Mechanical Vibrations of Spring Systems

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1 Introduction

The goal of this project is to demonstrate the mathematical and physical properties of mechanical vibrations. These vibrations are visualized through a mass-spring system, where a hanging solid has its position, velocity and acceleration determined by a set of environmental variables. In order to determine the position, velocity, and acceleration of a vibrating object, we developed a computer program that takes physical constant variables as inputs and resolves these into a graphical simulation of a spring system.

Mechanical vibrations are described with a set of differential equations, which are solved for the position equation $y(t)$. The most general situation occurs by solving the equation $ay'' + by' + cy = g(t)$. In this project, we explored three types of equations, each one including different environmental factors:

1. Undamped Free Vibration
2. Damped Free Vibration
3. Forced Vibration

The simplest, undamped free vibration, occurs when an object vibrates freely and without damping. This situation satisfies the differential equation $ay'' + by' + cy = g(t)$ where $b = 0$ and $g(t) = 0$. This equation has a simple solution, solely depending on the values of a and c.

Damped free vibration includes a damping term in the equation, which satisfies the equation $ay''+by'+cy = g(t)$ where $g(t) = 0$. This case requires different solutions depending on the value of b.

Forced vibrations describes the situation where the object is acted upon by an external force $g(t)$. This satisfies the equation $ay'' + by' + cy = g(t)$. As with free vibrations, there are one or multiple solutions depending on the value of b. Although a programmatic solution for all possible functions could possibly exist, in our simulation we only allowed for $g(t)$ to be a polynomial, exponential, or sinusoidal function, or a product of the three. Such a general solution would be outside the scope of this project.

We chose to implement this program in JavaScript, so that it would be accessible to most people through their browser. Additional information, such as the velocity and acceleration were calculated by finding the derivatives with Wolfram Alpha. The results were visualized with a JavaScript graphing library, Flot, and the spring-mass system was drawn using HTML5 Canvas.

2 Methods for Solving Second Order Constant Coefficient Differential Equations

A linear second order differential equation is defined as $ay''+by'+cy = g(t)$. There are two types of equations associated with this form.

When $g(t) = 0$, this equation is called homogenous. A homogenous equation has a characteristic equation which follows the form $a\lambda^2 + b\lambda + c = 0$. The solution $y(t)$ to the differential equation depends on the number of roots in the characteristic equation.

When $b^2 - 4ac > 0$, the equation has two real roots λ_1 and λ_2. The general solution to this is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \tag{1}$$

When $b^2 - 4ac = 0$, then the equation has one real root λ, as $\lambda_1 = \lambda_2$. The general solution follows the form

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \tag{2}$$

When $b^2 - 4ac < 0$, then the equation has two complex roots $\lambda_1 = \mu + iv$ and $\lambda_2 = \mu - iv$. The general solution is

$$y(t) = c_1 e^{vt} cos(vt) + c_2 t e^{vt} sin(vt) \tag{3}$$

When $g(t) \neq 0$, the equation is called nonhomogeneous. The general solution to the nonhomogoneous equation is comprised of two parts: a homogenous solution $y_h(t)$, which is the solution to $ay'' + by' + c = 0$, and a particular solution $y_p(t)$ for the whole equation. The solution for the entire nonhomogenous equation is $y(t) = y_h(t) + y_p(t)$. The particular solution, for the functions that we allow, is defined as

$$\begin{aligned} y_p(t) = \\ t^s[(A_0 + A_1 t + A_2 t^2 + ... + A_n t^n) e^{\alpha t} cos(\beta t) \\ +(B_0 + B_1 t + B_2 t^2 + ... + B_n t^n) e^{\alpha t} sin(\beta t)] \end{aligned} \tag{4}$$

In this equation, s is determined to be the number that would ensure that the homogeneous solution does not equal the particular solution; therefore, $s \in \{0, 1, 2\}$.

To determine the undetermined constants, find the first and second derivatives of $y_p(t)$, and substitute $y_p(t)$, $y_p'(t)$ and $y_p''(t)$ for y, y' and y'', respectively. This should result in a real solution for A and B.

