Proposition 2.5. *For all $m \in \mathbb{Z}$, $m \cdot 0 = 0 = 0 \cdot m$.*

Proof. Given an $m \in \mathbb{Z}$, we use Axiom 2.4: there exists $-m \in \mathbb{Z}$ such that $m + (-m) = 0$. Now by Axiom 2.3, $m = m \cdot 1 = m \cdot (1 + 0) = m \cdot 1 + m \cdot 0$, using Axioms 2.2 and 2.1(iii). Again appealing to Axiom 2.3, this gives

$$m = m + m \cdot 0 \ .$$

To cancel m on both sides we add $-m$ to both sides:

$$m + (-m) = (m + m \cdot 0) + (-m)$$

and use Axiom 2.1(i) and (ii) on the right hand side to obtain $(m + m \cdot 0) + (-m) = m + (m \cdot 0 + (-m)) = m + (-m + m \cdot 0) = (m + (-m)) + m \cdot 0$. This gives

$$m + (-m) = (m + (-m)) + m \cdot 0$$

and we can use Axiom 2.4 to simplify this equation to

$$0 = 0 + m \cdot 0 \ .$$

Finally, we use Axioms 2.1(i) and 2.2 to simplify the right-hand side: $0 + m \cdot 0 = m \cdot 0 + 0 = m \cdot 0$, which gives

$$0 = m \cdot 0 \ .$$

The other identity follows with Axiom 2.1(iv): $m \cdot 0 = 0 \cdot m$. $\square$