IDENTIFYING 1-RECTIFIABLE MEASURES IN CARNOT GROUPS

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ABSTRACT. We continue to develop a program in geometric measure theory that seeks to identify how measures in a space interact with canonical families of sets in the space. In particular, extending a theorem of the first author and R. Schul in Euclidean space, for an arbitrary *locally finite Borel measure* in an arbitrary *Carnot group*, we develop tests that identify the part of the measure that is carried by rectifiable curves and the part of the measure that is singular to rectifiable curves. Our main result is entwined with an extension of the *Analyst's Traveling Salesman Theorem*, which characterizes subsets of rectifiable curves in $\mathbb{R}^2$ (P. Jones, 1990), in $\mathbb{R}^n$ (K. Okikolu, 1992), or in an arbitrary Carnot group (the second author) in terms of local geometric least squares data called *Jones' β-numbers*. In a secondary result, we implement the Garnett-Killip-Schul construction of a doubling measure in $\mathbb{R}^n$ that charges a rectifiable curve in an arbitrary *complete, quasiconvex, doubling metric space*.

CONTENTS

1. INTRODUCTION

Rectifiability is a concept in geometric measure theory that supplies a finer notion of regularity of measure than dimension [9, 56]. Given any metric space X, family $\mathcal{A}$ of

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Borel subsets of X, and Borel measure μ on X, we say that μ *charges* $\mathcal{A}$ if $\mu(A) > 0$ for some $A \in \mathcal{A}$. We also say that μ is *carried by* $\mathcal{A}$ if

$$\mu\left(X \setminus \bigcup_1^\infty A_i\right) = 0 \quad \text{for some sequence } A_1, A_2, \cdots \in \mathcal{A}.$$

At the other extreme, we say that μ is *singular to* $\mathcal{A}$ if $\mu(A) = 0$ for every $A \in \mathcal{A}$. Rectifiable measures are those that are carried by canonical families of lower dimensional sets such as rectifiable curves, Lipschitz graphs, or smooth submanifolds. In particular, a measure μ is said to be *(countably) 1-rectifiable* if μ is carried by rectifiable curves and *purely 1-unrectifiable* if μ is singular to rectifiable curves [13, 32].

The *(upper) Hausdorff dimension* of μ is the infimum of all $q \geq 0$ such that μ is carried by sets of Hausdorff dimension q. Every 1-rectifiable measure has Hausdorff dimension at most 1, as rectifiable curves have Hausdorff dimension 0 or 1, but the converse is not true. An archetypical example of a purely 1-unrectifiable measure of Hausdorff dimension 1 is the restriction $\mathcal{H}^1 \llcorner E$ of the 1-dimensional Hausdorff measure $\mathcal{H}^1$ to any self-similar Cantor set $E \subset \mathbb{R}^n$ of similarity dimension 1 (see [39]). For examples of 1-rectifiable measures on $\mathbb{R}^n$ with Hausdorff dimension $0 < q < 1$, see [16, 34, 53].

Under the *a priori* restriction $\mu \ll \mathcal{H}^m$, i.e. for measures such that $\mu(E) = 0$ whenever $\mathcal{H}^m(E) = 0$, there is a rich theory of *m-rectifiable measures* on $\mathbb{R}^n$ that are carried by Lipschitz images of $\mathbb{R}^m$; see [54] for an exposition of results through the last century and [8, 23, 63] for more recent developments. For emphasis, we note that "absolutely continuous" m-rectifiable measures are "top dimensional": if $\mu \neq 0$ is m-rectifiable and $\mu \ll \mathcal{H}^m$, then μ has Hausdorff dimension m. To read about the emerging theory of higher-order rectifiability, i.e. measures carried by $C^{k,\alpha}$ submanifolds, see [5, 29, 35, 60]. There is also much interest in understanding the rectifiability of sets and measures in non-Euclidean metric spaces; see [3, 4, 17, 19, 42, 55, 57] for a short sample.

It turns out that detecting m-dimensional rectifiability is more subtle for measures of Hausdorff dimension less than m than it is for measures of Hausdorff dimension m. Pointwise characterizations of locally finite measures on X that are carried by a family $\mathcal{A}$ (without restriction on dimension, doubling properties, or null sets of μ!) are presently available in two situations: (i) for measures on $\mathbb{R}^n$ carried by rectifiable curves [15], and (ii) for measures on $\mathbb{R}^n$ carried by m-dimensional Lipschitz graphs [11]. These results are made possible by a thorough understanding of subsets of rectifiable curves or Lipschitz graphs in $\mathbb{R}^n$ and the incorporation of ideas from harmonic analysis. More on this below.

In this paper, extending the main theorem of [15] for measures in $\mathbb{R}^n$, we identify the 1-rectifiable and purely 1-unrectifiable parts of an arbitrary locally finite measure on an arbitrary Carnot group. To be concrete, let G be a step s Carnot group, equipped with a Hebisch-Sikora norm (see §2). For every locally finite Borel measure μ, we define the *lower 1-density* $\underline{D}^1(\mu, \cdot) : G \to [0, \infty]$ by

$$(1.1) \qquad \underline{D}^1(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} \quad \text{for all } x \in G,$$

where $B(x, r)$ is the closed ball with center $x \in G$ and radius $r > 0$. Further, we define the *density-normalized Jones function* $J^*(\mu, \cdot) : G \to [0, \infty]$ by

$$(1.2) \qquad J^*(\mu, x) := \sum_{\substack{Q \in \Delta \\ \text{side } Q \leq 1}} \beta^*(\mu, Q)^{2s} \operatorname{diam} Q \, \frac{\chi_Q(x)}{\mu(Q)} \quad \text{for all } x \in G,$$

where Δ is a fixed system of "dyadic cubes" for G and $\beta^*(\mu, Q)$ is a certain *anisotropic* measurement of the deviation of μ in a neighborhood of Q from being a measure supported on a horizontal line in G based on the *stratified β numbers* of [49]. We postpone the precise definitions of Δ and $\beta^*(\mu, Q)$ to §§2 and 3, but remark that the definition of $\beta^*(\mu, Q)$ involves the step of G and recall that $s = 1$ when $G = \mathbb{R}^n$. Further, let us note that horizontal lines are the natural class of tangents of rectifiable curves in Carnot groups. Thus, $\beta^*(\mu, Q)$ may be viewed as quantifying the likelihood that μ has a 1-dimensional tangent at points near Q. When $\mu(Q) > 0$ and $\beta^*(\mu, Q) = 0$, the existence of tangents is certain; however, when $\beta^*(\mu, Q)$ is large, the existence of tangents is less likely. Here is our main result:

Theorem 1.1. *Each locally finite Borel measure μ on G admits a unique decomposition $\mu = \mu_{\text{rect}} + \mu_{\text{pu}}$ as a sum of locally finite Borel measures such that μ_{rect} is 1-rectifiable and μ_{pu} is purely 1-unrectifiable. Moreover, the component measures μ_{rect} and μ_{pu} are identified by the pointwise behavior of $\underline{D}^1(\mu, \cdot)$ and $J^*(\mu, \cdot)$:*

$$(1.3) \qquad \mu_{\text{rect}} = \mu \llcorner \left\{ x \in G \, : \, \underline{D}^1(\mu, x) > 0 \text{ and } J^*(\mu, x) < \infty \right\},$$

$$(1.4) \qquad \mu_{\text{pu}} = \mu \llcorner \left\{ x \in G \, : \, \underline{D}^1(\mu, x) = 0 \text{ or } J^*(\mu, x) = \infty \right\}.$$

Existence and uniqueness of the decomposition $\mu = \mu_{\text{rect}} + \mu_{\text{pu}}$ is an easy variation on the usual proof of the Lebesgue decomposition theorem (see Lemma 2.1). The content of Theorem 1.1 is the identification of the rectifiable and purely unrectifiable components of the measure given by (1.3) and (1.4). The following consequences are immediate.

Corollary 1.2. *A locally finite Borel measure μ on G is 1-rectifiable if and only if $\underline{D}^1(\mu, x) > 0$ and $J^*(\mu, x) < \infty$ at μ-a.e. $x \in G$.*

Corollary 1.3. *A locally finite Borel measure μ charges a rectifiable curve if and only if there exists $E \subset G$ with $\mu(E) > 0$ such that $\underline{D}^1(\mu, x) > 0$ and $J^*(\mu, x) < \infty$ for all $x \in E$.*

Underpinning the main theorem is a characterization of subsets of rectifiable curves, with estimates on the length of the shortest curve containing a given set, usually called the *analyst's traveling salesman theorem*. First established in $\mathbb{R}^n$ by Jones [43], when $n = 2$, and by Okikiolu [59], when $n \geq 3$, the analyst's traveling salesman theorem was recently extended to arbitrary Carnot groups by the second author [49] (for earlier necessary or sufficient conditions, see [20, 33, 44, 50, 51]). A key insight in [49] is that to obtain a full characterization of subsets of rectifiable curves, with effective estimates on length, the local deviation of the set from a horizontal line should incorporate distance in each layer

of the Carnot group. Following [49], for any nonempty set $E \subset G$ and ball $B(x, r)$, define the *stratified β number* for $E \cap B(x, r)$ by setting

$$(1.5) \qquad \beta_E(x, r)^{2s} = \inf_L \sum_{i=1}^s \sup_{z \in E \cap B(x,r)} \left(\frac{d_i(\pi_i(z), \pi_i(L))}{r} \right)^{2i},$$

where L ranges over all horizontal lines in G, $\pi_i : G \to G_i$ is the projection of G onto a layer $G_i = G/G^{(i+1)}$ of G (see §2), and $d_i(x, A) = \inf\{d_i(x, a) : a \in A\}$ for some choice of metric d_i associated to a Hebisch-Sikora norm on G_i. When $G = \mathbb{R}^n$, the step $s = 1$, horizontal lines are 1-dimensional affine subspaces (i.e. tangent lines to rectifiable curves), π_1 is the identity, and the stratified β number reduces to the usual Jones β number.

Theorem 1.4 (see [49, Theorem 1.5]). *Let G be a step s Carnot group with Hausdorff dimension q. For any set $E \subset G$, define the quantity*

$$(1.6) \qquad \beta(E) = \int_0^\infty \int_G \beta_E(x, r)^{2s} \operatorname{diam} B(x, r) \frac{dx}{r^q} \frac{dr}{r}.$$

Then E lies in a rectifiable curve if and only if $\operatorname{diam} E + \beta(E)$ is finite. More precisely, there exists a constant $C > 1$ depending only on G and its underlying metrics d_i so that:

(1) If Γ is any curve containing E, then $\operatorname{diam} E + \beta(E) \le C\mathcal{H}^1(\Gamma)$.
(2) If $\operatorname{diam} E + \beta(E) < \infty$, then there exists a curve Γ containing E for which $\mathcal{H}^1(\Gamma) \le C(\operatorname{diam} E + \beta(E))$.

Alternatively, Theorem 1.4 holds with the number $\beta_E(x, r)$ replaced by the quantity

$$(1.7) \qquad \inf_L \inf\{\varepsilon > 0 : E \cap B(x, r) \subset L \cdot \delta_r(B_{\mathbb{R}^n}(\varepsilon^s))\},$$

where $B(x, r)$ is the ball in G, $B_{\mathbb{R}^n}(\varepsilon^s)$ is a Euclidean ball about the origin on underlying manifold, and ε represents the "width" of a tubular neighborhood $L \cdot \delta_r(B_{\mathbb{R}^n}(\varepsilon^s))$ of the horizontal line L, formed using the group multiplication, the group dilation, and the step of the group. See [49, Proposition 1.6].

To promote Theorem 1.4 to a characterization of 1-rectifiable measures on G, we need to first extend the algorithm for constructing a rectifiable curve through E when $\beta(E) < \infty$, which traces back to [43] when $G = \mathbb{R}^n$ and to [33] when G is the (first) Heisenberg group, to an algorithm for drawing a curve through the Hausdorff limit of a sequence (X_k) of point clouds. In the original setting of the analyst's traveling salesman theorem, we can simply take (X_k) to be a *nested* sequence of 2^{-k}-nets for E. However, in the setting of the main theorem, when trying to build a rectifiable curve charged by μ, we only know how to locate families X_k of 2^{-k}-separated points that are nearby, but not necessarily on a set with positive measure, and we must allow X_k to *float* as $k \to \infty$. This issue was resolved when $G = \mathbb{R}^n$ by the first author and Schul [15] by introducing "extensions" to "bridges" and reproving Jones' traveling salesman algorithm from first principles. In Appendix A, we integrate ideas from [15] and [49] and establish a flexible traveling salesman algorithm in arbitrary Carnot groups (see Propsition A.1). There are additional technical challenges along the way. To name one, the numbers $\beta^*(\mu, Q)$ appearing in Theorem 1.1 are designed

so that we can extract enough data points lying nearby a set with positive measure to which we can apply the traveling salesman algorithm. In [15], the extraction process involves a nice idea of Lerman [48]: convexity of the distance of a point to a Euclidean line L and Jensen's inequality controls the distance of the μ-center-of-mass z_Q in a bounded window Q to the line L. Unfortunately, we cannot use this observation in a higher step Carnot groups, and in §4, we must employ a different argument using the Chebyshev inequality.

Our methods also lead to a new necessary condition for 1-rectifiable measures such that $\mu \ll \mathcal{H}^1$ in terms of β numbers for balls on G. For any locally finite Borel measure μ on G and ball $B(x, r)$, define the *homogeneous stratified β number* for $\mu \llcorner B(x, r)$ by setting

$$(1.8) \qquad \beta(\mu, x, r)^{2s} = \inf_L \sum_{i=1}^{s} \int_{z \in E \cap B(x,r)} \left(\frac{d_i(\pi_i(z), \pi_i(L))}{r} \right)^{2i} \frac{d\mu(x)}{r},$$

where as usual L ranges over all horizontal lines. Homogeneous refers to the normalization $r^{-1} \, d\mu$ of μ in the integral, which is the natural one for measures satisfying $\mu(B(x, r)) \sim r$.

Theorem 1.5. *Let G be a step s Carnot group. A locally finite Borel measure μ on G is 1-rectifiable and $\mu \ll \mathcal{H}^1$ only if*

$$(1.9) \qquad 0 < \liminf_{r \downarrow 0} \frac{\mu(B(x, r))}{2r} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{2r} < \infty \quad \text{at } \mu\text{-a.e. } x \in G$$

and

$$(1.10) \qquad J(\mu, x) = \int_0^1 \beta(\mu, x, r)^{2s} \frac{dr}{r} < \infty \quad \text{at } \mu\text{-a.e. } x \in G.$$

Recall the function J^* defined in (1.2). To distinguish between J and J^*, we call $J(\mu, x)$ the *ordinary Jones function*. The numbers $\beta(\mu, x, r)$ appearing the definition of J were introduced by David and Semmes [24, 27] in their work connecting *uniform rectifiability* and *boundedness of singular integral operators*. It would be interesting to know whether the techniques of [8], [62], or [31] could be used to prove the sufficiency of (1.9) and (1.10) for 1-rectifiability. Answering a question in [15], Martikainen and Orponen [52] produce an example, which shows that the ordinary Jones function *cannot be used* to characterize rectifiability of Radon measures that are non-doubling in the sense that $\limsup_{r \downarrow 0} \mu(B(x, 2r))/\mu(B(x, r)) = \infty$ on a set of positive measure. It is only by using the stratified, anisotropic beta numbers $\beta^*(\mu, Q)$ defined in §3 that we are able to identify the 1-rectifiable and purely 1-unrectifiable parts of arbitrary measures on G.

Even in the presence of doubling, the ordinary Jones function $J(\mu, x)$ may not be suited to detecting rectifiability. In [34], Garnett, Killip, and Schul produce a family of measures μ on $\mathbb{R}^n$ that are doubling in the strong sense that

$$(1.11) \qquad 0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty \quad \text{for all } x \in \mathbb{R}^n \text{ and all } r > 0$$

and charge rectifiable curves. In fact, their measures are carried by rectifiable curves, but are singular with respect to the Hausdorff measure $\mathcal{H}^1$ and (see [14]) have both

$J(\mu, x) = \infty$ μ-a.e and $\underline{D}^1(\mu, x) = \infty$ μ-a.e. In this context, on arbitrary metric spaces, Azzam and Morgoglou [6] characterize 1-rectifiable doubling measures with σ-*compact connected supports*, but leave open the question of existence of such measures. To close the paper, we extend the Garnett-Killip-Schul existence theorem to a broad class of metric spaces, including arbitrary Carnot groups and Riemannian manifolds.

Theorem 1.6. *If X is a complete, quasiconvex, doubling metric space, then there exists a doubling measure ν on X with $\operatorname{spt} \nu = X$ such that ν charges a rectifiable curve.*

It is still an open problem to characterize subsets of rectifiable curves in an arbitrary Banach or metric space. See [10, 28, 36] for some partial results and discussion of the main difficulties. On the other hand, Schul [61] successfully reformulated the analyst's traveling salesman problem to hold in an arbitrary Hilbert space and Naples [58] has implemented a version of Theorem 1.1 for pointwise doubling measures on the sequence space ℓ_2. Also see [46]. Progress on traveling salesman theorems for higher-dimensional objects has been made in [7, 12, 40, 64].

The rest of the paper is arranged as follows. In §2, we collect background results in geometric measure theory and metric geometry, including definitions of Hausdorff and packing measures, metric cubes, and Carnot groups. In §3, we define the anisotropic, stratified beta numbers $\beta^*(\mu, Q)$. In §4, we show how positivity of the lower density $\underline{D}^1(\mu, x)$ and finiteness of the Jones function $J^*(\mu, x)$ for $x \in A$ yield rectifiability of $\mu \llcorner A$. In §5, we show that J^* is locally integrable on any rectifiable curve, which yields necessary conditions for 1-rectifiability. The proof of Theorems 1.1 and 1.5 are recorded in §6 using results from §§4 and 5. The proof of Theorem 1.6, in §7, may be read independently of §§3–6. The extension of the analyst's traveling salesman algorithm to floating point clouds in a Carnot group is deferred to Appendix A.

2. Preliminaries

2.1. **Implicit constants.** When working on a fixed metric space X (in §§3–6) on a Carnot group G, in §7 on a quasiconvex doubling space X), we may write $a \lesssim b$ to indicate that $a \leq Cb$ for some positive and finite constant C that may depend on X, including its metric and dimensions, but (without further qualification) is otherwise independent of a choices of particular sets or measures on X. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We may specify the dependence of implicit constants on additional parameters $c, d, \dots$ by writing $a \lesssim_{c,d,\dots} b$ and $a \sim_{c,d,\dots} b$.

2.2. **Measures and the identification problem.** To set our conventions, we recall that a *measurable space* $(X, \mathcal{M})$ is a nonempty set X paired with a σ-algebra $\mathcal{M}$ on X, i.e. a nonempty collection of subsets of X that is closed taking complements and countable unions; a *measure* on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$ whenever $A_1, A_2, \dots \in \mathcal{M}$ are pairwise disjoint. In particular, a *Borel measure* μ on a metric space X is a measure defined on some measurable space $(X, \mathcal{M})$ such that $\mathcal{M}$ contains every Borel set in X. Given a measure μ on $(X, \mathcal{M})$

and a set $E \in \mathcal{M}$, the *restriction* of μ to E is the measure $\mu \llcorner E$ defined by the rule $\mu \llcorner E(A) = \mu(A \cap E)$ for all $A \in \mathcal{M}$. We need the following convenient form of the Lebesgue decomposition theorem; a detailed proof is written in the appendix of [16].

Lemma 2.1. *Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{N}$ be a nonempty collection of sets in $\mathcal{M}$. For every σ-finite measure μ on $(X, \mathcal{M})$, there is a unique decomposition $\mu = \mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$ as a sum of measures on $(X, \mathcal{M})$ such that $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$ in the sense that $\mu_{\mathcal{N}}(X \setminus \bigcup_1^{\infty} N_i) = 0$ for some $N_1, N_2, \cdots \in \mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}(N) = 0$ for every $N \in \mathcal{N}$. Moreover, there exists a set $A \in \mathcal{M}$ that is a countable union of sets in $\mathcal{N}$ such that $\mu_{\mathcal{N}} = \mu \llcorner A$ and $\mu_{\mathcal{N}}^{\perp} = \mu \llcorner X \setminus A$. If A' is another set with this property, then $\mu(A \setminus A') + \mu(A' \setminus A) = 0$.*

Remark 2.2. The proof of Lemma 2.1 is abstract and does not provide any concrete method to produce sets $N_1, N_2, \cdots \in \mathcal{N}$ such that $\mu_{\mathcal{N}}(X \setminus \bigcup_1^{\infty} N_i) = 0$. The *identification problem* (see [9]) is to find pointwise defined properties $P(\mu, x)$ and $Q(\mu, x)$ such that

$$\mu_{\mathcal{N}} = \mu \llcorner \{x \in X : P(\mu, x) \text{ holds}\} \quad \text{and} \quad \mu_{\mathcal{N}}^{\perp} = \mu \llcorner \{x \in X : Q(\mu, x)\}$$

for every (locally) finite measure μ on X. An ideal solution should involve the geometry of the space X and the sets in $\mathcal{N}$.

