MATH 725 Midterm Exam 10/19/11

Part I (in-class exam 1:10–2:00 p.m.)

(1) Suppose V and W are vector spaces.
   (a) Define what it means for a set $S \subseteq V$ to be a *basis* of V.
   (b) Define the *dimension* of V.
   (c) Suppose V and W are finite dimensional. Show that if $\dim V > \dim W$ then there exists a surjective linear map $V \to W$.

Proof of (c). Fix bases $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of V and $(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m)$ for W; by assumption, we have $n > m$. We know that a linear map $T : V \to W$ is determined by the images of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, so we define T through

$$T(\mathbf{v}_j) := \begin{cases} \mathbf{w}_j & \text{if } j \geq m, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Given $\sum_{j=1}^{m} a_j \mathbf{w}_j \in W$,

$$T\left(\sum_{j=1}^{m} a_j \mathbf{v}_j\right) = \sum_{j=1}^{m} a_j T(\mathbf{v}_j) = \sum_{j=1}^{m} a_j \mathbf{w}_j,$$

i.e., T is surjective. □

(2) Suppose V and W are vector spaces.
   (a) Define what it means for a map $T : V \to W$ to be *linear*.
   (b) Define the *null space* and the *range* of T.
   (c) Give an example of a linear map (e.g., by giving a matrix) that has a two-dimensional null space and a three-dimensional range.

Solution for (c). Let $T \in L\left(\mathbf{R}^5\right)$ given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$ □

(3) Suppose V is a vector space and $T \in L(V)$.
   (a) Define the notion of *eigenvalue* and *eigenvector* of T.
   (b) Give an example of a linear map on a real vector space that has no eigenvalues.
   (c) State a condition on V or T that guarantees that T has an eigenvalue.

Solution. (b) Let $T \in L\left(\mathbf{R}^2\right)$ given by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c) V is a complex vector space. (Alternative I: V has a basis with respect to which T is upper triangular.) (Alternative II: T is self adjoint.) □

(4) Suppose V is a complex inner-product space and $T \in L(V)$.
   (a) Define T^*.
   (b) Define what it means for T to be *self adjoint* and what it means to be *normal*.
   (c) Show that any self-adjoint operator is normal.

Proof of (c). Suppose $T = T^*$. Then $TT^* = T^2 = T^*T$, i.e., T is normal. $\square$

Part II (take-home exam, due on 10/21/11 at 1:00 p.m.)

You are welcome to use book and internet sources, but you are not allowed to discuss this exam with anyone (including your class mates).

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the text book and homework problems.

(1) Suppose $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a linearly independent list of vectors in some vector space.
   (a) Is the list $(\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2, \ldots, \mathbf{v}_n - \mathbf{v}_{n-1})$ linearly independent?
   (b) Is the list $(\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_2, \ldots, \mathbf{v}_n - \mathbf{v}_{n-1}, \mathbf{v}_1 - \mathbf{v}_n)$ linearly independent?

   Proof. (a) Yes: if $a_1, a_2, \ldots, a_{n-1} \in \mathbf{F}$ satisfy

$$\begin{aligned} 0 &= a_1(\mathbf{v}_2 - \mathbf{v}_1) + a_2(\mathbf{v}_3 - \mathbf{v}_2) + \cdots + a_{n-1}(\mathbf{v}_n - \mathbf{v}_{n-1}) \\ &= -a_1\mathbf{v}_1 + (a_1 - a_2)\mathbf{v}_2 + (a_2 - a_3)\mathbf{v}_3 + \cdots + (a_{n-1} - a_n)\mathbf{v}_n, \end{aligned}$$

   then (since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent) $a_1 = 0$, $a_1 - a_2 = 0$ (and thus $a_2 = 0$), $a_2 - a_3 = 0$ (and thus $a_3 = 0$), etc., down to $a_n = 0$.
   (b) No, because the sum of these vectors is zero. $\square$

(2) Let M be the vector space of all real $n \times n$ matrices, for some fixed $n \in \mathbf{Z}_{>0}$. For $A = (a_{jk}) \in M$, define the *trace* of A as

$$\operatorname{tr}(A) := \sum_{j=1}^{n} a_{jj}.$$

   (a) Show that $U := \{A \in M : \operatorname{tr}(A) = 0\}$ is a subspace of M.
   (b) Compute the dimension of U.

