MATH 435/735 Midterm Exam 10/19/22 12:00–12:50 a.m.

Show complete work—that is, all the steps needed to completely justify your answer. Simplify your answers as much as possible. You may refer to theorems in the book and class notes (without having to reference theorem numbers etc.).

1. (a) State one of the isomorphism theorems.
   (b) Suppose A and B are groups, and $C \trianglelefteq A$ and $D \trianglelefteq B$. Show that $C \times D \ \trianglelefteq\ A \times B$.
   (c) Prove that $(A \times B)/(C \times D) \simeq (A/C) \times (B/D)$.
   (*Hint:* there is a way to set up (c) so that (b) will follow as a side product.)

Proof.

(b) & (c) Let $\phi : A \times B \to (A/C) \times (B/D)$ be defined through $\phi(a, b) = (a\,C, b\,D)$. Then ϕ is a homomorphism and onto (both easy to check), and

$$\ker(\phi) \;=\; \{(a, b) \in A \times B :\, (a\,C, b\,D) = (C, D)\} \;=\; C \times D\,.$$

Thus $C \times D \ \trianglelefteq\ A \times B$, and by the First Isomorphism Theorem, $(A \times B)/(C \times D) \simeq (A/C) \times (B/D)$. $\square$

2. Suppose the group G acts on a set A.
   (a) Given $a \in A$, define its orbit $\mathrm{orb}(a)$ and stabilizer G_a.
   (b) Prove that $|\mathrm{orb}(a)| = [G : G_a]$. (*Hint:* construct a bijection between $\mathrm{orb}(a)$ and the left cosets of G_a.)
   (c) Show that a group of order p^k, for some prime p, has a nontrivial center. (*Hint:* consider G acting on itself by conjugation and partitioning G into the orbits of this group action.)

Proof.

(b) Define $\phi : \mathrm{orb}(a) \to \{g\,G_a :\, g \in G\}$ by $\phi(g \cdot a) := g\,G_a$. This map is well defined and injective:

$$g\,G_a = h\,G_a \quad\Longleftrightarrow\quad g^{-1}h \in G_a \quad\Longleftrightarrow\quad \left(g^{-1}h\right) \cdot a = a$$
$$\Longleftrightarrow\quad g \cdot a = g\left(g^{-1}h\right) \cdot a = \left(gg^{-1}\right) h \cdot a = h \cdot a\,.$$

Since ϕ is clearly surjective, it is thus a bijection, and so there are $|\mathrm{orb}(a)|$ left cosets of G_a, whose number is also $[G : G_a]$.

(c) The stabilizer of $a \in G$ under conjugation is the centralizer $C_G(a)$, and $\mathrm{orb}(a) = \{a\}$ if and only if $a \in Z(G)$. In particular, if $a \notin Z(G)$ then $|\mathrm{orb}(a)| = [G : C_G(a)]$ is divisible by p. We know that we can partition G into its orbits, from which we can write $|G|$ as the number of single-element orbits (which is $|Z(G)|$) plus some number divisible by p. Because $|G|$ is also divisible by p, so is $|Z(G)|$. $\square$

3. (a) State one of the three parts of Sylow's Theorem.
   (b) Find all Sylow 2-subgroups of A_4 and determine their isomorphism type.
   (c) Find all Sylow 3-subgroups of A_4 and determine their isomorphism type.

Proof.

(b) There is only one subgroup of A_4 of order 4, namely $\langle (12)(34), (13)(24) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) There are four subgroups of A_4 of order 3, namely those generated by a 3-cycle; all are isomorphic to $\mathbb{Z}_3$. $\square$