MATH 725 Homework Set 7 due 10/10/11

(1) Consider the vector space $\mathscr{P}(\mathbf{C})$ with the inner product $\langle f,g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}\,dx$.
  (a) Show that this defines an inner-product space.
  (b) Compute the norm of x^n, where n is a nonnegative integer.
  (c) Compute an orthonormal basis for $\mathscr{P}_2(\mathbf{C})$.

Proof. (a) First, $\langle f,f\rangle = \int_{-1}^{1} f(x)\overline{f(x)}\,dx = \int_{-1}^{1} |f(x)|^2\,dx$, and so this real integral over a nonnegative function is ≥ 0 and equals 0 if and only if the integrand is the zero function (which is equivalent to f being the zero function).
Second, $\langle af_1+f_2,g\rangle = \int_{-1}^{1} (af_1(x)+f_2(x))\,\overline{g(x)}\,dx = a\int_{-1}^{1} f_1\overline{g(x)}\,dx + \int_{-1}^{1} f_2\overline{g(x)}\,dx$.
Third, $\langle f,g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx = \int_{-1}^{1} \overline{\overline{f(x)}g(x)}\,dx = \overline{\int_{-1}^{1} \overline{f(x)}g(x)\,dx} = \overline{\langle g,f\rangle}$.
(b)

$$||x^n|| = \sqrt{\langle x^n,x^n\rangle} = \sqrt{\int_{-1}^{1} |x^n|^2\,dx} = \sqrt{\int_{-1}^{1} |x|^{2n}dx} = \sqrt{2\int_{0}^{1} x^{2n}dx} = \sqrt{\frac{2}{2n+1}}\,.$$

(c) We apply Gram–Schmidt to the basis $\left(1,x,x^2\right)$ of $\mathscr{P}_2(\mathbf{C})$. From part (b) we know $||1|| = \sqrt{2}$, so the first basis vector (polynomial) is $\mathbf{e}_1 = \frac{1}{\sqrt{2}}$. To compute $\mathbf{e}_2$, we calculate

$$x - \langle x,\mathbf{e}_1\rangle\,\mathbf{e}_1 = x - \frac{1}{2}\int_{-1}^{1} x\,dx = x$$

and so (using part (a)) $\mathbf{e}_2 = \frac{x}{||x||} = \sqrt{\frac{3}{2}}\,x$. To compute $\mathbf{e}_3$,

$$x^2 - \left\langle x^2,\mathbf{e}_1\right\rangle\mathbf{e}_1 - \left\langle x^2,\mathbf{e}_2\right\rangle\mathbf{e}_2 = x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,dx - \frac{3}{2}x\int_{-1}^{1} x^3\,dx = x^2 - \frac{1}{3}$$

and so $\mathbf{e}_3 = \frac{x^2-\frac{1}{3}}{\left|\left|x^2-\frac{1}{3}\right|\right|} = \sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)$.
Remark: We have just computed the first three *Legendre polynomials*. □

(2) Suppose that V is an inner-product space over $\mathbf{F}$, and $\mathbf{v},\mathbf{w}\in V$.
  (a) If $\mathbf{F}=\mathbf{R}$, show that $\langle \mathbf{v},\mathbf{w}\rangle = \frac{1}{4}\left(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2\right)$.
  (b) If $\mathbf{F}=\mathbf{C}$, show that $\langle \mathbf{v},\mathbf{w}\rangle = \frac{1}{4}\left(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i\,||\mathbf{v}+i\,\mathbf{w}||^2 - i\,||\mathbf{v}-i\,\mathbf{w}||^2\right)$.

