THE SCHUBERT ALGEBRA AND THE RING OF RC GRAPHS

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1. Some preliminaries

1.1. Notation.

Definition 1.1.1. For a permutation v' such that $v' \in S_n$, define

$$\varphi_{i,n}(v')(j) = \begin{cases} v'(j) & \text{if } j < i \\ n+1 & \text{if } j = i \\ v'(j-1) & \text{if } i < j \leq n+1 \\ j & \text{if } j > n+1 \end{cases}$$

Now fix $v \in S_n$ and $i \geq 1$ an integer, we define a relation $\overset{i}{\searrow}$ by declaring that $v \overset{i}{\searrow} v'$ if

$$v \overset{i}{\Rightarrow} \varphi_{i,n}(v')$$

Equivalently, $v \overset{i}{\searrow} v'$ if whenever $v \in S_n$ and n is minimal, we have that v satisfies the relation $\overset{i}{\Rightarrow}$ with respect to the permutation obtained from v' by inserting $n+1$ at position i. We note that this concept was introduced by Bergeron and Sottile in [5].

If $v \overset{i}{\searrow} v'$, we define a set of integers $Q_i(v', v)$ by

$$Q_i(v', v) = \{v(j) \mid j > i \text{ and } v'(j-1) = v(j)\}$$

Given the fact that any element of S_∞ fixes all but finitely many positive integers, it follows that $Q_i(v', v)$ is a finite set.

We then define $\mathcal{D}_i(v)$ to be all permutations $v' \in S_\infty$ such that $v \overset{i}{\searrow} v'$. The permutations $\mathcal{D}_i(v)$ arise when pulling a variable out of a Schubert polynomial and expressing the coefficients of the powers of this variable as Schubert polynomials in the remaining variables, as is done in [4], from which this definition essentially comes.

For a permutation $v \in S_\infty$, we define $\uparrow v$ to be the permutation defined by

$$\uparrow v(i) = \begin{cases} v(i-1)+1 & \text{if } i > 1 \\ 1 & \text{if } i = 1 \end{cases}$$

Recursively, we define $\uparrow^k(v)$ to be $\uparrow(\uparrow^{k-1}(v))$.

For a permutation w, we define $\mathfrak{m}(w)$ to be the maximum right descent of w. That is

$$\mathfrak{m}(w) = \max\{i \mid w(i) > w(i+1)\}$$

Define

$$\Pi\,(a \mid B) = \prod_{b \in B} (a - b)$$

Proposition 1.1.2 (Special case of Pieri formula [8, Theorem 7.1])**.** *Suppose $k \geq 1$ and $u, w \in S_\infty$. If $u \overset{k}{\not\to} w$, then*

$$_{(y)}\partial_u^w \Pi\left(x_1 \mid y_{[k]}\right) = 0$$

If $u \xrightarrow{k} w$, define

$$Q = \{u(i) \mid i \leq k \text{ and } u(i) = w(i)\}$$

then

$$_{(y)}\partial_u^w \Pi\left(x_1 \mid y_{[k]}\right) = \Pi\left(x_1 \mid y_Q\right)$$

Proof. This is simply a change of variables from the original theorem. □

The next proposition specializes, in the case of ordinary Schubert polynomial $\mathfrak{S}_v(x)$, to a formular that isolate a chosen index i: one can express $\mathfrak{S}_v(x)$ as a sum of terms of the form $x_i^p\,\mathfrak{S}_{v'}(x^{(i)})$, thereby effectively extracting the variable x_i and leaving Schubert polynomials in the remaining variables [5, Theorem 5.1]. Proposition 1.1.3 is the double Schubert polynomial version of this, which, as far as we know, is new.

Proposition 1.1.3. *Let $v \in S_\infty$ and let $i > 0$ be an integer. Then we have*

$$\mathfrak{S}_v(x;y) = \sum_{v \overset{i}{\searrow} v'} \Pi\left(x_i \mid y_{Q_i(v',v)}\right) \mathfrak{S}_{v'}(x^{(i)};y)$$

Proof. Suppose $v \in S_n$. We have

$$\mathfrak{S}_v(x;y) = {}_{(y)}\partial^{vw_0(n)}(\mathfrak{S}_{w_0(n)}(x;y))$$

This is equal to

$${}_{(y)}\partial^{vw_0(n)}(\mathfrak{S}_{s_{n+1-i}\cdots s_1 w_0(n)}(x^{(i)};y)\Pi\left(x_i \mid y_{[n+1-i]}\right))$$

and, applying the Leibniz formula, is also equal to

$$\sum_{\substack{v' \in S_\infty \\ \ell(v'w_0(n)s_1\cdots s_{n+1-i}) = \ell(w_0(n)s_1\cdots s_{n+1-i}) - \ell(v')}} \mathfrak{S}_{v'}(x^{(i)};y)\,{}_{(y)}\partial^{vw_0(n)}_{v'w_0(n)s_1\cdots s_{n+1-i}}\Pi\left(x_i \mid y_{[n+1-i]}\right)$$

By the restricted Pieri formula (Proposition 1.1.2), for this to be nonzero necessarily $v'w_0(n)s_1\cdots s_{n+1-i} \xrightarrow{n+1-i} vw_0(n)$. We note that $v'w_0(n)s_1\cdots s_{n+1-i} \xrightarrow{n+1-i} vw_0(n)$ if and only if

$$v \overset{i}{\Rightarrow} v'w_0(n)s_1\cdots s_{n+1-i}w_0(n) = v's_ns_{n-1}\cdots s_i$$

We have that $v's_n\cdots s_i$ is exactly $\varphi_{i,n}(v')$ since $v' \in S_n$, so we require that

$$v \overset{i}{\Rightarrow} \varphi_{i,n}(v')$$

so the sum is over all $v' \in \mathcal{D}_i(v)$.

Applying Proposition 1.1.2, we obtain that the result is equal to

$$\sum_{v' \in \mathcal{D}_i(v)} \Pi\left(x_i \mid y_{A(v',v)}\right) \mathfrak{S}_{v'}(x^{(i)};y)$$

where $A(v',v)$ is the set of all $vw_0(n)(j)$ such that $1 \le j \le n+1-i$ and $v'w_0(n)s_1\cdots s_{n+1-i}(j) = vw_0(n)(j)$. These values are the same as at the indices that comprise the set of all $1 \le j \le n+1-i$ such that

$$v'(n+2-s_1\cdots s_{n+1-i}(j)) = v(n+2-j)$$

Applying the $s_1\cdots s_{n+1-i}$ to j, since $1 \le j \le n+1-i$ we have that

$$s_1\cdots s_{n+1-i}(j) = j+1$$

Hence we need

$$v'(n+2-(j+1)) = v'(n+1-j) = v(n+2-j)$$

Replacing j with $n+2-p$, the indices are the set of all p such that $i < p \le n+1$ and

$$v'(p-1) = v(p)$$

Thus $A(v',v)$ is the set of all $v(p)$ such that $p > i$ and $v'(p-1) = v(p)$, which is exactly $Q_i(v',v)$, and we are done. □

2. The Schubert Algebra and its dual

2.1. **Definition.** We define a commutative algebra $\mathcal{A}$ over the integers as follows. For each n, define $\mathcal{A}_n$ to be the polynomial ring over $\mathbb{Z}$ in the variables $x_1, \ldots, x_n$. Then define

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

The multiplication within $\mathcal{A}_n$ is as usually defined for the polynomial ring. However, if $a \in \mathcal{A}_m$ and $b \in \mathcal{A}_n$ with $m \neq n$ and $m, n > 0$, then

$$ab = 0$$

Otherwise, the component for $n = 0$ is identified with the coefficient ring. Note that the "identity element" for positive n is not an identity element of $\mathcal{A}$ (we may sometimes refer to it as a "fat identity").

Each $\mathcal{A}_n$ has a basis consisting of elements $x_a^{(n)}$, where a is a sequence of n nonnegative integers, and the notation indicates that

$$x_a = x_1^{a_1} \cdots x_n^{a_n}$$

The direct sum therefore has a basis that can canonically be identified with union of these.

We define a coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ on the basis $x_c^{(n)}$ by

$$\Delta(x_c^{(n)}) = \sum_{\substack{p+q=n \\ ab=c}} x_a^{(p)} \otimes x_b^{(q)}$$

where the equation $ab = c$ indicates that a concatenated with b is equal to c. We also define a counit $\varepsilon : \mathcal{A} \to \mathbb{Z}$ by $\varepsilon(x_a^{(n)}) = 0$ unless $n = 0$.

Lemma 2.1.1. *With Δ and ε, $\mathcal{A}$ is a coassociative, counital coalegebra.*

Proof. We have

$$\Delta(x_d^{(n)}) = \sum x_a^{(p)} \otimes x_c^{(q)}$$

Applying Δ to either tensor factor results in

$$\sum x_a^{(p)} \otimes x_b^{(q)} \otimes x_c^{(r)}$$

The symmetry of this is exactly the coassociativity condition. Seeing that we may choose $p = n$ or $q = n$, the definition of the counit gives us the result that $\mathcal{A}$ is counital as well under Δ and ε. □

Lemma 2.1.2. *$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a homomorphism of rings.*

Proof. This is where the condition that $x_a^{(p)} x_b^{(q)} = 0$ unless $p = q$ when both $p, q > 0$ comes in. It ensures that only monomials of the same length have nonzero products and preserves the structure of the coproduct as a homomorphism of rings. □

Lemma 2.1.3. *$\nabla : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a homomorphism of coalgebras.*

Proof. □

Corollary 2.1.4. *$\mathcal{A}$ is a bialgebra over $\mathbb{Z}$.*

$\mathcal{A}$ is afforded a grading into finite dimensional components by observing that each $\mathcal{A}_n$ itself is a graded ring with each homogeneous component being a finitely generated free module. Considering the pair (n, d), where n is the number of variables and d is the degree, as a $\mathbb{Z}^2$ grading, we may take the graded dual module $\mathcal{D}$, which, by virtue of the grading, is isomorphic as a free module to $\mathcal{A}$. We identify the dual basis element x_a^* with the sequence of nonnegative integers a. That is to say,

$$\langle a, x_b \rangle = \delta_{ab}$$

Inspection of the coproduct reveals that the product of a and b in $\mathcal{D}$ is simply the concatenation ab. Hence $\mathcal{D}$ is isomorphic to the free associative algebra on a countable set indexed by nonnegative integers.

