MATH 380 Midterm Exam 11/5/07

Show complete work—that is, all the steps needed to completely justify your answer. You may refer to theorems proved in class and in the text.

(1) Suppose $G \subseteq \mathbb{C}$ is open, $f : G \to \mathbb{C}$, and $z \in G$. Define:

(a) f is differentiable at z.

(b) f is analytic at z.

Solution:

(a) f is differentiable at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

(b) f is analytic at z_0 if f is differentiable for all points in $\{z \in \mathbb{C} : \ |z - z_0| < r\}$ for some $r > 0$.

(2) For **one** of the following functions, give the subset of $\mathbb{C}$ where the function is differentiable, respectively analytic, and find its derivative. (As usual, $z = x + iy$.)

(a) $f(z) = e^x(\cos y + i \sin y)$

(b) $f(z) = |z|^2 = x^2 + y^2$

(c) $f(z) = \frac{1}{\sin z}$

(d) $f(z) = (z - 1 + 2i)^{3i}$

(e) $f(z) = (3i)^{z-1+2i}$

Solution:

(a) $f(z) = e^x(\cos y + i \sin y) = e^x e^{iy} = \exp z$ is entire (as proved in class).

(b) The real part of f is $u = x^2 + y^2$ and the imaginary part is $v = 0$. For the Cauchy-Riemann equations to hold, we need

$$u_x = 2x = 0 = v_y \qquad \text{and} \qquad u_y = 2y = 0 = -v_x \ ,$$

and these equations are only satisfied for $x = 0$ and $y = 0$. Hence the first part of the Cauchy-Riemann Theorem 2.4 says that f is not differentiable for all $z \in \mathbb{C}\backslash\{0\}$. Since u_x, u_y, v_x, v_y are continuous, the second part of the Cauchy-Riemann Theorem 2.4 implies that f is differentiable at 0. Since a point contains no disk, f is nowhere analytic.

(c) sin is entire, so the only points where $1/\sin z$ is not analytic are the zeros of the sine. These can be computed as follows:

$$\begin{aligned}\sin z &= \frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right) = \frac{1}{2i}\left(e^{-y}e^{ix} - e^{y}e^{-ix}\right) \\ &= \frac{1}{2i}\left(\cos x\left(e^{-y} - e^{y}\right) + i \sin x\left(e^{-y} + e^{y}\right)\right) = 0\end{aligned}$$

means (after cancelling $2i$)

$$\cos x \left(e^{-y} - e^{y}\right) = 0 \qquad \text{and} \qquad \sin x \left(e^{-y} + e^{y}\right) = 0 \; .$$

Since $e^{-y} + e^{y} > 0$, the second equation implies $x = \pi k,\ k \in \mathbb{Z}$. For any of those x, $\cos x = \pm 1 \neq 0$, so that the first equation can only hold if $e^{-y} - e^{y} = 0$, which means $y = 0$. Hence the zeros of the sine are precisely at $z = \pi k,\ k \in \mathbb{Z}$, which in turn means that $1/\sin z$ is analytic on $\mathbb{C} \setminus \{\pi k :\ k \in \mathbb{Z}\}$.

(d) By definition,

$$(z-1+2i)^{3i} = \exp\left((3i)\,\mathrm{Log}(z-1+2i)\right).$$

exp is an entire function, so $(z-1+2i)^{3i}$ is analytic wherever $\mathrm{Log}(z-1+2i)$ is. As we showed many times, Log is analytic everywhere but the nonpositive real axis ($z = x+iy$ with $x \leq 0$ and $y = 0$), which implies that $\mathrm{Log}(z-1+2i)$ is analytic everywhere but when $z-1+2i$ is real and nonpositive, that is, for $z = x+iy$ with $x \leq 1$ and $y = -2$. Hence $(z-1+2i)^{3i}$ is analytic on $\mathbb{C} \setminus \{x+iy \in \mathbb{C} :\ x \leq 1,\ y = -2\}$.

(e) By definition,

$$\begin{aligned}(3i)^{z-1+2i} &= \exp\left((z-1+2i)\,\mathrm{Log}(3i)\right) = \exp\left((z-1+2i)\left(\ln|3i| + \mathrm{Arg}(3i)\right)\right) \\ &= \exp\left((z-1+2i)\left(\ln 3 + i\pi/2\right)\right).\end{aligned}$$

This is the exponential function applied to a polynomial. Both are entire functions, so $(3i)^{z-1+2i}$ is entire.

(3) Prove: If f is entire and real valued (that is, $\mathrm{Im}(f(z)) = 0$ for all $z \in \mathbb{C}$) then f is constant.

Solution: Let $f = u + iv$, then the conditions imply that $v = 0$. Hence by the Cauchy-Riemann equations,

$$u_x = v_y = 0 \qquad \text{and} \qquad u_y = -v_x = 0 \; ,$$

so u and thus $f = u + iv$ have to be constant.

(4) Integrate **one** of the following functions over the circle $|z| = 2$, oriented counterclockwise.

(a) $\overline{z}$

(b) $\dfrac{1}{z^4}$

(c) $\left(\dfrac{\exp z}{z}\right)^2$

(d) $\mathrm{Log}(z+3)$

(e) $\dfrac{\sin z}{(z-1)(z-3)}$

(f) $\dfrac{1}{z^3 + 34z}$

Solution: A parametrization of the circle γ is $\gamma(t) = 2\,e^{it} = 2\cos t + 2i\sin t,\ 0 \le t \le 2\pi$. Note that $\gamma'(t) = 2i\,e^{it}$.

