Chapter 1: Review

Section 1: Algebra Review

The following is a list of some algebra skills you are expected to have. The example problems here are only a brief review. This is not enough to teach you these skills. If you need more review, you can look in any book called Intermediate Algebra.

Laws of Exponents

The Laws of Exponents let you rewrite algebraic expressions that involve exponents. The last three listed here are really definitions rather than rules.

Laws of Exponents:

All variables here represent real numbers and all variables in denominators are nonzero.

$x^a \cdot x^b = x^{a+b}$ $\qquad \dfrac{x^a}{x^b} = x^{a-b}$ $\qquad \left(x^a\right)^b = x^{ab}$

$(xy)^a = x^a y^a$ $\qquad \left(\dfrac{x}{y}\right)^b = \dfrac{x^b}{y^b}$ $\qquad x^0 = 1$, provided $x \neq 0$

$x^{-n} = \dfrac{1}{x^n}$, provided $x \neq 0$ $\qquad x^{1/n} = \sqrt[n]{x}$, provided this is a real number

Example: Simplify as much as possible and write your answer using only positive exponents: $\left(\dfrac{x^{-2}}{y^{-3}}\right)^2$

Solution:

$$\left(\frac{x^{-2}}{y^{-3}}\right)^2 = \frac{\left(x^{-2}\right)^2}{\left(y^{-3}\right)^2} = \frac{x^{-4}}{y^{-6}} = \frac{y^6}{x^4}$$

Example: Rewrite using only positive exponents:

$\left(\sqrt{p^5}\right)^{-1/3}$

Solution:

$$\left(\sqrt{p^5}\right)^{-1/3} = \left(\left(p^5\right)^{1/2}\right)^{-1/3} = p^{-5/6} = \frac{1}{p^{5/6}}$$

Writing Equations of Lines

Identify the independent and dependent variables. The independent variable is the one that you think explains the relationship, and the dependent variable is the one that you think responds. If you are counting cricket chirps per minute at various temperatures, the temperature could affect how the crickets chirp, but the cricket chirps are unlikely to affect the temperature. In this example, the independent variable will be temperature and the dependent variable will be the number of chirps per minute.

Slope is a number that tells you which direction the line points. If the slope is positive, the line points uphill as you read from left to right. If the slope is negative, the line points downhill. Horizontal lines have a slope of zero. The closer the slope is to zero, the closer the line is to horizontal. The further the slope is from zero, either positive or negative, the steeper the line is. Vertical lines have undefined slope – because the "run" in the rise over run calculation is zero. One way to define a straight line is as a curve with a constant slope. You can calculate slope using any two points on the line. Parallel lines have the same slope. Perpendicular lines have negative reciprocal slopes (that is, their slopes multiply to make –1).

Slope is a rate of change. The units of slope are fractional, *y*-units over *x*-units, like miles per hour or dollars per day. In an application problem, look for the fractional units to help you find the slope. The identifying feature of a linear equation is that the slope is constant.

Equations of lines: There are several different forms of an equation of a line that you might encounter. (Here I'm assuming *x* is the independent variable and *y* is the dependent variable.)

Slope-Intercept form, $y = mx + b$: This is the favorite of most students. The form is easy to remember. You can read the slope *m* and *y*-intercept *b* right off the equation. If you don't have the *y*-intercept, you will have to do some algebra to use this form.

Point-Slope form, $y - y_1 = m(x - x_1)$: This is my favorite form. The slope *m* is visible, and (x_1, y_1) is some known point on the line. I like this form the best because there is no algebra required – just plop the slope and one point into place and you're done.

Standard form, $Ax + By = C$: This form is useful for comparing different types of equations. But it's not a very helpful form for graphing or writing the equation of a line. You have to do algebra to find either the slope or any point on the line.

All you need in order to write the equation of a line is the slope and one point. The slope might be given to you (look for fractional units!), or you might compute it from two points, or perhaps get it from another line that is parallel or perpendicular to it. The one point is usually given to you, or you could need to find the intersection of some curves to get the point.

Find and interpret the rate of change (slope). If you have two points, whether they are given to you numerically or if you read them off a graph, you can compute the slope using that familiar rise-over-run formula (the difference in the y's over the difference in the x's). If you have an equation, you can algebraically maneuver it into slope-intercept form and read the slope right off. If the situation is described in English, then the constant rate of change is the slope.

Remember the units (y-units over x-units)! Simply writing down a sentence like "the rate of change is 15 dollars per year" is the biggest step in interpreting the slope.

If the slope is positive, then the function is increasing. If the slope is negative, then the function is decreasing. If the slope is zero, the function is neither increasing or decreasing (staying constant).

You can compare the rates of change for two functions by comparing their slopes. The function whose graph is steeper, whose slope is further from zero, is changing more rapidly.

Find and interpret various points of the linear function (for example, the *y*-intercept). The points that are on the line, the points that satisfy the algebraic equation, are individual examples that fit your situation. The x- and y-intercepts are usually important, but they are not the only important points.