$\mathcal{D}$ also has a coproduct compatible with its product, namely

$$c \mapsto \sum_{a+b=c} a \otimes b$$

Thus $\mathcal{A}$ and $\mathcal{D}$ are dual bialgebras.

Calling $\mathcal{A}$ the "Schubert algebra" may seem unnecessarily grandiose, however the reason will become clear below.

2.2. **The Schubert basis.** In $\mathcal{A}$, each $\mathcal{A}_n$ has a basis of Schubert polynomials $\mathfrak{S}_u^{(n)}$, where the largest right descent of u is at most n, ensuring that $\mathfrak{S}_u(x)$ has at most n variables. Schubert polynomials limited to a specific number of variables have well-defined structure constants $c_{u,v}^w$, independent of n, given by

$$\mathfrak{S}_u^{(n)}\,\mathfrak{S}_v^{(n)} = \sum_w c_{u,v}^w\,\mathfrak{S}_w^{(n)}$$

These are known to be nonnegative integers, however except in special cases no positive combinatorial formula is known.

Schubert polynomials have nonnegative coefficients in terms of the $x_a^{(n)}$ basis, for which many formulas are known. There is also a less well-understood unique expansion of the Schubert polynomials into sums of products of elementary symmetric polynomials with at most n variables.

2.3. **The elementary basis.** We say that a weak composition α is *n-elementary* if $\alpha_i \leq i$ for all $i \leq n$ and $\alpha_i \leq n$ for all $i > n$, satisfying the additional restriction that $\alpha_i \geq \alpha_{i+1}$ if $i \geq n$.

An n-elementary monomial is a an element of the polynomial ring of the following form:

$$e_{\alpha_1}^1 \cdots e_{\alpha_n}^n\, e_{\alpha_{n+1}}^n \cdots e_{\alpha_m}^n$$

where α is an n-elementary weak composition. Define $\mathcal{E}_n$ to be the set of n-elementary monomials.

Theorem 2.1. *$\mathcal{E}_n$ forms a $\mathbb{Z}$-basis for $\mathcal{A}_n$, and $\bigcup_n \mathcal{E}_n$ forms a basis for $\mathcal{A}$.*

Proof. It is a theorem of Macdonald that the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ is a free module over Λ_n with basis the Schubert polynomials $\mathfrak{S}_u(x)$ such that $u \in S_n$. These Schubert polynomials are uniquely expressed in terms of strict elementary symmetric monomials with fewer than n variables. Since Λ_n is a polynomial ring in the $e_{i,n}$ by the fundamental theorem of elementary symmetric polynomials, Λ_n has a basis of monomials $e_{\lambda,n}$ as λ ranges over all partitions with parts bounded by n. The multiplicative combination of these two bases is therefore a $\mathbb{Z}$-basis of $\mathcal{A}_n$. This is exactly the description of the monomials e_α^n for α an n-elementary weak composition. □

We can ask for transition formulas between the Schubert and the elementary basis. From elementary to Schubert, we may use Sottile's Pieri formula. That is,

$$e_\alpha^n = \sum_{1 \xrightarrow{1,2,\ldots,n}_\alpha u} \mathfrak{S}_u^n$$

For the reverse transition, suppose we want to express $\mathfrak{S}_w^n$ in the elementary basis. Let $\lambda = \mu_n^*(w)$ and consider the double Schubert polynomial

$$\mathfrak{S}_\lambda(x; y) = \prod_i E_{\lambda_i}(x; y_i)$$

We have that

$$\mathfrak{S}_w(x; y) = \partial_y^{w\lambda^{-1}}\,\mathfrak{S}_\lambda(x; y)$$

Applying the Leibniz formula, this is

$$\sum_{u_0 \xrightarrow{1} u_1 \cdots} \prod_i \partial_y^{u_i/u_{i-1}} E_{\lambda_i}(x; y_i)$$

It is possible to find a combinatorial formula for each factor in the product, keeping the y variables, but we only need the sign. We define $\sigma_i(u_{i-1}, u_i)$ to be the number of indexes $j < i$ such that $u_{i-1}(j) \neq u_i(j)$. Then we define

$$\sigma(P) = \sum_i \sigma_i(u_{i-1}, u_i)$$

Then we define

$$E_w^{\alpha;n} = \sum_{\substack{P:1 \xrightarrow{1,2,\ldots,n} w\lambda^{-1} \\ \alpha(P)=\alpha}} (-1)^{|\lambda|-\ell(w)+\sigma(P)}$$

Theorem 2.2. *For each w, we have*

$$\mathfrak{S}_w^n = \sum_\alpha E_w^\alpha e_\alpha^n$$

These numbers are stable for fixed w as soon as the number of variables is at least as large as the value of n such that $w \in S_n$, and are well-studied but poorly understood.

Example 2.3.1. Let $n = 5$ and let w be the permutation such that $\mathfrak{c}(w) = (0,2,0,2,3)$. Let $\lambda = (5,5,5,4,3,2,1)$. Then

$$\begin{aligned}\mathfrak{S}_w^n &= e_{(0,0,0,0,5,1,1)} - e_{(0,0,0,0,4,2,1)} + e_{(0,0,0,0,4,3)} - e_{(0,1,0,0,4,1,1)} + e_{(0,0,1,0,3,2,1)} \\ &\quad + e_{(0,1,0,0,5,1)} + e_{(0,1,0,0,3,3)}\end{aligned}$$

Suppose α and β are weak compositions. We define $\alpha\|_p\beta$ to be the weak composition defined by

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \leq p \\ \alpha_i + \beta_{i-p} & \text{if } i > p \end{cases}$$

Theorem 2.3.

$$\Delta(e_\gamma^n) = \sum_{\substack{\alpha\|_p\beta=\gamma \\ p+q=n}} e_{\alpha^{\overline{p}}}^p \otimes e_\beta^q$$

where α ranges over all weak compositions such that $\alpha^{\overline{p}}$ is p-elementary and β ranges over all q-elementary weak compositions.

2.4. **The Schubert-Schur and separated descents basis.** $\mathcal{A}_n$ has a basis of the form $S_{\lambda,u}^n = s_\lambda(x_1,\ldots,x_n)\,\mathfrak{S}_u(x)$, where $u \in S_n$ and s_λ is the Schur polynomial corresponding to λ in Λ_n. This is called the *Schubert-Schur* basis. There is a generalization that this is an easy special case of that can be described as follows.

Let $K_{u,v;k}^n = \mathfrak{S}_u(x_1,\ldots,x_n)\,\mathfrak{S}_v(x_1,\ldots,x_{k-1})$, where $v \in S_k$ and u is a permutation such that $\ell(us_i) > \ell(u)$ for all $i < k$. We will call this the *separated descents basis* for the descent k.

Theorem 2.4. *For fixed k, $K_{u,v;k}^n$ form a basis of $\mathcal{A}_n$ for all $n \geq k$.*

Proof. First we show that these elements additively generate the subring. Note first that $\mathcal{E}_n$ is a basis. Let $e_{\alpha_1}^{\lambda_1}\cdots e_{\alpha_m}^{\lambda_m} \in \mathcal{E}_n$. Let j be the index such that $\lambda_j = k$. Let

$$P = \prod_{i=1}^{j} e_{\alpha_i}^{\lambda_i}$$

and let

$$Q = \prod_{i=j+1}^{m} e_{\alpha_i}^{\lambda_i}$$

By the fact that $\lambda_{i-1} = \lambda_i + 1$ for all $i > j$, it follows that

$$Q = \sum c_v\,\mathfrak{S}_v(x)$$

for integers c_v with $v \in S_k$. Given the fact that P is symmetric in $x_1,\ldots,x_k$, it can be shown that

$$P = \sum d_u\,\mathfrak{S}_u(x)$$

for integers d_u and permutations $u \in S_\infty$ with $\ell(us_i) > \ell(u)$ for all $i < k$. Thus the n-elementary monomial can be expressed as

$$PQ = \sum c_u d_v\,\mathfrak{S}_u(x_1,\ldots,x_n)\,\mathfrak{S}_v(x_1,\ldots,x_{k-1})$$

Since the n-elementary monomials form a basis, this proves that $K_{u,v;k}^n$ spans $\mathcal{A}_n$.

To prove independence, first note that if

$$\sum_u c_u K^n_{u,1;k} = \sum_u c_u \mathfrak{S}_u(x) = 0$$

then $c_u = 0$ for all u by independence of Schubert polynomials. Assume the inductive hypothesis that for some $m > 0$ we have that if

$$\sum_u c_{uv} K^n_{u,v;k} = 0$$

and for all v in the sum with $\ell(v) \geq m$ we have $c_{uv} = 0$, then $c_{uv} = 0$ for all u and v. For a sum with possibly nonzero terms such that $\ell(v) = m$ at most, suppose

$$\sum_{u,v} c_{u,v} K^n_{u,v;k} = 0$$

Let i be a positive integer. Then by applying the divided difference ∂^{s_i} we see that

$$\sum_{u,v} c_{u,v} K^n_{u,vs_i;k} = 0$$

for all v with $\ell(vs_i) < \ell(v)$. In particular, the maximum length with a possibly nonzero coefficient has decreased by at least 1. By the inductive hypothesis, all terms in this sum are 0, hence $c_{uv} = 0$ for all u and v with $\ell(vs_i) < \ell(v)$. Iterating over all i, we have the result. $\square$

For transition coefficients, expressing $K^n_{u,v;k}$ in terms of Schubert polynomials is "easy" and combinatorially known to have nonnegative coefficients. While positive formulas are known, their discovery is quite recent.

