Proposition 11.13. *Suppose* $A = \lim_{n\to\infty} a_n$ *and* $B = \lim_{n\to\infty} b_n$.

(i) *For any* $c \in \mathbb{R}$, $\lim\limits_{n\to\infty} (c\, a_n) = c\, A$.

(ii) $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$.

(iii) $\lim\limits_{n\to\infty} (a_n b_n) = AB$.

(iv) *If* $A \neq 0$, *then* $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.

Proof. Suppose $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$.

(ii) By assumption,

$$\text{for all } \epsilon > 0, \text{ there exists } N_1 \in \mathbb{N}, \text{ such that for all } n \geq N_1, \ |a_n - A| < \frac{\epsilon}{2} \ ,$$

and

$$\text{for all } \epsilon > 0, \text{ there exists } N_2 \in \mathbb{N}, \text{ such that for all } n \geq N_2, \ |b_n - B| < \frac{\epsilon}{2} \ .$$

Now given $\epsilon > 0$, choose $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

Here the penultimate inequality follows from Proposition 11.6(iv), and the last inequality follows from our assumptions.

(iii) By Proposition 11.12, (b_n) is bounded because it converges. So suppose $M > 0$ is a bound for (b_n), that is, $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Now we distinguish two cases, namely $A = 0$ and $A \neq 0$. In the first case $A = 0$, we use that

$$\text{for all } \epsilon > 0, \text{ there exists } N_1 \in \mathbb{N}, \text{ such that for all } n \geq N_1, \ |a_n| < \frac{\epsilon}{M} \ .$$

Now given $\epsilon > 0$, choose $N = N_1$. Then for all $n \geq N$,

$$|a_n b_n - AB| = |a_n b_n| = |a_n|\,|b_n| < \frac{\epsilon}{M}\, M = \epsilon \ .$$

In the second case $A \neq 0$, we use that

$$\text{for all } \epsilon > 0, \text{ there exists } N_1 \in \mathbb{N}, \text{ such that for all } n \geq N_1, \ |a_n - A| < \frac{\epsilon}{2M} \ ,$$

and

$$\text{for all } \epsilon > 0, \text{ there exists } N_2 \in \mathbb{N}, \text{ such that for all } n \geq N_2, \ |b_n - B| < \frac{\epsilon}{2|A|} \ .$$

Now given $\epsilon > 0$, choose $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - A b_n + A b_n - AB| = |(a_n - A)\, b_n + (b_n - B)\, A| \\ &\leq |a_n - A|\,|b_n| + |b_n - B|\,|A| < \frac{\epsilon}{2M} M + \frac{\epsilon}{2|A|} |A| = \epsilon \ . \end{aligned}$$

□