On a metric space X, we let $U(x, r)$ and $B(x, r)$ denote the open and closed balls with center $x \in X$ and radius $r > 0$, respectively. Let $E \subset X$ and let $\delta > 0$. A δ-*cover* of E is a finite or infinite sequence of sets $E_1, E_2, \cdots \subset X$ such that $E \subset \bigcup_i E_i$ and $\operatorname{diam} E_i \leq \delta$ for all i, where $\operatorname{diam} A$ denotes the diameter of a set A. A δ-*packing* in E is a finite or infinite sequence $B_1, B_2, \ldots$ of pairwise disjoint closed balls centered in E such that $2 \operatorname{rad} B_i \leq \delta$ for all i, where $\operatorname{rad} B$ denotes the radius of a ball B. For any $E \subset X$, $s \geq 0$, and $\delta > 0$, we define

$$\mathcal{H}_{\delta}^s(E) = \inf \left\{ \sum_i \operatorname{diam} E_i : E_1, E_2, \ldots \text{ is a } \delta\text{-cover of } E \right\},$$

$$\mathcal{H}^s(E) = \lim_{\delta \downarrow 0} \mathcal{H}_{\delta}^s(E) = \sup_{\delta > 0} \mathcal{H}_{\delta}^s(E),$$

$$P_{\delta}^s(E) = \sup \left\{ \sum_i (2 \operatorname{rad} B_i)^s : B_1, B_2, \ldots \text{ is a } \delta\text{-packing in } E \right\},$$

$$P^s(E) = \lim_{\delta \downarrow 0} P^s(E) = \inf_{\delta > 0} P^s(E),$$

$$\mathcal{P}^s(E) = \inf \left\{ \sum_i P^s(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

We call $\mathcal{H}^s$ the *s-dimensional Hausdorff measure* and call $\mathcal{P}^s$ the *s-dimensional packing measure*; both $\mathcal{H}^s$ and $\mathcal{P}^s$ are Borel regular metric outer measures on X, and in particular, $\mathcal{H}^s$ and $\mathcal{P}^s$ are measures when restricted to the σ-algebra of Borel sets. The auxiliary quantity P^s is called the *s-dimensional packing premeasure*. We caution the reader that the premeasure P^s is generally not an outer measure—it is monotone, but is not countably

subadditive. Note that we have adopted the "radius" definition of the packing measure instead of the "diameter" definition. The next estimate (valid on any metric space!) is a special case of [22, Theorem 3.16].

Theorem 2.3 (see Cutler [22])**.** *Let μ be a finite Borel measure on a metric space X, let $E \subset X$ be Borel, and let $s > 0$. If $a \leq \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) \leq b$ for all $x \in E$, then*

$$a\mathcal{P}^s(E) \leq \mu(E) \leq 2^s b \mathcal{P}^s(E),$$

where we take the left hand side to be 0 if $a = 0$ or $\mathcal{P}^s(E) = 0$ and take the right hand side to be ∞ if $b = \infty$ or $\mathcal{P}^s(E) = \infty$.

We can now use Cutler's theorem to solve an instance of the identification problem.

Corollary 2.4. *Let X be a metric space, let $s > 0$, and let $\mathcal{N}$ be the collection of all Borel sets $E \subset X$ with $\mathcal{P}^s(E) < \infty$. For every Borel measure μ on X such that μ is finite on bounded sets, the parts $\mu_\mathcal{N}$ carried by $\mathcal{N}$ and $\mu_\mathcal{N}^\perp$ singular to $\mathcal{N}$ (see Lemma 2.1) are identified pointwise by the positivity of the lower s-density:*

$$\mu_\mathcal{N} = \mu \llcorner \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) > 0\},$$

$$\mu_\mathcal{N}^\perp = \mu \llcorner \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) = 0\}.$$

Proof. Fix any base point $x_0 \in X$. The set $A = \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) > 0\}$ can be written as a countable union of sets

$$A_{k,l} = \{x \in B(x_0, l) : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) > 1/k\},$$

where k and l range over all positive integers. Using Cutler's theorem, we determine that $\mathcal{P}^s(A_{k,l}) \leq k \, \mu(A_{k,l}) \leq k \, \mu(B(x_0, l)) < \infty$ for each k and l. Therefore, $\mu \llcorner A$ is carried by sets of finite $\mathcal{P}^s$ measure. Similarly, let $B = \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) = 0\}$ and suppose $\mathcal{P}^s(E) < \infty$. Then

$$\mu \llcorner B(E) = \lim_{l \to \infty} \mu \llcorner (B \cap U(x_0, l))(E) \leq 2^s \cdot 0 \cdot \mathcal{P}^s(E) = 0,$$

by continuity from below and the upper bound in Cutler's theorem with $b = 0$. Thus, $\mu \llcorner B$ is singular to sets of finite $\mathcal{P}^s$ measure. Clearly $\mu = \mu \llcorner A + \mu \llcorner B$. By uniqueness of the decomposition in Lemma 2.1, we confirm that $\mu_\mathcal{N} = \mu \llcorner A$ and $\mu_\mathcal{N}^\perp = \mu \llcorner B$. $\square$

Corollary 2.5. *Let X be a metric space, let $s > 0$, and let $\mathcal{N}$ be the collection of all Borel sets $E \subset X$ with $\mathcal{P}^s(E) = 0$. For every Borel measure μ on X such that μ is finite on bounded sets, the parts $\mu_\mathcal{N}$ carried by $\mathcal{N}$ and $\mu_\mathcal{N}^\perp$ singular to $\mathcal{N}$ (see Lemma 2.1) are identified pointwise by the magnitude of the lower s-density:*

$$\mu_\mathcal{N} = \mu \llcorner \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) = \infty\},$$

$$\mu_\mathcal{N}^\perp = \mu \llcorner \{x \in X : \liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) < \infty\}.$$

In particular, $\mu \ll \mathcal{P}^s$ if and only if $\liminf_{r \downarrow 0} (2r)^{-s} \mu(B(x,r)) < \infty$ μ-a.e.

Proof. We leave the proof that $\mu_\mathcal{N}$ and $\mu_\mathcal{N}^\perp$ are identified by the given formulas to the reader. (Just mimick the proof of Corollary 2.4.) For the last remark, notice that $\mu \ll \mathcal{P}^s$ if and only if $\mu(E) = 0$ whenever $\mathcal{P}^s(E) = 0$. Thus, $\mu \ll \mathcal{P}^s$ if and only if μ is singular to sets of zero $\mathcal{P}^s$ measure. $\qquad\square$

Remark 2.6. Analogous results hold with the *Hausdorff measures* replacing the packing measures and *upper densities* defined using $\limsup$ replacing lower densities defined using $\liminf$. The proof of Theorem 2.3 for Hausdorff measures is considerably easier and can be proved using Vitali's 5r-covering lemma (see [54] or [38]) and the definition of $\mathcal{H}^s$.

2.3. **Rectifiable curves.** The length of a curve in a metric space can be defined either *intrinsically* in terms of the variation of a parameterization of the curve or *extrinsically* using the 1-dimensional Hausdorff measure of the trace of the curve. It is well known that a curve has finite extrinsic length if and only if it admits a parameterization with finite intrinsic length; for a detailed explanation, see [1]. The following theorem originated in the 1920s (see [1] for a reference).

Theorem 2.7 (Ważewski's Theorem). *Let X be a metric space. For any nonempty set $\Gamma \subset X$, the following are equivalent:*

(1) Γ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$;
(2) $\Gamma = f([0,1])$ for some continuous map $f : [0,1] \to X$ such that $\mathrm{var}(f) = \sup_{t_0 < t_1 < \cdots < t_n} \sum_1^n \mathrm{dist}(f(t_{i-1}), f(t_i)) < \infty$;
(3) $\Gamma = f([0,1])$ for some Lipschitz continuous map $f : [0,1] \to X$.

Moreover, any set Γ satisfying (1), (2), or (3) is the image of a Lipschitz continuous map $f : [0,1] \to X$ with $|f(t) - f(s)| \le L|t - s|$ for all $s, t \in [0,1]$, where f is essentially 2-to-1 and $L = \mathrm{var}(f) = 2\mathcal{H}^1(\Gamma)$.

A *rectifiable curve* Γ in a metric space X is any nonempty set satisfying one of the three conditions in Ważewski's theorem. To test whether a given set Γ is a rectifiable curve it is usually easiest to check (1). However, once we know that a set Γ is a rectifiable curve, it may be convenient to choose a Lipschitz parameterization of Γ as in (3).

Remark 2.8. Since every rectifiable curve Γ admits a Lipschitz parameterization, it follows that $\mathcal{P}^1(\Gamma) \lesssim_L \mathcal{P}^1([0,1]) < \infty$ (e.g. see [13, Lemma 2.8]). Thus, every 1-rectifiable measure μ on X is automatically carried by sets of finite $\mathcal{P}^1$ measure. Therefore, if μ is a Borel measure on X that is finite on bounded sets, then the 1-rectifiable part of μ (cf. Theorem 1.1) satisfies

$$\mu_{\mathrm{rect}} \le \mu \llcorner \{x \in X : \liminf_{r\downarrow 0}(2r)^{-1}\mu(B(x,r)) > 0\}$$

by Corollary 2.4. In particular, if μ is a 1-rectifiable measure on a metric space and μ is finite on bounded sets, then the lower 1-density $\underline{D}^1(\mu, x) = \liminf_{r\downarrow 0}(2r)^{-1}\mu(B(x,r)) > 0$ at μ-a.e. $x \in X$. This observation significantly generalizes [54, Theorem 7.9], which says that $\underline{D}^1(\mathcal{H}^1 \llcorner \Gamma, x) > 0$ at $\mathcal{H}^1$-a.e. $x \in \Gamma$ for any rectifiable curve Γ in $\mathbb{R}^n$.

2.4. **Carnot groups.** A connected, simply connected Lie group G is called a *step s Carnot group* if its associated Lie algebra $\mathfrak{g}$ satisfies

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_i] = V_{i+1} \text{ for } i = 1, \ldots, s-1, \quad [V_1, V_s] = \{0\},$$

where $V_1, \ldots, V_s$ are non-zero subspaces of $\mathfrak{g}$. We call this a *stratification* of the Lie algebra $\mathfrak{g}$. Choose a basis $\{X_1, \ldots, X_N\}$ of $\mathfrak{g}$ so that

$$\left\{ X_{\sum_{j=1}^{i-1}(\dim V_j)+1}, \ldots, X_{\sum_{j=1}^{i}(\dim V_j)} \right\} \text{ is a basis of } V_i \text{ for each } i \in \{1, \ldots, s\}.$$

For any $x \in G$, we may use the exponential map $\exp : \mathfrak{g} \to G$ to uniquely write $x = \exp(x_1 X_1 + \cdots + x_N X_N)$ for some $(x_1, \ldots, x_N) \in \mathbb{R}^N$. In other words, we can identify G with $\mathbb{R}^N$ via the relationship $x \leftrightarrow (x_1, \ldots, x_N)$. These are called the *exponential coordinates* of G. We will actually group coordinates by the layer that the corresponding basis elements are in. Thus, we will actually write

$$x = (x_1, ..., x_s),$$

where $x_i \in \mathbb{R}^{n_i}$ where $n_i = \dim V_i$. Under this identification, we have $p^{-1} = -p$ for any $p \in G$. Denote by $|\cdot|$ the Euclidean norm in $G = \mathbb{R}^N$ relative to the above choice of basis.

For each $r \in \{2, ..., s\}$, we also define the normal subgroups

$$G^{(r)} = \exp(V_r \oplus ... \oplus V_s).$$

In terms of exponential coordinates, these are the subspaces of $\mathbb{R}^N$ spanned by the coordinates corresponding to vectors $X_i \in V_r \oplus ... \oplus V_s$. For a general discussion of Carnot groups, see [18].

We can express group multiplication in G on the level of the Lie algebra using the Baker-Campbell-Hausdorff (BCH) formula:

$$\log(\exp(X)\exp(Y)) = \sum_{k>0} \frac{(-1)^{k-1}}{k} \sum_{\substack{r_i+s_i>0, \\ r_i,s_i \geq 0, \\ 1 \leq i \leq k}} a(r_1, s_1, ..., r_k, s_k)[X^{r_1} Y^{s_1} \cdots X^{r_k} Y^{s_k}]$$

Here the bracket term denotes iterated Lie brackets

$$[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}] = [\underbrace{X, [X, \cdots [X}_{r_1}, \underbrace{Y, [Y, \cdots [Y}_{s_1}, \cdots \underbrace{X, [X, \cdots [X}_{r_n}, \underbrace{Y, [Y, \cdots Y}_{s_n}]\cdots].$$

We have explicit formulas for group multiplication in terms of exponential coordinates:

$$(x_1, ..., x_s) \cdot (y_1, ..., y_s) = (x_1 + y_1, x_2 + y_2 + P_2, ..., x_s + y_s + P_s).$$

Here each P_i is a polynomial of $(x_1, ..., x_{i-1})$ and $(y_1, ..., y_{i-1})$, where x_i and y_i are vectors in $\mathbb{R}^{n_i}$. We call the P_is the BCH polynomials. We need the following lemma in §A.

Lemma 2.9 ([49, Lemma 4.1]). *There exists some constant $C > 0$ depending only on G so that if $|y_i| \leq \eta$ and $|x_i| \leq 1$ for all $i \in \{1, ..., k-1\}$ and any $\eta \in (0, 1)$, then*

$$|P_k(x_1, ..., x_{k-1}, y_1, ..., y_{k-1})| \leq C\eta.$$

There is a natural family of automorphisms known as *dilations* on G indexed by $t > 0$. Given $t > 0$, we define

$$\delta_t(x) = \delta_t(x_1, ..., x_s) = \left(tx_1, t^2x_2, \ldots, t^s x_s\right).$$

It follows that $\{\delta_t\}_{t>0}$ is a one parameter family, *i.e.* $\delta_u \circ \delta_t = \delta_{ut}$.

A *homogeneous norm* $N : G \to [0, \infty)$ is a function satisfying the following properties:

(1) $N(g) = 0 \iff g = 0$,
(2) $N(g^{-1}) = N(g)$,
(3) $N(gh) \leq N(g) + N(h)$.
(4) $N(\delta_t(g)) = tN(g)$ for all $t > 0, g \in G$.

The first three properties ensure that if we define $d(g, h) = N(g^{-1}h)$, then d is a left-invariant metric on G. The last property ensures that the metric scales with dilations, *i.e.* for all $t > 0$ and $g, h \in G$ we have

$$d(\delta_t(g), \delta_t(h)) = td(g, h).$$

Thus, we see that dilations and homogeneous norms on Carnot groups behave like scalar multiplication and linear norms. That is to say, Carnot groups may be viewed as non-abelian generalizations of vector spaces. In fact, the class of abelian Carnot groups are precisely the Euclidean spaces. Finally, we mention that it is well known that any two metrics on a Carnot group G induced by homogeneous norms are bi-Lipschitz equivalent.

We now define a family of homogeneous norms that exist for all Carnot groups. Given a parameter $\eta > 0$, consider $B_{\mathbb{R}^N}(\eta)$, the Euclidean ball around 0 in G with respect to the Euclidean norm $|\cdot|$. We then define an associated Minkowski gauge on G by

$$N_\eta(g) = \inf\{r > 0 : g \in \delta_r(B_{\mathbb{R}^N}(\eta))\}.$$

It is a theorem of Hebisch and Sikora [37] that, for any Carnot group G, there exists $\eta_0 > 0$ such that N_η is a homogeneous norm for all $0 < \eta < \eta_0$. As Euclidean balls of different radii are not homothetic under the dilations of G, we obtain a family of non-isometric norms $\{N_\eta\}_{0<\eta<\eta_0}$. We call these the *Hebisch-Sikora norms* on G.

Define $\pi : G \to \mathbb{R}^{n_1}$ to be the projection of G onto its first layer. Further, for each $r = 1, \ldots, s-1$, we let $\pi_r : G \to G_r := G/G^{(r+1)}$. We endow G with a metric d that arises from a Hebisch-Sikora norm N chosen so that the projected unit ball of N in each G_r also forms the unit ball of a Hebisch-Sikora norm. In particular, this choice ensures that each projection π_r is 1-Lipschitz. We note that the norms may be considered "nested" in the following sense: if N and N' are norms of G_r and G_{r+1}, then

$$N(x_1, ..., x_r) = N'(x_1, ..., x_r, 0)$$

by the convexity of balls centered at 0. By abusing notation, we will use N to denote all of these norms. We now record a lemma, which will be important in §A.

Lemma 2.10 ([49, Lemma 6.6]). *For any $\alpha \in (0,1)$ and $r \in \{1, \ldots, s-1\}$, there exists a constant $C > 0$ so that if $N(x_1, ..., x_{r-1}, 0) \in [\alpha, 1]$ and $|y| \leq 1/C$, then*

$$N(x_1, ..., x_r, y) \leq N(x_1, ..., x_r) + C|y|^2.$$

Finally, a set $L \subset G$ is called a *horizontal line* if it is a coset of a 1-dimensional subspace in $\mathbb{R}^{n_1} \times \{0\} \subset G$. In other words,

$$L = x \cdot \{(sv, 0, \ldots, 0) \ : \ s \in \mathbb{R}\} \quad \text{for some } x \in G,$$

where $v \in \mathbb{R}^{n_1}$. By the definition of the norm on G, horizontal lines are isometric copies of $\mathbb{R}$ in G. Using the BCH formulas, one can show that the Jacobian of left translation on G is 1. This tells us that the Lebesgue measure on the underlying manifold $\mathbb{R}^N$ of G is a Haar measure. By looking at the anisotropic scaling of the dilation δ_λ, we see that the Lebesgue measure of balls in G satisfy

(2.1) $$|B(x,r)| = c_G r^q \quad \text{for all } x \in G \text{ and } r > 0,$$

where $c_G = |B(0,1)|$ is the Lebesgue measure of the unit ball and $q = \sum_{k=1}^s k \dim V_k$ is the *homogeneous dimension* of G. Therefore, the Lebesgue measure on any Carnot group G is q-uniform, Ahlfors q-regular, and doubling. Furthermore, it follows from a standard packing argument that any ball in G of radius r may be covered by at most $C(q, \varepsilon)$ balls of radius εr.

2.5. **Dyadic cubes in "finite-dimensional" metric spaces.** We shall need access to a certain decomposition of an arbitrary Carnot group into a system of "dyadic cubes", where cubes of the same "side length" are pairwise disjoint. In the harmonic analysis literature, such systems are often called *Christ* or *Christ-David cubes* after constructions by David [25] and Christ [21] (see e.g. [41]), but similar decompositions in a metric space were given earlier by Larman [47]. Here we quote (a special case of) a recent streamlined construction of cubes by Käenmäki, Rajala, and Suomola [45], which can be carried out in any metric space which is "finite-dimensional" in the weak sense that every ball B is totally bounded, i.e. for every $r > 0$, B can be covered by a finite number of balls of radius r. For simplicity, we record the KRS construction with the scaling parameter $1/2$; see [45] for the general case, which allows for *any* scaling parameter between 0 and 1.

A set $Y \subset X$ is said to be *r-separated* if $\mathrm{dist}(y, z) \geq r$ for all $y, z \in Y$. If, in addition, $\mathrm{dist}(x, Y) < r$ for all $x \in X$, then we call Y a *r-net* for X. Recall that we write $U(x, r)$ and $B(x, r)$ to denote open and closed balls in X, respectively.

Theorem 2.11 ([45, Theorem 2.1, Remark 2.2]). *Let X be any metric space with totally bounded balls. Suppose that we are given $x_0 \in X$ and a family $(X_k)_{k \in \mathbb{Z}}$ of 2^{-k}-nets for X such that $x_0 \in X_k \subset X_{k+1}$ for all $k \in \mathbb{Z}$. Then there exist a family of collections $\Delta_k = \{Q_{k,i} \ : \ i \in N_k \subset \mathbb{N}\}_{k \in \mathbb{Z}}$ of Borel sets ("cubes") with the following properties:*

(1) partitioning: $X = \bigcup_i Q_{k,i}$ for every $k \in \mathbb{Z}$,
(2) nesting: $Q_{k,i} \cap Q_{m,j} = \emptyset$ or $Q_{k,i} \subset Q_{m,j}$ whenever $k \geq m$,

(3) *centers and roundness:* for every $Q_{k,i}$, there is a point $x_{k,i} \in X_k$ such that

$$U(x_{k,i}, \tfrac{2}{3} \cdot 2^{-k}) \subset Q_{k,i} \subset B(x_{k,i}, \tfrac{4}{3} \cdot 2^{-k}),$$

(4) *inheritance:* $\{x_{k,i} : i \in N_k\} \subset \{x_{k+1,i} : i \in N_{k+1}\}$ for all $k \in \mathbb{Z}$.
(5) *origin:* there exists $x_0 \in X$ so that for every $k \in \mathbb{Z}$, there exists $Q_{k,i}$ such that

$$U(x_0, \tfrac{2}{3} \cdot 2^{-k}) \subset Q_{k,i}.$$

(To derive Theorem 2.11 as stated, invoke the theorem in [45] with $r = 1/4$ and throw out odd generations of 4-adic cubes. The cubes that remain are the dyadic cubes.)