Expressing the Schubert basis in terms of $K^n_{u,v;k}$ can be done as follows.

Recall we can express $\mathfrak{S}^n_w$ as

$$\mathfrak{S}^n_w = \sum_\alpha E^{\alpha;n}_w e^n_\alpha$$

To transition to the basis $K^n_{u,v;k}$, let a be the index such that $\lambda_a = k$. Then

$$\mathfrak{S}^n_w = \sum_\alpha E^{\alpha;n}_w \left(\prod_{i=1}^{a} e_{\alpha_i,\lambda_i}\right)\left(\prod_{j=a+1}^{\ell(\lambda)} e_{\alpha_j,\lambda_j}\right)$$

These products can be expanded with the Pieri formula as

$$\mathfrak{S}^n_w = \sum_\alpha E^{\alpha;n}_w c^u_{\alpha',\lambda'} c^v_{\alpha'',\lambda''} K^n_{u,v;k}$$

Example 2.4.1. Let $n = 5$, let $k = 3$, and let w be the permutation such that $\mathfrak{c}(w) = (0, 2, 0, 2, 3)$. Then

$$\mathfrak{S}^n_w = K^5_{12468357,132;3} - K^5_{13468257,1;3}$$

If instead we let $k = 4$, we obtain

$$\mathfrak{S}^n_w = -K^5_{124563,132;4} + K^5_{1246735,132;4} + K^5_{134562,1;4} - K^5_{1346725,1;4} + K^5_{234561,132;4}$$

2.5. **The dual algebra.** We examine the graded dual algebra $\mathcal{D}$ more closely now. This is a graded ring generated by countably many elements that we denote by $[i]$, where i is a nonnegative integer. The product, as mentioned previously, is concatenation of sequences. Thus

$$[a_1 \cdots a_p][b_1 \cdots b_q] = [a_1 \cdots a_p b_1 \cdots b_q]$$

It is not hard to see that $\mathcal{D}$ is a free algebra on these generators. Thus the set of sequences $[a_1 \cdots a_n]$ forms a $\mathbb{Z}$-basis for $\mathcal{D}$. We may realize this as the dual of $\mathcal{A}$ by declaring that

$$\langle [a_1 \cdots a_n], x_1^{b_1} \cdots x_n^{b_n} \rangle = \prod_i \delta_{a_i,b_i}$$

We have a $\mathbb{Z} \times \mathbb{Z}$ grading such that

$$\deg([a_1 \cdots a_n]) = (n, -a_1 - a_2 \cdots - a_n)$$

The set of elements such that $\deg(a) = (n, -)$ is $\mathcal{D}_n$, dual as a graded module to $\mathcal{A}_n$.

2.6. **The dual Schubert basis.** There is a basis Ξ_u^n dual to the Schubert basis for $\mathcal{D}_n$. Specifically, with the unique pairing $\langle -,- \rangle : \mathcal{D} \times \mathcal{A} \to \mathbb{Z}$ such that

$$\langle \alpha, x_\beta \rangle = \delta_{\alpha\beta}$$

we define Ξ_u^n to be the unique basis of $\mathcal{D}_n$ such that

$$\langle \Xi_u^n, \mathfrak{S}_v^n \rangle = \delta_{uv}$$

We characterize it with an explicit formula.

Theorem 2.5. *For each permutation u and integer n with $\ell(us_i) > \ell(u)$ for all $i > n$ we have the equation*

$$\Xi_u^n = \sum_{\ell(\alpha)=n} E_{u\mu^{-1}}^{\mathfrak{c}(\mu)-\alpha,\mathfrak{c}(\mu)} \alpha$$

where μ is any strict dominant permutation such that $0 \neq \mathfrak{c}_n(\mu) \geq \ell(u)$ and $\mathfrak{c}_{n+1}(\mu) = 0$.

Proof. By definition, the coefficient of α in Ξ_u^n is the coefficient of $\mathfrak{S}_u^n$ in x_α. This can be derived from the Cauchy formula for double Schubert polynomials. Note that for any permutation μ as laid out in the statement of the theorem, $\ell(u\mu^{-1}) = \ell(\mu) - \ell(u)$. Thus,

$$\partial_y^{u\mu^{-1}} \mathfrak{S}_\mu(x;-y) = \mathfrak{S}_u(x;-y)$$

We have that

$$\mathfrak{S}_\mu(x;-y) = \sum_u \mathfrak{S}_u(x)\, \mathfrak{S}_{u\mu^{-1}}(y)$$

Expressing the second factor in the $e_{\alpha,\lambda}(y)$ basis, we have

$$\mathfrak{S}_\mu(x;-y) = \sum_{u,\alpha} \mathfrak{S}_u(x) E_{u\mu^{-1}}^{\mathfrak{c}(\mu)-\alpha,\mathfrak{c}(\mu)} e_{\mathfrak{c}(\mu)-\alpha,\mathfrak{c}(\mu)}(y)$$

An alternative expression for $\mathfrak{S}_\mu(x;-y)$ is

$$\mathfrak{S}_\mu(x;-y) = \sum_\alpha x_\alpha e_{\mathfrak{c}(\mu)-\alpha,\mathfrak{c}(\mu)}(y)$$

from which we see that the coefficient is correct. $\square$

This is not stable for fixed u as n increases, and this is expected.

Lemma 2.6.1. *Let $u, v \in S_\infty$ and $p, q > 0$ be integers. Write*

$$\Xi_u^p \Xi_v^q = \sum_w d_{u,v}^w(p,q) \Xi_w^{p+q}$$

Then for each u, v, w the coefficient $d_{u,v}^w(p,q)$ is the coefficient of $\mathfrak{S}_u(x_1,\dots,x_p)\,\mathfrak{S}_v(x_{p+1},\dots,x_{p+q})$ in the expansion of $\mathfrak{S}_w(x_1,\dots,x_{p+q})$ in terms of the basis of products of Schubert polynomials in $x_1,\dots,x_p$ and Schubert polynomials in x_{p+1} onward.

Proof. This is true by examination of the definition of the coproduct of $\mathcal{A}$, since this is the coefficient of $\mathfrak{S}_u^p \otimes \mathfrak{S}_v^q$ in the coproduct of $\mathfrak{S}_w^{p+q}$. $\square$

Thus the product structure of $\mathcal{D}$ encodes splitting of the Schubert polynomial into two sets of variables. In particular,

Lemma 2.6.2. *Let $a \geq 0$ be an integer and $w \in S_\infty$. Then we have*

$$[a] \cdot \Xi_w^n = \sum_{\substack{w \in \mathcal{D}_1(w') \\ \ell(w,w')=a}} \Xi_{w'}^{n+1}$$

Observation 2.1. *In the formula*

$$\Delta(\Xi_w^n) = \sum_{u,v} c_{u,v}^w \Xi_u^n \otimes \Xi_v^n$$

the coefficients $c_{u,v}^w$ are the structure constants of Schubert polynomials. That is, the entire multiplicative structure of Schubert polynomials is encoded in the coproduct.

3. The ring of bounded RC graphs

3.1. The module of bounded RC graphs.

Definition 3.1.1. Let $R \subseteq \mathbb{P} \times \mathbb{P}$ be a finite set. To each such set we associate an element w_R of S_∞ as follows. Given a pair (i, j) where $i, j > 0$ are integers, define $s(i, j) = s_{i+j-1}$. Totally order the grid such that $(i, j) < (a, b)$ if and only if $i < a$ or $i = a$ and $j > b$ (in other words, lexicographical order, except that the order on the second coordinate is reversed). By this ordering, index R as $r_1, r_2, \ldots, r_m$ in increasing order. Then

$$w_R = s(r_1) \cdots s(r_m)$$

If $\ell(w_R) = m$, then we say that R is an *RC-graph.*

A pair (R, n) such that R is an RC-graph and w_R has no right descent larger than n is called a *bounded RC-graph*. A bounded RC graph has an associated vector $\mathtt{wt}(R, n)$ such that $\mathtt{wt}_i(R, n)$ is the number of elements of R with first coordinate i (the number of elements in row i).

To a bounded RC graph (R, n), we define an associated bounded RC graph $\mathtt{trim}(R, n) = (R', n-1)$ such that R' is the set of all $(i-1, j)$ such that $(i, j) \in R$ and $i > 1$. We also define

$$\uparrow^m R = \{(i+m, j) | (i, j) \in R\}$$

Let $\mathcal{BRC}$ be the free abelian group spanned by all bounded RC graphs. If $\mathcal{BRC}_n$ is the subgroup spanned by all bounded RC graphs of the form (R, n), then we have a grading

$$\mathcal{BRC} = \bigoplus_{n=0}^{\infty} \mathcal{BRC}_n$$

There is an evident evaluation map $\phi : \mathcal{BRC} \to \mathcal{A}$ defined on basis elements as

$$\phi(R, n) = x_{\mathtt{wt}(R,n)}$$

which is a surjective homomorphism of graded modules. There is also a map $\alpha : \mathcal{BRC} \to \mathcal{D}$ defined by

$$\alpha(R, n) = \Xi^n_{w_R}$$

which is also a surjective homomorphism of graded modules. In addition, there is $\omega : \mathcal{BRC} \to \mathcal{D}$ defined by

$$\omega(R, n) = [\mathtt{wt}(R, n)]$$

which is similarly surjective.

We can define elements $\mathcal{S}_w(n)$ as

$$\mathcal{S}_w(n) = \sum_{w_R = w} (R, n)$$

For a generating element $[a]$ of $\mathcal{D}$ and a bounded RC graph (R, n), we define

$$[a] \cdot (R, n) = \sum_{\mathtt{trim}(R', n+1) = (R, n)} (R', n+1)$$

which is an element of $\mathcal{BRC}$. By virtue of the fact that $\mathcal{D}$ is a free algebra generated by these elements, it is nearly a trivial observation that this is a left module action on $\mathcal{BRC}$. The consequences of this, however, are not at all trivial.

