

# The Frobenius Problem, Rational Polytopes, and Fourier-Dedekind Sums<sup>1</sup>

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We study the number of lattice points in integer dilates of the rational polytope

$$\mathcal{P} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n : \sum_{k=1}^n x_k a_k \leq 1 \right\},$$

where  $a_1, \dots, a_n$  are positive integers. This polytope is closely related to the *linear Diophantine problem of Frobenius*: given relatively prime positive integers  $a_1, \dots, a_n$ , find the largest value of  $t$  (the *Frobenius number*) such that  $m_1 a_1 + \dots + m_n a_n = t$  has no solution in positive integers  $m_1, \dots, m_n$ . This is equivalent to the problem of finding the largest dilate  $t\mathcal{P}$  such that the facet  $\{\sum_{k=1}^n x_k a_k = t\}$  contains no lattice point. We present two methods for computing the Ehrhart quasipolynomials  $L(\mathcal{P}, t) := \#(t\mathcal{P} \cap \mathbb{Z}^n)$  and  $L(\mathcal{P}^\circ, t) := \#(t\mathcal{P}^\circ \cap \mathbb{Z}^n)$ . Within the computations a Dedekind-like finite Fourier sum appears. We obtain a reciprocity law for these sums, generalizing a theorem of

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Gessel. As a corollary of our formulas, we rederive the reciprocity law for Zagier's higher-dimensional Dedekind sums. Finally, we find bounds for the Fourier-Dedekind sums and use them to give new bounds for the Frobenius number.

*Key Words:* rational polytopes, lattice points, the linear diophantine problem of Frobenius, Ehrhart quasipolynomial, Dedekind sums

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## 1. INTRODUCTION

Let  $a_1, \dots, a_n$  be positive integers,  $\mathbb{Z}^n \subset \mathbb{R}^n$  be the  $n$ -dimensional integer lattice, and

$$\mathcal{P} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_k \geq 0, \sum_{k=1}^n a_k x_k \leq 1, \right\}, \quad (1)$$

a rational polytope with vertices

$$(0, \dots, 0), \left( \frac{1}{a_1}, 0, \dots, 0 \right), \left( 0, \frac{1}{a_2}, 0, \dots, 0 \right), \dots, \left( 0, \dots, 0, \frac{1}{a_n} \right).$$

For a positive integer  $t \in \mathbb{N}$ , let  $L(\overline{\mathcal{P}}, t)$  be the number of lattice points in the dilated polytope  $t\mathcal{P} = \{tx : x \in \mathcal{P}\}$ . Denote further the relative interior of  $\mathcal{P}$  by  $\mathcal{P}^\circ$  and the number of lattice points in  $t\mathcal{P}^\circ$  by  $L(\mathcal{P}^\circ, t)$ . Then  $L(\mathcal{P}^\circ, t)$  and  $L(\overline{\mathcal{P}}, t)$  are quasipolynomials in  $t$  of degree  $n$  [11], i.e. expressions

$$c_n(t) t^n + \dots + c_1(t) t + c_0(t),$$

where  $c_0, \dots, c_n$  are periodic functions in  $t$ . In fact, if the  $a_k$ 's are pairwise relatively prime then  $c_1, \dots, c_n$  are constants, so only  $c_0$  will show this periodic dependency on  $t$ .

Let  $A = \{a_1, \dots, a_n\}$  be a set of relatively prime positive integers, and

$$p'_A(t) = \# \left\{ (m_1, \dots, m_n) \in \mathbb{N}^n : \sum_{k=1}^n m_k a_k = t \right\}. \quad (2)$$

The function  $p'_A(t)$  can be described as the number of *restricted partitions of  $t$  with parts in  $A$* , where we require that each part is used at least once. (We reserve the name  $p_A$  for the enumeration function of those partitions which do not have this restriction.) Geometrically,  $p'_A(t)$  enumerates the lattice points on the skewed facet of  $\mathcal{P}$ . Define  $f(a_1, \dots, a_n)$  to be the largest value of  $t$  for which

$$p'_A(t) = 0.$$

In the 19th century, Frobenius inaugurated the study of  $f(a_1, \dots, a_n)$ . For  $n = 2$ , it is known (probably at least since Sylvester [28]) that  $f(a_1, a_2) = a_1 a_2$ . For  $n > 2$ , all attempts for explicit formulas have proved elusive.

Here we focus on the study of  $p'_A(t)$ , and show that it has an explicit representation as a quasipolynomial. Through the discussion of  $p'_A(t)$ , we gain new insights into Frobenius's problem.

Another motivation to study  $p'_A(t)$  is the following trivial reduction formula to lower dimensions:

$$p'_{\{a_1, \dots, a_n\}}(t) = \sum_{m \geq 0} p'_{\{a_1, \dots, a_{n-1}\}}(t - ma_n). \quad (3)$$

Here we use the convention that  $p'_A(t) = 0$  if  $t \leq 0$ . This identity can be easily verified by viewing  $p'_A(t)$  as

$$p'_A(t) = \# \left\{ (m_1, \dots, m_n) \in \mathbb{N}^n : \sum_{k=1}^{n-1} m_k a_k = t - m_n a_n \right\}.$$

Hence, precise knowledge of the values of  $t$  for which  $p'_A(t) = 0$  in lower dimensions sheds additional light on the Frobenius number in higher dimensions.

The number  $p'_A(t)$  appears in the lattice point count of  $\mathcal{P}$ . It is for this reason that we decided to focus on this particular rational polytope. We present two methods (Sections 2 and 3) for computing the terms appearing in  $L(\mathcal{P}^\circ, t)$  and  $L(\overline{\mathcal{P}}, t)$ . Both methods are refinements of concepts that were earlier introduced by the authors [2, 9]. In contrast to the mostly algebraic-geometric and topological ways of computing  $L(\mathcal{P}^\circ, t)$  and  $L(\overline{\mathcal{P}}, t)$  [1, 6, 7, 14, 17, 18], our methods are analytic. In passing, we recover the Ehrhart-Macdonald reciprocity law relating  $L(\mathcal{P}^\circ, t)$  and  $L(\overline{\mathcal{P}}, t)$  [11, 20]. Within the computations a Dedekind-like finite Fourier sum appears, which shares some properties with its classical siblings, discussed in Section 4. In particular, we prove two reciprocity laws for these sums: a rederivation of the reciprocity law for Zagier's higher-dimensional Dedekind sums [30], and a new reciprocity law that generalizes a theorem of Gessel [13]. Finally, in Section 5 we give bounds on these generalized Dedekind sums and apply our results to give new bounds for the Frobenius number. The literature on such bounds is vast—see, for example, [4, 8, 12, 16, 25, 26, 27, 29].

