MATH 725 **Homework Set 11** **due 11/14/11**

(1) Compute the singular values of the differentiation operator on $\mathscr{P}_2(\mathbf{R})$ equipped with our usual inner product $\langle f,g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}\,dx$.

Solution. We computed (in an earlier homework) the orthonormal basis

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \qquad \mathbf{e}_2 = \sqrt{\tfrac{3}{2}}\,x \qquad \mathbf{e}_3 = \sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)$$

for this inner-product space. With respect to this basis, the differentiation operator D has the matrix

$$D = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since our basis is orthonormal, we can compute D^* by transposing this matrix:

$$D^* = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{pmatrix},$$

and so

$$\sqrt{D^*D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \end{pmatrix}.$$

Thus the singular values of D (which are the eigenvalues of $\sqrt{D^*D}$) are 0, $\sqrt{3}$, and $\sqrt{15}$. □

(2) Consider $T \in L(\mathbf{C}^3)$ given by $T(x,y,z) = (y,z,0)$.
 (a) Compute the eigenvalues and generalized eigenspaces of T.
 (b) Prove that T has no square root.[1]

Proof. (a) Writing T in terms of the standard basis,

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

we see that $\dim\operatorname{null}(T^2) = 2$ and $\dim\operatorname{null}(T^3) = 3$, and so 0 is the only eigenvalue of T and it has multiplicity 3, i.e., the corresponding generalized eigenspace is all of $\mathbf{C}^3$.

(b) Suppose $S \in L(\mathbf{C}^3)$ is a square root of T, i.e., $S^2 = T$. By part (a), $\dim\operatorname{null}(S^6) = \dim\operatorname{null}(T^3) = 3$, and so (by a proposition proved in class)

$$\dim\operatorname{null}(S^3) = \dim\operatorname{null}(S^4) = \dim\operatorname{null}(S^5) = \dim\operatorname{null}(S^6) = 3.$$

But this means that $\dim\operatorname{null}(T^2) = \dim\operatorname{null}(S^4) = 3$, contrary to what we showed in part (a). □

(3) Suppose $T \in L(V)$ and $m \in \mathbf{Z}_{\geq 0}$ such that $\operatorname{range}(T^m) = \operatorname{range}(T^{m+1})$. Prove that $\operatorname{range}(T^k) = \operatorname{range}(T^m)$ for all $k \geq m$.

Proof. Suppose that $\operatorname{range}(T^m) = \operatorname{range}(T^{m+1})$. We will prove that $\operatorname{range}(T^{m+1}) = \operatorname{range}(T^{m+2})$; the general statement will then follow by induction.

We already know that $\operatorname{range}(T^{m+2}) \subseteq \operatorname{range}(T^{m+1})$. To prove the other inclusion, assume that $\mathbf{v} \in \operatorname{range}(T^{m+1})$, i.e., $T^{m+1}(\mathbf{u}) = \mathbf{v}$ for some $\mathbf{u} \in V$. Let $\mathbf{w} := T^m(\mathbf{u})$; be definition $\mathbf{w} \in \operatorname{range}(T^m)$. By the first line of our proof, $\mathbf{w} \in \operatorname{range}(T^{m+1})$, and so $T^{m+1}(\mathbf{x}) = \mathbf{w}$ for some $\mathbf{x} \in V$. Thus

$$\mathbf{v} = T^{m+1}(\mathbf{u}) = T(\mathbf{w}) = T^{m+2}(\mathbf{x}),$$

[1] *Hint:* start by showing that if $S^2 = T$ then $S^3 = 0$.

which implies that $\mathbf{v} \in \text{range}(T^{m+2})$. $\square$

(4) Suppose $T \in L(V)$ where $\dim V = n$. Show that, while in general it is not true that $V = \text{null}(T) \oplus \text{range}(T)$, we always have
$$V = \text{null}(T^n) \oplus \text{range}(T^n)\,.$$

Proof. The example $V = \mathbf{R}^2$, $T(x,y) = (y,0)$ shows that $V = \text{null}(T) \oplus \text{range}(T)$ is not always true (in this example, $\text{null}(T) = \text{range}(T) = \{(x,0) : x \in \mathbf{R}\}$).