Given a fixed system of KRS cubes $(\Delta_k)_{k \in \mathbb{Z}}$ and $Q = Q_{k,i} \in \Delta_k$, we let $x_Q = x_{k,i}$ denote its *center* and let $\operatorname{side} Q = 2^{-k}$ denote its *side length*. Furthermore, we define

$$\lambda U_Q = U(x_Q, \tfrac{2}{3}\lambda \cdot 2^{-k}) \quad \text{and} \quad \lambda B_Q = B(x_Q, \tfrac{4}{3}\lambda \cdot 2^{-k})$$

for all $\lambda > 0$. Given $Q \in \Delta_k$ and $R \in \Delta_{k+1}$, we say that R is a *child* of Q if $R \subset Q$. Let $\Delta_1(Q)$ denote the set of all children of Q. Extending this metaphor, we may define *grandchildren*, *descendents*, *parents*, *grandparents*, *ancestors*, and *siblings* in the natural way as convenient.

Definition 2.12. We say that $\mathcal{T} \subset \Delta$ is a *tree of cubes* if $\mathcal{T}$ has a unique maximal element $\operatorname{Top}(\mathcal{T})$ such that if $Q \in \mathcal{T}$, then $P \in \mathcal{T}$ for all $P \in \Delta$ with $Q \subset P \subset \operatorname{Top}(\mathcal{T})$. For each level $l \geq 0$, let $\mathcal{T}_l$ denote the collection of all cubes $Q \in \mathcal{T}$ with $\operatorname{side} Q = 2^{-l} \operatorname{side} \operatorname{Top}(\mathcal{T})$. An *infinite branch* of $\mathcal{T}$ is a chain $\operatorname{Top}(\mathcal{T}) \equiv Q_0 \supset Q_1 \supset Q_2 \supset \cdots$ with $Q_l \in \mathcal{T}_l$ for all $l \geq 0$. We define the *set of leaves* of $\mathcal{T}$ by

$$\operatorname{Leaves}(\mathcal{T}) := \bigcup \left\{ \bigcap_{l=0}^{\infty} Q_l : Q_0 \supset Q_1 \supset Q_2 \supset \cdots \text{ is an infinite branch of } \mathcal{T} \right\}.$$

Remark 2.13. Because X has totally bounded balls, $\#\mathcal{T}_l < \infty$ for all $l \geq 0$. Using König's lemma (i.e. in a graph with infinitely many vertices, each of which has finite degree, there exists an infinite path), it can thus be shown that $\operatorname{Leaves}(\mathcal{T}) = \bigcap_{l=0}^{\infty} \bigcup \mathcal{T}_l$. In particular, $\operatorname{Leaves}(\mathcal{T})$ is a Borel set, since cubes in Δ are Borel.

Definition 2.14 (see [15, p. 18]). For any locally finite Borel measure μ on X, tree of cubes $\mathcal{T}$, and function $b : \mathcal{T} \to [0, \infty)$, we define the *$\mu$-normalized sum function*

$$S_{\mathcal{T},b}(\mu, x) := \sum_{Q \in \mathcal{T}} b(Q) \frac{\chi_Q(x)}{\mu(Q)} \in [0, \infty] \quad \text{for all } x \in X,$$

where we interpret $0/0 = 0$ and $1/0 = \infty$.

The following lemma is a slight variation on the Hardy-Littlewood maximal theorem for dyadic cubes in $\mathbb{R}^n$. The proof in [15] works *mutatis mutandis*, because the system of cubes Δ satisfies properties (1) and (2) in Theorem 2.11.

Lemma 2.15 (localization [15, Lemma 5.6])**.** *Let μ be a locally finite Borel measure on X, let $\mathcal{T}$ be a tree, and let $b : \mathcal{T} \to [0, \infty)$. Fix $0 < M < \infty$ and define*

$$A := \{x \in \mathsf{Top}(\mathcal{T}) : S_{\mathcal{T},b}(\mu, x) \leq M\}.$$

For every $\varepsilon > 0$, there is a set $\mathcal{G} \subset \mathcal{T}$ such that

(1) either $\mathcal{G} = \emptyset$ or $\mathcal{G}$ is a tree of cubes with $\mathsf{Top}(\mathcal{G}) = \mathsf{Top}(\mathcal{T})$,
(2) $\mu(A \cap \mathsf{Leaves}(\mathcal{G})) \geq (1 - \varepsilon)\mu(A)$, and
(3) $\sum_{Q \in \mathcal{G}} b(Q) < (M/\varepsilon)\mu(\mathsf{Top}(\mathcal{T}))$.

Mimicking the usual construction of Whitney cubes in $\mathbb{R}^n$, we may use the system of KRS cubes to build Whitney cubes in the complement of any closed set in X.

Lemma 2.16. *If $E \subset X$ is a nonempty closed set, then there exists a family $\mathcal{W}$ of cubes in Δ with the following properties.*

(1) partitioning: $X \setminus E = \bigcup_{W \in \mathcal{W}} W$ and $W_1 \cap W_2 \neq \emptyset$ if and only if $W_1 = W_2$;
(2) size and location: $\operatorname{diam} W \leq \operatorname{dist}(W, E) < 4 \operatorname{diam} W$ for all $W \in \mathcal{W}$, where $\operatorname{dist}(W, E) = \inf_{w \in W} \inf_{x \in E} \operatorname{dist}(w, x)$.

Proof. Given E, take $\mathcal{W}$ to be any maximal family of cubes $W \in \Delta$ such that $\operatorname{dist}(W, E) \geq \operatorname{diam} W$. The partitioning property follows from maximality and properties (2) and (3) of Theorem 2.11. Let $W \in \mathcal{W}$. One one hand, $\operatorname{dist}(W, E) \geq \operatorname{diam} W$ by definition of the family. On the other hand, let V be the parent of W in Δ. Then $\operatorname{dist}(V, E) < \operatorname{diam} V$ by maximality. Hence

$$\operatorname{dist}(W, E) \leq \operatorname{dist}(V, E) < \operatorname{diam} V \leq \operatorname{diam} B_V \leq 4 \operatorname{diam} U_W \leq 4 \operatorname{diam} W. \qquad \square$$

Remark 2.17. Suppose that X is a doubling metric measure space in the sense that there is a Borel measure μ on X and constant $C > 0$ such that (1.11) holds for all $x \in X$ and $r > 0$. By (2) and (3) in Theorem 2.11, for any $Q \in \Delta_k$ and $R \in \Delta_1(Q)$, we have

$$Q \subset B(x_R, \operatorname{diam} B_Q) \subset B(x_R, \tfrac{8}{3} \cdot 2^{-k}) \text{ and } B(x_R, \tfrac{1}{3} \cdot 2^{-k} - \varepsilon) \subset U(x_R, \tfrac{2}{3} \cdot 2^{-(k+1)}) \subset U_R.$$

Doubling of the measure at x_R yields $\mu(Q) \leq C^4 \mu(U_R)$ for all $R \in \Delta_1(Q)$. Hence

$$\mu(Q) = \sum_{R \in \Delta_1(Q)} \mu(R) \geq \sum_{R \in \Delta_1(Q)} \mu(U_R) \geq C^{-4} \mu(Q) \cdot \#\Delta_1(Q).$$

That is, $\#\Delta_1(Q) \leq C^4$ for every KRS cube Q.

3. Stratified β numbers for locally finite measures

From here through the end of §6, we let G be a fixed Carnot group of step s and choose metrics d_i associated to a Hebisch-Sikora norm on $G_i = G/G^{(i+1)}$ for all $1 \leq i \leq s$. Furthermore, let $\Delta = \bigcup_{k \in \mathbb{Z}} \Delta_k$ be a fixed system of dyadic cubes on G given by Theorem 2.11 with underlying 2^{-k}-nets $(X_k)_{k \in \mathbb{Z}}$. Motivated by [15] and [49], we wish to design a useful gauge of how close a locally finite measure μ on G is to being supported on a horizontal line in a neighborhood of a cube $Q \in \Delta$, which both allows for the possibility of non-doubling measures and incorporates distance in each of the layers $G_1, \ldots, G_s$ of G.
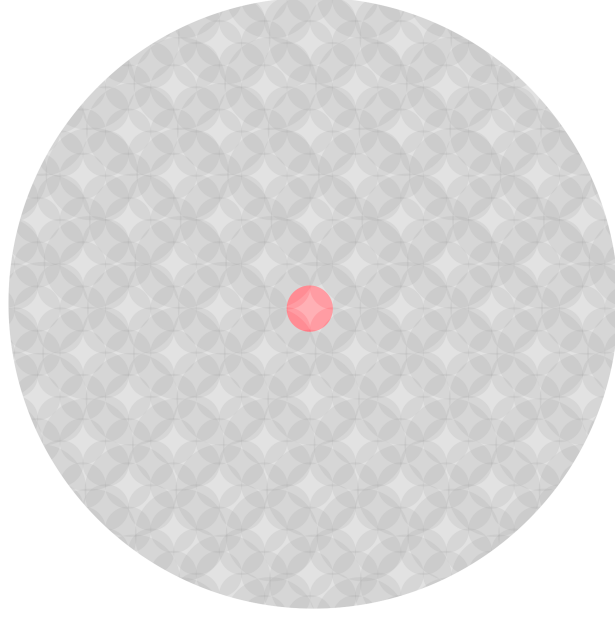
FIGURE 3.1. In $G = \mathbb{R}^2$: Illustration of pattern formed by overlapping balls $2B_R$ with $R \in \mathsf{Near}(Q)$ inside of the window $40B_Q$. Central region $2B_Q$ highlighted red.

Definition 3.1. For all $x, y \in G$ and $r > 0$, define
$$\widetilde{\beta}(x, y; r)^{2s} := \sum_{i=1}^{s} \left(\frac{d_i(\pi_i(x), \pi_i(y))}{r} \right)^{2i}.$$
Further, define $\widetilde{\beta}(x, E; r) := \inf_{y \in E} \widetilde{\beta}(x, y; r)$ for all nonempty $E \subset G$.

Definition 3.2 (non-homogeneous stratified β numbers)**.** Let μ be a locally finite Borel measure on G. For any Borel set Q, with $0 < \operatorname{diam} Q < \infty$, and any horizontal line L, define
$$\beta(\mu, Q, L)^{2s} := \int_Q \widetilde{\beta}(z, L; \operatorname{diam} Q)^{2s} \frac{d\mu(z)}{\mu(Q)}.$$
Further, define $\beta(\mu, Q) := \inf_L \beta(\mu, Q, L)$, where L runs over all horizontal lines in G.

Definition 3.3. For $Q \in \Delta_k$, $k \in \mathbb{Z}$, we define the family $\mathsf{Near}(Q)$ of cubes *near* Q by
$$\mathsf{Near}(Q) := \{R \in \Delta_{k-1} \cup \Delta_k \ : \ 2B_R \cap 636B_Q \neq \emptyset\},$$
where $636B_Q = B(x_Q, 848 \cdot 2^{-k})$ and x_Q is the center of Q.

Definition 3.4 (anisotropic stratified β numbers)**.** Let μ be a locally finite Borel measure on G. For every $Q \in \Delta$, define
$$\beta^*(\mu, Q)^{2s} := \inf_L \max_{R \in \mathsf{Near}(Q)} \beta(\mu, 2B_R, L)^{2s} \min \left\{ 1, \frac{\mu(2B_R)}{\operatorname{diam} 2B_R} \right\}$$
where the infimum is over the set of all horizontal lines in G.

Remark 3.5. The numbers $\beta^*(\mu, Q)$ are a rough gauge of how far $\mu \llcorner 636 B_Q$ is to a measure supported on a horizontal line. They are *anisotropic* insofar as the normalizations

$$\frac{1}{\mu(2B_R)} \min \left\{ 1, \frac{\mu(2B_R)}{\operatorname{diam} 2B_R} \right\}$$

of the integral of the scale-invariant stratified distance of points in $2B_R$ to a horizontal line L against the measure μ, i.e.

$$\sum_{i=1}^{s} \int_{2B_R} \left(\frac{d_i(\pi_i(z), \pi_i(L))}{\operatorname{diam} 2B_R} \right)^{2i} d\mu(z),$$

vary independently in the regions $2B_R$ that emanate in *different directions and distances* from the central region $2B_Q$ inside of the window $636 B_Q$. See Figure 3.1.

Remark 3.6. In Definition 3.2, *non-homogeneous* refers to normalizing the integral of the stratified distance by $\mu(Q)^{-1}$ (the measure in the window Q). Contrast this with homogeneous β numbers (1.8), where we used the normalization r^{-1} (the radius of the window $Q = B(x, r)$). The definition of β^* chooses one normalization or the other in each region $2B_R$ of the window $636 B_Q$ depending on whether the density $\mu(2B_R)/\operatorname{diam} 2B_R$ is big or small.

Remark 3.7. Let $x \in G$, let $\mathcal{T}$ denote the tree of cubes $Q \in \Delta$ such that $x \in Q$ and $\operatorname{side} Q \leq 1$, and let $b(Q) = \beta^*(\mu, Q)^{2s} \operatorname{diam} Q$ for all $Q \in \mathcal{T}$. Then $J^*(\mu, x) = S_{\mathcal{T}, b}(\mu, x)$, where $S_{\mathcal{T}, b}(\mu, \cdot)$ is given by Definition 2.14.

Remark 3.8. Let $Q \in \Delta_k$ and let $R \in \mathsf{Near}(Q) \cap \Delta_{k-1}$. Then

$$U(x_R, \tfrac{4}{3} \cdot 2^{-k}) = U_R \subset R \subset 2B_R \subset B(x_R, \tfrac{16}{3} \cdot 2^{-k}).$$

Because $2B_R \cap 636 B_Q \neq \emptyset$, we conclude that

$$(3.1) \qquad 2B_R \subset B(x_Q, 848 \cdot 2^{-k} + \operatorname{diam} 2B_R) \subset B(x_Q, 860 \cdot 2^{-k}) = 645 B_Q.$$

Further, since cubes in $\mathsf{Near}(Q) \cap \Delta_k$ are pairwise disjoint, a volume doubling argument yields $\#\mathsf{Near}(Q) \cap \Delta_{k-1} \lesssim 1$, where the implicit constant depends only on G. A similar computation shows that $2B_R \subset 645 B_Q$ for all $R \in \mathsf{Near}(Q) \cap \Delta_k$ and $\#\mathsf{Near}(Q) \cap \Delta_k \lesssim 1$, as well.

Remark 3.9. Midpoint convexity of $x \mapsto x^p$ when $p > 1$ gives us a quasitriangle inequality for the stratified distance:

$$(3.2) \qquad \widetilde{\beta}(x, y; r)^{2s} \leq 2^{2s-1} \left(\widetilde{\beta}(x, z; r)^{2s} + \widetilde{\beta}(z, y; r)^{2s} \right).$$

We also have change of scales inequalities:

$$(3.3) \qquad \widetilde{\beta}(x, y; s) \leq \widetilde{\beta}(x, y; r) \leq \frac{s}{r} \widetilde{\beta}(x, y; s) \quad \text{whenever } s \geq r > 0.$$

4. Rectifiability of sets on which the Jones function is finite

Suppose that μ is a locally finite Borel measure on G. For each cutoff $c > 0$, we define the truncated beta number $\beta^{*,c}(\mu, Q)$ for $Q \in \Delta$ by ignoring cubes $R \in \mathsf{Near}(Q)$ on which μ has small 1-dimensional density. That is,

$$(4.1) \quad \beta^{*,c}(\mu, Q)^{2s} := \inf_L \max \left\{ \beta(\mu, 2B_R, L)^{2s} \min\{c, 1\} : R \in \mathsf{Near}(Q), \frac{\mu(2B_R)}{\operatorname{diam} 2B_R} \geq c \right\},$$

where as usual the infimum runs over all horizontal lines in G and $\beta(\mu, 2B_R, L)^{2s}$ appears in Definition 3.2. If there are no $R \in \mathsf{Near}(Q)$ with $\mu(2B_R) \geq c \operatorname{diam} 2B_R$, simply assign $\beta^{*,c}(\mu, Q) = 0$. The associated density-normalized Jones function is defined by

$$(4.2) \qquad J^{*,c}(\mu, x) := \sum_{Q \in \Delta_+} \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam}(Q) \frac{\chi_Q(x)}{\mu(Q)} \quad \text{for all } x \in G,$$

where Δ_+ is the set of cubes of side length at most 1. It is immediate from the definitions that $\beta^{*,c}(\mu, Q) \leq \beta^*(\mu, Q)$ for all $Q \in \Delta$ and $J^{*,c}(\mu, x) \leq J^*(\mu, x)$ for all $x \in G$.

This section is devoted to the proof of the following theorem.

Theorem 4.1. *Let μ be a locally finite Borel measure on G. For every $c > 0$,*

$$(4.3) \qquad \mu \llcorner \{x \in G : \underline{D}^1(\mu, x) > 2c \text{ and } J^{*,c}(\mu, x) < \infty\}$$

is 1-rectifiable.

Our main tool for constructing a rectifiable curve passing through a set of points is Proposition A.1. In order to find (countably many) rectifiable curves covering the set where $\underline{D}^1(\mu, x)$ is positive and $J^{*,c}(\mu, x)$ is finite, we need to extract enough data to input to the proposition. In [15], the convexity of the Euclidean distance of a point to a line was used to find points z_Q (centers of mass) for each $Q \in \Delta$ for which we could control the distance of z_Q to any line L using β numbers. This approach is not available in an arbitrary Carnot group G, so we reverse the process. First, we associate a special line ℓ_Q to each $Q \in \Delta$. In particular, with μ and $c > 0$ fixed, for each $Q \in \Delta$, choose any horizontal line ℓ_Q so that

$$(4.4) \quad \max \left\{ \beta(\mu, 2B_R, \ell_Q)^{2s} \min\{c, 1\} : R \in \mathsf{Near}(Q), \frac{\mu(2B_R)}{\operatorname{diam} 2B_R} \geq c \right\} \leq 2\beta^{*,c}(\mu, Q)^{2s}.$$

If there are no $R \in \mathsf{Near}(Q)$ such that $\mu(2B_R) \geq c \operatorname{diam} 2B_R$, choose ℓ_Q arbitrarily or leave ℓ_Q undefined—we will never refer to it. Once we have fixed these lines, we may show that there exist points $\{z_R\}_{R \in \Delta}$ for which we can control the distance of z_R to ℓ_Q whenever $R \in \mathsf{Near}(Q)$ and $\mu(2B_R) \geq c \operatorname{diam} 2B_R$.

Lemma 4.2. *There exist points $\{z_R\}_{R \in \Delta}$ such that $z_R \in 2B_R$ for each $R \in \Delta$ and*

$$(4.5) \qquad \widetilde{\beta}(z_R, \ell_Q; \operatorname{diam} 2B_Q) \lesssim \widetilde{\beta}(z_R, \ell_Q; \operatorname{diam} 2B_R) \lesssim \beta(\mu, 2B_R, \ell_Q)$$

for each R and Q in Δ with $R \in \mathsf{Near}(Q)$ and $\mu(2B_R) \geq c \operatorname{diam} 2B_R$.

