TWO SUFFICIENT CONDITIONS FOR RECTIFIABLE MEASURES

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Abstract. We identify two sufficient conditions for locally finite Borel measures on $\mathbb{R}^n$ to give full mass to a countable family of Lipschitz images of $\mathbb{R}^m$. The first condition, extending a prior result of Pajot, is a sufficient test in terms of L^p affine approximability for a locally finite Borel measure μ on $\mathbb{R}^n$ satisfying the global regularity hypothesis

$$\limsup_{r\downarrow 0} \mu(B(x,r))/r^m < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n$$

to be m-rectifiable in the sense above. The second condition is an assumption on the growth rate of the 1-density that ensures a locally finite Borel measure μ on $\mathbb{R}^n$ with

$$\lim_{r\downarrow 0} \mu(B(x,r))/r = \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n$$

is 1-rectifiable.

1. Introduction

In the treatise [Fed69] on geometric measure theory, Federer supplies the following general notion of rectifiability with respect to a measure. Let $1 \le m \le n-1$ be integers. Let μ be a *Borel measure* on $\mathbb{R}^n$, i.e. a Borel regular outer measure on $\mathbb{R}^n$. Then $\mathbb{R}^n$ *is countably* (μ, m) *rectifiable* if there exist countably many Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$ such that μ assigns full measure to the images sets $f_i([0,1]^m)$, i.e.

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1]^m)\right) = 0.$$

When $m = 1$, each set $\Gamma_i = f_i([0,1])$ is a *rectifiable curve*. Below we shorten Federer's terminology, saying that μ *is m-rectifiable* if $\mathbb{R}^n$ is countably (μ, m) rectifiable.

Two well-studied subclasses of rectifiable measures are rectifiable sets and absolutely continuous rectifiable measures. Given any Borel measure μ on $\mathbb{R}^n$ and Borel set $E \subseteq \mathbb{R}^n$, define the measure $\mu \llcorner E$ ("μ restricted to E") by the rule $\mu \llcorner E(F) = \mu(E \cap F)$ for all Borel sets $F \subseteq \mathbb{R}^n$. We call a Borel set $E \subseteq \mathbb{R}^n$ an *m-rectifiable set* if $\mathcal{H}^m \llcorner E$ is an m-rectifiable measure, where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure on $\mathbb{R}^n$. One may think of an m-rectifiable set E as an m-rectifiable measure by identifying E with the measure $\mathcal{H}^m \llcorner E$. More generally, we say that an m-rectifiable measure μ on $\mathbb{R}^n$ is *absolutely continuous* if $\mu \ll \mathcal{H}^m$, i.e. $\mu(E) = 0$ whenever $E \subset \mathbb{R}^n$ and $\mathcal{H}^m(E) = 0$.

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It is a remarkable fact that rectifiable sets and absolutely continuous rectifiable measures can be identified by the asymptotic behavior of the measures on small balls.

Definition 1.1 (Hausdorff density)**.** Let $B(x,r)$ denote the closed ball in $\mathbb{R}^n$ with center $x\in\mathbb{R}^n$ and radius $r>0$. For each positive integer $m\geq 1$, let $\omega_m=\mathcal{H}^m(B^m(0,1))$ denote the volume of the unit ball in $\mathbb{R}^m$. For all locally finite Borel measures μ on $\mathbb{R}^n$, we define the *lower Hausdorff m-density* $\underline{D}^m(\mu,\cdot)$ and *upper Hausdorff m-density* $\overline{D}^m(\mu,\cdot)$ by

$$\underline{D}^m(\mu,x):=\liminf_{r\to 0}\frac{\mu(B(x,r))}{\omega_m r^m}\in[0,\infty]$$

and

$$\overline{D}^m(\mu,x):=\limsup_{r\to 0}\frac{\mu(B(x,r))}{\omega_m r^m}\in[0,\infty]$$

for all $x\in\mathbb{R}^n$. If $\underline{D}^m(\mu,x)=\overline{D}^m(\mu,x)$ for some $x\in\mathbb{R}^n$, then we write $D^m(\mu,x)$ for the common value and call $D^m(\mu,x)$ the *Hausdorff m-density of μ at x.*

