A Meshalkin Theorem for Projective Geometries[1]

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Dedicated to the memory of Lev Meshalkin.

Abstract: Let $\mathcal{M}$ be a family of sequences $(a_1, \ldots, a_p)$ where each a_k is a flat in a projective geometry of rank n (dimension $n-1$) and order q, and the sum of ranks, $r(a_1) + \cdots + r(a_p)$, equals the rank of the join $a_1 \vee \cdots \vee a_p$. We prove upper bounds on $|\mathcal{M}|$ and corresponding LYM inequalities assuming that (i) all joins are the whole geometry and for each $k < p$ the set of all a_k's of sequences in $\mathcal{M}$ contains no chain of length l, and that (ii) the joins are arbitrary and the chain condition holds for all k. These results are q-analogs of generalizations of Meshalkin's and Erdős's generalizations of Sperner's theorem and their LYM companions, and they generalize Rota and Harper's q-analog of Erdős's generalization.

Keywords: Sperner's theorem, Meshalkin's theorem, LYM inequality, antichain, r-family, r-chain-free

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1. Introducing the Players

We present a theorem that is at once a q-analog of a generalization, due to Meshalkin, of Sperner's famous theorem on antichains of sets and a generalization of Rota and Harper's q-analog of both Sperner's theorem and Erdős's generalization.

Sperner's theorem [12] concerns a subset $\mathcal{A}$ of $\mathcal{P}(S)$, the power set of an n-element set S, that is an *antichain*: no member of $\mathcal{A}$ contains another. It is part (b) of the following theorem. Part (a), which easily implies (b) (see, e.g., [1, Section 1.2]) was found later by Lubell [9], Yamamoto [13], and Meshalkin [10] (and Bollobás independently proved a generalization [4]); consequently, it and similar inequalities are called *LYM inequalities*.

Theorem 1. *Let $\mathcal{A}$ be an antichain of subsets of S. Then:*

(a) $\displaystyle\sum_{A\in\mathcal{A}} \frac{1}{|A|} \leq 1$ *and*
(b) $|\mathcal{A}| \leq \binom{n}{\lfloor n/2\rfloor}$.
(c) *Equality occurs in (a) and (b) if $\mathcal{A}$ consists of all subsets of S of size $\lfloor n/2\rfloor$, or all of size $\lceil n/2\rceil$.*

The idea of Meshalkin's insufficiently well known generalization[3] (an idea he attributes to Sevast'yanov) is to consider ordered p-tuples $A = (A_1, \ldots, A_p)$ of pairwise disjoint sets whose union is S. We call these *weak compositions of S into p parts.*

Theorem 2. *Let $\mathcal{M}$ be a family of weak compositions of S into p parts such that each set $\mathcal{M}_k = \{A_k : A \in \mathcal{M}\}$ is an antichain.*

(a) $\displaystyle\sum_{A\in\mathcal{M}} \frac{1}{\binom{n}{|A_1|,\ldots,|A_p|}} \leq 1.$
(b) $\displaystyle|\mathcal{M}| \leq \max_{\alpha_1+\cdots+\alpha_p=n} \binom{n}{\alpha_1, \ldots, \alpha_p} = \binom{n}{\lceil\frac{n}{p}\rceil, \ldots, \lceil\frac{n}{p}\rceil, \lfloor\frac{n}{p}\rfloor, \ldots, \lfloor\frac{n}{p}\rfloor}.$
(c) *Equality occurs in (a) and (b) if, for each k, $\mathcal{M}_k$ consists of all subsets of S of size $\lceil\frac{n}{p}\rceil$, or all of size $\lfloor\frac{n}{p}\rfloor$.*

Part (b) is Meshalkin's theorem [10]; the corresponding LYM inequality (a) was subsequently found by Hochberg and Hirsch [7]. (In expressions like the multinomial coefficient in (b), since the lower numbers must sum to n, the number of them that equal $\lceil\frac{n}{p}\rceil$ is the least nonnegative residue of n modulo $p+1$.)

In [2] Wang and we generalized Theorem 2 in a way that simultaneously also generalizes Erdős's theorem on *l-chain-free families*: subsets of $\mathcal{P}(S)$ that contain no chain of length l. (Such families have been called "r-families" and "k-families", where r or k is the forbidden length. We believe a more suggestive name is needed.)

Theorem 3 ([2, Corollary 4.1])**.** *Let $\mathcal{M}$ be a family of weak compositions of S into p parts such that each $\mathcal{M}_k$, for $k < p$, is l-chain-free. Then:*

(a) $\displaystyle\sum_{A\in\mathcal{M}} \frac{1}{\binom{n}{|A_1|,\ldots,|A_p|}} \leq l^{p-1}$, *and*

[3] We do not find it in books on the subject [1, 5] but only in [8].

