MATH 725 Homework Set 3 due 9/12/11

(1) Show that $\mathscr{P}(\mathbf{F})$, the set of all polynomials with coefficients in $\mathbf{F}$, is not finite dimensional.

Proof. If B is a finite basis for $\mathscr{P}(\mathbf{F})$ then there will be a polynomial of largest degree d among the polynomials in B. But then there is no way to write x^{d+1} as a linear combination of polynomials in B. Thus $\mathscr{P}(\mathbf{F})$ cannot have a finite basis. $\square$

(2) The purpose of this exercise is to prove that every non-trivial vector space has a basis (not just finite-dimensional ones), assuming Zorn's Lemma (which we will give below). To do so, we need the following definition: Let V be a vector space over $\mathbf{F}$. A *linear combination* is a sum of the form $\sum_{\mathbf{v}\in V} a_{\mathbf{v}}\mathbf{v}$ where all but finitely many of the $a_{\mathbf{v}}$ are 0. Having settled this, we repeat the same definitions what it means for a non-empty subset $S \subseteq V$ to *span* V, what it means for S to be *linearly independent*, and what it means for S to be a *basis* of V as in the Axler book.

We also need the definition of a *partially ordered set* (a *poset*): this is a set P equipped with a relation $\preceq$ that satisfies *reflexivity*, *antisymmetry*, and *transitivity*; that is, for all $a, b, c \in P$,

$$\begin{aligned} &a \preceq a \\ a \preceq b \text{ and } b \preceq a \quad &\Longrightarrow \quad a = b \\ a \preceq b \text{ and } b \preceq c \quad &\Longrightarrow \quad a \preceq c\,. \end{aligned}$$

A subset $C \subseteq P$ is a *chain* if it is totally ordered, i.e., $a, b \in C \Longrightarrow a \preceq b$ or $b \preceq a$. We call $u \in P$ an *upper bound* for the subset $S \subseteq P$ if $a \preceq u$ for all $a \in S$, and we call u *maximal* if it is an upper bound for P.

Zorn's Lemma (which we will assume for this exercise) says that if P is a partially ordered set in which every chain has an upper bound, then P has a maximal element.[1]

(a) Show that $P(V)$, the *power set* of V consisting of all subsets of V, is partially ordered under set containment (i.e., we take $\preceq$ to be $\subseteq$).
(b) Let $L \subseteq P(V)$ consist of all linearly independent subsets of V. Show that every chain in L has an upper bound.
(c) According to Zorn's Lemma, L contains a maximal element B. Show that B is a basis for V.

Proof. (a) Let A, B, C be subsets of V. Then we have $A \subseteq A$ by definition, $A \subseteq B \subseteq C \Longrightarrow A \subseteq C$ because the statements $x \in A \Longrightarrow x \in B$ and $x \in B \Longrightarrow x \in C$ imply the statement $x \in A \Longrightarrow x \in C$, and $A \subseteq B \supseteq A \Longrightarrow A = B$ by definition of set equality.

(b) Suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be a chain in L. Then $A := \bigcup_{j=1}^{\infty} A_j$ (if the chain is finite, make this a finite union) is a subset of V that by construction contains every A_j as a subset, and so A is an upper bound for the chain.

(c) We need to show that B spans V and is linearly independent. The latter statement follows because $B \in L$. To prove that B spans V, assume the contrary, i.e., there exists a vector $\mathbf{v} \in V$ that is not in $\operatorname{span}(B)$. But then the set $B \cup \{\mathbf{v}\}$ is linearly independent, hence $B \cup \{\mathbf{v}\} \in L$, contradicting the fact that B is the maximum element of L. $\square$

(3) Suppose V and W are finite-dimensional vector spaces. Show that the following statements are equivalent:
(a) $\dim(V) \geq \dim(W)$.
(b) There exists a surjective linear map $V \to W$.
(c) There exists an injective linear map $W \to V$.

Proof. Fix a basis $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of V and a basis $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ of W. We will prove (a) $\Longleftrightarrow$ (b) and (a) $\Longleftrightarrow$ (c), in two steps each.