Theorem 1.2 ([Mat75])**.** *Let $1\leq m\leq n-1$. Suppose $E\subset\mathbb{R}^n$ is Borel and $\mu=\mathcal{H}^m\llcorner E$ is locally finite. Then μ is m-rectifiable if and only if the Hausdorff m-density of μ exists and $D^m(\mu,x)=1$ at μ-a.e. $x\in\mathbb{R}^n$.*

Theorem 1.3 ([Pre87])**.** *Let $1\leq m\leq n-1$. If μ is a locally finite Borel measure on $\mathbb{R}^n$, then μ is m-rectifiable and $\mu\ll\mathcal{H}^m$ if and only if the Hausdorff m-density of μ exists and $0<D^m(\mu,x)<\infty$ at μ-a.e. $x\in\mathbb{R}^n$.*

Remark 1.4. For any locally finite Borel measure μ on $\mathbb{R}^n$:

$$\mu\ll\mathcal{H}^m \iff \overline{D}^m(\mu,x)<\infty \text{ at } \mu\text{-a.e. } x\in\mathbb{R}^n\text{; and,}$$

$$\mu \text{ is } m\text{-rectifiable} \implies \underline{D}^m(\mu,x)>0 \text{ at } \mu\text{-a.e. } x\in\mathbb{R}^n. \tag{1.1}$$

See [Mat95, Chapter 6] and [BS14, Lemma 2.7].

There are several other characterizations of rectifiable sets and absolutely continuous rectifiable measures (e.g. in terms of projections or tangent measures); see Mattila [Mat95] for a full survey of results through 1993. Further investigations on rectifiable sets and absolutely continuous rectifiable measures include [Paj96, Paj97, Lég99, Ler03, Tol12, CGLT14, TT14, Tol14, ADT14a, BL14, Bue14, ADT14b, AT, Tol].

The first result of this note is an extension of Pajot's theorem on rectifiable sets [Paj97] to absolutely continuous rectifiable measures. To state these results, we must recall the notion of an L^p beta number from the theory of quantitative rectifiability.

Definition 1.5 (L^p beta numbers)**.** Let $1\leq m\leq n-1$ and let $1\leq p<\infty$. For every locally finite Borel measure μ on $\mathbb{R}^n$ and bounded Borel set $Q\subset\mathbb{R}^n$, define $\beta_p^{(m)}(\mu,Q)$ by

$$\beta_p^{(m)}(\mu,Q)^p:=\inf_\ell\int_Q\left(\frac{\operatorname{dist}(x,\ell)}{\operatorname{diam}Q}\right)^p\frac{d\mu(x)}{\mu(Q)}\in[0,1],$$

where ℓ in the infimum ranges over all m-dimensional affine planes in $\mathbb{R}^n$. If $\mu(Q)=0$, then we interpret (1.5) as $\beta_p^{(m)}(\mu,Q)=0$.

Remark 1.6. Beta numbers (of sets) were introduced by Jones [Jon90] to characterize subsets of rectifiable curves in the plane and are now often called *Jones beta numbers.* The L^p variant in Definition 1.5 originated in the fundamental work of David and Semmes on uniformly rectifiable sets [DS91, DS93] with the normalization appearing in (1.2). The normalization of $\beta_p^{(m)}(\mu, Q)$ presented in Definition 1.5 is not new; see e.g. [Ler03].

When $Q = B(x,r)$, some sources (e.g. [DS91, DS93, Paj97]) define L^p beta numbers using the alternate normalization

$$\widetilde{\beta}_p^{(m)}(\mu, B(x,r))^p := \inf_{\ell} \int_{B(x,r)} \left(\frac{\mathrm{dist}(x,\ell)}{r}\right)^p \frac{d\mu(x)}{r^m} \in [0,\infty), \tag{1.2}$$

where ℓ in the infimum again ranges over all m-dimensional affine planes in $\mathbb{R}^n$. However, $\beta_p^{(m)}(\mu, B(x,r))$ and $\widetilde{\beta}_p^{(m)}(\mu, B(x,r))$ are quantitatively equivalent at locations and scales where $\mu(B(x,r)) \sim r^m$. We have freely translated beta numbers in theorem statements quoted from other sources to the convention of Definition 1.5, which is better suited for generic locally finite Borel measures.

