SUBSETS OF RECTIFIABLE CURVES IN BANACH SPACES: SHARP EXPONENTS IN SCHUL-TYPE THEOREMS

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ABSTRACT. The Analyst's Traveling Salesman Problem is to find a characterization of subsets of rectifiable curves in a metric space. This problem was introduced and solved in the plane by Jones in 1990 and subsequently solved in higher-dimensional Euclidean spaces by Okikiolu in 1992 and in the infinite-dimensional Hilbert space ℓ_2 by Schul in 2007. In this paper, we establish sharp extensions of Schul's necessary and sufficient conditions for a bounded set $E \subset \ell_p$ to be contained in a rectifiable curve from $p = 2$ to $1 < p < \infty$. While the necessary and sufficient conditions coincide when $p = 2$, we demonstrate that there is a strict gap between the necessary condition and sufficient condition when $p \neq 2$. This investigation is partly motivated by recent work of Edelen, Naber, and Valtorta on Reifenberg-type theorems in Banach spaces and complements work of Hahlomaa and recent work of David and Schul on the Analyst's TSP in general metric spaces.

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1. INTRODUCTION

Given a set in a path-connected metric space, we may ask whether or not the set is contained in a curve of finite length (also called a *rectifiable curve*), and if so, ask how to find a curve containing the set that is (essentially) as short as possible. This problem was introduced and solved in the Euclidean plane by Jones [Jon90] and is now commonly known as the *Analyst's Traveling Salesman Problem*. While it is immediate that a set contained in a rectifiable curve is necessarily bounded and has finite one-dimensional Hausdorff measure $\mathcal{H}^1$, this pair of conditions is not sufficient. To decide when a set is

contained in a rectifiable curve requires additional information about the local geometry of the set within the space (see below). Full solutions to the Analyst's Traveling Salesman Problem are currently available in $\mathbb{R}^n$ [Oki92], Carnot groups [Li19], infinite-dimensional Hilbert space ℓ_2 [Sch07c], graph inverse limit spaces [DS17], and for Radon measures in $\mathbb{R}^n$ [BS17]; partial solutions are available in general metrics spaces [Hah05, DS19] as well as for higher-dimensional curves [BNV19, BZ19] and surfaces [DT12, AS18, ENV19, Ghi19, AV19, Vil19]. Beyond the intrinsic interest of the Analyst's TSP in metric geometry, finding tests to determine when a set is contained in a rectifiable curve or "nice" surface has led to applications in complex analysis, dynamics and probability, geometric analysis, and harmonic analysis. For a sample of applications, see [BJ94, BJ97, BJPP97, Tol03, AS12, McN13, AT15, NV17, Bad19, BV19a, Azz19, BV19b, GM19, Nap20].

In this paper, we establish sharp extensions of Schul's necessary and sufficient conditions for a bounded set $E \subset \ell_p$ to be contained in a rectifiable curve from $p = 2$ to $1 < p < \infty$ (see Theorem 1.6). While the necessary and sufficient conditions coincide when $p = 2$, we demonstrate that there is a strict gap between the necessary condition and sufficient condition when $p \neq 2$. En route, we prove that the classes of rectifiable curves in the infinite-dimensional spaces ℓ_p and ℓ_q differ when $p \neq q$:

Proposition 1.1. *Let $1 < p < \infty$. Every rectifiable curve in ℓ_p is a rectifiable curve in ℓ_q for all $q \geq p$. However, there exists a curve Γ in ℓ_p such that Γ is rectifiable (i.e. $\mathcal{H}^1(\Gamma) < \infty$) in ℓ_q for every $q > p$, but Γ is not rectifiable (i.e. $\mathcal{H}^1(\Gamma) = \infty$) in ℓ_p.*

Proposition 1.1 and Theorem 1.6 capture a special infinite-dimensional phenomenon. In particular, they imply that a solution of the Analyst's TSP in ℓ_p cannot be neatly derived from the solution in ℓ_2. By contrast, bi-Lipschitz equivalence of finite-dimensional Banach spaces ensures that a set in $\mathbb{R}^n$ is (a subset of) a rectifiable curve independent from the choice of underlying norm, even though the actual length of the curve depends on the norm. This paper serves to clarify the difference between the finite and infinite-dimensional settings. An essential reason for us to study (subsets of) rectifiable curves in ℓ_p for $1 < p < \infty$ is that the spaces interpolate between ℓ_2, where the Analyst's TSP is solved, and ℓ_∞, which contains an isometric copy of any separable metric space. Thus, a resolution of the Analyst's TSP in ℓ_p may provide insight into the Analyst's TSP in general metric spaces. For further discussion and description of related research, see §1.3.

