MATH 725 **Homework Set 14** **due 12/12/11**

(1) Suppose V is a complex inner-product space and $\mathbf{v},\mathbf{w} \in V$. Define the linear operator $T(\mathbf{u}) := \langle \mathbf{u},\mathbf{v}\rangle \mathbf{w}$. Find a formula for $\operatorname{tr}(T)$.

Solution. Fix an orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n)$ of V, and express $\mathbf{v}$ and $\mathbf{w}$ in terms of this basis:

$$\mathbf{v} = \sum_{j=1}^{n} a_j\,\mathbf{e}_j\,, \qquad \mathbf{w} = \sum_{j=1}^{n} b_j\,\mathbf{e}_j\,,$$

where $a_j := \langle \mathbf{v},\mathbf{e}_j\rangle$ and $b_j := \langle \mathbf{w},\mathbf{e}_j\rangle$. Now

$$T(\mathbf{e}_k) = \langle \mathbf{e}_k,\mathbf{v}\rangle\,\mathbf{w} = \overline{a_k}\sum_{j=1}^{n} b_j\,\mathbf{e}_j$$

and so the (j,k)-entry of the matrix of T (with respect to our basis) is $\overline{a_k}b_j$. Thus

$$\operatorname{tr}(T) = \sum_{j=1}^{n} \overline{a_j}b_j = \langle \mathbf{w},\mathbf{v}\rangle\,. \qquad \square$$

(2) Suppose V is a complex inner-product space, and $T \in L(V)$.
(a) Prove that if $(\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n)$ is an orthonormal basis of V then

$$\operatorname{tr}(T^*T) = ||T(\mathbf{e}_1)||^2 + ||T(\mathbf{e}_2)||^2 + \cdots + ||T(\mathbf{e}_n)||^2\,.$$

(b) Show that if T is positive and $\operatorname{tr}(T) = 0$ then $T = 0$, the zero operator.

Proof. (a) By definition,

$$\operatorname{tr}(T^*T) = \sum_{j=1}^{n} \left\langle T^*T(\mathbf{e}_j),\mathbf{e}_j\right\rangle = \sum_{j=1}^{n} \left\langle T(\mathbf{e}_j),T(\mathbf{e}_j)\right\rangle = \sum_{j=1}^{n} \left|\left|T(\mathbf{e}_j)\right|\right|^2.$$

(b) Suppose T is positive and $\operatorname{tr}(T) = 0$. We proved in class (some time ago) that, since T is positive, there exists $S \in L(V)$ such that $T = S^*S$. Thus, by part (a),

$$0 = \operatorname{tr}(T) = \operatorname{tr}(S^*S) = \sum_{j=1}^{n} \left|\left|S(\mathbf{e}_j)\right|\right|^2,$$

and so $S(\mathbf{e}_j) = 0$ for $j = 1,\dots,n$. But then S is the zero operator and thus $T = 0$. $\square$

(3) Construct counterexamples to the following claims about $S,T \in L(V)$:
(a) $\operatorname{tr}(ST) = \operatorname{tr}(S)\operatorname{tr}(T)$.
(b) $\det(S+T) = \det(S) + \det(T)$.

Solution. (a) Let $S = T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (with respect to the standard basis of $\mathbf{C}^2$). Then $\operatorname{tr}(S) = \operatorname{tr}(T) = 0$ but $\operatorname{tr}(ST) = \operatorname{tr}(\mathrm{I}) = 2$.

(b) Let S and T be the identity operator on $\mathbf{C}^2$. Then $S+T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ with determinant 4, whereas $\det(S) = \det(T) = 1$. $\square$

(4) Suppose V is a complex inner-product space.
(a) Prove that $\langle S,T\rangle := \operatorname{tr}(ST^*)$ defines an inner product on $L(V)$.

(b) Fix an orthonormal basis of V and let $\left(a_{jk}\right)_{1\leq j,k\leq n}$ be the matrix of $T \in L(V)$ with respect to this basis. Show that

$$\langle T,T\rangle = \sum_{1\leq j,k\leq n} \left|a_{jk}\right|^2,$$

the square of the standard norm on $L(V)$ (identifying operators with their matrices with respect to the chosen orthonormal basis). Conclude that $\langle S,T\rangle$ coincides with the standard inner product on $L(V)$.

Proof. (a) $\langle T,T\rangle \geq 0$ and $\langle T,T\rangle = 0 \iff T = 0$ follow with Exercise 2(a). Second,

$$\begin{aligned}\langle aR+S,T\rangle &= \operatorname{tr}\left((aR+S)T^*\right) = \operatorname{tr}\left(aRT^* + ST^*\right) = \operatorname{tr}\left(aRT^*\right) + \operatorname{tr}\left(ST^*\right)\\ &= a\operatorname{tr}\left(RT^*\right) + \operatorname{tr}\left(ST^*\right) = a\operatorname{tr}\langle R,T^*\rangle + \langle S,T^*\rangle\,.\end{aligned}$$

Here we have used that for any linear operator T and $a \in \mathbf{C}$, $\operatorname{tr}(aT) = a\operatorname{tr}(T)$, which is easily seen to be true by writing T in terms of a matrix.

For the final property that will determine that $\langle S,T\rangle$ is indeed a inner product, we will need that for any $T \in L(V)$,

$$\operatorname{tr}(T^*) = \sum_{j=1}^{n} \left\langle T^*(\mathbf{e}_j),\mathbf{e}_j\right\rangle = \sum_{j=1}^{n} \left\langle \mathbf{e}_j,T(\mathbf{e}_j)\right\rangle = \sum_{j=1}^{n} \overline{\left\langle T(\mathbf{e}_j),\mathbf{e}_j\right\rangle} = \overline{\sum_{j=1}^{n} \left\langle T(\mathbf{e}_j),\mathbf{e}_j\right\rangle} = \overline{\operatorname{tr}(T)}\,.$$

Thus

$$\langle S,T\rangle = \operatorname{tr}(ST^*) = \overline{\operatorname{tr}(TS^*)} = \overline{\langle T,S\rangle}\,,$$

and thus $\langle S,T\rangle$ satisfies all the properties of an inner product.

(b) By Exercise 2(a),

$$\langle T,T\rangle = \operatorname{tr}(TT^*) = \operatorname{tr}(T^*T) = \sum_{k=1}^{n} ||T(\mathbf{e}_k)||^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} \left|a_{jk}\right|^2.$$

As the statement of the exercise says, this is the square of the standard norm on $L(V)$. Thus (by Exercise 2 of Homework Set 7) this norm induces the standard inner product on $L(V)$. $\square$

(5) Suppose V is a complex inner-product space and $T \in L(V)$.

(a) Show that $\det(T^*) = \overline{\det(T)}$.

(b) Show that $|\det(T)| = \det\sqrt{T^*T}$.

Proof. (a) Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of T (listed with repetition according to multiplicities). We proved earlier that $\overline{\lambda_1},\overline{\lambda_2},\ldots,\overline{\lambda_n}$ are then the eigenvalues of T^*, and so

$$\det(T^*) = \overline{\lambda_1}\,\overline{\lambda_2}\cdots\overline{\lambda_n} = \overline{\lambda_1\,\lambda_2\cdots\lambda_n} = \overline{\det(T)}\,.$$

(b) We know that $T = S\sqrt{T^*T}$ for some isometry S. Note that $|\det(S)| = 1$ and $\sqrt{T^*T}$ is positive (and thus has a nonnegative determinant), and so

$$|\det(T)| = \left|\det(S)\det\sqrt{T^*T}\right| = \det\sqrt{T^*T}\,. \qquad \square$$