## 2. THE RESIDUE METHOD

In [2], the first author used the residue theorem to count lattice points in a lattice polytope, that is, a polytope with integer vertices. Here we extend these methods to the case of *rational* vertices.

We are interested in the number of lattice points in the tetrahedron  $\mathcal{P}$  defined by (1) and integral dilates of it. We can interpret

$$L(\overline{\mathcal{P}}, t) = \# \left\{ (m_1, \dots, m_n) \in \mathbb{Z}^n : m_k \geq 0, \sum_{k=1}^n m_k a_k \leq t \right\},$$

as the Taylor coefficient of  $z^t$  of the function

$$\begin{aligned} & (1 + z^{a_1} + z^{2a_1} + \dots) \cdots (1 + z^{a_n} + z^{2a_n} + \dots) (1 + z + z^2 + \dots) \\ &= \frac{1}{1 - z^{a_1}} \cdots \frac{1}{1 - z^{a_n}} \frac{1}{1 - z} . \end{aligned}$$

Equivalently

$$L(\overline{\mathcal{P}}, t) = \text{Res} \left( \frac{z^{-t-1}}{(1 - z^{a_1}) \cdots (1 - z^{a_n}) (1 - z)}, z = 0 \right) . \quad (4)$$

If this expression counts the number of lattice points in  $t\overline{\mathcal{P}}$ , then the remaining task is to compute the other residues of

$$F_{-t}(z) := \frac{z^{-t-1}}{(1 - z^{a_1}) \cdots (1 - z^{a_n}) (1 - z)} ,$$

and use the residue theorem for the sphere  $\mathbb{C} \cup \{\infty\}$ .  $F_{-t}$  has poles at 0 and all  $a_1^{\text{th}}, \dots, a_n^{\text{th}}$  roots of unity. It is particularly easy to get precise formulas if the poles at the nontrivial roots of unity are simple. For this reason, assume in the following that  $a_1, \dots, a_n$  are *pairwise relatively prime*. Then the residues for the  $a_1^{\text{th}}, \dots, a_n^{\text{th}}$  roots of unity are not hard to compute: Let  $\lambda^{a_1} = 1 \neq \lambda$ , then

$$\begin{aligned} & \text{Res}(F_{-t}(z), z = \lambda) = \\ &= \frac{\lambda^{-t-1}}{(1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n}) (1 - \lambda)} \text{Res} \left( \frac{1}{1 - z^{a_1}}, z = \lambda \right) \\ &= \frac{\lambda^{-t-1}}{(1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n}) (1 - \lambda)} \lim_{z \rightarrow \lambda} \frac{z - \lambda}{1 - z^{a_1}} \\ &= - \frac{\lambda^{-t}}{a_1 (1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n}) (1 - \lambda)} . \end{aligned}$$

If we add up all the nontrivial  $a_1^{\text{th}}$  roots of unity, we obtain

$$\begin{aligned} & \sum_{\lambda^{a_1}=1 \neq \lambda} \text{Res}(F_{-t}(z), z = \lambda) = \\ &= \frac{-1}{a_1} \sum_{\lambda^{a_1}=1 \neq \lambda} \frac{\lambda^{-t}}{(1 - \lambda^{a_2}) \cdots (1 - \lambda^{a_n}) (1 - \lambda)} \\ &= \frac{-1}{a_1} \sum_{k=1}^{a_1-1} \frac{\xi^{-kt}}{(1 - \xi^{ka_2}) \cdots (1 - \xi^{ka_n}) (1 - \xi^k)} , \end{aligned}$$

where  $\xi$  is a primitive  $a_1^{\text{th}}$  root of unity. This motivates the following

DEFINITION 1. Let  $c_1, \dots, c_n \in \mathbb{Z}$  be relatively prime to  $c \in \mathbb{Z}$ , and  $t \in \mathbb{Z}$ . Define the **Fourier-Dedekind sum** as

$$\sigma_t(c_1, \dots, c_n; c) = \frac{1}{c} \sum_{\lambda^c=1 \neq \lambda} \frac{\lambda^t}{(\lambda^{c_1}-1) \cdots (\lambda^{c_n}-1)}.$$

Some properties of  $\sigma_t$  are discussed in Section 4. With this notation, we can now write

$$\sum_{\lambda^{a_1}=1 \neq \lambda} \text{Res}(F_{-t}(z), z = \lambda) = (-1)^{n+1} \sigma_{-t}(a_2, \dots, a_n, 1; a_1).$$

We get similar residues for the  $a_2^{\text{th}}, \dots, a_n^{\text{th}}$  roots of unity. Finally, note that  $\text{Res}(F_{-t}, z = \infty) = 0$ , so that the residue theorem allows us to rewrite (4):

THEOREM 1. *Let  $\mathcal{P}$  be given by (1), with  $a_1, \dots, a_n$  pairwise relatively prime. Then*

$$L(\overline{\mathcal{P}}, t) = R_{-t}(a_1, \dots, a_n) + (-1)^n \sum_{j=1}^n \sigma_{-t}(a_1, \dots, \hat{a}_j, \dots, a_n, 1; a_j)$$

where  $R_{-t}(a_1, \dots, a_n) = -\text{Res}(F_{-t}(z), z = 1)$ , and  $\hat{a}_j$  means we omit the term  $a_j$ .

