Ajtai commitment expansion

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What we are trying to do

- We start off with a relation made hard to 'break' based on the MSIS problem
- We will show that this relation is equivalent to two other relations, that can be used for folding
- We then bind these two other relations to the Customisable Constraint System (CCS)
- This allows for a folding scheme based on ajtai commitments and compatible with ccs

Ajtai Commitments

- Ajtai commitments allow us to commit to a vector of polynomials
- We commit to an a vector $\overrightarrow{x} \in \mathcal{R}^m$ by multiplying it with a random matrix $\mathbf{A} \in \mathcal{R}_{\boldsymbol{q}}^{\kappa \times \mathbf{m}}$
- $||\overrightarrow{x}||_\infty < B$ where B is the norm bound
- Output of commitment is $cm := \mathbf{A} \cdot \overrightarrow{x} \,\mathbf{mod\ q} \in \mathcal{R}_{\boldsymbol{q}}^{\kappa}$
- This commitment is considered binding because of the assumed hardness of MSIS

Ajtai commitments as a relation

- We define relation $\mathcal{R}^{B}_{MSIS^{\infty}}$ between an ajtati commitment and the $\overrightarrow{x}$
- $\mathcal{R}^{B}_{MSIS^{\infty}} := (pp,\ cm \in \mathcal{R}^{\kappa}_{q}\,;\ \overrightarrow{x} \in \mathcal{R}^{m} : (cm\ =\ \mathbf{A} \cdot \overrightarrow{x}\ \mathbf{mod\ q}) \wedge ||\overrightarrow{x}||_{\infty} < \mathbf{B})$
- $pp\ :=\ (\kappa, m, B, \mathbf{A})$ are the public parameters of the relation
- Public parameters define the 'meta' information of the relation:
    1. The size of the vectors and matrices
    2. The norm limit of $\overrightarrow{x}$
    3. The random matrix $\mathbf{A}$

$\overrightarrow{x} \in \mathcal{R}_q^m$

- We now rewrite $\overrightarrow{x} \in \mathcal{R}_q^m$
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  - $||\overrightarrow{x}||_\infty < B$ and $B < \frac{q}{2}$
  - $\overrightarrow{x} \in \mathcal{R}^m$ can be uniquely represented in $\mathcal{R}_q^m$
  - We define $||\overrightarrow{x}||_{|infty} < B$ as the norm after lifting $\overrightarrow{x} \in \mathcal{R}_q^m to \mathcal{R}$
- We can rewrite our commitment as $\mathcal{R}^B_{MSIS\infty} := (pp,\ cm \in \mathcal{R}_q^\kappa;\ \overrightarrow{x} \in \mathcal{R}_q^m : (cm\ =\ \mathbf{A} \cdot \overrightarrow{x}) \wedge ||\overrightarrow{x}||_\infty < \mathbf{B})$

Coefficient Embeddings and Rotational Matrices

- For $a \in \mathcal{R}_q$, $vec(a) \in \mathbb{Z}_q^d$ represents the vectors of coefficients
- For a vector $\overrightarrow{a} \in \mathcal{R}_q^m$, $vec(\overrightarrow{a}) \in \mathbb{Z}^{m \times d}$ represents the coefficient vectors in $\overrightarrow{a}$
- $fvec(\overrightarrow{a}) \in \mathbb{Z}^{md}$ is the vector that concatenates the rows of $\overrightarrow{a}$
- $\mathrm{Rot}(\mathbf{a}) := (\mathrm{vec}(\mathbf{a}), \mathrm{vec}(\mathbf{X} \cdot \mathbf{a}), \ldots, \mathrm{vec}(\mathbf{X}^{d-1} \cdot \mathbf{a})) \in \mathbb{Z}_q^{d \times d}$.
- For a matrix $\mathbf{A} \in \mathbb{R}_q^{\kappa \times m}$, we define the rotation matrix $\mathrm{Rot}(\mathbf{A}) \in \mathbb{Z}_q^{\kappa d \times md}$ as

$$\mathrm{Rot}(\mathbf{A}) := \begin{bmatrix} \mathrm{Rot}(\mathbf{A}_{1,1}) & \cdots & \mathrm{Rot}(\mathbf{A}_{1,m}) \\ \vdots & \ddots & \vdots \\ \mathrm{Rot}(\mathbf{A}_{\kappa,1}) & \cdots & \mathrm{Rot}(\mathbf{A}_{\kappa,m}) \end{bmatrix}$$

- $\mathrm{fvec}(\mathbf{Af}) = \mathrm{Rot}(\mathbf{A})\mathrm{fvec}(\mathbf{f})$ for any $\mathbf{A} \in \mathbb{R}_q^{\kappa \times m}$ and $\mathbf{f} \in \mathbb{R}_q^m$.

