Sperner's Theorem and its generalizations

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"Everything should be made as simple as possible, but not simpler."

Albert Einstein

"The worst thing you can do to a problem is solve it completely."

Daniel Kleitman

Matroid theory

Ehrhart theory

Department

Lattice!

Define a weak partial composition into p parts as an ordered p-tuple $(A_1, \ldots, A_p)$ such that $A_1, \ldots, A_p$ are pairwise disjoint sets.

Theorem Suppose $(A_{j1}, \ldots, A_{jp})$ for $j = 1, \ldots, m$ are different weak set compositions into p parts with the condition that, for all $1 \le k \le p$ and all $I \subseteq [m]$ with $|I| = r+1$, there exist distinct $i, j \in I$ such that either $A_{ik} = A_{jk}$ or

$$A_{ik} \cap \bigcup_{l \neq k} A_{jl} \neq \varnothing \neq A_{jk} \cap \bigcup_{l \neq k} A_{il} .$$

Then

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_{j1}|+\cdots+|A_{jp}|}{|A_{j1}|,\ldots,|A_{jp}|}} \le r^p$$

and m is bounded by the sum of the r^p largest p-multinomial coefficients for integers less than or equal to

$$\max_{1 \le j \le m} \left(|A_{j1}| + \cdots + |A_{jp}|\right) .$$

Let S be an n-element set.

Sperner's Theorem (1928)
Suppose $A_1, \ldots, A_m \subseteq S$ such that $A_k \not\subseteq A_j$ for $k \neq j$. Then

$$m \leq \binom{n}{\lfloor n/2 \rfloor} .$$

LYM Inequality
(Lubell, Yamamoto, Meshalkin, 1960±6)
Suppose $A_1, \ldots, A_m \subseteq S$ such that $A_k \not\subseteq A_j$ for $k \neq j$. Then

$$\sum_{k=1}^{m} \frac{1}{\binom{n}{|A_k|}} \leq 1 .$$

Both bounds can be attained for any n.

Theorem (Erdős, 1945)
Suppose $\{A_1, \ldots, A_m\} \subseteq \mathcal{P}(S)$ contains no chains with $r+1$ elements. Then m is bounded by the sum of the r largest binomial coefficients $\binom{n}{k}, 0 \le k \le n$.

Theorem (Rota–Harper, 1970)
Suppose $\{A_1, \ldots, A_m\} \subseteq \mathcal{P}(S)$ contains no chains with $r+1$ elements. Then

$$\sum_{k=1}^{m} \frac{1}{\binom{n}{|A_k|}} \le r\ .$$

Both bounds can be attained for any n and r.

Theorem (Griggs–Stahl–Trotter, 1984)
Suppose $\{A_{j0}, \ldots, A_{jq}\}$ are m different chains in $\mathcal{P}(S)$ such that $A_{ji} \not\subseteq A_{kl}$ for all i and l and all $j \neq k$. Then

$$m \leq \binom{n-q}{\lfloor (n-q)/2 \rfloor} .$$

Theorem (Bollobás, 1965)
Suppose (A_j, B_j) are m pairs of sets such that $A_j \cap B_j = \varnothing$ for all j and $A_j \cap B_k \neq \varnothing$ for all $j \neq k$. Then

$$\sum_{j=1}^{m} \frac{1}{\binom{|A_j|+|B_j|}{|A_j|}} \leq 1 .$$

Both bounds can be attained for any n and q.

Define a weak composition of S into p parts as an ordered p-tuple $A = (A_1, \ldots, A_p)$ of sets A_k such that $A_1, \ldots, A_p$ are pairwise disjoint and $A_1 \cup \cdots \cup A_p = S$.

Theorem (Meshalkin, 1963)
Suppose $\mathcal{M} = \{A^1, \ldots, A^m\}$ is a class of weak compositions of S into p parts such that for all $1 \le k \le p$ the set $\{A_k^j\}_{j=1}^m$ forms an antichain. Then $m = |\mathcal{M}|$ is bounded by the largest p-multinomial coefficient for n.

Theorem (Hochberg–Hirsch, 1970)

$$\sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|, \ldots, |A_p|}} \le 1 .$$

Both bounds can be attained for any n and p.

Proof. $p = 2$: LYM.

For general p, let

$\mathcal{M}(F) = \{(A_2, \ldots, A_p) : (F, A_2, \ldots, A_p) \in \mathcal{M}\}$.