Remarks. 1.  $R_{-t}$  can be easily calculated via

$$\begin{aligned} \text{Res}(F_{-t}(z), z = 1) &= \text{Res}(e^z F_{-t}(e^z), z = 0) \\ &= \text{Res}\left(\frac{e^{-tz}}{(1 - e^{a_1 z}) \cdots (1 - e^{a_n z})(1 - e^z)}, z = 0\right). \end{aligned}$$

To facilitate the computation in higher dimensions, one can use mathematics software such as **Maple** or **Mathematica**. It is easy to see that  $R_{-t}(a_1, \dots, a_n)$  is a polynomial in  $t$  whose coefficients are rational expressions in  $a_1, \dots, a_n$ . The first values for  $R_{-t}$  are

$$\begin{aligned} R_{-t}(a) &= \frac{t}{a} + \frac{1}{2a} + \frac{1}{2} \\ R_{-t}(a, b) &= \frac{t^2}{2ab} + \frac{t}{2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \right) + \frac{1}{4} \left( 1 + \frac{1}{a} + \frac{1}{b} \right) + \\ &\quad + \frac{1}{12} \left( \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) \\ R_{-t}(a, b, c) &= \frac{t^3}{6abc} + \frac{t^2}{4} \left( \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{abc} \right) + \\ &\quad + \frac{t}{12} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{3}{ab} + \frac{3}{ac} + \frac{3}{bc} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} + \frac{1}{abc} \right) + \\ &\quad + \frac{1}{24} \left( 3 + \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right). \end{aligned}$$

2. If  $a_1, \dots, a_n$  are *not* pairwise relatively prime, we can get similar formulas for  $L(\overline{\mathcal{P}}, t)$ . In this case we do not have only simple poles, so that the computation of the residues gets slightly more complicated.

For the computation of  $L(\mathcal{P}^\circ, t)$  (the number of lattice points in the *interior* of our tetrahedron  $t\mathcal{P}$ ), we similarly write

$$L(\mathcal{P}^\circ, t) = \# \left\{ (m_1, \dots, m_n) \in \mathbb{Z}^n : m_k > 0, \sum_{k=1}^n m_k a_k < t \right\} .$$

So now we can interpret  $L(\mathcal{P}^\circ, t)$  as the Taylor coefficient of  $z^t$  of the function

$$\begin{aligned} & (z^{a_1} + z^{2a_1} + \dots) \cdots (z^{a_n} + z^{2a_n} + \dots) (z + z^2 + \dots) \\ &= \frac{z^{a_1}}{1 - z^{a_1}} \cdots \frac{z^{a_n}}{1 - z^{a_n}} \frac{z}{1 - z} , \end{aligned}$$

or equivalently as

$$\begin{aligned} & \text{Res} \left( \frac{z^{a_1}}{1 - z^{a_1}} \cdots \frac{z^{a_n}}{1 - z^{a_n}} \frac{z}{1 - z} z^{-t-1}, z = 0 \right) \\ &= \text{Res} \left( \frac{-1}{z^2} \frac{1}{z^{a_1} - 1} \cdots \frac{1}{z^{a_n} - 1} \frac{1}{z - 1} z^{t+1}, z = \infty \right) . \end{aligned}$$

To be able to use the residue theorem, this time we have to consider the function

$$-\frac{1}{z^{a_1} - 1} \cdots \frac{1}{z^{a_n} - 1} \frac{1}{z - 1} z^{t-1} = (-1)^n F_t(z)$$

The residues at the finite poles of  $F_t$  can be computed as before, with  $t$  replaced by  $-t$ , and the proof of the following theorem is completely analogous to Theorem 1:

**THEOREM 2.** *Let  $\mathcal{P}$  be given by (1), with  $a_1, \dots, a_n$  pairwise relatively prime. Then*

$$L(\mathcal{P}^\circ, t) = (-1)^n R_t(a_1, \dots, a_n) + \sum_{j=1}^n \sigma_t(a_1, \dots, \hat{a}_j, \dots, a_n, 1; a_j)$$

As an immediate consequence we get the remarkable

**COROLLARY 1** (Ehrhart-Macdonald Reciprocity Law).

$$L(\mathcal{P}^\circ, -t) = (-1)^n L(\overline{\mathcal{P}}, t) .$$

This result was conjectured for convex rational polytopes by Ehrhart [11], and first proved by Macdonald [20].

Of particular interest is the number of lattice points on the boundary of  $t\mathcal{P}$ . Besides computing  $L(\mathcal{P}^\circ, t)$  and  $L(\overline{\mathcal{P}}, t)$  and taking differences, we

can also adjust our method to this situation, especially if we are interested in only *parts* of the boundary. As an example, we will compute  $p'_A(t)$  as defined in the introduction (2), which appears in the context of the Frobenius problem. Again, for reasons of simplicity we assume in the following that  $a_1, \dots, a_n$  are *pairwise coprime* positive integers.

This time we interpret

$$p'_A(t) = \# \left\{ (m_1, \dots, m_n) \in \mathbb{N}^n : \sum_{k=1}^n m_k a_k = t \right\}$$

as the Taylor coefficient of  $z^t$  of the function

$$\begin{aligned} & (z^{a_1} + z^{2a_1} + \dots) \dots (z^{a_n} + z^{2a_n} + \dots) \\ &= \frac{z^{a_1}}{1 - z^{a_1}} \dots \frac{z^{a_n}}{1 - z^{a_n}}. \end{aligned}$$

That is,

$$\begin{aligned} p'_A(t) &= \text{Res} \left( \frac{z^{a_1}}{1 - z^{a_1}} \dots \frac{z^{a_n}}{1 - z^{a_n}} z^{-t-1}, z = 0 \right) \\ &= \text{Res} \left( \frac{-1}{z^2} \frac{1}{z^{a_1} - 1} \dots \frac{1}{z^{a_n} - 1} z^{t+1}, z = \infty \right). \end{aligned}$$

