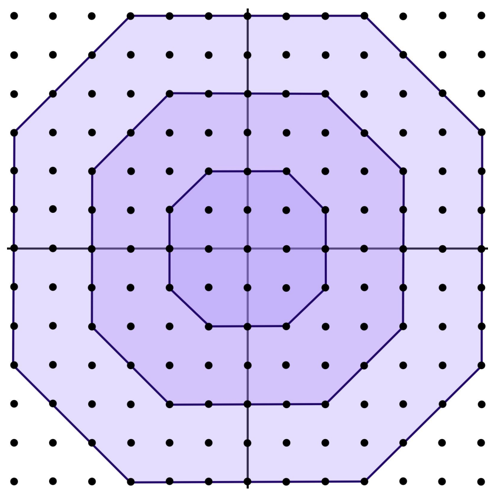
q-polynomials

Matthias Beck
San Francisco State University
matthbeck.github.io



Esme Bajo
Art of Problem Solving

Ben Braun
University of Kentucky

Alvaro Cornejo
University of Kentucky

Thomas Kunze
UC Irvine

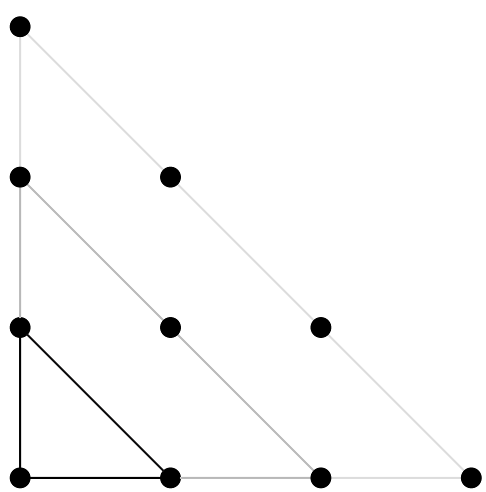
Andrés Vindas-Meléndez
Harvey Mudd College

Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $\mathrm{ehr}_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $\mathrm{ehr}_{\mathcal{P}}(t)$ is a polynomial in t. Furthermore, $\mathrm{ehr}_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} \# \left(t\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.



Example $\Delta = \mathrm{conv}\{(0,0),\ (1,0),\ (0,1)\}$

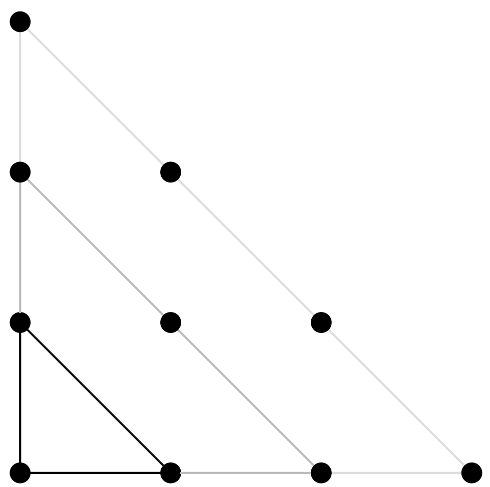
$$\mathrm{ehr}_{\Delta}(t) \ = \ \frac{1}{2} t^2 + \frac{3}{2} t + 1$$

Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

For $t \in \mathbb{Z}_{>0}$ let $\operatorname{ehr}_\mathcal{P}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$

Theorem (Ehrhart 1962, Macdonald 1971) $\operatorname{ehr}_\mathcal{P}(t)$ is a polynomial in t.
Furthermore, $\operatorname{ehr}_\mathcal{P}(-t) = (-1)^{\dim \mathcal{P}} \# \left(t\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.