Proof. Fix $R \in \Delta$. Since $\operatorname{diam} 2B_Q \leq \operatorname{diam} 2B_R \leq 2 \operatorname{diam} 2B_Q$, the first inequality in (4.5) follows from (3.3). Thus, it suffices to prove the second inequality in (4.5). By definition, for any horizontal line L,

$$\beta(\mu, 2B_R, L)^{2s} = \int_{2B_R} \widetilde{\beta}(z, L; \operatorname{diam} 2B_R)^{2s} \frac{d\mu(z)}{\mu(2B_R)}.$$

Thus, for each horizontal line ℓ_Q associated to some $Q \in \Delta$, Chebyshev's inequality gives

$$\mu\left(\left\{z \in 2B_R : \widetilde{\beta}(z, \ell_Q; \operatorname{diam} 2B_R)^{2s} \geq C\beta(\mu, 2B_R, \ell_Q)^{2s}\right\}\right) \leq \frac{\mu(2B_R)}{C} \quad \text{for all } C > 1.$$

By an argument similar to Remark 3.8, there exists a constant $N = N(G) < \infty$ such that $\#\{Q \in \Delta : R \in \mathsf{Near}(Q)\} \leq N$. Choosing $C = 2N > 1$, it follows that

$$\mu\left(\bigcup_{\{Q:R\in\mathsf{Near}(Q)\}} \{z \in 2B_R : \widetilde{\beta}(z, \ell_Q; \operatorname{diam} 2B_R)^{2s} \geq 2N\beta(\mu, 2B_R, \ell_Q)^{2s}\}\right) \leq \frac{1}{2}\mu(2B_R).$$

Therefore, as long as $\mu(2B_R) > 0$, there exists $z_R \in 2B_R$ such that

$$(4.6) \qquad\qquad \widetilde{\beta}(z_R, \ell_Q; \operatorname{diam} 2B_R)^{2s} \leq 2N\beta(\mu, 2B_R, \ell_Q)^{2s}$$

for all $Q \in \Delta$ such that $R \in \mathsf{Near}(Q)$. Pick one such point for each $R \in \Delta$ such that $\mu(2B_R) > 0$. (This includes all cubes $R \in \Delta$ such that $\mu(2B_R) \geq c \operatorname{diam} 2B_R$. For any $R \in \Delta$ with $\mu(2B_R) = 0$, choose $z_R = x_R$ if desired.) $\qquad\square$

The following lemma describe a scenario when the whole set of leaves of a tree is contained in a rectifiable curve. Moreover, the length of such a curve can be controlled by the diameter or side length of the top cube and a sum involving $\beta^{*,c}(\mu, Q)^{2s}$.

Lemma 4.3. *Let μ and c be fixed as above. Suppose that $\mathcal{T}$ is a tree of cubes such that*

$$(4.7) \qquad\qquad \mu(2B_Q) \geq c \operatorname{diam}(2B_Q) \quad \text{for all } Q \in \mathcal{T} \qquad \text{and}$$

$$(4.8) \qquad\qquad S_{\mathcal{T}} = \sum_{Q \in \mathcal{T}} \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam}(Q) < \infty.$$

Then there exists a rectifiable curve Γ with $\mathsf{Leaves}(\mathcal{T}) \subset \Gamma$ such that

$$(4.9) \qquad\qquad \mathcal{H}^1(\Gamma) \lesssim \operatorname{side} \mathsf{Top}(\mathcal{T}) + \max\{c^{-1}, 1\}S_{\mathcal{T}}.$$

Proof. If the set of leaves of the tree is empty, the conclusion is trivial. Thus, we assume that $\mathsf{Leaves}(\mathcal{T}) \neq \emptyset$. Without loss of generality, we may further assume that every cube in $\mathcal{T}$ intersects $\mathsf{Leaves}(\mathcal{T})$. (Delete any cubes without this property.) Let $\{\ell_Q\}_{Q\in\Delta}$ be given by (4.4) and let $\{z_R\}_{R\in\Delta}$ be given by Lemma 4.2.

We employ a traveling salesman algorithm for constructing rectifiable curves in Carnot groups from Appendix A. In particular, we will apply Proposition A.1 with parameters

$$C^{\star} = 13 \quad \text{and} \quad r_0 = \operatorname{side} \mathsf{Top}(\mathcal{T}).$$

To do so, we must identify a sequence $(V_k)_{k \geq 0}$ of point clouds satisfying conditions (V_I), (V_{II}), (V_{III}) of the proposition and sequences $(\ell_{k,v})_{k \geq 0, v \in V_k}$ of lines and $(\alpha_{k,v})_{k \geq 0, v \in V_k}$ of linear approximation errors satisfying (A.1) and (A.2).

Point Clouds. For each $k \geq 0$, define $Z_k := \{z_Q : Q \in \mathcal{T} \text{ and side } Q = 2^{-k}r_0\}$ and choose V_k to be a maximal $2^{-k}r_0$-separated subset of Z_k. By definition, V_k satisfies (V_I).

Suppose that $v_k \in V_k$ for some $k \geq 0$. Then $v_k = z_Q$ for some $Q \in \mathcal{T}$ with side $Q = 2^{-k}r_0$. Because every cube in $\mathcal{T}$ is part of an infinite chain, there exists $R \in \mathcal{T}$ with side $R = (1/2)$ side Q and $R \subset Q$. By maximality of V_{k+1} in Z_{k+1}, there is $S \in \mathcal{T}$ with side $S =$ side R such that $z_S \in V_{k+1}$ and $d(z_S, z_R) \leq 2^{-(k+1)}r_0$. Hence $v_{k+1} := z_S$ satisfies

$$d(v_k, v_{k+1}) = d(z_Q, z_S) \leq d(z_Q, x_Q) + d(x_Q, x_R) + d(x_R, z_R) + d(z_R, z_S)$$
$$\leq \left(\tfrac{8}{3} + \tfrac{4}{3} + \tfrac{4}{3} + \tfrac{1}{2}\right) \cdot 2^{-k}r_0 < 6 \cdot 2^{-k}r_0.$$

Therefore, (V_{II}) holds.

Similarly, suppose that $v_k \in V_k$ for some $k \geq 1$, again say that $v_k = z_Q$ for some $Q \in \mathcal{T}$ with side $Q = 2^{-k}r_0$. Let $P \in \mathcal{T}$ be the parent of Q, which satisfies side $P = 2$ side Q and $Q \subset P$. By maximality of V_{k-1} in Z_{k-1}, there is $O \in \mathcal{T}$ with side $O =$ side P such that $z_O \in V_{k-1}$ and $d(z_O, z_P) \leq 2^{-(k-1)}r_0$. Hence $v_{k-1} := z_O$ satisfies

$$d(v_k, v_{k-1}) = d(z_Q, x_O) \leq d(z_Q, x_Q) + d(x_Q, x_P) + d(x_P, z_P) + d(z_P, z_O)$$
$$\leq \left(\tfrac{8}{3} + \tfrac{8}{3} + \tfrac{16}{3} + 2\right) \cdot 2^{-k}r_0 < 13 \cdot 2^{-k}r_0$$

Therefore, (V_{III}) holds.

Horizontal Lines and Linear Approximation Errors. Next, we will describe how to choose the horizontal lines $\ell_{k,v}$ and errors $\alpha_{k,v}$ for use in Proposition A.1. For each $k \geq 0$ and $v \in V_k$, let $Q_{k,v}$ denote the cube $Q \in \mathcal{T}$ such that side $Q = 2^{-k}r_0$ and $v = z_Q$. Then let $\ell_{k,v} = \ell_{Q_{k,v}}$ be the horizontal line chosen just before Lemma 4.2 to satisfy (4.4).

Suppose that $k \geq 1$, let $v \in V_k$, let $Q = Q_{k,v}$, and let

$$x \in (V_{k-1} \cup V_k) \cap B(v, 65C^\star 2^{-k}r_0) = (V_{k-1} \cup V_k) \cap B(v, 845 \cdot 2^{-k}r_0).$$

We must bound the distance of x to $\ell_{k,v}$. Since $x \in V_{k-1} \cup V_k$, we can express $x = z_R$ for some $R = R_x \in \mathcal{T}$ with side $Q \leq$ side $R \leq 2$ side Q. Note that $x \in 2B_R$ and

$$d(x, x_Q) \leq d(x, v) + d(v, x_Q) \leq 845 \cdot 2^{-k}r_0 + \tfrac{8}{3} \cdot 2^{-k}r_0 < 848 \cdot 2^{-k}r_0.$$

Thus, $x \in 2B_R \cap 636B_Q$, whence $R \in \mathsf{Near}(Q)$. By Lemma 4.2 and (3.3), we obtain

$$\widetilde{\beta}(x, \ell_{k,v}; 2^{-k}r_0)^{2s} \sim \widetilde{\beta}(x, \ell_{k,v}; \operatorname{diam} 2B_Q)^{2s} = \widetilde{\beta}(z_R, \ell_Q; \operatorname{diam} 2B_Q)^{2s} \lesssim \beta(\mu, 2B_R, \ell_Q)^{2s}.$$

Taking the maximum over all admissible x and invoking (4.4) and (4.7), we obtain

$$\sup_{x \in (V_{k-1}, V_k) \cap B(v, 65C^\star 2^{-k}r_0)} \widetilde{\beta}(x, \ell_{k,v}; 2^{-k}r_0)^{2s} \lesssim \beta^{*,c}(\mu, Q)^{2s} \max\{c^{-1}, 1\}.$$

By [49, Proposition 1.6] or [49, Lemma 6.2], it follows that there exists $\alpha_{k,v}$ such that $\alpha_{k,v}^{2s} \lesssim \beta^{*,c}(\mu, Q)^{2s} \max\{c^{-1}, 1\}$ and

$$x \in \ell_{k,v} \cdot \delta_{2^{-k}r_0}(B_{\mathbb{R}^n}(\alpha_{k,v}^s)) \quad \text{for all } x \in (V_{k-1}, V_k) \cap B(v, 65C^\star 2^{-k}r_0).$$

In other words, the errors $\alpha_{k,v}$ satisfy A.1. Moreover,

$$\sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^{2s} 2^{-k} r_0 \lesssim \max\{c^{-1}, 1\} \sum_{Q \in \mathcal{T}} \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam}(Q) \sim \max\{c^{-1}, 1\} S_{\mathcal{T}} < \infty$$

by (4.8). This verifies (A.2).

The Rectifiable Curve. Therefore, by Proposition A.1, there exists a rectifiable curve Γ in G such that the Hausdorff distance limit $V = \lim_{k \to \infty} V_k$ is contained in Γ. Moreover,

$$\mathcal{H}^1(\Gamma) \lesssim r_0 + \sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^{2s} 2^{-k} \sim \operatorname{side} \mathsf{Top}(\mathcal{T}) + \max\{c^{-1}, 1\} S_{\mathcal{T}}.$$

In other words, (4.9) holds. It remains to prove that $\mathsf{Leaves}(\mathcal{T}) \subset \Gamma$ and suffices to show that $\mathsf{Leaves}(\mathcal{T}) \subset V$. Pick $y \in \mathsf{Leaves}(\mathcal{T})$ so that $y = \lim_{k \to \infty} y_k$ for some sequence of points $y_k \in Q_k$, for some infinite chain $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$ in $\mathcal{T}$. By maximality of V_k in Z_k, for each $k \geq 0$ we may find $v_k \in V_k$ such that $d(v_k, z_{Q_k}) < 2^{-k} r_0$. Hence

$$d(y, V) \leq d(y, y_k) + d(y_k, z_{Q_k}) + d(z_{Q_k}, v_k) \leq d(y, y_k) + \operatorname{diam} 2B_{Q_k} + 2^{-k} r_0 \to 0$$

as $k \to \infty$, since $\lim_{k \to \infty} y_k = y$. Thus, $y \in V$, and therefore, $\mathsf{Leaves}(\mathcal{T}) \subset V \subset \Gamma$. $\qquad \square$

We are ready to prove the theorem.

Proof of Theorem 4.1. Let μ be a locally finite Borel measure on G and $c > 0$ be given. We wish to show that the measure μ_c defined by (4.3) is 1-rectifiable. That is, we wish to find a sequence $\Gamma_1, \Gamma_2, \dots$ of rectifiable curves such that $\mu_c(G \setminus \bigcup_1^{\infty} \Gamma_i) = 0$.

Suppose that $x \in G$ has $\underline{D}^1(\mu, x) > 2c$. Then there is some radius $r_x > 0$ such that

$$\mu(B(x, r)) > 4cr \quad \text{for all } 0 < r \leq r_x.$$

Thus, for any $Q \in \Delta$ containing x with $\frac{4}{3} \operatorname{side} Q \leq r_x$, we have $B(x, \frac{4}{3} \operatorname{side} Q) \subset 2B_Q$ and

$$\mu(2B_Q) \geq \mu(B(x, \tfrac{4}{3} \operatorname{side} Q)) \geq \tfrac{16}{3} \operatorname{side} Q = c \operatorname{diam} 2B_Q.$$

Choose $Q_x \in \Delta$ to be the maximal cube containing x with $\frac{4}{3} \operatorname{side} Q \leq r_x$ and $\operatorname{side} Q \leq 1$. Then $x \in \mathsf{Leaves}(\mathcal{T}_x)$, where

$$\mathcal{T}_x := \{Q \in \Delta \,:\, Q \subset Q_x \text{ and } \mu(2B_R) \geq c \operatorname{diam}(2B_R) \text{ for all } R \in \Delta \text{ with } Q \subset R \subset Q_x\}.$$

Note that the collection $\{Q_x : \underline{D}^1(\mu, x) > 2c\}$ of cubes is countable, since it is a subset of the countable family Δ. Thus, we may choose a sequence $\{x_i\}_1^{\infty}$ of points in G such that $\underline{D}^1(\mu, x_i) > 2c$ for each $i \geq 1$ and $\{x \in G : \underline{D}^1(\mu, x) > 2c\} \subset \bigcup_{i=1}^{\infty} Q_{x_i}$. Therefore,

$$\{x \in G : \underline{D}^1(\mu, x) > 2c \text{ and } J^{*,c}(\mu, x) < \infty\} \subset \bigcup_{i=1}^{\infty} \bigcup_{M=1}^{\infty} \{x \in Q_{x_i} : J^{*,c}(\mu, x) \leq M\}.$$

This shows that to prove the measure μ_c defined in (4.3) is 1-rectifiable, it suffices to prove that each measure $\mu \llcorner \{x \in Q_{x_i} : J^{*,c}(\mu, x) \leq M\}$ is 1-rectifiable.

Fix $i \geq 1$ and $M \geq 1$. Since $\operatorname{side} Q_{x_i} \leq 1$, the set $\{x \in Q_{x_i} : J^{*,c}(\mu, x) \leq M\}$ is contained in

$$A := \left\{ x \in Q_{x_i} : \sum_{Q \in \mathcal{T}_{x_i}} \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam} Q \, \frac{\chi_Q(x)}{\mu(Q)} \leq M \right\}.$$

To complete the proof of the theorem, it is enough to prove that $\mu \llcorner A$ is 1-rectifiable. If $\mu(A) = 0$, we are done. Suppose that $\mu(A) > 0$. By Lemma 2.15, applied with the function $b(Q) \equiv \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam} Q$, for each $k \geq 2$, there is a subtree $\mathcal{G}_k$ of $\mathcal{T}_{x_i}$ such that $\mu(A \cap \mathsf{Leaves}(\mathcal{G}_k)) \geq (1 - 1/k)\mu(A)$ and

$$\sum_{Q \in \mathcal{G}_k} \beta^{*,c}(\mu, Q)^{2s} \operatorname{diam}(Q) < kM \, \mu(Q_{x_i}) < \infty.$$

Since the tree $\mathcal{T}_{x_i}$ satisfies (4.7) and (4.8), Lemma 4.3 produces a rectifiable curve Γ_k in G such that $\mathsf{Leaves}(\mathcal{G}_k) \subset \Gamma_k$ and

$$\mu(A \setminus \Gamma_k) = \mu(A) - \mu(A \cap \Gamma_k) \leq \mu(A) - \mu(A \cap \mathsf{Leaves}(\mathcal{G}_k)) \leq (1/k) \cdot \mu(A)$$

Therefore, $\mu \llcorner A$ is 1-rectifiable:

$$\mu\left(A \setminus \bigcup_2^\infty \Gamma_k\right) \leq \inf_{k \geq 2} \mu(A \setminus \Gamma_k) \leq \inf_{k \geq 2} (1/k) \cdot \mu(A) = 0. \qquad \square$$

By repeating the arguments above, making minor changes as necessary, one can obtain the following two variants of Theorem 4.1. For some sample details, see [15, Lemmas 5.4 and 7.3]. For all $Q \in \Delta$, define $\beta^{**}(\mu, Q) = \inf_L \max_{R \in \mathsf{Near}(Q)} \beta(\mu, 2B_R, L)$, where the infimum is over all horizontal lines in G. Also define

$$(4.10) \qquad J^{**}(\mu, x) = \sum_{Q \in \Delta_+} \beta^{**}(\mu, Q)^{2s} \operatorname{diam} Q \, \frac{\chi_Q(x)}{\mu(Q)} \quad \text{for all } x \in G.$$

Theorem 4.4. *If μ is a locally finite Borel measure on G, then the measure given by $\mu \llcorner \{x \in G : J^{**}(\mu, x) < \infty\}$ is 1-rectifiable.*

With $\beta(\mu, Q)$ as in Definition 3.2, define

$$(4.11) \qquad \widetilde{J}(\mu, x) = \sum_{Q \in \Delta_+} \beta(\mu, 2B_Q)^{2s} \operatorname{diam} Q \, \frac{\chi_Q(x)}{\mu(Q)} \quad \text{for all } x \in G.$$

Theorem 4.5. *If μ is a locally finite Borel measure on G, then the measure*

$$\mu \llcorner \left\{ x \in G : \limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty \text{ and } \widetilde{J}(\mu, x) < \infty \right\}$$

is 1-rectifiable.

5. Finiteness of the Jones function on rectifiable curves

In this section, we show that finiteness of the Jones function defined in (1.2) is necessary for a measure to be carried by rectifiable curves; cf. [15, §4].

Theorem 5.1. *If μ is a locally finite Borel measure on a Carnot group G and Γ is a rectifiable curve in G, then the function $J^*(\mu, \cdot) \in L^1(\mu \llcorner \Gamma)$. In particular, $J^*(\mu, x) < \infty$ for μ-a.e. $x \in \Gamma$.*

At the core of Theorem 5.1 is the following computation, which incorporates and extends the necessary half of Theorem 1.4. Recall Δ_+ is the set of $Q \in \Delta$ with side $Q \leq 1$.

Lemma 5.2. *If ν is a finite Borel measure on G and Γ is a rectifiable curve in G, then*

$$(5.1) \qquad \sum_{\substack{Q \in \Delta_+ \\ \nu(Q \cap \Gamma) > 0}} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q \lesssim \mathcal{H}^1(\Gamma) + \nu(G \backslash \Gamma).$$

Proof. Given two sets $E, U \subset G$, define

$$\widetilde{\beta}_E(U) = \inf_L \sup_{z \in E \cap U} \widetilde{\beta}(z, L; \operatorname{diam} U),$$

where as usual the infimum is over all horizontal lines in G. In particular, recalling (1.5), we have $\widetilde{\beta}_E(B(x, r)) \leq \beta_E(x, r) \leq 2\widetilde{\beta}_E(B(x, r))$ for all $x \in G$ and $r > 0$ by (3.3).

By Remark 3.1, $2B_R \subset 645B_Q$ for all $R \in \mathsf{Near}(Q)$. Fix an absolute constant $A = 1300$ and a small constant $\varepsilon > 0$ depending only on the step s of G to be determined later. Partition the set of cubes $Q \in \Delta_+$ that intersect the curve Γ in a set of positive measure into two classes:

$$\Delta_\Gamma = \{Q \in \Delta_+ : \nu(\Gamma \cap Q) > 0 \text{ and } (\varepsilon/2A)\beta^*(\nu, Q) \leq \widetilde{\beta}_\Gamma(AB_Q)\},$$

$$\Delta_\nu = \{Q \in \Delta_+ : \nu(\Gamma \cap Q) > 0 \text{ and } (\varepsilon/2A)\beta^*(\nu, Q) > \widetilde{\beta}_\Gamma(AB_Q)\}.$$

Then

$$\sum_{\substack{Q \in \Delta_+ \\ \nu(Q \cap \Gamma) > 0}} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q = \sum_{Q \in \Delta_\Gamma} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q + \sum_{Q \in \Delta_\nu} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q.$$

From the definition of Δ_Γ, the Analyst's Traveling Salesman Theorem in Carnot groups (Theorem 1.4), and (2.1), it follows that

$$\sum_{Q \in \Delta_\Gamma} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q \leq \sum_{Q \in \Delta_\Gamma} (\varepsilon/2A)^{-2s} \widetilde{\beta}_\Gamma(AB_Q)^{2s} \operatorname{diam} B_Q$$

$$\leq (\varepsilon/2A)^{-2s} \sum_{Q \in \Delta} \beta_\Gamma(x_Q, (4A/3)\operatorname{side} Q)^{2s} \operatorname{diam} B_Q \lesssim \mathcal{H}^1(\Gamma).$$

To complete the proof of (5.1), we will show that $\sum_{Q \in \Delta_\nu} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q \lesssim \nu(G \setminus \Gamma)$.