Theorem 1.7 ([Paj97])**.** *Let $1 \leq m \leq n-1$ and let*

$$\begin{cases} 1 \leq p < \infty & \text{if } m = 1 \text{ or } m = 2, \\ 1 \leq p < 2m/(m-2) & \text{if } m \geq 3. \end{cases} \tag{1.3}$$

Assume that $K \subset \mathbb{R}^n$ is compact and $\mu = \mathcal{H}^m \llcorner K$ is a finite measure. If $\underline{D}^m(\mu, x) > 0$ at μ-a.e. $x \in \mathbb{R}^n$ and

$$\int_0^1 \beta_p^{(m)}(\mu, B(x,r))^2 \, \frac{dr}{r} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n, \tag{1.4}$$

then μ is m-rectifiable.

In §2, we note the following extension of Pajot's theorem. Also, see Theorem 2.1.

Theorem A. *Let $1 \leq m \leq n-1$ and let $1 \leq p < \infty$ satisfy* (1.3)*. Assume that μ is a locally finite Borel measure on $\mathbb{R}^n$ such that $\mu \ll \mathcal{H}^m$. If $\underline{D}^m(\mu, x) > 0$ at μ-a.e. $x \in \mathbb{R}^n$ and* (1.4) *holds, then μ is m-rectifiable.*

In a forthcoming paper, Tolsa [Tol] proves that (1.4) is a necessary condition for an absolutely continuous measure to be rectifiable. Together with Theorem A and (1.1), this result provides a full characterization of absolutely continuous rectifiable measures in terms of the beta numbers and lower Hausdorff density of a measure.

Theorem 1.8 ([Tol])**.** *Let $1 \leq m \leq n-1$ and let $1 \leq p \leq 2$. If μ is m-rectifiable and $\mu \ll \mathcal{H}^m$, then* (1.4) *holds.*

Corollary 1.9. *Let $1 \leq m \leq n-1$ and let $1 \leq p \leq 2$. If μ is a locally finite Borel measure on $\mathbb{R}^n$ such that $\mu \ll \mathcal{H}^m$, then the following are equivalent:*

- *μ is m-rectifiable;*
- *$\underline{D}^m(\mu, x) > 0$ at μ-a.e. $x \in \mathbb{R}^n$ and* (1.4) *holds.*

In a companion paper to [Tol], Azzam and Tolsa [AT] prove that in the case $p = 2$, Theorem A holds with the hypothesis $\underline{D}^m(\mu, x) > 0$ at μ-a.e. $x \in \mathbb{R}^n$ on the lower density replaced by a weaker assumption $\overline{D}^m(\mu, x) > 0$ at μ-a.e. $x \in \mathbb{R}^n$ on the upper density.

For general m-rectifiable measures that are allowed to be singular with respect to $\mathcal{H}^m$, the following basic problem in geometric measure theory is still open.

Problem 1.10. For all $1 \leq m \leq n-1$, find necessary and sufficient conditions in order for a locally finite Borel measure μ on $\mathbb{R}^n$ to be m-rectifiable. (Do not assume that $\mu \ll \mathcal{H}^m$.)

Partial progress on Problem 1.10 has recently been made in [GKS10, BS14] in the case $m = 1$. In [GKS10], Garnett, Killip, and Schul exhibit a family $(\nu_\delta)_{0<\delta\leq\delta_0}$ of self-similar locally finite Borel measures on $\mathbb{R}^n$, which are

- *doubling*: $0 < \nu_\delta(B(x,r)) \leq C_\delta\, \nu_\delta(B(x,r/2)) < \infty$ for all $x \in \mathbb{R}^n$ and $r > 0$;
- *badly linearly approximable*: $\beta_2^{(1)}(\nu_\delta, B(x,r)) \geq c_\delta > 0$ for all $x \in \mathbb{R}^n$ and $r > 0$;
- *singular*: $D^1(\nu_\delta, x) = \infty$ at ν_δ-a.e. $x \in \mathbb{R}^n$ (hence $\nu_\delta \perp \mathcal{H}^1$); and,
- *1-rectifiable*: $\nu_\delta(\mathbb{R}^n \setminus \bigcup_i \Gamma_i) = 0$ for some countable family of rectifiable curves Γ_i.