The y-intercept is the place where the line crosses the y-axis. This is the y-value when $x = 0$.

The x-intercept is the place where the line crosses the x-axis. This is the x-value that makes $y = 0$.

There may be other important points that arise because of the applied setting for your problem.

The units can help you decide what the important points mean in each particular situation.

> **Example:** The cost of a diet program is \$299 to join, plus \$70 per week for the food. Describe this function and tell how much it would cost to join for ten weeks.
>
> **Solution:** You can tell this is a linear function, because the rate of change (look for those fractional units – dollars per week) is a constant. The slope of this function is 70 dollars a week – this means that each week I

will spend another 70 dollars on the food. The independent variable (x) is weeks and the dependent variable (y) is dollars. The y-intercept is the \$299 fee to join – I will pay this no matter how many weeks I belong. The x-intercept for the line doesn't make sense in this situation. The line itself has an x-intercept, with x some negative number of weeks. But it's not possible to join the diet program for a negative number of weeks and pay zero dollars. To stay with the program for ten weeks I would pay \$299 + 10(\$70) = \$999. This is the point $(x, y) = (10, 999)$, which lies on the line.

If you know the function is linear, two points are enough to write the formula. Use the two points to find the slope, and then you can solve to find the y-intercept.

Example: A faucet is dripping water at a constant rate into a bowl. At 1:00, there was ½ cup of water in the bowl. At 1:45, there was ¾ cup of water in the bowl. How much water will be in the bowl at 3:30?

Solution: This is a linear function, because the faucet is dripping at a constant rate. The domain is the set of times (hours past noon). The range is the set of volumes in cups (numbers ≥ 0). Let t be the time, measured in hours past noon, and let W be the amount of water in the bowl, measured in cups. There are two points given: when $t = 1$, $W = 0.5$, and when $t = 1.75$, $W = 0.75$.
The slope is rise/run, $\Delta W/\Delta t = (0.75 - 0.5) / (1.75 - 1) = .25 / .75 = 1/3$ cups per hour. So the equation will be

$$W = \frac{1}{3}t + b.$$

To find the W-intercept, just plug in one of the points you know and solve for b:

$$\frac{1}{2} = \frac{1}{3}\cdot 1 + b, \text{ or } b = \frac{1}{6}.$$

The function that tells us how much water is in the bowl after t hours is given by

$$W = \frac{1}{3}t + \frac{1}{6}.$$

As a check, let's make sure this gives us the right answer at the other known point – if I plug in $t = 1.75$, I get $W = 0.75$, which is right. At 3:30, $t = 3.5$, and $W = 4/3$ cup.

Factoring and the Quadratic Formula

By this time, you should have seen how to solve quadratic equations in several different ways. In this class, you will only need a couple: you will be expected to factor easy things, use the quadratic formula, and to approximate the solutions using technology (such as your calculator).

Factor out common monomial factors.
Example: $70b^3 + 49b^6 = 7b^3(10 + 7b^3)$

Recognize and be able to factor a sum or difference of squares and perfect square trinomials. These “special products” come up all the time, and you should be able to handle them automatically.

Factor quadratic trinomials with leading coefficient of 1 – if they’re easy! This is a guess-and-check process; look for two numbers whose sum is the coefficient of the linear term (exponent = 1) and whose product is the constant term.

> **Example:** Factor $x^2 - 11x + 30$
>
> **Solution:** I’m looking for two numbers who add to –11 and multiply to 30; –5 and –6 work. So the factorization is $x^2 - 11x + 30 = (x-5)(x-6)$.

Unfortunately, factoring this way can take a long time, and it’s hard to know if you should stop. If you can’t find a factorization, is it because you didn’t try the right factors yet, or maybe the factors involve square roots, or maybe the quadratic is already fully factored?

I usually don’t spend very long searching for a factorization of a quadratic trinomial. If I can’t see a factorization quickly, I turn to the quadratic formula (see below).

> **Quadratic formula:** The solutions of the quadratic equation $ax^2 + bx + c = 0$ (where a, b, and c are real numbers and $a \neq 0$ are given by
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Understand and frequently use that deep connection between roots and factors. This is often called the Zero Product Property. This is the primary reason we factor – to find the roots, solutions, of an equation. But remember that it goes the other way, also. If you know the solutions of a polynomial equation, you can use them to construct the factors.

Use the quadratic formula to give you the roots, and use them to construct the factors. If you can't easily factor a quadratic, you can always exploit that deep connection between roots and factors. This takes a little bit of time, too, but it will always give you an answer. This is also the easiest way to find factors that involve square roots.

Example: Factor $10x^2+14x+4$.

Solution:

I don't immediately see a factorization, but I can use the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-14\pm\sqrt{(14)^2-4(10)(4)}}{2(10)}=\frac{-14\pm\sqrt{196-160}}{20}=\frac{-14\pm 6}{20}.$$

The two roots are -1 and $-\frac{2}{5}$, so the factors are $(x+1)$ and $\left(x+\frac{2}{5}\right)$. I'll need to multiply by 10 so that the leading coefficient is right: $10x^2+14x+4=10(x+1)\left(x+\frac{2}{5}\right)$.