Lemma 3.1.2 ([3, Corollary 3.11]). *We have that $\mathcal{D}_1(w)$ is equal to the set of permutations w' such that there exists a permutation $v = s_{a_1} \cdots s_{a_m}$ for some integers $a_1 > a_2 > \cdots > a_m \geq 1$ such that $w = v \uparrow w'$.*

Lemma 3.1.3. *Let v be a permutation. If $v \overset{1}{\searrow} v'$, let $a_1, \ldots, a_k$ be the elements of $Q_1(v', v)$ in decreasing order. Then*

$$v = s_{a_1} \cdots s_{a_k} \uparrow v'$$

Proof. Suppose $v \in S_n$. We have by definition that there is a sequence of integers $b_1, \ldots, b_p$, all distinct and greater than 1, such that

$$v t_{1,b_1} \cdots t_{1,b_p}(1) = n + 1$$

and

$$v t_{1,b_1} \cdots t_{1,b_p}(i + 1) = v'(i)$$

for all $i < n-1$. This means that

$$vt_{1,b_1}\cdots t_{1,b_p} = s_n s_{n-1}\cdots s_1 {\uparrow} v'$$

This is because if $v'' = {\uparrow} v'$, then

$$v''(1) = 1$$

and

$$v''(i) = v(i-1)+1$$

The cycle $s_n \cdots s_1$ sends $1 \mapsto n+1$ and $i \mapsto i-1$ if $1 < i \leq n+1$, hence

$$vt_{1,b_1}\cdots t_{1,b_p} = s_n s_{n-1}\cdots s_1 {\uparrow} v'$$

In particular,

$$v = s_n s_{n-1}\cdots s_1 {\uparrow} v' t_{1,b_p}\cdots t_{1,b_1}$$

This results in the factorization

$$v = s_n \cdots s_1 t_{1,v'(b_p-1)+1}\cdots t_{1,v'(b_1-1)+1} {\uparrow} v'$$

The value $v'(b_j - 1)$ necessarily decreases as j decreases, since applying the corresponding reflection in the reverse order strictly increases the length with each application. Consequently, multiplying $s_n \cdots s_1$ on the right by these reflections removes the simple reflections $s_{v'(b_p-1)}, \ldots, s_{v'(b_1-1)}$. The indices removed are precisely the complement of the elements of $Q_1(v', v)$, hence the elements of $Q_1(v', v)$ in decreasing order are what remain, as desired. □

Definition 3.1.4. For a set of positive integers $A = \{a_1, \ldots, a_m\}$ with $a_1 > a_2 > \cdots > a_m \geq 1$ and a positive integer i, define

$$\mathrm{row}_i(A) = \{(i, a_j) \mid a_j \in A\}$$

Theorem 3.1. *Let $a \geq 0$ be an integer and let $(R, n) \in \mathcal{BRC}$. Then*

$$[a]\cdot(R,n) = \sum_{\substack{w_R \in \mathcal{D}_1(w') \\ \ell(w_R, w') = a}} (\mathrm{row}_1(Q_1(w_R, w')) \cup {\uparrow} R, n+1)$$

Proof. If $\mathtt{trim}(R', n+1) = (R, n)$, then by definition $w_{R'} = s_{a_1}\cdots s_{a_m} {\uparrow} w_R$. It follows by Lemma 3.1.2 then $w_R \in \mathcal{D}_1(w_{R'})$. By Lemma 3.1.3, the integers $a_1, \ldots, a_m$ are precisely the elements of $Q_1(w_R, w_{R'})$ in decreasing order. This establishes the result. □

Lemma 3.1.5. *Let $a \geq 0$ be an integer and let $w \in S_\infty$. For any valid n, we have*

$$[a]\cdot \mathcal{S}_w(n) = \sum_{\substack{w \in \mathcal{D}_1(w') \\ \ell(w, w') = a}} \mathcal{S}_{w'}(n+1)$$

Let t be an indeterminate commuting with all elements of $\mathcal{D}$. Write

$$\mathfrak{S}(t) = \sum_{a=0}^{\infty} [a] t^a$$

Define

$$\mathfrak{S}(x_1, x_2, \ldots, x_n) = \mathfrak{S}(x_1)\,\mathfrak{S}(x_2)\cdots\mathfrak{S}(x_n)$$

Theorem 3.2. *Let $(\emptyset, 0) \in \mathcal{BRC}$. Then*

$$\mathfrak{S}(x_1, \ldots, x_n)\cdot(\emptyset, 0) = \sum_{w \in S_\infty} \mathfrak{S}_w(x_1, \ldots, x_n)\mathcal{S}_w(n)$$

Proof. The result for $n = 1$ is Lemma 3.1.5 together with Theorem 3.1. The general result follows by induction on n. Consider the inductive hypothesis

$$\mathfrak{S}(x_2, \ldots, x_n)\cdot(\emptyset, 0) = \sum_{w \in S_\infty} \mathfrak{S}_w(x_2, \ldots, x_n)\mathcal{S}_w(n-1)$$

Then applying $\mathfrak{S}(x_1)$ to both sides, we have

$$\mathfrak{S}(x_1)\,\mathfrak{S}(x_2,\dots,x_n)\cdot(\emptyset,0)=\sum_{w\in S_\infty}\mathfrak{S}_w(x_2,\dots,x_n)\,\mathfrak{S}(x_1)\mathcal{S}_w(n-1)$$

which is equal to

$$\sum_{a=0}^{\infty}\sum_{w\in S_\infty}x_1^a\,\mathfrak{S}_w(x_2,\dots,x_n)\,\mathfrak{S}(x_1)[a]\mathcal{S}_w(n-1)$$

Applying Lemma 2.6.2 to each term, we have

$$\sum_{a=0}^{\infty}\sum_{w\in S_\infty}x_1^a\,\mathfrak{S}_w(x_2,\dots,x_n)\sum_{\substack{w\in\mathcal{D}_1(w')\\ \ell(w,w')=a}}\mathcal{S}_{w'}(n)$$

Bringing the polynomial inside the inner sum, this is

$$\sum_{w'\in S_\infty}\left(\sum_{w\in\mathcal{D}_1(w')}x_1^{\ell(w,w')}\,\mathfrak{S}_w(x_2,\dots,x_n)\right)\mathcal{S}_{w'}(n)$$

which simplifies to

$$\sum_{w'\in S_\infty}\mathfrak{S}_{w'}(x_1,\dots,x_n)\mathcal{S}_{w'}(n)$$

as desired. □

Corollary 3.1.6. *For each $w\in S_\infty$ and valid n, we have*

$$\phi(\mathcal{S}_w(n))=\mathfrak{S}^n_w(x_1,\dots,x_n)$$

3.2. **The pipe dream visualization and roots.** Given a pair of positive integers i,j and an RC graph R, define

$$R[i,j]=\{(a,b)\in R\mid (a,b)>(i,j)\}$$

Given an RC graph R and a pair of positive integers i,j, we define an ordered pair

$$\mathtt{rt}_R(i,j)=(w_{R[i,j]}^{-1}(i+j-1),w_{R[i,j]}^{-1}(i+j))$$

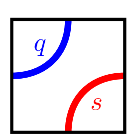
There is a common visualization of RC graphs as pipe dreams. In this visualization, we draw an infinite grid in the first quadrant, and in each position (i,j) we draw either a crossing (if $(i,j)\in R$) or an elbow (if $(i,j)\notin R$). Then we draw pipes entering from the left edge of the grid, with the pipe entering at row i labeled i. The pipes travel through the grid, turning at elbows and crossing at crossings, and exit at the top of the grid, which is labeled with the same number.

A modification to this common visualization that we use has the following additional features:

- We write the index of the simple reflections corresponding to the crossings in the grid area itself. Thus, at position (i,j) we write the number $i+j-1$ if there is a crossing at that position.
- For a bounded RC graph (R,n), we only draw the first n pipes entering from the left side of the grid and clip features outside of the first n rows. The pipes exiting at the top are still labeled since the width is not limited.

See Figure 1 for an example of this visualization.

The main benefit of this is that visualizing $\mathtt{rt}_R(i,j)$ is easy. The pipes that pass through position (i,j) are labeled s and q for some $s,q>0$. For a positive root in an unoccupied square, we will have the following labeling, where $q<s$:



We observe that in the above RC graph, the position $(1,5)$ does not have this configuration. Placing a crossing there would create a negative root, and the pipes would cross twice. *Note that this results in a collection of ordered pairs that is not an RC graph, if such a crossing exists.* For a valid RC graphs, only positive roots occur as crossings, and this happens if and only if pipes cross at most once.

