Project 8.9. *For $B \subseteq \mathbb{R}$, $B \neq \emptyset$ we can define the greatest lower bound (*$\inf B$ *for "infimum") of B. Give the precise definition for* $\inf B$ *and prove that it is unique if it exists.*

Solution. The greatest lower bound $\inf B$ is a lower bound of B that is larger than or equal to every lower bound of B.

To prove that $\inf B$ is unique, we will show that, if b_1 and b_2 are both greatest lower bounds of B then $b_1 = b_2$. Namely, suppose b_1 and b_2 are both greatest lower bounds of B. That is, they are both lower bounds of B, and $b_1 \geq b_2$ (since b_1 is a greatest lower bound) and $b_2 \geq b_1$ (since b_2 is a greatest lower bound). In summary, we have $b_1 \leq b_2 \leq b_1$, and Proposition 8.2(v) implies that $b_1 = b_2$. $\square$