminus \bigcup \mathcal{H}$, or the closure of such a component.

The *multiplicity of* $x \in \mathbb{R}^d$ *with respect to* $\mathcal{H}$ is

$$m_{\mathcal{H}}(x) := \text{the number of closed regions of } \mathcal{H} \text{ that contain } x.$$

The *multiplicity with respect to* $(C, \mathcal{H})$ is

$$m_{C,\mathcal{H}}(x) := \begin{cases} \text{the number of closed regions of } \mathcal{H} \text{ in } C \text{ that contain } x, & \text{if } x \in C, \\ 0, & \text{if } x \notin C. \end{cases}$$

This may not equal $m_{\mathcal{H}}(x)$ for $x \in C$, unless one assumes transversality. The *closed* and *open* D*-enumerators* of $(C, \mathcal{H})$ are

$$E_{C,\mathcal{H}}(D) := \sum_{x \in D} m_{C,\mathcal{H}}(x)$$

and

$$E^\circ_{C,\mathcal{H}}(D) := \#\big(D \cap C \setminus \bigcup \mathcal{H}\big).$$

Theorem 3.1. *Let C be a bounded, convex subset of $\mathbb{R}^d$, $\mathcal{H}$ a hyperplane arrangement not containing the degenerate hyperplane, and D a discrete set in $\mathbb{R}^d$. Then*

$$E^\circ_{C,\mathcal{H}}(D) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) \#(D \cap C \cap u) \tag{3.1}$$

and if $\mathcal{H}$ is transverse to C,

$$E_{C,\mathcal{H}}(D) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| \#(D \cap C \cap u), \tag{3.2}$$

where μ is the Möbius function of $\mathcal{L}(\mathcal{H})$.

Proof of Equation (3.1). We begin with the observation that, for any flat r of $\mathcal{H}$,

$$\#(D \cap C \cap r) = \sum_{u \in \mathcal{L}: u \geq r} E^\circ_{C \cap u, \mathcal{H}^u}(D).$$

The reason for this is that $C \cap r$ is the disjoint union of all the open faces of $\mathcal{H}$ in $C \cap r$, $C \cap u \setminus \bigcup \mathcal{H}^u$ is the disjoint union of the open faces of $\mathcal{H}$ that span u, and counting points of D is an additive function on open faces (a *valuation* in technical language). By Möbius inversion,

$$E^{\circ}_{C\cap r,\mathcal{H}^r}(D) = \sum_{u\in\mathcal{L}:u\geq r} \mu(r,u)\#(D\cap C\cap S).$$

Setting $r = \hat{0}$ gives the desired formula unless $\mathcal{H}$ contains the degenerate hyperplane. In that case, however, both sides of (3.1) equal zero. ∎

The proof of the second equation depends on two lemmas about transversality and an algebraic expression for the multiplicity of a point.

Lemma 3.2. *Let C be a convex set and $\mathcal{H}$ a transverse hyperplane arrangement. If u is a flat of $\mathcal{H}$, then $\overline{u\cap C^{\circ}} = u \cap \bar{C}$.*

Proof. We need to prove that every neighborhood of a point $x \in u\cap\partial C$ intersects $u\cap C^{\circ}$. By transversality u intersects C°, say in a point y. Then the segment $\mathrm{conv}(x, y)$ lies in $u \cap C^{\circ}$ except possibly for x. This implies our desideratum. ∎

Lemma 3.3. *Let C be a convex set and $\mathcal{H}$ a transverse hyperplane arrangement. If F is a face of $\mathcal{H}$, then $\overline{F\cap C^{\circ}} = \bar{F} \cap \bar{C}$.*

Proof. The question reduces to proving that $\overline{F\cap C^{\circ}} \supseteq \bar{F}\cap\bar{C}$ when F is an open face of $\mathcal{H}$. Take $x \in (\bar{F}\cap\bar{C}) \setminus (F\cap C^{\circ})$. Then $x \in F(x)\cap\partial C$. By Lemma 3.2 with $u = \operatorname{aff} F(x)$, every neighborhood of x intersects $u\cap C^{\circ}$. Because $F(x)$ is open in u, every small neighborhood of x in u is contained in $F(x)$. Since $F(x) \subseteq \bar{F}$, every small neighborhood of x in $\mathbb{R}^d$ meets $F \cap C^{\circ}$ and therefore $x \in \overline{F\cap C^{\circ}}$. ∎

Lemma 3.4. *Let C be a convex set and $\mathcal{H}$ a transverse hyperplane arrangement. The multiplicity of $x \in \mathbb{R}^d$ with respect to $\mathcal{H}$ is given by*

$$m_{\mathcal{H}}(x) = (-1)^{\operatorname{codim} s(x)} p_{\mathcal{H}(x)}(-1).$$

Proof. x belongs to the unique open face $F(x)$, which is an open region of $\mathcal{H}^{s(x)}$. There is an obvious bijection between (closed) regions R of $\mathcal{H}$ that contain $F(x)$ and regions R' of $\mathcal{H}(x)$: $R' \leftrightarrow R$ if $R' \subseteq R$. In fact, each region R' contains $s(x)$ and is dissected by $\mathcal{H}\setminus\mathcal{H}(x)$ into regions of $\mathcal{H}$, of which one and only one contains x. Therefore, the number of regions of $\mathcal{H}$ that contain x equals the number of regions of $\mathcal{H}(x)$, which is $(-1)^{\operatorname{codim} s(x)} p_{\mathcal{H}(x)}(-1)$. ∎