In [BS14], Badger and Schul identify a pointwise necessary condition for an arbitrary locally finite Borel measure μ on $\mathbb{R}^n$ to be 1-rectifiable.

Theorem 1.11 ([BS14, Theorem A]). *Let $n \geq 2$ and let Δ be a system of closed or half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If μ is a locally finite Borel measure on $\mathbb{R}^n$ and μ is 1-rectifiable, then*

$$\sum_{Q\in\Delta} \beta_2^{(1)}(\mu, 3Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

The second result of this note is a sufficient condition for a measure μ with $D^1(\mu, x) = \infty$ at μ-a.e. $x \in \mathbb{R}^n$ to be 1-rectifiable.

Theorem B. *Let $n \geq 2$ and let Δ be a system of half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If μ is a locally finite Borel measure on $\mathbb{R}^n$ and*

$$\sum_{Q\in\Delta} \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n, \tag{1.5}$$

then μ is 1-rectifiable, and moreover, there exist a countable family of rectifiable curves Γ_i and Borel sets $B_i \subseteq \Gamma_i$ such that $\mathcal{H}^1(B_i) = 0$ for all $i \geq 1$ and $\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty B_i\right) = 0$.

Together Theorem 1.11 and Theorem B provide a full characterization of 1-rectifiability of measures with "pointwise large beta number" (1.6). Examples of measures that satisfy this beta number condition include the measures $(\nu_\delta)_{0<\delta\leq\delta_0}$ from [GKS10].

Corollary 1.12. *Let $n \geq 2$ and let Δ be a system of half-open dyadic cubes in $\mathbb{R}^n$ of side length at most 1. If μ is a locally finite Borel measure such that*

$$\liminf_{r\downarrow 0} \beta_2^{(1)}(\mu, B(x,r)) > 0 \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n, \tag{1.6}$$

then μ is 1-rectifiable if and only if (1.5) *holds.*

The remainder of this note is split into two sections. We prove Theorem A in §2 and we prove Theorem B in §3.

2. Proof of Theorem A

We show how to reduce Theorem A to Theorem 1.7 using standard geometric measure theory techniques; see Chapters 1, 2, 4, and 6 of [Mat95] for general background. In fact, we will establish the following "localized version" of Theorem A.

Theorem 2.1. *Let* $1 \leq m \leq n-1$ *and let*

$$\begin{cases} 1 \leq p < \infty & \text{if } m = 1 \text{ or } m = 2, \\ 1 \leq p < 2m/(m-2) & \text{if } m \geq 3. \end{cases} \tag{2.1}$$

If μ *is a locally finite Borel measure on* $\mathbb{R}^n$ *such that*

$$J_p(\mu, x) := \int_0^1 \beta_p^{(m)}(\mu, B(x,r))^2 \frac{dr}{r} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

then $\mu \llcorner \left\{x \in \mathbb{R}^n : 0 < \underline{D}^m(\mu, x) \leq \overline{D}^m(\mu, x) < \infty\right\}$ *is* m*-rectifiable.*

Proof. Without loss of generality, we assume for the duration of the proof that $\mathcal{H}^m$ is normalized so that $\omega_m = \mathcal{H}^m(B^m(0,1)) = 2^m$. This is the convention used in [Mat95].

Suppose that $1 \leq m \leq n-1$, let p belong to the range (2.1), and let μ be a locally finite Borel measure on $\mathbb{R}^n$ such that $J_p(\mu, x) < \infty$ at μ-a.e. $x \in \mathbb{R}^n$. Define

$$A := \left\{x \in \mathbb{R}^n : 0 < \underline{D}^m(\mu, x) \leq \overline{D}^m(\mu, x) < \infty\right\}.$$

Also, for each pair of integers $j, k \geq 1$, define

$$A(j,k) := \left\{x \in B(0, 2^k) : 2^{-j} r^m \leq \mu(B(x,r)) \leq 2^j r^m \text{ for all } 0 < r \leq 2^{-k}\right\}.$$

Then $\overline{A(j,k)}$ is compact and $\overline{A(j,k)} \subseteq A(j+1, k+1)$ for all $j, k \geq 1$. Also note that

$$A = \bigcup_{j,k=1}^{\infty} A(j,k) = \bigcup_{j,k=1}^{\infty} \overline{A(j,k)},$$

Thus, to prove that $\mu \llcorner A$ is m-rectifiable, it suffices to verify that $\mu \llcorner \overline{A(j,k)}$ is m-rectifiable for all $j, k \geq 1$.