(b) $|\mathcal{M}|$ *is no greater than the sum of the* l^{p-1} *largest multinomial coefficients of the form* $\binom{n}{\alpha_1,\dots,\alpha_p}$.

Erdős's theorem [6] is essentially the case $p = 2$, in which $A_2 = S \setminus A_1$ is redundant. The upper bound is then the sum of the l largest binomial coefficients $\binom{n}{j}$, $0 \le j \le n$, and is attained by taking a suitable subclass of $\mathcal{P}(S)$. In general the bounds in Theorem 3 cannot be attained [2, Section 5].

Rota and Harper began the process of q-analogizing by finding versions of Sperner's and Erdős's theorems for finite projective geometries [11]. We think of a projective geometry $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(q)$ of order q and rank n (i.e., dimension $n-1$) as a lattice of flats, in which $\hat{0} = \varnothing$ and $\hat{1}$ is the whole set of points. The *rank* of a flat a is $r(a) = \dim a + 1$. The *q-Gaussian coefficients* (usually the "q" is omitted) are the quantities

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!_q}{k!_q (n-k)!_q} \qquad \text{where} \qquad n!_q = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1) \ .$$

They are the q-analogs of the binomial coefficients. Again, a family of projective flats is *l-chain-free* if it contains no chain of length l. Let $\mathcal{L}_k$ be the set of all flats of rank k in $\mathbb{P}^{n-1}(q)$.

Theorem 4 ([11, p. 200])**.** *Let $\mathcal{A}$ be an l-chain-free family of flats in $\mathbb{P}^{n-1}(q)$.*

(a) $\displaystyle\sum_{a \in \mathcal{A}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix}} \le l.$

(b) $|\mathcal{A}|$ *is at most the sum of the l largest Gaussian coefficients* $\begin{bmatrix} n \\ j \end{bmatrix}$ *for* $0 \le j \le n$.

(c) *There is equality in (a) and (b) when $\mathcal{A}$ consists of the l largest classes $\mathcal{L}_k$, if $n - l$ is even, or the $l - 1$ largest classes and one of the two next largest classes, if $n - l$ is odd.*

Our q-analog theorem concerns the projective analogs of weak compositions of a set. A *Meshalkin sequence of length p* in $\mathbb{P}^{n-1}(q)$ is a sequence $a = (a_1, \dots, a_p)$ of flats whose join is $\hat{1}$ and whose ranks sum to n. The submodular law implies that, if $a_J := \bigvee_{j \in J} a_j$ for an index subset $J \subseteq [p] = \{1, 2, \dots, p\}$, then $a_I \wedge a_J = \hat{0}$ for any disjoint $I, J \subseteq [p]$; so the members of a Meshalkin sequence are highly disjoint.

To state the result we need a few more definitions. If $\mathcal{M}$ is a set of Meshalkin sequences, then for each $k \in [p]$ we define $\mathcal{M}_k := \{a_k : (a_1, \dots, a_p) \in \mathcal{M}\}$. If $\alpha_1, \dots, \alpha_p$ are nonnegative integers whose sum is n, we define the *(q-)Gaussian multinomial coefficient* to be

$$\begin{bmatrix} n \\ \alpha \end{bmatrix} = \begin{bmatrix} n \\ \alpha_1, \dots, \alpha_p \end{bmatrix} = \frac{n!_q}{\alpha_1!_q \cdots \alpha_p!_q} \ ,$$

where $\alpha = (\alpha_1, \dots, \alpha_p)$. We write

$$s_2(\alpha) = \sum_{i<j} \alpha_i \alpha_j$$

for the second elementary symmetric function of α. If a is a Meshalkin sequence, we write

$$r(a) = (r(a_1), \dots, r(a_p))$$

for the sequence of ranks. We define $\mathbb{P}^{n-1}(q)$ to be empty if $n = 0$, a point if $n = 1$, and a line of $q + 1$ points if $n = 2$.