Lemma 3.5. *Let C be a full-dimensional convex subset of $\mathbb{R}^d$ and $\mathcal{H}$ a transverse hyperplane arrangement. The multiplicity of $x \in \mathbb{R}^d$ with respect to C and $\mathcal{H}$ is given by*

$$m_{C,\mathcal{H}}(x) = \begin{cases} (-1)^{\operatorname{codim} s(x)} p_{\mathcal{H}(x)}(-1) & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

Proof. We may assume $x \in C$. In relation to C, x lies in the open face $F(x)\cap C$ of $\mathcal{H}$ in C. We must prove that every region R of $\mathcal{H}$ that contains x corresponds to a region of $\mathcal{H}$ in C, so that $m_{C,\mathcal{H}}(x) = m_{\mathcal{H}}(x)$. This follows from Lemma 3.3 with $F = R^{\circ}$: since $x \in R\cap\bar{C}$, $R^{\circ}\cap C^{\circ}$ is nonempty, so $R \cap C$ is full-dimensional. ∎

Proof of Equation (3.2). We apply Lemma 3.5, the definition of the characteristic polynomial, and Rota's sign theorem [24, Section 7, Theorem 4].

$$\begin{aligned}
E_{C,\mathcal{H}}(D) &= \sum_{x\in C\cap D} |p_{\mathcal{H}(x)}(-1)| \\
&= \sum_{x\in C\cap D} \sum_{u\leq s(x)} |\mu(\hat{0},u)| \\
&= \sum_{u\in\mathcal{L}} |\mu(\hat{0},u)| \#\{x\in C\cap D : s(x)\subseteq u\} \\
&= \sum_{u\in\mathcal{L}} |\mu(\hat{0},u)| \#(D\cap C\cap u).
\end{aligned}$$

■

There is also a proof of (3.2) by inversion. See Section 7.

4. In which we arrange Ehrhart theory with hyperplanes

We are interested in counting those points of a discrete lattice D that are contained in a D-fractional convex polytope but not in a hyperplane arrangement. A *discrete lattice* is a set of points in an affine space (such as $\mathbb{Z}^d$ in $\mathbb{R}^d$) that is locally finite and is invariant under any translation that carries some lattice point to another lattice point. (Any discrete lattice in $\mathbb{R}^d$, which is $\mathbb{R}^d$ with coordinates, is a translate of a *lattice group*, which is a discrete lattice that contains the origin.) We call a polytope *D-integral* if its vertices all lie in D and *D-fractional* if the vertices lie in the contracted lattice $t^{-1}D$ for some positive integer t. The *denominator* of P [12] is the smallest such t. (To define $t^{-1}D$ here we assume coordinates chosen so that $0 \in D$. Later, at Corollary 4.3, we deal with a more general situation.) According to Ehrhart [10], if P is an d-dimensional D-fractional polytope and

$$E_P(t) := \#(D\cap tP) = \#(P\cap t^{-1}D),$$

then E_P is a quasipolynomial whose degree is d and whose period divides the denominator of P, whose leading coefficient equals $\mathrm{vol}_D P$, the volume of P normalized with respect to D (that is, we take the volume of a fundamental domain of D to be 1; in the case of the integer lattice $\mathbb{Z}^d$ this is the ordinary volume), and whose *constant term*, defined as $E_P(0)$ even when E_P is not a true polynomial, equals $\varepsilon(P)$, the combinatorial Euler characteristic of P. (Ehrhart proved this for the integer lattice; it follows in general because every discrete lattice is affinely equivalent to $\mathbb{Z}^d$. It applies to a partly closed polytope as long as the polytope contains every open face of which it contains a point.) Ehrhart [10] then conjectured and he, Macdonald [20], and McMullen [22] proved the reciprocity law

$$E_{P^\circ}(t) = (-1)^{\dim P} E_{\bar{P}}(-t).$$

This is ordinary Ehrhart theory.

Our theory begins with a rational, closed convex polytope P and an arrangement $\mathcal{H}$ of rational hyperplanes that is transverse to P. Rationality means that the vertices of P are rational points and the hyperplanes in $\mathcal{H}$ are specified by equations with rational coefficients. We call $(P,\mathcal{H})$ a *rational inside-out polytope* of *dimension* $\dim P$. (Omitting the requirement of rationality we would have a general *inside-out polytope*.) More generally we have any discrete lattice D, a D-fractional convex polytope P, and a D-fractional hyperplane arrangement $\mathcal{H}$ (transverse to P): that is, each hyperplane in $\mathcal{H}$ is spanned by the D-fractional points it contains. Then $(P, \mathcal{H})$ is a *D-fractional inside-out polytope*; we call it *D-integral* if all the vertices of P and all the intersection points formed by the hyperplanes and the facets of P lie in D. We always assume that P is closed.

Because P is convex, $(P, \mathcal{H})$ has two kinds of boundary as described in Section 3. It also has two kinds of faces. There are the faces of $\mathcal{H}$ in P, as in Section 3. Then there are the faces of $(P, \mathcal{H})$. A *closed face of* $(P, \mathcal{H})$ is an intersection of a closed face of P with a closed face of $\mathcal{H}$. An *open face of* $(P, \mathcal{H})$ is the relative interior of a closed face. If we think of P as dissected by $\mathcal{H}$ into $\dim P$-dimensional pieces, the open and closed faces of $(P, \mathcal{H})$ are the open and closed faces of those pieces. Note that a face F of $\mathcal{H}$ in P need not be a face of $(P, \mathcal{H})$, though $\bar{F}$ is a closed face and F° is an open face of $(P, \mathcal{H})$.