Fix any $j, k \geq 1$ and set $K := \overline{A(j,k)}$, $\nu := \mu \llcorner K$, and $\sigma := \mathcal{H}^m \llcorner K$. In order to prove that ν is m-rectifiable, it is enough to show that $\nu \ll \sigma \ll \nu$ and σ is m-rectifiable. By Theorem 6.9 in [Mat95], since $2^{-j-1-m} \leq \overline{D}^m(\mu, x) \leq 2^{j+1-m}$ for all $x \in K$, we have

$$\nu(B(x,r)) = \mu(K \cap B(x,r)) \leq 2^{j+1} \mathcal{H}^m(K \cap B(x,r)) = 2^{j+1} \sigma(B(x,r)) \tag{2.2}$$

and

$$\sigma(B(x,r)) = \mathcal{H}^m(K \cap B(x,r)) \leq 2^{j+1+m} \mu(K \cap B(x,r)) = 2^{j+1+m} \nu(B(x,r)) \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Note that

$$\sigma(\mathbb{R}^n) = \sigma(B(0, 2^k)) \leq 2^{j+1+m} \mu(B(0, 2^k)) < \infty,$$

since μ is locally finite. That is, σ is a finite measure. Thus, ν and σ are mutually absolutely continuous by (2.2), (2.3), and Lemma 2.13 in [Mat95]. Now,

$$\sigma(B(x,r)) \leq 2^{j+1+m} \mu(B(x,r)) \leq 2^{2j+2+m} r^m \quad \text{for all } x \in K \text{ and } 0 < r \leq 2^{-k-1}. \tag{2.4}$$

On the other hand, let K' denote the set of $x \in K$ such that

$$2\nu(B(x,r)) = 2\mu(K \cap B(x,r)) \geq \mu(B(x,r)) \quad \text{for all } 0 < r \leq r_x$$

for some $r_x \leq 2^{-k-1}$. Then $\sigma(\mathbb{R}^n \setminus K') = 0$, because $\nu(\mathbb{R}^n \setminus K') = \mu(K \setminus K') = 0$, and

$$\sigma(B(x,r)) \geq 2^{-j-2}\mu(B(x,r)) \geq 2^{-2j-3}r^m \quad \text{for all } x \in K' \text{ and } 0 < r \leq r_x. \tag{2.5}$$

In particular, $\underline{D}^m(\sigma,x) \geq c(m,j) > 0$ at σ-a.e. $x \in \mathbb{R}^n$. To conclude that σ is m-rectifiable using Theorem 1.7, it remains to verify $J_p(\sigma,x) < \infty$ at σ-a.e. $x \in \mathbb{R}^n$.

By (2.4) and (2.5), there exists a constant $C = C(m,j) < \infty$ such that

$$C^{-1} \leq \frac{\nu(B(x,r))}{\sigma(B(x,r))} \leq C \quad \text{for all } 0 < r \leq r_x \text{ at } \sigma\text{-a.e. } x \in \mathbb{R}^n.$$

Thus, by differentiation of Radon measures, we can write $d\nu = f\,d\sigma$, where $f \in L^1_{\rm loc}(d\sigma)$ and $C^{-1} \leq f(x) \leq C$ at σ-a.e. $x \in \mathbb{R}^n$. Therefore, at σ-a.e. $x \in \mathbb{R}^n$, for every $0 < r \leq r_x$ and for every m-dimensional affine plane ℓ,

$$\begin{aligned}\int_{B(x,r)} \left(\frac{\operatorname{dist}(y,\ell)}{\operatorname{diam} B(x,r)}\right)^p \frac{d\sigma(y)}{\sigma(B(x,r))} &\leq C^2 \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,\ell)}{\operatorname{diam} B(x,r)}\right)^p \frac{d\nu(y)}{\nu(B(x,r))}\\ &\leq 2C^2 \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,\ell)}{\operatorname{diam} B(x,r)}\right)^p \frac{d\mu(y)}{\mu(B(x,r))}.\end{aligned}$$