   Proof. (a) The zero matrix has trace zero, and for $A = (a_{jk})$, $B = (b_{jk})$, and $r \in \mathbf{R}$, then

$$\operatorname{tr}(rA + B) = \sum_{j=1}^{n} (r a_{jj} + b_{jj}) = r \sum_{j=1}^{n} a_{jj} + \sum_{j=1}^{n} b_{jj} = r \operatorname{tr}(A) + \operatorname{tr}(B), \qquad (\star)$$

   and so if $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$, we have $\operatorname{tr}(rA + B) = 0$.
   (b) Considering the trace as a map $\operatorname{tr} : M \to \mathbf{R}$, we showed in $(\star)$ that tr is linear. But U is, by definition, the null space of tr. Since tr is surjective (we can reach any $r \in \mathbf{R}$ by considering a matrix with $a_{11} = r$ and all other entries 0), we have

$$n^2 = \dim M = \dim \operatorname{null}(\operatorname{tr}) + \operatorname{range}(\operatorname{tr}) = \dim U + 1,$$

   and thus $\dim U = n^2 - 1$. $\square$

(3) Find a polynomial $p \in \mathscr{P}_2(\mathbf{R})$ such that

$$\int_{-1}^{1} \sin(x)\, q(x)\, dx = \int_{-1}^{1} p(x)\, q(x)\, dx$$

for all $q \in \mathscr{P}_2(\mathbf{R})$.

Proof. Consider the inner product $\langle p, q\rangle := \int_{-1}^{1} p(x)\, q(x)\, dx$ on $\mathscr{P}_2(\mathbf{R})$. In HW 7, we computed the orthonormal basis

$$\left(\tfrac{1}{\sqrt{2}}, \sqrt{\tfrac{3}{2}}\, x, \sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)\right).$$

By the linearity properties of an integral, we can view

$$\varphi(q) := \int_{-1}^{1} \sin(x)\, q(x)\, dx$$

as a linear functional on $\mathscr{P}_2(\mathbf{R})$, and so we can apply (the proof of) Theorem 6.45 in the book (or from the lecture notes):

$$\varphi(q) = \left\langle q, \varphi\left(\tfrac{1}{\sqrt{2}}\right)\tfrac{1}{\sqrt{2}} + \varphi\left(\sqrt{\tfrac{3}{2}}\, x\right)\sqrt{\tfrac{3}{2}}\, x + \varphi\left(\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)\right)\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)\right\rangle.$$

We compute

$$\begin{aligned}
\varphi\left(\tfrac{1}{\sqrt{2}}\right) &= \tfrac{1}{\sqrt{2}} \int_{-1}^{1} \sin(x)\, dx = 0 \\
\varphi\left(\sqrt{\tfrac{3}{2}}\, x\right) &= \sqrt{\tfrac{3}{2}} \int_{-1}^{1} x \sin(x)\, dx = \sqrt{6}\,(\sin(1) - \cos(1)) \\
\varphi\left(\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)\right) &= \sqrt{\tfrac{45}{8}} \int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right) \sin(x)\, dx = 0\,,
\end{aligned}$$

and so

$$\varphi(q) = \left\langle q, \sqrt{6}\,(\sin(1) - \cos(1))\sqrt{\tfrac{3}{2}}\, x\right\rangle = 3\,(\sin(1) - \cos(1)) \int_{-1}^{1} x\, q(x)\, dx\,.$$

Thus $p(x) = 3\,(\sin(1) - \cos(1))\, x$. □

(4) A linear operator $T \in L(V)$ is *nilpotent* if there exists $k \in \mathbf{Z}_{>0}$ such that $T^k = 0$.[1]

(a) Prove that if T is nilpotent, then 0 is the only eigenvalue of T.

(b) Show that if T is nilpotent and self adjoint, then $T = 0$.

Proof. (a) Suppose $T^k = 0$ and k is the smallest positive integer with this property. If $k = 1$, T is the zero map, which certainly has 0 as an eigenvalue. If $k > 1$, then there exists $\mathbf{v} \in V$ such that $T^{k-1}(\mathbf{v}) \neq \mathbf{0}$, and so (because $T\left(T^{k-1}(\mathbf{v})\right) = \mathbf{0} = 0\,\mathbf{v}$) 0 is an eigenvalue of T.

Now suppose λ is another eigenvalue with eigenvector $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{0} = T^k(\mathbf{v}) = \lambda^k \mathbf{v}$ and so $\lambda = 0$.[2]

(b) Suppose T is nilpotent and self adjoint. By the spectral theorem, there exists an orthonormal basis with respect to which T has a diagonal matrix, and the diagonal entries are the eigenvalues of T. By part (a), these entries are zero, and so the matrix is the zero matrix. □

[1] Here 0 is the linear operator that returns $\mathbf{0}$ for every input vector.

[2] Technically, here it is important that V is a vector space over $\mathbf{R}$ or $\mathbf{C}$, since otherwise we cannot conclude from $\lambda^k = 0$ that $\lambda = 0$.