A *vertex* of $(P, \mathcal{H})$ is a vertex of any face. The *denominator* of $(P, \mathcal{H})$ (with respect to the discrete lattice D) is the smallest positive integer t for which $t^{-1}D$ contains every vertex of $(P, \mathcal{H})$.

The Ehrhart quasipolynomials of $(P, \mathcal{H})$ (recall that we always assume P is closed) are the (*closed*) *Ehrhart quasipolynomial*,

$$E_{P,\mathcal{H}}(t) := E_{P,\mathcal{H}}(t^{-1}D) = \sum_{x \in t^{-1}D} m_{P,\mathcal{H}}(x),$$

and the *open Ehrhart quasipolynomial*,

$$E^\circ_{P,\mathcal{H}}(t) := E^\circ_{P,\mathcal{H}}(t^{-1}D) = \#(t^{-1}D \cap [P \setminus \bigcup \mathcal{H}]),$$

both defined for positive integers t in terms of the D-enumerators of Section 3 with D replaced by $t^{-1}D$. Thus if P is full-dimensional and $R_1, \ldots, R_k$ are the closed regions of $(P, \mathcal{H})$,

$$E_{P,\mathcal{H}}(t) = \sum_{i=1}^{k} E_{R_i}(t) \quad \text{and} \quad E^\circ_{P^\circ,\mathcal{H}}(t) = \sum_{i=1}^{k} E^\circ_{R_i}(t) \tag{4.1}$$

by the definition of multiplicity and ordinary Ehrhart quasipolynomials. We use the term "quasipolynomial" in anticipation of the theorem:

Theorem 4.1. *If D is a full-dimensional discrete lattice and $(P, \mathcal{H})$ is a closed, full-dimensional, D-fractional inside-out polytope in $\mathbb{R}^d$ such that $\mathcal{H}$ does not contain the degenerate hyperplane, then $E_{P,\mathcal{H}}(t)$ and $E^\circ_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t, with period equal to a divisor of the denominator of $(P, \mathcal{H})$, with leading term $c_d t^d$ where $c_d = \operatorname{vol}_D P$, and with the constant term $E_{P,\mathcal{H}}(0)$ equal to the number of regions of $(P, \mathcal{H})$. Furthermore,*

$$E^\circ_{P^\circ,\mathcal{H}}(t) = (-1)^d E_{P,\mathcal{H}}(-t). \tag{4.2}$$

Proof. By (4.1), standard Ehrhart theory, and the fact that a closed region has Euler characteristic 1. ■

The periodically varying *quasiconstant term* $c_0(t)$ has no presently known interpretation, save at $t \equiv 0$.

It is easy to prove as well that $E^\circ_{P,\mathcal{H}}(t)$ and $E_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t with some of the same properties, e.g., the leading term, although we do not know they have the same period as each other or as $E_{P,\mathcal{H}}(t)$.

The first theorem does not require transversality, but the next one does, in part.

Theorem 4.2. *If D, P, and $\mathcal{H}$ are as in Theorem 4.1 and furthermore P is closed, then*

$$E^{\circ}_{P,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, u) E_{P\cap u}(t), \tag{4.3}$$

and if $\mathcal{H}$ is transverse to P,

$$E_{P,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, u)| E_{P\cap u}(t). \tag{4.4}$$

Proof. A special case of Theorem 3.1. ■

If it so happens that, as in the graph coloring examples, $E_{P\cap u}(t) = f(t)^{\dim u}$, then the right side of (4.4) becomes $(-1)^d p_{\mathcal{H}}(-f(t))$ and that of (4.3) becomes $p_{\mathcal{H}}(f(t))$. For instance, $f(t)$ may be t or $t-1$.

Theorem 4.2 gives a second proof of inside-out Ehrhart reciprocity, Equation (4.2). Apply ordinary reciprocity in flats and Rota's sign theorem:

$$\begin{aligned} E^{\circ}_{P^\circ,\mathcal{H}}(t) &= \sum_{u\in\mathcal{L}} \mu(\hat{0}, u) E_{P^\circ\cap u}(t) \\ &= \sum_{u\in\mathcal{L}} \mu(\hat{0}, u)(-1)^{\dim u} E_{P\cap u}(-t) \\ &= (-1)^d \sum_{u\in\mathcal{L}} |\mu(\hat{0}, u)| E_{P\cap u}(-t) \\ &= (-1)^d E_{P,\mathcal{H}}(-t). \qquad ■ \end{aligned}$$

Often the polytope is not full-dimensional; its affine span, aff P, might not even intersect the discrete lattice. Suppose, then, that D is a discrete lattice in $\mathbb{R}^d$ and s is any affine subspace. The *period* $p(s)$ of s with respect to D is the smallest positive integer p for which $p^{-1}D$ meets s. Then Theorem 4.1 implies

Corollary 4.3. *Let D be a discrete lattice in $\mathbb{R}^d$, P a D-fractional convex polytope, and $\mathcal{H}$ a hyperplane arrangement in $s :=$ aff P that does not contain the degenerate hyperplane. Then $E_{P,\mathcal{H}}(t)$ and $E^{\circ}_{P^\circ,\mathcal{H}}(t)$ are quasipolynomials in t that satisfy the reciprocity law $E^{\circ}_{P^\circ,\mathcal{H}}(t) = (-1)^{\dim s} E_{P,\mathcal{H}}(-t)$. Their period is a multiple of $p(s)$ and a divisor of the denominator of $(P, \mathcal{H})$. If $t \equiv 0 \bmod p(s)$, the leading term of $E_{P,\mathcal{H}}(t)$ is $(\mathrm{vol}_{p(s)^{-1}D} P) t^{\dim s}$ and its constant term is the number of regions of $(P, \mathcal{H})$; but if $t \not\equiv 0 \bmod p(s)$, then $E_{P,\mathcal{H}}(t) = E^{\circ}_{P^\circ,\mathcal{H}}(t) = 0$.* ■

The period's being greater than one suggests that we should renormalize, multiplying s, P, and $\mathcal{H}$ by $p(s)$. This divides both the denominator of the inside-out polytope and the period of the Ehrhart quasipolynomials by $p(s)$ and eliminates the zero constituents of the quasipolynomials. The 3×3 magic squares are a perfect example [4, Section 3.2].

