THE NUMBER OF NOWHERE-ZERO FLOWS IN GRAPHS AND SIGNED GRAPHS

MATTHIAS BECK AND THOMAS ZASLAVSKY

ABSTRACT. The existence of an integral flow polynomial that counts nowhere-zero k-flows on a graph, due to Kochol, is a consequence of a general theory of inside-out polytopes. The same holds for flows on signed graphs. We develop these theories, as well as the related counting theory of nowhere-zero flows on a signed graph with values in an abelian group of odd order.

Note to publisher:
This paper does NOT have a "corresponding author" or "senior author".
All authors are EQUAL.
All authors are able to answer correspondence from readers.

For editorial purposes ONLY, contact the writer of the cover letter of submission.

Date: Version of 19 September 2003.

2000 *Mathematics Subject Classification. Primary* 05C99, 05C22; *Secondary* 05B35, 52B20, 52C35.

Key words and phrases. nowhere-zero flow, bidirected graph, signed graph, integral flow polynomial, modular flow polynomial, lattice-point counting, rational convex polytope, arrangement of hyperplanes, Tutte polynomial.

This paper was written while the first author was a Robert Riley Assistant Professor at Binghamton University, SUNY. The author thanks the Department of Mathematical Sciences at Binghamton for its hospitality.

The research of the second author was partially supported by National Science Foundation grant DMS-0070729.

1. In which we go with the nowhere-zero flow

A *nowhere-zero flow* on a graph $\Gamma = (V, E)$, with values in an abelian group A, is a mapping $x : E \to A$ such that, for every node $v \in V$,

$$\sum_{h(e)=v} x(e) = \sum_{t(e)=v} x(e), \tag{1.1}$$

and which never takes the value 0. Here $h(e)$ and $t(e)$ are respectively the head and tail of the edge e in a (fixed) orientation of Γ. (In a certain sense, described below, x is independent of the chosen orientation.) A *nowhere-zero k-flow* is an integral flow (i.e. $A = \mathbb{Z}$) with values in $[k-1] := \{1, 2, \ldots, k-1\}$. Nowhere-zero flows (nicely surveyed in [7] and [16]) are dual to colorings, and partly for that reason the existence of nowhere-zero integral flows of small width k is a major open question in graph theory, whose solution, if it is as conjectured, would imply the Four-Color Theorem. However, our interest is in a somewhat different direction.

It has long been known that the number of nowhere-zero flows with values in a finite abelian group of order k is a polynomial function of k (Tutte [19]). Recently Kochol [9] discovered that the number of nowhere-zero k-flows is also a polynomial in k, although not the same polynomial. Here we show that this fact is a consequence of a general theory of counting lattice points in inside-out polytopes [3]. Furthermore, we extend Kochol's theorem to signed graphs (in which each edge is positive or negative), where the polynomial becomes a quasipolynomial of period two: that is, two polynomials, one for odd values of k and the other for even k (Theorem 4.5); and we partially extend to signed graphs Tutte's counting theory for nowhere-zero flows in abelian groups (Theorem 4.1). Still further, the concept of reciprocity in lattice-point counting leads us to a geometrical interpretation of the number of totally cyclic orientations that are compatible with a given k-flow (Theorems 3.1 and 4.5 and Corollary 4.4), a fact which parallels Stanley's theorem [17] that the chromatic polynomial of Γ evaluated at negative integers counts acyclic orientations compatible with a node-labelling of Γ.

2. In which we explain our ways

The theory of inside-out polytopes was motivated by the problem of counting the integral points of a rational convex polytope that do not lie in any of the members of a particular *hyperplane arrangement* (a finite set of homogeneous hyperplanes in $\mathbb{R}^d$). A hyperplane arrangement is *rational* if each of its hyperplanes is spanned by the rational points it contains. Suppose we are given a rational convex polytope P spanning $\mathbb{R}^d$ and a rational hyperplane arrangement $\mathcal{H}$. Then $(P, \mathcal{H})$ is a *rational inside-out polytope*. More generally, P and $\mathcal{H}$ may lie in a rational (homogeneous) subspace z spanned by P. P will always be closed.

A *region* of $\mathcal{H}$ is a connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$. The arrangement *induced* by $\mathcal{H}$ in a subspace s of $\mathbb{R}^d$ is

$$\mathcal{H}^s := \{h \cap s : h \in \mathcal{H}, h \not\supseteq s\}.$$

The *intersection lattice* of $\mathcal{H}$ is

$$\mathcal{L}(\mathcal{H}) := \left\{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H}\right\},$$

ordered by reverse inclusion [23]. $\mathcal{L}$ is a *geometric lattice* with $\hat{0} = \bigcap \varnothing = \mathbb{R}^d$ and $\hat{1} = \bigcap \mathcal{H}$. (For matroids and geometric lattices we refer to [13], [18], et al.) The *Möbius function* of $\mathcal{L}$

is the function $\mu : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ defined recursively by

$$\mu(r,s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r,u) & \text{if } r < s. \end{cases}$$

(Sources for the Möbius function are, *inter alia*, [15] and [18].)

As for $(P, \mathcal{H})$, a *region* of it is the nonempty intersection of a region of $\mathcal{H}$ with P. A *vertex* of $(P, \mathcal{H})$ is a vertex of any such region. The *denominator* of $(P, \mathcal{H})$ is the smallest positive integer t for which every vertex of $(tP, t\mathcal{H})$ is integral.

Inside-out Ehrhart theory. The fundamental counting functions associated with $(P, \mathcal{H})$ are the (*closed*) *Ehrhart quasipolynomial*,

$$E_{P,\mathcal{H}}(t) := \sum_{x \in t^{-1}\mathbb{Z}^d} m_{P,\mathcal{H}}(x),$$

where the *multiplicity* $m_{P,\mathcal{H}}(x)$ of $x \in \mathbb{R}^d$ with respect to $\mathcal{H}$ is defined through

$$m_{P,\mathcal{H}}(x) := \begin{cases} \text{the number of closed regions of } \mathcal{H} \text{ in } P \text{ that contain } x, & \text{if } x \in P, \\ 0, & \text{if } x \notin P, \end{cases}$$

and the *open Ehrhart quasipolynomial*,

$$E^{\circ}_{P,\mathcal{H}}(t) := \# \left(t^{-1}\mathbb{Z}^d \cap \left[P \setminus \bigcup \mathcal{H} \right] \right).$$

Note that if P spans a subspace z, $\mathcal{H}$ also lies in z so the multiplicity of x is 0 if $x \notin z$.

We denote by P° the relative interior of P, and by $\operatorname{vol} P$ the volume of P normalized with respect to $z \cap \mathbb{Z}^d$ where z is the subspace spanned by P; that is, we take the volume of a fundamental domain of the integer lattice in z to be 1 (so when $z = \mathbb{R}^d$ this is the ordinary volume).

