Proposition 5.6.

(i) *Let* $k \in \mathbb{Z}$ *and let* $(x_j)_{j \in \mathbb{N}}$ *be a sequence in* $\mathbb{Z}$*. Then* $k \cdot \left(\sum_{j=1}^{n} x_j \right) = \sum_{j=1}^{n} (kx_j)$*.*

(ii) *If* $x_j = 1$ *for all* $j \in \mathbb{N}$ *then* $\sum_{j=1}^{n} x_j = n$ *for all* $n \in \mathbb{N}$*.*

(iii) *If* $x_j = r \in \mathbb{Z}$ *for all* $j \in \mathbb{N}$ *then* $\sum_{j=1}^{n} x_j = rn$ *for all* $n \in \mathbb{N}$*.*

Proof. (i) We proceed by induction on n. For $n = 1$, the left-hand side of the identity to be proven is $k \cdot \left(\sum_{i=1}^{1} x_i \right) = k \cdot x_1$, by definition, and the right-hand side is $\sum_{i=1}^{1} (kx_i) = kx_1$, so both sides match.

For the induction step, assume that $k \cdot (\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} (kx_i)$. Then, again by the recursive definition of sums,

$$k \cdot \left(\sum_{i=1}^{n+1} x_i \right) = k \cdot \left(\sum_{i=1}^{n} x_i + x_{n+1} \right) = k \sum_{i=1}^{n} x_i + kx_{n+1} \ ;$$

in the last step we used the distributivity axiom. Now by induction hypothesis,

$$k \sum_{i=1}^{n} x_i + kx_{n+1} = \sum_{i=1}^{n} (kx_i) + kx_{n+1} = \sum_{i=1}^{n+1} (kx_i) \ ;$$

the last equation follows once more by the recursive definition of sums. Comparing the last two math lines yields

$$k \cdot \left(\sum_{i=1}^{n+1} x_i \right) = \sum_{i=1}^{n+1} (kx_i) \, ,$$

and our induction is complete.

(ii) follows as a special case ($r = 1$) of (iii), so it suffices to prove (iii).

(iii) Again we use induction on n. For $n = 1$, we have $\sum_{i=1}^{1} x_1 = x_1 = r$, as desired. For the induction step, assume that $\sum_{i=1}^{n} x_i = rn$. Then

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1} = rn + r \ .$$

Here the last equation follows with the induction hypothesis and the fact that $x_i = r$ for all i. Since $rn + r = r(n+1)$, we concluded the induction step. $\square$