MATH 725 **Homework Set 2** **due 9/7/11**

(1) Consider the subspaces

$$U := \left\{(x,y,z,w) \in \mathbf{R}^4 : x+y+z+w = x-y+z-w = 0\right\} \qquad \text{and}$$
$$V := \left\{(x,y,z,w) \in \mathbf{R}^4 : x+z = y-w = 0\right\}$$

of $\mathbf{R}^4$. Is $\mathbf{R}^4 = U \oplus V$? If so, prove it; if not, construct a subspace W such that $\mathbf{R}^4 = U \oplus W$.

Solution. First note that the equations $x+y+z+w = x-y+z-w = 0$ are equivalent to the equations $x+y+z+w = 0$ and $y+w = -y-w$, the latter of which is equivalent to $y+w = 0$, which in turn implies that the former equation can be simplified to $x+z = 0$. So

$$U = \left\{(x,y,z,w) \in \mathbf{R}^4 : x+z = y+w = 0\right\}.$$

The vector $(1,0,-1,0)$ is in both U and V, and so $U+V$ is not a direct product (because then we would have $U \cap V = \{(0,0,0,0)\}$).

Let $W := \left\{(x,y,z,w) \in \mathbf{R}^4 : z = w = 0\right\}$. We claim that $\mathbf{R}^4 = U \oplus W$; we will prove this by showing $\mathbf{R}^4 = U+W$ and $U \cap W = \{(0,0,0,0)\}$. First, any $(a,b,c,d) \in \mathbf{R}^4$ can be written as

$$(a,b,c,d) = (-c,-d,c,d) + (a+c,b+d,0,0);$$

note that $(-c,-d,c,d) \in U$ and $(a+c,b+d,0,0) \in W$, so this proves $\mathbf{R}^4 = U+W$. Now suppose $(a,b,c,d) \in U \cap W$; then

$$a = -c = 0 \qquad \text{and} \qquad b = -d = 0,$$

that is, $(a,b,c,d) = (0,0,0,0)$. This proves $U \cap W = \{(0,0,0,0)\}$. □

(2) Let $U := \left\{(x_1,x_2,\ldots,x_5) \in \mathbf{R}^5 : x_1 = 5x_2 = 6x_3\right\}$.
  (a) Construct a basis of U (and prove that it is a basis).
  (b) Construct a basis of $\mathbf{R}^5$ that extends your basis in (a).

Solution. (a) We claim that $B := ((30,6,5,0,0),(0,0,0,1,0),(0,0,0,0,1))$ is a basis for U. The list B is linearly independent because

$$a(30,6,5,0,0) + b(0,0,0,1,0) + c(0,0,0,0,1) = (30a,6a,5a,b,c) = (0,0,0,0,0)$$

implies that $a = b = c = 0$. The list B spans U because any vector in U is by definition of the form $\left(x_1, \frac{1}{5}x_1, \frac{1}{6}x_1, x_4, x_5\right)$, and

$$\left(x_1, \tfrac{1}{5}x_1, \tfrac{1}{6}x_1, x_4, x_5\right) = x_1(30,6,5,0,0) + x_4(0,0,0,1,0) + x_5(0,0,0,0,1).$$

(b) We claim that $C := ((30,6,5,0,0),(0,0,0,1,0),(0,0,0,0,1),(1,0,0,0,0),(0,1,0,0,0))$ is such a basis for $\mathbf{R}^5$. The list C is linearly independent because

$$\begin{aligned} &a(30,6,5,0,0) + b(0,0,0,1,0) + c(0,0,0,0,1) + d(1,0,0,0,0) + e(0,1,0,0,0) \\ &\quad = (30a+d, 6a+e, 5a, b, c) = (0,0,0,0,0) \end{aligned}$$

implies that $a = b = c = d = e = 0$. The list C spans $\mathbf{R}^5$ because

$$\begin{aligned} (x_1,x_2,x_3,x_4,x_5) = &\tfrac{1}{5}x_3(30,6,5,0,0) + x_4(0,0,0,1,0) + x_5(0,0,0,0,1) + \\ &+ (x_1 - 6x_3)(1,0,0,0,0) + \left(x_2 - \tfrac{6}{5}x_3\right)(0,1,0,0,0). \end{aligned}$$ □

(3) Suppose $S_1, S_2 \subseteq V$. Let $U_1 := \operatorname{span}(S_1)$ and $U_2 := \operatorname{span}(S_2)$.
  (a) Show that $U_1 = U_2$ if and only if $S_1 \subseteq U_2$ and $S_2 \subseteq U_1$.
  (b) Show that $\operatorname{span}(S_1 \cup S_2) = U_1 + U_2$.