Let $Q \in \Delta_\nu$. By change of scales (3.3), the definition of $\widetilde{\beta}_\Gamma(AB_Q)$, and the definition of the family Δ_ν, we can find a horizontal line L in G so that

$$(5.2) \qquad \sup_{z \in \Gamma \cap AB_Q} \widetilde{\beta}(z, L; \operatorname{diam} 2B_Q) \leq A\widetilde{\beta}_\Gamma(AB_Q) < (\varepsilon/2)\beta^*(\nu, Q).$$

For the same horizontal line L, we have that

$$\beta^*(\nu, Q)^{2s} \leq \max_{R \in \mathsf{Near}(Q)} \beta(\nu, 2B_R, L)^{2s} \min\left\{1, \frac{\mu(2B_R)}{\mathrm{diam}\, 2B_R}\right\} =: \max_{R \in \mathsf{Near}(Q)} \beta(\nu, 2B_R, L)^{2s} m_R.$$

Fix $R \in \mathsf{Near}(Q)$ and divide $2B_R$ into two sets:

$$N_R = \{y \in 2B_R : \widetilde{\beta}(y, L; \mathrm{diam}\, 2B_R) \leq \varepsilon \beta^*(\nu, Q)\}, \quad F_R = 2B_R \setminus N_R.$$

Note that $F_R \subset G \setminus \Gamma$ by (5.2). To proceed, write

$$\beta(\nu, 2B_R, L)^{2s} m_R = \int_{N_R \cup F_R} \widetilde{\beta}(y, L; \mathrm{diam}\, 2B_R)^{2s} m_R \frac{d\nu(y)}{\nu(2B_R)}$$

$$\leq \varepsilon^{2s} \beta^*(\nu, Q)^{2s} + \int_{F_R} \widetilde{\beta}(y, L; \mathrm{diam}\, 2B_R)^{2s} m_R \frac{d\nu(y)}{\nu(3R)}.$$

The point is now that because $Q \in \Delta_\nu$, if ε is very small, then $\widetilde{\beta}_\Gamma(AB_Q)$ is very small relative to $\beta^*(\mu, Q)$. This will allow us to effectively replace the horizontal line L appearing in the expression $\int_{F_R} \widetilde{\beta}(y, L; \mathrm{diam}\, 2B_R)^2 s \cdots$ with Γ. For any $y \in 2B_R$, the inequalities (3.2), (3.3), and (5.2), the fact that $2B_R \subset 645B_Q$ and $\nu(\Gamma \cap Q) > 0$, and the choice $A = 1300 > 2 \cdot 645 + (\mathrm{diam}\, Q)/(\mathrm{side}\, Q)$ give us

$$\widetilde{\beta}(y, L; \mathrm{diam}\, 2B_R)^{2s} \leq 2^{2s-1}\left(\widetilde{\beta}(y, \Gamma \cap AB_Q; \mathrm{diam}\, 2B_R)^{2s} + \sup_{z \in \Gamma \cap AB_Q} \widetilde{\beta}(z, L; \mathrm{diam}\, 2B_R)^{2s}\right)$$

$$< 2^{2s-1}\widetilde{\beta}(y, \Gamma \cap AB_Q; \mathrm{diam}\, 2B_R)^{2s} + (1/2)\varepsilon^{2s}\beta^*(\nu, Q)^{2s}$$

$$= 2^{2s-1}\widetilde{\beta}(y, \Gamma; \mathrm{diam}\, 2B_R)^{2s} + (1/2)\varepsilon^{2s}\beta^*(\nu, Q)^{2s}.$$

Combining the previous two displays and using $m_R \leq \nu(2B_R)/\mathrm{diam}\, 2B_R$, we have

$$\beta(\nu, 2B_R, L)^{2s} m_R \leq (3/2)\varepsilon^{2s}\beta^*(\nu, Q)^{2s} + 2^{2s-1}\int_{F_R} \widetilde{\beta}(y, \Gamma; \mathrm{diam}\, 2B_R)^{2s} m_R \frac{d\nu(y)}{\nu(2B_R)}$$

$$\leq (3/2)\varepsilon^{2s}\beta^*(\nu, Q)^{2s} + 2^{2s-1}\int_{F_R} \widetilde{\beta}(y, \Gamma; \mathrm{diam}\, 2B_R)^{2s} \frac{d\nu(y)}{\mathrm{diam}\, 2B_R}.$$

Taking the maximum over all cubes $R \in \mathsf{Near}(Q)$, choosing ε sufficiently small depending only on s, rearranging, and using $\mathrm{diam}\, Q \leq \mathrm{diam}\, 2B_R$, we obtain

$$(5.3) \qquad \beta^*(\nu, Q)^{2s} \mathrm{diam}\, Q \leq 2^{2s} \max_{R \in \mathsf{Near}(Q)} \int_{F_R} \widetilde{\beta}(y, \Gamma; \mathrm{diam}\, 2B_R)^{2s} d\nu(y).$$

As we already noted each $F_R \subset G \setminus \Gamma$. Thus, by Remark 3.8 and (3.3),

$$(5.4) \qquad \beta^*(\nu, Q)^{2s} \mathrm{diam}\, Q \lesssim \int_{645B_Q \setminus \Gamma} \widetilde{\beta}(y, \Gamma; \mathrm{side}\, Q)^{2s} d\nu(y)$$

Let $\mathcal{W}$ be a Whitney decomposition of $G \setminus \Gamma$ given by Lemma 2.16. For each $j \in \mathbb{Z}$, let

$$\mathcal{W}_j = \{W \in \mathcal{W} : 2^{-(j+1)} < \mathrm{dist}(W, \Gamma) \leq 2^{-j}\}.$$

For any set I, also define $\mathcal{W}(I) = \{W \in \mathcal{W} : \nu(I \cap W) > 0\}$ and $\mathcal{W}_j(I) = \mathcal{W}_j \cap \mathcal{W}(I)$. Then, continuing from (5.4),

$$\beta^*(\nu, Q)^{2s} \operatorname{diam} Q \lesssim \sum_{W \in \mathcal{W}(645B_Q)} \sup_{y \in W} \widetilde{\beta}(y, \Gamma, \operatorname{side} Q)^{2s} \nu(W \cap 645B_Q)$$

$$\lesssim \sum_{i=1}^{s} \sum_{W \in \mathcal{W}(645B_Q)} \sup_{y \in W} \left(\frac{d_i(\pi_i(y), \pi_i(\Gamma))}{\operatorname{side} Q} \right)^{2i} \nu(W \cap 645B_Q).$$

Suppose that $\operatorname{side} Q = 2^{-k}$. If $W \in \mathcal{W}_j(645B_Q)$, then by bounding the distance of a point in $W \cap 645B_Q$ to a point in $\Gamma \cap Q$, we have

$$2^{-(j+1)} \leq \operatorname{dist}(W, \Gamma) \leq \operatorname{diam} 645B_Q \leq 1720 \cdot 2^{-k},$$

which implies that $j \geq k - 11$. Also if $W \in \mathcal{W}_j$ and $y \in W$, then $d_i(\pi_i(y), \pi_i(\Gamma)) \leq \operatorname{dist}(y, \Gamma) \leq \operatorname{dist}(W, \Gamma) + \operatorname{diam} W \leq 5 \operatorname{dist}(W, \Gamma) \leq 5 \cdot 2^{-j}$, where the first inequality holds because the projections $\pi_i : G \to G_i$ are 1-Lipschitz and the penultimate inequality is by property (2) of Lemma 2.16. Therefore,

$$(5.5) \quad \beta^*(\nu, Q)^{2s} \operatorname{diam} Q \lesssim \sum_{i=1}^{s} \sum_{j=\log_2(\operatorname{side} Q)-11}^{\infty} \sum_{W \in \mathcal{W}_j(645B_Q)} \left(\frac{2^{-j}}{\operatorname{side} Q} \right)^{2i} \nu(W \cap 645B_Q).$$

This estimate is valid for every $Q \in \Delta_\nu$.

Equation (5.5) is analogous to [13, (3.8)] (with step $s = 1$). Because the cubes in $\mathcal{W}$ are pairwise disjoint and each of the families $\{645B_Q : Q \in \Delta$ and $\operatorname{side} Q = 2^{-k}\}$ have bounded overlap, we may repeat the computation in [13] *mutatis mutandis* s times to obtain $\sum_{Q \in \Delta_\nu} \beta^*(\nu, Q)^{2s} \operatorname{diam} Q \lesssim \nu(G \setminus \Gamma)$. □

We now apply the lemma to prove that $J^*(\mu, \cdot)$ is integrable on any rectifiable curve.

Proof of Theorem 5.1. Let $\Gamma \subset G$ be a rectifiable curve. Integrating the Jones function,

$$\int_\Gamma J^*(\mu, x) \, d\mu(x) = \sum_{Q \in \Delta_+} \beta^*(\mu, Q)^{2s} \frac{\operatorname{diam}(Q)}{\mu(Q)} \int_\Gamma \chi_Q(x) d\mu(x)$$

$$= \sum_{\substack{Q \in \Delta_+ \\ \mu(Q \cap \Gamma) > 0}} \beta^*(\mu, Q)^{2s} \operatorname{diam}(Q) \frac{\mu(Q \cap \Gamma)}{\mu(Q)} \leq \sum_{\substack{Q \in \Delta_+ \\ \mu(Q \cap \Gamma) > 0}} \beta^*(\mu, Q)^{2s} \operatorname{diam}(Q).$$

Let $K = \overline{\bigcup \{Q \in \Delta_+ : \mu(Q \cap \Gamma) > 0\}}$ and put $\nu := \mu \llcorner K$. Since the set K is compact and μ is locally finite, we have $\nu(G) = \mu(K) < \infty$. Furthermore, $\mu \llcorner Q = \nu \llcorner Q$ whenever $Q \in \Delta_+$ and $\mu(Q \cap \Gamma) > 0$. Thus, by Lemma 5.2,

$$\int_\Gamma J^*(\mu, x) \, d\mu(x) \leq \sum_{\substack{Q \in \Delta_+ \\ \nu(Q \cap \Gamma) > 0}} \beta^*(\nu, Q)^{2s} \operatorname{diam}(Q) \lesssim \mathcal{H}^1(\Gamma) + \nu(G \setminus \Gamma) < \infty. \quad □$$

Corollary 5.3. *Let μ be any locally finite Borel measure on G. Then the measure $\mu \llcorner \{x \in G : J^*(\mu, x) = \infty\}$ is purely 1-unrectifiable.*

Proof. If Γ is a rectifiable curve in G, then $J^*(\mu, x) < \infty$ at μ-a.e. $x \in \Gamma$ by Theorem 5.1. That is to say, $\mu(\Gamma \cap \{x \in G : J^*(\mu, x) = \infty\}) = 0$ for every rectifiable curve Γ. $\square$

.

6. PROOF OF THEOREMS 1.1 AND 1.5

In this section, we gather the main results of §§4 and 5 to prove Theorem 1.1. We then derive Theorem 1.5.

6.1. **Proof of Theorem 1.1.** Let μ be a locally finite Borel measure on G. Both the lower density $\underline{D}^1(\mu, \cdot)$ and the Jones function $J^*(\mu, \cdot)$ are Borel measurable. Hence

$$R = \left\{x \in G : \underline{D}^1(\mu, x) > 0 \text{ and } J^*(\mu, x) < \infty\right\} \text{ and}$$

$$P = \left\{x \in G : \underline{D}^1(\mu, x) = 0 \text{ or } J^*(\mu, x) = \infty\right\}$$

are Borel sets and $G = R \cup P$. By the uniqueness clause of Lemma 2.1, if we show that $\mu \llcorner R$ is 1-rectifiable and $\mu \llcorner P$ is purely 1-unrectifiable, then

$$\mu_{\text{rect}} = \mu \llcorner R \quad \text{and} \quad \mu_{\text{pu}} = \mu \llcorner P.$$

On the one hand, $J^{*,c}(\mu, x) \leq J^*(\mu, x)$ for all $x \in G$ and $c > 0$ (see §4). Thus,

$$R = \left\{x \in G : \underline{D}^1(\mu, x) > 0 \text{ and } J^*(\mu, x) < \infty\right\}$$

$$\subset \bigcup_{n=1}^{\infty} \left\{x \in G : \underline{D}^1(\mu, x) > 2/n \text{ and } J^{*,1/n}(\mu, x) < \infty\right\} =: \bigcup_{n=1}^{\infty} R_n.$$

By Theorem 4.1, $\mu \llcorner R_n$ is 1-rectifiable for each $n \geq 1$. Therefore, $\mu \llcorner R \leq \sum_1^{\infty} \mu \llcorner R_n$ is 1-rectifiable, as well. On the other hand, we can write

$$P = \{x \in G : J^*(\mu, x) = \infty\} \cup \left\{x \in G : \underline{D}^1(\mu, x) = 0\right\} =: P_1 \cup P_2.$$

The measure $\mu \llcorner P_1$ is purely 1-unrectifiable by Corollary 5.3 and the measure $\mu \llcorner P_2$ is purely 1-unrectifiable by Corollary 2.4 and Remark 2.8. Since $\mu \llcorner P \leq \mu \llcorner P_1 + \mu \llcorner P_2$, $\mu \llcorner P$ is also purely 1-unrectifiable. This completes the proof of Theorem 1.1.

6.2. **Proof of Theorem 1.5.** Let μ be a locally finite Borel measure μ on G. First of all, we note that $\mu \ll \mathcal{H}^1$ if and only if

$$(6.1) \qquad \overline{D}^1(\mu, x) := \limsup_{r \downarrow 0} \frac{\mu(B(x, r))}{2r} < \infty \quad \mu\text{-a.e.}$$

by Corollary 2.5 and Remark 2.6.

Suppose that μ is 1-rectifiable and $\mu \ll \mathcal{H}^1$. On the one hand, $\underline{D}^1(\mu, x) > 0$ μ-a.e. by Theorem 1.1 and $\overline{D}^1(\mu, x) < \infty$ μ-a.e. as noted above. Theorem 1.1 also ensures that $J^*(\mu, x) < \infty$ μ-a.e. Suppose that at some $x \in G$, we have $ar \leq \mu(B(x, r)) \leq br$ for all $0 < r \leq 10$ and $J^*(\mu, x) < \infty$. Since $Q \in \mathsf{Near}(Q)$, we have

$$\beta(\mu, 2B_Q)^{2s} m_Q \leq \beta^*(\mu, Q)^{2s} \quad \text{for all } Q \in \Delta,$$

where $m_Q = \min\{1, \mu(2B_Q)/\operatorname{diam} 2B_Q\}$. If $Q_k \in \Delta_+$ contains x and side $Q_k = 2^{-k}$, then $Q_k \subset B_{Q_k} \subset B(x, \frac{8}{3} \cdot 2^{-k})$ and $B(x, \frac{4}{3} \cdot 2^{-k}) \subset 2B_{Q_k} \subset B(x, \frac{16}{3} \cdot 2^{-k})$. It follows that $\mu(Q_k) \lesssim b \operatorname{diam} Q_k$ and $a \operatorname{diam} 2B_{Q_k} \lesssim \mu(2B_Q) \lesssim b \operatorname{diam} 2B_{Q_k}$. Thus,

$$
J(\mu, x) = \sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \beta(\mu, x, r)^{2s} \frac{dr}{r}
$$

$$
\lesssim \sum_{k=0}^{\infty} \beta(\mu, 2B_{Q_k})^{2s} \frac{\mu(2B_{Q_k})}{2^{-(k+1)}} \lesssim_{a,b} \sum_{k=0}^{\infty} \beta(\mu, 2B_{Q_k})^{2s} m_{Q_k} \frac{\operatorname{diam} Q_k}{\mu(Q_k)} \lesssim J^*(\mu, x) < \infty.
$$

We conclude that $J(\mu, x) < \infty$ μ-a.e. This completes the proof of Theorem 1.5.

Remark 6.1. Bounds $0 < \underline{D}^1(\mu, x) \leq \overline{D}^1(\mu, x) < \infty$ μ-a.e. on the lower and upper density implies *pointwise doubling*: $\limsup_{r \downarrow 0} \mu(B(x, 2r))/\mu(B(x, r)) < \infty$ μ-a.e. Thus, to prove the converse of Theorem 1.5, it is natural to try using Theorem 4.5. An obstruction to this approach is the possibility that $0 < \mu(Q) \ll \operatorname{diam} Q$ on arbitrarily small cubes $Q \in \Delta$. To overcome this challenge, one could try to first build a *David-Mattila lattice* of cubes with *thin boundaries* (see [26]) in the Carnot or metric setting and then adapt the proof of the main theorem of [8]. We leave this as an open problem.

7. GARNETT-KILLIP-SCHUL-TYPE MEASURES IN QUASICONVEX METRIC SPACES

Let us prove Theorem 1.6. Suppose that (X, d) is a complete metric space such that

- X is *doubling*, i.e. there exists a constant $C_{db} \geq 1$ such that every bounded set of diameter D can be covered by C_{db} or fewer sets of diameter $D/2$; and,
- X is *quasiconvex*, i.e. there exists a constant $C_q \geq 1$ such that for every $x, y \in X$ with $x \neq y$, there exists a parameterized curve $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{var}(\gamma) \leq C_q \, d(x, y)$.

Because X is complete and doubling, there exists a doubling measure μ on X, i.e. a measure satisfying (1.11) for all $x \in X$ and $r > 0$; for a proof, see [45, Theorem 3.1] or [38, Theorem 13.3]. Our goal is to construct a doubling measure ν on X and a rectifiable curve $\Gamma \subset X$ such that $\nu(\Gamma) > 0$. We will explicitly construct ν and Γ in a similar manner to [34], which handled $X = \mathbb{R}^n$ with μ equal to the Lebesgue measure.

Fix any system $(\Delta_k)_{k \in \mathbb{Z}}$ of dyadic cubes on X given by Theorem 2.11. We freely adopt the notation of §2.5. In particular, to each $Q \in \Delta \equiv \bigcup_{k \in \mathbb{Z}} \Delta_k$, we may refer to the center x_Q, side length side Q, inner ball U_Q, and outer ball B_Q associated to Q. For any $j \geq 1$, let $\Delta_j(Q) = \{R \in \Delta_{k+j} : R \subset Q\}$ denote the collection of all j-th generation descendents of Q. Note that $\mu(Q) \geq \mu(U_Q) > 0$ for all $Q \in \Delta$, because μ is doubling. We already proved the following pair of facts in Remark 2.17.

Lemma 7.1. *There exists $C_1 > 0$ depending only on the doubling constant of μ such that $\mu(R) \geq C_1 \mu(Q)$ for all $R \in \Delta_1(Q)$.*

Corollary 7.2. *There exists $M \geq 1$ depending only on the doubling constant of μ such that $\#\Delta_j(Q) \leq M^j$ for all $Q \in \Delta$ and $j > 0$.*
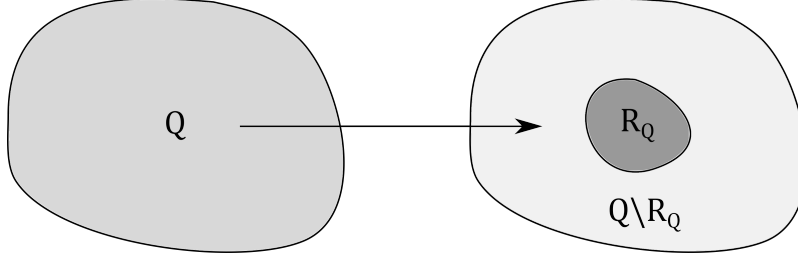
FIGURE 7.1. To define $f_Q\,d\mu$, redistribute the mass $\mu(Q)$ so that more mass is assigned to R_Q and less mass is assigned to $Q \setminus R_Q$.

Next, let us show that each cube contains a descendent—within a few generations—that is far away from the complement of the cube, quantitatively.

Lemma 7.3. *For any $n \in \mathbb{Z}$ and $Q \in \Delta_n$, there exists some $R \in \Delta_9(Q)$ such that $d(R, Q^c) \equiv \inf_{x \in R} \inf_{y \notin Q} d(x, y) > 2^{-(n+2)}$.*

Proof. Fix $n \in \mathbb{Z}$ and $Q \in \Delta_n$. By property (4) of Theorem 2.11, there exists $R \in \Delta_{n+9}$ such that $x_R = x_Q$. Therefore,

$$d(R, Q^c) \geq d(B_R, U_Q^c) \geq d(x_Q, U_Q^c) - \sup_{z \in B_R} d(z, x_Q) \geq \tfrac{1}{3} \cdot 2^{-n} - \tfrac{4}{3} \cdot 2^{-(n+9)} > 2^{-(n+2)}. \quad \square$$

It will be convenient to thin Δ by skipping generations and to restrict to cubes starting from a fixed generation. For each integer $k \geq 0$, define

$$(7.1) \qquad\qquad D_k = \Delta_{9k} \quad \text{and} \quad D = \bigcup_{k=0}^{\infty} D_k.$$

For all $Q \in D$ and $k \geq 0$, define $D_k(Q)$ to be the k-th generation descendents of Q in D. By Lemma 7.3, for each $Q \in D_n$, we may choose some $R_Q \in D_1(Q)$ such that

$$(7.2) \qquad\qquad d(R_Q, Q^c) > 2^{-(9n+2)}.$$

Let $0 < \delta \ll 1$ be a constant whose value will be fixed later. For each $Q \in D$, we define a Borel measure ν_Q on X that is absolutely continuous with respect to μ by defining its Radon-Nikodym as a sum of indicator functions:

$$(7.3) \qquad\qquad f_Q := \frac{d\nu_Q}{d\mu} = a_Q \chi_{R_Q} + \delta \chi_{Q \setminus R_Q},$$

where $a_Q > 0$ is chosen so that $\nu_Q(Q) = \mu(Q)$. Note that $\nu_Q(Q^c) = 0$. See Figure 7.1.