Thus, we have to find the other residues of

$$G_t(z) := \frac{z^{t-1}}{(z^{a_1} - 1) \dots (z^{a_n} - 1)} = (z - 1) F_t(z),$$

since

$$p'_A(t) = - \text{Res} (G_t(z), z = \infty). \quad (5)$$

$G_t$  has its other poles at all  $a_1^{\text{th}}, \dots, a_n^{\text{th}}$  roots of unity. Again, note that  $G_t$  has *simple* poles at all the nontrivial roots of unity. Let  $\lambda$  be a nontrivial  $a_1^{\text{th}}$  root of unity, then

$$\begin{aligned} \text{Res} (G_t(z), z = \lambda) &= \frac{\lambda^{t-1}}{(\lambda^{a_2} - 1) \dots (\lambda^{a_n} - 1)} \text{Res} \left( \frac{1}{z^{a_1} - 1}, z = \lambda \right) \\ &= \frac{\lambda^t}{a_1 (\lambda^{a_2} - 1) \dots (\lambda^{a_n} - 1)}. \end{aligned}$$

Adding up all the nontrivial  $a_1^{\text{th}}$  roots of unity, we obtain

$$\begin{aligned} \sum_{\lambda^{a_1}=1 \neq \lambda} \text{Res} (G_t(z), z = \lambda) &= \frac{1}{a_1} \sum_{\lambda^{a_1}=1 \neq \lambda} \frac{\lambda^t}{(\lambda^{a_2} - 1) \dots (\lambda^{a_n} - 1)} \\ &= \sigma_t(a_2, \dots, a_n; a_1). \end{aligned}$$

Together with the similar residues at the other roots of unity, (5) gives us

THEOREM 3.

$$p'_A(t) = R'_t(a_1, \dots, a_n) + \sum_{j=1}^n \sigma_t(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) ,$$

where  $R'_t(a_1, \dots, a_n) = \text{Res}(G_t(z), z = 1)$ .

$R'$  is as easily computed as before, the first values are

$$\begin{aligned} R'_t(a, b) &= \frac{t}{ab} - \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) \\ R'_t(a, b, c) &= \frac{t^2}{2abc} - \frac{t}{2} \left( \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) + \\ &\quad + \frac{1}{12} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \\ R'_t(a, b, c, d) &= \frac{t^3}{6abcd} - \frac{t^2}{4} \left( \frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd} \right) \\ &\quad + \frac{t}{12} \left( \frac{3}{ab} + \frac{3}{ac} + \frac{3}{ad} + \frac{3}{bc} + \frac{3}{bd} + \frac{3}{cd} + \frac{a}{bcd} + \frac{b}{acd} + \frac{c}{abd} + \frac{d}{abc} \right) \\ &\quad - \frac{1}{24} \left( \frac{a}{bc} + \frac{a}{bd} + \frac{a}{cd} + \frac{b}{ad} + \frac{b}{ac} + \frac{b}{cd} + \frac{c}{ab} + \frac{c}{ad} + \frac{c}{bd} + \frac{d}{ab} + \frac{d}{ac} + \frac{d}{bc} \right) \\ &\quad - \frac{1}{8} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) . \end{aligned}$$

A general formula for  $R'_t(a_1, \dots, a_n)$  was recently discovered in [3].

For generalizations, note that we can apply our method to any tetrahedron given in the form (1), with the  $a_k$ 's replaced by any rational numbers. Moreover, any convex rational polytope (that is, a convex polytope whose vertices have rational coordinates) can be described by a finite number of inequalities over the rationals. In other words, a convex lattice polytope  $\mathcal{P}$  is an intersection of finitely many half-spaces. This description of the polytope leads to an integral in several complex variables, as discussed in [2, Theorem 8] for lattice polytopes.

### 3. THE FOURIER METHOD

In this section we outline a Fourier-analytic method that achieves the same results. Although the theory is a little harder, the method is of independent interest. It draws connections to Brion's theorem on generating functions [5] and to the basic results of [9].

To be concrete, we illustrate the general case with the 2-dimensional rational triangle  $\mathcal{P}$  whose vertices are  $v_0 = (0, 0)$ ,  $v_1 = (\frac{t}{a}, 0)$ , and  $v_2 = (0, \frac{t}{b})$ . As before, the number of lattice points in the 1-dimensional hypotenuse of



this right triangle is

$$p'_{\{a,b\}}(t) = \# \{ (m, n) \in \mathbb{N}^2 : am + bn = t \} .$$

We denote the tangent cone to  $\mathcal{P}$  at the vertex  $v_i$  by  $K_i$ . We recall that the exponential sum attached to the cone  $K$  (with vertex  $v$ ) is by definition

$$\sigma_K(s) = \sum_{m \in \mathbb{Z}^n \cap K} e^{-2\pi \langle s, m \rangle} , \quad (6)$$

where  $s$  is any complex vector that makes the infinite sum (6) converge. An equivalent formulation of (6) which appears more combinatorial is

$$\sigma_K(x) = \sum_{m \in \mathbb{Z}^n \cap K} x^m , \quad (7)$$

where  $x^m = x_1^{m_1} \cdots x_n^{m_n}$  and  $x_j = e^{-2\pi s_j}$ .

In general dimension, let the vertices of the rational polytope  $\mathcal{P}$  be  $v_1, \dots, v_l$ . Let the corresponding tangent cone at  $v_j$  be  $K_j$ . Finally, let the *finite* exponential sum over  $\mathcal{P}$  be

$$\sigma_{\mathcal{P}}(s) = \sum_{m \in \mathbb{Z}^n \cap \mathcal{P}} e^{-2\pi \langle s, m \rangle} . \quad (8)$$

Then there is the basic result that each exponential sum (7) is a rational function of  $x$ , and the following theorem relates these rational functions [5]:

THEOREM 4 (Brion). *For a generic value of  $s \in \mathbb{C}^n$ ,*

$$\sigma_{\mathcal{P}}(s) = \sum_{i=1}^l \sigma_{K_i}(s) . \quad (9)$$

This result allows us to transfer the enumeration of lattice points in  $\mathcal{P}$  to the enumeration of lattice points in the tangent cones  $K_i$  at the vertices of  $\mathcal{P}$ , an easier task. In the theorem above, ‘generic value of  $s$ ’ means any  $s \in \mathbb{C}^n$  for which these rational functions do not blow up to infinity.