FIGURE 1. The pipe dream visualization of the bounded RC graph $(R, 5)$ where $R = \{(1,1),(1,2),(2,1),(3,1),(3,3)\}$

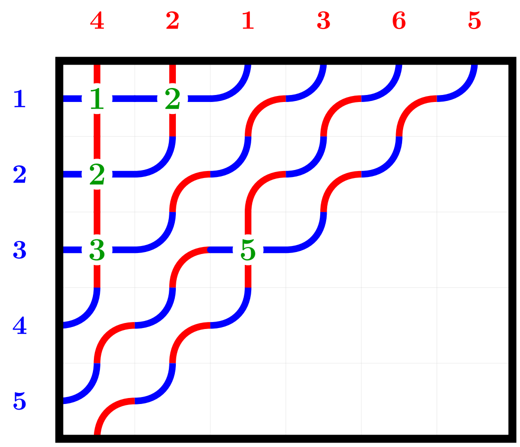
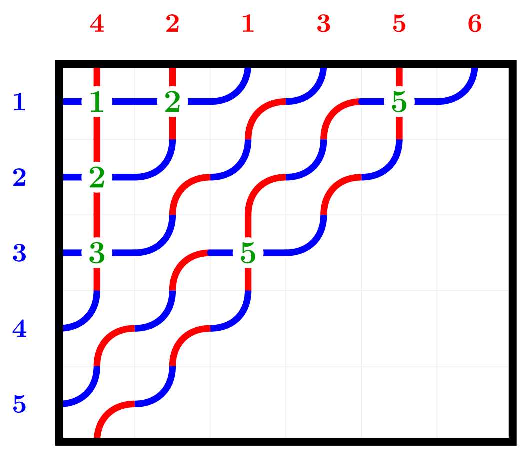


FIGURE 2. An invalid set of crossings causing pipes to cross more than once, caused by inserting a crossing at a negative root



3.3. **Bijection with inversion tableaux.** [2]

Definition 3.3.1. An *inversion tableau* is a function $T : (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0})^+ \to \mathbb{Z}_{>0} \cup \{-\infty\}$ such that only finitely many values are nonzero. Note $-\infty < a$ for all a, including $a = -\infty$. In addition, we have the following requirements:

(1) For each $i < j < k$ we have either
$$T(i,j) \leq T(i,k) < T(j,k)$$
or
$$T(j,k) < T(i,k) \leq T(i,j)$$
(2) For each $i \geq 1$ we have
$$T(i, i+1) \leq i$$

If R is an RC graph, we define an associated inversions tableau T_R as in the proof of [2, Theorem 3.13] by
$$T_R(a,b) = \begin{cases} i & \text{if } (i,j) \in R \text{ and } (a,b) = \texttt{rt}_R^{-1}(i,j) \\ -\infty & \text{otherwise} \end{cases}$$

Lemma 3.3.2. *We have the following for an RC graph R:*

(1) *Suppose $(i,j),(i,j+1) \in R$. Then $\texttt{rt}_R(i,j) = (a,c)$ and $\texttt{rt}_R(i,j+1) = (a,b)$ for some $a < b < c$.*

(2) *Suppose* $(i, j), (i+1, j) \in R$. *Then* $\mathtt{rt}_R(i, j) = (a, c)$ *and* $\mathtt{rt}_R(i+1, j) = (b, c)$ *for some* $a < b < c$.

3.4. **Zeroing out the last row.** We proceed now to define a product on $\mathcal{BRC}$ turning it into a ring. To do this, we need to be able to define a function $Z : \mathcal{BRC} \to \mathcal{BRC}$ trimming empty rows from the bottom instead of from the top. This is far more complicated.

Algorithm 1 RC insertion algorithm

1: **Input:** An RC graph R, parameter k, and sequence $k \geq i_1 \geq i_2 \geq \cdots \geq i_m \geq 1$.
2: **Output:** A modified RC graph R'.
3: **Initialize:** Let $L \leftarrow []$ (empty list of pairs (a, b) where $a \leq k < b$), and $r \leftarrow []$ (empty list).
4: **for** each $i \in (i_1, \ldots, i_m)$ **do**
5:     Find leftmost position $(i, j) \notin R$ in row i satisfying $j > j'$ for all $(i, j') \in r$ such that either:
6:       a) $\mathtt{rt}_R(i, j) = (s, q)$ where $s \leq k < q$ and $q \notin \{b \mid (a, b) \in L\}$.
7:       b) $\mathtt{rt}_R(i, j) = (b_r, q)$ where $q > k$, $b_r < q$, and $q \notin \{b \mid (a, b) \in L\}$.
8:     $R \leftarrow R \cup \{(i, j)\}$ and $r \leftarrow r \cup \{(i, j)\}$
9:     Update L by adding (s, q) or replacing (a_r, b_r) with (a_r, q) and (a_r, b_r).
10:     $(R, L) \leftarrow$ Rectify$(R, L, i-1)$ (Algorithm 2)
11: **end for**
12: **return** R

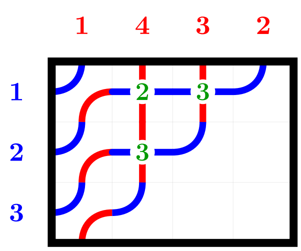
Algorithm 2 Subroutine: Rectify

1: **procedure** Rectify(R, L, i)
2:     **if** R is an RC graph or $i = 0$ **then**
3:         **return** (R, L)
4:     **end if**
5:     **for** minimal j such that $(i, j) \in R$ and $\mathtt{rt}_R(i, j) \in \Phi^-$ **do**
6:         $R \leftarrow R \setminus \{(i, j)\}$
7:         Let $\mathtt{rt}_R(i, j) = (q, s)$
8:         **if** $(s, q) \in L$ **then**
9:             Remove (s, q) from L
10:         **else**
11:             Remove (a_r, q) from L, where $(a_r, q), (a_r, s) \in L$
12:         **end if**
13:         Perform new insertion step at row i using criteria from Algorithm 1 Step 5.
14:     **end for**
15:     **return** Rectify$(R, L, i-1)$
16: **end procedure**

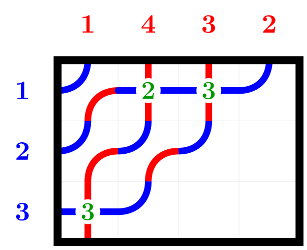
Algorithm 3 Map $Z(R, n)$ zeroing out row n

Input: A bounded RC graph (R, n) with row n empty.
Output: A bounded RC graph $(R', n-1)$ such that $|R'| = |R|$ and $w_R \overset{n}{\searrow} w_{R'}$.
if $(R, n-1)$ is a valid bounded RC graph **then**
  return $(R, n-1)$
else
  Let $R_0 \leftarrow R$.
  Find the maximal p and sequence $\{(i_m, j_m)\}_{m=1}^{p}$ such that:
   $\mathtt{rt}_{R_{m-1}}(i_m, j_m) = (n+m-1, n+m)$ for each m, where $R_m = R_{m-1} \setminus \{(i_m, j_m)\}$ for each $m \geq 1$.
  $R^- \leftarrow R \setminus \{(i_1, j_1), \ldots, (i_p, j_p)\}$.
  Let $I \leftarrow (i_1, i_2, \ldots, i_p)$.
  $D \leftarrow R^- \cup \{(n, 1), (n, 2), \ldots, (n, p)\}$.
  $D' \leftarrow \textsc{Insert}(D, n-1, I)$ ▷ Using Algorithm 1
  $R' \leftarrow \{(i, j) \in D' \mid i < n\}$.
  return $(R', n-1)$.
end if

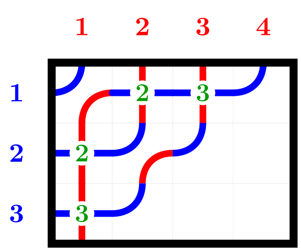
See Figure 3 for an example of the zeroing operation (Algorithm 3).



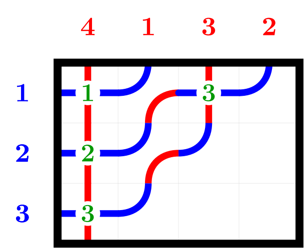
(a) Initial RC graph R



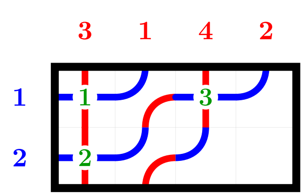
(b) After deleting $(2, 2)$ with $\mathtt{rt}_R(2, 2) = \alpha_3$ and moving to row 3



(c) After insertion with $i_1 = 2$



(d) After rectification



(e) Final result $\mathcal{Z}(R, 3) = (R', 2)$ after removing row 3

FIGURE 3. Step-by-step computation of $\mathcal{Z}(R, 3)$ for $R = \{(1, 2), (1, 3), (2, 2)\}$.

Lemma 3.4.1. *Given an RC graph R, an integer k, and $k \geq i_1 \geq \cdots \geq i_m \geq 1$, Algorithm 1 produces a valid RC graph R' satisfying*

$$\mathtt{wt}_i(R') = \mathtt{wt}_i(R) + \#\{j \mid i_j = i\}$$

and

$$w_R \overset{k}{\Longrightarrow} w_{R'}$$

Proof. Set $R_0 = R$ to be the initial RC graph. We claim that given R and L at the start of iteration p of the main loop, we have that R is a valid RC graph satisfying

$$\mathtt{wt}_i(R) = \mathtt{wt}_i(R_0) + \#\{j \leq p-1 \mid i_j = i\}$$

and

$$w_R = w_{R_0} t_{a_1 b_1} \cdots t_{a_{p-1} b_{p-1}}$$

where $L = [(a_1, b_1), \ldots, (a_p, b_p)]$. This is clear for $p = 1$. For larger p, suppose we are inserting at row i_p. If we are in case (a) of Step 5, then adding (i_p, j) to R adds the root $t_s - t_q$ to the inversion set of w_R, and multiplying by t_{sq} gives the desired result. If we are in case (b) of Step 5, then adding (i_p, j) to R results in multiplication by $t_{b_r q}$. This commutes past the other reflections to meet $t_{a_r b_r}$. We thus have the situation

$$t_{a_r b_r} t_{b_r q} = t_{a_r q} t_{a_r b_r}$$

This ensures that the product remains as desired if the RC graph is valid. However, the condition that the RC graph R be valid is not guaranteed after this step, so we must rectify. Suppose we must rectify at row $i-1$. We find a negative root $(i-1, j')$ which must have been created by the insertion, and hence by the strong exchange property equal to $\mathtt{rt}_R(i,j)$ that was inserted. Removing this root (from both the graph and from L) returns us to the original permutation at the beginning of the iteration, and performing the insert at row $i-1$ adds a new positive root, giving the desired result on the permutation. The weight is unchanged since we removed and added one crossing in row $i-1$. If the algorithm ends here, we have instead multiplied by the new reflection obtained in the ultimate insert step. Otherwise, repeating this process for all necessary rectifications completes the proof of the claim. The claim iterated to $p = m+1$ yields the desired result, and the method of updating L ensures that the b_j are all distinct, so that $w_R \overset{k}{\Longrightarrow} w_{R'}$. ☐

Lemma 3.4.2. *Given a bounded RC graph (R, n) such that row n is empty, Algorithm 3 produces a valid bounded RC graph $(R', n-1)$ satisfying*

$$\mathtt{wt}(R') = \mathtt{wt}(R)$$

Proof. Stripping out the crossings at roots $(n, n+1), (n+1, n+2), \ldots, (n+p-1, n+p)$ removes exactly one crossing from each of rows $i_1, i_2, \ldots, i_p$, and moving them to the last row to obtain R_1 ensures that $w_R = w_{R_1}$. By construction, the portion of the RC graph in rows with index less than n is valid (meaning its max descent is at most $n-1$). Applying Algorithm 1 to insert at rows $i_1, i_2, \ldots, i_p$ adds back the removed crossings to preserve the weights of each row without adding descents past index n in the permutation, by Lemma 3.4.1. Trimming off row n then yields a valid bounded RC graph $(R', n-1)$ with the desired properties. ☐

Theorem 3.3. *Let w be a permutation with last descent at most n. Let $\mathcal{RC}(w,n)^0$ be the set of bounded RC graphs (R, n) such that $w_R = w$ and row n is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w,n)^0 \to \bigcup_{w \overset{n}{\searrow} w'} \mathcal{RC}(w', n-1)$$

is a bijection.

