MATH 725 Homework Set 13 due 12/5/11

(1) Give an example of an operator $T \in L(V)$ whose characteristic polynomial is $x(x-29)^3(x-34)$ and whose minimal polynomial is $x(x-29)^2(x-34)$.

Solution. One example is $T \in L\left(\mathbf{C}^5\right)$ given by the matrix (with respect to the standard basis of $\mathbf{C}^5$)

$$T := \begin{pmatrix} 0 & & & & \\ & 29 & 1 & & \\ & & 29 & & \\ & & & 29 & \\ & & & & 34 \end{pmatrix}. \qquad \square$$

(2) Fix $a_0, a_1, \ldots, a_{n-1} \in \mathbf{C}$, and let

$$T := \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

(with respect to the standard basis of $\mathbf{C}^n$). Compute the minimal and characteristic polynomial of T.[1]

Solution. Denote the standard basis vectors of $\mathbf{C}^n$ by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. By looking at the matrix, we see that

$$\begin{aligned} T(\mathbf{e}_1) &= \mathbf{e}_2 \\ T^2(\mathbf{e}_1) &= T(\mathbf{e}_2) = \mathbf{e}_3 \\ &\vdots \\ T^{n-1}(\mathbf{e}_1) &= T(\mathbf{e}_{n-1}) = \mathbf{e}_n \\ T^n(\mathbf{e}_1) &= T(\mathbf{e}_n) = -a_0\,\mathbf{e}_1 - a_1\,\mathbf{e}_2 - \cdots - a_{n-1}\,\mathbf{e}_n\,. \qquad (\star) \end{aligned}$$

From this we deduce that $\mathbf{e}_1, T(\mathbf{e}_1), T^2(\mathbf{e}_1), \ldots, T^{n-1}(\mathbf{e}_1)$ is a linearly independent list; in particular the minimal polynomial $m(x)$ of T has degree n (otherwise, $m(T)(\mathbf{e}_1) \neq 0$), and so it equals the characteristic polynomial of T. Rewriting $(\star)$ as

$$T^n(\mathbf{e}_1) = -a_0\,\mathbf{e}_1 - a_1 T(\mathbf{e}_1) - a_2 T^2(\mathbf{e}_1) - \cdots - a_{n-1} T^{n-1}(\mathbf{e}_1)\,,$$

we see that the polynomial $p(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_0$ satisfies $p(T) = 0$. But since $m(x)$ is monic and of degree n, we must have $m(x) = p(x)$. $\square$

(3) Suppose $T \in L(V)$ and $\mathbf{v} \in V$. Prove that, if $p(x)$ be the monic polynomial of smallest degree such that $p(T)(\mathbf{v}) = \mathbf{0}$, then $p(x)$ divides the minimal polynomial of T.

Proof. Fix $\mathbf{v} \in V$, let $p(x)$ be the monic polynomial of smallest degree such that $p(T)(\mathbf{v}) = \mathbf{0}$, and denote the minimal polynomial of T by $m(x)$. By the division algorithm, there exist polynomials $q(x)$ and $r(x)$ with

$$m(x) = p(x)\,q(x) + r(x)$$

and the degree of $r(x)$ is less than the degree of $p(x)$. But then

$$m(T)(\mathbf{v}) = p(T)(\mathbf{v})\,q(T)(\mathbf{v}) + r(T)(\mathbf{v})\,,$$

[1] This shows that every monic polynomial over $\mathbf{C}$ is the characteristic polynomial of some linear operator.

which gives $r(T)(\mathbf{v}) = \mathbf{0}$. Since we can divide this equation by the leading coefficient of $r(x)$ (thus making $r(x)$ monic), this contradicts the minimality of $p(x)$, unless $r(x) = 0$. But this means that $p(x)$ divides $m(x)$. $\square$

(4) Suppose V is an inner-product space, and $T \in L(V)$ is normal. Prove that the minimal polynomial of T has no repeated roots.[2]

Proof. Denote the minimal polynomial of T by $m(x)$ and suppose λ is an eigenvalue of T. Then $m(x) = (x-\lambda)^k q(x)$ for some polynomial $q(x)$ which does not have λ as a root. Thus $m(T) = (T - \lambda \mathrm{I})^k q(T) = 0$, which means that

$$\operatorname{range} q(T) \subseteq \operatorname{null}(T - \lambda \mathrm{I})^k = \operatorname{null}(T - \lambda \mathrm{I}),$$

where for the last equality we used that $T - \lambda \mathrm{I}$ is normal (because T is) and so this equality follows from Homework #3 on Set 9. Thus $(T - \lambda \mathrm{I})q(T) = 0$, and so $k > 1$ would contradict the minimality of $m(x)$. So we must have $k = 1$, which means that any root of $m(x)$ is simple. $\square$

(5) If $T \in L(V)$ has minimal polynomial $(x-1)^2(x+1)$ and characteristic polynomial $(x-1)^6(x+1)^2$, what are the possible different Jordan normal forms for T?

Solution. All the Jordan forms must have a 2×2 Jordan block with eigenvalue 1 (of the form $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$) and two 1×1 blocks with eigenvalue -1. The possible variations are

(a) three 2×2 Jordan blocks with eigenvalue 1 and two 1×1 blocks with eigenvalue -1,
(b) two 2×2 Jordan blocks with eigenvalue 1, two 1×1 blocks with eigenvalue 1, and two 1×1 blocks with eigenvalue -1, and
(c) one 2×2 Jordan block with eigenvalue 1, four 1×1 blocks with eigenvalue 1, and two 1×1 blocks with eigenvalue -1. $\square$

[2]*Hint:* start by observing that, if λ is an eigenvalue of T, then $T - \lambda \mathrm{I}$ is also normal.