1.1. **Analyst's TSP in Euclidean space and ℓ_2.** To solve the Analyst's TSP in the plane, Jones introduced unilateral linear approximation numbers, now universally called *Jones' beta numbers*, which measure how "flat" a set is in a given window. The Jones' beta numbers are well-defined in any Banach space. Let $\mathbb{X}$ be a Banach space[1], let $E \subset \mathbb{X}$ be a nonempty set, and let $Q \subset \mathbb{X}$ be a set of positive diameter. If $E \cap Q \neq \emptyset$, define

$$(1.1) \qquad\qquad \beta_E(Q) = \inf_L \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, L)}{\operatorname{diam} Q} \in [0, 1],$$

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[1]All Banach spaces in this paper are real Banach spaces of dimension at least 2.

where the infimum ranges over all one-dimensional affine subspaces (lines) $L \subset \mathbb{X}$; and, if $E \cap Q = \emptyset$, define $\beta_E(Q) = 0$. At one extreme, if $\beta_E(Q) = 0$, then $E \cap Q$ is contained in a line. At the other extreme, if $\beta_E(Q) \gtrsim 1$, then the set $E \cap Q$ is uniformly far away from every line passing through Q. From the definition, it immediately follows that

$$(1.2) \qquad \beta_E(R) \le \frac{\operatorname{diam} Q}{\operatorname{diam} R} \beta_F(Q) \quad \text{for all } E \subset F \text{ and } R \subset Q.$$

In view of the fact that rectifiable curves (having parameterizations of bounded variation) admit tangent lines $\mathcal{H}^1$-a.e., one may expect that sets contained in a finite length curve have "vanishing beta numbers" at typical points of those sets. The *Analyst's Traveling Salesman Theorem* makes this idea precise and provides a characterization of subsets of rectifiable curves with an estimate on the shortest length of a curve containing the set:

Theorem 1.2 (Jones [Jon90] in $\mathbb{R}^2$; Okikiolu [Oki92] in $\mathbb{R}^n$). *Let $n \ge 2$ and let $E \subset \mathbb{R}^n$. Then E is contained in a rectifiable curve if and only if*

$$(1.3) \qquad S_E(\mathbb{R}^n) := \operatorname{diam} E + \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_E(3Q)^2 \operatorname{diam} Q < \infty,$$

where the sum ranges over all dyadic cubes Q in $\mathbb{R}^n$ and $3Q$ denotes the concentric dilate of the cube Q with scaling factor 3. More precisely, if $S_E(\mathbb{R}^n) < \infty$, then E is contained in a curve Γ in $\mathbb{R}^n$ with

$$(1.4) \qquad \mathcal{H}^1(\Gamma) \lesssim_n S_E(\mathbb{R}^n).$$

If $\Sigma \subset \mathbb{R}^n$ is a connected set, then

$$(1.5) \qquad S_\Sigma(\mathbb{R}^n) \lesssim_n \mathcal{H}^1(\Sigma).$$

The constant 3 in (1.3) can be replaced with any constant $A > 1$. Then (1.4) and (1.5) hold with implicit constants depending on n and A.