Lemma 7.4. *For all $Q \in D$, we have $\nu_Q(R_Q) \geq (1 - \delta)\nu_Q(Q)$.*

Proof. Because $\mu(Q) = \nu_Q(Q)$, we have

$$\nu_Q(R_Q) = \nu_Q(Q) - \nu_Q(Q \setminus R_Q) = \nu_Q(Q) - \delta\,\mu(Q \setminus R_Q)$$
$$\geq \nu_Q(Q) - \delta\,\mu(Q) = (1 - \delta)\nu_Q(Q). \quad \square$$
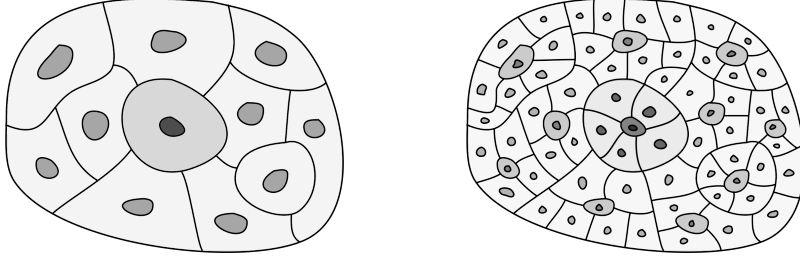
FIGURE 7.2. Possible densities $f_1 f_2|_Q$ (left) and $f_1 f_2 f_3|_Q$ (right).

Lemma 7.5. *There is a constant $C_2 > 0$ depending only on the doubling constant of μ such that $\sup_{Q \in D} a_Q \le C_2$.*

Proof. Since $D_1(Q) = \Delta_9(Q)$, iterating Lemma 7.1 gives $\mu(R_Q) \ge C_1^{-9} \mu(Q)$ for all $Q \in D$. We defined a_Q so that

$$\mu(Q) = \nu_Q(Q) = a_Q \mu(R_Q) + \delta\, \mu(Q \setminus R_Q) = a_Q \mu(R_Q) + \delta\, \mu(Q) - \delta\, \mu(R_Q).$$

Hence $a_Q = \delta + (1 - \delta)\mu(Q)/\mu(R_Q) \le 1 + C_1^9 =: C_2$. $\qquad\square$

To define the measure ν, we iterate the construction of $f_Q\, d\mu_Q$ and pass to a limit. Formally, for each $k \ge 0$, we define $f_k = \sum_{Q \in D_k} f_Q$. Using these weights, for each $n \ge 0$, we define a Borel measure ν_n the measures by setting

$$d\nu_n = \left(\prod_{k=0}^{n} f_k \right) d\mu.$$

See Figure 7.2. Finally, we define the measure ν to be the weak-$*$ limit of ν_n, whose (local) existence is an application of the martingale convergence theorem; e.g., see [30, Ch. 4].

Lemma 7.6. *There is a constant $C_3 \ge 1$ depending only on the doubling constant of μ and δ so that, for any $n \ge 0$, if $S, T \in D_n$ satisfy $d(S, T) \le 2^{-(9n-4)}$, then*

$$C_3^{-1} \nu(S) \le \nu(T) \le C_3 \nu(S).$$

Proof. Note that $\nu(Q) = \nu_n(Q)$ for all $Q \in D_n$ by construction. First suppose there is some largest integer $k \ge 0$ such that $S \subset Q_0$ and $T \subset Q_0$ for some $Q_0 \in D_k$. We claim that neither S nor T is contained in R_Q for any $Q \in \bigcup_{j=k+1}^{n} D_j$. Indeed, suppose first that $S \subset R_Q$ for some $Q \in D_j$ with $k < j < n$. Then $T \cap Q = \emptyset$ by our assumption on k, and so

$$d(R_Q, Q^c) \le d(S, Q^c) \le d(S, T) \le 2^{-(9n-4)} \le 2^{-(9(j+1)-4)} = 2^{-(9j+5)} < 2^{-(9j+2)}$$

which contradicts the definition of R_Q. Also, since $S \in D_n$, it is not possible that $S \subset R_Q$ for $Q \in D_n$. By construction, then, we have $f_j(x) = \delta = f_j(y)$ for all $x \in S$ and $y \in T$ when $k < j \le n$. Note also that $f_j(x) = f_j(y)$ for all $x \in S$ and $y \in T$ when $j < k$ as

$Q_0 \in D_k$ is a common ancestor of S and T. Thus, Lemma 7.5 gives

(7.4) $$\frac{\left(\prod_{j=0}^n f_j(x)\right)}{\left(\prod_{j=0}^n f_j(y)\right)} = \frac{f_k(x)}{f_k(y)} \in (\delta/C_2, C_2/\delta).$$

If S and T are never contained in some common ancestor then the same proof from above will give us that neither S not T is contained in R_Q for any $Q \in \bigcup_{j=0}^n D_j$. Then, $f_j(x) = \delta = f_j(y)$ for all $x \in S$ and $y \in T$ when $0 \le j \le n$ and so

(7.5) $$\frac{\left(\prod_{j=0}^n f_j(x)\right)}{\left(\prod_{j=0}^n f_j(y)\right)} = 1.$$

The result now follows from either (7.4) or (7.5) as μ is a doubling measure. $\square$

Lemma 7.7. *There is a constant $C_4 \ge 1$ depending only on δ and the doubling constant of μ so that*

(7.6) $$C_4^{-1}\mu(S) \le \nu(S) \le C_4\mu(S), \qquad \forall Q \in D_1.$$

Proof. We again note that $\nu(Q) = \nu_1(Q)$ for all $Q \in D_1$ by construction. Thus,

$$\nu(Q) = \int_Q f_0(x) f_1(x) \, d\mu(x).$$

As f_0 is constant on cubes of D_1, we can express this as

$$\nu(Q) = f_0(Q) \int_Q f_1(x) \, d\mu(x) = f_0(Q)\nu_Q(Q) = f_0(Q)\mu(Q).$$

Thus, we get the result by setting $C_4 = \max\{\delta^{-1}, C_2\}$ and using Lemma 7.5. $\square$

Proposition 7.8. *ν is doubling.*

Proof. Let $B(x,r)$ be a ball in X.

First assume that $r \le \frac{8}{3}$, and let j be the smallest integer satisfying $\frac{4}{3} \cdot 2^{-(9j-1)} \le r$. We get that $j \ge 0$. The collection $\left\{B(x_Q, \frac{4}{3} \cdot 2^{-9j}) : Q \in D_j\right\}$ is a cover of X, so there is some $Q \in D_j$ such that $d(x_Q, x) \le \frac{4}{3} \cdot 2^{-9j}$. We have by the triangle inequality that

$$Q \subset B(x_Q, \tfrac{4}{3} \cdot 2^{-9j}) \subset B(x, r).$$

In particular, $\nu(B(x,r)) \ge \nu(Q)$. Now let $\mathcal{S}$ denote the collection of all cubes in D_j that intersect $B(x, 2r)$. Thus, $\nu(B(x,2r)) \le \sum_{S \in \mathcal{S}} \nu(S)$. As (X, μ) is doubling, the minimality of j implies that $\#\mathcal{S}$ is bounded by some constant depending only on X. The proposition will therefore follow once we have proven that $\nu(S) \lesssim \nu(Q)$ for all $S \in \mathcal{S}$.

Since $Q \subset B(x_Q, \frac{4}{3} \cdot 2^{-9j})$, each $S \in D_j$ which intersects $B(x, 2r)$ satisfies

$$d(Q, S) \le d(x_Q, x) + d(x, S) \le \tfrac{4}{3} \cdot 2^{-9j} + 2r \le \tfrac{4}{3} \cdot 2^{-9j} + \tfrac{8}{3} \cdot 2^{-(9j-2)} < 2^{-(9j-4)}.$$

The bound we seek now follows from Lemma 7.6.

Now assume $r > \frac{8}{3}$. Let

$$S_1 = \bigcup \{Q \in D_1 : Q \cap B(x, 2r) \neq \emptyset\} \quad \text{and} \quad S_2 = \bigcup \{Q \in D_1 : Q \cap B(x, r/2) \neq \emptyset\}.$$

As elements of D_1 have diameters bounded by $4/3 \leq r/2$, we get the containments

$$B(x, 2r) \subset S_1 \subset B(x, 4r) \quad \text{and} \quad B(x, r/2) \subset S_2 \subset B(x, r).$$

We now can bound

$$\nu(B(x, 2r)) \leq \nu(S_1) \overset{(7.6)}{\leq} C_4 \mu(S_1) \leq C_4 \mu(B(x, 4r)) \leq C_4 C_\mu^3 \mu(B(x, r/2))$$

$$\leq C_4 C_\mu^3 \mu(S_2) \overset{(7.6)}{\leq} C_4^2 C_\mu^3 \nu(S_2) \leq C_4^2 C_\mu^3 \nu(B(x, r))$$

where C_μ is the doubling constant of μ. $\qquad \square$

For $0 \leq k \leq n$ and $Q \in D$, we define $\mathcal{K}_Q(n, k)$ to be the collection of cubes $S \in D_n(Q)$ for which there exist at least $n - k$ distinct cubes $T \in \bigcup_{j=0}^{n-1} D_j(Q)$ such that $S \subset R_T$.

Lemma 7.9. $\nu\left(\bigcup \mathcal{K}_Q(n, k)\right) \geq \left(1 - \exp\left[-\frac{n}{8}\left(\frac{k}{n} - \delta\right)^2\right]\right)\nu(Q).$

Proof. Without loss of generality, we will assume $Q \in D_0$ and $\nu(Q) = 1$. This will allow us to adopt a probabilistic view. We thus let $\mathbb{P}$ denote $\nu|_Q$.

For $j \geq 1$, define $D_j' := \{R_Q : Q \in D_{j-1}\}$. Consider the random variables $Y_j = \sum_{S \in D_j'} \mathbf{1}_S$. By Lemma 7.4, we have that $\mathbb{E}[Y_j] \geq 1 - \delta$. By the construction of Y_j and the nested nature of the D_k's, we get that

$$X_0 = 0, \qquad X_j = \sum_{i=1}^{j} Y_j - \mathbb{E}[Y_j] \quad \forall j \geq 1$$

is a martingale with respect to the filtration generated by $\{D_j\}$. We also have that $|X_j - X_{j-1}| \leq |Y_j - \mathbb{E}[Y_j]| \leq 2$ for all j. We can now bound

$$\mathbb{P}\left[\sum_{j=1}^{n} Y_j < n - k\right] = \mathbb{P}\left[X_n < n - k - \sum_{j=1}^{n} \mathbb{E}[Y_j]\right] \leq \mathbb{P}[X_n - X_0 < \delta n - k]$$

$$\leq \exp\left[-\frac{(\delta n - k)^2}{8n}\right],$$

where we used Azuma's inequality (see e.g. [2, Theorem 7.2.1]) for the last step. The lemma follows, because we have $\bigcup \mathcal{K}_Q(n, k) = \{\sum_{j=1}^{n} Y_j \geq n - k\}$. $\qquad \square$

Lemma 7.10. *There exists a constant $C_5 \geq 1$ depending only on X so that*

$$(7.7) \qquad\qquad\qquad \#\mathcal{K}_Q(n, k) \leq \left(\frac{C_5 n}{k}\right)^k, \qquad \forall Q \in D.$$

Proof. By Corollary 7.2, we can index each child in $D_1(Q)$ of a cube Q by a character in $1, ..., M^9$. (Recall that M depends only on X.) We will also make the convention that R_Q is indexed by 1. We can then continue indexing all descendants via strings of the characters $1, ..., M^9$ in the obvious way so that cubes in $D_n(Q)$ are length-n strings.

By our indexing convention and the definition of $\mathcal{K}_Q(n, k)$, we see that $\#\mathcal{K}_Q(n, k)$ is no greater than the number of length-n strings where at least $n - k$ of the characters are 1. We can bound this quantity by $\binom{n}{n-k} M^{9k}$ since $\binom{n}{n-k}$ equals the number ways in which $n - k$ 1's can be chosen and M^{9k} equals the number of all possible choices of characters in the other k positions. We therefore have

$$\#\mathcal{K}_Q(n, k) \leq \binom{n}{n - k} M^{9k} \leq \frac{n^k}{k!} M^{9k} \leq \left(\frac{neM^9}{k} \right)^k,$$

where we used the Taylor series of e^x to write $k^k/k! < e^k$. $\qquad\square$

Given $Q \in D$ and $0 \leq k \leq n$, we now define a curve $\Gamma_Q(n, k)$ as follows: for each $S \in \mathcal{K}_Q(n, k)$, connect x_Q to x_S with a curve of length at most $C_q \operatorname{diam}(Q)$, where C_q is the quasiconvexity constant of X. The set $\Gamma_Q(n, k)$ is then defined to be the union of these curves. We have the following bound

$$(7.8) \quad \mathcal{H}^1(\Gamma_Q(n, k)) \leq C_q \operatorname{diam}(Q) \#\mathcal{K}_Q(n, k) \overset{(7.7)}{\leq} \tfrac{8}{3} C_q \cdot 2^{-9m} \left(\frac{C_5 n}{k} \right)^k, \qquad \forall Q \in D_m.$$

Recalling that C_5 does not depend on δ, we may finally fix $\delta > 0$ sufficiently small and $n_1 \in \mathbb{N}$ so that

$$(7.9) \qquad\qquad \left(\frac{C_5}{2\delta} \right)^{2\delta} < 2^{1/2}$$

and such that $k_1 = 2\delta n_1$ is an integer. We now construct a sequence $(n_i, k_i)_{i=1}^\infty$ by defining $n_\ell = \ell n_1$ and $k_\ell = \ell k_1$. Note that $n_j/k_j = (2\delta)^{-1}$ for all $j \in \mathbb{N}$.

We will now construct Γ. Fix some $Q_0 \in D_0$ and define $\mathcal{K}_0 = \{Q_0\}$. Given $\mathcal{K}_j$, we define $\mathcal{K}_{j+1} = \bigcup_{Q \in \mathcal{K}_j} \mathcal{K}_Q(n_{j+1}, k_{j+1})$ and $K_j = \bigcup \mathcal{K}_j$. Note that $K_{j+1} \subset K_j$, and

$$\#\mathcal{K}_j \overset{(7.7)}{\leq} \#\mathcal{K}_{j-1} \left(\frac{C_5 n_j}{k_j} \right)^{k_j} = \#\mathcal{K}_{j-1} \left(\frac{C_5}{2\delta} \right)^{k_j}.$$

Iterating this estimate gives

$$(7.10) \qquad\qquad \#\mathcal{K}_j \leq \left(\frac{C_5}{2\delta} \right)^{k_1+...+k_j}.$$

We now define

$$\Gamma = \bigcup_{j=0}^\infty \bigcup_{Q \in \mathcal{K}_j} \Gamma_Q(n_{j+1}, k_{j+1}) \cup \bigcap_{j=1}^\infty K_j.$$

As $\Gamma_Q(n_{j+1}, k_{j+1})$ connects x_Q to x_S for each $S \in \mathcal{K}_Q(n_{j+1}, k_{j+1})$, this is a connected set.

The next two lemmas complete the proof of Theorem 1.6.

Lemma 7.11. $\mathcal{H}^1(\Gamma) < \infty$.

Proof. Fix $\ell \geq 1$ and $\varepsilon = \frac{8}{3} \cdot 2^{-(n_1+\ldots+n_\ell)}$.

$$\mathcal{H}^1_\varepsilon\left(\bigcap_{j=1}^\infty K_j\right) \leq \mathcal{H}^1_\varepsilon(K_\ell) \overset{(7.10)}{\leq} \frac{8}{3} \cdot 2^{-(n_1+\ldots+n_\ell)}\left(\frac{C_5}{2\delta}\right)^{k_1+\ldots+k_\ell} \overset{(7.9)}{\leq} \frac{8}{3} \cdot 2^{-(n_1+\ldots+n_\ell)/2}.$$

Since $\varepsilon \to 0$ and $n_1 + \ldots + n_\ell \to \infty$ as $\ell \to \infty$, we get that $\mathcal{H}^1\left(\bigcap_j K_j\right) = 0$. Thus, we can bound

$$\mathcal{H}^1(\Gamma) \leq \sum_{j=0}^\infty \sum_{Q \in \mathcal{K}_j} \mathcal{H}^1(\Gamma_Q(n_{j+1}, k_{j+1})).$$

As the cubes of $\mathcal{K}_j$ are in $D_{n_1+\ldots+n_j}$, we have a constant $C > 0$ such that

$$\mathcal{H}^1(\Gamma) \overset{(7.8)}{\leq} \frac{8}{3}C_q \sum_{j=0}^\infty \#\mathcal{K}_j 2^{-9(n_1+\ldots+n_j)}\left(\frac{C_5 n_{j+1}}{k_{j+1}}\right)^{k_{j+1}}$$

$$\overset{(7.10)}{\leq} \frac{8}{3}C_q \sum_{j=0}^\infty 2^{-9(n_1+\ldots+n_j)}\left(\frac{C_5}{2\delta}\right)^{k_1+\ldots+k_{j+1}} = \frac{8}{3}C_q \sum_{j=0}^\infty \left[2^{-9}\left(\frac{C_5}{2\delta}\right)^{2\delta}\right]^{n_1+\ldots+n_{j+1}} 2^{9n_{j+1}}$$

$$\overset{(7.9)}{<} \frac{8}{3}C_q \sum_{j=0}^\infty (2^{-17/2})^{n_1+\ldots+n_{j+1}} 2^{9n_{j+1}} \leq C + \frac{8}{3}C_q \sum_{j=6}^\infty 2^{-8n_{j+1}},$$

where, in the final inequality, we use the fact that $n_1 + \ldots + n_j \geq 3n_{j+1}$ when $j \geq 6$ since $n_j = jn_1$. Therefore, since the tail of the above series is bounded by a converging geometric series, this proves the lemma. $\square$

Lemma 7.12. $\nu(\Gamma) > 0$.

Proof. Note that, as $K_{j+1} \subset K_j$, we have by the dominated convergence theorem that

$$\nu(\Gamma) \geq \nu\left(\bigcap_{j=1}^\infty K_j\right) = \lim_{j\to\infty} \nu(K_j).$$

By the construction of $\mathcal{K}_j$ and Lemma 7.9, we have

$$\nu(K_j) \geq \left(1 - e^{-n_j\delta^2/8}\right)\nu(K_{j-1}) \geq \nu(Q_0)\prod_{i=1}^j(1 - e^{-n_i\delta^2/8}).$$

This product converges to a nonzero number as $\sum_i e^{-n_i\delta^2/8}$ is a convergent geometric series since $n_i = in_1$. This proves the lemma and thus the theorem. $\square$

Appendix A. Traveling salesman algorithm in Carnot groups

In this section, our goal is to prove the following traveling salesman type criterion for existence of a rectifiable curve passing through the Hausdorff limit of a sequence of point clouds. Crucially, the weak coherence condition (V_{II}) only requires that each cloud lie nearby, but not necessarily on the rectifiable curve. We used this flexibility in the proof

of Lemma 4.3. In the Euclidean setting, Proposition A.1 is due to the first author and Schul [15], based in part on earlier constructions in [43] and [48]. There are at least two difficulties in extending this criterion to arbitrary Carnot groups. The first challenge is in the statement of the criterion. The number $\alpha_{k,v}$ is a penalty term that bounds the *stratified distance* of points x in the clouds V_{k-1} *and* V_k that lie nearby the point v in V_k to a horizontal line $\ell_{k,v}$; the correct dependence on the step s in (A.1) and (A.2) was only recently identified by the second author [49]. Another challenge for higher step groups appears in the proof. We must bifurcate length estimates in the horizontal layer of the projection of abstract graphs Γ_k connecting V_k and estimates on the Hausdorff measure of geometric realizations $\widehat{\Gamma}_k$ of the graphs in G (see the proof of Proposition A.1 in §A.4).