To apply these results to our given rational triangle  $\mathcal{P}$ , we first employ the methods of [9] to get an explicit formula for the exponential sum for each tangent cone of  $\mathcal{P}$ . Then, by Brion’s theorem on tangent cones, the sum of the three exponential sums attached to the tangent cones equals the exponential sum over  $\mathcal{P}$ . Canceling the singularities arising from each tangent cone, and letting  $s \rightarrow 1$ , we get the explicit formula of the previous section for the number of lattice points in the rational triangle  $\mathcal{P}$ .

In our case,  $K_1$  is generated by the two rational vectors  $-v_1$  and  $v_2 - v_1$ . We form the matrix

$$A_1 = \begin{pmatrix} -\frac{t}{a} & -\frac{t}{a} \\ 0 & \frac{t}{b} \end{pmatrix} ,$$

whose columns are the vectors that generate the cone  $K_1$ . Once we compute  $\sigma_{K_1}(s)$ ,  $\sigma_{K_2}(s)$  will follow by symmetry. The easiest exponential sum to compute is

$$\begin{aligned}\sigma_{K_0}(s) &= \sum_{m \in \mathbb{Z}^2 \cap K_0} e^{-2\pi \langle s, m \rangle} = \sum_{\substack{m_1 \geq 0 \\ m_2 \geq 0}} e^{-2\pi(m_1 s_1 + m_2 s_2)} \\ &= \frac{1}{(1 - e^{-2\pi s_1})(1 - e^{-2\pi s_2})} .\end{aligned}$$

To compute  $\sigma_{K_i}(s)$  ( $i \neq 0$ ), we first translate the cone  $K_i$  by the vector  $-v_i$  so that its new vertex is the origin. We therefore let  $K = K_i - v_i$ , and the following elementary lemma illustrates how a translation affects the Fourier transform. Let

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K \end{cases}$$

denote the characteristic function of  $K$ .

LEMMA 1. *Let*

$$F_v(x) = \chi_{K+v}(x) e^{-2\pi \langle s, m \rangle}$$

for  $x \in \mathbb{R}^n, s \in \mathbb{C}^n$ . Then

$$\hat{F}_v(\xi) = \hat{\chi}_K(\xi + is) e^{-2\pi i \langle \xi + is, v \rangle}$$

*Proof.*

$$\begin{aligned}\hat{F}_v(\xi) &= \int_{\mathbb{R}^n} \chi_{K+v}(x) e^{-2\pi \langle s, m \rangle} e^{2\pi i \langle \xi, x \rangle} dx \\ &= \int_{\mathbb{R}^n} e^{2\pi i \langle \xi + is, x \rangle} \chi_{K+v}(x) dx \\ &= \int_{\mathbb{R}^n} e^{2\pi i \langle \xi + is, y - v \rangle} \chi_K(y) dy \\ &= e^{-2\pi i \langle \xi + is, v \rangle} \int_{\mathbb{R}^n} e^{2\pi i \langle \xi + is, y \rangle} \chi_K(y) dy \\ &= e^{-2\pi i \langle \xi + is, v \rangle} \hat{\chi}_K(\xi + is)\end{aligned}$$

■

This lemma also shows why it is useful to study the Fourier transform of  $K$  at *complex* values of the variable; that is, at  $\xi + is$ . We study  $F(x)$  because (6) can be rewritten as

$$\sigma_{K_0+v}(s) = \sum_{m \in \mathbb{Z}^n} \chi_{K_0+v}(m) e^{-2\pi \langle s, m \rangle} = \sum_{m \in \mathbb{Z}^n} F_v(m) .$$

All of the lemmas of [9] remain true in this rational polytope context. The idea is to apply Poisson summation to  $\sum_{m \in \mathbb{Z}^n} F_v(m)$  and write formally

$$\sum_{m \in \mathbb{Z}^n} F_v(m) = \sum_{m \in \mathbb{Z}^n} \hat{F}_v(m)$$

The right-hand side diverges, though, and some smoothing completes the picture. Because the steps are identical to those in [9], we omit the ensuing details. Let  $\xi_a = e^{\frac{2\pi i}{a}}$ . We get

$$\begin{aligned} \sigma_{K_1}(s_1, s_2) = \frac{\xi_a^{ts_1}}{4a} \sum_{r=0}^{a-1} \xi_a^{rt} \left( \coth \frac{\pi b}{t} \left( s_{1,2} + \frac{irt}{a} \right) - 1 \right) \\ \cdot \left( \coth \frac{\pi}{t} \left( s_{1,1} + \frac{irt}{a} \right) + 1 \right), \end{aligned} \quad (10)$$

where

$$s_{1,1} = \langle s, \text{generator 1 of } K_1 \rangle = \left\langle (s_1, s_2), \left( -\frac{t}{a}, 0 \right) \right\rangle = -\frac{ts_1}{a}$$

and

$$s_{1,2} = \langle s, \text{generator 2 of } K_1 \rangle = \left\langle (s_1, s_2), \left( -\frac{t}{a}, \frac{t}{b} \right) \right\rangle = -\frac{ts_1}{a} + \frac{ts_2}{b}.$$

By (9), we have

$$\# \{ \mathbb{Z}^2 \cap t\mathcal{P} \} = \sum_{m \in \mathbb{Z}^2 \cap t\mathcal{P}} 1 = \lim_{s \rightarrow 0} (\sigma_{K_0}(s) + \sigma_{K_1}(s) + \sigma_{K_2}(s)).$$

Using the explicit description of  $\sigma_{K_i}(s)$  in terms of cotangent functions, we can cancel their singularities at  $s = 0$  and simply add the holomorphic contributions to  $\sigma_{K_i}(s)$  at  $s = 0$ . The left-hand side of (9) is holomorphic in  $s$ , so that we are guaranteed that the singularities on the right-hand side cancel each other.