Thus, $\beta_p^{(m)}(\sigma,B(x,r))^2 \leq (2C^2)^{2/p}\,\beta_p^{(m)}(\mu,B(x,r))^2$ for all $0 < r \leq r_x$ at σ-a.e. $x \in \mathbb{R}^n$. Since $J_p(\mu,x) < \infty$ at μ-a.e. $x \in \mathbb{R}^n$ and $\sigma \ll \mu$, it follows that $J_p(\sigma,x) < \infty$ at σ-a.e. $x \in \mathbb{R}^n$. Finally, since K is compact, $\sigma = \mathcal{H}^m \llcorner K$ is finite, and $\underline{D}^m(\sigma,x) > 0$ and $J_p(\sigma,x) < \infty$ at σ-a.e. $x \in \mathbb{R}^n$, we conclude that σ is m-rectifiable by Theorem 1.7. As noted above, this implies that $\nu = \mu \llcorner \overline{A(j,k)}$ is m-rectifiable for all $j,k \geq 1$, and therefore, $\mu \llcorner A$ is m-rectifiable. □

3. Proof of Theorem B

For every Borel measure μ on $\mathbb{R}^n$, define the quantity

$$S(\mu,x) := \sum_{Q\in\Delta} \frac{\operatorname{diam} Q}{\mu(Q)}\chi_Q(x) \in [0,\infty] \quad \text{for all } x \in \mathbb{R}^n,$$

where Δ denotes any system of *half-open* dyadic cubes in $\mathbb{R}^n$ of side length at most 1. Theorem B is a special case of the following statement.

Theorem 3.1. *Let $n \geq 2$. If μ is a locally finite Borel measure on $\mathbb{R}^n$, then*

$$\rho := \mu \llcorner \{x \in \mathbb{R}^n : S(\mu,x) < \infty\}$$

is 1-rectifiable. Moreover, there exists a countable family of rectifiable curves $\Gamma_i \subset \mathbb{R}^n$ and Borel sets $B_i \subseteq \Gamma_i$ such that $\mathcal{H}^1(B_i) = 0$ for all $i \geq 1$ and $\rho(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty B_i) = 0$.

We start with a lemma, which will be used to organize the proof of Theorem 3.1.

Lemma 3.2. *Let $n \geq 1$ and let μ be a locally finite Borel measure on $\mathbb{R}^n$. Given $Q_0 \in \Delta$ such that $\eta := \mu(Q_0) > 0$ and $N < \infty$, let*

$$A := \{x \in Q_0 : S(\mu,x) \leq N\}.$$

For all $0 < \varepsilon < 1/\eta$, the set of dyadic cubes $Q \subseteq Q_0$ can be partitioned into good cubes *and* bad cubes *with the following properties:*

(1) *every child of a bad cube is a bad cube;*

(2) *the set* $B := A \setminus \bigcup\{Q : Q \subseteq Q_0$ *is a bad cube*$\}$ *satisfies* $\mu(B) \geq (1-\varepsilon\eta)\mu(A)$*;*
(3) $\sum \operatorname{diam} Q < N/\varepsilon$*, where the sum ranges over all good cubes* $Q \subseteq Q_0$*.*

Proof. Suppose that n, μ, Q_0, η, N, and A are given as above and let $\varepsilon > 0$. If $\mu(A) = 0$, then we may declare every dyadic cube $Q \subseteq Q_0$ to be a bad cube and the conclusion of the lemma hold trivially. Thus, suppose that $\mu(A) > 0$. Declare that a dyadic cube $Q \subseteq Q_0$ is a *bad cube* if there exists a dyadic cube $R \subseteq Q_0$ such that $Q \subseteq R$ and $\mu(A \cap R) \leq \varepsilon\mu(A)\mu(R)$. We call a dyadic cube $Q \subseteq Q_0$ a *good cube* if Q is not a bad cube. Property (1) is immediate. To check property (2), observe that