$\overrightarrow{x} \in \mathbb{Z}^{\kappa d}$

- We can uniquely represent $\overrightarrow{x} \in \mathcal{R}_q^m$ as $\overrightarrow{x} \in \mathbb{Z}^{\kappa d}$ by taking $fvec(\overrightarrow{x})$
- $\overline{\mathbf{A}} = rot(\mathbf{A})$
- $\overline{cm}$ is the coefficient embedding of cm
- $\overline{cm} = \overline{\mathbf{A}} \cdot fvec(\overrightarrow{x})$

$$\mathcal{R}^B_{MSIS^\infty} := (pp, \overline{cm} \in \mathbb{Z}^{\kappa d}; \overrightarrow{x} \in \mathbb{Z}^{md} : (\overline{cm} = \overline{\mathbf{A}} \cdot \overrightarrow{x}) \wedge ||\overrightarrow{x}||_\infty < B)$$

Representing $||\overrightarrow{x}||_\infty < B$ as an hadamard product

$$\mathcal{R}^{B}_{\mathsf{MSISProd}} := \left\{ (pp, \overline{\mathsf{cm}} \in \mathbb{Z}^{\kappa d}; \overrightarrow{x} \in \mathbb{Z}^{md} \mid \begin{array}{c} \overline{\mathsf{cm}} = \overline{\mathbf{A}} \cdot \overrightarrow{x} \\ \wedge \, \| \overrightarrow{x} \| \circ \left[\bigcirc_{i=1}^{B-1} (\overrightarrow{x} - \overrightarrow{i}) \circ (\overrightarrow{x} + \overrightarrow{i}) \right] = \overrightarrow{0} \end{array} \right\}$$

- To see this see that the biggest coefficient in any of the x matrices is less than B

$\mathcal{R}^{B}_{cm}$

- We can look at $\overrightarrow{x}$ in two ways
- $\overrightarrow{x}$ is a NTT representation of a $\hat{f} \in \mathcal{R}_q^m$
- $\overrightarrow{x}$ is coefficient embedding of a $\overrightarrow{f} \in \mathcal{R}_q^m$
- The Hadamard product of two NTT representation is equivalent to the multiplication of the two elements
- i.e $\overrightarrow{x} \circ \overrightarrow{x} \cong \hat{f} \circ \hat{f}$

$$\mathcal{R}^{B}_{\mathrm{cm}} := \left\{ (pp, \mathrm{cm} \in \mathcal{R}_q^{\kappa};\ \overrightarrow{f} \in \mathcal{R}_q^m \mid \begin{array}{c} \overline{\mathrm{cm}} = \overline{\mathbf{A}} \cdot \overrightarrow{f} \\ \wedge \, \|\hat{f}\| \circ \left[\bigcirc_{i=1}^{B-1} (\hat{f} - \hat{i}) \circ (\hat{f} + \hat{i}) \right] = \hat{0} \end{array} \right\}$$

$\mathcal{R}^B_{eval}$

- Essentially the same as before, with an added evaluation statement
- We supply the relation with variables and an evaluation of the $\overrightarrow{f}$ at those variable

$$\mathcal{R}^B_{\text{eval}} = \left\{ (pp;\ (r, v, cm) \in \mathcal{R}_q^{\log m} \times \mathcal{R}_q \times \mathcal{R}_q^{\kappa};\ \overrightarrow{f} \in \mathcal{R}_q^m) \mid \begin{array}{c} (pp;\ cm;\ \overrightarrow{f}) \in \mathcal{R}^B_{cm} \\ \wedge\, \mathsf{mle}[\hat{f}](\overrightarrow{r}) = v \end{array} \right\}$$

Let's take this to CCS

- We introduce an insane amount of notation
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  - Public Paramers ($\mathbf{pp}$) $:=(n_r,\ n_c,\ t,\ n_s,\ \deg, l_{in})$
  - $\overline{\mathcal{R}}$ is an arbitrary ring
  - $\mathbb{i}$ consists of
    1. t matrices $M_1..M_t \in \overline{\mathcal{R}}^{n_r \times n_c}$ with $\mathcal{O}(n_r + n_c)$ non-zero entries
    2. n_s multisets $S_1..S_{n_s} \subseteq [t]$ with $|S_i| < \deg$ for all $i \in [n_s]$
    3. n_s scalars $cn_1, ..., cn_s \in \overline{\mathcal{R}}$
- We then introduce the relation $\mathcal{R}_{\text{ccs}}$
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  1. $\mathbf{pp}_{\text{ccs}} := (\mathbf{pp}, \mathbb{i})$
  2. $(\mathbf{pp}_{\text{ccs}}, \mathbb{x} \in \overline{\mathcal{R}}^{l_{\text{in}}}, \mathbb{w} \in \overline{\mathcal{R}}^{n_c - l_{\text{in}} - 1})$
  3. $\bar{\mathbf{z}} := (\mathbb{x}, 1, \mathbb{w}) \in \mathcal{R}^{n_c}$
  4. The condition for the relation is $\sum_{i=1}^{n_s} c_i \cdot \bigcirc_{j \in S_i}(M_j \cdot \vec{z}) = 0^{n_r}$

$$\mathcal{R}_{\text{ccs}} := \left\{ (\mathbf{pp}_{\text{ccs}}, \mathbb{x} \in \overline{\mathcal{R}}^{l_{\text{in}}}, \mathbb{w} \in \overline{\mathcal{R}}^{n_c - l_{\text{in}} - 1}) \text{ such that } \sum_{i=1}^{n_s} c_i \cdot \bigcirc_{j \in S_i}(M_j \cdot \vec{z}) = 0^{n_r} \right\}$$

Let's bind $\mathcal{R}_{\mathsf{cm}}$ and $\mathcal{R}_{\mathsf{ccs}}$ together!