A *quasipolynomial* in t is a function $Q(t) = \sum_0^d c_i t^i$ with coefficients c_i that are periodic functions of t (so that Q is a polynomial on each residue class modulo some integer, called the *period*; these polynomials are the *constituents* of Q). The names of our counting functions are justified by the fact that in the absence of $\mathcal{H}$ we recover Ehrhart's classical theory of lattice-point enumeration in polytopes [5, 18], and by one of the main results in [3]:

Theorem 2.1 ([3, Theorem 4.1])**.** *If $(P, \mathcal{H})$ is a closed, full-dimensional, rational inside-out polytope in $z \subseteq \mathbb{R}^d$, then $E_{P,\mathcal{H}}(t)$ and $E^{\circ}_{P^{\circ},\mathcal{H}}(t)$ are quasipolynomials in t, with period equal to a divisor of the denominator of $(P, \mathcal{H})$, with leading term $(\operatorname{vol} P)t^{\dim P}$, and with constant term $E_{P,\mathcal{H}}(0)$ equal to the number of regions of $(P, \mathcal{H})$. Furthermore,*

$$E^{\circ}_{P^{\circ},\mathcal{H}}(t) = (-1)^d E_{P,\mathcal{H}}(-t). \tag{2.1}$$

In particular, if $(P, \mathcal{H})$ is integral then $E_{P,\mathcal{H}}$ and $E^{\circ}_{P^{\circ},\mathcal{H}}$ are polynomials. The proof, though more general (and arrived at independently), is similar to Kochol's proof of Theorem 3.1(a): we sum the Ehrhart quasipolynomials of the pieces into which $\mathcal{H}$ dissects P.

For the second theorem of [3] we will use here, we need the notion of transversality: $\mathcal{H}$ is *transverse* to P if every *flat* $u \in \mathcal{L}(\mathcal{H})$ that intersects P also intersects P°, and P does not lie in any of the hyperplanes of $\mathcal{H}$. Let $E_P(t) := \#(tP \cap \mathbb{Z}^d)$, the standard Ehrhart counting function (without any hyperplanes present).

Theorem 2.2 ([3, Theorem 4.2])**.** *If P and $\mathcal{H}$ are as in Theorem 2.1, then*

$$E^\circ_{P,\mathcal{H}}(t) = \sum_{u\in\mathcal{L}(\mathcal{H})} \mu(\hat{0},u) E_{P\cap u}(t), \tag{2.2}$$

and if $\mathcal{H}$ is transverse to P,

$$E_{P,\mathcal{H}}(t) = \sum_{u\in\mathcal{L}(\mathcal{H})} |\mu(\hat{0},u)| E_{P\cap u}(t). \tag{2.3}$$

Transversality, defined in [3], is always satisfied in the applications here.

Matrix matroids. We shall want a general lemma about matroids of hyperplane arrangements induced by coordinate arrangements. Any homogeneous hyperplane arrangement $\mathcal{H}$ has a matroid $M(\mathcal{H})$ whose ground set is the set of hyperplanes and whose rank function is $\operatorname{rk} \mathcal{S} = \operatorname{codim} \bigcap \mathcal{S}$ for $\mathcal{S} \subseteq \mathcal{H}$. This matroid is simply the linear dependence matroid of normal vectors of the hyperplanes. The *column matroid* of a matrix A, $M(A)$, is the matroid of linear dependence of its columns; to keep the notation correct we take the ground set to be the set of indices of columns. The *chain-group matroid* of a subspace $s \subseteq F^m$ is the matroid $N(s)$ on $[m]$ whose circuits are the minimal nonempty supports of vectors in s. $\operatorname{Lat} M$ denotes the lattice of closed sets of a matroid M. Thus $\operatorname{Lat} M(\mathcal{H}) \cong \mathcal{L}(\mathcal{H})$.

Lemma 2.3. *Let A be an $n \times m$ matrix with entries in a field F, let $\mathcal{H}_{[m]} = \{h_e : e \in [m]\}$ be the arrangement of coordinate hyperplanes in F^m, and let $u = \operatorname{Row} A$, the row space, and $z = \operatorname{Nul} A$, the null space.*

(a) The mapping $e \mapsto h_e \cap u$ is a matroid isomorphism from $M(A)$ to $M((\mathcal{H}_{[m]})^u)$. Also, $e \mapsto h_e \cap z$ is a matroid isomorphism of $N(\operatorname{Row} A)$ with $M((\mathcal{H}_{[m]})^z)$.

(b) The mapping

$$s \mapsto E_u(s) := \{e \in [m] : h_e \supseteq s\}$$

is the isomorphism of $\mathcal{L}((\mathcal{H}_E)^u)$ with $\operatorname{Lat} M(A)$ induced by the first mapping in (a). The mapping

$$s \mapsto E_z(s) := \{e \in [m] : h_e \supseteq s\}$$

is the isomorphism of $\mathcal{L}((\mathcal{H}_E)^z)$ with $\operatorname{Lat} N(\operatorname{Row} A)$ induced by the second mapping in (a).

(c) If F is an ordered field, then the regions of $(\mathcal{H}_{[m]})^u$ correspond bijectively to the acyclic orientations of the oriented matroid of columns of A, and those of $(\mathcal{H}_{[m]})^z$ correspond to the totally cyclic orientations.

Proof. (a) Each coordinate hyperplane h_e is $b_e^\perp$ for a basis vector b_e along the e-axis. Therefore, $h_e \cap u$ is the orthogonal complement in u of a_e, the orthogonal projection of b_e into u. Let $B := \{b_e : e \in [m]\}$. We may take A to be the matrix whose columns are the vectors a_e, since it has the same row space as the original matrix A. By linear duality, $M(A) \cong M((\mathcal{H}_{[m]})^u)$ under the correspondence $e \mapsto h_e \cap u$.

We treat $(\mathcal{H}_{[m]})^z$ by taking A^*, a matrix whose row space is $\operatorname{Nul} A$, and applying the first part to $\operatorname{Row} A^*$. $N(\operatorname{Row} A)$ is dual to $M(A)$; hence it is $M(A^*)$ because $\operatorname{Row} A$ and $\operatorname{Nul} A$ are dual chain-groups. (See, e.g., Tutte [22, Chapter VIII] on primitive chain-groups.)

(b) Obvious from (a) and the definitions.

(c) Write $\mathcal{M}(A)$ for the oriented matroid. The proof is as in part (a), but relying on the dual relationship between $\mathcal{M}(A)$ and $\mathcal{M}(A^*)$ [28, Section 6.3(c)]. Our $\mathcal{M}(A)$ is Ziegler's

$\mathcal{M}(\mathrm{Nul}\, A)$, so our $\mathcal{M}(A^*) = \mathcal{M}(\mathrm{Row}\, A)$, which is dual to $\mathcal{M}(\mathrm{Nul}\, A)$ because the row and null spaces are orthogonal complements. ■

3. In which the flows are graphic

A *flow* (or *1-cycle*) on a graph Γ with values in an abelian group A, called an *A-flow*, is a function $x : E \to A$ which satisfies (1.1) for every node $v \in V$ (so it is like a nowhere-zero flow, but the flow value zero is allowed). This definition requires that the edges be oriented in a fixed way (which we call the *reference orientation*). The reference orientation is arbitrary; it is an artifact of notation, and to overcome this artificiality we define, for an oriented edge e, e^{-1} to be the same edge in the opposite orientation and $x(e^{-1}) := -x(e)$. With this law for flows, the validity of Equation 1.1 is independent of the choice of the orientation of Γ.