The only term in the finite sum (10) that contributes a singularity at  $s = 0$  is the  $r = 0$  term. We expand the three exponential sums  $\sigma_{K_i}(s)$  into their Laurent expansions about  $s = 0$ . Here we only require the first 3 terms of their Laurent expansions. In dimension  $n$  we would require the first  $n + 1$  terms; otherwise every step is the same in general dimension  $n$ .

We make use of the Laurent series

$$\frac{1}{1 - e^{-\alpha s}} = \frac{1}{\alpha s} + \frac{1}{2} + \frac{\alpha s}{12} + O(s^2)$$

near  $s = 0$ , as well as the Laurent series for  $\cot \pi s$  near  $s = 0$ . After expanding each cotangent in (10) for  $\sigma_{K_0}(s)$ ,  $\sigma_{K_1}(s)$  and  $\sigma_{K_2}(s)$  and letting

$s \rightarrow 0$ , we obtain Theorem 1 above as

$$\begin{aligned} L(\overline{\mathcal{P}}, t) &= \frac{t^2}{2ab} + \frac{t}{2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \right) \\ &+ \frac{1}{4} \left( 1 + \frac{1}{a} + \frac{1}{b} \right) + \frac{1}{12} \left( \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right) \\ &+ \frac{1}{a} \sum_{r=1}^{a-1} \frac{\xi_a^{rt}}{(1 - \xi_a^{rb})(1 - \xi_a^r)} + \frac{1}{b} \sum_{r=1}^{b-1} \frac{\xi_b^{rt}}{(1 - \xi_b^{ra})(1 - \xi_b^r)} . \end{aligned}$$

Note that, as before, the periodic portion of  $L(\overline{\mathcal{P}}, t)$  is entirely contained in the “constant”  $t$  term. By Ehrhart’s reciprocity law (Corollary 1, [11]), there is a similar expression for  $L(\mathcal{P}, t)$ , and taking

$$L(\overline{\mathcal{P}}, t) - L(\mathcal{P}, t) - \left\lfloor \frac{t}{a} \right\rfloor - \left\lfloor \frac{t}{b} \right\rfloor - 1$$

gives us  $p_{\{a,b\}}(t)$ . The same analysis gives us Theorem 1 in  $\mathbb{R}^n$ .

#### 4. THE FOURIER-DEDEKIND SUM

In the derivation of the various lattice count formulas, we naturally arrived at the Fourier-Dedekind sum

$$\sigma_t(c_1, \dots, c_n; c) = \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{\lambda^t}{(\lambda^{c_1} - 1) \cdots (\lambda^{c_n} - 1)} .$$

This expression is a generalization of the classical Dedekind sum  $\mathfrak{s}(h, k)$  [23] and its various generalizations [10, 13, 21, 22, 30]. In fact, an easy calculation shows

$$\begin{aligned} \sigma_0(a, 1; c) &= \frac{1}{c} \sum_{\lambda^c = 1 \neq \lambda} \frac{1}{(\lambda^a - 1)(\lambda - 1)} = \\ &= \frac{1}{4} - \frac{1}{4c} - \frac{1}{4c} \sum_{k=1}^{c-1} \cot \frac{\pi k a}{c} \cot \frac{\pi k}{c} = \frac{1}{4} - \frac{1}{4c} - \mathfrak{s}(a, c) . \end{aligned}$$

In general, note that  $\sigma_t(c_1, \dots, c_n; c)$  is a rational number: It is an element of the cyclotomic field of  $c^{\text{th}}$  roots of unity, and invariant under all Galois transformations of this field.

Some obvious properties are

$$\begin{aligned} \sigma_t(c_1, \dots, c_n; c) &= \sigma_t(c_{\pi(1)}, \dots, c_{\pi(n)}; c) \quad \text{for any } \pi \in S_n \\ \sigma_t(c_1, \dots, c_n; c) &= \sigma_{(t \bmod c)}(c_1 \bmod c, \dots, c_n \bmod c; c) \\ \sigma_t(c_1, \dots, c_n; c) &= \sigma_{bt}(bc_1, \dots, bc_n; c) \quad \text{for any } b \in \mathbb{Z} \text{ with } (b, c) = 1 \end{aligned} \tag{11}$$

We can get more familiar-looking formulas for  $\sigma_t$  in certain dimensions. For example, counting points in dimension 1, we find that

$$L(\overline{\mathcal{P}}, t) = \# \{m \in \mathbb{Z} : m \geq 0, mc \leq t\} = \left\lfloor \frac{t}{c} \right\rfloor + 1 ,$$

so that Theorem 1 implies

$$\sigma_{-t}(1; c) = \frac{1}{c} \sum_{\lambda^c=1 \neq \lambda} \frac{\lambda^{-t}}{(\lambda-1)} = \frac{t}{c} - \left\lfloor \frac{t}{c} \right\rfloor - \frac{1}{2} + \frac{1}{2c} = \left( \left( \frac{t}{c} \right) \right) + \frac{1}{2c} . \quad (12)$$

Here,  $((x)) = x - \lfloor x \rfloor - 1/2$  is a sawtooth function (differing slightly from the one appearing in the classical Dedekind sums). This restates the well-known finite Fourier expansion of the sawtooth function (see, e.g., [23]).