Throughout the appendix, we let G be a fixed Carnot group of step s and choose metrics d_i associated to a Hebisch-Sikora norm on $G_i = G/G^{(i+1)}$ for all $1 \le i \le s$ (see §2.4). Recall that we let $B(x,r)$ denote the closed ball in G with center $x \in G$ and radius $r > 0$.

Proposition A.1 (traveling salesman criterion for point clouds)**.** *Let $x_0 \in G$, let $C^\star \ge 1$, and let $r_0 > 0$. Suppose that $(V_k)_{k=0}^\infty$ is a sequence of nonempty finite subsets of $B(x_0, C^\star r_0)$ such that*

(V_I) $d(v, v') \ge 2^{-k} r_0$ for all distinct points $v, v' \in V_k$,
(V_{II}) for all $v_k \in V_k$, there exists $v_{k+1} \in V_{k+1}$ such that $d(v_{k+1}, v_k) \le C^\star 2^{-k} r_0$,
(V_{III}) for all $v_k \in V_k$, there exists $v_{k-1} \in V_{k-1}$ such that $d(v_{k-1}, v_k) \le C^\star 2^{-k} r_0$.

Suppose also that, for all $k \ge 1$ and all $v \in V_k$, there is a horizontal line $\ell_{k,v}$ in G and a number $\alpha_{k,v} \ge 0$ such that

$$(A.1) \qquad x \in \ell_{k,v} \cdot \delta_{2^{-k} r_0}(B_{\mathbb{R}^n}(\alpha_{k,v}^s)) \quad \text{for all } x \in (V_{k-1} \cup V_k) \cap B(v, 65 C^\star 2^{-k} r_0).$$

Finally, suppose that

$$(A.2) \qquad \sum_{k=1}^\infty \sum_{v \in V_k} \alpha_{k,v}^{2s} 2^{-k} r_0 < \infty.$$

Then the sets V_k converge in the Hausdorff metric to a compact set $V \subset B(x_0, C^\star r_0)$ and there exists a rectifiable curve $\Gamma \subset B(x_0, C^\star r_0)$ such that $V \subset \Gamma$ and

$$(A.3) \qquad \mathcal{H}^1(\Gamma) \lesssim_{G, C^\star} r_0 + \sum_{j=k_0}^\infty \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j} r_0.$$

The following result will be an essential bilipschitz property of projections near those points which are relatively "flat", i.e. close to a horizontal line relative to their scale of separation. It replaces [15, Lemma 8.3].

Proposition A.2. *Assume G is a Carnot group of step s, and let $\pi : G \to \mathbb{R}^{n_1}$ be the projection to the first layer of G. For any $\alpha > 1$, there exist positive constants C and ε_0 depending on G and α so that if $L \subset G$ is a horizontal line, $P : G \to \pi(L)$ is the composition of π with the orthogonal projection in $\mathbb{R}^{n_1}$ onto $\pi(L)$, and $a, b \in L \cdot B_{\mathbb{R}^n}(\varepsilon^s)$*

for some $\varepsilon < \varepsilon_0$ so that $d(a,b) \in [1, \alpha]$ then

$$\frac{d(a,b)}{1 + C\varepsilon^{2s}} \leq |P(a) - P(b)| \leq d(a,b).$$

Proof. The right hand inequality is trivial as the projections which comprise P are 1-Lipschitz. We will prove the left hand inequality. We may without loss of generality assume that the horizontal line L contains the origin. In particular, this means that L has the form $\{(ut, 0, ..., 0) : t \in \mathbb{R}\}$ for some $u \in \mathbb{R}^{n_1}$. We also suppose that $a \in 0 \cdot B_{\mathbb{R}^n}(\varepsilon^s)$ and u was chosen so that $b \in (u, 0, ..., 0) \cdot B_{\mathbb{R}^n}(\varepsilon^s)$. Note then that

(A.4) $$\pi(a), \pi(b) \subset \pi(L) + B_{\mathbb{R}^{n_1}}(\varepsilon^s).$$

By choosing ε_0 sufficiently small, we can use the triangle inequality to guarantee that $|\pi(b) - \pi(a)| \geq 1/2$, $|P(b) - P(a)| \geq 1/4$, and $|u| \leq 2\alpha$.

We first prove that there exists a constant $C_0 > 0$ so that $a^{-1}b = (\pi(b) - \pi(a), \delta_2, ..., \delta_s)$ where $\delta_i \in \mathbb{R}^{n_i}$ have norm $|\delta_i| \leq C_0 \varepsilon^s$. We will actually prove the statement for $\delta_{1/2\alpha}(a^{-1}b)$ (with the first layer properly rescaled) as it will allow us to use Lemma 2.9. Rescaling back by $\delta_{2\alpha}$ then gives the corresponding statement for $a^{-1}b$.

That the coordinate in the first layer of $\delta_{1/2\alpha}(a^{-1}b)$ is $\frac{1}{2\alpha}(\pi(b) - \pi(a))$ is clear by the Baker-Campbell-Hausdorff formula. By our assumptions on a, b, we have that

$$\delta_{1/2\alpha}(a^{-1}b) = (x_1, ..., x_s) \cdot (u', 0, ..., 0) \cdot (y_1, ..., y_s)$$

where $|x_i|, |y_i| \leq \varepsilon^s/2\alpha$ and s $|u'| = |u|/2\alpha \leq 1$. Now two applications of Lemma 2.9 gives our needed result.

Now, by Lemma 2.10 we have

$$d(a,b) = N(a^{-1}b) \leq N(\pi(b) - \pi(a), \delta_2, ..., \delta_{s-1}) + C_1 \varepsilon^{2s}$$

for some constant $C_1 > 0$. Iterating this gives a constant $C_2 > 0$ so that

$$d(a,b) \leq N(\pi(b) - \pi(a)) + C_2 \varepsilon^{2s} = |\pi(b) - \pi(a)| + C_2 \varepsilon^{2s}.$$

Recalling (A.4), the Pythagorean theorem gives $|\pi(b) - \pi(a)| \leq |P(a) - P(b)| + 10\varepsilon^{2s}$. Altogether, we get a constant $C_3 > 0$ such that

$$d(a,b) \leq |P(a) - P(b)| + C_3 \varepsilon^{2s}.$$

Since $|P(a) - P(b)| \geq 1/4$, we have proven the desired inequality. $\square$

A.1. **Start of the proof of Proposition A.1.** The rest of this section is devoted to the proof of Proposition A.1. We follow the general outline of the proof in the Euclidean case (see [15, §8.1]). We shall refer the reader to the original proof for arguments that are essentially metric and highlight the changes that are necessary for the Carnot setting.

Without loss of generality, we can rescale the metric on G using a dilation so that $r_0 = 1$. By (the proof of) Lemma 8.2 of [15], the sets V_k converge in the Hausdorff metric to a compact set $V \subset B(x_0, C^\star)$. Note that, if $\#V_k = 1$ for all k, then V is a singleton, and so the result trivially holds. Assume, therefore, that there is some least $k_0 \geq 0$ so that that $\#V_k \geq 2$ for all $k \geq k_0$.

A.2. **The construction.** We will inductively construct a sequence of abstract graphs Γ_k on the vertices of $\bigcup_j V_j$. The abstract edges will simply be unordered pairs of vertices. On occasion, we may refer to connected families of edges as "curves". (In the Euclidean case [15], the edges in Γ_k were realized geometrically as line segments.)

To begin, we will define the *extension* of a vertex. Given $v \in V_k$, we define $E[k, v]$ in the following way. Let $v_0 = v$. Once $v_i \in V_{k+i}$ has been defined, choose v_{i+1} to be a closest point in V_{k+i+1} to v_i. The extension $E[k, v]$ is then defined as $E[k, v] = \{(v_i, v_{i+1})\}_{i=0}^{\infty}$. Given distinct vertices $v, v' \in V_k$, define the *bridge*

$$B[k, v, v'] = E[k, v] \cup \{(v, v')\} \cup E[k, v'].$$

Bridges will be used to span large "gaps" between vertices in V_k.

A.2.1. *Initial curve* Γ_{k_0}. We remark that either $k_0 = 0$ and $V_0 \subset B(x_0, C^\star)$ by assumption, or $k_0 \geq 1$ and $V_{k_0} \subset B(x, C^\star 2^{-k_0})$ by (V_{III}), where $V_{k_0-1} = \{x\}$. We construct the initial graph Γ_{k_0} by including every edge (v', v'') with $v', v'' \in V_{k_0}$. That is,

$$\text{(A.5)} \qquad \Gamma_{k_0} := \bigcup_{v', v'' \in V_{k_0}} (v', v'').$$

A.2.2. *Future curves* Γ_k. Suppose that $\Gamma_{k_0}, \ldots, \Gamma_{k-1}$ have been defined for some $k \geq k_0+1$. In order to define the next set Γ_k, we first describe the edge set in Γ_k locally nearby each vertex $v \in V_k$. We will then declare Γ_k to be the union of new parts of the curve together with the bridges from previous generations. That is, if $\Gamma_{k,v}$ denotes the new part of Γ_k nearby v, then

$$\text{(A.6)} \qquad \Gamma_k := \bigcup_{v \in V_k} \Gamma_{k,v} \cup \bigcup_{j=k_0}^{k-1} \bigcup_{B[j,w',w''] \subset \Gamma_j} B[j, w', w''].$$

For each $k \geq k_0$ and $v \in V_k$, define $B_{k,v} := B(v, 65C^\star 2^{-k})$. According to (V_I), there is some constant $M > 0$ such that $\#(V_k \cap B_{k,v}) \leq M$ for all $k \geq k_0$ and every $v \in V_k$. Let $\varepsilon > 0$ be a small parameter, depending only on G, chosen according to various needs below. In particular, when $\varepsilon > 0$ is sufficiently small, we will be able to invoke Proposition A.2.

Fix an arbitrary vertex $v \in V_k$. We will define $\Gamma_{k,v}$ in two cases.

Case I: Suppose $\alpha_{k,\widehat{v}} \geq \varepsilon$ for some $\widehat{v} \in V_k \cap B_{k,v}$.

To construct $\Gamma_{k,v}$, consider each pair of vertices $v', v'' \in V_k \cap B_{k,v}$. If $|\pi(v') - \pi(v'')| < 30C^\star 2^{-k}$, include the edge (v', v'') in $\Gamma_{k,v}$. Otherwise, include the bridge $B[k, v', v'']$. In other words,

$$\Gamma_{k,v} = \bigcup_{v', v'' \in V_k} \left(\bigcup_{|\pi(v') - \pi(v'')| < 30C^\star 2^{-k}} (v', v'') \cup \bigcup_{|\pi(v') - \pi(v'')| \geq 30C^\star 2^{-k}} B[k, v', v''] \right)$$

This ends the description of $\Gamma_{k,v}$ in **Case I**.

Case II: Suppose $\alpha_{k,\widehat{v}} < \varepsilon$ for every $\widehat{v} \in V_k \cap B_{k,v}$.

Identify the projected horizontal line $\pi(\ell_{k,v})$ with $\mathbb{R}$. (In particular, pick directions "left" and "right".) Let $\pi_{k,v} : G \to \mathbb{R}$ denote the projection P defined in Proposition A.2 composed with this identification. By (A.1), (V_I), and Proposition A.2, the map $\pi_{k,v}$ is bi-Lipschitz on $(V_k \cup V_{k-1}) \cap B_{k,v}$ with

$$(A.7) \qquad d(z', z'') \leq (1 + C\varepsilon^{2s})|\pi_{k,v}(z') - \pi_{k,v}(z'')| \qquad \forall z', z'' \in (V_k \cup V_{k-1}) \cap B_{k,v}.$$

In particular, both $V_k \cap B_{k,v}$ and $V_{k-1} \cap B_{k,v}$ can be arranged linearly along $\ell_{k,v}$. That is, if we set $v_0 = v \in V_k$, we can write

$$v_{-l}, \ldots, v_{-1}, v_0, v_1, \ldots, v_m$$

to denote the vertices in $V_k \cap B_{k,v}$ arranged from left to right according to the relative order of $\pi_{k,v}(v_i)$ in $\mathbb{R}$, where $l, m \geq 0$. In other words,

$$\pi_{k,v}(v_{-l}) < \cdots < \pi_{k,v}(v_{-1}) < \pi_{k,v}(v_0) < \pi_{k,v}(v_1) < \cdots < \pi_{k,v}(v_m).$$

We start by describing the "right half" $\Gamma_{k,v}^R$ of $\Gamma_{k,v}$. Starting from v_0 and working to the right, include each edge (v_i, v_{i+1}) in $\Gamma_{k,v}^R$ until $|\pi(v_{i+1}) - \pi(v_i)| \geq 30C^\star 2^{-k}$, $v_{i+1} \notin B(v, 30C^\star 2^{-k})$, or v_{i+1} is undefined (because $i = m$). Let $t \geq 0$ denote the number of edges that were included in $\Gamma_{k,v}^R$.

Case II-NT: If $t \geq 1$ (that is, at least one edge was included), then we say that the vertex v *is not terminal to the right*, and we are done describing $\Gamma_{k,v}^R$.

Case II-T1 and Case II-T2: If $t = 0$ (that is, no edges were included), then we say that the vertex v *is terminal to the right* and continue our description of $\Gamma_{k,v}^R$, splitting into subcases depending on how Γ_{k-1} looks near v. Let w_v be a vertex in V_{k-1} that is closest to v. As mentioned above, we may enumerate the vertices in $V_{k-1} \cap B_{k,v}$ starting from w_v and moving right (with respect to the identification of $\ell_{k,v}$ with $\mathbb{R}$) by

$$w_v = w_{v,0}, w_{v,1}, \ldots, w_{v,s}$$

i.e. $\pi_{k,v}(w_{v,0}) < \cdots < \pi_{k,v}(w_{v,s})$. Let $w_{v,r}$ denote the rightmost vertex that appears in $V_{k-1} \cap B(v, C^\star 2^{-(k-1)})$. There are two alternatives:

T1: If $r = s$ or if $|\pi(w_{v,r}) - \pi(w_{v,r+1})| \geq 30C^\star 2^{-(k-1)}$, then we set $\Gamma_{k,v}^R = \{v\}$.
T2: If $|\pi(w_{v,r}) - \pi(w_{v,r+1})| < 30C^\star 2^{-(k-1)}$, then v_1 exists by (V_{II}) (and $|\pi(v) - \pi(v_1)| \geq 30C^\star 2^{-k}$). In this case, we set $\Gamma_{k,v}^R = B[k, v, v_1]$.

The first alternative defines **Case II-T1**. The second alternative defines **Case II-T2**. This concludes the description of $\Gamma_{k,v}^R$.

We define the "left half" $\Gamma_{k,v}^L$ of $\Gamma_{k,v}$ symmetrically. Also, define the terminology v *is not terminal to the left* and v *is terminal to the left* by analogy with the corresponding terminology to the right. Having separately defined both the "left half" $\Gamma_{k,v}^L$ and the "right half" $\Gamma_{k,v}^R$ of $\Gamma_{k,v}$, we now declare

$$\Gamma_{k,v} := \Gamma_{k,v}^L \cup \Gamma_{k,v}^R.$$

This concludes the construction of $\Gamma_{k,v}$ in **Case II**.

A.3. **Connectedness.** The graph Γ_{k_0} is connected as it is the complete graph on V_{k_0}. The graphs Γ_k are locally connected nearby each vertex in V_k by construction of the $\Gamma_{k,v}$. Together with the fact that Γ_k includes all bridges appearing in Γ_{k-1} and that bridges include extensions to all future generations, it can be shown that Γ_k is globally connected. See [15, §8.3] for sample details.

A.4. **Start of the length estimates.** Let $\pi : G \to \mathbb{R}^{n_1}$ be the horizontal projection. Given E, a nonempty collection of abstract edges of $\bigcup_{k=k_0}^{\infty} V_k$ (for example Γ_k), we define its *projected length* $\ell(E)$ by

$$(A.8) \qquad \ell(E) := \sum_{(u,v) \in E} |\pi(u) - \pi(v)|.$$

We remark that the projected length may be larger than the length of the curve in $\mathbb{R}^{n_1}$ formed by projecting $\bigcup_{k=k_0}^{\infty} V_k$ into $\mathbb{R}^{n_1}$ and connecting pairs of points whose vertices in E are contained in an edge. The difference is that the quantity above might over-count the length since the projected line segments may not be disjoint.

Our primary task is to verify the following bound on $\ell(\Gamma_k)$:

Lemma A.3. *There exists a constant $C > 0$ depending only on G and $C^\star$ so that*

$$(A.9) \qquad \ell(\Gamma_k) \le C \left(2^{-k_0} + \sum_{j=k_0+1}^{k} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j} \right) \quad \text{for all } k \ge k_0 + 1,$$

For convenience, in the sequel we write $a \lesssim b$ to denote $a \lesssim_{G,C^\star} b$. Let us first see how Proposition A.1 follows from this lemma.

Proof of Proposition A.1 given Lemma A.3. First, assume that for some constant $C_1 > 0$ depending on at most G and $C^\star$, we know that for all $k \ge k_0 + 1$,

$$(A.10) \qquad \sum_{(u,v) \in \Gamma_k} d(u, v) \le C_1 \left(\ell(\Gamma_k) + \sum_{j=k_0+1}^{k} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j} \right).$$

Let $\widehat{\Gamma}_k$ be a geometric realization of Γ_k in G formed by drawing a geodesic in G for each edge $(u, v) \in \Gamma_k$ and taking the closure of the union of these geodesics. Together, (A.2), (A.9), and (A.10) yield

$$(A.11) \qquad \mathcal{H}^1(\widehat{\Gamma}_k) \le C_2 \left(2^{-k_0} + \sum_{j=k_0+1}^{\infty} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j} \right) < \infty \quad \text{for all } k \ge k_0 + 1,$$

where C_2 is a constant depending on at most G and $C^\star$. Let $(\widehat{\Gamma}_{k_j})_{j=1}^{\infty}$ be any subsequence of $(\widehat{\Gamma}_k)_{k=k_0}^{\infty}$ that converges in the Hausdorff metric, say $\Gamma = \lim_{j\to\infty} \widehat{\Gamma}_{k_j}$. Then by Gołab's semicontinuity theorem, which is valid in any metric space (see [1]), Γ is a rectifiable curve and $\mathcal{H}^1(\Gamma) \le \liminf_{j\to\infty} \mathcal{H}^1(\widehat{\Gamma}_{k_j}) < \infty$ by (A.11). That is to say, Γ satisfies (A.3). Back in §A.1, we noted that V_{k_j} converges in the Hausdorff metric to a compact set $V \subset B(x_0, C^\star)$.

Since $V_k \subset \widehat{\Gamma}_k$, it follows that $V \subset \Gamma$, as well. Therefore, we have reduced the proof of Proposition A.1, given Lemma A.3, to verifying (A.10).