Solving Exponential Equations

You will need to remember logarithms, but you won't have to do a lot of algebra with them. You won't have to simplify expressions involving logarithms, so you won't need many of the laws of logarithms. Here they are, just in case you want to look at them – the only ones you are likely to need are Law 3 and Law 4.

Laws of Exponents:

1: $\log_a(xy)=\log_a(x)+\log_a(y)$. In English: The log of a product is the sum of the logs.

2: $\log_a\left(\frac{x}{y}\right)=\log_a(x)-\log_a(y)$. In English: The log of a quotient is the difference of the logs.

3: $\log_a(x^n)=n\log_a(x)$. In English: When you take the log of a power, the exponent comes down in front.

4: $\log_a(a) = 1$ and $\log_a(1) = 0$

5: Change of Base Formula: $\log_a(x)=\frac{\log_b(x)}{\log_b(a)}$

An exponential equation is any equation that involves an exponential function. This is the technique you can use to solve for the exponent.

1. Do as much ordinary algebra (adding, subtracting, multiplying or dividing – always to both sides of the equation) as you can in order to isolate the exponent.
2. Take a logarithm of both sides. You can use any base you want here. If you intend to get a calculator approximation, your life will be easier if you use common log or natural log.
3. Use the 3rd Law of Logarithms to bring the exponent down in front. This is the whole point of using logarithms – it gets the exponent on ground level where you can do ordinary algebra to it.
4. Use ordinary algebra to solve for the exponent.

Example: A bacteria colony doubles every 20 minutes. It starts with 3 million bacteria at noon. When will there be 8 million bacteria in the colony?

Solution: If t is in hours and $A(t)$ is in millions of bacteria, the function that tells how many bacteria in the colony is $A(t) = 3(2)^{3t}$. (Review on your own if you don't remember how to find this function.) So the equation we want to solve is

$$8 = 3(2)^{3t}$$

First, we do as much ordinary algebra as possible to isolate the exponent. For this example, that means dividing both sides of the equation by 3:

$$\frac{8}{3} = 2^{3t}$$

That's as much as we can do without logarithms. Now it's time to take the log of both sides. I want a calculator approximation when I'm done here (so I can write down a time), so I'll use natural log. You can use any log you like, as long as you do the same thing to both sides.

$$\ln\left(\frac{8}{3}\right) = \ln\left(2^{3t}\right) = 3t\ln(2)$$

Taking the log brings the exponent down in front (third Law of Logarithms), which is just what we want. Now we have an equation of the form number = number times t; it's time to do ordinary algebra again to solve for t. Divide both sides of the equation by $3\ln(2)$ to get

$$t = \frac{\ln(8/3)}{3\ln(2)}$$

This is the exact answer. I can't just look at this answer and see how big it is, though, so I want a calculator answer.

$$t = \frac{\ln(8/3)}{3\ln(2)} \cong 0.4717$$

This tells me that the colony will have 8 million bacteria about 0.47 hours, or a bit more than 28 minutes, past noon. Does this make sense? By counting up we can see that the colony would have 6 million bacteria at 12:20 and 12 million at 12:40, so this is reasonable. There will be 8 million bacteria at about 12:28.

Section 2: What is a Function?

Functions

The notion of a function is one of the most powerful in mathematics. It's a surprisingly simple idea, though. The reason students are so often confused when they encounter functions for the first time in an algebra class is the notation. Before we get to the notation, we'll concentrate on the core idea.

Our lives are full of relationships and correspondences between sets, although we don't always think of them in these terms. For example, we know that the number of plates we take out of the cupboard corresponds to the number of people we're expecting at the table. We know that each telephone number we know corresponds to one person that we want to reach. We know that the size of our electric bill corresponds to the amount of electricity we use. A function is just a special type of correspondence.

Definition: A *function* is a correspondence between two sets that assigns to each element of the first set **exactly one** element of the second set. The first set, the set of inputs, is called the *domain*. The second set, the set of outputs, is called the *range*.

Functions do not have to have anything to do with numbers. The key point is those words "**exactly one**." That makes them predictable, and that's the reason they're so important.

Example: Every person has a birthday. This is an example of a function. Notice that each person gets exactly one birthday. Notice also that lots of people can have the same birthday – that doesn't affect whether this relationship is a function or not. The "exactly one" only needs to work the one direction. In this example, the domain is the set of all people, and the range is the set of all possible birthdays (the days of the year).

Example: Every number has a square. This is also an example of a function. Again, notice that every number has exactly one square – if you give me a number, I can give you its square (a function is predictable). In this case, the domain is the set of all numbers, and the range is the set of all possible squares.

The point of a function is to be predictable, so it's nicest if we can write down a rule. There are several different ways to write a function:

- A function could come as a table. The income tax tables in the back of the tax booklet are examples of this kind of function. There's one such function every year for each type of taxpayer: single, married filing jointly, etc. Within each of these tables, the assignment of a tax amount to a taxable income amount is the function, and the information comes from a table. In this example, the domain is the set of possible taxable income amounts and the range is the set of possible tax amounts.