$$\mu(A \setminus B) \leq \sum_{\text{maximal bad } Q\subseteq Q_0} \mu(A\cap Q) \leq \varepsilon\mu(A) \sum_{\text{maximal bad } Q\subseteq Q_0} \mu(Q) \leq \varepsilon\mu(A)\mu(Q_0),$$

where the last inequality follows because the maximal bad cubes are pairwise disjoint (since Δ is composed of half-open cubes). Recalling $\mu(Q_0) = \eta$, it follows that

$$\mu(B) = \mu(A) - \mu(A \setminus B) \geq (1 - \varepsilon\eta)\mu(A).$$

Thus, property (2) holds. Finally, since $S(\mu, x) \leq N$ for all $x \in A$,

$$N\mu(A) \geq \int_A S(\mu,x)\, d\mu(x) \geq \sum_{Q\subseteq Q_0} \operatorname{diam} Q\, \frac{\mu(A\cap Q)}{\mu(Q)} > \varepsilon\mu(A) \sum_{\text{good } Q\subseteq Q_0} \operatorname{diam} Q,$$

where we interpret $\mu(A \cap Q)/\mu(Q) = 0$ if $\mu(Q) = 0$. Because $\mu(A) > 0$, it follows that

$$\sum_{\text{good } Q\subseteq Q_0} \operatorname{diam} Q < \frac{N}{\varepsilon}.$$

This verifies property (3). □

Lemma 3.3. *Let* $n \geq 2$ *and let* μ *be a locally finite Borel measure on* $\mathbb{R}^n$*. If*

$$\mu(\{x \in Q_0 : S(\mu,x) \leq N\}) > 0 \quad \textit{for some } Q_0 \in \Delta \textit{ and } N < \infty,$$

then for all $0 < \varepsilon < 1/\mu(Q_0)$ *the set* $B = B(\mu, Q_0, N, \varepsilon)$ *described in Lemma 3.2 lies in a rectifiable curve* Γ *with* $\mathcal{H}^1(\Gamma) < N/2\varepsilon$ *and* $\mathcal{H}^1(B) = 0$*.*

Proof. Let $n \geq 2$ and let μ be a locally finite Borel measure on $\mathbb{R}^n$. Suppose $\mu(A) > 0$ for some $Q_0 \in \Delta$ and $N < \infty$, where $A = \{x \in Q_0 : S(\mu,x) \leq N\}$. Then $\eta := \mu(Q_0) > 0$, as well. Given any $0 < \varepsilon < 1/\eta$, let $B = B(\mu, Q_0, N, \varepsilon)$ denote the set from Lemma 3.2. Since ε is small enough such that $\mu(B) \geq (1-\varepsilon\eta)\mu(A) > 0$, the cube Q_0 is a good cube. Construct a connected set $T \subset \mathbb{R}^n$ by drawing a (closed) straight line segment ℓ_Q from the center of each good cube $Q \subsetneq Q_0$ to the center of its parent, which is also a good cube. Let $\overline{T}$ denote the closure of T. For all $\delta > 0$,

$$\overline{T} \subseteq \bigcup_{\substack{\text{good } Q\subsetneq Q_0\\ \operatorname{diam} Q>\delta}} \ell_Q \cup \bigcup_{\substack{\text{good } Q\subseteq Q_0\\ \operatorname{diam} Q\leq\delta}} \overline{Q},$$

whence

$$\mathcal{H}^1_\delta(\overline{T}) \leq \sum_{\substack{\text{good } Q\subsetneq Q_0\\ \operatorname{diam} Q>\delta}} \operatorname{diam} \ell_Q + \sum_{\substack{\text{good } Q\subseteq Q_0\\ \operatorname{diam} Q\leq\delta}} \operatorname{diam} \overline{Q} = \sum_{\substack{\text{good } Q\subsetneq Q_0\\ \operatorname{diam} Q>\delta}} \frac{1}{2}\operatorname{diam} Q + \sum_{\substack{\text{good } Q\subseteq Q_0\\ \operatorname{diam} Q\leq\delta}} \operatorname{diam} Q.$$