Suppose first that $(u, v) \in \Gamma_k$ is a pair which is not part of an extension $E[i, z]$ included in Γ_k. If this edge was added to $\Gamma_{j,w}$ in **Case I** above (noting that it is only possible for $j < k$ when (u, v) is the "central span" of a bridge $B[j, u, v]$), then $u, v \in V_j \cap B_{j,w}$ and $\alpha_{j,\widehat{v}} \geq \varepsilon$ for some $\widehat{v} \in V_j \cap B_{j,w}$. Thus,

$$d(u, v) \leq \operatorname{diam} B_{j,w} \leq 130 C^\star 2^{-j} \leq 130 C^\star \varepsilon^{-2s} \alpha_{j,\widehat{v}}^{2s} 2^{-j}.$$

Since each $B_{j,w}$ contains boundedly many pairs (u, v) depending only on G and $C^\star$, and further, each $\widehat{v}$ is selected by a bounded number of points w, we may choose C_1 large enough so that the sum of $d(u, v)$ over all such pairs (u, v) is bounded from above by

$$C_1 \varepsilon^{-2s} \sum_{j=k_0+1}^{k} \sum_{\widehat{v} \in V_j} \alpha_{j,\widehat{v}}^{2s} 2^{-j}.$$

If (u, v) was added in **Case II**, then we get from (A.7) that

$$d(u, v) \leq (1 + C\varepsilon^{2s})|\pi(u) - \pi(v)|.$$

Choosing $C_1 \geq 1 + C\varepsilon^{2s}$ ensures that the sum of $d(u, v)$ over all pairs (u, v) discussed here is bounded from above by

$$C_1 \sum_{(u,v) \in \Gamma_k} |\pi(u) - \pi(v)| = C_1 \ell(\Gamma_k).$$

We now bound the length of all extensions $E[j, w]$ in Γ_k. If $E[j, w]$ was added to $\Gamma_{j,v}$ in **Case I** for some $v \in V_j$, then there is some $\widehat{v} \in V_j \cap B_{j,v}$ so that $\alpha_{j,\widehat{v}} \geq \varepsilon$. We then get

$$(A.12) \qquad \sum_{(u',u'') \in E[j,w]} d(u', u'') \leq C^\star 2^{-j+1} \leq 2C^\star \varepsilon^{-2s} \alpha_{j,\widehat{v}}^{2s} 2^{-j}.$$

As each $\Gamma_{j,v}$ can only have boundedly many such extensions and each $V_j \cap B_{j,v}$ has boundedly many elements, we may conclude that the sum of $d(u', u'')$ over all edges (u', u'') in such extensions is bounded by

$$2C^\star \varepsilon^{-2s} \sum_{j=k_0}^{k} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j}.$$

For extensions contained in a bridge $B[j, w, w']$ that were added in **Case II**, we get a bound as follows:

$$\sum_{(u',u'') \in E[j,w]} d(u', u'') + \sum_{(u',u'') \in E[j,w']} d(u', u'') \overset{\text{(A.12)}}{\leq} 4C^\star 2^{-j} \leq \frac{4}{30}|\pi(w) - \pi(w')|.$$

Thus, by increasing the lower bound $C_1 \geq 1 + C\varepsilon^{2s}$ to $C_1 \geq 2 + C\varepsilon^{2s}$, we can account for all such extensions. This completes the proof of (A.10). $\qquad \square$

The rest of this section is now dedicated to proving Lemma A.3. Roughly speaking, we would like to bound the length of Γ_{k_0} by $C2^{-k_0}$ and to bound $\ell(\Gamma_k)$ by $\ell(\Gamma_{k-1}) + C\sum_{v\in V_k}\alpha_{k,v}^{2s}2^{-k}$ for all $k \geq k_0$ and some C independent of k. At each step, we will "pay" for the length of Γ_k with the length of Γ_{k-1} plus some extra accumulation $C\sum_{v\in V_k}\alpha_{k,v}^{2s}2^{-k}$. The main difficulty arises when attempt to "pay" for an edge (v', v'') in Γ_k when either of its vertices is close to a terminal vertex from **Case II** of the construction. This is because, in this case, the old curve may not be long enough to "pay" for $|\pi(v') - \pi(v'')|$. To address this issue, we will take advantage of a "prepayment" technique called phantom length originating in Jones' original traveling salesman construction [43] (also see [48]).

A.5. **Phantom length.** Below, it will be convenient to have notation to refer to the vertices appearing in a bridge. For each extension $E[k,v] = \bigcup_{i=0}^{\infty}(v_i, v_{i+1})$, we define the corresponding *extension index set* $I[k,v]$ by

$$I[k,v] = \{(k+i, v_i) : i \geq 0\}.$$

For each bridge $B[k, v', v'']$, we define the corresponding *bridge index set* $I[k, v', v'']$ by

$$I[k, v', v''] = I[k, v'] \cup I[k, v''].$$

Following [15], for all $k \geq k_0$ and $v \in V_k$, we define the *phantom length associated with the pair* (k, v) as $p_{k,v} := 3C^\star 2^{-k}$. If $B[k, v', v'']$ is a bridge between vertices $v', v'' \in V_k$, then the totality $p_{k,v',v''}$ of phantom length associated to pairs in $I[k, v', v'']$ is given by

$$p_{k,v',v''} := 3C^\star\left(2^{-k} + 2^{-(k+1)} + \cdots\right) + 3C^\star\left(2^{-k} + 2^{-(k+1)} + \cdots\right) = 12C^\star 2^{-k}.$$

During the proof, we will track phantom length at certain pairs (k, v) with $v \in V_k$ as we now describe. For the initial generation, define the *index set* $\mathsf{Phantom}(k_0)$ by

$$\mathsf{Phantom}(k_0) := \{(k_0, v) : v \in V_{k_0}\}.$$

Suppose that $\mathsf{Phantom}(k_0), \ldots, \mathsf{Phantom}(k-1)$ have been defined for some $k \geq k_0 + 1$, where the index sets already defined satisfy the following two properties.

- *Bridge property:* For all $j \in \{k_0, \ldots, k-1\}$, if a bridge $B[j, w', w'']$ was introduced in Γ_j, then $\mathsf{Phantom}(j)$ contains $I[j, w', w'']$.
- *Terminal vertex property:* Let $w \in V_{k-1}$ and suppose ℓ is a horizontal line with

$$y \in \ell \cdot \delta_{2^{-(k-1)}}\left(B_{\mathbb{R}^n}(\varepsilon^s)\right) \quad \text{for all } y \in V_{k-1} \cap B(w, 30C^\star 2^{-(k-1)}).$$

Let $\pi_\ell : \mathbb{G} \to \mathbb{R}$ be the composition of π with the orthogonal projection in $\mathbb{R}^{n_1}$ onto ℓ and the identification of ℓ with $\mathbb{R}$ as before. If there does not exist

$$w' \in V_{k-1} \cap B(w, 30C^\star 2^{-(k-1)}) \quad \text{with } \pi_\ell(w') < \pi_\ell(w)$$

or there does not exist

$$w'' \in V_{k-1} \cap B(w, 30C^\star 2^{-(k-1)}) \quad \text{with } \pi_\ell(w'') > \pi_\ell(w),$$

then $(k-1, w) \in \mathsf{Phantom}(k-1)$.

(Note that $\mathsf{Phantom}(k_0)$ satisfies both properties trivially, since by definition Γ_{k_0} does not include and $\mathsf{Phantom}(k_0)$ does include (k_0, v) for every $v \in V_{k_0}$.) We will form $\mathsf{Phantom}(k)$ via $\mathsf{Phantom}(k-1)$ as follows. Initialize the set $\mathsf{Phantom}(k)$ to be equal to $\mathsf{Phantom}(k-1)$. Next, delete all pairs $(k-1, w)$ and (k, z) appearing in $\mathsf{Phantom}(k-1)$ from $\mathsf{Phantom}(k)$. Lastly, for each vertex $v \in V_k$, include additional pairs in $\mathsf{Phantom}(k)$ according to the following rules.

- **Case I:** Suppose that $v \in V_k$ and $\alpha_{k,w} \geq \varepsilon$ for some $w \in V_k \cap B_{k,v}$. Include (k, v') in $\mathsf{Phantom}(k)$ for all vertices $v' \in V_k \cap B_{k,v}$ and include $I[k, v', v'']$ as a subset of $\mathsf{Phantom}(k)$ for every bridge $B[k, v', v'']$ in $\Gamma_{k,v}$.
- **Case II:** Suppose that $v \in V_k$ and $\alpha_{k,w} < \varepsilon$ for all $w \in V_k \cap B_{k,v}$.
    - **Case II-NT:** Suppose $\Gamma_{k,v}^R$ or $\Gamma_{k,v}^L$ is defined by **Case II-NT**. Do nothing.
    - **Case II-T1:** Suppose $\Gamma_{k,v}^R$ or $\Gamma_{k,v}^L$ is defined by **Case II-T1**. Include $(k, v) \in \mathsf{Phantom}(k)$.
    - **Case II-T2:** Suppose $\Gamma_{k,v}^R$ or $\Gamma_{k,v}^L$ is defined by **Case II-T2**. When $\Gamma_{k,v}^R$ is defined by **Case II-T2**, include $I[k, v, v_1]$ as a subset of $\mathsf{Phantom}(k)$. When $\Gamma_{k,v}^L$ is defined by **Case II-T2**, include $I[k, v_{-1}, v]$ as a subset of $\mathsf{Phantom}(k)$. In particular, note that (k, v) is included in $\mathsf{Phantom}(k)$.

The phantom length associated to deleted pairs will be available to pay for the length of edges in Γ_k near terminal vertices in V_k and to pay for the phantom length of pairs in $\mathsf{Phantom}(k) \setminus \mathsf{Phantom}(k-1)$. Verification that $\mathsf{Phantom}(k)$ satisfies the bridge and terminal vertex properties is the same as the Euclidean case. See [15, p. 30] for details.

A.6. **Proof of** (A.9) **given** (A.13). The projected length of a set of edges is defined in (A.8). Suppose that there exists $C = C(G, C^\star)$ such that for all $k \geq k_0 + 1$,

$$
\text{(A.13)}\quad
\begin{aligned}
\ell(\mathsf{Edges}(k)) &+ \ell(\mathsf{Bridges}(k)) + \sum_{(j,u) \in \mathsf{Phantom}(k)} p_{j,u} \\
&\leq \ell(\mathsf{Edges}(k-1)) + \sum_{(j,u) \in \mathsf{Phantom}(k-1)} p_{j,u} + C \sum_{v \in V_k} \alpha_{k,v}^{2s} 2^{-k} + \frac{5}{6} \ell(\mathsf{Bridges}(k)),
\end{aligned}
$$

where $\mathsf{Edges}(k)$ denotes the set of all pairs (v', v'') included in Γ_k that are not part of a bridge $B[j, w', w'']$ included in Γ_k, $\mathsf{Bridges}(k)$ denotes the union of all bridges $B[k, v', v'']$ included in Γ_k, and $\mathsf{Phantom}(k)$ is defined in §A.5. Recall the definition of Γ_k in (A.6) and also that Γ_{k_0} contains no bridges. Applying (A.13) telescopically $k - k_0$ times yields

$$
\ell(\Gamma_k) = \ell(\mathsf{Edges}(k)) + \sum_{j=k_0+1}^{k} \ell(\mathsf{Bridges}(j))
$$

$$
\leq \underbrace{\ell(\mathsf{Edges}(k_0)) + \sum_{(j,u) \in \mathsf{Phantom}(k_0)} p_{j,u}}_{I} + C \sum_{j=k_0+1}^{k} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j} + \underbrace{\frac{5}{6} \sum_{j=k_0+1}^{k} \ell(\mathsf{Bridges}(j))}_{II}.
$$

Since $V_{k_0} \subset B(x, C^\star 2^{-k_0})$ for some x and V_{k_0} is 2^{-k_0}-separated, the number of points in V_{k_0} is bounded, depending only on G and $C^\star$. It follows that $I \lesssim_{G,C^\star} 2^{-k_0}$. Also, since Γ_k includes all bridges introduced in $\Gamma_{k_0+1}, \ldots, \Gamma_k$, we have $II \leq \frac{5}{6}\ell(\Gamma_k)$. Thus,

$$\frac{1}{6}\ell(\Gamma_k) \lesssim_{G,C^*} 2^{-k_0} + \sum_{j=k_0+1}^{k} \sum_{v \in V_j} \alpha_{j,v}^{2s} 2^{-j}.$$

This proves (A.9) given (A.13).

A.7. **Proof of** (A.13). This section corresponds to [15, §9.4]. Fix $k \geq k_0 + 1$. Our goal is to prove (A.13). As the projection $\pi : G \to \mathbb{R}^{n_1}$ is 1-Lipschitz, we have from (A.1) that

(A.14)
$$\sup_{x \in (V_k \cup V_{k-1}) \cap B_{k,v}} \text{dist}_{\mathbb{R}^{n_1}}(\pi(x), \pi(\ell_{k,v})) \leq \alpha_{k,v}^s 2^{-k}.$$

By an abuse of notation, we will refer to the projected line $\pi(\ell_{k,v})$ in $\mathbb{R}^{n_1}$ as $\ell_{k,v}$. It should always be clear from context to which line we are referring. Moreover, we will write $\pi_{k,v} : \mathbb{R}^{n_1} \to \mathbb{R}$ to denote orthogonal projection onto $\pi(\ell_{k,v})$ composed with identification of the line with $\mathbb{R}$. By (A.14), the sets $\pi(V_k)$ satisfy [15, (8.1)] with "error" $\alpha_{k,v}^s$. Thus, the estimate (A.13) is almost a direct application of the proof of [15, Proposition 8.1], except for the fact that $\pi(V_k)$ is not necessarily 2^{-k} separated. In [15], the separation condition is primarily used to get a bound on $\#\pi(V_k)$, but in our context this conclusion follows from a bound on $\#V_k$. We sketch some details for the reader's convenience.

It follows from the construction that for all $k \geq k_0$,

$$(v', v'') \in \mathsf{Edges}(k) \implies |\pi(v') - \pi(v'')| < 30C^\star 2^{-k},$$

$$B[k, v', v''] \subset \mathsf{Bridges}(k) \implies 30C^\star 2^{-k} \leq |\pi(v') - \pi(v'')| < 130C^\star 2^{-k}.$$

Furthermore, if $B[k, v', v''] \subset \mathsf{Bridges}(k)$, then

$$\ell(B[k, v', v'']) = |\pi(v') - \pi(v'')| + \ell(E[k, v']) + \ell(E[k, v''])$$
$$\overset{\text{(A.12)}}{\leq} |\pi(v') - \pi(v'')| + 4C^\star 2^{-k} < 1.14|\pi(v') - \pi(v'')|,$$

where in addition to (A.12) we used the fact that π is 1-Lipschitz.

Each graph Γ_k gives rise to a geometric realization of $\pi(\Gamma_k)$ in $\mathbb{R}^{n_1}$ by taking a union of line segments in $\mathbb{R}^{n_1}$ corresponding to abstract edges:

$$\mathcal{E}_k := \bigcup_{(u,v) \in \Gamma_k} [\pi(u), \pi(v)].$$

Since Γ_k is connected, $\mathcal{E}_k$ is as well. The length of an edge in Γ_k agrees with the Hausdorff measure $\mathcal{H}^1$ of the corresponding line segment in $\mathcal{E}_k$. We will call line segments in $\mathcal{E}_k$ "edges" and unions of line segments with the extensions at their endpoints "bridges" using the same classification as in §A.2. Given $v \in V_k$, we let $\mathcal{E}_{k,v}$ denote the associated line segments from $\Gamma_{k,v}$.

Edges and bridges forming $\mathcal{E}_k$ and "new" phantom length associated to pairs in the set $\mathsf{Phantom}(k) \setminus \mathsf{Phantom}(k-1)$ may enter the local picture $\mathcal{E}_{k,v}$ of $\mathcal{E}_k$ near $\pi(v)$ for several

vertices $v \in V_k$, but they each only need to be accounted for once to estimate the left hand side of (A.13). Continuing to follow [15], we prioritize as follows:

1. **Case I** edges, **Case I** bridges, **Case I** phantom length.
2. **Case II-T1** phantom length and edges that are near **Case II-T1** terminal vertices (where here and below *near* means at a distance at most $2C^{\star}2^{-k}$);
3. **Case II-T2** bridges, **Case II-T2** phantom length, and (parts of) edges that are near **Case II-T2** terminal vertices;
4. remaining (parts of) edges, which are necessarily not near **Case I** vertices and **Case II-T1** and **Case II-T2** terminal vertices.

First Estimate (Case I): This is analogous to the estimates on [15, p. 33]. Since $\#(V_k \cap B_{k,v}) \lesssim_{G,C^*} 1$, we may charge the length of edges, new bridges, and new phantom length appearing in $B_{k,v}$ to $\alpha_{k,u}^{2s}2^{-k}$ for some vertex $u \in B_{k,v}$ with $\alpha_{k,u} \geq \epsilon$.

Second Estimate (Case II-T1): As long as we choose ε to be small enough so that $2(1 + C\varepsilon^{2s}) < 2.5$, where C is the constant in Proposition A.2, this estimate is the same as the one on [15, p. 33]. Use Proposition A.2 in place of [15, Lemma 8.3].

Third Estimate (Case II-T2): This estimate introduces the term $\frac{5}{6}\ell(\mathsf{Bridges}(k))$ in (A.13). While it is similar to the estimate on [15, pp. 33–34], the proof there uses a notion of the "core" of a bridge, which we have not introduced. Thus, we record some details. Suppose that $\alpha_{k,u} < \varepsilon$ for all $u \in V_k \cap B_{k,v}$ and v is **T2** terminal to the right. (The case when v is terminal to the left can be handled analogously.) Let $v_1 \in V_k$ and $w_{v,r}, w_{v,r+1} \in V_{k-1}$ denote vertices appearing in the definition of $\Gamma_{k,v}^R$. We will pay for p_{k,v,v_1}, the projected length of the bridge $B[k,v,v_1]$, and the length (Hausdorff measure) of the part of any segments in $\mathcal{E}_k$ inside of $B_{\mathbb{R}^{n_1}}(\pi(v), 2C^{\star}2^{-k}) \cup B_{\mathbb{R}^{n_1}}(\pi(v_1), 2C^{\star}2^{-k}) =: BB$ with at least one endpoint in $B(v, 2C^{\star}2^{-k})$ or $B(v_1, 2C^{\star}2^{-k})$.

First, the totality p_{k,v,v_1} of phantom length associated to all vertices in $B[k,v,v_1]$ is $12C^{\star}2^{-k}$. Second,

$$\ell(B[k,v,v_1]) \overset{(\text{A.12})}{\leq} 4C^{\star}2^{-k} + |\pi(v) - \pi(v_1)| \leq 8C^{\star}2^{-k} + |\pi(w_{v,r}) - \pi(w_{v,r+1})|$$

because $d(v, w_{v,r}) < 2C^{\star}2^{-k}$ and $d(v_1, w_{v,r+1}) < 2C^{\star}2^{-k}$. Finally, by our choice of ε in the **Second Estimate** as before, since $\alpha_{k,v} < \varepsilon$ and $\alpha_{k,v_1} < \varepsilon$, the total length of parts of edges inside BB does not exceed $5C^{\star}2^{-k}$. Altogether,

$$\ell(B[k,v,v_1]) + p_{k,v,v_1} + \sum_{(v',v'')\in\mathsf{Edges}(k)} \mathcal{H}^1([\pi(v'),\pi(v'')] \cap BB) \cup B_{\mathbb{R}^{n_1}}(\pi(v_1), 2C^{\star}2^{-k})))$$

$$\leq |\pi(w_{v,r}) - \pi(w_{v,r+1})| + 8C^{\star}2^{-k} + 12C^{\star}2^{-k} + 5C^{\star}2^{-k}$$

$$\leq |\pi(w_{v,r}) - \pi(w_{v,r+1})| + \frac{25}{30}|\pi(v) - \pi(v_1)| = \frac{5}{6}|\pi(v) - \pi(v_1)|.$$

In the last inequality, we used $|\pi(v)-\pi(v_1)| \geq 30C^{\star}2^{-k}$. In fact, this is the entire rationale for the requiring bridges to have large spans. We remark that $(w_{v,r}, w_{v,r+1}) \in \mathsf{Edges}(k-1)$.

We have now paid for all phantom length, all bridges, and those parts of edges that are within a ball of radius $2C^\star 2^{-k}$ from the projection of a **Case II-T1** and **Case II-T2** terminal vertex. The next estimate will pay for all remaining edge lengths.

Fourth Estimate (Case II-NT): Suppose $(v', v'') \in \mathsf{Edges}(k)$ is an edge for which the length of $[\pi(v'), \pi(v'')]$ has not yet been fully paid, and fix a point $y \in V_{k-1}$ so that $d(y, v') < C^\star 2^{-k}$. Then $\alpha_{k,v'} < \varepsilon$ and $\alpha_{k,v''} < \varepsilon$, and there are $u', u'' \in \mathbb{R}^{n_1}$ such that $[u', u'']$ is the largest closed subinterval of $[\pi(v'), \pi(v'')]$ and u' and u'' lie at distance at least $2C^\star 2^{-k}$ from the projections of **II-T1** and **II-T2** terminal vertices of $V_k \cap B_{k,v'}$. Only $\mathcal{H}^1([u', u''])$ remains to be paid for as we have already paid for the rest of the length of $[\pi(v'), \pi(v'')]$ in the **Second** and **Third Estimate**. By Proposition A.2 and (A.14),

$$\begin{aligned}
|u' - u''| &\le (1 + C\alpha_{k,v'}^{2s})|\pi_{k,v'}(u') - \pi_{k,v'}(u'')| \\
&\le \mathcal{H}^1([\pi_{k,v'}(u'), \pi_{k,v'}(u'')]) + C\alpha_{k,v'}^{2s}|\pi(v') - \pi(v'')| \\
&\le \mathcal{H}^1([\pi_{k,v'}(u'), \pi_{k,v'}(u'')]) + 30C^\star C\alpha_{k,v'}^{2s}2^{-k}.
\end{aligned}$$

This is analogous to the first displayed equation in the Fourth Estimate on [15, p. 34], except that we have replaced $90 = 3 \cdot 30$ with $30C$, where C is from Proposition A.2. The argument on [15, pp. 34–35] shows how to efficiently charge $\mathcal{H}^1([\pi_{k,v'}(u'), \pi_{k,v'}(u'')])$ to $\ell(\mathsf{Edges}(k-1))$ and $\sum_{u \in V_k} \alpha_{k,u}^{2s} 2^{-k}$.

Carefully tallying the four estimates above, one obtains (A.13).

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