Example $\Delta = \operatorname{conv}\{(0,0),\ (1,0),\ (0,1)\}$

$$\operatorname{ehr}_\Delta(t) \ = \ \frac{1}{2} t^2 + \frac{3}{2} t + 1$$

Philosophy We do not need limits for

$$\operatorname{vol}(\mathcal{P}) = \lim_{t \to \infty} \frac{1}{t^d} \operatorname{ehr}_\mathcal{P}(t)$$

Some Motivation

- Linear systems are everywhere, and so polyhedra are everywhere.

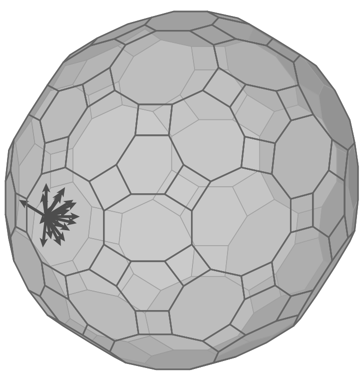
- In applications, the volume of the polytope represented by a linear system measures some fundamental data of this system ("average").

- Many discrete problems in various areas are linear problems, thus they ask for the discrete volume of a polytope in disguise.

- Much discrete geometry can be modeled using polynomials and, conversely, many combinatorial polynomials can be modeled geometrically.

- Volume computation is hard.

- Also, polytopes are cool.



♡ Polynomials

- Computation

Class of Ehrhart polynomials $\longrightarrow$ two main research problems:

- Classification — which polynomials are Ehrhart polynomials?
  (open in dimension 3)

- Detection — does a given polynomial determine the polytope?
  (fails somewhwat spectacularly)

q-Ehrhart Polynomials

Lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ — convex hull of finitely points in $\mathbb{Z}^d$

$$\operatorname{ehr}_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbb{Z}^d \right)$$

Theorem (Ehrhart 1962, Macdonald 1971) $\operatorname{ehr}_{\mathcal{P}}(t)$ is a polynomial in t. Furthermore, $\operatorname{ehr}_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} \# \left(t\mathcal{P}^\circ \cap \mathbb{Z}^d \right)$.

Now fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

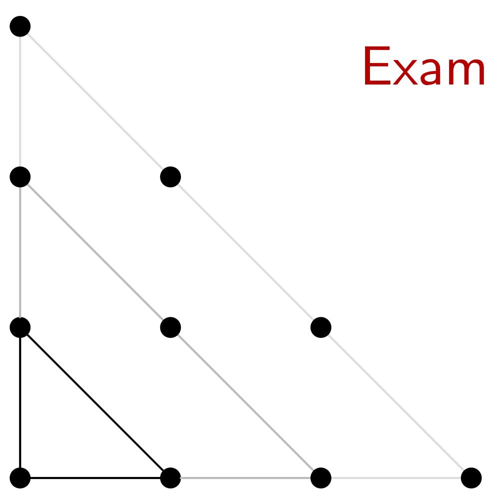
Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\operatorname{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \operatorname{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

q-Ehrhart Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\operatorname{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \operatorname{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.



Example $\Delta = \operatorname{conv}\{(0,0),\ (1,0),\ (0,1)\}$ and $\lambda = (1,2)$

$$\operatorname{cha}_{\Delta}^{\lambda}(q,x) = \frac{q^3}{q+1}\,x^2 + \frac{q(2q+1)}{q+1}\,x + 1$$

Chapoton Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $m := \max\{\lambda(\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$ and all the poles of the coefficients of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ are roots of unity of order $\leq m$.

Furthermore, $(-1)^{\dim \mathcal{P}} \, \mathrm{cha}_{\mathcal{P}}^{\lambda}\left(\frac{1}{q}, -qx\right) = \mathrm{cha}_{\mathcal{P}^\circ}^{\lambda}(q,x)$.

Chapoton Polynomials

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $m := \max\{\lambda(\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$ and all the poles of the coefficients of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ are roots of unity of order $\leq m$.

Furthermore, $(-1)^{\dim \mathcal{P}}\, \mathrm{cha}_{\mathcal{P}}^{\lambda}\left(\frac{1}{q}, -qx\right) = \mathrm{cha}_{\mathcal{P}^\circ}^{\lambda}(q,x)$.

