Proposition 9.11. *The function e preserves multiplication: that is, $e\left(m \cdot_{\mathbb{Z}} n\right) = e(m) \cdot_{\mathbb{R}} e(n)$, where $\cdot_{\mathbb{Z}}$ and $\cdot_{\mathbb{R}}$ are the two multiplication operations.*

Proof. Fix $m \in \mathbb{Z}$. We will prove that for all $n \in \mathbb{Z}$, $e\left(m \cdot n\right) = e(m) \cdot e(n)$ in three steps. First, the result holds for $n = 0$ since $e(0) = 0$ and we have $x \cdot 0 = 0$, both in $\mathbb{Z}$ and $\mathbb{R}$.

Next, we prove $P(n):\ e\left(m \cdot n\right) = e(m) \cdot e(n)$ by induction on $n \in \mathbb{N}$, which will establish the proposition for *positive* n. The base case $P(1)$ follows with the fact that $e(1) = 1$ and the axioms about the numbers 1 in $\mathbb{Z}$ and $\mathbb{R}$. For the induction step, assume $P(n)$. Then, by Proposition 9.10,

$$\begin{aligned} e\left(m \cdot (n+1)\right) &= e\left(m \cdot n + m\right) = e\left(m \cdot n\right) + e\left(m\right) \overset{(\star)}{=} e(m) \cdot e(n) + e(m) \\ &= e(m) \cdot (e(n) + 1) = e(m) \cdot e(n+1)\ . \end{aligned}$$

Here $(\star)$ follows from the induction hypothesis.

Finally, we prove $Q(n):\ e\left(m \cdot (-n)\right) = e(m) \cdot e(-n)$ by induction on $n \in \mathbb{N}$, which will establish the proposition for *negative* n. The base case $Q(1)$ follows with $e(-1) = -e(1) = -1$ and Proposition 9.8(iii):

$$e\left(m \cdot (-1)\right) = e(-m) = -e(m) = (-1) \cdot e(m) = e(-1) \cdot e(m)\ .$$

For the induction step, assume $Q(n)$. Then, with Proposition 9.10 and the base case $Q(1)$,

$$\begin{aligned} e\left(m \cdot (-(n+1))\right) &= e\left(m \cdot (-n) + (-m)\right) = e\left(m \cdot (-n)\right) + e(-m) \overset{(\star)}{=} e(m) \cdot e(-n) + e(-m) \\ &= e(m) \cdot e(-n) - e(m) = e(m) \cdot (e(-n) - 1) = e(m) \cdot e(-n-1), \end{aligned}$$

where $(\star)$ follows from the induction hypothesis. $\square$