Remark 1.3. If E is a subset of a rectifiable curve in $\mathbb{R}^n$, then the Analyst's Traveling Salesman Theorem ensures

$$(1.6) \qquad \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \beta_E(3Q) \ge \varepsilon}} \operatorname{diam} Q \le \varepsilon^{-2} S(E) < \infty \quad \text{for all } \varepsilon > 0.$$

It follows that

$$(1.7) \qquad \lim_{\substack{Q \in \Delta(\mathbb{R}^n) \\ Q \downarrow x}} \beta_E(3Q) = 0 \quad \text{at } \mathcal{H}^1\text{-a.e. } x \in E,$$

or equivalently by (1.2),

$$(1.8) \qquad \lim_{r \downarrow 0} \beta_E(B(x, r)) = 0 \quad \text{at } \mathcal{H}^1\text{-a.e. } x \in E.$$

Thus, subsets of rectifiable curves in $\mathbb{R}^n$ have "vanishing beta numbers" at typical points in the sense of (1.8). It is possible to construct examples of generalized von Koch snowflake curves (with carefully chosen angles) to show that (1.8) is also satisfied by certain curves of infinite length. By contrast, the Analyst's Traveling Salesman Theorem guarantees that
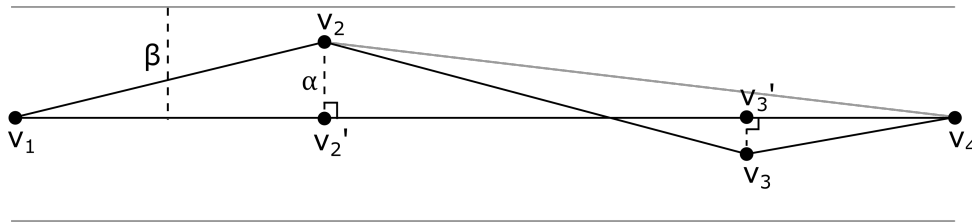
FIGURE 1. To use the Pythagorean theorem to verify (1.9) and (1.10) (shown with $k = 4$), first draw right triangles formed from line segments $[v_i, v_{i+1}]$ between consecutive points and the line passing through v_1 and v_k.

every curve Γ in $\mathbb{R}^n$ of infinite length satisfies $S_\Gamma(\mathbb{R}^n) = \infty$. In other words, finiteness of $S_\Gamma(\mathbb{R}^n)$ is a perfect test to determine rectifiability of a curve Γ in $\mathbb{R}^n$.

Remark 1.4. Let V be a finite set of points in $\mathbb{R}^n$ (equipped with the Euclidean norm) that is *1-separated* in the sense $|v - w| \geq 1$ for all distinct $v, w \in V$. Assume that ℓ is a line in $\mathbb{R}^n$ such that $\mathrm{dist}(v, \ell) \leq \beta \ll 1$ for all $v \in V$. Then the set $V = \{v_1, \ldots, v_k\}$ may be enumerated according to its orthogonal projection onto ℓ. For simplicity, let us further assume that $v_1, v_k \in \ell$. By the triangle inequality and a simple computation with the Pythagorean theorem (see Figure 1),

$$(1.9) \qquad |v_1 - v_k| \leq |v_1 - v_2| + \cdots + |v_{k-1} - v_k| \leq (1 + C_1\beta^2)|v_1 - v_k|$$

for some universal constant C_1. Conversely, if $\mathrm{dist}(v_i, \ell) \geq \alpha$ for some $1 \leq i \leq k$, then the Pythagorean theorem yields

$$(1.10) \qquad |v_1 - v_i| + |v_i - v_k| \geq (1 + C_2\alpha^2)|v_1 - v_k|$$

for some universal constant C_2. At a high level, the estimates (1.9) and (1.10) correspond to (1.4) and (1.5) in Analyst's Traveling Salesman Theorem, respectively. Informally, we say that the Pythagorean theorem is responsible for the exponent 2 on $\beta_E(3Q)^2$ in (1.3).

The dependence on the ambient dimension in the implicit constants in (1.4) and (1.5) is ultimately a consequence of using dyadic cubes in the sum $S_E(\mathbb{R}^n)$ to index "all" locations and scales in $\mathbb{R}^n$. To formulate a dimension independent version of the Analyst's Traveling Salesman Theorem in ℓ_2, Schul [Sch07c] replaced $S_E(\mathbb{R}^n)$ with a sum $S_E(\mathscr{G})$ indexed over a multiresolution family $\mathscr{G}$ of "all" locations and scales in the set E.