Theorem (Robins 2023, Sanyal @ FPSAC 2025) The set of all $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$, where λ ranges over all generic and positive integral forms, determines $\mathcal{P}$.

Some More Motivation

- $\mathrm{ehr}_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$ has polynomial structure, but sometimes we need to understand the integer point transform

$$\sigma_{\mathcal{P}}(\mathbf{z}) := \sum_{\mathbf{m} \in \mathcal{P} \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$$

- For fixed λ,

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})} = \sigma_{t\mathcal{P}}\left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d}\right)$$

  still has polynomial structure.

- Chapoton polynomials contain interesting number theory, connection to partition functions, . . .

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

- $\square = [0,1]^d$ and $\lambda = \mathbf{1} := (1,1,\dots,1)$

$$\mathrm{ehr}_{\square}^{\mathbf{1}}(q,t) \;=\; [t+1]_q^d \qquad \longrightarrow \qquad \mathrm{cha}_{\square}^{\mathbf{1}}(q,x) \;=\; (1+qx)^d$$

Carlitz identity (really due to MacMahon)

$$\sum_{t \geq 0} [t+1]_q^n \, x^t \;=\; \frac{\sum_{\pi \in S_n} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^{n} (1 - xq^j)}$$

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

- $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d_{\geq 0} : \, x_1 + x_2 + \cdots + x_d = 1 \right\}$

$$\mathrm{ehr}_{\Delta}^{\lambda}(q,t) \ = \sum_{\mathbf{m} \in t\Delta} q^{\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_d m_d}$$

is the generating function for partitions with exactly t parts in the set $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$

$$\mathrm{cha}_{\Delta}^{\lambda}(q,x) \ = \sum_{j=1}^{d} \frac{1}{\prod_{k \neq j} \left(1 - q^{\lambda_k - \lambda_j}\right)} \big((q-1)x + 1\big)^{\lambda_j}$$

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

- $\Delta = \left\{ \mathbf{x} \in \mathbb{R}^d : \, 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1 \right\}$ and $\lambda = \mathbf{1}$

$$\mathrm{ehr}_{\Delta}^{\mathbf{1}}(q,t) \, = \sum_{\mathbf{m} \in t\Delta} q^{m_1 + m_2 + \cdots + m_d} \, = \begin{bmatrix} t+d \\ d \end{bmatrix}_q$$

is the generating function for partitions with $\leq d$ parts, each of which $\leq t$

$$\mathrm{cha}_{\Delta}^{\mathbf{1}}(q,x) \, = \sum_{j=0}^{d} \frac{1}{\prod_{k \neq j} \left(1 - q^{k-j}\right)} \big((q-1)x + 1\big)^j$$

Familiar Faces

Fix a linear form λ that is generic ($\lambda(\mathbf{v}) \neq \lambda(\mathbf{w})$ for adjacent vertices $\mathbf{v}$ and $\mathbf{w}$ of $\mathcal{P}$) and positive ($\lambda(\mathbf{v}) \geq 0$ for any vertex $\mathbf{v}$), and let

$$\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})}$$

- $\mathcal{P}$ — order polytope of $[m] \times [n]$

MacMahon (1909) $\operatorname{cha}_{\mathcal{P}}^{\mathbf{1}}(q,x) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{[i+j-1]_q + x\, q^{i+j-1}}{[i+j-1]_q}$

Familiar Faces

- Lecture hall simplex $\Delta_n := \left\{ \mathbf{x} \in [0,1]^n : \, x_1 \le \frac{x_2}{2} \le \frac{x_3}{3} \le \cdots \le \frac{x_n}{n} \right\}$

