Proposition 5.16. *Let p be prime and $0 < r < p$. Then $\binom{p}{r}$ is divisible by p.*

Proof. Suppose p is prime and $0 < r < p$. Then none of the numbers $1, 2, \ldots, r$ and $1, 2, \ldots, p-r$ divides p. On the other hand, we know that

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} = \frac{p \cdot (p-1)!}{r!(p-r)!}$$

is an integer. If $1, 2, \ldots, r$ and $1, 2, \ldots, p-r$ do not divide p, then they have to divide $(p-1)!$, i.e., $\frac{(p-1)!}{r!(p-r)!}$ is an integer. But then $\binom{p}{r} = p\,\frac{(p-1)!}{r!(p-r)!}$ implies that p divides $\binom{p}{r}$. $\square$