Solution. (a) Assume $U_1 = U_2$. Any $\mathbf{s} \in S_1$ certainly lies in $\operatorname{span}(S_1) = U_2$, and so $S_1 \subseteq U_2$. Switching the subscripts yields $S_2 \subseteq U_1$.
Conversely, assume $S_1 \subseteq U_2$ and $S_2 \subseteq U_1$. Any $\mathbf{u} \in U_1$ can be written as a linear combination of vectors in S_1, and since $S_1 \subseteq U_2$, this linear combination is in U_2, i.e., $\mathbf{u} \in U_2$. This gives $U_1 \subseteq U_2$. Switching the subscripts yields $U_2 \subseteq U_1$.

(b) Since $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$, any vector in $\operatorname{span}(S_1 \cup S_2)$ is in $U_1 + U_2$, i.e., $\operatorname{span}(S_1 \cup S_2) \subseteq U_1 + U_2$. Now let $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \in U_1 + U_2$. Then we can write $\mathbf{u}_1$ and $\mathbf{u}_2$ as linear combinations of vectors in $S_1 = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m)$ and $S_2 = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$, respectively, say

$$\mathbf{u}_1 = \sum_{j=1}^{m} a_j \mathbf{s}_j \qquad \text{and} \qquad \mathbf{u}_2 = \sum_{j=1}^{n} b_j \mathbf{t}_j .$$

The lists S_1 and S_2 might have vectors in common; upon possibly renumbering the two lists, we may assume that the common vectors are $\mathbf{s}_1 = \mathbf{t}_1, \mathbf{s}_2 = \mathbf{t}_2, \ldots, \mathbf{s}_k = \mathbf{t}_k$. Thus we can rewrite

$$\mathbf{u}_1 = \sum_{j=1}^{k} a_j \mathbf{s}_j + \sum_{j=k+1}^{m} a_j \mathbf{s}_j \qquad \text{and} \qquad \mathbf{u}_2 = \sum_{j=1}^{k} b_j \mathbf{s}_j + \sum_{j=k+1}^{n} b_j \mathbf{t}_j$$

and so

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 = \sum_{j=1}^{k} (a_j + b_j) \mathbf{s}_j + \sum_{j=k+1}^{m} a_j \mathbf{s}_j + \sum_{j=k+1}^{n} b_j \mathbf{t}_j$$

is in $\operatorname{span}(S_1 \cup S_2) = \operatorname{span}(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m, \mathbf{t}_{k+1}, \mathbf{t}_{k+2}, \ldots, \mathbf{t}_n)$. This proves $U_1 + U_2 \subseteq \operatorname{span}(S_1 \cup S_2)$. $\square$

(4) Recall the definition of the binomial coefficient

$$\binom{x}{n} := \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$

for an arbitrary x (e.g., $x \in \mathbf{C}$ or x a variable) and $n \in \mathbf{Z}_{\geq 0}$. Show that $\left(\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}\right)$ is a basis of $\mathscr{P}_n(\mathbf{F})$, the set of all polynomials of degree $\leq n$ with coefficients in $\mathbf{F}$.

Proof. Let $S_1 := \left(1, x, x^2, \ldots, x^n\right)$ and $S_2 := \left(\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}\right)$. The list S_1 is a basis of $\mathscr{P}_n(\mathbf{F})$ by inspection, so by Exercise (3a), we only need to show that (viewed as sets) $S_1 \subseteq \operatorname{span}(S_2)$ and $S_2 \subseteq \operatorname{span}(S_1)$. The latter set inclusion is clear by expanding $\binom{x}{j}$ in terms of x.

To prove $S_1 \subseteq \operatorname{span}(S_2)$, we need to show that $x^k \in \operatorname{span}(S_2)$ for any $0 \leq k \leq n$. We will prove this by induction on n. The base case is $n = 0$: $1 \in \operatorname{span}\left\{\binom{x}{0}\right\} = \operatorname{span}\{1\}$. For the induction step, assume that $x^j \in \operatorname{span}\left(\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n-1}\right)$ for any $0 \leq j \leq n-1$. Now given $0 \leq k \leq n$, if $k \neq n$ then $x^k \in \operatorname{span}\left(\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}\right)$ by induction hypothesis. If $k = n$ then we use the fact that

$$\binom{x}{n} = \frac{1}{n!} x^n + p(x)$$

for some polynomial of degree $n-1$. Thus

$$x^n = n! \binom{x}{n} + n!\, p(x)$$

and the last summand can be written as a linear combination of $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n-1}$, by induction hypothesis. This gives a linear combination for x^n in terms of $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n}$. $\square$

(5) Suppose that U is a subspace of the finite-dimensional vector space V, and $\dim U = \dim V$. Prove that $U = V$.

Proof. Let $n = \dim U = \dim V$. Since V (and hence also U) are finite-dimensional, there exists a basis $B := (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of U that can be extended to a basis of V. But since (by a theorem proved in class) such an extended basis also has to contain n vectors, B must already span V, that is, $U = V$. $\square$