To tell if a table represents a function, you need to check whether any input has two outputs. Remember, a function associates **exactly one** output to each input. In our income tax example, you can tell it's a function because no matter how many times you look it up, the amount you owe the government doesn't change. Notice that it doesn't matter that several taxable income amounts yield the same tax amount – it's OK for many different inputs to give the same output.

- A function could come as a graph. For example, the graph that shows the Dow Jones average in the newspaper represents a function. The domain is along the horizontal axis (in my newspaper, that represents the set of the last five business days), and the range is represented vertically (the Dow Jones average for that day). The information about this function comes from the graph. In order to find the Dow Jones average for last Friday, say, you read the graph. Every time you read this week's graph for last Friday, you'll see the same Dow Jones average – the graph is predictable.

  To tell if a graph represents a function, you need to check whether any input (along the horizontal axis) has two outputs (values above or below it on the graph). An easy way to tell is to use the vertical line test. If any vertical line hits the graph more than once, then the graph does not represent a function.

  Example: The graph of a circle is not a function, because there are lots of vertical lines that cross the circle more than once . This graph fails the vertical line test. The graph of the top half of a circle is a function.

- A function could come as an algebraic rule. This is the way most students think about functions (which may be why so many people become confused about functions). This is a great shorthand way to write a function that has to do with numbers. For example, our square number example from above could be written this way:

$$f(x) = x^2$$

  This is read "*f* of *x* equals *x* squared."

  The *f* here is the name of the function. You'll often see *f* used for function, because *f* is the first letter in the word "function." But any letter or combination of letters would be fine. In fact, it's a good idea to pick a letter that will remind you of what you're doing.

  The parentheses here do not denote multiplication. They're read aloud as "of." The fact that they're right next to the name of the function tells you that this is a function, and you should look inside them to see what the variable will be.

  The *x* here is the variable name. Again, *x* is very commonly used, but there's nothing magic about it. You could use any letter or symbol that you like. The point is to look within the parentheses to see what letter is there, because that's what will stand for the input in the rule.

The algebraic stuff on the right hand side of the equals sign is the rule. This is the part that tells you what to do with your input. Your input goes exactly in place of the variable (which you identified right above). This rule says "take the input and square it."

Example: In the function $C = \frac{5}{9}(F - 32)$, the function name is *C*, the variable name is *F*, and the rule says "first subtract 32 from your input, then multiply the result by 5/9." This is the algebraic representation of the function that associates degrees Celsius to degrees Fahrenheit. The domain here is the set of all possible temperatures, measured in degrees Fahrenheit, and the range is the set of all possible temperatures, measured in degrees Celsius. This is a function, because there is exactly one Celsius measurement corresponding to each Fahrenheit measurement.

One convenient thing about having an algebraic representation for a function is that you don't have to check whether the "exactly one" condition is satisfied. Algebra has that property built in – you always get the same answer when you plug in the same input.

The Rule of Four

There are four ways that mathematical information can be communicated to you.

- Numerically- as a list of numbers in a table, for example.
- Algebraically or analytically - as a formula.
- Graphically or geometrically, as a graph or a picture.
- In English - the story or word problem.

Each of these ways has distinct advantages and disadvantages. Depending on what kind of mathematical information you need to communicate, you might choose just one of these ways, or some combination of these ways.

Many students are most familiar with algebra and formulas. And many math textbooks seem to focus on formulas. But all of these ways of looking at mathematical information are important. We'll be communicating mathematics in all four ways during this course.

Numerically

Advantages

You get precise information - actual numbers. This is often how real-world information comes to you, as numerical data that's been collected.

Disadvantages
There's no information about anything that isn't already on your list. Patterns and trends are difficult or impossible to find.

Algebraically or Analytically (with formulas)

Advantages
You get precise information - you can solve for an actual number. You can use a formula to predict information about any number you're interested in. Patterns in the situation may be revealed by what we know about the formula.

Disadvantages
Trends may be difficult or impossible to find. Formulas are mathematical models only -- the real world is usually not as neat and tidy as the formula suggests.

Communicating Graphically or Geometrically

Advantages
You get big picture information - you can easily see trends, change, and growth. You can easily approximate the interesting points on the curve. It’s a quick way to see what’s really going on.

Disadvantages
You can only approximate numbers, except for certain known and labeled points.

Communicating in English

Advantages
This is how real-world problems come. Nobody outside of a math class will ever ask you to solve a quadratic equation. Instead, they'll ask you how many pounds of salmon you'll need to feed a dinner party of eight, or how much is their share of the phone bill, or what's the most efficient speed for running the machinery on the factory floor.

Disadvantages
You usually need to use one of the other ways to solve such a problem. Translation can be difficult. English is a fluid language with many meanings. Sometimes there are legitimate but contradictory interpretations of the same English statements.

Section 3: Library of Functions

There are a few functions that you should be completely familiar with. By this time, you should have seen linear, quadratic, and exponential functions many times.