Usually $\mathcal{H}$ will be induced by an arrangement $\mathcal{H}_0$ in $\mathbb{R}^d$. It is easy to see that $\mathcal{H}$ is transverse to P if and only if $\mathcal{H}_0$ is.

5. In which we color graphs and signed graphs

The best way to approach graph coloring geometrically is not precisely the one sketched in the introduction. For unsigned graphs that is sufficient, but for signed graphs we do better with two subtle changes. For one, rather than inflating the fundamental polytope we should contract the lattice; but the main difference is that we should realign and rescale the whole picture so that the fundamental polytope, even for signed graphs, is $[0,1]^n$ and the hyperplanes center on the point $\frac{1}{2}\mathbf{1}$, where $\mathbf{1} := (1,1,\dots,1)$. This enables us to unify unsigned and signed graph coloring and, in addition, the two kinds of signed chromatic polynomials.

We take up unsigned graphs first. Our purpose is to see how much information we can extract purely from inside-out Ehrhart theory; let the reader therefore forget everything he or she knows about the chromatic polynomial except the definition:

$$\chi_\Gamma(c) := \text{ the number of proper } c\text{-colorings of } \Gamma.$$

An *ordinary graph* is a graph whose edges are links (with two distinct endpoints) and loops (with two coinciding endpoints); multiple edges are permitted. We treat, always, only finite graphs. The *order* is the number of nodes; we write n for the order of Γ.

Theorem 5.1. *Let Γ be an ordinary graph and let $P = [0,1]^n$. The closed and open Ehrhart quasipolynomials of $(P, \mathcal{H}[\Gamma])$ satisfy*

$$(-1)^n E_{P,\mathcal{H}[\Gamma]}(-t) = E^\circ_{P^\circ,\mathcal{H}[\Gamma]}(t) = \chi_\Gamma(t-1).$$

Proof. In $t^{-1}\mathbb{Z}^n$ the points that are counted by $E^\circ_{P^\circ,\mathcal{H}[\Gamma]}(t)$ are those of $(t^{-1}\{1,2,\dots,t-1\})^n$ that do not lie in any forbidden hyperplane. The number of such points is the number of proper $t-1$-colorings of Γ. ■

Corollary 5.2 (Birkhoff [6] for maps, Whitney [34] for graphs)**.** *For an ordinary graph Γ with no loops, χ_Γ is a monic polynomial of degree n. If Γ has a loop, $\chi_\Gamma = 0$.*

Proof. Since P is full-dimensional in $\mathbb{R}^n$ and has volume 1, the leading term of χ_Γ is $1x^n$ by Ehrhart theory. It remains to prove that $(P, \mathcal{H}[\Gamma])$ has denominator 1, or in other words that $(P, \mathcal{H}[\Gamma])$ has integer vertices. This is the next lemma. ■

Lemma 5.3. *If Γ is an ordinary graph, $(P, \mathcal{H}[\Gamma])$ has integer vertices.*

Proof. The proof is simple; we supply it because the lemma is so important. Because, as is well known, $\mathcal{H}[\Gamma]$ is a hyperplanar representation of the graphic matroid $G(\Gamma)$, the flats of $\mathcal{H}(\Gamma)$ correspond to closed subgraphs of Γ, i.e., to partitions π of V into blocks that induce connected subgraphs. The flat $s(\pi)$ corresponding to π is described by $x_i = x_j$ if i and j belong to the same block of π (we write $i \underset{\pi}{\sim} j$). A vertex of $(P, \mathcal{H}(\Gamma])$ is determined by a flat $s(\pi)$ of dimension k, say, together with k facet hyperplanes of P that have the form $x_i = a_i \in \{0,1\}$. Obviously, such points are integral. ■

Given an orientation α of Γ and a c-coloring $x : V \to [c]$, Stanley calls them *compatible* if $x_j \geq x_i$ whenever there is a Γ-edge oriented from i to j, and *proper* if $x_j > x_i$ under the same condition [27]. An orientation is *acyclic* if it has no directed cycles. From Theorem 5.1 we derive a more unified version of Stanley's second proof of his famous result.

Corollary 5.4 (Stanley [27])**.** *The number of pairs (α, x) consisting of an acyclic orientation of an ordinary graph Γ and a compatible c-coloring equals $(-1)^n\chi_\Gamma(-c)$. In particular, $(-1)^n\chi_\Gamma(-1) =$ the number of acyclic orientations of Γ.*

Proof. From Theorem 5.1,

$$E_{P,\mathcal{H}[\Gamma]}(t) = (-1)^n\chi_\Gamma(-(t+1)).$$

What $E_{P,\mathcal{H}[\Gamma]}(t)$ counts is the number of pairs (x, R) where x is a coloring with color set $\{0, 1, \dots, t\}$, R is a closed region of $\mathcal{H}[\Gamma]$, and $x \in R$. Greene observed that regions R correspond with acyclic orientations α in the following way: R° is determined by converting each equation $x_i = x_j$ corresponding to an edge of Γ into an inequality $x_i < x_j$; then in α the edge ij is directed from i to j. (See [14] or [16, Section 7].) The orientation is acyclic because $R^\circ \neq \varnothing$. Thus x is compatible with α if and only if $x \in R$. The final assertion is an instance of the evaluation $E(0)$ in Theorem 4.1. ∎

This proof generalizes Greene's geometrical approach to the case $c = 1$ (see [14] or [16, Section 7]).