Proof. (a) Computing

$$||\mathbf{v}+\mathbf{w}||^2 = \langle \mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{w}\rangle + \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle$$
$$||\mathbf{v}-\mathbf{w}||^2 = \langle \mathbf{v}-\mathbf{w},\mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}\rangle - \langle \mathbf{v},\mathbf{w}\rangle - \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle$$

yields

$$||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 = 2\,\langle \mathbf{v},\mathbf{w}\rangle + 2\,\langle \mathbf{w},\mathbf{v}\rangle = 4\,\langle \mathbf{v},\mathbf{w}\rangle\,,$$

where the last equation follows from $\langle \mathbf{w},\mathbf{v}\rangle = \overline{\langle \mathbf{v},\mathbf{w}\rangle} = \langle \mathbf{v},\mathbf{w}\rangle$ since everything is in $\mathbf{R}$.
(b) Combining the first two equations in (a) with

$$||\mathbf{v}+i\,\mathbf{w}||^2 = \langle \mathbf{v}+i\,\mathbf{w},\mathbf{v}+i\,\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}\rangle - i\,\langle \mathbf{v},\mathbf{w}\rangle + i\,\langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle$$
$$||\mathbf{v}-i\,\mathbf{w}||^2 = \langle \mathbf{v}-i\,\mathbf{w},\mathbf{v}-i\,\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}\rangle + i\,\langle \mathbf{v},\mathbf{w}\rangle - i\,\langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle$$

yields

$$||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2 + i\,||\mathbf{v}+i\,\mathbf{w}||^2 - i\,||\mathbf{v}-i\,\mathbf{w}||^2 = 4\,\langle \mathbf{v},\mathbf{w}\rangle\,.$$ □

(3) Suppose that V is an inner-product space over $\mathbf{F}$, and $\mathbf{v},\mathbf{w}\in V$. Prove that $\langle \mathbf{v},\mathbf{w}\rangle = 0$ if and only if $||\mathbf{v}|| \leq ||\mathbf{v}+a\,\mathbf{w}||$ for all $a\in\mathbf{F}$.

Proof. Since norms are nonnegative, $||\mathbf{v}|| \leq ||\mathbf{v}+a\,\mathbf{w}||$ is equivalent to $||\mathbf{v}||^2 \leq ||\mathbf{v}+a\,\mathbf{w}||^2$.

Assume $\langle \mathbf{v},\mathbf{w}\rangle = 0$, and let $a \in \mathbf{F}$. Then $\mathbf{v}$ and $a\,\mathbf{w}$ are orthogonal, and so by the Pythagorean theorem,

$$||\mathbf{v}+a\,\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||a\,\mathbf{w}||^2 \geq ||\mathbf{v}||^2 .$$

Conversely, assume $||\mathbf{v}||^2 \leq ||\mathbf{v}+a\,\mathbf{w}||^2$ for all $a \in \mathbf{F}$. Then

$$\begin{aligned} ||\mathbf{v}||^2 &\leq \langle \mathbf{v}+a\,\mathbf{w}, \mathbf{v}+a\,\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{v}, a\,\mathbf{w}\rangle + \langle a\,\mathbf{w},\mathbf{v}\rangle + \langle a\,\mathbf{w}, a\,\mathbf{w}\rangle \\ &= ||\mathbf{v}||^2 + \overline{a}\,\langle \mathbf{v},\mathbf{w}\rangle + a\overline{\langle \mathbf{v},\mathbf{w}\rangle} + |a|^2\,||\mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\,\mathrm{Re}\,(\overline{a}\,\langle \mathbf{v},\mathbf{w}\rangle) + |a|^2\,||\mathbf{w}||^2 , \end{aligned}$$

i.e.,

$$0 \leq 2\,\mathrm{Re}\,(\overline{a}\,\langle \mathbf{v},\mathbf{w}\rangle) + |a|^2\,||\mathbf{w}||^2 .$$

If $\mathbf{w} = \mathbf{0}$ then $\langle \mathbf{v},\mathbf{w}\rangle = 0$ (as desired), so we may assume that $\mathbf{w} \neq \mathbf{0}$. Then choosing $a = -\frac{\langle \mathbf{v},\mathbf{w}\rangle}{||\mathbf{w}||^2}$ yields