- We introduce the gadget matrix
- $l := \frac{m}{n_c}$
- $\mathbf{G} := \mathsf{I}_{n_c} \otimes [1, B, \ldots B^{l-1}] \in \mathbb{Z}_q^{n_c \times m}$
- We then define $\mathcal{R}^B_{\mathsf{cmcss}}$

We then define

$$\mathcal{R}^B_{\mathsf{cmcss}} := \left\{ \begin{array}{l} (\mathsf{pp}, \mathbb{x} := (cm \in \mathcal{R}_q^{\kappa}, \mathbb{x}_{\mathsf{ccs}} \in \mathcal{R}_q^{l_{\mathsf{in}}})); \mathbb{w} := (\overrightarrow{\mathbf{f}} \in \mathcal{R}_q^{m}, \mathbb{w}_{\mathsf{ccs}} \in \mathcal{R}^{n-l_{\mathsf{in}}-1}) \text{ s.t.} \\ (\mathsf{pp}, \mathsf{cm}, \overrightarrow{\mathbf{f}} \in \mathcal{R}^B_{\mathsf{cm}}) \wedge (\mathsf{pp}_{\mathsf{ccs}}, \mathbb{x}_{\mathsf{ccs}}; \mathbb{w}_{\mathsf{ccs}}) \in \mathcal{R}_{\mathsf{ccs}} \wedge (\mathbf{z}_{\mathsf{ccs}} = \mathbf{G}\overrightarrow{\mathbf{f}}) \end{array} \right\}$$

$$\overline{\mathbf{z}}_{\mathsf{ccs}} := (\mathbb{x}_{\mathsf{ccs}}, 1, \mathbb{w}_{\mathsf{ccs}}) \in \mathcal{R}^{n_c}$$

$\mathcal{R}^{B}_{\text{lccs}}$

- We take the mutltilinear ccs constraint matrices and provide some values and give expected evaluations
- We first make sure that the number of constraints n_r as the same as the size of our Ajtai commitment vector m
- $\overrightarrow{\mathbf{r}} \in \mathcal{R}_q^{\text{log m}}$ is the values to be evaluated
- $[\mathbf{u}_i]_{i=1}^{t}$ are the expected evaluations
- $\mathcal{R}\text{lcss} := (\text{pp}_{\text{ccs}}, (\overrightarrow{\mathbf{r}}, [\mathbf{u}_i]_{i=1}^{t}, \varkappa_{\text{ccs}}, \mathbf{h}))$
- For all $i \in [t]$ it holds that $\mathbf{u}_i = \sum_{\overrightarrow{\mathbf{b}} \in \{0,1\}^{log\ n_c}} \text{mle}[M_i](\overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{b}}) \cdot \text{mle}[\mathbf{z}_{\text{ccs}}](\overrightarrow{\mathbf{b}})$

$\mathcal{R}^B_{\text{evalccs}}$

- We can now combine $\mathcal{R}^B_{eval}$ and $\mathcal{R}^B_{\text{lccs}}$ to yield our final relation $\mathcal{R}^B_{\text{evalccs}}$

$$\mathcal{R}^B_{\text{evalccs}} = \left\{ \begin{array}{l} (\mathsf{pp}, \mathbb{x} := (\overrightarrow{\mathbf{r}} \in \mathcal{R}_q^{\log\ m},\ cm,\ \mathbf{v},\ [\mathbf{u}_i]_{i=1}^t,\ \mathbb{x}_{\text{ccs}}, \mathbf{h})); \\ \mathbb{w} := (\overrightarrow{\mathbf{f}} \in \mathcal{R}_q^m, \mathbb{w}_{\text{ccs}} \in \mathcal{R}^{n-l_{\text{in}}-1})\ \text{s.t.} \\ (\mathbf{z}_{\text{ccs}} = \mathbf{G}\,\overrightarrow{\mathbf{f}}) \wedge (\mathsf{pp}_{\text{cm}}, (\text{cm}, \overrightarrow{\mathbf{r}}, \mathbf{v});\ \overrightarrow{\mathbf{f}}) \in \mathcal{R}^B_{eval} \\ \wedge\ (\mathsf{pp}_{\text{ccs}}, (\overrightarrow{\mathbf{r}}_{\text{ccs}}, [\mathbf{u}_i]_{i=1}^t\ \mathbb{x}_{\text{ccs}}, \mathbf{h});\ \mathbb{w}_{\text{ccs}}) \in \mathcal{R}_{\text{lccs}} \end{array} \right\}$$