To prove Theorem 3.3 for $n = 2$, we may characterize the bounded RC graphs $(R, 2)$ such that row 2 is empty and $\mathfrak{m}(w_R) = 2$ as those for permutations $w = s_k s_{k-1} \cdots s_2$ for some $k > 1$. Algorithm 3 in this case consists moves the entirety of the crossings in row 1 to row 2. The procedure then inserts $k-1$ crossings into row 1 in order from left to right, which must have roots $t_q - t_s$ such that $q = 1$. Thus R' is precisely $\{(1,1), (1,2), (1,3), \ldots, (1, k-1)\}$, which is the unique bounded RC graph for the permutation $w' = s_{k-1} s_{k-2} \cdots s_1$ with one row. This establishes the base case.

For the induction step, we require the following lemmas.

Lemma 3.4.3. *Let (R, n) be a bounded RC graph with $n \geq 2$ such that row n is empty. Then if $\mathcal{Z}(R, n) = (R', n-1)$, we have*

$$w_R \overset{n}{\searrow} w_{R'}$$

Proof. Let $w = w_R$ and suppose $\mathfrak{c}_n(w) = m$. By construction, the second to last step of Algorithm 3 yields a bounded RC graph $(\widetilde{R}, n)$ such that if $\widetilde{w} = w_{\widetilde{R}}$, then

$$\widetilde{w}(n) > \widetilde{w}(n+1) < \widetilde{w}(n+2) < \cdots < \widetilde{w}(n+m)$$

and

$$w \xRightarrow{n-1} \widetilde{w}$$

via reflections t_{ab} such that $n \leq b \leq n+m$. Let $w' = w_{R'}$. We also have

$$\widetilde{w} \overset{n}{\Rightarrow} \varphi_{n,N}(w')$$

via only reflections t_{nb} such that $b > n+m$. We claim that

$$w \overset{n}{\Rightarrow} \varphi_{n,N}(w')$$

This can only fail if $\widetilde{w}(n) \neq w(n)$ by the observations above. However, the pipe labeled n, by virtue of the fact that we have $(n, 1), \ldots, (n, m)$ populated, will always lie to the right of pipes $n+1, n+2, \ldots, n+m$ in the wiring diagram for $\widetilde{w}$ or any intermediate stage. Inserting into any row that contained a crossing $(n, n+p)$, there must be a valid empty space to the left of the pipe labeled n since we have deleted these crossings. By virtue of the fact that the leftmost is always chosen, no reflection (i, n) will ever be inserted. This ensures that $\widetilde{w}(n) = w(n)$, as desired. □

Lemma 3.4.4. *Let (R, n) be a bounded RC graph with $n \geq 2$ such that row n is empty. Then*

$$\mathcal{Z}(\mathtt{trim}(R, n)) = \mathtt{trim}(\mathcal{Z}(R, n))$$

Proof. This is almost a trivial observation. In terms of the word of R, $\mathtt{trim}(R, n)$ preserves a suffix. Removing all of the initial roots before trimming is therefore the same as removing the initial roots that still remain after trimming, by the exchange property. Afterwards, in perfoming the insertion algorithm, there is no dependency on rows with a lower index, the modification only proceeds downward in row number. Therefore, trimming the first row at any point only stops the process earlier, and does not change rows with higher index than 1 in the outcome. □

Lemma 3.4.5. *Let w be a permutation with last descent at most n. Let $\mathcal{RC}(w, n)^0$ be the set of bounded RC graphs (R, n) such that $w_R = w$ and row n is empty. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \to \mathcal{RC}(n-1)$$

is injective.

Proof. If $n = 2$ this is clear. Suppose now that $n > 2$ and that the result holds for $n-1$. Let $(R, n) \in \mathcal{RC}(w, n)^0$. If $(R', n) \in \mathcal{RC}(w, n)^0$ is another bounded RC graph such that

$$\mathcal{Z}(R, n) = \mathcal{Z}(R', n)$$

then by Lemma 3.4.4, we have

$$\mathcal{Z}(\mathtt{trim}(R, n)) = \mathcal{Z}(\mathtt{trim}(R', n))$$

hence R and R' agree on all rows except possibly row 1 by the inductive hypothesis. Since $w_R = w_{R'}$, it follows that $R = R'$, establishing injectivity. □

Proof of Theorem 3.3. By the previous lemma, it suffices to show that $\mathcal{Z}$ is surjective. However, this follows from the pigeonhole principle and the well-known transition formula for Schubert polynomials. □

We may extend $\mathcal{Z}$ to an endomorphism $\mathcal{BRC} \to \mathcal{BRC}$ by defining $\mathcal{Z}(R, n) = 0$ if row n in R is not empty. Then we have the following result.

Theorem 3.4. *Let w be a permutation with last descent at most n. Suppose*

$$\mathfrak{S}_w(x_1,\ldots,x_{n-1},0)=\sum_{w'}\mathfrak{S}_{w'}(x_1,\ldots,x_{n-1})$$

Then

$$\mathcal{Z}(\mathcal{S}_w(n))=\sum_{w'}\mathcal{S}_{w'}(n-1)$$

Algorithm 4 Map $\mathcal{Z}_k(R,n)$ zeroing out row k

Input: A bounded RC graph (R,n) with row k empty such that s_k is a descent of w_R.
Output: A bounded RC graph $(R',n-1)$ such that $|R'|=|R|$ and $w_R \overset{k}{\searrow} w_{R'}$.
Let $R_0 \leftarrow R$.
Find the maximal p and sequence $\{(i_m,j_m)\}_{m=1}^{p}$ such that $i_m<k$ and:
  $\mathtt{rt}_{R_{m-1}}(i_m,j_m)=(k+m-1,k+m)$ for each m, where $R_m=R_{m-1}\setminus\{(i_m,j_m)\}$ for each $m\geq 1$.
$R^- \leftarrow R\setminus\{(i_1,j_1),\ldots,(i_p,j_p)\}$.
Let $I\leftarrow(i_1,i_2,\ldots,i_p)$.
$D\leftarrow R^-\cup\{(k,1),(k,2),\ldots,(k,p)\}$.
$D'\leftarrow \textsc{Insert}(D,k-1,I)$ ▷ Using Algorithm 1
$R'\leftarrow\{(i,j)\in D' \mid i<k\}\cup\{(i-1,j)\in D' \mid i>k\}$.
return $(R',n-1)$.

3.5. **Preservation of the Assaf-Schilling Demazure crystal structure.** RC-graphs have a Demazure crystal structure defined by Assaf and Schilling in [1]. Namely, they define operators $e_i:\mathcal{RC}(w)\to\mathcal{RC}(w)\cup\{\emptyset\}$ and $f_i:\mathcal{RC}(w)\to\mathcal{RC}(w)\cup\{\emptyset\}$ for each $i\geq 1$ as follows.

Definition 3.5.1. Consider elements $(i,j)\in R$ and $(i+1,k)\in R$. The pairing algorithm starts with the largest j in row i and pairs (i,j) with $(i+1,k)$ where k is the smallest such that $k\geq j$. If no such k exists, then (i,j) is unpaired. The algorithm continues by considering the next largest j in row i and repeating the process until all elements in row i have been considered.

Define R_i to be the set of unpaired elements in row i after applying the pairing algorithm between rows i and $i+1$, and define L_i to be the set of unpaired elements in row $i+1$. Then $f_i(R)$ is defined by removing the leftmost element (i,j) from R_i and adding the element $(i+1,k)$ to R, where k is the largest value such that $(i+1,k)\notin R$ and $k<j$. If R_i is empty or k does not exist, then $f_i(R)=\emptyset$. Similarly, $e_i(R)$ is defined by removing the rightmost element $(i+1,k)$ from L_i and adding the element (i,j) to R, where j is the smallest value such that $(i,j)\notin R$ and $j>k$. If L_i is empty, then $e_i(R)=\emptyset$.

We may transport this structure to bounded RC graphs as operators $e_i:\mathcal{RC}(w,n)\to\mathcal{RC}(w,n)\cup\{\emptyset\}$ and $f_i:\mathcal{RC}(w,n)\to\mathcal{RC}(w,n)\cup\{\emptyset\}$ by

$$e_i(R,n)=(e_i(R),n)$$

and

$$f_i(R,n)=(f_i(R),n)$$

Definition 3.5.2. An RC graph R has a uniquely associated reduced word $\mathrm{word}(R)=(s_{i_1},\ldots,s_{i_N})$. Recall the Coxeter-Knuth relation $\widetilde{\mathrm{eg}}$ defined by

$$j\ i\ k\sim j\ k\ i$$

and

$$k\ i\ j\sim i\ k\ j$$

if $i<j<k$, and

$$i\ i+1\ i\sim i+1\ i\ i+1$$

For an RC graph R, define $N(R)$ to be the maximum letter in R. Then define

$$\widetilde{\mathrm{eg}}(R)=(N(R)+1-i_1,\ldots,N(R)+1-i_N)$$

Via the Edelman-Greene insertion algorithm, we may define a pair

$$(P, Q)$$

of tableaux of the same shape such that the reading word of P is CK-equivalent to $\widetilde{\mathrm{eg}}(R)$ and Q, the recording tableau, is standard. Define $\texttt{tab}(R)$ to be the tableau of the same shape as Q such that the entry in box (i, j) is the row index of the box in R that was inserted to create box (i, j) in P.