$$\begin{aligned} 0 &\leq 2\,\mathrm{Re}\left(-\frac{\overline{\langle \mathbf{v},\mathbf{w}\rangle}}{||\mathbf{w}||^2}\,\langle \mathbf{v},\mathbf{w}\rangle\right) + \left|\frac{\langle \mathbf{v},\mathbf{w}\rangle}{||\mathbf{w}||^2}\right|^2 ||\mathbf{w}||^2 = 2\,\mathrm{Re}\left(-\frac{|\langle \mathbf{v},\mathbf{w}\rangle|^2}{||\mathbf{w}||^2}\right) + \frac{|\langle \mathbf{v},\mathbf{w}\rangle|^2}{||\mathbf{w}||^2} \\ &= -2\frac{|\langle \mathbf{v},\mathbf{w}\rangle|^2}{||\mathbf{w}||^2} + \frac{|\langle \mathbf{v},\mathbf{w}\rangle|^2}{||\mathbf{w}||^2} \end{aligned}$$

which is equivalent to

$$2\,|\langle \mathbf{v},\mathbf{w}\rangle|^2 \leq |\langle \mathbf{v},\mathbf{w}\rangle|^2$$

which in turn can only hold if $\langle \mathbf{v},\mathbf{w}\rangle = 0$. □

(4) Let V be a real[1] vector space. A *norm* on V is a function $||\ \ || : V \to \mathbf{R}_{\geq 0}$ with the following properties:
   (i) $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$;
   (ii) for all $a \in \mathbf{F}$ and all $\mathbf{v} \in V$, $||a\,\mathbf{v}|| = |a|\,||\mathbf{v}||$;
   (iii) for all $\mathbf{v},\mathbf{w} \in V$, $||\mathbf{v}+\mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.

   Prove that if a $||\ \ ||$ satisfies the "parallelogram equality"

$$||\mathbf{v}+\mathbf{w}||^2 + ||\mathbf{v}-\mathbf{w}||^2 = 2\left(||\mathbf{v}||^2 + ||\mathbf{w}||^2\right),$$

   then it comes from an inner product, i.e., there is an inner product $\langle\ ,\ \rangle$ on V such that for all $||\mathbf{v}|| = \sqrt{\langle \mathbf{v},\mathbf{v}\rangle}$ for all $\mathbf{v} \in V$.

Proof. Suppose $||\ \ ||$ is a norm on V that satisfies the "parallelogram equality." Inspired by (2), we define

$$\langle \mathbf{v},\mathbf{w}\rangle := \tfrac{1}{4}\left(||\mathbf{v}+\mathbf{w}||^2 - ||\mathbf{v}-\mathbf{w}||^2\right).$$

We will prove that this is an inner product on V. First,

$$\langle \mathbf{v},\mathbf{v}\rangle = \tfrac{1}{4}\,||2\mathbf{v}||^2 = ||\mathbf{v}||$$

which is ≥ 0 with equality if and only if $\mathbf{v} = \mathbf{0}$. Second,

$$\begin{aligned} \langle a\,\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}\rangle &= \tfrac{1}{4}\left(||a\,\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{w}||^2 - ||a\,\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{w}||^2\right) \\ a\,\langle \mathbf{v}_1,\mathbf{w}\rangle &= \tfrac{1}{4}\left(||a\,\mathbf{v}_1 + \mathbf{w}||^2 - ||a\,\mathbf{v}_1 - \mathbf{w}||^2\right) \\ \langle \mathbf{v}_2,\mathbf{w}\rangle &= \tfrac{1}{4}\left(||\mathbf{v}_2 + \mathbf{w}||^2 - ||\mathbf{v}_2 - \mathbf{w}||^2\right) \end{aligned}$$

and so to prove that $\langle a\,\mathbf{v}_1 + \mathbf{v}_2,\mathbf{w}\rangle = a\,\langle \mathbf{v}_1,\mathbf{w}\rangle + \langle \mathbf{v}_2,\mathbf{w}\rangle$, we need to show that

$$||a\,\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{w}||^2 - ||a\,\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{w}||^2 = ||a\,\mathbf{v}_1 + \mathbf{w}||^2 - ||a\,\mathbf{v}_1 - \mathbf{w}||^2 + ||\mathbf{v}_2 + \mathbf{w}||^2 - ||\mathbf{v}_2 - \mathbf{w}||^2 ,$$

[1] The statement of this exercise also holds for *complex* vector spaces, with a similar but more involved proof.