Suppose $n \geq m$. Then $T : V \to W$ given by

$$T\left(\sum_{j=1}^{n} a_j \mathbf{v}_j\right) = \sum_{j=1}^{m} a_j \mathbf{w}_j$$

[1] Zorn's Lemma is equivalent to the *Axiom of Choice*, a remark that suggests that it is a nontrivial item in mathematical logic. Do yourself a favor and spend a few minutes googling these two terms.

is well defined, surjective, and linear by construction. Conversely, suppose there exists a surjective linear map $V \to W$. Then

$$\dim V = \dim \operatorname{null}(T) + \dim \operatorname{range}(T) = \dim \operatorname{null}(T) + \dim W \geq \dim W\,.$$

For the second double implication, again suppose $n \geq m$. Then $T : W \to V$ given by

$$T\left(\sum_{j=1}^{m} a_j \mathbf{w}_j\right) = \sum_{j=1}^{m} a_j \mathbf{v}_j$$

is well defined, injective, and linear, again by construction. Conversely, suppose there exists a injective linear map $W \to V$. Then

$$\dim W = \dim \operatorname{null}(T) + \dim \operatorname{range}(T) = \dim \operatorname{range}(T) \leq \dim V\,. \qquad \square$$

(4) Let $S, T : V \to V$ be linear maps on a vector space V.
  (a) Show that $\operatorname{null}(T) \subseteq \operatorname{null}(ST)$.
  (b) Give an example where $\operatorname{null}(S) \not\subseteq \operatorname{null}(ST)$.
  (c) Show that $\operatorname{range}(ST) \subseteq \operatorname{range}(S)$.
  (d) Give an example where $\operatorname{range}(ST) \not\subseteq \operatorname{range}(T)$.

Proof. (a) Suppose $\mathbf{v} \in \operatorname{null}(T)$, i.e., $T(\mathbf{v}) = \mathbf{0}$. Then $S(T(\mathbf{v})) = \mathbf{0}$, i.e., $\mathbf{v} \in \operatorname{null}(ST)$.

(b) Let $V = \mathscr{P}(\mathbf{R})$ and define the maps $S, T \in L(V)$ through

$$S(p(x)) = p'(x) \qquad \text{and} \qquad T(p(x)) = \int_0^x p(t)\,dt.$$

Then ST is the identity map with $\operatorname{null}(ST) = \{0\}$; however, $\operatorname{null}(S)$ consists of all constant polynomials.

(c) Suppose $\mathbf{w} \in \operatorname{range}(ST)$, i.e., there exists $\mathbf{v} \in V$ such that $S(T(\mathbf{v})) = \mathbf{w}$. Then $\mathbf{w} \in \operatorname{range}(S)$ because $T(\mathbf{v})$ is the pre-image of $\mathbf{w}$ under S.

(d) Let $V = \mathscr{P}(\mathbf{R})$, let T be the identity map, and $S(p(x)) = x\,p(x)$. Then $\operatorname{range}(T) = V$; however, $\operatorname{range}(ST) = \operatorname{range}(S)$ consists of all polynomials that do not have a nonzero constant term. $\square$

(5) As usual, let $\mathscr{P}(\mathbf{R})$ be the set of all polynomials with coefficients in $\mathbf{R}$.
  (a) Show that $\frac{d}{dx}$ is a linear map $\mathscr{P}(\mathbf{R}) \to \mathscr{P}(\mathbf{R})$. Is the map injective or surjective or both?
  (b) Fix $a \in \mathbf{R}$ and let $I_a : \mathscr{P}(\mathbf{R}) \to \mathscr{P}(\mathbf{R})$ be defined by $I_a(f) := \int_a^x f(t)\,dt$. Show that I_a is linear. Is I_a injective or surjective or both?
  (c) Is I_a a left or right inverse of $\frac{d}{dx}$? Is it possible to choose a value of a so that I_a is a two-sided inverse of $\frac{d}{dx}$?

Proof. (a) Given $p(x), q(x) \in \mathscr{P}(\mathbf{R})$ and $\lambda \in \mathbf{R}$,

$$\tfrac{d}{dx}\left(\lambda\, p(x) + q(x)\right) = \lambda\, p'(x) + q'(x)$$

by the rules of calculus, so $\frac{d}{dx}$ is linear. This map is surjective because by the Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_0^x p(t)\,dt = p(x)\,,$$

i.e., the polynomial $\int_0^x p(t)\,dt$ is a pre-image of $p(x)$. Differentiation is not injective because $\frac{d}{dx}(x+1) = \frac{d}{dx}(x+2) = 1$.

(b) Given $p(x), q(x) \in \mathscr{P}(\mathbf{R})$ and $\lambda \in \mathbf{R}$,

$$I_a(\lambda\, p(x) + q(x)) = \int_a^x \lambda\, p(t) + q(t)\,dt = \lambda \int_a^x p(t)\,dt + \int_a^x q(t)\,dt$$

by the rules of calculus, so I_a is linear. This map is injective because if

$$\int_a^x p(t)\,dt = \int_a^x q(t)\,dt$$

then we can differentiate both sides to conclude $p(x) = q(x)$. The map I_a is not surjective, because $I_a(p(x))$ is a polynomial of degree ≥ 1 (unless it is zero).

(c) The Fundamental Theorem of Calculus says that I_a is a right inverse of $\frac{d}{dx}$. It cannot be a two-sided inverse because then it would be surjective. $\square$