For our theorem about signed graphs we need the chromatic polynomials from the introduction; we take $P = [0, 1]^n$ (just as with unsigned graph coloring); and we replace $\mathcal{H}[\Sigma]$ by its translate $\mathcal{H}''[\Sigma] = \mathcal{H}[\Sigma] + \frac{1}{2}\mathbf{1}$; that is, we add $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ to every hyperplane. (Translating the hyperplane of a positive edge leaves it unchanged; thus, $\mathcal{H}''[\Gamma] = \mathcal{H}[\Gamma]$, which is why translation was unnecessary for Theorem 5.1.) A signed graph, defined in complete generality, is $\Sigma = (\Gamma, \sigma)$ where Γ is a graph that may have, besides links and loops, also *halfedges* (with only one endpoint) and *loose edges* (no endpoints), and σ is a signature defined only on links and loops (which are called *ordinary edges*). The hyperplane corresponding to a halfedge at node i is $x_i = 0$, the same as for a negative loop at i, and the "hyperplane" that corresponds to a loose edge is the degenerate hyperplane $0 = 0$, just as for a positive loop. The *order* of a signed graph is the number of nodes, written n.

Theorem 5.5. *Let Σ be a signed graph and let $P = [0, 1]^n$. The Ehrhart quasipolynomial of $(P, \mathcal{H}''[\Sigma])$ satisfies*

$$(-1)^n E_{P,\mathcal{H}''[\Sigma]}(-t) = E^\circ_{P^\circ,\mathcal{H}''[\Sigma]}(t) = \begin{cases} \chi_\Sigma(t-1) & \text{if } t \text{ is even,} \\ \chi^*_\Sigma(t-1) & \text{if } t \text{ is odd.} \end{cases}$$

Proof. An easy way to see the correctness of the expression for E° in terms of χ_Σ and χ^*_Σ is to translate the center of P to the origin and dilate by t. Then P becomes $\tilde{P} = [-\frac{t}{2}, \frac{t}{2}]^n$ and $\mathcal{H}''[\Sigma]$ becomes $\mathcal{H}[\Sigma]$. What happens to $t^{-1}\mathbb{Z}^n$ depends on the parity of t. If t is even, $t^{-1}\mathbb{Z}^n$ becomes $\mathbb{Z}^n$ and, much as in the introduction and the proof of Theorem 5.1, $E^\circ_{P^\circ,\mathcal{H}[\Sigma]}(t) = \chi_\Sigma(2c + 1)$ with $c = \frac{t}{2} - 1$. When t is odd, $t^{-1}\mathbb{Z}^n$ is transformed to $\mathbb{Z}^n + \frac{1}{2}\mathbf{1}$, in which no vector has an integral entry; the number of points of this lattice in $\tilde{P}^\circ$ and not in $\bigcup \mathcal{H}[\Sigma]$ equals $\chi^*_\Sigma(t)$ if we regard the latter as counting colorings with color set $\frac{1}{2}\{\pm 1, \pm 3, \dots, \pm(t-2)\}$, which is an acceptable color set because it consists of $\frac{1}{2}(t-1)$ colors, each with both signs, and does not contain 0. ∎

The way the parity of t affects the geometry is what prevents us from conveniently stating the theorem in terms of $\mathcal{H}[\Sigma]$.

Corollary 5.6. *For a signed graph with no positive loops or loose edges, χ_Σ and χ^*_Σ are monic polynomials of degree n. If Σ has a positive loop or a loose edge, $\chi_\Sigma = \chi^*_\Sigma = 0$.*

Proof. The leading terms are $1x^n$ because P is n-dimensional with volume 1. Polynomiality is a consequence of the next lemma, by which $(P, \mathcal{H}''[\Sigma])$ has denominator 1 or 2. ∎

Lemma 5.7. *If Σ is a signed graph, $(P, \mathcal{H}''[\Sigma])$ has half-integral vertices.*

FIGURE 3. Left: The $\frac{1}{2(k+1)}$-lattice points in $[0,1]^2$ that k-color the signed graph $\pm K_2^\circ$, with shifted hyperplanes and with $k = 8$. Right: The $\frac{1}{(2k+1)}$-lattice points that k-color it without 0.

Proof. The flats of $\mathcal{H}[\Sigma]$ correspond to partial signed partitions (π, σ) of V. (This description, drawn from [39], is based on [8, Section 3], interpreted in light of [36, Theorem 5.1(b)].) A *partial partition* is a partition of a subset of V. A *signed partition* is a partition π along with, for each block B, a pair $[\sigma] = \{\sigma, -\sigma\}$ where $\sigma : B \to \{+1, -1\}$ is a signature on B. In the correspondence $(\pi, \sigma) \mapsto s$, the flat s has the equations $x_i = 0$ for $i \notin \bigcup \pi$ and, for each block $B \in \pi$, $\sigma_i x_i = \sigma_j x_j$ if $i, j \in B$. (In general not all subspaces of this form are flats of $\mathcal{H}[\Sigma]$, the exception being the complete signed graph $\pm K_n^\bullet$ [36].)