Linear Functions

Linear functions are the simplest kind of functions to work with. Many relationships are truly linear, and many more can be approximated well enough with a linear function. Linear functions have many helpful features – their graphs are straight lines, which we know a lot about. Their rates of change are simply slopes, which we know how to find.

Recognize linear growth, no matter how the information is given to you. Remember that there are four ways quantitative information can be presented.

> **Numerically**: Linear functions have a constant change in y for every constant change in x. This reflects the graphical idea of a linear function – the change in y over the change in x, or $\Delta y/\Delta x$, is the constant slope of the line. One way to recognize a line is if you see a constant slope in a table of numbers.
>
> **Algebraically**: The formula for a linear function can always be algebraically maneuvered into one of the common forms given above. You can recognize that a function is linear if it has only one independent variable, which is raised to the first power only (no squares, no one-overs, no roots), and some constants.
>
> **Graphically**: Linear functions are the ones whose graphs are straight lines.
>
> **In English**: Linear functions have a constant rate of change. You can often recognize the slope by its units; look for fractional units, rise/run units, like miles per hour, or dollars per pound, or people per year. The y-intercept is like the fixed cost or the overhead – how much y you have when x is zero.

Quadratic Functions

Quadratic functions have lots of applications (for example, the height of a baseball can be modeled with a quadratic function). We already discussed how to solve quadratic equations.

> **Numerically**: The best way to tell if a table displays a quadratic function is to graph it.
>
> **Algebraically**: Quadratic functions can always be algebraically maneuvered to look like $f(x) = ax^2 + bx + c$. You can recognize that a function is quadratic if there is only one independent variable, and the only powers you see are 1 and 2.

Graphically: The graph of a quadratic function is a parabola, a sort of curvy V-shape. The formula can tell you a lot of information about the graph:

The sign of a tells you if the graph opens up ($a > 0$) or down ($a < 0$)

The y-intercept (where $x = 0$) is at $y = c$.

The solutions of the equation $f(x) = ax^2 + bx + c = 0$ are the zeros, the roots of the function – these are the x-intercepts of the parabola.

The vertex formula tells you where the high or low point of the parabola is:

$$x = \frac{-b}{2a}; y = \ldots \text{ plug it in.}$$

You can use the same kind of information to go from the graph to a formula – use the zeros of the function to find the factors, adjust the leading coefficient using the y-intercept.

In English: If the function is quadratic, they'll need to say so specifically. One of the most common applications is the height of a falling body.

Exponential Functions

Exponential functions are very common. For example, the compound interest formula is an exponential function. And many natural things grow (or decay) exponentially.

Numerically: Exponential functions show a constant *ratio* in y for a constant change in x. That is, if you increase x by 1, you multiply y by a constant multiplier. The multiplier is the easiest base to use for the exponential function.

Algebraically: An exponential function is of the form $f(x) = A_0 b^x$, where $b > 0$ and $b \neq 1$. The domain of the exponential function is the set of all real numbers – we can use any real number as the exponent. The range of the exponential function is the set of all positive real numbers. (If we raise a positive number to any power, we get a positive number back.)

Note – some books are totally e-happy. That is, they want every exponential function to have base e. Now, e is a lovely number, and it's the perfect base for some applications – for example, continuously compounded interest. But it's a better idea to let b be the multiplier. That is, if you have a quantity that doubles every hour, you'll be much happier if you use $b = 2$.

Example: Suppose a bacteria colony is growing in such a way that it doubles in size every 20 minutes. There are 3 million bacteria at noon.

a. How many will there be at 1:00 pm?
b. How many will there be at 1:30 pm?

Solution: Because the doubling time is constant, we know the bacteria are growing exponentially.

a. This part is easy to figure out without writing a formula, by just counting up. If they double every 20 minutes, then there are 6 million at 12:20, there are 12 million at 12:40, and there are 24 million at 1:00.

b. This part is not so easy – 1:30 isn't a whole number of 20-minute chunks after noon. So we will build the formula. Our units will be millions of bacteria and hours. The initial amount, the principal, A_0 is the 3 million bacteria we started with at noon. Our population is doubling every twenty minutes, so it's being multiplied by 2 every 1/3 hour. Over one hour, then, it will be multiplied by 2^3.
The formula that tells how many million bacteria there are in this colony *t* hours after noon is

$$A(t) = 3(2)^{3t}.$$

1:30 is $t = 1.5$ hours past noon, so there should be $A(1.5) = 3(2)^{3\times 1.5} \cong 67.89$ million bacteria. Does this make sense? Yes, by counting up we find that there should be 48 million at 1:20 and 96 million at 1:40, so this seems right.

Note: You can pick whatever units are convenient for you. Your formula may end up looking different, but your answers will be correct. In the bacteria example, you could have used the units of (single) bacterium and twenty-minute-intervals. Then the formula would look different: $A(t) = 3{,}000{,}000(2)^t$, but you'd use $t = 4.5$ (because 1:30 is 4.5 twenty-minute-intervals past noon) and you'd get the same answer – about 67.89 million bacteria.

