Proposition 9.3.

(i) *If $f : A \to B$ is injective and $g : B \to C$ is injective then $g \circ f : A \to C$ is injective.*

(ii) *If $f : A \to B$ is surjective and $g : B \to C$ is surjective then $g \circ f : A \to C$ is surjective.*

(iii) *If $f : A \to B$ is bijective and $g : B \to C$ is bijective then $g \circ f : A \to C$ is bijective.*

Proof. (i) Suppose $f : A \to B$ and $g : B \to C$ are injective functions. Then by definition, for any $a_1, a_2 \in A$,

$$f(a_1) = f(a_2) \qquad \text{implies} \qquad a_1 = a_2 \tag{9.1}$$

and, for any $b_1, b_2 \in B$,

$$g(b_1) = g(b_2) \qquad \text{implies} \qquad b_1 = b_2 . \tag{9.2}$$

Now suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$. To prove that $g \circ f$ in injective, we will show that $a_1 = a_2$. We know that $g(f(a_1)) = g(f(a_2))$. Then by applying (9.2) to $b_1 = f(a_1)$ and $b_2 = f(a_2)$,

$$b_1 = b_2 \qquad \text{that is,} \qquad f(a_1) = f(a_2) .$$

But now by (9.1), $a_1 = a_2$.

(ii) Suppose $f : A \to B$ and $g : B \to C$ are surjective functions. Then by definition,

$$\text{for any } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b \tag{9.3}$$

and

$$\text{for any } c \in C \text{ there exists } b \in B \text{ such that } g(b) = c . \tag{9.4}$$

We need to show that $g \circ f$ is surjective. So given $c \in C$, we will construct $a \in A$ such that $(g \circ f)(a) = c$. For the given c we can, by (9.4), find $b \in B$ such that $g(b) = c$. For this b we can, by (9.3), find $a \in A$ such that $f(a) = b$. This a will do, because

$$(g \circ f)(a) = g(f(a)) = g(b) = c .$$

(iii) follows immediately with (i) and (ii), by definition of bijectivity. □

Proposition 9.4.

(i) *$f : A \to B$ is injective if and only if f has a left inverse.*

(ii) *$f : A \to B$ is surjective if and only if f has a right inverse.*

(iii) *$f : A \to B$ is bijective if and only if f has an inverse.*

Proof of (i). Suppose $f : A \to B$ is injective. Then fix an $a_0 \in A$ and define the function $g : B \to A$ through

$$g(b) := \begin{cases} a & \text{if } b \text{ is in the range of } f \text{ and } f(a) = b, \\ a_0 & \text{otherwise.} \end{cases}$$

Then g is a well-defined function, because f is injective.

Conversely, suppose $f : A \to B$ has a left inverse g, that is, $g : B \to A$ is a function such that $g \circ f = \mathrm{id}_A$. Suppose $a_1, a_2 \in A$ satisfy $f(a_1) = f(a_2)$; to prove that f is injective we will show that this identity implies $a_1 = a_2$. Because g is a function, $f(a_1) = f(a_2)$ implies that

$$(g \circ f)(a_1) = g(f(a_1)) = g(f(a_2)) = (g \circ f)(a_2).$$

Comparing the left-hand side of this identity with the right-hand side yields $a_1 = a_2$, since $g \circ f = \mathrm{id}_A$. □

Proof of (ii). Suppose $f : A \to B$ is surjective. We will construct a function $g : B \to A$ as follows: Given $b \in B$, we can find an $a \in A$ such that $f(a) = b$ (we can possibly find more than one such a, in which case we choose one). We define this a to be the image of b under the function g, that is, we define $g(b) = a$. With this definition, we obtain $(f \circ g)(b) = f(g(b)) = f(a) = b$, that is, g is a right inverse to f.

Conversely, suppose $f : A \to B$ has a right inverse g, that is, $g : B \to A$ is a function such that $f \circ g = \mathrm{id}_B$. We need to show that f is surjective, that is, given $b \in B$, we need to find $a \in A$ such that $f(a) = b$. Given such a $b \in B$, we define $a = g(b)$. Then by construction, $f(a) = f(g(b)) = (f \circ g)(b) = b$. □