A flat of $\mathcal{H}''[\Sigma]$ therefore has the equations $x_i = \frac{1}{2}$ if $i \notin \bigcup \pi$ and $\sigma_j x_j = \sigma_i x_i + \frac{1}{2}(\sigma_j - \sigma_i)$ if $i \underset{\pi}{\sim} j$. The constant term in the latter is half integral. A vertex of $(P, \mathcal{H}''[\Sigma])$ is described by $n - k$ equations of these kinds, determining a k-flat s, and k equations of the form $x_i = a_i \in \{0, 1\}$; clearly, then, the vertex has half-integral coordinates. ∎

We say more about half integrality in relation to the incidence matrix in [3].

There is a stronger conclusion if Σ is *balanced*, that is, it has no halfedges and no circles with negative sign product. In that case Σ is obtained from an all-positive graph by reversing the signs of all edges of a cutset, an operation called *switching*. (This was proved by König [19, Theorem X.10]. See [36, Corollary 3.3] for more detail.) We represent switching by a function $\eta : V \to \{+, -\}$ such that the cutset consists of all edges whose endpoints have opposite signs. When Σ is balanced, obtained by switching $+\Gamma$ (where Γ is the underlying graph of Σ), an edge has sign $\sigma(ij) = \eta(i)\eta(j)$ and a flat of $\mathcal{H}[\Sigma]$ is specified by a partition π of V and equations $\eta(i)x_i = \eta(j)x_j$ when $i \underset{\pi}{\sim} j$.

Corollary 5.8 (Zaslavsky [37, Section 2.1]). *For a balanced signed graph,* $\chi_\Sigma = \chi_\Sigma^*$.

Proof. Σ is obtained through switching $+\Gamma$ by a switching function η. The effect of η on $\mathcal{H}''[\Sigma]$ is to reverse coordinates: $x_i \to 1 - x_i$ if $\eta(i) = -$, but $x_i \to x_i$ if $\eta(i) = +$. We apply η to P and $t^{-1}\mathbb{Z}^n$ in the same way so that switching does not alter the Ehrhart quasipolynomials. Therefore, we may apply Lemma 5.3 to $(P, \mathcal{H}''[\Sigma])$. ■

The switching equivalence of a balanced signed graph to an all-positive graph demonstrates that $(P, \mathcal{H}''[\Sigma])$ then has integral vertices. Another proof is by observing that its equations are totally unimodular, that is, every subdeterminant is 0 or ± 1, as shown in [36, Proposition 8A.5] and originally in [17, Theorem 1]. We omit the details.

Corollary 5.8 is not the whole story. Going beyond Ehrhart theory, we can prove that $\chi_\Sigma \neq \chi^*_\Sigma$ when Σ is unbalanced, by comparing the lattice $\operatorname{Lat} G(\Sigma)$ of closed subgraphs of Σ to the semilattice $\operatorname{Lat}^{\mathrm{b}} \Sigma$ of closed, balanced subgraphs [36, Section 5]. They are equal if and only if Σ is balanced, and by [37, Theorem 2.4] $\chi_\Sigma = \chi^*_\Sigma$ if and only if they are equal. Expressed in Ehrhartian terms: for a signed-graphic inside-out polytope the period of the Ehrhart quasipolynomial is equal to the denominator of $(P, \mathcal{H})$, which is not true in general.

The signed-graphic generalization of Stanley's theorem, Corollary 5.4, is also a consequence of Ehrhart theory.

Corollary 5.9 ([37, Theorem 3.5])**.** *The number of compatible pairs (α, x) consisting of an acyclic orientation α and a c-coloring of a signed graph Σ is equal to $(-1)^n \chi_\Sigma(-(2c+1))$. The number in which x is zero-free equals $(-1)^n \chi^*_\Sigma(-2c)$. In particular, $(-1)^n \chi_\Sigma(-1) =$ the number of acyclic orientations of Σ.*

Sketch of Proof. We omit the details of proof because they are as in our proof of Stanley's theorem. We omit the definitions because they are lengthy. Acyclic orientations and compatible pairs are defined in [37, Section 3]. Acyclic orientations are defined in [38] and their correspondence to regions of $\mathcal{H}[\Sigma]$ is proved in [38, Theorem 4.4]. ■

Problem 5.10. A combinatorial interpretation of $(-1)^n \chi^*_\Sigma(-1)$ would be a valuable contribution, since it would interpret the quasiconstant term $c_0(1)$ of the $t \equiv 1$ polynomial.

6. In which subspace arrangements put in their customary appearance

An *arrangement of subspaces* in $\mathbb{R}^d$ is an arbitrary finite set $\mathcal{A}$ of (affine) subspaces. (We assume all the subspaces are proper.) We wish to generalize our results to a polytope with a subspace arrangement, along the lines taken by Blass and Sagan [7] for graph coloring. This is possible in part. For instance, we can define the "multiplicity" of a point with respect to $\mathcal{A}$, but only algebraically; it need not count anything, in fact it could be negative.