Graphically: If $a > 1$, $f(x) = a^x$ represents exponential growth, and the graph of the function will be incredibly flat on the left, incredibly steep on the right. If $0 < a < 1$, $f(x) = a^x$ represents exponential decay, and the graph of the function will be the mirror image, left to right, of an exponential growth graph. It will be incredibly steep on the left and incredibly flat on the right.

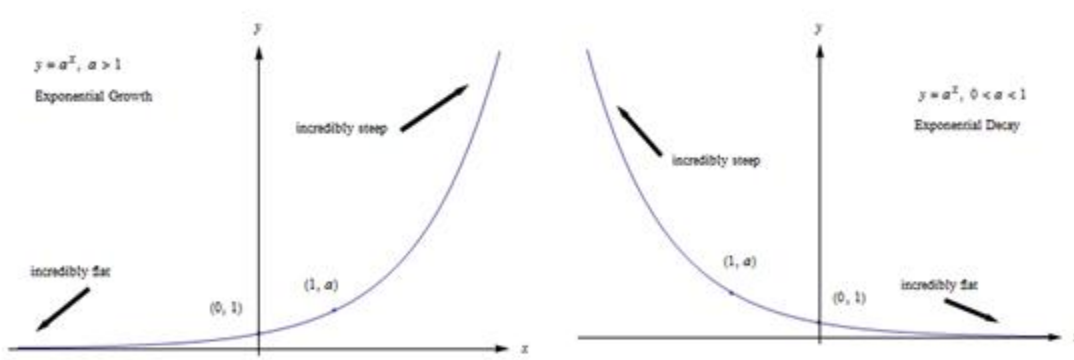

Figure 1

We always get two points for free on any simple exponential graph: (0, 1) and (1, *a*).

In English: Exponential functions show up when the increase depends on how much is already there. For example, compound interest (the additional interest depends on how much is in the account), or simple population growth (the number of additional babies depends on how many people are in the account).

Other functions

There are several other functions that you should know something about – you should recognize their formulas and their graphs. You should know

- the absolute value functions
- polynomial graphs in general, cubics (3^{rd} degree) in particular
- rational functions (remember vertical asymptotes?)
- power functions ($f(x) = x^n$ for some n), including the square root function
- logarithmic functions – with base 10 and the natural log, with base e

Section 4: New Functions from Old

Transformations

Changing the constants that appear in an algebraic formula changes the graph in some predictable ways.

Here are the principles:

- Changing the x (the input) changes the horizontal.
- Changing the y (the output) changes the vertical.
- Multiplying by a constant stretches (or squashes) the graph.
- Multiplying by –1 reflects the graph.
- Adding a constant shifts the graph.

Here are the details:

Start with the graph of $y = f(x)$. The graph of each of the following will have the same basic shape as $y = f(x)$, altered as noted. For all of these, a is a constant

- $y = af(x)$ is a times as tall.
- $y = -f(x)$ is reflected vertically across the x-axis (upside down).
- $y = f(x) + a$ is shifted up a units.

- $y = f(ax)$ is $\frac{1}{a}$ times as wide.
- $y = f(-x)$ is reflected horizontally across the y-axis.
- $y = f(x + a)$ is shifted to the left a units.
- Notice that for horizontal stretches or shifts, the effect is sort of backwards from what you might expect at first. These can be confusing – check your answers by plotting a couple of points.

You can handle these all at once. You can do horizontal and vertical changes independently. For each of these, follow the order of operations – stretch and reflect first, then shift. Use the origin as your anchor point, even if it's not on your graph.

> **Example:** The graph of $y = -4|x - 2| + 5$ has the same basic shape as $y = |x|$. It's been shifted to the right 2 units. It's upside down, 4 times as tall, and has been shifted up 5 units. So the new graph has its vertex at (2, 5), opens down, and looks stretched vertically compared to the original. You can plug in a point or two to confirm your answer. The new graph goes through the point (0, –3), which makes sense. If you move 2 units to the left of the vertex, the unchanged graph goes up 2 units. Here, it goes down (because the graph is upside down) 8 units (because the graph is stretched by a factor of 4.)

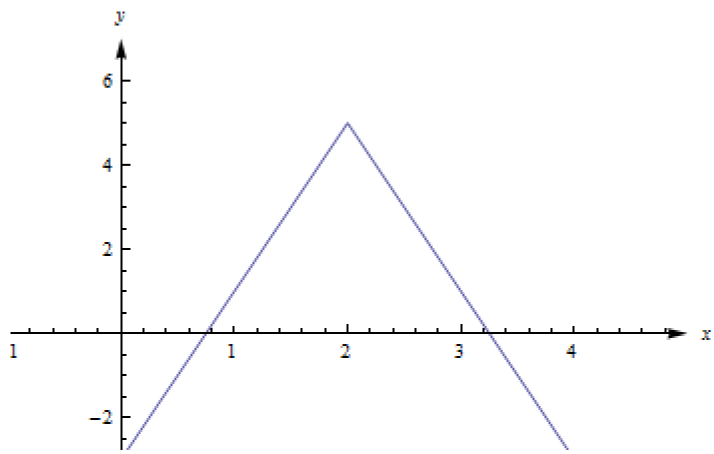

Figure 2

Composition

One of the most important ways to combine functions is to chain them together, using the output from one as the input into another. A simple example of this is unit conversion – we have one function that tells us how many meters high the ball is after t seconds, and another that tells how many feet are in a certain number of meters. We can use the output of the first function (meters) as the input to the second function to find how many feet high the ball is after t seconds. The chaining together of functions in this way is called composition:

The composition of f with g, written $f \circ g(x)$, is the function that takes x, first does g to it, and then does f to the output. That is, $f \circ g(x) = f(g(x))$.

Example: Let $f(x)=4x^2-13$ and $g(x)=\sqrt{x-2}$. Then their composition $f\circ g(x)=f(g(x))=f\left(\sqrt{x-2}\right)=4\left(\sqrt{x-2}\right)^2-13.$

Decomposition

Sometimes you will be given the composition and be asked to identify the component pieces. This is called "decomposition." It turns out to be a very useful skill in calculus. There are often several correct decompositions for a function, but usually only one of them is useful. It may take some practice before you can see which composition is the useful one.

Example: The function $G(x)=\dfrac{1}{2x+3}$ is a composition $f\circ g$. Identify the component functions *f* and *g*.

The most useful solution: In many cases, there is an "obvious" choice, which you can find by thinking about the inside and the outside. In $G(x)=\dfrac{1}{2x+3}$, the "inside function" is the denominator and the "outside function" is the reciprocal function (that is, "one over"). In the composition $f\circ g$, *g* is the inside function and *f* is the outside function. So this decomposition would be $f(x)=\dfrac{1}{x}$ and $g(x)=2x+3.$ Then $F(x)=f(g(x))=f(2x+3)=\dfrac{1}{2x+3}$. This is the most useful solution. This is the solution you would see in the answer pages of the textbook. This is the type of decomposition you should look for.

There are usually lots of correct solutions, some of which involve some creativity to find. In this class, you don't have to ever find any of these clever decompositions. If you do find one, it will be correct. But your teacher may suggest that you stick to the more useful decomposition.

Inverse Functions

The word "inverse" means backwards, and that's what inverse functions are about – going backwards. There are a few different and useful ways to think about inverse functions.

Swapping the roles of input and output

One important reason we care about inverse functions is that, in many cases, the same relationship can give two different functions, depending on what questions you're interested in answering. Which function you use depends on which quantity you want to use as your input.

Example: A private investigator charges a \$500 fee per case, plus \$80 per hour that she works on the case. There is a functional relationship between the hours she works and the amount she bills. But which is the input and which is the output?

If the number of hours she works is the input, then the number of dollars she bills is the output. And it's a function, because each possible number of hours is associated with exactly one billing amount. This might be the function you'd think of first. If we let *h* be the number of hours the detective works and *b* be the number of dollars she bills, then this function might be written as

$$b = f(h) = 500 + 80h.$$

You'd use this function if you knew how many hours she worked on your case and you wanted to know how much she would charge you.

But the very same relationship yields a different function, whose input is the billing amount and whose output is the number of hours she works. This is also a function, because each possible bill is associated with exactly one amount of time. Again, letting *h* be the number of hours and *b* be the amount she bills in dollars, we can write this function:

$$h = f^{-1}(b) = \frac{b-500}{80}$$

This would be a helpful function if you had a certain amount of money to spend and you wanted to know how many hours she would work on your case.

The two functions here are inverse functions. They model the same relationship, but the roles of input and output have been exchanged. That little –1 that looks like an exponent for the *f* in the second formula indicates it is the inverse function for *f*. (It is not an exponent.)

Undoing

The most important reason we want to study inverse functions is that they undo each other. Remember the algebraic definition of inverse functions:

$f(x)$ and $f^{-1}(x)$ are inverse functions means that their composition in either order is the identity function. That is, both

$$f\left(f^{-1}(x)\right) = x \text{ and } f^{-1}\left(f(x)\right) = x$$

The arrow diagram may be clearer:

$$x \xrightarrow{f^{-1}} f^{-1}(x) \xrightarrow{f} x \text{ and } x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} x$$

Graphically

If you graph a function and its inverse on the same axes, the inverse will be a reflection of the original across the line $y = x$. That's because the inverse function swaps the roles of input and output. On a graph, that means interchanging the order of the coordinates for every point. That is, if (x, y) is on the graph of $y = f(x)$, then (y, x) will be on the graph of $y = f^{-1}(x)$.