Here we used the fact that any straight line segment ℓ can be subdivided into finitely many line segments $\ell'_1, \ldots, \ell'_k$ such that $\operatorname{diam} \ell'_i \leq \delta$ for all i and $\sum_{i=1}^{k} \operatorname{diam} \ell'_i = \operatorname{diam} \ell$. Since $\sum_{\text{good } Q \subseteq Q_0} \operatorname{diam} Q < N/\varepsilon$, it follows that

$$\mathcal{H}^1(\overline{T}) = \lim_{\delta \downarrow 0} \mathcal{H}^1_\delta(\overline{T}) \leq \frac{1}{2} \sum_{\text{good } Q \subsetneq Q_0} \operatorname{diam} Q < \frac{N}{2\varepsilon}.$$

Now,

$$\begin{aligned} B &\subseteq Q_0 \setminus \bigcup_{\text{bad } Q \subset Q_0} Q \\ &= \left\{ \bigcap_{i=0}^{\infty} Q_i : Q_0 \supseteq Q_1 \supseteq \ldots \text{ is chain of good cubes, } \lim_{i\to\infty} \operatorname{diam} Q_i = 0 \right\} &\quad (3.1) \\ &\subseteq \left\{ \lim_{i\to\infty} x_i : x_i \in \ell_{Q_i} \text{ for some good cubes } Q_0 \supseteq Q_1 \supseteq \ldots,\ \lim_{i\to\infty} \operatorname{diam} Q_i = 0 \right\}. &\quad (3.2) \end{aligned}$$

Thus, $B \subseteq \overline{T}$ by (3.2). Moreover, since $\sum_{\text{good } Q \subseteq Q_0} \operatorname{diam} Q < \infty$, $\mathcal{H}^1(B) = 0$ by (3.1). Finally, because $\overline{T}$ is a continuum in $\mathbb{R}^n$ with $\mathcal{H}^1(\overline{T}) < \infty$, $\overline{T}$ coincides with the image $\Gamma = f([0,1])$ of some Lipschitz map $f : [0,1] \to \mathbb{R}^n$; e.g. see [Fal86, Chapter 3]. □

The proof of Theorem 3.1 uses Lemmas 3.2 and 3.3 repeatedly over a suitable, countable choice of parameters.

Proof of Theorem 3.1. Suppose $n \geq 2$ and let μ be a locally finite Borel measure on $\mathbb{R}^n$. Our goal is to show that $\mu \llcorner \{x \in \mathbb{R}^n : S(\mu, x) < \infty\}$ is 1-rectifiable. It suffices to prove that $\mu \llcorner \{x \in Q_0 : S(\mu, x) \leq N\}$ is 1-rectifiable for all $Q_0 \in \Delta$ and for all integers $N \geq 1$.

Fix $Q_0 \in \Delta$ and $N \geq 1$. Let $A = \{x \in Q_0 : S(\mu, x) \leq N\}$. If $\mu(A) = 0$, then there is nothing to prove. Thus, assume $\mu(A) > 0$. Then $\eta = \mu(Q_0) > 0$, as well. Pick any sequence $(\varepsilon_i)_{i=1}^\infty$ such that $0 < \varepsilon_i < 1/\eta$ for all $i \geq 1$ and $\varepsilon_i \to 0$ as $i \to \infty$. By Lemmas 3.2 and 3.3, there exist a Borel set $B_i = B(\mu, Q_0, N, \varepsilon_i) \subseteq A$ and a rectifiable curve $\Gamma_i \supseteq B_i$ such that $\mathcal{H}^1(B_i) = 0$ and $\mu(A \setminus B_i) \leq \varepsilon_i \eta \mu(A)$. Hence

$$\mu\left(A \setminus \bigcup_{i=1}^\infty \Gamma_i\right) \leq \mu\left(A \setminus \bigcup_{i=1}^\infty B_i\right) \leq \inf_{j\geq 1} \mu(A \setminus B_j) \leq \eta\mu(A) \inf_{j \geq 1} \varepsilon_j = 0.$$

Therefore, $\mu \llcorner A$ is 1-rectifiable, and moreover, $\mu \llcorner A\,(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty B_i) = 0$. □

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