To begin with we take the situation of Section 3 in which C is a bounded convex set, D is a discrete set, and $\mathcal{A}$ is an arrangement, now a subspace arrangement, that is transverse to C. We can take over most of the definitions from Sections 1–3 simply by changing $\mathcal{H}$ to $\mathcal{A}$. For one example, the *open D-enumerator* of $(C, \mathcal{A})$ is

$$E^\circ_{C,\mathcal{A}}(D) := \#\big(D \cap C \setminus \bigcup \mathcal{A}\big).$$

There are some complications, however. The semilattice $\mathcal{L}(\mathcal{A})$, still partially ordered by reverse inclusion, is not necessarily geometric or ranked; instead it is *extrinsically graded* by the rank function

$$\rho(u) = \operatorname{codim} u$$

and the total rank $\rho(\mathcal{L}) = d$, so that u has extrinsic corank $\rho(\mathcal{L}) - \rho(u) = \dim u$. (The notion of extrinsic grading, without a particular name, is common in writings on subspace arrangements.) The *multiplicity* of $x \in \mathbb{R}^d$ with respect to C and $\mathcal{A}$ is

$$m_{C,\mathcal{A}}(x) := \begin{cases} (-1)^d p_{\mathcal{A}(x)}(-1) = \displaystyle\sum_{u \in \mathcal{L}(\mathcal{A}): x \in u} \mu(\hat{0}, u)(-1)^{\rho(u)} & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

Lemma 3.5 ensures that this agrees with the definition for hyperplane arrangements, in Section 3. Now we can define the *closed D-enumerator* of $(C, \mathcal{A})$ as before:

$$E_{C,\mathcal{A}}(D) := \sum_{x \in D} m_{C,\mathcal{A}}(x).$$

Theorem 6.1. *Let C be a bounded, convex subset of $\mathbb{R}^d$, $\mathcal{A}$ a subspace arrangement that is transverse to C, and D a discrete set in $\mathbb{R}^d$. Then*

$$E_{C,\mathcal{A}}(D) = \sum_{u \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, u)(-1)^{\operatorname{codim} u} \#(D \cap C \cap u)$$

and

$$E^\circ_{C,\mathcal{A}}(D) = \sum_{u \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, u) \#(D \cap C \cap u).$$

Proof. That of Theorem 3.1, including Lemmas 3.2 and 3.3, goes through with obvious modifications and the understanding that an "open face" must be interpreted as a connected component of $u \setminus \bigcup \mathcal{A}^u$ but may not be simply connected, much less a cell. ∎

For the main result about subspace arrangements we adapt the notation of Section 4, in particular the *closed* and *open Ehrhart functions*,

$$E_{P,\mathcal{A}}(t) := E_{P,\mathcal{A}}(t^{-1}D) = \sum_{x \in t^{-1}D} m_{P,\mathcal{A}}(x)$$

and

$$E^\circ_{P,\mathcal{A}}(t) := E^\circ_{P,\mathcal{A}}(t^{-1}D) = \#(t^{-1}D \cap [P \setminus \bigcup \mathcal{A}]).$$

Theorem 6.2. *If D is a discrete lattice in $\mathbb{R}^d$, P is a full-dimensional D-fractional convex polytope, and $\mathcal{A}$ is a D-fractional subspace arrangement, then $E_{P,\mathcal{A}}(t)$ and $E^\circ_{P,\mathcal{A}}(t)$ are quasipolynomials in t, each with period equal to a divisor of the D-denominator of $(P, \mathcal{A})$ and with leading term $(\mathrm{vol}_P\, D)t^d$. We have*

$$E^\circ_{P^\circ,\mathcal{A}}(t) = (-1)^d E_{\bar{P},\mathcal{A}}(-t). \tag{6.1}$$

Furthermore,

$$E_{P,\mathcal{A}}(t) = \sum_{u \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, u)(-1)^{\operatorname{codim} u} E_{P \cap u}(t) \tag{6.2}$$

and

$$E^\circ_{P,\mathcal{A}}(t) = \sum_{u \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, u) E_{P \cap u}(t) \tag{6.3}$$

Proof. The two latter equations are special cases of Theorem 6.1. The reciprocity law (6.1) follows from (6.2), (6.3), and standard Ehrhart reciprocity. ∎

Problem 6.3. The constant term $E(0)$ does not seem to have an obvious combinatorial interpretation except in special cases, as for instance if the arrangement leaves P connected, when $E(0) = \varepsilon(P)$ as in ordinary Ehrhart theory.

7. In which we prove a general valuation formula

A *normalized valuation* on the faces of a hyperplane arrangement is a function v on finite unions of open faces, with values in an abelian group, such that $v(A \cup B) + v(A \cap B) = v(A) + v(B)$ for any two such unions, or more simply

$$v(F_1 \cup \cdots \cup F_k) = v(F_1) + \cdots + v(F_k)$$

for distinct open faces $F_1, \ldots, F_k$, and also

$$v(\varnothing) = 0$$

(the normalization). For example, if D_0 is a finite subset of $\mathbb{R}^d$, $v(F) = \#(D_0 \cap F)$ is a valuation. Specializing further, if D is a discrete set and C is a bounded convex set, then $v(F) = \#(D \cap C \cap F)$ is a valuation. For a flat u of $\mathcal{H}$, set

$$E_v(u) = \sum_R v(R)$$

summed over closed regions R of $\mathcal{H}^u$. If $v(F) = \#(D \cap C \cap F)$, and $\mathcal{H}$ is transverse to C, this is simply $E_{C \cap u, \mathcal{H}^u}(D)$.

Theorem 7.1. *For $u \in \mathcal{L}(\mathcal{H})$ and v a normalized valuation on the faces of $\mathcal{H}$,*

$$E_v(u) = \sum_{s \in \mathcal{L} : s \geq u} |\mu(s,u)| v(s).$$

Equation (3.2) is a special case. What is different about this theorem compared to Theorem 3.1, besides its general statement, is the proof by Möbius inversion. The proof is more complicated, but we think it is interesting.