Tutte [19, Section 6] proved that the number of nowhere-zero A-flows on Γ is a polynomial in $|A|$, independent of the actual group. We shall write $\bar{\varphi}_\Gamma$ for this polynomial and call it the *(strict) modular flow polynomial* of Γ. (Usually $\bar{\varphi}_\Gamma$ is called just the "flow polynomial" but we need to distinguish it from other flow polynomials.) The modular flow polynomial is the evaluation $(-1)^{\xi(\Gamma)} t_\Gamma(0, 1-k)$ of the Tutte polynomial of Γ, as Tutte showed, $\xi(\Gamma)$ being the cyclomatic number $|E| - |V| + c(\Gamma)$ where $c(\Gamma)$ is the number of connected components; thus, $\bar{\varphi}_\Gamma$ is the characteristic polynomial of the bond matroid of Γ. (We should mention that the total number of A-flows is the simple polynomial $|A|^{\xi(\Gamma)}$.) Tutte further proved (in [21, pp. 83–84], based on [20, Theorem VI]) that a nowhere-zero $\mathbb{Z}_k$-flow exists if and only if there is a nowhere-zero k-flow. However, the actual number of nowhere-zero k-flows, which we write $\varphi_\Gamma(k)$, does not equal the number of nowhere-zero $\mathbb{Z}_k$-flows and indeed was never known to be a polynomial until the recent work of Kochol [9]. Kochol employed standard Ehrhart theory combined with a special construction to prove this. We shall show that Kochol's theorem is a natural consequence of inside-out polytopes applies to k-flows and can be extended to a reciprocity theorem that interprets $\varphi_\Gamma(k)$ at negative arguments.

An orientation of Γ is *totally cyclic* if every edge lies in a *coherent circle* (that is, where the edges are oriented in a consistent direction around the circle). We call a totally cyclic orientation τ and a flow x *compatible* if $x \geq 0$ when it is expressed in terms of τ. Taking the standpoint of the flow x, the nonzero edge set $\mathrm{supp}\, x$ has a preferred orientation, the one in which $x \geq 0$, and the zero edges are free to take up any orientation that makes Γ totally cyclic. Just orienting $\mathrm{supp}\, x$, we have the *partial orientation* of Γ *induced by* x, written $\tau(x)$. An *isthmus* of a graph is an edge whose deletion increases the number of connected components. There is no totally cyclic orientation if Γ has an isthmus.

The *real cycle space*[1] z is defined in $\mathbb{R}^E$ by Equation (1.1). To this space z we associate the polytope $P := z \cap [-1, 1]^E$.

Theorem 3.1. *Given: a graph Γ and its real cycle space z.*

(a) (Kochol [9]) $\varphi_\Gamma(k)$ is a polynomial function of k for $k = 1, 2, 3, \ldots$. It has leading term $(\mathrm{vol}\, P)k^{\xi(\Gamma)}$ if Γ has no isthmi; otherwise it is identically zero.

(b) Furthermore, $(-1)^{\xi(\Gamma)}\varphi_\Gamma(-k)$ equals the number of $(k+1)$-flows counted with multiplicity equal to the number of compatible totally cyclic orientations of Γ.

(c) In particular, the constant term $\varphi_\Gamma(0)$ equals the number of totally cyclic orientations of Γ, that is, $(-1)^{\xi(\Gamma)}\bar{\varphi}_\Gamma(-1)$.

---

[1] So called because it actually is the space of 1-cycles of Γ in cellular homology with real coefficients. For a similar reason, no doubt, Tutte modernized the term 'flow' to 'cycle' in his textbook [22, Chapter VIII].

(d) *Finally, the total number of k-flows, nowhere-zero or not, is a polynomial $\varphi^0_\Gamma(k)$ whose leading term is the same as that of $\varphi_\Gamma(k)$ and whose constant term is 1, if Γ has no isthmi; otherwise it is identically zero.*

We call φ_Γ the *(strict) integral flow polynomial* of Γ and $\varphi^0_\Gamma(k)$ the *weak integral flow polynomial.* $\varphi^0_\Gamma(k)$ already gives rise to a number of interesting computational problems, as discussed, for example, in [2].

Proof. For (a) we apply Theorem 2.1 in z with polytope P and arrangement $\mathcal{H} = (\mathcal{H}_E)^z$, where $\mathcal{H}_E$ is the arrangement of coordinate hyperplanes in $\mathbb{R}^E$. A nowhere-zero k-flow is then precisely a point $x \in z \cap \mathbb{Z}^d$ such that $\frac{1}{k}x \in P^\circ \setminus \bigcup \mathcal{H}$. Consequently,

$$\varphi_\Gamma(k) = E^\circ_{P^\circ,\mathcal{H}}(k). \tag{3.1}$$

We call upon the total unimodularity of the matrix of the cycle equations (1.1) to deduce that φ_Γ is a polynomial. Also due to total unimodularity, P is a convex hull of lattice points.

Since $E_{P,\mathcal{H}}(k)$ counts pairs (x, R) where $x \in \mathbb{Z}^d \cap P$ and R is a closed region of $\mathcal{H}$ that contains x, part (b) follows if we show that the regions of $\mathcal{H}$ correspond to the totally cyclic orientations of Γ and a region of $\mathcal{H}$ whose closure contains a chosen point $x \in z \cap \mathbb{Z}^d$ corresponds to a totally cyclic orientation that is compatible with x. The first statement was demonstrated by Greene and Zaslavsky in [6, Section 8], based on the obvious bijection (given the fixed reference orientation of Γ) between orthants of $\mathbb{R}^E$ and orientations of Γ. The second is then obvious.

Thus the constant term is the number of totally cyclic orientations. The fact that this equals $t_\Gamma(0, 2)$ is a theorem originally due to Las Vergnas (see [10, Proposition 8.1] and [11, remark after Theorem $1'$, p. 296]) and independently proved by Greene and Zaslavsky [6, Corollary 8.2].

Part (d) is standard Ehrhart theory, because a k-flow is simply a point $x \in z \cap \mathbb{Z}^d$ such that $\frac{1}{k-1}x \in P$. The constant term is the Euler characteristic of P. ■

Problem 3.2. Find a formula for, or a combinatorial interpretation of, the leading coefficient $\operatorname{vol} P$ of the integral flow polynomials.

Problem 3.3. Is there a combinatorial interpretation of $\bar{\varphi}_\Gamma(-k)$ for $k \geq 2$?