Corteel–Lee–Savage (FPSAC 2005) For any $j \ge 0$ and $1 \le i \le n$

$$\operatorname{ehr}_{\Delta_n}^{\mathbf{1}}(q, jn+i) \, = \, \operatorname{ehr}_{\Delta_n}^{\mathbf{1}}(q, jn+i-1) + q^{jn+i} \operatorname{ehr}_{\Delta_{n-1}}^{\mathbf{1}}(q, j(n-1)+i-1)$$

Familiar Faces

- Lecture hall simplex $\Delta_n := \left\{ \mathbf{x} \in [0,1]^n : \, x_1 \le \frac{x_2}{2} \le \frac{x_3}{3} \le \cdots \le \frac{x_n}{n} \right\}$

Corteel–Lee–Savage (FPSAC 2005) For any $j \ge 0$ and $1 \le i \le n$

$$\mathrm{ehr}^{\mathbf{1}}_{\Delta_n}(q, jn+i) \; = \; \mathrm{ehr}^{\mathbf{1}}_{\Delta_n}(q, jn+i-1) + q^{jn+i} \, \mathrm{ehr}^{\mathbf{1}}_{\Delta_{n-1}}(q, j(n-1)+i-1)$$

Chapoton polynomials, anyone?

$$\mathrm{cha}_{1,0}(x) \; := \; 1 + qx \qquad \text{and} \qquad \mathrm{cha}_{1,1}(x) \; := \; 1 + q + q^2 x$$

and for $j \ge 0$ and $1 \le i \le n$

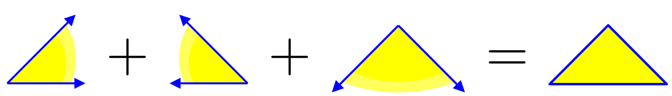
$$\mathrm{cha}_{n,i}(x) \; = \; \mathrm{cha}_{n,i-1}(x) + q^i \big((q-1)x + 1 \big)^n \, \mathrm{cha}_{n-1,i-1}(x)$$

Brion Magic

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

When S is a rational polyhedron, $\sigma_S(\mathbf{z})$ evaluates to a rational function.

Given a vertex $\mathbf{v}$ of P, let $\mathcal{K}_\mathbf{v} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w} - \mathbf{v})$



Theorem (Brion 1988) If $\mathcal{P}$ is a rational polytope, then

$$\sigma_\mathcal{P}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathbf{v}+\mathcal{K}_\mathbf{v}}(\mathbf{z}) \, .$$

Brion $\longrightarrow$ Chapoton

Integer point transform $\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}$

Given a vertex $\mathbf{v}$ of P, let $\mathcal{K}_\mathbf{v} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\ge 0}(\mathbf{w} - \mathbf{v})$

Theorem (Brion 1988) $\sigma_\mathcal{P}(\mathbf{z}) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{\mathbf{v}+\mathcal{K}_\mathbf{v}}(\mathbf{z})\,.$

$$
\begin{aligned}
\mathrm{ehr}_\mathcal{P}^\lambda(q,t) &= \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})} = \sigma_{t\mathcal{P}}\left(q^{\lambda_1}, q^{\lambda_2}, \ldots, q^{\lambda_d}\right) \\
&= \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \sigma_{t\mathbf{v}+\mathcal{K}_\mathbf{v}}\left(q^{\lambda_1}, q^{\lambda_2}, \ldots, q^{\lambda_d}\right) \\
&= \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} q^{t\lambda(\mathbf{v})}\, \sigma_{\mathcal{K}_\mathbf{v}}\left(q^{\lambda_1}, q^{\lambda_2}, \ldots, q^{\lambda_d}\right)
\end{aligned}
$$

Chapoton Polynomials Revisited

Theorem (Chapoton 2015) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form, there exists a polynomial $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \in \mathbb{Z}(q)[x]$ such that $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q)$, where $[t]_q := 1 + q + \cdots + q^{t-1}$.