Let $\mathbb{X}$ be a metric space and let $E \subset \mathbb{X}$ be a nonempty set. For any $\rho > 0$, a *ρ-net* X_ρ in $\mathbb{X}$ is a set such that $\mathrm{dist}(y, z) \geq \rho$ for all distinct $y, z \in X_\rho$ and $\mathrm{dist}(x, X_\rho) < \rho$ for all $x \in \mathbb{X}$. Following [Sch07c], we define a *multiresolution family* $\mathscr{G}$ for E with *inflation factor* $A_\mathscr{G} > 1$ to be a collection of closed balls of the form

$$(1.11) \qquad \mathscr{G} = \{B(x, A_\mathscr{G}2^{-k}) : x \in X_k, k \in \mathbb{Z}\},$$

where $(X_k)_{k \in \mathbb{Z}}$ is a nested family of 2^{-k}-nets for E.

For any nonempty set $E \subset \ell_2$ and multiresolution family $\mathscr{G}$ for E, define the sum

$$(1.12) \qquad S_E(\mathscr{G}) := \operatorname{diam} E + \sum_{Q \in \mathscr{G}} \beta_E(Q)^2 \operatorname{diam} Q.$$

Theorem 1.5 (Schul [Sch07c]). *If $E \subset \ell_2$ and $S_E(\mathscr{G}) < \infty$ for some multiresolution family $\mathscr{G}$ for E with inflation factor $A_{\mathscr{G}} > 200$, then E is contained in a rectifiable curve Γ in ℓ_2 with*

$$(1.13) \qquad \mathcal{H}^1(\Gamma) \lesssim_{A_{\mathscr{G}}} S_E(\mathscr{G}).$$

If $\Sigma \subset \ell_2$ is a connected set and $\mathscr{H}$ is a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$, then

$$(1.14) \qquad S_\Sigma(\mathscr{H}) \lesssim_{A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

Note that the implicit constants in (1.13) and (1.14) depend on the inflation factor of the multiresolution family, but are otherwise dimension free. Once again, the Pythagorean theorem in ℓ_2 determines the exponent 2 on $\beta_E(Q)^2$ in Theorem 1.5 *à la* Remark 1.4.

1.2. **Schul's theorem in ℓ_p.** For any nonempty set $E \subset \ell_p$ and multiresolution family $\mathscr{G}$ for E, define the sums

$$(1.15) \qquad S_{E,r}(\mathscr{G}) := \operatorname{diam} E + \sum_{Q \in \mathscr{G}} \beta_E(Q)^r \operatorname{diam} Q \quad \text{for all } 0 < r < \infty.$$

Note that in the notation of the previous section, $S_{E,2}(\mathscr{G}) \equiv S_E(\mathscr{G})$. Moreover, by (1.2),

$$(1.16) \qquad S_{E,r}(\mathscr{G}) \leq S_{F,s}(\widehat{\mathscr{G}}) \quad \text{for all } E \subset F \text{ and } s \leq r,$$

where $\widehat{\mathscr{G}}$ is a multiresolution family for F extending $\mathscr{G}$.

The following theorem, extending Schul's theorem from ℓ_2 to ℓ_p with $1 < p < \infty$, is our main result. We emphasize that when $p \neq 2$, there is a strict gap between the necessary and sufficient conditions for a set to be contained in a rectifiable curve.

Theorem 1.6 (sharp necessary and sufficient conditions in ℓ_p). *Let $1 < p \leq 2$. If $E \subset \ell_p$ and $S_{E,p}(\mathscr{G}) < \infty$ for some multiresolution family $\mathscr{G}$ for E with inflation factor $A_{\mathscr{G}} \geq 240$, then E is contained in a curve Γ in ℓ_p with*

$$(1.17) \qquad \mathcal{H}^1(\Gamma) \lesssim_{p,A_{\mathscr{G}}} S_{E,p}(\mathscr{G}).$$

If $\Sigma \subset \ell_p$ is a connected set and $\mathscr{H}$ is a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$, then

$$(1.18) \qquad S_{\Sigma,2}(\mathscr{H}) \lesssim_{p,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

The exponents p and 2 in (1.17) and (1.18) are sharp.