Example 3.4 (Small graphs). We calculated the integral flow polynomials of some small graphs by counting integral k-flows on a computer for $1 \leq k \leq \xi(\Gamma) + 2$ and interpolating to get the polynomial. The graphs were mK_2, the graph of m parallel links, for $m = 3, 4, 5, 6$, and K_4. We state our results along with the modular flow polynomials for comparison; the latter are $\bar{\varphi}^0_\Gamma(k) = k^{\xi(\Gamma)}$ and $\bar{\varphi}_\Gamma(k) = \chi_{\Gamma^*}(k)/k$, Γ^* being the planar dual graph. First, $3K_2$:

$$\bar{\varphi}^0(k) = k^2, \qquad \bar{\varphi}(k) = (k-1)(k-2),$$
$$\varphi^0(k) = 3k^2 - 3k + 1, \qquad \varphi(k) = 3(k-1)(k-2).$$

Next, $4K_2$:

$$\bar{\varphi}^0(k) = k^3, \qquad \bar{\varphi}(k) = (k-1)(k^2 - 3k + 3),$$
$$\varphi^0(k) = \frac{(2k-1)(8k^2 - 8k + 3)}{3}, \qquad \varphi(k) = \frac{2(k-1)(8k^2 - 22k + 21)}{3}.$$

Next, $5K_2$:

$$\bar\varphi^0(k) = k^4, \qquad \bar\varphi(k) = (k-1)(k^3-4k^2+6k-4),$$
$$\varphi^0(k) = \frac{115k^4-230k^3+185k^2-70k+12}{12},$$
$$\varphi(k) = \frac{5(k-1)(k-2)(23k^2-41k+36)}{12}.$$

Next, $6K_2$:

$$\bar\varphi^0(k) = k^5, \qquad \bar\varphi(k) = (k-1)(k^4-5k^3+10k^2-10k+5),$$
$$\varphi^0(k) = \frac{2(2k-1)(44k^4-88k^3+71k^2-27k+5)}{10},$$
$$\varphi(k) = \frac{(k-1)(176k^4-839k^3+1571k^2-1404k+620)}{10}.$$

Finally, K_4:

$$\bar\varphi^0(k) = k^3, \qquad \bar\varphi(k) = (k-1)(k-2)(k-3),$$
$$\varphi^0(k) = (2k-1)(2k^2-2k+1), \qquad \varphi(k) = 4(k-1)(k-2)(k-3).$$

Problem 3.5. Is there any general reason why in some of these examples ($3K_2$ and K_4) both of the integral flow polynomials have integral coefficients and the integral nowhere-zero flow polynomial is a multiple of the modular polynomial?

The number of closed regions that contain a flow x is equal to the number of totally cyclic extensions to Γ of $\tau(x)$. Clearly, we get such an extension by orienting the contraction $\Gamma/\operatorname{supp} x$ in a totally cyclic manner. Whence:

Lemma 3.6. *The multiplicity of an integral flow x with respect to $(\mathcal{H}_E)^z$ equals $(-1)^{\xi(\Gamma/\operatorname{supp} x)}\bar\varphi_{\Gamma/\operatorname{supp} x}(-1)$.* ■

Corollary 3.7. $|\varphi_\Gamma(-k)| = \sum_{x\in P\cap\mathbb{Z}^E} |\bar\varphi_{\Gamma/\operatorname{supp} x}(-1)|$. ■

Other formulas arise from the intersection expansions of Theorem 2.2, but as we need its Möbius function, first we have to find the lattice $\mathcal{L}(\mathcal{H})$ explicitly. We do so in the more general context of signed graphs.

4. In which signed graphs get with the flow

The best way to understand the cycle equations (1.1) is in terms of the incidence matrix, which in turn is best expounded in terms of signed, or bidirected, graphs. A graph is *bidirected* when each end of each edge is independently oriented. (A loop or link has two ends, a halfedge one, and a loose edge none.) We express the bidirection by means of an *incidence function* η defined on edge ends: the function is $+1$ if the arrow on that end points into the incident node, and -1 otherwise. For an edge end ε, let $e(\varepsilon)$ denote the edge and $v(\varepsilon)$ the node incident to ε. We define

$$\eta(v,e) := \sum \big\{\eta(\varepsilon) : v(\varepsilon) = v \text{ and } e(\varepsilon) = e\big\}.$$

Thus, $\eta(v,e) = 0$ if v and e are not incident or e is a loose edge or positive loop. The *bidirected incidence matrix* is

$$\mathrm{H}(\Gamma,\eta) := \big(\eta(v,e)\big)_{V\times E}.$$

A bidirection is really an orientation of a signed graph. An ordinary edge e with ends ε_1 and ε_2 has sign

$$\sigma(e) := -\eta(\varepsilon_1)\eta(\varepsilon_2).$$

In plain language, if the two arrows on e point in the same direction, then e is positive, but if they conflict, e is negative. We call η an *orientation of the signed graph* Σ. (See [26]. This notion corresponds to the ordinary notion of graph orientation if we identify an unsigned graph Γ with the all-positive graph $+\Gamma$.) We call an *(oriented) incidence matrix of* Σ any incidence matrix of an orientation of Σ, writing it $\mathrm{H}(\Sigma)$ (and $\mathrm{H}(\Gamma) := \mathrm{H}(+\Gamma)$). The ambiguity arising from the unspecified orientation can usually be ignored, since the only effect of changing the orientation is to negate some columns. Recall that reorienting e to e^{-1} also has the effect of replacing $x(e)$ by $x(e^{-1}) = -x(e)$ for every $x \in A^E$ (A an abelian group). With these conventions we define a *flow* on Σ *with values in* A as any $x \in A^E$ for which

$$\mathrm{H}(\Sigma)x = 0, \tag{4.1}$$

in other words, for which $x \in \operatorname{Nul}\mathrm{H}(\Sigma)$. This definition generalizes that of a flow on a graph; naturally, then, we generalize theorems about flows. A *k-flow* is an integral flow x for which every $|x(e)| < k$, just as before. Such flows on signed graphs were studied by Bouchet [4] and Khelladi [8], searching for the smallest possible width k. Again, our own interest is enumerative rather than existential.

4.1. **In which flows are group-valued.** We begin signed graphs with the analog of the modular flow polynomial, since as far as we know it has not been published. We define the *cyclomatic number* of Σ to be $|E| - |V| + b(\Sigma)$ where $b(\Sigma)$ is the number of balanced components of Σ (ignoring any loose edges). This is the rank of the dual matroid $G^{\perp}(\Sigma)$ of the bias matroid, or signed-graph matroid, $G(\Sigma)$ [24, Section 5]. A *coloop* of $G(\Sigma)$, and of Σ, is an edge e whose deletion makes an unbalanced component balanced, or which is an isthmus connecting two components of $\Sigma \setminus e$ of which at least one is balanced (all this by [24, Theorem 5.1 as corrected]). The *Tutte polynomial* $t_\Sigma(x, y)$ of Σ is defined to be that of $G(\Sigma)$.

Theorem 4.1. *For each signed graph Σ there is a polynomial $\bar\varphi_\Sigma(k)$ such that the number of nowhere-zero flows on Σ with values in a finite abelian group A of odd order is $\bar\varphi_\Sigma(|A|)$. In fact,*

$$\bar\varphi_\Sigma(k) = (-1)^{\xi(\Sigma)} t_\Sigma(0, 1-k).$$

Proof. We assume acquaintance with deletion and contraction of edges in signed graphs, as in [24]. We fix A and an orientation η of Σ and write $\bar\varphi_\Sigma(A)$ for the number of nowhere-zero A-flows on Σ. The proof is by induction on $|E|$.

It is clear that if Σ has components $\Sigma_1, \ldots, \Sigma_c$, then $\bar\varphi_\Sigma(A) = \bar\varphi_{\Sigma_1}(A) \cdots \bar\varphi_{\Sigma_c}(A)$. Analogously, $t_\Sigma = t_{\Sigma_1} \cdots t_{\Sigma_c}$. Therefore, we may assume Σ connected.