Chapter 1 Exercises

1. Use the rules of exponents to simplify the following. Write your answer using only positive exponents. Assume all variables represent non-zero numbers.

a. $\dfrac{4y^3}{12y^7}$ b. $\left(2xy^{-3}z^0\right)^3$ c. $\left(m^{-2}\right)^8$

d. $(xyz)^0$ e. $\left(5x^3\right)\left(-7x^5\right)$ f. $\sqrt[4]{\left(4ab^2\right)^4\left(b^{-3}\right)^{-2}}$

2. Write the equation of the line that is parallel to $y = 1.5x - 11$ and has y-intercept at $y = 3$.

3. Write the equation of the line that passes through the points (1, 5) and (3, 11).

4. A coffee supplier finds that it costs \$850 to roast and package 100 pounds of coffee in a day and \$2850 to produce 500 pounds of coffee in a day. Assume that the cost function is linear. Express the cost as a function of the number of pounds of coffee they produce each day.

5. Factor $y^5 - 6y^4 + 9y^3$.

6. Factor $a^4 - 5a^2 - 36$.

7. The two points $\left(-3, \dfrac{5}{8}\right)$ and $(2,20)$ both lie on the graph of a function $y = g(x)$.

a. Find the formula for $y = g(x)$ if it is a linear function.

b. Find the formula for $y = g(x)$ if it is of the form $g(x) = Ca^x$.

8. An object is thrown into the air. Its height in feet above the ground t seconds later is given by $h(t) = -16t^2 + 30t + 25$.

a. Find the time when the object reaches its maximum height.
b. How high does it get?
c. When does it land?

d. The graph of $y = h(t)$ is a parabola. What is the physical meaning of the y-intercept of this parabola? (That is, what does the y-intercept of the parabola tell us about the object?)

9. Suppose $u \circ v(x) = \dfrac{1}{x^2} + 1$ and $v \circ u(x) = \dfrac{1}{(x+1)^2}$. Find possible formulas for $u(x)$ and $v(x)$.

10. Suppose $f(x) = x^2 - 2x$.

a. Compute $\dfrac{f(x) - f(3)}{x-3}$. Simplify your answer.

b. In English, what is the meaning of your answer to part a?

c. Graphically, what is the meaning of your answer to part a?

11. Here is the graph of $y = f(x)$. Use this graph to sketch the graph ***on graph paper*** of $y = -3f(x-2)$. Be careful and neat, and remember to label your graph. Briefly explain what you did to find the graph.

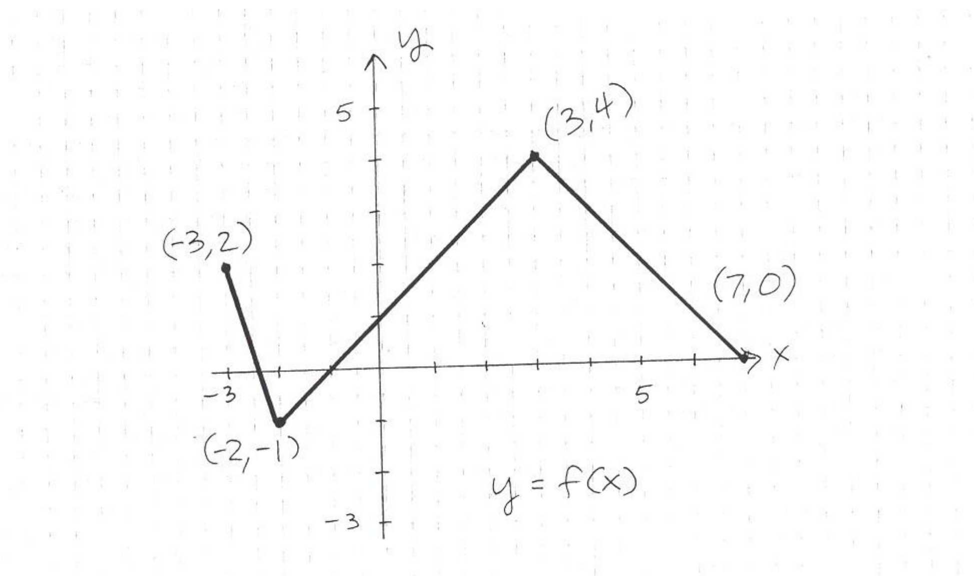

Figure 3

12. In 2000, the number of people infected by a virus was P_0. Due to a new vaccine, the number of infected people has decreased by 14% each year since 2000.

a. Find a formula for $P = f(n)$, the number of infected people n years after 2000.

b. When will there be (or when were there) just half as many people infected as there were in 2000?

13. Here is the graph of an exponential or logarithmic function.

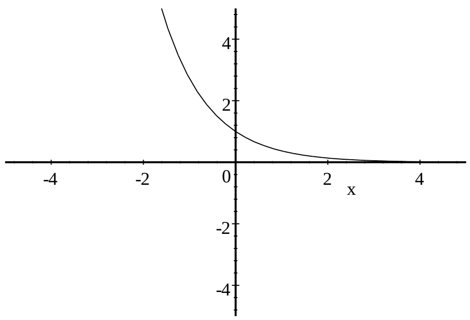

Figure 4

a. Is this an exponential function with base $b > 1$, an exponential function with $0 < b < 1$, or a logarithmic function with base $b > 1$?

b. What is the value of b (the base) for this graph? How do you know?