For $2 \leq p < \infty$, the same conclusion holds, but with the exponents in (1.17) and (1.18) reversed. That is,

$$(1.19) \qquad \mathcal{H}^1(\Gamma) \lesssim_{p,A_{\mathscr{G}}} S_{E,2}(\mathscr{G})$$
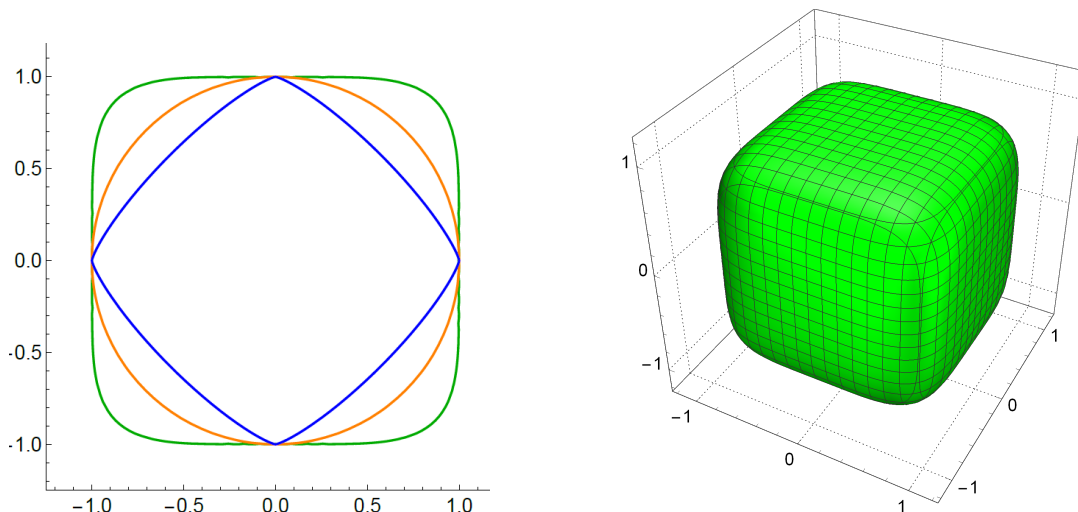
FIGURE 2. Unit balls in $\ell_{5/4}^2$ (blue), $\mathbb{R}^2$ (orange), ℓ_5^2 and ℓ_5^3 (green), where ℓ_p^n denotes $(\mathbb{R}^n, |\cdot|_p)$.

and

$$S_{\Sigma,p}(\mathscr{H}) \lesssim_{p,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma). \tag{1.20}$$

The exponents 2 and p in (1.19) *and* (1.20) *are sharp.*

Remark 1.7. More generally, we prove analogues of (1.17) and (1.19) hold in all uniformly smooth Banach spaces (see Theorem 2.26) and analogues of (1.18) and (1.20) hold in all uniformly convex Banach spaces (see Theorem 3.14). The ℓ_p spaces are uniformly smooth and convex for all $1 < p < \infty$. In addition, we prove that a universal sufficient condition with exponent 1 is valid in arbitrary Banach spaces (see Theorem 2.10).

An essential feature of ℓ_2 is that the unit ball and induced distance are *rotationally-invariant*. In particular, to compute the distance of a point x to a line L in Hilbert space, one may first translate and rotate so that $L = \mathrm{span}(e_1)$ and $x \in \mathrm{span}(e_1, e_2)$ if convenient. In ℓ_p, when $p \neq 2$, rotational-invariance is no longer available and computation of the distance of a point to a line is sensitive to the position of the line and geometry of the unit ball. We remark that the gain in complexity witnessed when moving from ℓ_p^2 to ℓ_p^3 (e.g. consider the shape of slices of their unit balls, see Figure 2) continues to increase when moving from the finite-dimensional spaces ℓ_p^n to the infinite-dimensional space ℓ_p. For instance, although the norms in ℓ_2^n and ℓ_p^n are $C(n,p)$-bi-Lipschitz equivalent for each pair n and p, the bi-Lipschitz constant $C(n,p)$ degenerates as $n \to \infty$ for each $p \neq 2$.