Suppose e is a positive link of Σ, oriented from v to w, such that deleting e from Σ does not increase the number of balanced components; that is, e is not a coloop. Let x'' be a nowhere-zero A-flow on Σ/e. We can define a flow on Σ by

$$\begin{aligned} x(f) &= x''(f) && \text{if } f \neq e, \\ x(e) &= \sum_{f \neq e} \eta(v, f) x(f). \end{aligned}$$

Then $\mathrm{H}(\eta)x = 0$. If $x(e) \neq 0$, x is a nowhere-zero flow on Σ; if $x(e) = 0$, x is a nowhere-zero flow on $\Sigma \setminus e$. Conversely, any nowhere-zero flow on $\Sigma \setminus e$ or Σ gives rise to one on Σ/e. Since the assumption about e guarantees that Σ, Σ/e, and $\Sigma \setminus e$ all have the same number of balanced components,

$$(-1)^{\xi(\Sigma)}\bar{\varphi}_\Sigma(A) = (-1)^{\xi(\Sigma/e)}\bar{\varphi}_{\Sigma/e}(A) + (-1)^{\xi(\Sigma\setminus e)}\bar{\varphi}_{\Sigma\setminus e}(A). \tag{4.2}$$

If e is a negative link, subject to the same hypothesis on balance, with endpoints v and v', we *switch* v so e becomes positive. That means changing η to η^v defined by

$$\eta^v(\varepsilon) = \begin{cases} \eta(\varepsilon) & \text{if } v(\varepsilon) \neq v, \\ -\eta(\varepsilon) & \text{if } v(\varepsilon) = v. \end{cases}$$

The associated switched signed graph is denoted Σ^v. Because a flow on Σ remains a flow after switching, $\bar{\varphi}_{\Sigma^v}(A) = \bar{\varphi}_\Sigma(A)$. Thus, (4.2) is valid for a negative link.

The analogous Tutte-polynomial formula, $t_\Sigma = t_{\Sigma/e} + t_{\Sigma\setminus e}$, is valid as long as e is not a coloop in $G(\Sigma)$, but that is our balance assumption. In this case, therefore, the theorem is valid for Σ by induction.

If on the other hand an edge e is a coloop, then $t_\Sigma(0, y) = 0$ and, by the proofs of Bouchet's Lemmas 2.4 and 2.5 [4], also $\bar{\varphi}_\Sigma(A) = 0$. (The oddness of A, so that $2a = 0$ implies $a = 0$ in A, is needed at this point.)

The preceding arguments reduce the theorem to the case of a signed graph with a single node v. The flow in a loose edge or positive loop can be any nonzero value in A, independent of all other flow values. Therefore, $\bar{\varphi}_\Sigma(A) = (|A| - 1)^l \bar{\varphi}_{\Sigma_0}(A)$ where l is the number of positive loops and loose edges and Σ_0 is Σ with all such edges deleted. Now let us assume Σ has one node and its edges are halfedges $e_1, \ldots, e_i$ and negative loops $f_1, \ldots, f_j$. We may assume they are oriented into v, so the total inflow at v is

$$x(e_1) + \cdots + x(e_i) + 2x(f_1) + \cdots + 2x(f_j) = 0$$

by the cycle condition (4.1). If $i + j = 0$, there is one such flow, in agreement with $t_\Sigma = 1$. If $i + j = 1$, there is none (again we require $|A|$ odd) so $\bar{\varphi}_\Sigma = 0$, in agreement with $t_\Sigma(0, y) = 0$. If $i + j > 1$, we get a nowhere-zero A-flow by taking arbitrary nonzero values for x except on one edge, say $x(e)$, which is determined by the rest through the cycle equations. The number of such choices in which $x(e) \neq 0$ is $\bar{\varphi}_\Sigma(A)$; the number in which $x(e) = 0$ is $\bar{\varphi}_{\Sigma\setminus e}(A)$. Therefore,

$$\begin{aligned} \bar{\varphi}_\Sigma(A) &= (-1)^{i+j-1}(1 - |A|)^{i+j-1} - \bar{\varphi}_{\Sigma\setminus e}(A) \\ &= (-1)^{\xi(\Sigma/e)} t_{\Sigma/e}(0, 1 - |A|) - (-1)^{\xi(\Sigma\setminus e)} t_{\Sigma\setminus e}(0, 1 - |A|) \end{aligned}$$

by the fact that $G((\Sigma/e)$ has rank 0 and induction on $i + j = |E|$,

$$= (-1)^{\xi(\Sigma)} t_\Sigma(0, 1 - |A|)$$

by the fact that e is not a loop or coloop in $G(\Sigma)$. Our theorem now follows by induction. ∎

Problem 4.2. What is the significance of $\bar{\varphi}^0_\Sigma$ or $\bar{\varphi}_\Sigma$ evaluated at even natural numbers?

The theorem means that there is a polynomial $\bar{\varphi}_\Sigma(k)$, which we call the *(strict) modular flow polynomial*, such that for any odd positive number k, $\bar{\varphi}_\Sigma(k)$ is the number of nowhere-zero flows on Σ with values in any fixed abelian group of order k. This is remarkably reminiscent of signed-graph coloring, where only odd values of λ can be interpreted in the chromatic polynomial $\chi_\Sigma(\lambda)$ [25].

Example 4.3. The number of A-flows depends on the group A when it has even order. Consider the signed graph consisting of two negative loops at a node v. Orient both loops into v. The cycle equation (1.1) is $2x(e) = 2x(f)$. We compare two groups of order 4. If A is the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$, then every $x : E \to A$ satisfies the cycle equations and we have

$$\bar{\varphi}^0_\Sigma(A) = 16, \qquad \bar{\varphi}_\Sigma(A) = 9.$$

If $A = \mathbb{Z}_4$, then the A-flows are $(x(e), x(f)) \in \{0, 2\}^2 \cup \{1, 3\}^2$ and

$$\bar{\varphi}^0_\Sigma(A) = 8, \qquad \bar{\varphi}_\Sigma(A) = 5.$$

For comparison, at odd values of k we have

$$\bar{\varphi}^0_\Sigma(k) = k, \qquad \bar{\varphi}_\Sigma(k) = k - 1.$$

Corollary 4.4. *The number of totally cyclic orientations of Σ equals $(-1)^{\xi(\Sigma)}\bar{\varphi}_\Sigma(-1)$.*

Proof. The number of totally cyclic reorientations of an orientation of a matroid M is $t_M(0, 2)$ [10, 11]. Since cycles in an orientation of Σ are the same as cycles in the corresponding orientation of $G(\Sigma)$ [26], the number of totally cyclic orientations of Σ equals $t_{G(\Sigma)}(0, 2) = (-1)^{\xi(\Sigma)}\bar{\varphi}_\Sigma(-1)$. (The term 'cycle' as used here refers to oriented cycles of an oriented matroid or oriented signed graph, not to a flow. We regret the confusion, which is embedded in the literature and to which the second author made some contribution.) ∎

4.2. **In which signed graphs integrally k-flow.** Now it is time for integral flows. Let

$$\varphi_\Sigma(k) := \text{ the number of nowhere-zero } k\text{-flows on } \Sigma.$$

As with abelian-group flows, $\varphi_\Sigma = 0$ if there is a matroid coloop of $G(\Sigma)$. Let

$$\varphi^0_\Sigma(k) := \text{ the number of all } k\text{-flows on } \Sigma.$$

We take z to be the real cycle space $\operatorname{Nul} \mathrm{H}(\Sigma)$ and $P = z \cap [-1, 1]^E$. As with ordinary graphs, a flow x and an orientation η are *compatible* if $x \geq 0$ when expressed in terms of η.