Theorem 5. *Let $n \geq 0$, $l \geq 1$, $p \geq 2$, and $q \geq 2$. Let $\mathcal{M}$ be a family of Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that, for each $k \in [p-1]$, $\mathcal{M}_k$ contains no chain of length l. Then*

(a) $\displaystyle\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix} q^{s_2(r(a))}} \leq l^{p-1}$, *and*

(b) $|\mathcal{M}|$ *is at most equal to the sum of the l^{p-1} largest amongst the quantities $\begin{bmatrix} n \\ \alpha \end{bmatrix} q^{s_2(\alpha)}$ for $\alpha = (\alpha_1, \dots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$.*

The antichain case (where $l = 1$), the analog of Meshalkin and Hochberg and Hirsch's theorems, is captured in

Corollary 6. *Let $\mathcal{M}$ be a family of Meshalkin sequences of length $p \geq 2$ in $\mathbb{P}^{n-1}(q)$ such that each $\mathcal{M}_k$ for $k < p$ is an antichain. Then*

(a) $\displaystyle\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix} q^{s_2(r(a))}} \leq 1$, *and*

(b) $\displaystyle|\mathcal{M}| \leq \max_{\alpha} \begin{bmatrix} n \\ \alpha \end{bmatrix} q^{s_2(r(a))} = \begin{bmatrix} n \\ \lceil \frac{n}{p} \rceil, \dots, \lceil \frac{n}{p} \rceil, \lfloor \frac{n}{p} \rfloor, \dots, \lfloor \frac{n}{p} \rfloor \end{bmatrix} q^{s_2(\lceil n/p \rceil, \dots, \lceil n/p \rceil, \lfloor n/p \rfloor, \dots, \lfloor n/p \rfloor)}.$

(c) *Equality holds in (a) and (b) if, for each k, $\mathcal{M}_k$ consists of all flats of rank $\lceil \frac{n}{p} \rceil$ or all of rank $\lfloor \frac{n}{p} \rfloor$.* ■

We believe—but without proof—that the largest families $\mathcal{M}$ described in (c) are the only ones.

Notice that we do not place any condition in either the theorem or its corollary on $\mathcal{M}_p$.

Our theorem is not exactly a generalization of that of Rota and Harper because a flat in a projective geometry has a variable number of complements, depending on its rank. Still, our result does imply this and a generalization, as we shall demonstrate in Section 4.

2. Proof of Theorem 5

The proof of Theorem 5 is adapted from the short proof of Theorem 3 in [3]. It is complicated by the multiplicity of complements of a flat, so we require the powerful lemma of Harper, Klain, and Rota ([8, Lemma 3.1.3], improving on [11, Lemma on p. 199]; for a short proof see [2, Lemmas 3.1 and 5.2]) and a count of the number of complements.

Lemma 7. *Suppose given real numbers $m_1 \geq m_2 \geq \cdots \geq m_N \geq 0$, other real numbers $q_1, \dots, q_N \in [0, 1]$, and an integer P with $1 \leq P \leq N$. If $\sum_{k=1}^{N} q_k \leq P$, then*

$$q_1 m_1 + \cdots + q_N m_N \leq m_1 + \cdots + m_P \ . \tag{1}$$

Let $m_{P'+1}$ and $m_{P''}$ be the first and last m_k's equal to m_P. Assuming $m_P > 0$, there is equality in (1) if and only if

$$q_k = 1 \text{ for } m_k > m_P, \qquad q_k = 0 \text{ for } m_k < m_P, \qquad \text{and} \qquad q_{P'+1} + \cdots + q_{P''} = P - P' \ .$$

■

Lemma 8. *A flat of rank k in $\mathbb{P}^{n-1}(q)$ has $q^{k(n-k)}$ complements.*

Proof. The number of ways to extend a fixed ordered basis $(P_1, \dots, P_k)$ of the flat to an ordered basis $(P_1, \dots, P_n)$ of $\mathbb{P}^{n-1}(q)$ is

$$\frac{q^n - q^k}{q-1} \, \frac{q^n - q^{k+1}}{q-1} \cdots \frac{q^n - q^{n-1}}{q-1} \, .$$

Then $P_{k+1} \vee \cdots \vee P_n$ is a complement and is generated by the last $n-k$ points in

$$\frac{q^{n-k} - 1}{q-1} \, \frac{q^{n-k} - q}{q-1} \cdots \frac{q^{n-k} - q^{n-k-1}}{q-1}$$

of the extended ordered bases. Dividing the former by the latter, there are

$$q^{\left(\binom{n}{2} - \binom{k}{2}\right) - \binom{n-k}{2}} = q^{k(n-k)}$$

complements. ■

Proof of (a). We proceed by induction on p. For a flat f, define

$$\mathcal{M}(f) := \{(a_2, \dots, a_p) : \ (f, a_2, \dots, a_p) \in \mathcal{M}\}$$

and also, letting c be another flat, define

$$\mathcal{M}^c(f) := \{(a_2, \dots, a_p) \in \mathcal{M}(f) : \ a_2 \vee \cdots \vee a_p = c\} \, .$$