Example 1.8. To illustrate the essential idea behind the exponents in Theorem 1.6, let's estimate the length gain in $\ell_5^2 = (\mathbb{R}^2, |\cdot|_5)$ of isosceles triangles

$$\begin{aligned} & T_h : a_h = (0,0), \ b_h = (l/2, h), \ c_h = (l, 0), \ \text{and} \\ & T_d : a_d = (0,0), \ b_d = (2^{-6/5}l - 2^{-1/5}h, 2^{-6/5}l + 2^{-1/5}h), \ c_d = (2^{-1/5}l, 2^{-1/5}l) \end{aligned} \tag{1.21}$$

with horizontal and diagonal bases $\overline{ac}$ of length l and height $\mathrm{dist}(b, \overline{ac}) = h$. On one hand,

$$(1.22) \qquad \begin{aligned} |\overline{a_h b_h}|_5 + |\overline{b_h c_h}|_5 - |\overline{a_h c_h}|_5 &= 2\left((l/2)^5 + h^5\right)^{1/5} - l \\ &= l\left((1 + 32(h/l)^5)^{1/5} - 1\right) \simeq l(h/l)^5 \end{aligned}$$

for $h \ll l$ by Taylor's theorem for $x \mapsto x^{1/5}$ at $x = 1$. On the other hand,

$$(1.23) \qquad \begin{aligned} |\overline{a_d b_d}|_5 + |\overline{b_d c_d}|_5 - |\overline{a_d c_d}|_5 &= 2\left((2^{-6/5}l - 2^{-1/5}h)^5 + (2^{-6/5}l + 2^{-1/5}h)^5\right)^{1/5} - l \\ &= l\left((1 + 40(h/l)^2 + 80(h/l)^4)^{1/5} - 1\right) \simeq l(h/l)^2 \end{aligned}$$

when $h \ll l$ by Taylor's theorem, again. This computation indicates that on the axial directions the excess in the triangle inequality for "flat" triangles is comparable to $l(h/l)^5$, while on the diagonal directions, the excess is comparable to $l(h/l)^2$. In fact, as long as the base of the triangle is neither horizontal nor vertical, the excess is comparable to $l(h/l)^2$ provided that $h \ll l$ (depending on proximity of the base to an axial direction). In higher dimensions, the excess in the triangle inequality also depends on the direction of the altitude of a triangle in addition to the direction of its base.

The strict gap in the exponents witnessed in Example 1.8 is an essential feature of ℓ_p geometry when $p \neq 2$. To prove Theorem 1.6, we employ tools from functional analysis such as modulus of smoothness, modulus of convexity, and Alber's generalized projections to carry out the estimates outlined in Remark 1.4 and Example 1.8 in the setting of ℓ_p in full generality. In fact, we work in arbitrary uniformly smooth and uniformly convex Banach spaces, which include the ℓ_p spaces when $1 < p < \infty$. To establish the sufficient conditions (1.17) and (1.19), we modify the "parametric proof" of the sufficient condition in ℓ_2 recently developed by Badger, Naples, and Vellis [BNV19]. This approach also yields new sufficient criteria for a set in a Banach space to be contained in a $(1/s)$-Hölder curve with $s > 1$ (see Theorem 2.7). To verify the necessary conditions (1.18) and (1.20), we follow the proof of the necessary condition in ℓ_2 originally introduced by Schul [Sch07c], indicating which parts of the proof are metric, which parts are Banach, and which parts rely on the uniform convexity of the norm. The adaptation of the Hilbert space proofs to ℓ_p and related Banach spaces is non-trivial, but largely technical; see §2 (sufficient conditions) and §3 (necessary conditions) for details. To show that the exponents in Theorem 1.6 are sharp and to prove Proposition 1.1, we construct Koch-snowflake-like curves in §4. While the examples of sharpness for (1.18) and (1.19) can be built inside any two-dimensional subspace of ℓ_p, the examples for sharpness of (1.17) and (1.20) require curves that extend outside of every finite-dimensional subspace of ℓ_p. This is plausible, because we know the critical exponent in any finite-dimensional Banach space is 2 by the Analyst's Traveling Salesman Theorem in $\mathbb{R}^n$ and bi-Lipschitz equivalence of norms in finite dimensions. Thus, it is natural to expect examples showing sharpness of an exponent $p \neq 2$ to live in infinite dimensions.