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} q^{t\lambda(\mathbf{v})} \, \sigma_{\mathcal{K}_{\mathbf{v}}} \left(q^{\lambda_1}, q^{\lambda_2}, \dots, q^{\lambda_d} \right)$$

Now use $q^{kt} \ = \ \left((q-1)[t]_q + 1 \right)^k$ . . .

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q) \left((q-1)x + 1 \right)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) \ := \ \sigma_{\mathcal{K}_{\mathbf{v}}} \left(q^{\lambda_1}, \ q^{\lambda_2}, \ \dots, \ q^{\lambda_d} \right)$

Chapoton Polynomials Revisited

$$\operatorname{ehr}_{\mathcal{P}}^{\lambda}(q,t) \ := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})} \ = \ \operatorname{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} \ := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\operatorname{cha}_{\mathcal{P}}^{\lambda}(q,x) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q)\big((q-1)x+1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) \ := \ \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1}, \ q^{\lambda_2}, \ \ldots, \ q^{\lambda_d}\right)$.

Corollary Each pole of $\rho_{\mathbf{v}}^{\lambda}(q)$ is an nth root of unity where $n = |\lambda(g(\mathbf{w}-\mathbf{v}))|$ for some adjacent vertex $\mathbf{w}$, where $g(\mathbf{w}-\mathbf{v})$ is primitive.

Corollary The leading coefficient of $\operatorname{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is $(q-1)^{\lambda(\mathbf{v})}\rho_{\mathbf{v}}^{\lambda}(q)$ where $\mathbf{v}$ is the vertex of $\mathcal{P}$ that maximizes $\lambda(\mathbf{v})$.

Chapoton Polynomials Revisited

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) \ := \sum_{\mathbf{m}\in t\mathcal{P}} q^{\lambda(\mathbf{m})} \ = \ \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} \ := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\geq 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) \ = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q)\big((q-1)x+1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) \ := \ \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1}, \ q^{\lambda_2}, \ \dots, \ q^{\lambda_d}\right)$.

Chapoton: compute $\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t)$ in the limit as $t \to \infty$ . . .

Corollary

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}\left(q,\frac{1}{1-q}\right) = \begin{cases} \rho_{\mathbf{0}}^{\lambda}(q) & \text{if } \mathbf{0} \text{ is a vertex of } \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Chapoton Polynomials Revisited

$$\mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,t) := \sum_{\mathbf{m} \in t\mathcal{P}} q^{\lambda(\mathbf{m})} = \mathrm{cha}_{\mathcal{P}}^{\lambda}(q,[t]_q) \qquad \mathcal{K}_{\mathbf{v}} := \sum_{\mathbf{w} \text{ adjacent to } \mathbf{v}} \mathbb{R}_{\ge 0}(\mathbf{w}-\mathbf{v})$$

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a lattice polytope and λ is a generic and positive integral form,

$$\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x) = \sum_{\mathbf{v} \text{ vertex of } \mathcal{P}} \rho_{\mathbf{v}}^{\lambda}(q) \big((q-1)x+1\big)^{\lambda(\mathbf{v})}$$

where $\rho_{\mathbf{v}}^{\lambda}(q) := \sigma_{\mathcal{K}_{\mathbf{v}}}\left(q^{\lambda_1},\ q^{\lambda_2},\ \ldots,\ q^{\lambda_d}\right)$.

Corollary The constant term of $\mathrm{cha}_{\mathcal{P}}^{\lambda}(q,x)$ is 1.

Chapoton Quasipolynomials

Theorem (MB–Kunze 2025+) If $\mathcal{P}$ is a rational polytope with denominator p and λ is an integral form that is generic and positive, then there exist polynomials $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x) \in \mathbb{Q}(q)[x]$ such that

$$\mathrm{cha}_{\mathcal{P}}^{\lambda,r}\left(q,[k]_q\right) \,=\, \mathrm{ehr}_{\mathcal{P}}^{\lambda}(q,kp+r)$$

for all integers $k \ge 0$ and all $0 \le r < p$.