Then

$$\sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|,\ldots,|A_p|}} = \sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|}\binom{n-|A_1|}{|A_2|,\ldots,|A_p|}}$$

$$= \sum_{F \in \mathcal{M}_1} \frac{1}{\binom{n}{|F|}} \sum_{A' \in \mathcal{M}(F)} \frac{1}{\binom{n-|F|}{|A_2|,\ldots,|A_p|}}$$

where $A' = (A_2, \ldots, A_p)$,

$$\leq \sum_{F \in \mathcal{M}_1} \frac{1}{\binom{n}{|F|}} \cdot 1$$

by the induction hypothesis,

$$\leq 1$$

by LYM. ■

Sperner

E GST M

Theorem

Suppose $\mathcal{M} = \{A^1, \ldots, A^m\}$ is a class of weak compositions of S into p parts such that for all $1 \le k < p$ the set

$$\{A_k^j : 1 \le j \le m\}$$

contains no chain of length r. Then

(a) $\displaystyle\sum_{A \in \mathcal{M}} \frac{1}{\binom{n}{|A_1|,\ldots,|A_p|}} \le r^{p-1}$

(b) $m = |\mathcal{M}|$ is bounded by the sum of the r^{p-1} largest p -multinomial coefficients for n.

Projective geometry $\mathbb{P}^{n-1}(q)$

Rank of a flat $r(a) = \dim a + 1$

q-Gaussian coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q (n-k)!_q},$$

where

$$n!_q = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$$

Theorem (Rota–Harper, 1970)
Suppose $\{a_1, \ldots, a_m\} \subseteq \mathbb{P}^{n-1}(q)$ contains no chains with $r+1$ elements.

(a) $\displaystyle\sum_{j=1}^{m} \frac{1}{\begin{bmatrix} n \\ r(a_j) \end{bmatrix}_q} \leq r$

(b) m is at most the sum of the r largest Gaussian coefficients $\begin{bmatrix} n \\ j \end{bmatrix}_q$ for $0 \leq j \leq n$

A Meshalkin sequence of length p in $\mathbb{P}^{n-1}(q)$ is a sequence $a = (a_1, \ldots, a_p)$ of flats whose join is $\hat{1}$ and whose ranks sum to n .

If a is a Meshalkin sequence, we write

$$r(a) = (r(a_1), \ldots, r(a_p))$$

for the sequence of ranks.

For $\alpha = (\alpha_1, \ldots, \alpha_p)$ we write

$$s_2(\alpha) = \sum_{i<j} \alpha_i \alpha_j$$

and define the $(q$-)Gaussian multinomial coefficient as

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \frac{n!_q}{\alpha_1!_q \cdots \alpha_p!_q} \ .$$

Theorem

Suppose $\mathcal{M} = \{a^1, \ldots, a^m\}$ is a family of Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that for all $1 \leq k < p$ the set

$$\left\{a_k^j : 1 \leq j \leq m\right\}$$

contains no chain of length r . Then

(a) $$\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix}_q q^{s_2(r(a))}} \leq r^{p-1}$$

(b) $m = |\mathcal{M}|$ is at most equal to the sum of the r^{p-1} largest amongst the quantities

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q q^{s_2(\alpha)}$$

for $\alpha = (\alpha_1, \ldots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$

Corollary

Suppose $\mathcal{M} = \{a^1, \ldots, a^m\}$ is a family of Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that for all $1 \leq k < p$ the set

$$\left\{a_k^j : 1 \leq j \leq m\right\}$$

is an antichain. Then

(a) $$\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix}_q q^{s_2(r(a))}} \leq 1$$

(b) $$m = |\mathcal{M}| \leq \max_{\alpha} \begin{bmatrix} n \\ \alpha \end{bmatrix}_q q^{s_2(\alpha)}$$

(c) The bounds in (a) and (b) can be achieved for any n and p .

A "stranger" LYM inequality is

Corollary

Suppose $\mathcal{M} = \{a^1, \ldots, a^m\}$ is a family of Meshalkin sequences of length p in $\mathbb{P}^{n-1}(q)$ such that for all $1 \leq k < p$ the set

$$\left\{a_k^j : 1 \leq j \leq m\right\}$$

contains no chain of length r . Then

$$\sum_{a \in \mathcal{M}} \frac{1}{\begin{bmatrix} n \\ r(a) \end{bmatrix}_q}$$

is bounded by the sum of the r^{p-1} largest expressions $q^{s_2(\alpha)}$ for $\alpha = (\alpha_1, \ldots, \alpha_p)$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_p = n$.