**Proposition 11.5.** *For $x, y \in \mathbb{R}_{\geq 0}$, $x < y$ if and only if $x^2 < y^2$, and $x = y$ if and only if $x^2 = y^2$.*

*Proof.* Suppose $x, y \in \mathbb{R}_{\geq 0}$.

If $x < y$, then we have $y - x \in \mathbb{R}_{>0}$. If $x = 0$, then $y > 0$ implies $y^2 > 0$ (by Axiom 8.6(ii)). If $x > 0$, then $y^2 - x^2 = (y - x)(y + x) \in \mathbb{R}_{>0}$, by Axiom 8.6(i) and (ii).

Conversely, if $x^2 < y^2$, then $y^2 - x^2 = (y - x)(y + x) \in \mathbb{R}_{>0}$. We cannot have $x = y = 0$, so at least one of $x$ and $y$ is in $\mathbb{R}_{>0}$, which implies that $y + x \in \mathbb{R}_{>0}$. There are three cases for $y - x$, namely $y - x < 0$ or $y - x = 0$ or $y - x > 0$. The first case leads to a contradiction of Axiom 8.6(ii) (since $y^2 - x^2 > 0$ and $y + x > 0$), and the second case gives $x = y$, which implies $x^2 = y^2$, and this would contradict our assumption. Hence $y - x > 0$.

For the second part of the statement, $x = y$ implies $x^2 = y^2$, so we only need to prove the converse. Suppose $x^2 = y^2$. Then $0 = x^2 - y^2 = (x - y)(x + y)$, which means that either $x - y = 0$ or $x + y = 0$. If the former holds, we obtain the desired identity $x = y$, so now assume that $x + y = 0$. If both $x, y > 0$, then $x + y > 0$ by Axiom 8.6(i), which cannot happen if $x + y = 0$. So at least one of $x, y$ has to be 0; let's say $x = 0$ (we can always switch $x$ and $y$ if necessary). Then $0 = x + y = 0 + y = y$, so in both cases we get $x = y$. $\square$