i.e., that

$$||a\mathbf{v}_1+\mathbf{v}_2+\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2-\mathbf{w}||^2+\Big(||a\mathbf{v}_1-\mathbf{w}||^2+||\mathbf{v}_2-\mathbf{w}||^2\Big)-\Big(||a\mathbf{v}_1+\mathbf{w}||^2+||\mathbf{v}_2+\mathbf{w}||^2\Big)=0\,.$$

To prove this, we will use the "parallelogram equality" (read from right to left) for the expressions in the two pairs of parantheses, which gives

$$\begin{aligned}
&||a\mathbf{v}_1+\mathbf{v}_2+\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2-\mathbf{w}||^2+\Big(||a\mathbf{v}_1-\mathbf{w}||^2+||\mathbf{v}_2-\mathbf{w}||^2\Big)-\Big(||a\mathbf{v}_1+\mathbf{w}||^2+||\mathbf{v}_2+\mathbf{w}||^2\Big)\\
&\quad=||a\mathbf{v}_1+\mathbf{v}_2+\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2-\mathbf{w}||^2+\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2-2\mathbf{w}||^2+||a\mathbf{v}_1-\mathbf{v}_2||^2\Big)\\
&\qquad\quad-\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2+2\mathbf{w}||^2+||a\mathbf{v}_1-\mathbf{v}_2||^2\Big)\\
&\quad=||a\mathbf{v}_1+\mathbf{v}_2+\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2-\mathbf{w}||^2+\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2-2\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2+2\mathbf{w}||^2\Big)\\
&\quad=\Big(||a\mathbf{v}_1+\mathbf{v}_2+\mathbf{w}||^2+||\mathbf{w}||^2\Big)-\Big(||a\mathbf{v}_1+\mathbf{v}_2-\mathbf{w}||^2+||\mathbf{w}||^2\Big)\\
&\qquad\quad+\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2-2\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2+2\mathbf{w}||^2\Big)\\
&\quad=\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2+2\mathbf{w}||^2+||a\mathbf{v}_1+\mathbf{v}_2||^2\Big)-\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2||^2+||a\mathbf{v}_1+\mathbf{v}_2-2\mathbf{w}||^2\Big)\\
&\qquad\quad+\tfrac{1}{2}\Big(||a\mathbf{v}_1+\mathbf{v}_2-2\mathbf{w}||^2-||a\mathbf{v}_1+\mathbf{v}_2+2\mathbf{w}||^2\Big)\\
&\quad=0\,.
\end{aligned}$$

(In the penultimate equation, we have used the "parallelogram equality" once more, again read from right to left.) Third,

$$\langle\mathbf{v},\mathbf{w}\rangle=\tfrac{1}{4}\Big(||\mathbf{v}+\mathbf{w}||^2-||\mathbf{v}-\mathbf{w}||^2\Big)=\tfrac{1}{4}\Big(||\mathbf{w}+\mathbf{v}||^2-||\mathbf{w}-\mathbf{v}||^2\Big)=\langle\mathbf{w},\mathbf{v}\rangle\,,$$

and this finishes our proof that $\langle\mathbf{v},\mathbf{w}\rangle$ is an inner product. □

(5) Let M be an real $n\times n$ matrix. Show that the space spanned by the rows of M is the orthogonal complement of $\mathrm{null}(M)$.

Proof. Suppose the rows of M are $\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_n$. Let $\mathbf{v}\in\mathrm{span}\,(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_n)$, i.e., $\mathbf{v}=\sum_{j=1}^n a_j\mathbf{r}_j$ for some $a_1,a_2,\dots,a_n$, and let $\mathbf{w}\in\mathrm{null}(M)$, i.e., $M\mathbf{w}=\mathbf{0}$, i.e., $\langle\mathbf{r}_j,\mathbf{w}\rangle=0$ for all j. Then

$$\langle\mathbf{v},\mathbf{w}\rangle=\left\langle\sum_{j=1}^n a_j\mathbf{r}_j,\mathbf{w}\right\rangle=\sum_{j=1}^n a_j\left\langle\mathbf{r}_j,\mathbf{w}\right\rangle=0\,.$$ □