Lemma 3.5.3. *We have that* $\texttt{tab}(R)$ *is a semi-standard Young tableau of shape* $\mathrm{shape}(Q)$*, and the map*

$$R \mapsto \texttt{tab}(R)$$

induces a morphism of crystal graphs into $B(\mathrm{shape}(Q))$*.*

Proof. Suppose R is a highest weight. Then $\texttt{wt}(R)$ is a partition. By the highest weight condition, the longest increasing subsequence of $\widetilde{\mathrm{eg}}(R)$ is of length $\texttt{wt}(R)_1$ and is precisely the first row read in the grid order. By the Edelman-Greene correspondence [6], the first row of P has length equal to the length of the longest increasing subsequence of $\widetilde{\mathrm{eg}}(R)$. Hence, the first row of P has length $\texttt{wt}(R)_1$. By similar reasoning on the subsequent rows, we have that $\mathrm{shape}(P) = \texttt{wt}(R)$. We then have that $\texttt{tab}(R)$ is the unique Yamanouchi tableau of shape $\texttt{wt}(R)$, establishing the claim for highest weights. The general case follows from the rigidity of the Demazure crystal structure [1]; since the function is weight-preserving, it is a morphism of crystal graphs. □

Theorem 3.5. *The distinct Demazure crystals that make up* $\mathcal{RC}(w)$ *are in bijection with the Coxeter-Knuth equivalence classes of reduced words for* $w_0 w w_0$ *via the map* $R \mapsto \widetilde{\mathrm{eg}}(R)$*, with* $R \mapsto \texttt{tab}(R)$ *being a homomorphism of crystal graphs into* $B(\lambda)$*, where* $\lambda = \mathrm{shape}(P(\widetilde{\mathrm{eg}}(R)))$*.*

Proof. By [7, Theorem 4.11], if $e_i(R) \neq \emptyset$, then R and $e_i(R)$ have the same Edelman-Greene insertion tableau. Since conjugation by w_0 is an automorphism of Coxeter-Knuth equivalence, so do $\widetilde{\mathrm{eg}}(R)$ and $\widetilde{\mathrm{eg}}(e_i(R))$. However, the shape □

Definition 3.5.4. Let $\mathcal{RC}(w, n)^0$ be the set of bounded RC graphs (R, n) such that $w_R = w$ and row n is empty. We define a function $\mathcal{Z}_*^R : I(w) \to I(w')$, where $\mathcal{Z}(R, n) = (R', n-1)$ and $w' = w_{R'}$, as follows. Suppose $\mathfrak{c}_n(w) = m$. Let $c = s_{n+m-1} \cdots s_n$. Then there exist unique reflections $t_{a_1 b_1}, \ldots, t_{a_m b_m}$ such that

$$wc^{-1} = w' t_{a_m b_m} \cdots t_{a_1 b_1}$$

and each reflection satisfies $a_i \leq n-1 < b_i$, inducing a Bruhat covering relation. At this base level, there is a sub-RC-graph of each of R. and R', say R'', on which we may partially define the map of inversion sets. That is, if $(i, j) \in R''$, then

$$\mathcal{Z}_*^R(\texttt{rt}_R(i, j)) = \texttt{rt}_{R'}(i, j)$$

In the word of R', the right roots $(a_1, b_1), \ldots, (a_m, b_m)$ occur in a specific order, say $r_1, \ldots, r_m$. On the remaining roots in R, we define

$$\mathcal{Z}_*^R(n, n+m+1-i) = r_i$$

Lemma 3.5.5. *The map* $\mathcal{Z}_*^R : I(w) \to I(w')$ *depends only on* w *and* w'*, not on the choice of* R *(besides the fact that it determines* w'*).*

Proof. This is clear if $\mathfrak{m}(R) < n$. Otherwise, we prove this by induction on $\mathfrak{c}_n(w_R)$. If $\mathfrak{c}_n(w_R) = 1$, then given w' the $(n, n+1)$ root from $I(w)$ can only map to one root in $I(w')$, namely the root (a, b) such that $ws_n = w' t_{ab}$. Now suppose the result holds for $\mathfrak{c}_n(w_R) < m$ for some $m > 1$, and let $\mathfrak{c}_n(w_R) = m$. Let $R_1 = R \setminus \{(i, j)\}$ where $\texttt{rt}_R(i, j) = (n, n+m)$, and let $\mathcal{Z}(R_1, n) = (\widetilde{R}, n-1)$. By the inductive hypothesis, the map $\mathcal{Z}_*^{R_1} : I(w_{R_1}) \to I(w_{\widetilde{R}})$ is independent of the choice of R_1. By construction of the algorithm, we have that $\widetilde{R} = R' \setminus \{(i, k)\}$ for some k. Hence $\mathcal{Z}_*^R$ is determined by $\mathcal{Z}_*^{R_1}$ along with the image of $(n, n+m)$, which is determined by w and w' alone. □

Theorem 3.6. *Let* $w \in S_\infty$ *and* $n > 0$ *be such that* $\mathfrak{m}(w) \leq n$*. Then*

$$\mathcal{Z} : \mathcal{RC}(w, n)^0 \to \bigcup_{w \overset{n}{\searrow} w'} \mathcal{RC}(w', n-1)$$

is an isomorphism of crystal graphs.

Proof. Suppose $\mathcal{Z}(R,n)=(R',n-1)$. We claim that for any $R''\in\mathcal{RC}(w_{R'},n-1)$, we may define a function $T^*_{R''}$ by

$$T^*_{R''}(i,j)=T_{R''}(\mathcal{Z}^R_*(i,j))$$

and $T^*_{R''}$ is an inversion tableau for w_R. Since e_i and f_i preserve the permutation $w_{R'}$, it follows that there is at least a crystal raising operator $e^*_i:\mathcal{RC}(w,n)^0\to\mathcal{RC}(w,n)^0$ defined by

$$T_{e_i(R'')}=e^*_i(T^*_{R''})$$

and similarly for f^*_i. It is clear that these operators are inverses of each other when defined, and hence we have a crystal structure on $\mathcal{RC}(w,n)^0$ transported from $\bigcup_{w\overset{n}{\searrow}w'}\mathcal{RC}(w',n-1)$. Since $\mathcal{Z}$ is weight-preserving bijection and crystals do not have any nontrivial automorphisms that preserve weight, it follows that in fact $e_i=e^*_i$ for all i, and hence $\mathcal{Z}$ is an isomorphism of crystal graphs. □

3.6. **Definition of the ring product.**

Definition 3.6.1. Suppose we have two bounded RC graphs (R_1,m) and (R_2,n). The product of these is a sum of bounded RC graphs defined as follows. Define $\mathscr{P}_{m,n}(R_1,v)$ to be the set of RC graphs R' such that there exists an $N\geq n$ for which

$$\mathcal{Z}^N(R',m+N)=(R_1,m)$$

and $w_{R'}\uparrow^m v$ is a reduced product with $\mathfrak{m}(w_{R'}\uparrow^m v)\leq m+n$. Then the product is defined by

$$(R_1,m)\diamond(R_2,n)=\sum_{R'\in\mathscr{P}_{m,n}(R_1,w_{R_2})}(R'\cup\uparrow^m(R_2),m+n)$$

Theorem 3.7. *The product $\diamond$ turns $\mathcal{BRC}$ into a ring, and $\omega\otimes\alpha:\mathcal{BRC}\to\mathcal{D}\otimes\mathcal{D}$ is a homomorphism of rings.*

Proof. What is in question is associativity. Let $(R,m+n)$ be a bounded RC graph, and let

$$R_{\leq m}=\{(i,j)\in R\mid i\leq m\}$$

also let

$$R_{>m}=\{(i-m,j)\in R\mid i>m\}$$

Then there is a unique R' such that $R_{\leq m}\in\mathscr{P}_{m,n}(R',w_{R_{>m}})$, namely

$$R'=\mathcal{Z}^N(R_{\leq m+N},m+N)$$

for sufficiently large N. Hence there is a well-defined function

$$S_{p,q,r}:\mathcal{RC}(p+q+r)\to\mathcal{RC}(p)\times\mathcal{RC}(q)\times\mathcal{RC}(r)$$

such that

$$S_{p,q,r}(R)=(\mathcal{Z}^N(R_{\leq p+N},p+N),\mathcal{Z}^M(R_{>p,\leq p+q+M},q+M),R_{>p+q})$$

for which

$$(R_1,p)\diamond((R_2,q)\diamond(R_3,r))=\sum_{S_{p,q,r}(R)=(R_1,R_2,R_3)}(R,p+q+r)=((R_1,p)\diamond(R_2,q))\diamond(R_3,r)$$

Associativity follows. $\omega:\mathcal{BRC}\to\mathcal{D}$ is easily seen to be a homomorphism of rings. Proving that $\alpha:\mathcal{BRC}\to\mathcal{D}$ is a homomorphism of rings is nontrivial. However, by Theorem 3.4, we have the identity

