MATH 725 Homework Set 9 due 10/31/11

(1) A linear operator $T \in L(V)$ is *unitary* if T^*T is the identity map. Show that all eigenvalues of a unitary operator have absolute value 1.

Proof. Suppose T is a unitary operator and $T(\mathbf{v}) = \lambda \mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$. Then

$$\langle \mathbf{v}, \mathbf{v} \rangle = \langle T^*T(\mathbf{v}), \mathbf{v} \rangle = \langle T(\mathbf{v}), T(\mathbf{v}) \rangle = \langle \lambda \mathbf{v}, \lambda \mathbf{v} \rangle = |\lambda|^2 \langle \mathbf{v}, \mathbf{v} \rangle ,$$

and so (since $\langle \mathbf{v}, \mathbf{v} \rangle \neq 0$) $|\lambda| = 1$. □

(2) Suppose $S, T \in L(V)$ are self adjoint.
(a) Give an example that shows that ST might not be self adjoint.
(b) Prove that ST is self adjoint if and only if $ST = TS$.

Solution. (a) Let $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, both expressed in terms of the standard basis of $\mathbf{R}^2$. Then $ST = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not self adjoint.
(b) Since S and T are self adjoint, we have $(ST)^* = T^*S^* = TS$. Thus ST is self adjoint if and only if $ST = TS$. □

(3) If $T \in L(V)$ is normal and $k \in \mathbf{Z}_{>0}$ then

$$\operatorname{null}\left(T^k\right) = \operatorname{null}(T) \qquad \text{and} \qquad \operatorname{range}\left(T^k\right) = \operatorname{range}(T) .$$

Hint: for one set inclusion, assuming $\mathbf{v} \in \operatorname{null}\left(T^k\right)$, consider $\left\langle T^*T^{k-1}(\mathbf{v}), T^*T^{k-1}(\mathbf{v}) \right\rangle$.

Proof. Suppose $\mathbf{v} \in \operatorname{null}\left(T^k\right)$. Then

$$0 = \left|\left|T^k(\mathbf{v})\right|\right|^2 = \left\langle T^k(\mathbf{v}), T^k(\mathbf{v}) \right\rangle = \left\langle T^{k-1}(\mathbf{v}), T^*T^k(\mathbf{v}) \right\rangle = \left\langle T^{k-1}(\mathbf{v}), TT^*T^{k-1}(\mathbf{v}) \right\rangle$$
$$= \left\langle T^*T^{k-1}(\mathbf{v}), T^*T^{k-1}(\mathbf{v}) \right\rangle ,$$

which implies that $T^*T^{k-1}(\mathbf{v}) = \mathbf{0}$. But then

$$0 = \left\langle T^*T^{k-1}(\mathbf{v}), T^{k-2}(\mathbf{v}) \right\rangle = \left\langle T^{k-1}(\mathbf{v}), T^{k-1}(\mathbf{v}) \right\rangle ,$$

which implies, in turn, that $T^{k-1}(\mathbf{v}) = \mathbf{0}$, i.e., $\mathbf{v} \in \operatorname{null}\left(T^{k-1}\right)$. We can repeat this proof to conclude that $\mathbf{v} \in \operatorname{null}\left(T^{k-2}\right)$, $\mathbf{v} \in \operatorname{null}\left(T^{k-3}\right)$, etc., and after a finite number of steps we conclude that $\mathbf{v} \in \operatorname{null}(T)$. This proves $\operatorname{null}\left(T^{k-1}\right) \subseteq \operatorname{null}(T)$.

Conversely, suppose $\mathbf{v} \in \operatorname{null}(T)$. Then $T^k(\mathbf{v}) = T^{k-1}(T(\mathbf{v})) = T^{k-1}(\mathbf{0}) = \mathbf{0}$. This proves $\operatorname{null}(T) \subseteq \operatorname{null}\left(T^{k-1}\right)$.

To see that $\operatorname{range}\left(T^k\right) = \operatorname{range}(T)$, we note that $\operatorname{range}\left(T^k\right) \subseteq \operatorname{range}(T)$ (since $\mathbf{v} \in \operatorname{range}\left(T^k\right)$ means that there exists $\mathbf{u} \in V$ such that $T^k(\mathbf{u}) = \mathbf{v}$; but then $T\left(T^{k-1}(\mathbf{u})\right) = \mathbf{v}$, i.e., $\mathbf{v} \in \operatorname{range}(T)$). But these two subspaces have the same dimension:

$$\dim \operatorname{range}\left(T^k\right) = \dim V - \dim \operatorname{null}\left(T^k\right) = \dim V - \dim \operatorname{null}(T) = \dim \operatorname{range}(T) ,$$

and thus $\operatorname{range}\left(T^k\right) = \operatorname{range}(T)$. □

(4) Let V be a complex vector space. Prove that a normal operator in $L(V)$ is self adjoint if and only if all its eigenvalues are real.

Proof. We proved in class that all eigenvalues of a self-adjoint operator are real.

Conversely, suppose all eigenvalues of $T \in L(V)$ are real. By the (complex version of the) spectral theorem, there exists an orthonormal basis of eigenvectors of T, and with respect to this basis, T has a diagonal matrix whose diagonal entries are the eigenvalues. But since these entries are real, $T^* = T$ (neither conjugation nor transposing changes the matrix), i.e., T is self adjoint. $\square$

(5) Prove that if $T \in L(V)$ is positive, then so is T^k, for any $k \in \mathbf{Z}_{>0}$.

Proof. Suppose $T \in L(V)$ is positive. Note that this implies automatically that T is self adjoint. We will prove that T^k is positive by (strong) induction on $k \in \mathbf{Z}_{>0}$. The base case $k = 1$ is given, and for $k = 2$ we have for any $\mathbf{v} \in V$

$$\left\langle T^2(\mathbf{v}), \mathbf{v} \right\rangle = \langle T(\mathbf{v}), T(\mathbf{v}) \rangle \geq 0\,.$$

For the induction step, assume that $k \geq 3$ and T^k is positive, i.e., $\left\langle T^k(\mathbf{v}), \mathbf{v} \right\rangle \geq 0$ for any $\mathbf{v} \in V$. Then for any $\mathbf{v} \in V$

$$\left\langle T^{k+1}(\mathbf{v}), \mathbf{v} \right\rangle = \left\langle T^k(\mathbf{v}), T(\mathbf{v}) \right\rangle = \left\langle T^{k-1}\left(T(\mathbf{v})\right), T(\mathbf{v}) \right\rangle \geq 0$$

because T^{k-1} is positive by induction hypothesis. $\square$