Notation: if $S \subseteq E$, then $\Gamma|S$ or $\Sigma|S$ denotes the spanning subgraph whose edge set is S.

Theorem 4.5. *(a) For any signed graph Σ, $\varphi_\Sigma(k)$ is a quasipolynomial function of k for $k = 1, 2, 3, \ldots$. Its period is 1 or 2, and is 1 if Σ is balanced. It has leading term $(\operatorname{vol} P)k^{\xi(\Sigma)}$ if Σ has no coloops; otherwise it is identically zero.*

(b) Furthermore, $(-1)^{\xi(\Sigma)}\varphi_\Sigma(-k)$ equals the number of $(k+1)$-flows counted with multiplicity equal to the number of compatible totally cyclic orientations of Σ.

(c) In particular, the constant term $\varphi_\Sigma(0)$ equals the number of totally cyclic orientations of Σ, that is, $(-1)^{\xi(\Sigma)}\bar{\varphi}_\Sigma(-1)$.

(d) Finally, if Σ has no coloops, $\varphi^0_\Sigma(k)$ is a quasipolynomial of period 1 or 2 (period 1 if Σ is balanced) whose leading term is the same as that of $\varphi_\Sigma(k)$ and whose constant term is 1; otherwise, $\varphi^0_\Sigma(k)$ is identically zero.

Lemma 4.6. *The vertices of $(P, \mathcal{H})$ are half integral.*

Proof. We may as well assume Σ has a halfedge at every node; thus $\mathrm{H}(\Sigma)$ contains an identity matrix I_n. A vertex is a solution of $\mathrm{H}(\Sigma)x = 0$ with $|E| - n$ coordinates of x set equal to fixed values in $\{0, 1, -1\}$. Let B be the edge set whose coordinates in x are left undetermined, let $B^c := E \setminus B$, and write $x = (x_B, x_{B^c})^{\mathrm{T}}$. Then x is the unique solution of $\mathrm{H}(\Sigma|B)x_B = -\mathrm{H}(\Sigma|B^c)x_{B^c}$.

The remainder of the proof is based on work of Jon Lee. Lee proved that the null space $\mathrm{Nul}\,\mathrm{H}(\Sigma)$ is 2-regular [12, Proposition 9.1] and that if A is a nonsingular square matrix for which $\mathrm{Nul}[I \mid A]$ is 2-regular, then $A^{-1}b$ is half integral for every integral vector b (a special case of [12, Proposition 6.1]; the definition of 2-regularity need not concern us). These facts applied to $A = \mathrm{H}(\Sigma|B)$ imply that the solution of $\mathrm{H}(\Sigma|B)x = b$ is half integral for any $b \in \mathbb{Z}^B$. Apply this to $b = -\mathrm{H}(\Sigma|B^c)x_{B^c}$. ■

Proof of Theorem 4.5. The proof is similar to that of Theorem 3.1. In (a) and (d), instead of total unimodularity we have Lemma 4.6 to tell us that the denominator of $(P, \mathcal{H})$, hence the period of the Ehrhart quasipolynomials, divides 2.

For (b) we need to show that the regions of $\mathcal{H}$ correspond to the totally cyclic orientations of Σ. The latter are the totally cyclic reorientations of the natural orientation of $G(\Sigma)$, which is the oriented matroid of columns of $\mathrm{H}(\Sigma)$ [26, Theorem 3.3]. Now we apply Lemma 2.3(b).

For (c) we use Corollary 4.4. ■

Example 4.7 (Some signed graphs). We calculated some small flow polynomials and modular flow polynomials. First, the signed graph with two negative loops at one node, for comparison with the modular flow polynomials in Example 4.3:

$$\varphi^0_\Sigma(k) = 2k - 1, \qquad \varphi_\Sigma(k) = 2(k-1).$$

Another signed graph, only slightly larger, has two nodes joined by positive and negative edges and at each node either a halfedge or a negative loop. We write $\pm K_2^{(i,j)}$ for this graph if i nodes have halfedges and j other nodes have negative loops and $\varphi_{(i,j)}$ for the polynomials; the examples we calculated are where $i + j = 2$. The modular nowhere-zero flow polynomial is the characteristic polynomial of the dual matroid of $G(\pm K_2^{(i,j)})$, which is the four-point line.

$$\begin{aligned} \bar\varphi^0_{(i,j)}(k) &= k^2, \\ \bar\varphi_{(i,j)}(k) &= (k-1)(k-3). \end{aligned}$$

The integral polynomials are

$$\begin{aligned} \varphi^0_{(0,2)}(k) = \varphi^0_{(2,0)}(k) &= 2k^2 - 2k + 1, \\ \varphi_{(0,2)}(k) = \varphi_{(2,0)}(k) &= \begin{cases} 2(k-1)(k-3) & \text{if } k \text{ is odd,} \\ 2(k-2)^2 & \text{if } k \text{ is even} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \varphi^0_{(1,1)}(k) &= \begin{cases} \frac{1}{2}(3k^2 - 2k + 1) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(3k^2 - 4k + 2) & \text{if } k \text{ is even,} \end{cases} \\ \varphi_{(1,1)}(k) &= \begin{cases} \frac{1}{2}(3k^2 - 12k + 9) & \text{if } k \text{ is odd,} \\ \frac{1}{2}(3k^2 - 14k + 16) & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Problem 4.8. We do not understand why $\pm K_2^{(0,2)}$ and $\pm K_2^{(2,0)}$ have the same flow quasipolynomials but $\pm K_2^{(1,1)}$ does not, nor why $\varphi^0_{(0,2)}$ has period one, nor why the odd constituent of $\varphi_{(0,2)}$ is an integral multiple of $\bar\varphi_{(i,j)}$.

4.3. **Half integrality and the incidence matrix.** In our geometric treatment of graph coloring in [3, Section 5] we noticed an important half-integrality property of the signed-graphic hyperplane arrangement $\mathcal{H}[\Sigma]$, which consists of the dual hyperplanes to all the columns in the incidence matrix of Σ. It is this:

Lemma 4.9 ([3, Lemma 5.7])**.** *If Σ is a signed graph, $(P, \mathcal{H}''[\Sigma])$ has half-integral vertices.*

Despite their appearance, not only are Lemmas 4.6 and 4.9 similar in statement, both also involve the incidence matrix. A *basic solution* of a linear program $\mathrm{H}(\Sigma)^{\mathrm{T}}x \leq b$, $x \geq 0$ is the solution of $\mathrm{H}(\Sigma|B)^{\mathrm{T}}x = b$ for some set B of n linearly independent columns of $\mathrm{H}(\Sigma)$.