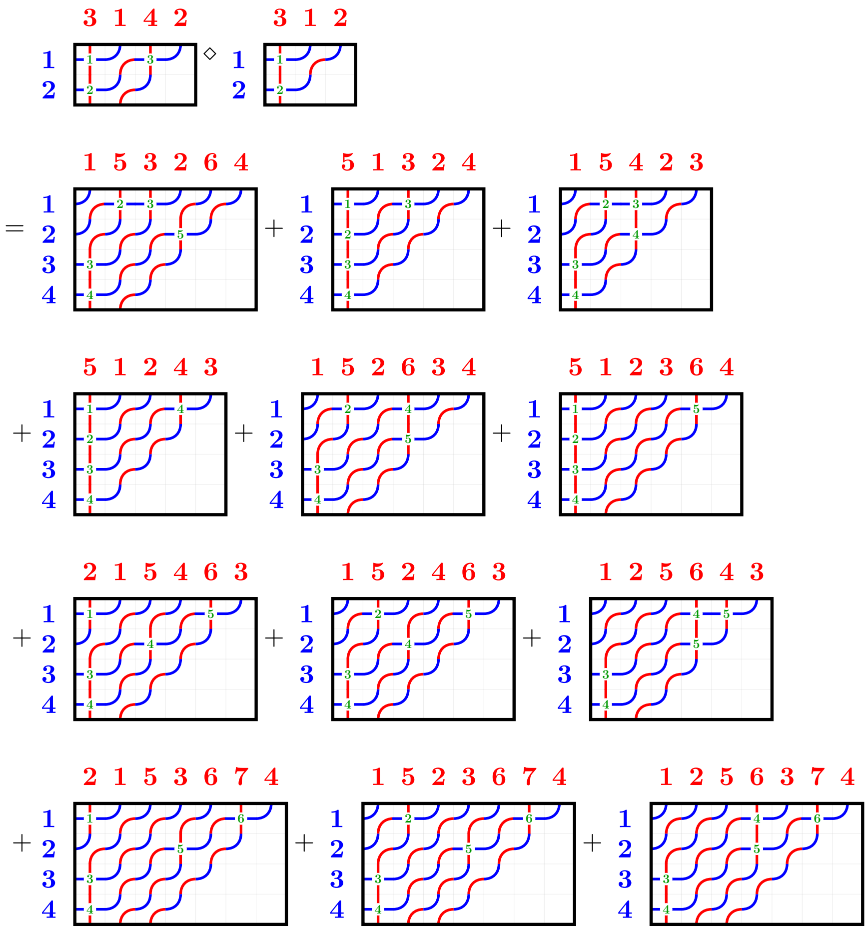
$$(\phi\otimes\phi)\circ S=\Delta\circ\phi$$

where $\Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is the coproduct on $\mathcal{A}$. Since the product in $\mathcal{D}$ is dual to this and

$$(R_1,m)\diamond(R_2,n)=\sum_{S_{m,n}(R)=(R_1,R_2)}(R,m+n)$$

the property of being a homomorphism follows. □

Example 3.6.2.



3.7. **Crystal divided differences and the dominant Pieri formula.** Let R be an RC graph and let $i > 0$ be an integer. We define $\mathfrak{F}^i R$, an element of $\mathcal{RC}$, as follows. Define

$$\mathfrak{F}^i R = 0$$

unless s_i is a right descent of w_R and there exists $(i,j) \in R$ such that $\mathtt{rt}_R(i,j) = (i,i+1)$. If these latter two conditions are satisfied, define

$$R' = R \setminus \{(i,j)\}$$

where $\mathtt{rt}_R(i,j) = (i,i+1)$. If $e_i(R') \neq \emptyset$, define $\mathfrak{F}^i R' = 0$. Otherwise, define

$$\mathfrak{F}^i R = \sum_{p=0}^{\varphi_i(R')} f_i^p R'$$

Lemma 3.7.1. *Let R be an RC graph and let $i > 0$ be an integer. If s_i is not a right descent of w_R, then there is a unique R' such that the coefficient of R in $\mathfrak{F}^i R'$ is 1, and for other R' the coefficient is 0.*

Proof. Suppose s_i is not a right descent of w_R. Assume without loss of generality that $e_i R = \emptyset$. If $(i+1,1) \notin R$, then $(i,1) \notin R$ because s_i is not a right descent of w_R. In that case, let $R' = R \cup \{(i,1)\}$. Then

$$\mathfrak{F}^i R' = R + \text{other terms}$$

If instead $(i+1,1) \in R$, since $e_i(R) = \emptyset$ it follows that $(i,1) \in R$, and if j is the maximum value such that $(i+1,j') \in R$ for all $j' < j$, it follows that $(i,j') \in R$ for all $j' < j$. We must have that $(i,j+1) \notin R$, because $\mathtt{rt}_R(i,j+1) = (i,i+1)$. Setting $R' = R \cup \{(i,j+1)\}$ gives us the result. $\square$

Theorem 3.8. *Suppose $w \in S_\infty$ and $i > 0$. If i is not a right descent of w, then*

$$\mathfrak{F}^i \mathcal{S}_w(n) = 0$$

Otherwise,

$$\mathfrak{F}^i \mathcal{S}_w(n) = \mathcal{S}_{ws_i}(n)$$

Proof. The result is trivial if i is not a right descent of w. Suppose i is a right descent of w. By Lemma 3.7.1, for each RC graph R such that $w_R = ws_i$, there is a unique RC graph R' such that the coefficient of R in $\mathfrak{F}^i R'$ is 1. Since $w_{R'} = w$, the result follows. □

Definition 3.7.2. For a permutation w, define the *principal RC graph* $R^0(w)$ of w to be

$$R^0(w) = \{(i,j) \mid 1 \leq j \leq \mathfrak{c}_i^*(w)\}$$

Proposition 3.7.3. *Let R be an RC graph and let $w = w_R$. Then for any reduced word $(s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ for w, we have*

$$\mathfrak{F}^{i_1}\mathfrak{F}^{i_2}\cdots\mathfrak{F}^{i_m} R = \delta_{R,R^0(w)}\emptyset$$

Let $\mathcal{RC}^C$ be the submodule of $\mathcal{RC}$ consisting of formal sums of RC graphs

$$\sum_R c_R R$$

such that if $e_i R \neq \emptyset$, then $c_{e_i R} = c_R$, and if $f_i R \neq \emptyset$, then $c_{f_i R} = c_R$, for all $i > 0$.

Lemma 3.7.4. *The submodule $\mathcal{RC}^C$ is stable under the crystal divided difference operators $\mathfrak{F}^i$ for all $i > 0$.*

Proof. □

Theorem 3.9. *The crystal divided difference operators satisfy the relations:*

$$\mathfrak{F}^i\mathfrak{F}^i = 0$$

$$\mathfrak{F}^i\mathfrak{F}^j = \mathfrak{F}^j\mathfrak{F}^i \quad \textit{for } |i-j| > 1$$

and on the submodule $\mathcal{RC}^C$,

$$\mathfrak{F}^i\mathfrak{F}^{i+1}\mathfrak{F}^i = \mathfrak{F}^{i+1}\mathfrak{F}^i\mathfrak{F}^{i+1}$$

Proof. By Proposition 3.7.3, these identities hold when applied to any principal RC graph. Ason □

Definition 3.7.5. $\widetilde{s}_i$ is the operator on bounded RC graphs defined by

For an RC graph R with row k empty, suppose k is not a descent of w_R. Let $m > k$ be the smallest integer such that m is a descent of w_R. Define

$$\widetilde{R} = \widetilde{s}_{m-1}\widetilde{s}_{m-2}\cdots\widetilde{s}_k R$$

Then define

$$\mathcal{Z}_k(R,n) = \mathcal{Z}_m(\widetilde{R},n)$$

Definition 3.7.6. Let (R,n) be a bounded RC graph. We define a $\{0,1\}$-valued function $\delta_\mu^w(R,n)$ for a dominant permutation μ and an arbitrary permutation w via the following construction. For integers p,q, let $d[p,q] = s_p s_{p+1}\cdots s_{p+q-1}$ be a product of simple reflections.

Let m be the length of $\mathfrak{c}^*(\mu)$. Initialize $(R_0, n_0) = (R,n)$. For each i from 1 to m:

(1) If $\mathfrak{c}_i^*(\mu) > \mathfrak{c}_i^*(w)$, terminate and set $\delta_\mu^w(R,n) = 0$.
(2) Otherwise, let $k = \mathfrak{c}_i^*(w) - \mathfrak{c}_i^*(\mu)$ and apply the crystal divided difference operator:

$$R' = \mathfrak{F}^{d[\mathfrak{c}_i^*(\mu)+1,\,k]}(R_{i-1})$$

(3) If row $\mathfrak{c}_i^*(\mu)+1$ is not empty in R', terminate and set $\delta_\mu^w(R,n) = 0$.
(4) If the row is empty, define the next state:

$$(R_i, n_i) = \mathcal{Z}_{(\mathfrak{c}_i^*(\mu)+1)}(R', n_{i-1})$$

If the loop completes, for any remaining indices $i > m$ where $\mathfrak{c}_i^*(w) > 0$, incrementally update the graph:

$$R_i = \mathfrak{F}^{d[i+1-m,\,\mathfrak{c}_i^*(w)]}(R_{i-1})$$

Finally, define $\delta_\mu^w(R,n) = 1$ if $(R_N, n_N) = (\emptyset, 0)$ for sufficiently large N, and $\delta_\mu^w(R,n) = 0$ otherwise.

Theorem 3.10. *Let μ be a dominant permutation and let v, w be permutations such that $\ell(\mu)+\ell(v) = \ell(w)$. Then*

$$c_{\mu,v}^w = \sum_{w_R = v} \delta_\mu^w(R)$$

4. Quantum rules

Theorem 4.1. *We have*

$$\mathfrak{S}_u^q(x;y)E_{p,n_k}^q(x;z) = \sum_{(d_1,\ldots,d_n)\in \mathrm{Pie}_{k,p}(u)} \sum_{u\tau_{\mathbf{d}} \xrightarrow{k} w\phi_{\mathbf{d}}} E_{p-F_{n_k}(u,w),n_k-F_{n_k}(u,w)}(y_{P_{n_k}(u,w)};z)q^{d_1}\cdots q^{d_n}\,\mathfrak{S}_w^q(x;y)$$

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