1.3. **Related work.** The inception for this investigation is a recent paper of Edelen, Naber, and Valtorta [ENV19] that extends *Reifenberg's Topological Disk Theorem* [Rei60]

(also see [DT12]) from the Euclidean to infinite-dimensional Hilbert and Banach spaces. The original formulation of the theorem says that if a closed set $\Sigma \subset \mathbb{R}^n$ is uniformly bilaterally $\delta(k,n)$-close[2] to some k-dimensional affine plane at all locations in Σ and on all sufficiently small scales, then Σ is locally homoeomorphic to open subsets of $\mathbb{R}^k$ (that is, Σ is locally a topological disk). Reifenberg proved the Topological Disk Theorem to establish existence and regularity of the Plateau problem in arbitrary dimension and codimension. Underpinning the theorem is an algorithm that takes a collection of planes approximating the set Σ and patches them together using orthogonal projections and partitions of unity to construct a parameterization. Edelen, Naber, and Valtorta solve the problem of how to implement this algorithm in a Banach space with dimension independent estimates. The main application of the Reifenberg algorithm in [ENV19] is a structure theorem for measures in Banach spaces, which we now briefly describe.

Following the convention used in [ENV19], for every Borel regular measure μ on a Banach space $\mathbb{X}$, location $x \in \mathbb{X}$, and scale $r > 0$, define the *k-dimensional L^2 Jones beta number* $\beta_\mu^k(x,r)$ by

$$(1.24) \qquad \beta_\mu^k(x,r)^2 := \inf_{p+V^k} r^{-k-2} \int_{B(x,r)} \mathrm{dist}(z, p+V)^2 \, d\mu(z),$$

where the infimum ranges over all k-dimensional affine subspaces of $\mathbb{X}$. Beta numbers associated to measures were originally introduced by David and Semmes [DS91, DS93] to build a bridge between singular integral operators and quantitative rectifiability of sets.

Theorem 1.9 (Edelen, Naber, Valtorta [ENV19, Theorem 2.1]). *Let $\mathbb{X}$ be a Banach space, let μ be a finite Borel measure supported in $B(0,1)$, let $S \subset B(0,1)$ be a set with $\mu(B(0,1) \setminus S) = 0$, and let $r : S \to (0,1)$. Assume that μ satisfies*

$$(1.25) \qquad \int_{r(x)}^2 \beta_\mu^k(x,s)^\alpha \, \frac{ds}{s} \leq M^{\alpha/2} \quad \text{for all } x \in S,$$

where the exponent α is determined as follows:

- *if $\mathbb{X}$ is a generic Banach space, then $\alpha = 1$;*
- *if $\mathbb{X}$ is a Hilbert space, then $\alpha = 2$;*
- *if $\mathbb{X}$ is a uniformly smooth Banach space and $k = 1$, then α is the smoothness power of $\mathbb{X}$ (e.g. the smoothness power of ℓ_p is $\min(p,2)$ for $1 < p < \infty$).*

Then there is a subset $S' \subset S$ so that we have the following packing/measure estimate:

$$(1.26) \qquad \mu\left(B(0,1) \setminus \bigcup_{x \in S'} B(x, r(x))\right) \lesssim_{k,\rho_\mathbb{X}} M \quad \text{and} \quad \sum_{x \in S'} r(x)^k \lesssim_{k,\rho_\mathbb{X}} 1,$$

where $\rho_\mathbb{X}$ denotes the modulus of smoothness of $\mathbb{X}$ (see §2 below).

Roughly speaking, condition (1.25) allows one to control the tilt of the approximating planes in the Reifenberg algorithm and construct local bi-Lipschitz parameterizations.

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[2]For a comparison of unilateral (Jones) versus bilateral (Reifenberg) flatness of a set, see [BL15].