As another example, we reformulate

$$\sigma_t(a, b; c) = \frac{1}{c} \sum_{\lambda^c=1 \neq \lambda} \frac{\lambda^t}{(\lambda^a-1)(\lambda^b-1)}$$

by means of finite Fourier series. Consider

$$\begin{aligned} \sigma_t(a; c) &= \frac{1}{c} \sum_{\lambda^c=1 \neq \lambda} \frac{\lambda^{-t}}{(\lambda^a-1)} = \frac{1}{c} \sum_{k=1}^{c-1} \frac{\xi^{kt}}{(\xi^{ka}-1)} = \frac{1}{c} \sum_{k=1}^{c-1} \frac{\xi^{ka^{-1}t}}{(\xi^k-1)} \\ &= \left( \left( \frac{-a^{-1}t}{c} \right) \right) + \frac{1}{2c} , \end{aligned} \quad (13)$$

where  $\xi$  is a primitive  $c^{\text{th}}$  root of unity and  $aa^{-1} \equiv 1 \pmod{c}$ ; here, the last equality follows from (12). We use the well-known convolution theorem for finite Fourier series:

**THEOREM 5.** *Let  $f(t) = \frac{1}{N} \sum_{k=0}^{N-1} a_k \xi^{kt}$  and  $g(t) = \frac{1}{N} \sum_{k=0}^{N-1} b_k \xi^{kt}$ , where  $\xi$  is a primitive  $N^{\text{th}}$  root of unity. Then*

$$\frac{1}{N} \sum_{k=0}^{N-1} a_k b_k \xi^{kt} = \sum_{m=0}^{N-1} f(t-m)g(m) .$$

Hence by (13),

$$\begin{aligned} \sigma_t(a, b; c) &= \sum_{m=0}^{c-1} \sigma_{t-m}(a; c) \sigma_m(b; c) \\ &= \sum_{m=0}^{c-1} \left[ \left( \left( \frac{-a^{-1}(t-m)}{c} \right) \right) + \frac{1}{2c} \right] \left[ \left( \left( \frac{-b^{-1}m}{c} \right) \right) + \frac{1}{2c} \right] \\ &= \sum_{m=0}^{c-1} \left( \left( \frac{a^{-1}(m-t)}{c} \right) \right) \left( \left( \frac{-b^{-1}m}{c} \right) \right) - \frac{1}{4c} . \end{aligned}$$

Here,  $aa^{-1} \equiv bb^{-1} \equiv 1 \pmod{c}$ . The last equality follows from

$$\sum_{m=0}^{c-1} \left( \left( \frac{m}{c} \right) \right) = -\frac{1}{2} .$$

Furthermore, by the periodicity of  $((x))$ ,

$$\sigma_t(a, b; c) = \sum_{m=0}^{c-1} \left( \left( \frac{-a^{-1}(bm+t)}{c} \right) \right) \left( \left( \frac{m}{c} \right) \right) - \frac{1}{4c} . \quad (14)$$

The expression on the right is, up to a trivial term, a special case of a *Dedekind-Rademacher sum* [10, 19, 21, 22]. It is a curious fact that the function  $\sigma_t(a, b; c)$  is the nontrivial part of a multiplier system of a weight-0 modular form [24, p. 121].

We conclude this section by proving two reciprocity laws for Fourier-Dedekind sums. The first one is equivalent to Zagier's reciprocity law for his *higher dimensional Dedekind sums* [30]. They are essentially Fourier-Dedekind sums with  $t = 0$ , that is, trivial numerators.

**THEOREM 6.** *For pairwise relatively prime integers  $a_1, \dots, a_n$ ,*

$$\sum_{j=1}^n \sigma_0(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) = 1 - R'_0(a_1, \dots, a_n) ,$$

where  $R'_t$  is the rational function given in Theorem 3.

*Proof.* It is well known [11] that the constant term of a *lattice polytope* (that is, a polytope with integral vertices) equals the Euler characteristic of the polytope. Consider the polytope

$$\left\{ (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n : \sum_{k=1}^n x_k a_k = 1 \right\} ,$$

whose dilates correspond to the quantor  $p'_A(t)$  of Theorem 3. If we dilate this polytope only by multiples of  $a_1 \cdots a_n$ , say  $t = a_1 \cdots a_n w$ , we obtain the dilates of a lattice polytope. Theorem 3 simplifies for these  $t$  to

$$p'_A(a_1 \cdots a_n w) = R'_{a_1 \cdots a_n w}(a_1, \dots, a_n) + \sum_{j=1}^n \sigma_0(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) ,$$

using the periodicity of  $\sigma_t$  (11). On the other hand, we know that the constant term (in terms of  $w$ ) is the Euler characteristic of the polytope and hence equals 1, which yields the identity

$$1 = R'_0(a_1, \dots, a_n) + \sum_{j=1}^n \sigma_0(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) .$$

■

The second one is a new reciprocity law, which generalizes the following [13]

**THEOREM 7** (Gessel). *Let  $m$  and  $n$  be relatively prime and suppose that  $0 \leq r < m + n$ . Then*

$$\begin{aligned} & \frac{1}{m} \sum_{\lambda^m=1 \neq \lambda} \frac{\lambda^{r+1}}{(\lambda^n-1)(\lambda-1)} + \frac{1}{n} \sum_{\lambda^n=1 \neq \lambda} \frac{\lambda^{r+1}}{(\lambda^m-1)(\lambda-1)} \\ &= -\frac{1}{12} \left( \frac{m}{n} + \frac{n}{m} + \frac{1}{mn} \right) + \frac{1}{4} \left( \frac{1}{m} + \frac{1}{n} - 1 \right) \\ & \quad + \frac{r}{2} \left( \frac{1}{m} + \frac{1}{n} - \frac{1}{mn} \right) - \frac{r^2}{2mn} . \end{aligned}$$

It is not hard to see that Gessel's theorem follows as the two-dimensional case of

**THEOREM 8.** *Let  $a_1, \dots, a_n$  be pairwise relatively prime integers and  $0 < t < a_1 + \dots + a_n$ . Then*

$$\sum_{j=1}^n \sigma_t(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) = -R'_t(a_1, \dots, a_n) ,$$

where  $R'_t$  is the rational function given in Theorem 3.

*Proof.* By definition,  $p'_A(t) = 0$  if  $0 < t < a_1 + \dots + a_n$ . Hence Theorem 3 yields an identity for these values of  $t$ :

$$0 = R'_t(a_1, \dots, a_n) + \sum_{j=1}^n \sigma_t(a_1, \dots, \hat{a}_j, \dots, a_n; a_j) .$$

■

It is worth noticing that both Theorems 6 and 7 imply the reciprocity law for the classical Dedekind sum  $\mathfrak{s}(a, b)$ . It should be finally mentioned that in special cases there are other reciprocity laws, for example, for the sum appearing on the right-hand side in (14) [10, 22]. We note that, as a consequence, we can compute  $\sigma_t(a, b; c)$  in polynomial time.