3 Generalized Solutions

3.1 Undamped Free Vibration

Undamped free vibrations are described by the homogenous differential equation $ay''+by'+cy = 0$ where b = 0. The general equation for this type of vibration is $y'' + \omega^2 y = 0$ where $\omega^2 = k/m$ with a spring constant, k, and a mass, m.

The corresponding characteristic equation for this general equation has two roots: $\lambda_1 = \omega i$ and $\lambda_2 = -\omega i$.

Thus, the corresponding solution to the differential equation is $y(t) = Acos(\omega t) + Bsin(\omega t)$ where $A = y_0$ and $B = v_0/\omega$. Given an initial position, an initial velocity, a mass, and a spring constant, a complete general solution can be determined for undamped free vibrations.

$$y(t) = y_0 cos(\omega t) + \frac{v_0}{\omega} sin(\omega t) \tag{5}$$

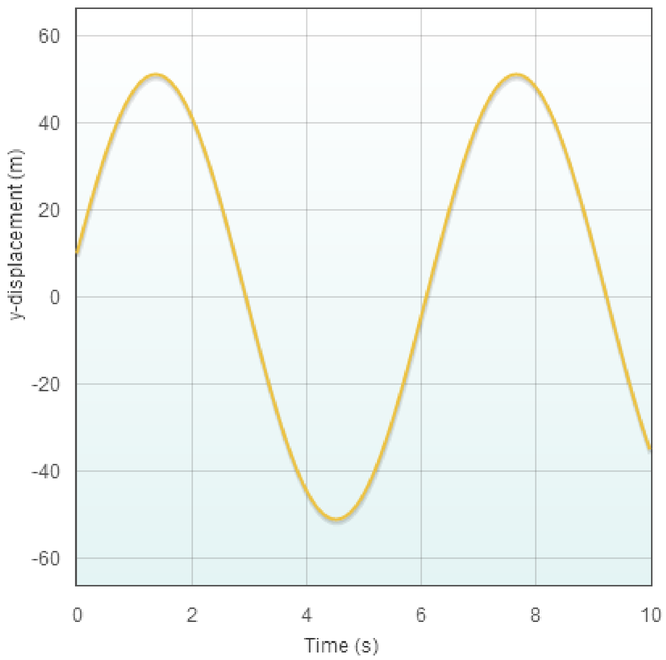

Figure 1: Simulation of undamped free vibration with $m = 100$, $k = 100$, $y_0 = 10$, $v_0 = 50$, over the period $t = [0, 10]$

As seen in Figure 1, an undamped free vibration is a sinusoidal wave which does not decay over time.

3.2 Damped Free Vibration

Damped free vibrations are described by the homogenous differential equation $ay'' + by' + cy = 0$, and the general form of the vibration is $my'' + \gamma y' + ky = 0$. In this case, γ is a positive damping term, m is the mass, and k is the spring constant.

Because the characteristic equation is found by solving for the roots of the equation $m\lambda^2 + \gamma\lambda + k = 0$, there are three possible forms of the characteristic equation.

3.2.1 Underdamped Harmonic Motion

If $\gamma^2 - 4mk < 0$, then the equation has a solution with complex roots $\frac{-\gamma}{2m} \pm iv$, where $v = \frac{\sqrt{4mk-\gamma^2}}{2m}$. Therefore by (1), the characteristic equation in this case is:

$$y(t) = e^{\frac{-\gamma}{2m}}(Acos(vt) + Bsin(vt)) \tag{6}$$

3.2.2 Critically Damping Harmonic Motion

In this case, $\gamma^2 = 4mk$, which has a single real root $\frac{-\gamma}{2m}$. Therefore by (2), the characterstic equation in this case is simply:

$$y(t) = (A + Bt)e^{\frac{-\gamma}{2m}} \tag{7}$$

3.2.3 Overdamped Harmonic Motion

In the final case when $\gamma^2 - 4mk > 0$, the general equation has two real roots λ_1, λ_2 of $\frac{-\gamma}{2m} \pm \frac{\sqrt{4mk-\gamma^2}}{2m}$. The characteristic equation is therefore:

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{8}$$

3.3 Forced Vibration