MATH 435/735 Final Exam 12/16/16 Name:

(1) Let G be a finite group.

(a) Define what it means for a subgroup $H \subseteq G$ to be a Sylow p-subgroup.

(b) Explain why each subgroup of S_4 of order 8 is isomorphic to D_8.

(c) Show that S_4 has three subgroups of order 8, none of which is normal.

Solution:

(b) D_8 is the symmetry group of a square and as such a Sylow 2-subgroup of S_4. Sylow's Theorem says that all other Sylow 2-subgroups can be obtained via conjugation and are thus isomorphic to D_8.

(c) Sylow's Theorem says that there are either one or three Sylow 2-subgroup of S_4, so it suffices to find two copies of D_8 among the subgroups of S_4, e.g.,

$$\langle (12), (1234) \rangle \qquad \text{and} \qquad \langle (12), (1324) \rangle .$$

Since there is more than one Sylow 2-subgroup, they cannot be normal by Sylow's Theorem.

(2) Let R be a ring with 1.

(a) Define what it means for $a \in R$ to be a zero divisor, and what it means for a to be a unit.

(b) Now let $R = \mathbb{Z}_2[x]/\langle x^3+1\rangle$. Show that $x^2+x+1+\langle x^3+1\rangle$, $x+1+\langle x^3+1\rangle$, $x^2+x+\langle x^3+1\rangle$, and $x^2+1+\langle x^3+1\rangle$ are zero divisors in R.

(c) Show that R^* is cyclic of order 3.

Solution:

(b) To ease notation, let $\bar{x} = x + \langle x^3+1\rangle \in R$. Since $x^3+1 = (x^2+x+1)(x+1)$, we conclude that $\bar{x}^2+\bar{x}+1$ and $\bar{x}+1$ are zero divisors in R, and thus so are $\bar{x}^2+\bar{x} = \bar{x}(\bar{x}+1)$ and $\bar{x}^2+1 = \bar{x}(\bar{x}^2+\bar{x})$.

(c) We just found four zero divisors. Because $|R| = 8$, there are three nonzero elements in R, namely 1, $\bar{x}$, and $\bar{x}^2$; they are all units since $\bar{x}\cdot\bar{x}^2 = 1$ in R. Hence R^* is a group of order three, which has to be cyclic.

(3) Let R be an integral domain.

(a) Define what it means for R to be a principal ideal domain.

(b) Define what it means for R to be Euclidean.

(c) Sketch a proof that $\mathbb{R}[x]$ is Euclidean.

Solution: See class notes.

(4) Let R, S, and T be rings, and let M and N be R-modules.

(a) Define what it means for $\phi : S \to T$ to be a ring homomorphism.

(b) Define what it means for $\psi : M \to N$ to be an R-module homomorphism.

(c) Show that $\mathbb{Z}_3[x]/\langle x^2+1\rangle$ and $\mathbb{Z}_3[x]/\langle x^2+x\rangle$ are

- isomorphic as $\mathbb{Z}_3$-modules,
- non-isomorphic as rings.

(*Hint for the last part:* one of $\mathbb{Z}_3[x]/\langle x^2+1\rangle$ and $\mathbb{Z}_3[x]/\langle x^2+x\rangle$ is a field.)

Solution: (c) Let $f = x^2+1$ and $g = x^2+x$. Then any polynomial $h \in \mathbb{Z}_3[x]$ can be written as $h = a_1x + a_0 + \langle f\rangle$ and $h = b_1x + b_0 + \langle g\rangle$ for some $a_1, a_0, b_1, b_0 \in \mathbb{Z}_3$.

The map $\phi : \mathbb{Z}_3[x]/\langle f\rangle \to \mathbb{Z}_3[x]/\langle g\rangle$ given by

$$\phi\left(cx + d + \langle f\rangle\right) = cx + d + \langle g\rangle$$

is a module isomorphism: If $c_1x + d_1 + \langle f\rangle, c_2x + d_2 + \langle f\rangle \in \mathbb{Z}_3[x]/\langle f\rangle$, then for any $\lambda \in \mathbb{Z}_3$,

$$\begin{aligned}
\phi\left(\lambda\left(c_1x + d_1 + \langle f\rangle\right) + \left(c_2x + d_2 + \langle f\rangle\right)\right) &= \phi\left(\left(\lambda c_1 + c_2\right)x + \left(\lambda d_1 + d_2\right) + \langle f\rangle\right) \\
&= \left(\lambda c_1 + c_2\right)x + \left(\lambda d_1 + d_2\right) + \langle g\rangle \\
&= \lambda\left(c_1x + d_1 + \langle g\rangle\right) + \left(c_2x + d_2 + \langle g\rangle\right) \\
&= \lambda\,\phi\left(c_1x + d_1 + \langle f\rangle\right) + \phi\left(c_2x + d_2 + \langle f\rangle\right)
\end{aligned}$$

and $\ker(\phi) = \langle f\rangle$. Since ϕ is apparently onto, the module isomorphism theorem gives the statement.

The polynomial f is irreducible over $\mathbb{Z}_3$ (because it has no roots), so $\mathbb{Z}_3[x]/\langle f\rangle$ is a field. On the other hand, $\mathbb{Z}_3[x]/\langle g\rangle$ has zero divisors (e.g., $\left(x + \langle g\rangle\right)\left(x + 1 + \langle g\rangle\right) = \langle g\rangle$), and so $\mathbb{Z}_3[x]/\langle g\rangle$ cannot be ring-isomorphic to $\mathbb{Z}_3[x]/\langle f\rangle$.