Theorem 7.1 can be interpreted in terms of the Möbius algebra of $\mathcal{L}(\mathcal{H})$. The Möbius algebra $M(L)$ of a poset L, introduced by Solomon [26] and developed by Greene [15], is the algebra (over any nice ring) generated by the elements of L as orthogonal idempotents. For $u \in L$ we define $\hat{u} = \sum_{s \geq u} \mu(u,s)s$. (Technically, this defines the Möbius algebra of the dual poset L^*; but that is a difference without a difference.) Let ε denote the combinatorial Euler characteristic given by $\varepsilon(u) = (-1)^{\dim u}$ and let εf denote the pointwise product with a function f. A function defined on $\mathcal{L}(\mathcal{H})$ naturally extends by linearity to the Möbius algebra of $\mathcal{L}(\mathcal{H})$. Theorem 7.1 says that, if v is a normalized valuation on $\mathcal{F}(\mathcal{H})$, extended in the obvious way to $\mathcal{L}(\mathcal{H})$ and then to the Möbius algebra, then $\varepsilon E_v(u) = \varepsilon v(\hat{u})$.

Proof. In effect, we use Möbius inversion twice.

The first time is in the *semilattice of faces* of $\mathcal{H}$,

$$\mathcal{F}(\mathcal{H}) = \{F : F \text{ is an open face of } \mathcal{H}\}$$

ordered by inclusion of the closures. The maximal elements of $\mathcal{F}(\mathcal{H})$ are the open regions; let $\mathcal{R}(\mathcal{H})$ be the set of open regions. We show that

$$(-1)^{\dim u} v(u) = \sum_{s \in \mathcal{L} : s \geq u} (-1)^{\dim s} E_v(s) \qquad \text{for } u \in \mathcal{L}(\mathcal{H}). \tag{7.1}$$

Multiplying by $(-1)^{\dim u}$, the left side equals

$$\sum_{F \in \mathcal{F}(\mathcal{H}^u)} v(F). \tag{7.2}$$

The right side equals

$$\begin{aligned}
&\sum_{s \geq u} (-1)^{\dim u - \dim s} \sum_{R \in \mathcal{R}(\mathcal{H}^s)} v(\bar{R}) \\
&\qquad = \sum_{s \geq u} (-1)^{\dim u - \dim s} \sum_{R \in \mathcal{R}(\mathcal{H}^s)} \sum_{F \in \mathcal{F}(\mathcal{H}^s): F \leq R} v(F) \\
&\qquad = \sum_{F \in \mathcal{F}(\mathcal{H}^u)} (-1)^{\dim u - \dim F} v(F) \sum_{\substack{R \in \mathcal{F}(\mathcal{H}^u) \\ R \geq F}} (-1)^{\dim R - \dim F}.
\end{aligned} \tag{7.3}$$

The *lattice of faces* of $\mathcal{H}$, $\hat{\mathcal{F}}(\mathcal{H})$, is $\mathcal{F}(\mathcal{H})$ with an extra top element $\hat{1}$ adjoined. It is known that $\hat{\mathcal{F}}(\mathcal{H})$ is Eulerian, that is, $\mu(x, y) = (-1)^{\operatorname{rk} y - \operatorname{rk} x}$ if $x \leq y$. Thus when $x \leq y < \hat{1}$, $\mu(x, y) = (-1)^{\dim y - \dim x}$. The inner sum in (7.3) is therefore

$$\sum_{R \geq F} \mu_{\hat{\mathcal{F}}}(F, R) = -\mu_{\hat{\mathcal{F}}(\mathcal{H}^u)}(F, \hat{1}) = (-1)^{\dim u - \dim F},$$

so (7.3) equals (7.2).

Having established (7.1) we invert to obtain

$$(-1)^{\dim u} E_v(u) = \sum_{s \geq u} \mu_{\mathcal{L}}(u, s)(-1)^{\dim s} v(s).$$

Multiplying this by $(-1)^{\dim u}$ and applying Rota's sign theorem, we have the theorem. ■

For completeness we sketch a proof that $\hat{\mathcal{F}}(\mathcal{H})$ is Eulerian. Let $\mathcal{H}_{\mathbb{P}}$ be the projectivization of $\mathcal{H}$, that is, $\mathcal{H} \cup \{h_\infty\}$ in $\mathbb{P}^d$ with the affine hyperplanes extended into infinity. $\mathcal{H}_{\mathbb{P}}$ is the projection of a homogeneous hyperplane arrangement $\mathcal{H}'$ in $\mathbb{R}^{d+1}$, whose face lattice is dual to that of a zonotope, whose face lattice is Eulerian because a zonotope is a convex polytope. Faces of $\mathcal{H}'$ other than the 0-face, $F'_0 = \bigcap \mathcal{H}'$, come in opposite pairs, F' and $-F'$, which project to a single face F of $\mathcal{H}_{\mathbb{P}}$. The interval $[F_1, \hat{1}]$ in $\hat{\mathcal{F}}(\mathcal{H}_{\mathbb{P}})$ is isomorphic to $[F'_1, \hat{1}]$ in $\hat{\mathcal{F}}(\mathcal{H}')$ if F'_i projects to F_i. As for F'_0, it projects to an infinite face. Therefore, for any face F of $\mathcal{H}$, which is necessarily a finite face of $\mathcal{H}_{\mathbb{P}}$, the interval $[F, \hat{1}]$ in $\hat{\mathcal{F}}(\mathcal{H})$ is equal to $[F, \hat{1}]_{\hat{\mathcal{F}}(\mathcal{H}_{\mathbb{P}})}$, which is isomorphic to $[F', \hat{1}]_{\hat{\mathcal{F}}(\mathcal{H}')}$. It follows that $\mu(F, \hat{1})$ in $\hat{\mathcal{F}}(\mathcal{H})$ equals $(-1)^{d+1-\dim F}$.

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