Proposition 4.10. *Every basic solution of $\mathrm{H}(\Sigma)^{\mathrm{T}}x \leq b$, $x \geq 0$ is half integral for any $b \in \mathbb{Z}^n$.*

Proof that Lemma 4.9 and Proposition 4.10 are equivalent. We may assume that Σ contains a negative loop at every node. Write a_e for the column of the incidence matrix that belongs to e. The equations of hyperplanes in $\mathcal{H}''[\Sigma]$ are $a_e^{\mathrm{T}}x = \frac{1}{2}a_e^{\mathrm{T}}\mathbf{1}$, which is 0 or 1 if e is a loop or link; but it is $\frac{1}{2}$ if e is a halfedge at node v_i and in that case the equation simplifies to $x_i = \frac{1}{2}$. In the situation of Lemma 4.9, the vertices of $(P, \mathcal{H}''[\Sigma])$ are obtained by choosing from among the equations $x_i = 0$, $x_i = 1$ (from P) and $x_i = \frac{1}{2}$, $a_e^{\mathrm{T}}x = 0$ or 1 (from $\mathcal{H}''[\Sigma]$) a subset of n equations whose coefficient matrix is nonsingular. An equation $x_i = 0$ or 1 has the form $a_e^{\mathrm{T}}x = 0$ or 1 with e the negative loop at v_i, so we need not consider it separately. Thus, half integrality of vertices of $([0,1]^n, \mathcal{H}''[\Sigma])$ implies half integrality of the solution of $\mathrm{H}(\Sigma|B)^{\mathrm{T}}x = b$ for any $b \in \mathbb{Z}^n$. The converse is obvious. ∎

Our attention was drawn to the relationship between Lemma 4.9 and half integrality (and Lee's work) by a recent manuscript of Appa and Kotnyek [1].

4.4. **In which nowhere-zero flows reduce to flows with Möbius complications.** The last main result expresses the nowhere-zero integral flow polynomial in terms of the weak integral flow polynomials of subgraphs. We begin with structural lemmas. For a flat $s \in \mathcal{L}(\mathcal{H})$ we define

$$E(s) := \{e \in E : s \subseteq h_e\} = \{e \in E : x(e) = 0 \text{ for all } x \in s\}.$$

This is the $E_z(s)$ of Lemma 2.3(b). We see that $E(s)^c$ is the union of the suppports of the vectors in s.

Lemma 4.11. *The lattice of flats of $\mathcal{H}$ is isomorphic to the lattice of closed sets of the dual bias matroid $G^{\perp}(\Sigma)$. The isomorphism is given by $s \mapsto E(s)$. The corresponding matroid isomorphism $G^{\perp}(\Sigma) \cong M((\mathcal{H}_E)^z)$ is given by $e \mapsto h_e \cap z$.*

Proof. An application of Lemma 2.3. The matrix is $\mathrm{H}(\Sigma)$. We know $M(\mathrm{H}(\Sigma))$ is the bias matroid $G(\Sigma)$ by [24, Theorem 8A.1], so $G^{\perp}(\Sigma)$ is the chain-group matroid of $\mathrm{Row}\,\mathrm{H}(\Sigma)$. The lemma applies because the real cycle space $z = \mathrm{Nul}\,\mathrm{H}(\Sigma)$. ∎

This lemma holds good for the canonical hyperplane representation of any F^*-gain graph Φ, for any field F [27, Section 2]. Denoting an incidence matrix by $\mathrm{H}(\Phi)$ and taking $z = \mathrm{Nul}\,\mathrm{H}(\Phi)$, we have $M((\mathcal{H}_E)^z) = G^{\perp}(\Phi)$. — But we digress.

Lemma 4.12. *φ_Σ is identically zero if and only if Σ has a coloop.*

Proof. This is an application of Bouchet's theorem on integral chain-group matroids [4, Proposition 3.1]: the chain-group has a nowhere-zero chain if and only if the matroid has no coloop. In our case the chain-group is the group of integral flows, $\operatorname{Nul}\mathrm{H}(\Sigma)$. Its chain-group matroid is dual to that of $\operatorname{Row}\mathrm{H}(\Sigma)$, which is dual to the column matroid of $\mathrm{H}(\Sigma)$, which is $G(\Sigma)$. ■

Recall that a *coloop* of Σ is a matroid coloop in $G(\Sigma)$.

Lemma 4.13. *A flat s of $(\mathcal{H}_E)^z$ can be represented as* $[\operatorname{Nul}\mathrm{H}(\Sigma|E(s)^c)] \times \{0\}^{E(s)}$.

Proof. Obvious from the definitions of z and $E(s)$. ■

Theorem 4.14. *Take a signed graph Σ. Letting S range over all subsets of E, or merely over all for which $\Sigma|S$ has no coloops (that is, all complements of flats of the dual bias matroid $G^\perp(\Sigma)$),*

$$\varphi_\Sigma(-k) = \sum_S |\mu(\varnothing, S^c)| \varphi^0_{\Sigma|S}(k+1) \tag{4.3}$$

and

$$\varphi_\Sigma(k) = \sum_S \mu(\varnothing, S^c) \varphi^0_{\Sigma|S}(k+1), \tag{4.4}$$

where μ is the Möbius function of $G^\perp(\Sigma)$.

Proof. We write $\mathcal{H} := (\mathcal{H}_E)^z$. The polytope and arrangement are transverse because $\bigcap \mathcal{H}$ intersects P°.

Since $\varphi_{\Sigma|S} = 0$ if $\Sigma|S$ has a coloop by Lemma 4.12, the two ranges of summation are equivalent. For $s \in \mathcal{L}(\mathcal{H})$, by Lemma 4.13 we know that

$$P \cap s = [-1,1]^E \cap \big([\operatorname{Nul}\mathrm{H}(\Sigma|E(s)^c)] \times \{0\}^{E(s)}\big).$$

Take $S = E(s)^c$. Then

$$P \cap s = \big([-1,1]^S \cap z'\big) \times \{0\}^{S^c}$$

where z' is the real cycle space of $\Sigma|S$. Its Ehrhart polynomial equals $\varphi^0_{\Sigma|S}(k+1)$.

Now the result follows from Lemma 4.11, Theorem 2.2, Equation (3.1), and Theorem 3.1 (d). ■

A reminder: to apply the theorem to a graph Γ, take $\Sigma = +\Gamma$.

Thus the strict integral flow polynomial can be expressed in terms of the weak polynomial and invariants of $\Gamma^\perp(\Sigma)$. If the weak polynomial were as simple as in the case of colorings, where it is a monomial [3, Section 5], we would have a nice formula for the number of nowhere-zero k-flows. But such is not the case.

It may be helpful to list some characterizations of the edge sets that support nowhere-zero integral flows.

Proposition 4.15. *For $S \subseteq E := E(\Sigma)$, the following properties are equivalent.*

(i) *$\Sigma|S$ has no coloops.*
(ii) *$\Sigma|S$ has a totally cyclic orientation.*
(iii) *$\Sigma|S$ has a nowhere-zero integral flow.*
(iv) *$\Sigma|S$ has a nowhere-zero real flow.*
(v) *S^c is closed in the dual bias matroid $G^\perp(\Sigma)$.*

(vi) $S = E(s)^c$ *for some flat* $s \in \mathcal{L}((\mathcal{H}_E)^z)$.