For $a \in \mathcal{M}$, we write $r_1 = r(a_1)$. Finally, $\mathcal{C}(a_1)$ is the set of complements of a_1. If $p > 2$, then

$$\begin{aligned}
\sum_{a \in \mathcal{M}} \frac{1}{\left[{n \atop r(a)}\right] q^{s_2(r(a))}} &= \sum_{a_1 \in \mathcal{M}_1} \frac{1}{\left[{n \atop r_1}\right] q^{r_1(n-r_1)}} \sum_{a' \in \mathcal{M}(a_1)} \frac{1}{\left[{n-r_1 \atop r(a')}\right] q^{s_2(r(a'))}} \\
&= \sum_{a_1 \in \mathcal{M}_1} \frac{1}{\left[{n \atop r_1}\right] q^{r_1(n-r_1)}} \sum_{c \in \mathcal{C}(a_1)} \sum_{a' \in \mathcal{M}^c(a_1)} \frac{1}{\left[{n-r_1 \atop r(a')}\right] q^{s_2(r(a'))}} \\
&\le \sum_{a_1 \in \mathcal{M}_1} \frac{1}{\left[{n \atop r_1}\right] q^{r_1(n-r_1)}} \sum_{c \in \mathcal{C}(a_1)} l^{p-2}
\end{aligned}$$

by induction, because $\mathcal{M}^c(a_1)$ is a Meshalkin family in $c \cong \mathbb{P}^{r(c)-1} = \mathbb{P}^{n-r_1-1}$ and each $\mathcal{M}^c_k(a')$ for $k < p-1$, being a subset of $\mathcal{M}_{k+1}$, is l-chain-free,

$$= \sum_{a_1 \in \mathcal{M}_1} \frac{1}{\left[{n \atop r_1}\right] q^{r_1(n-r_1)}} q^{r_1(n-r_1)} l^{p-2}$$

by Lemma 8,

$$\le l \cdot l^{p-2}$$

by the theorem of Rota and Harper.

The initial case, $p = 2$, is similar except that the innermost sum in the second step equals 1. ■

Lemma 9. *Let $\alpha = (\alpha_1, \dots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$. The number of all Meshalkin sequences a in $\mathbb{P}^{n-1}$ with $r(a) = \alpha$ is $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right] q^{s_2(\alpha)}$.*

Proof. If $p = 1$, then $a = \hat{1}$ so the conclusion is obvious. If $p > 1$, we get a Meshalkin sequence of length p in $\mathbb{P}^{n-1}$ with rank sequence $r(a) = \alpha$ by choosing a_1 to have rank α_1, then a complement c of a_1, and finally a Meshalkin sequence a' of length $p-1$ in $c \cong \mathbb{P}^{r(c)-1} = \mathbb{P}^{n-\alpha_1-1}$ whose rank sequence is $\alpha' = (\alpha_2, \dots, \alpha_p)$. The first choice can be made in $\left[\begin{smallmatrix} n-\alpha_1 \\ \alpha' \end{smallmatrix}\right]$ ways, the second in $q^{\alpha_1(n-\alpha_1)}$ ways, and the third, by induction, in $\left[\begin{smallmatrix} n-\alpha_1 \\ \alpha' \end{smallmatrix}\right] q^{s_2(\alpha')}$ ways. Multiply. ■

Proof of (b). Let $N(\alpha)$ be the number of $a \in \mathcal{M}$ for which $r(a) = \alpha$. In Lemma 7 take

$$q_\alpha = \frac{N(\alpha)}{\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right] q^{s_2(\alpha)}} \qquad \text{and} \qquad m_\alpha = \begin{bmatrix} n \\ \alpha \end{bmatrix} q^{s_2(\alpha)} ,$$

and number all possible α so that $m_{\alpha^1} \geq m_{\alpha^2} \geq \cdots$.

Lemma 9 shows that all $q_\alpha \leq 1$ so Lemma 7 does apply. The conclusion is that

$$|\mathcal{M}| = \sum_{i=1}^{N} q_{\alpha^i} m_{\alpha^i} \leq \begin{bmatrix} n \\ \alpha^1 \end{bmatrix} q^{s_2(\alpha^1)} + \cdots + \begin{bmatrix} n \\ \alpha^P \end{bmatrix} q^{s_2(\alpha^P)} ,$$

where $N = \binom{n+p-1}{p-1}$, the number of sequences α, and $P = \min(l^{p-1}, N)$. ■

3. Strangeness of the LYM Inequality

There is something odd about the LYM inequality in Theorem 5(a). A normal LYM inequality would be expected to have denominator $\left[\begin{smallmatrix} n \\ r(a) \end{smallmatrix}\right]$ without the extra factor $q^{s_2(r(a))}$. Such an LYM inequality does exist; it is a corollary of Theorem 5(a); but it is not strong enough to give the upper bound on $|\mathcal{M}|$. We prove this weaker inequality here.