Next we'll prove that $V = \text{null}(T^n) + \text{range}(T^n)$. Given $\mathbf{v} \in V$, we first remember that we proved in class $\text{range}(T^n) = \text{range}(T^{2n})$, and so $T^n(\mathbf{v})$ (which is by definition in $\text{range}(T^n)$) is in $\text{range}(T^{2n})$, i.e.,
$$T^n(\mathbf{v}) = T^{2n}(\mathbf{u})$$
for some $\mathbf{u} \in V$. But this can be rewritten as $T^n(\mathbf{v} - T^n(\mathbf{u})) = \mathbf{0}$, i.e., $\mathbf{v} - T^n(\mathbf{u}) \in \text{null}(T^n)$, i.e., $v \in \text{null}(T^n) + \text{range}(T^n)$.

Finally, we'll prove that $\text{null}(T^n) + \text{range}(T^n)$ is a direct sum. Suppose $\mathbf{v} \in \text{null}(T^n) \cap \text{range}(T^n)$, i.e.,
$$T^n(\mathbf{v}) = \mathbf{0} \qquad \text{and} \qquad T^n(\mathbf{u}) = \mathbf{v}$$
for some $\mathbf{u} \in V$. But then $T^{2n}(\mathbf{u}) = T^n(\mathbf{v}) = \mathbf{0}$, i.e., $\mathbf{u} \in \text{null}(T^{2n}) = \text{null}(T^n)$ (using the same result from class again). This means $\mathbf{v} = T^n(\mathbf{u}) = \mathbf{0}$, and so $\text{null}(T^n) \cap \text{range}(T^n) = \{\mathbf{0}\}$ and $V = \text{null}(T^n) \oplus \text{range}(T^n)$. $\square$

(5) Suppose $T \in L(V)$ is invertible. Prove that T and T^{-1} have the same generalized eigenspaces (even though their eigenvalues are different). What does this imply for the characteristic polynomials of T and T^{-1}?

Proof. Let $n = \dim V$, and consider an eigenvalue λ of T. Then
$$\text{null}(T - \lambda \mathrm{I})^n = \text{null}\left(\lambda T\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)\right)^n = \text{null}\left((\lambda T)^n \left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n\right)$$
for some $j \in \mathbf{Z}_{>0}$. (Here the last equality holds because T commutes with both I and T^{-1}.) But since λT is invertible,
$$\text{null}\left((\lambda T)^n \left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n\right) = \text{null}\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n,$$
and so we have
$$\text{null}(T - \lambda \mathrm{I})^n = \text{null}\left(\tfrac{1}{\lambda}\mathrm{I} - T^{-1}\right)^n,$$
in words: the generalized eigenspace of T corresponding to λ equals the generalized eigenspace of T^{-1} corresponding to $\frac{1}{\lambda}$.

Let's write $\text{mult}_T(\lambda)$ for the multiplicity of the eigenvalue λ of T. What we just proved implies
$$\text{mult}_T(\lambda) = \text{mult}_{T^{-1}}\left(\tfrac{1}{\lambda}\right),$$
and this implies, in turn, that the characteristic polynomial of T,
$$c_T(x) := \prod_{\lambda \text{ eigenvector of } T} (x - \lambda)^{\text{mult}_T(\lambda)},$$
is related to the following evaluation of the characteristic polynomial of T^{-1}:
$$c_{T^{-1}}\left(\tfrac{1}{x}\right) = \prod_{\lambda \text{ eigenvector of } T^{-1}} \left(\tfrac{1}{x} - \lambda\right)^{\text{mult}_{T^{-1}}(\lambda)} = \prod_{\lambda \text{ eigenvector of } T} \left(\tfrac{1}{x} - \tfrac{1}{\lambda}\right)^{\text{mult}_T(\lambda)}.$$
More precisely, $x^n c_{T^{-1}}\left(\frac{1}{x}\right)$ is a polynomial in x of degree n, which has the same roots (including multiplicities) as the polynomial $c_T(x)$. Thus these two polynomials only differ in a constant factor; e.g., we can state their relationship as
$$c_T(0)\, x^n\, c_{T^{-1}}\left(\tfrac{1}{x}\right) = c_T(x)\,.$$
$\square$