All problem sets are optional. If you turn in solutions to problems, I will read and return them to you with feedback.

Problem A: Let μ be a Radon measure on $\mathbb{R}^n$, $n \geq 2$. Prove that there exists at most countably many *parallel* $(n-1)$-dimensional affine planes L such that $\mu(L) > 0$.

Problem B: Let μ be a Radon measure on $\mathbb{R}^n$ and let $\Delta(\mathbb{R}^n)$ denote the system of closed dyadic cubes in $\mathbb{R}^n$, i.e. all sets of the form

$$Q = \left[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\right] \times \cdots \times \left[\frac{j_n}{2^k}, \frac{j_n+1}{2^k}\right], \qquad j_1, \dots, j_n, k \in \mathbb{Z}.$$

Prove there exists $x \in \mathbb{R}^n$ such that $\Delta_x(\mathbb{R}^n) = \{x + Q : Q \in \Delta(\mathbb{R}^n)\}$ has the property $\mu(\partial R) = 0$ for every $R \in \Delta_x(\mathbb{R}^n)$.

Problem C: The *support* of a Borel measure μ on $\mathbb{R}^n$ is the set defined by

$$\operatorname{spt}\mu = \{x \in \mathbb{R}^n : \mu(B(x,r)) > 0 \text{ for all } r > 0\}.$$

Equivalently, $\operatorname{spt}\mu$ is the smallest closed set $F \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus F) = 0$. A Radon measure μ on $\mathbb{R}^n$ is called a *doubling measure* if there exists a constant $0 < C < \infty$ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \quad \text{for all } x \in \operatorname{spt}\mu,\ r > 0.$$

Use the Vitali covering lemma to prove that if μ is a doubling measure on $\mathbb{R}^n$, then for every fine cover $\mathcal{F}$ of a set $A \subset \mathbb{R}^n$ by closed balls, there exists a disjoint countable collection $\mathcal{G} \subset \mathcal{F}$ such that

$$\mu\left(A \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$