The degree of $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x)$ is $\max\{\lambda(p\mathbf{v}) : \mathbf{v} \text{ vertex of } \mathcal{P}\}$. Each pole of a coefficient of $\mathrm{cha}_{\mathcal{P}}^{\lambda,r}(q,x)$ is an nth root of unity where $n = |\lambda(g(p(\mathbf{w}-\mathbf{v})))|$ for some adjacent vertices $\mathbf{v}$ and $\mathbf{w}$.

For any $0 \le r < p$ and $k > 0$

$$(-1)^{\dim \mathcal{P}}\, \mathrm{cha}_{\mathcal{P}}^{\lambda,r}\left(\tfrac{1}{q},[-k]_{\frac{1}{q}}\right) \,=\, \mathrm{ehr}_{\mathcal{P}^\circ}^{\lambda}(q,kp-r)\,.$$

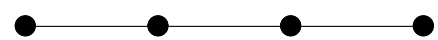
Chromatic Polynomials and Symmetric Functions

$G = (V, E)$ — graph (without loops)

Proper n-coloring — $\kappa : V \to [n] := \{1, 2, \ldots, n\}$ such that $\kappa(i) \neq \kappa(j)$ for any edge $ij \in E$

Chromatic polynomial — $\chi_G(n) := \#$ (proper n-colorings of G)

Example $\chi_{P_4}(n) = n\,(n-1)^3$



Chromatic symmetric function

$$X_G(x_1, x_2, \ldots) \ := \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

We recover $\chi_G(n) = X_G(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots)$

q-Chromatic Polynomials

Chromatic polynomial — $\chi_G(n) := \#\,(\text{proper } n\text{-colorings of } G)$

Chromatic symmetric function

$$X_G(x_1, x_2, \dots) \;:=\; \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

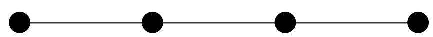
Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \dots, q^n, 0, 0, \dots)$

q-Chromatic Polynomials

Definition $\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)}$ where $\lambda \in \mathbb{Z}_{>0}^V$ is fixed

We recover $\chi_G(n) = \chi_G^{\mathbf{1}}(1,n)$ and $\chi_G^{\mathbf{1}}(q,n) = X_G(q, q^2, \dots, q^n, 0, 0, \dots)$

Example 

$$\chi_{P_4}^{\mathbf{1}}(q,n) = \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times$$
$$\Big(8q^{10}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}(1-q)^{n-3}$$
$$+(4q^9+6q^8+4q^7)(1-q)^{n+1}(1-q)^n(1-q)^{n-1}(1-q)^{n-2}$$
$$+2q^6(1-q)^{n+2}(1-q)^{n+1}(1-q)^n(1-q)^{n-1} \Big)$$

q-Chromatic Polynomial Structure

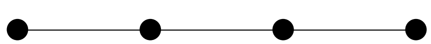
$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings} \\ \kappa:V\to[n]}} q^{\sum_{v\in V} \lambda_v \kappa(v)}$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+) There exists a (unique) polynomial $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Z}(q)[x]$ such that

$$\chi_G^\lambda(q,n) \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q) \qquad \text{where} \qquad [n]_q := 1+q+\cdots+q^{n-1}$$

Example

$$\widetilde{\chi}_{P_4}^{\mathbf{1}}(q,x) \ = \ \frac{1}{(1+q)(1+q+q^2)(1+q+q^2+q^3)} \times \\ \Big(\left(2q^8+4q^7+6q^6+4q^5+8q^4\right) x^4 \\ - \left(6q^8+10q^7+18q^6+18q^5+20q^4\right) x^3 \\ + \left(4q^8+10q^7+20q^6+22q^5+16q^4\right) x^2 \\ - \left(4q^7+8q^6+8q^5+4q^4\right) x \Big)$$



Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

$$\chi_G^{\mathbf{1}}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} q^{\sum_{v \in V} \kappa(v)}$$

$$\chi_G(n) = \#\,(\text{proper } n\text{-colorings of } G)$$

More Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^{\lambda}(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa : V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

More Motivation

$$X_G(x_1, x_2, \ldots) = \sum_{\text{proper colorings } \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots$$

$$\chi_G^\lambda(q, n) = \sum_{\substack{\text{proper colorings} \\ \kappa: V \to [n]}} \left(q^{\lambda_1}\right)^{\kappa(1)} \cdots \left(q^{\lambda_{|V|}}\right)^{\kappa(|V|)}$$

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Remarks $\chi_G^{\mathbf{1}}(q, n)$ was previously studied by Loebl (2007).

$\chi_G^\lambda(q, n)$ is a special evaluation (with polynomial structure) of Crew–Spirkl's (2020) weighted chromatic symmetric function.

q-Chromatic Polynomial Formulas

$$\chi_G^\lambda(q,n) \;:=\; \sum_{\substack{\text{proper colorings}\\ \kappa:V\to[n]}} q^{\sum_{v\in V}\lambda_v\kappa(v)} \;=\; \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Theorem (Bajo–MB–Vindas-Meléndez 2025+)

$$\widetilde{\chi}_G^\lambda(q,x) \;=\; q^{\Lambda_V} \sum_{\text{flats } S\subseteq E} \mu(\varnothing,S) \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

where $P(S)$ denotes the collection of vertex sets of the connected components induced by S and $\Lambda_W := \sum_{v\in W}\lambda_v$. In particular, for a tree

$$\widetilde{\chi}_T^\lambda(q,x) \;=\; q^{\Lambda_V} \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}$$

$\longrightarrow$ highly-structured formulas for paths, stars, . . .

The Leading Coefficient for Trees

$$\chi_G^\lambda(q,n) \ := \sum_{\substack{\text{proper colorings}\\ \kappa: V \to [n]}} q^{\sum_{v \in V} \lambda_v \kappa(v)} \ = \ \widetilde{\chi}_G^\lambda(q,[n]_q)$$

Corollary Given a tree T, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals

$$\begin{aligned} c_T^{\mathbf{1}}(q) &= (q-q^2)^d \sum_{S \subseteq E} \prod_{C \in P(S)} \frac{1}{1-q^{\Lambda_C}} \\ &= \frac{1}{[d]_q!} \sum_{(\rho,\sigma)} q^{d+\operatorname{maj}\sigma} \qquad\qquad d := |V| \end{aligned}$$

where the sum ranges over all pairs of acyclic orientations ρ of T and linear extensions σ of the poset induced by ρ

Corollary $c_T^{\mathbf{1}}(q) \ = \ (-q)^d \, X_T\left(\frac{1}{q}, \frac{1}{q^2}, \frac{1}{q^3}, \dots\right)$

Stanley's Tree Conjecture Revisited

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Theorem (MB–Braun–Cornejo 2026+) Fix $k \geq d$ and $\lambda_j := k^j$. Then $\widetilde{\chi}_G^{\lambda}(q, x)$ distinguishes graphs on d nodes.

Stanley's Tree Conjecture Revisited

Conjecture (Stanley 1995) $X_G(x_1, x_2, \ldots)$ distinguishes trees.

Conjecture (Loehr–Warrington 2024) $X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) = \chi_G^{\mathbf{1}}(q, n)$ distinguishes trees.

Conjecture (Bajo–MB–Vindas-Meléndez 2025+) The leading coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$ distinguishes trees.

Theorem (MB–Braun–Cornejo 2026+) Fix $k \ge d$ and $\lambda_j := k^j$. Then $\widetilde{\chi}_G^{\lambda}(q, x)$ distinguishes graphs on d nodes.

- Play with different polynomial bases
- G-partitions
- Other coefficients of $\widetilde{\chi}_G^{\mathbf{1}}(q, x)$?