Proof. (i) $\iff$ (ii) for graphs is Robbins' theorem [14]. We could prove it for signed graphs, but the simplest approach is via oriented matroids. We know that the number of totally cyclic reorientations of an orientation of a matroid M is $t_M(0, 2)$ [10, 11] and that this equals 0 if and only if M has a coloop. Apply to the natural orientation of $G(\Sigma)$.

(i) $\iff$ (iii) by Lemma 4.12.

(iii) $\implies$ (iv) is trivial.

(iv) $\implies$ (i) by the proofs of [4, Lemmas 2.4 and 2.5], which amount to saying that any flow on Σ with values in an abelian group where $2a = 0 \implies a = 0$ must be zero on every coloop. Here the group is the additive group of $\mathbb{R}$.

(v) $\iff$ (i): By matroid duality the complements of the closed sets in $G^\perp(\Sigma)$ are the edge sets that do not contain a coloop of $G(\Sigma)$.

(v) $\iff$ (vi): This is Lemma 4.11. ∎

References

[1] G. Appa and B. Kotnyek, A bidirected generalization of network matrices. Submitted.

[2] W. Baldoni-Silva, J.A. De Loera, and M. Vergne, Counting integer flows in networks. Submitted.

[3] Matthias Beck and Thomas Zaslavsky, Inside-out polytopes. Submitted. Available on the Web at `http://arXiv.org/abs/math.CO/0309330`

[4] A. Bouchet, Nowhere-zero integral flows on a bidirected graph. *J. Combin. Theory Ser. B* **34** (1983), 279–292. MR 85d:05109. Zbl. 518.05058.

[5] Eugène Ehrhart, *Polynômes arithmétiques et Méthode des Polyèdres en Combinatoire*. Internat. Ser. Numerical Math., Vol. 35. Birkhäuser, Basel, 1977. MR 55 #5544. Zbl. 337.10019.

[6] Curtis Greene and Thomas Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.* **280** (1983), 97–126. MR 84k:05032. Zbl. 539.05024.

[7] François Jaeger, Nowhere-zero flow problems. Chapter 4, pp. 71–95, in Lowell W. Beineke and Robin J. Wilson, eds., *Selected Topics in Graph Theory, 3*. Academic Press, London, 1988.

[8] A. Khelladi, Nowhere-zero integral chains and flows in bidirected graphs. *J. Combin. Theory Ser. B* **43** (1987), 95–115. MR 88h:05045. Zbl. 617.90026.

[9] Martin Kochol, Polynomials associated with nowhere-zero flows. *J. Combin. Theory Ser. B* **84** (2002), 260–269. MR 2002k:05193.

[10] Michel Las Vergnas, Matroïdes orientables. *C. R. Acad. Sci. Paris Sér. A–B* **280** (1975), Ai, A61–A64. MR 51 #7910. Zbl. 304.05013.

[11] ——, Sur les activités des orientations d'une géométrie combinatoire. Colloque Mathmatiques Discrètes: Codes et Hypergraphes (Brussels, 1978). *Cahiers Centre Études Rech. Opér.* **20** (1978), 293–300. MR 81d:05019. Zbl. 404.05018.

[12] Jon Lee, Subspaces with well-scaled frames. *Linear Algebra Appl.* **114/115** (1989), 21–56. MR 90k:90111. Zbl. 675.90061.

[13] J.G. Oxley, *Matroid Theory*. Oxford Univ. Press, Oxford, 1992. MR 94d:05033. Zbl. 784.05002.

[14] H.E. Robbins, A theorem on graphs, with an application to a problem of traffic control. *Amer. Math. Monthly* **46** (1939), 281–283. Zbl. 21, 357c (e: 021.35703).

[15] Gian-Carlo Rota, On the foundations of combinatorial theory: I. Theory of Möbius functions. *Z. Wahrsch. verw. Gebiete* **2** (1964), 340–368. MR 30 #4688. Zbl. 121, 24f (e: 121.02406). Many times reprinted.

[16] P.D. Seymour, Nowhere-zero flows. Appendix to Ch. 4, pp. 289–299, in R.L. Graham, M. Grötschel, and L. Lovász, eds., *Handbook of Combinatorics*, Vol. 1. North-Holland (Elsevier), Amsterdam, 1995. MR 96m:05091. Zbl. 845.05035.

[17] Richard P. Stanley, Acyclic orientations of graphs. *Discrete Math.* **5** (1973), 171–178. MR 47 #6537. Zbl. 258.05113.

[18] ——, *Enumerative Combinatorics*, Vol. I. Wadsworth & Brooks/Cole, Monterey, Calif., 1986. MR 87j:05003. Corrected reprint: Cambridge Stud. Adv. Math., Vol. 49. Cambridge University Press, Cambridge, Eng., 1997. MR 98a:05001. Zbl. 889.05001, 945.05006.

[19] W.T. Tutte, A ring in graph theory. *Proc. Cambridge Philos. Soc.* **43** (1947), 26–40. MR 8, 284k. Zbl. 31, 418c (e: 031.41803). Reprinted with annotations in *Selected papers of W. T. Tutte*, Vol. I, pp. 51–69. Edited by D. McCarthy and Ralph G. Stanton. Charles Babbage Research Centre, Winnipeg, Man., 1979. MR 81k:01058a. Zbl. 403.05028.

[20] ——, On the imbedding of linear graphs in surfaces. *Proc. London Math. Soc.* (2) **51** (1949), 474–483. MR 10, 616b. Zbl. 33, 308c (e: 033.30803).

[21] ——, A contribution to the theory of chromatic polynomials. *Canadian J. Math.* **6** (1954), 80–91. MR 15, 814c. Zbl. 55, 171a (e: 055.17101). Reprinted with annotations in *Selected papers of W. T. Tutte*, Vol. I, pp. 153–168. Edited by D. McCarthy and Ralph G. Stanton. Charles Babbage Research Centre, Winnipeg, Man., 1979. MR 81k:01058a. Zbl. 403.05028.

[22] ——, *Graph theory.* Encyc. Math. Appl., 21. Addison-Wesley, Reading, Mass., 1984. MR 87c:05001. Zbl. 554.05001. Reprinted by Cambridge University Press, Cambridge, 2001. MR 2001j:05002. Zbl. 964.05001.

[23] Thomas Zaslavsky, *Facing Up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes.* Mem. Amer. Math. Soc., No. 154. American Mathematical Society, Providence, R.I., 1975. MR 50 #9603. Zbl. 296.50010.

[24] ——, Signed graphs. *Discrete Appl. Math.* **4** (1982), 47–74. Erratum. *Discrete Appl. Math.* **5** (1983), 248. MR 84e:05095. Zbl. 503.05060.

[25] ——, Signed graph coloring. *Discrete Math.* **39** (1982), 215–228. MR 84h:05050a. Zbl. 487.05027.

[26] ——, Orientation of signed graphs. *European J. Combin.* **12** (1991), 361–375. MR 93a:05065. Zbl. 761.05095.

[27] ——, Biased graphs IV: Geometrical realizations. *J. Combin. Theory Ser. B*, to appear.

[28] Günther Ziegler, *Lectures on Polytopes.* Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995. MR 96a:52011. Zbl. 823.52002.

Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA 94720-5070, U.S.A.

Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902-6000, U.S.A.