Theorem 5.13 (Binomial Theorem for integers). *If $a, b \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$ then*

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}.$$

Proof. We prove $P(n): \ (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$ by induction on $n \geq 0$. The base case $P(0)$ follows with the definition $m^0 = 1$ and $\binom{0}{0} = 1$.

For the induction step, assume $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$. Then

$$\sum_{r=0}^{n+1} \binom{n+1}{r} a^r b^{n+1-r} = \binom{n+1}{0} a^0 b^{n+1} + \sum_{r=1}^{n} \binom{n+1}{r} a^r b^{n+1-r} + \binom{n+1}{n+1} a^{n+1} b^0$$

(by Proposition 5.7(i))

$$= b^{n+1} + \sum_{r=1}^{n} \left(\binom{n}{r-1} + \binom{n}{r} \right) a^r b^{n+1-r} + a^{n+1}$$

(by definition of exponents/binomial coefficients and Corollary 5.12)

$$= b^{n+1} + \sum_{r=1}^{n} \binom{n}{r-1} a^r b^{n+1-r} + \sum_{r=1}^{n} \binom{n}{r} a^r b^{n+1-r} + a^{n+1}$$

(by distributivity and Proposition 5.7(iii))

$$= \sum_{r=0}^{n-1} \binom{n}{r} a^{r+1} b^{n+1-(r+1)} + \sum_{r=0}^{n} \binom{n}{r} a^r b^{n+1-r} + a^{n+1}$$

(by Proposition 5.7(ii) applied to the first sum and combining b^{n+1} with the second sum)

$$= \sum_{r=0}^{n} \binom{n}{r} a^{r+1} b^{n+1-(r+1)} + \sum_{r=0}^{n} \binom{n}{r} a^r b^{n+1-r}$$

(by combining a^{n+1} with the first sum)

$$= a \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} + b \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

(by definition of exponents and Proposition 5.6(i))

$$= a\,(a+b)^n + b\,(a+b)^n$$

(by induction hypothesis)

$$= (a+b)\,(a+b)^n$$

(by distributivity)

$$= (a+b)^{n+1}$$

(by definition of exponents) $\square$