Worksheet 1: Euclidean Algorithm

1. In your group, remind each other about tests for divisibility by 2, 3, and 5. Prove that these tests work.

2. Let $a, b, c \in \mathbb{Z}$ with $c \neq 0$. Prove that if $c \mid a$ and $c \mid b$ then $c \mid (ax + by)$ for any $x, y \in \mathbb{Z}$.

3. (a) Why is the fraction $\frac{a}{0}$ "undefined" for $a \neq 0$?

   (b) Why is $\frac{0}{0}$ "indeterminate?"

4. The *division algorithm* says that every division problem has a unique quotient and remainder. Come up with a precise mathematical statement for the division algorithm and prove it.

5. Come up with a definition of the *greatest common divisor* of two integers. There are various ways to define the gcd; discuss advantages and disadvantages in your group.

6. Pick two 3-digit positive integers $a > b$ and run the division algorithm when b is divided into a. Run the algorithm again when the remainder is divided into b; repeat until you get remainder 0. What are you computing? Why?

7. Let $a, b \in \mathbb{Z}$, not both zero. Prove that there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = \gcd(a, b) \, .$$

   More generally, prove that

$$ax + by = c$$

   has a solution $(x, y) \in \mathbb{Z}^2$ if and only if $\gcd(a, b) \mid c$.

8. Andrews 2.3.1.

9. Experiment with the `sage` command `divmod`. Use it with two arguments, say a 6-digit and a 3-digit number, and check that `sage` gives the correct answer.

10. Experiment with the `sage` command `xgcd`. Use it with two 5-digit arguments and check that `sage` gives the correct answer.

11. Write down a precise statement for each definition we have given this week. For each definition, give an example and a non-example.