Proposition 10. *Assume the hypotheses of Theorem 5; that is: $n \geq 0$, $l \geq 1$, $p \geq 2$, and $q \geq 2$; and $\mathcal{M}$ is a family of Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that, for each $k \in [p-1]$, $\mathcal{M}_k$ contains no chain of length l. Then $\sum_{a \in \mathcal{M}} \left[\begin{smallmatrix} n \\ r(a) \end{smallmatrix}\right]^{-1}$ is bounded above by the sum of the l^{p-1} largest expressions $q^{s_2(\alpha)}$ for $\alpha = (\alpha_1, \dots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$.*

Proof. Again we apply Lemma 7, this time with $q_\alpha = N(\alpha) / \left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right] q^{s_2(\alpha)}$ and $M_\alpha = q^{s_2(\alpha)}$. ■

4. A "Partial" Corollary

We deduce Theorem 4(a) from the case $p = 2$ of Theorem 5(a). Our purpose is not to give a new proof of Theorem 4 but to show that we have a generalization of it.

The key to the proof is that $\mathcal{M}_2$ in our theorem is not required to be l-chain-free. Therefore if we have an l-chain-free set $\mathcal{A}$ of flats in $\mathbb{P}^{n-1}$, we can define

$$\mathcal{M} = \{(a, c) : \ a \in \mathcal{A} \text{ and } c \in \mathcal{C}(a)\} ;$$

and $\mathcal{M}$ will satisfy the requirements of Theorem 5. The LYM sum in Theorem 5(a) then equals the LYM sum in Theorem 4(a), and we are done.

The same argument gives a general corollary. A *partial Meshalkin sequence of length* p is a sequence $a = (a_1, \ldots, a_p)$ of flats in $\mathbb{P}^{n-1}(q)$ such that $r(a_1 \vee \cdots \vee a_p) = r(a_1) + \cdots + r(a_p)$. We simply do not require the join $\hat{a} = a_1 \vee \cdots \vee a_p$ to be $\hat{1}$. The generalized Rota–Harper theorem is:

Corollary 11. *Let $p \geq 1$, $l \geq 1$, $q \geq 2$, and $n \geq 0$. Let $\mathcal{M}$ be a family of partial Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that, for each $k \in [p]$, $\mathcal{M}_k$ contains no chain of length l. Then*

(a) $\displaystyle\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(\hat{a}) \end{bmatrix} \begin{bmatrix} r(\hat{a}) \\ r(a) \end{bmatrix} q^{s_2(r(a))}} \leq l^p$ *and*

(b) $|\mathcal{M}|$ *is at most equal to the sum of the l^p largest amongst the quantities $\begin{bmatrix} n \\ \alpha \end{bmatrix} q^{s_2(\alpha)}$ for $\alpha = (\alpha_1, \ldots, \alpha_{p+1})$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_{p+1} = n$.* ■

As a special case we generalize the q-analog of Sperner's theorem. (The q-analog is the case $p = 1$.)

Corollary 12. *Let $\mathcal{M}$ be a family of partial Meshalkin sequences of length $p \geq 1$ in $\mathbb{P}^{n-1}$ such that each $\mathcal{M}_k$ is an antichain. Then:*

(a) $\displaystyle\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(\hat{a}) \end{bmatrix} \begin{bmatrix} r(\hat{a}) \\ r(a) \end{bmatrix} q^{s_2(r(a))}} \leq 1.$

(b) $|\mathcal{M}| \leq \begin{bmatrix} n \\ \alpha \end{bmatrix} q^{s_2(\alpha)}$, *in which $\alpha = \left(\left\lceil \frac{n}{p+1} \right\rceil, \ldots, \left\lceil \frac{n}{p+1} \right\rceil, \left\lfloor \frac{n}{p+1} \right\rfloor, \ldots, \left\lfloor \frac{n}{p+1} \right\rfloor \right)$ where the number of terms equal to $\left\lceil \frac{n}{p+1} \right\rceil$ is the least nonnegative residue of n modulo $p+1$.*

(c) *Equality holds in (a) and (b) if, for each k, $\mathcal{M}_k$ consists of all flats of rank $\left\lceil \frac{n}{p+1} \right\rceil$ or all flats of rank $\left\lfloor \frac{n}{p+1} \right\rfloor$.* ■

We conjecture that the largest families $\mathcal{M}$ described in (c) are unique.

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