MATH 850 **Polynomial Method Homework**

(1) Fix $d \in \mathbb{Z}_{>0}$ and consider $\binom{x}{d}$ as a polynomial in x (over your favorite field of characteristic 0).

(a) Show that, as polynomials, $\binom{-x}{d} = (-1)^d \binom{x+d-1}{d}$.

(b) Prove that $\sum_{n \geq 0} \binom{n+d}{d} x^n = \frac{1}{(1-x)^{d+1}}$.

(c) Show that $p(x)$ is a polynomial of degree d if and only if

$$\sum_{n \geq 0} p(n)\, x^n \;=\; \frac{h(x)}{(1-x)^{d+1}}$$

for some polynomial $h(x)$ of degree at most d with $h(1) \neq 0$.

(2) A matrix is *unipotent* if it is the sum of the $d \times d$ identity matrix and a *nilpotent* matrix (i.e., a matrix $\mathbf{B}$ for which there exists a positive integer k such that $\mathbf{B}^k = \mathbf{0}$). Fix indices i and j, and consider the sequence $f(n) := (\mathbf{A}^n)_{ij}$ formed by the (i,j)-entries of the nth powers of a unipotent matrix $\mathbf{A}$. Prove that $f(n)$ agrees with a polynomial in n. (*Hint:* express $\mathbf{A}^n$ using the binomial theorem.)

(3) Let Δ be the *difference operator* defined by $(\Delta f)(n) := f(n+1) - f(n)$, for a given polynomial $f(n)$. Prove that $f(n)$ is of degree $\leq d$ if and only if $(\Delta^m f)(0) = 0$ for all $m > d$. (*Hint:* use the *shift operator* $(Sf)(n) := f(n+1)$ and express $(S^n f)(0)$ using the binomial theorem.)

(4) A $d \times d$ matrix with nonnegative integer coefficients is *n-magic* if each of its rows and columns sum to n.

(a) Prove that every n-magic matrix is the sum of n permutation matrices.

(b) Show that

$$\left\{ (m_{11}, m_{12}, \ldots, m_{dd}, n) \in \mathbb{Z}_{\geq 0}^{d^2+1} : \mathbf{m} \text{ is an } n\text{-magic matrix} \right\}$$

forms a semigroup, and deduce that the number $M_d(n)$ of n-magic $d \times d$ matrices is a polynomial in n.

(5) Compute $M_3(n)$.

(6) Let $\mathbf{m}$ be a magic labelling of a given graph G with "magic sum" n.

(a) Define the matrix $\mathbf{A} = (a_{ij})$ where a_{ij} is the label of $\mathbf{m}$ on the edge connecting the nodes i and j. Show that each row and column sum of $\mathbf{A}$ is n, and deduce with the Exercise (4) that

$$2\mathbf{A} \;=\; \sum_{\pi} \pi + \pi^T$$

where the sum is over a certain set of permutation matrices.

(b) Prove that $2\mathbf{m}$ is a sum of magic labellings with magic sum 2, and conclude that every completely fundamental magic labelling of G has magic sum 1 or 2.

(7) Show that $\dim \mathrm{Poly}_D(\mathbb{F}^n) = \binom{D+n}{n}$.

(8) We proved in class that, given a finite set $S \subset \mathbb{R}^3$, there is a nonzero polynomial of degree $\leq c\,|S|^{\frac{1}{3}}$ (for some constant c) that vanishes on S. Given n lines in $\mathbb{R}^3$, prove that there is a nonzero polynomial of degree $\leq c\,n^{\frac{1}{2}}$ (for some other constant c) that vanishes on all the lines. Generalize.