## 5. THE FROBENIUS PROBLEM

In this last section we apply Theorem 3 (the explicit formula for  $p'_A(t)$ ) to Frobenius's original problem. As an example, we will discuss the 3-dimensional case. Note that a bound for dimension 3 yields a bound for the general case: It can be easily verified that

$$f(a_1, \dots, a_n) \leq f(a_1, a_2, a_3) + a_4 + \dots + a_n \quad (15)$$

Furthermore, in dimension 3 it suffices to assume that  $a_1, a_2, a_3$  are pairwise coprime, due to Johnson's formula [15]: If  $g = (a_1, a_2)$ , then

$$f(a_1, a_2, a_3) = g \cdot f\left(\frac{a_1}{g}, \frac{a_2}{g}, a_3\right). \quad (16)$$

Now assume  $a, b, c$  pairwise relatively prime, and recall (14):

$$\sigma_t(a, b; c) = \sum_{m=0}^{c-1} \left( \left( \frac{-a^{-1}(bm+t)}{c} \right) \right) \left( \left( \frac{m}{c} \right) \right) - \frac{1}{4c},$$

where  $aa^{-1} \equiv 1 \pmod{c}$ . We will use the Cauchy-Schwartz inequality

$$\left| \sum_{k=1}^n a_k a_{\pi(k)} \right| \leq \sum_{k=1}^n a_k^2. \quad (17)$$

Here  $a_k \in \mathbb{R}$ , and  $\pi \in S_n$  is a permutation. Since  $(a^{-1}b, c) = 1$ , we can use (17) to obtain

$$\begin{aligned} \sigma_t(a, b; c) &\geq - \sum_{m=0}^{c-1} \left( \left( \frac{m}{c} \right) \right)^2 - \frac{1}{4c} = \sum_{m=0}^{c-1} \left( \frac{m}{c} - \frac{1}{2} \right)^2 - \frac{1}{4c} \\ &= -\frac{1}{c^2} \frac{(2c-1)(c-1)c}{6} + \frac{1}{c} \frac{c(c-1)}{2} - \frac{c}{4} - \frac{1}{4c} \\ &= -\frac{c}{12} - \frac{1}{12c}. \end{aligned}$$

This also restates Rademacher's bound on the classical Dedekind sums [23]. Using this in the formula for dimension 3 (remark after Theorem 3), we get

$$\begin{aligned} p'_{\{a,b,c\}}(t) &\geq \frac{t^2}{2abc} - \frac{t}{2} \left( \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) \\ &\quad + \frac{1}{12} \left( \frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \\ &\quad - \frac{1}{12}(a+b+c) - \frac{1}{12} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\ &= \frac{t^2}{2abc} - \frac{t}{2} \left( \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \right) + \frac{1}{12} \left( \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) \\ &\quad - \frac{1}{12}(a+b+c) + \frac{1}{6} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right). \end{aligned}$$

The larger zero of the right-hand side is an upper bound for the solution



of the Frobenius problem:

$$\begin{aligned}
f(a, b, c) &\leq abc \left( \frac{1}{2} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right) + \left[ \frac{1}{4} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right)^2 \right. \right. \\
&\quad \left. \left. - \frac{2}{abc} \left( \frac{1}{12} \left( \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \right) - \frac{1}{12}(a+b+c) + \frac{1}{6} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \right) \right]^{1/2} \right) \\
&\leq \frac{1}{2}(a+b+c) + abc \sqrt{\frac{1}{4} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right)^2 + \frac{1}{6} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right)} \\
&= \frac{1}{2}(a+b+c) + abc \sqrt{\frac{1}{2} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right) \left( \frac{1}{2} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right) + \frac{1}{3} \right)} \\
&\leq \frac{1}{2}(a+b+c) + abc \sqrt{\frac{1}{4} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \right)} .
\end{aligned}$$

For the last inequality, we used the fact that  $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} \leq \frac{1}{6} + \frac{1}{10} + \frac{1}{15} = \frac{1}{3}$ . This proves, using (15) and (16),

**THEOREM 9.** *Let  $a_1 \leq a_2 \leq \dots \leq a_n$  be relatively prime. Then*

$$f(a_1, \dots, a_n) \leq \frac{1}{2} \left( \sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)} + a_1 + a_2 + a_3 \right) + a_4 + \dots + a_n .$$

*Remarks.* 1. Sometimes the Frobenius problem is stated in a slightly different form: Given relatively prime positive integers  $a_1, \dots, a_n$ , find the largest value of  $t$  such that  $\sum_{k=1}^n m_k a_k = t$  has no solution in *nonnegative* integers  $m_1, \dots, m_n$ . This number is denoted by  $g(a_1, \dots, a_n)$ . It is, however, easy to see that

$$g(a_1, \dots, a_n) = f(a_1, \dots, a_n) - a_1 - \dots - a_n .$$

So we can restate Theorem 9 in a more compact form as

$$g(a_1, \dots, a_n) \leq \frac{1}{2} \left( \sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)} - a_1 - a_2 - a_3 \right) .$$

2. Bounds on the Frobenius number in the literature include results by Erdős and Graham [12]

$$g(a_1, \dots, a_n) \leq 2a_n \left\lfloor \frac{a_1}{n} \right\rfloor - a_1 ,$$

Selmer [27]

$$g(a_1, \dots, a_n) \leq 2a_{n-1} \left\lfloor \frac{a_n}{n} \right\rfloor - a_n ,$$

and Vitek [29]

$$g(a_1, \dots, a_n) \leq \left\lfloor \frac{1}{2}(a_2 - 1)(a_n - 2) \right\rfloor - 1 .$$

Theorem 9 is certainly of the same order. What might be more interesting, however, is the fact that the bound in Theorem 9 is of a different nature than the bounds stated above: namely, it involves three variables, and is thus—especially in terms of estimating  $g(a_1, a_2, a_3)$ —more symmetric.

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