(a) $\displaystyle\int_\gamma \overline{z}\,dz = \int_0^{2\pi} 2\,e^{-it}\,2i\,e^{it}\,dt = \int_0^{2\pi} 4i\,dt = 8\pi i\ .$

(b) $\frac{1}{z^4}$ has the antiderivative $-\frac{1}{3z^3}$ on $\mathbb{C}\setminus\{0\}$, which contains γ, and thus the integral is zero.

(c) Here we use the "extended Cauchy Formula" Theorem 5.1.: exp is entire, so we can choose $G = \mathbb{C}$, then γ is G-contractible, and $w = 0$ is inside γ. Hence

$$\int_\gamma \left(\frac{\exp z}{z}\right)^2 dz = \int_\gamma \frac{\exp^2 z}{z^2}\,dz = 2\pi i\,\left(\exp^2 z\right)'\Big|_{z=0} = 2\pi i\,\left(2\exp z\cdot\exp z\right)\big|_{z=0} = 4\pi i\ .$$

(d) The integrand $f(z) = \mathrm{Log}(z+3)$ is analytic in $G = \mathbb{C}\setminus\{z\in\mathbb{C}:\ \mathrm{Re}\,z \le -3\}$. However, γ is G-contractible, so by Corollary 4.5 (to Cauchy's Theorem)

$$\int_\gamma \mathrm{Log}(z+3)\,dz = 0\ .$$

(e) A partial fraction expansion gives $\displaystyle\frac{1}{(z-1)(z-3)} = \frac{1/2}{z-3} - \frac{1/2}{z-1}$, so

$$\int_\gamma \frac{\sin z}{(z-1)(z-3)}\,dz = \frac{1}{2}\int_\gamma \frac{\sin z}{z-3}\,dz - \frac{1}{2}\int_\gamma \frac{\sin z}{z-1}\,dz\ .$$

For the first integral, we can use Corollary 4.5 (to Cauchy's Theorem) with $G = \mathbb{C}\setminus\{3\}$, $f(z) = \sin z/(z-3)$: note that f is analytic in G and γ is closed and G-contractible, and so

$$\int_\gamma \frac{\sin z}{z-3}\,dz = 0\ .$$

For the second integral, we can use Cauchy's Integral Formula (Theorem 4.7) with $G = \mathbb{C}$, $f(z) = \sin z$, and $w = 1$ (note that w is inside γ and that γ is G-contractible):

$$\int_\gamma \frac{\sin z}{z-1}\,dz = 2\pi i\sin 1\ .$$

Putting it all together, we get

$$\int_\gamma \frac{\sin z}{(z-1)(z-3)}\,dz = -\pi i\sin 1\ .$$

(f) We write

$$\int_\gamma \frac{1}{z^3+34z}\,dz = \int_\gamma \frac{1}{(z^2+34)z}\,dz = \int_\gamma \frac{\frac{1}{z^2+34}}{z}\,dz\ ,$$

and use Cauchy's Integral Forumla (Theorem 4.7) with $f(z) = \frac{1}{z^2+34}$, $G = \mathbb{C}\setminus\{\pm i\sqrt{34}\}$, and $w = 0$ (note that 0 is inside γ and that γ is G-contractible). Hence

$$\int_\gamma \frac{1}{z^3+34z}\,dz = \int_\gamma \frac{\frac{1}{z^2+34}}{z}\,dz = 2\pi i\frac{1}{0^2+34} = \frac{\pi i}{17}\ .$$

(5) Suppose $(f_n(z))_{n=1}^{\infty}$ is a sequence of functions defined on $G \subseteq \mathbb{C}$. Define:

(a) $(f_n(z))_{n=1}^{\infty}$ converges pointwise on G.

(b) $(f_n(z))_{n=1}^{\infty}$ converges uniformly on G.

Solution: There exists a function $f : G \to \mathbb{C}$ such that:

(a) $\forall x \in G \; \forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; \; |f_n(z) - f(z)| < \epsilon$;

(b) $\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; \forall x \in G \; \; |f_n(z) - f(z)| < \epsilon$.

(6) Prove that the function sequence $(z^n)_{n=1}^{\infty}$ converges

(a) pointwise on $\{z \in \mathbb{C} : \; |z| < 1\}$,

(b) uniformly on $\{z \in \mathbb{C} : \; |z| \leq 1/2\}$.

Solution: We claim the limit function is $f(z) = 0$ for $|z| < 1$.

(a) Given any z with $|z| < 1$ and any $\epsilon > 0$, choose $N > \log \epsilon / \log |z|$. Then for all $n \geq N$

$$|z^n - 0| = |z|^n \leq |z|^N < \epsilon \; .$$

(b) Given any $\epsilon > 0$, choose N such that for all $n \geq N$

$$\left| \left(\tfrac{1}{2}\right)^n - 0 \right| = \left(\tfrac{1}{2}\right)^n < \epsilon \; .$$

(This N exists because of part (a).) Hence for all z with $|z| \leq 1/2$ and for all $n \geq N$

$$|z^n - 0| = |z|^n \leq \left(\tfrac{1}{2}\right)^n < \epsilon \; .$$