For our present discussion, the most interesting aspect of Theorem 1.9 is the dependence of the exponent α on the geometry of the Banach space $\mathbb{X}$ and the dimension k of the approximating planes. In any given space, one would like to identify the largest possible exponent such that the theorem holds. The exponent $\alpha = 1$ corresponds to the triangle inequality, which holds in any Banach space, and the exponent $\alpha = 2$ corresponds to the Pythagorean theorem, which holds in any Hilbert space. In an intermediate scenario, Edelen, Naber, and Valtorta prove that when $\mathbb{X}$ is a smooth Banach space *and* $k = 1$, the exponent α can be taken to be the smoothness power of the Banach space. For example, $\alpha = p$ when $\mathbb{X} = \ell_p$ and $1 < p < 2$. Furthermore, Edelen, Naber, and Valtorta prove that the restriction to $k = 1$ is necessary to obtain $\alpha > 1$ in non-Hilbert spaces. This is tied up with the existence of good projections onto lines when $k = 1$ and the absence of good projections onto subspaces when $k \geq 2$; see [ENV19, §3.6 and §5.3] for details.

A strength of the Reifenberg algorithm over the Analyst's Traveling Salesman Theorem is that it gives conditions to build parameterizations of *every* dimension k. An advantage of the Analyst's Traveling Salesman Theorem over the Reifenberg algorithm is that it provides *necessary and sufficient* conditions for parameterizations of dimension $k = 1$. Edelen, Naber, and Valtorta's successful implementation of the Reifenberg algorithm in smooth Banach spaces with $k = 1$ provided our initial motivation to look for an Analyst's Traveling Salesman Theorem in Banach spaces.

A separate vein of research by Hahlomaa [Hah05] and David and Schul [DS19] (also see [Sch07a, Hah08]) focuses on the Analyst's TSP in the setting of an arbitrary metric space. Because metric spaces are not necessarily path-connected, it is natural to reformulate the Analyst's TSP as stated above and instead ask which sets in a given metric space $\mathbb{X}$ are contained in *rectifiable curve fragments*, i.e. images of Lipschitz maps $f : S \to \mathbb{X}$ from some set $S \subset [0,1]$. Hahlomaa's original work in this direction established an analogue of the *sufficient half* of the Analyst's Traveling Salesman Theorem by redefining Jones' beta numbers using Menger curvature, or equivalently, using the excess in the triangle inequality. For different perspectives on rectifiability in measure metric spaces, see e.g. [PT92, Kir94, AK00, Bat15, BCW17, BL17, Nap20].

Following [DS19], for a given metric space E and ball $Q = B(p, r)$, define $\beta_\infty^E(Q)$ by

$$(1.27) \qquad \begin{aligned} \beta_\infty^E(Q)^2 := r^{-1} \sup \big\{ \, &\mathrm{dist}(x, y) + \mathrm{dist}(y, z) - \mathrm{dist}(x, z) : \\ &x, y, z \in E \cap Q \text{ and } \mathrm{dist}(x, y) \leq \mathrm{dist}(y, z) \leq \mathrm{dist}(x, z) \big\}. \end{aligned}$$

The *metric beta number* $\beta_\infty^E(Q)$ measures the normalized excess in the triangle inequality among triples of points in $E \cap Q$. The exponent 2 on the left hand side of (1.27) is a convention that is imposed to make the statement of Theorem 1.10 look similar to Theorem 1.2 when $E \subset \mathbb{R}^N$ is endowed with the Euclidean metric.

Theorem 1.10 (Hahlomaa [Hah05, Theorem 5.3]). *Let E be a metric space and let $\mathscr{G}$ be a multiresolution family for E with inflation factor $A_\mathscr{G} \simeq 1$. If*

$$(1.28) \qquad S_\infty^E(\mathscr{G}) := \mathrm{diam}\, E + \sum_{Q \in \mathscr{G}} \beta_\infty^E(Q)^2 \, \mathrm{diam}\, Q < \infty,$$

then there exists a set $A \subset [0,1]$ and a surjective Lipschitz map $f : A \to E$ with Lipschitz constant $\mathrm{Lip}(f) \lesssim S_\infty^E(\mathscr{G})$.

Remark 1.11. It is known that the converse to Hahlomaa's theorem is (quantitatively) false for certain rectifiable curves in $\ell_1^2 = (\mathbb{R}^2, |\cdot|_1)$; see [Sch07b, Example 3.3.1]. A similar phenomenon occurs in graph inverse limit spaces; see [DS17, §7].

David and Schul recently announced a partial converse to Hahlomaa's theorem, which is the first non-trivial necessary condition for the Analyst's TSP in a metric space. Together, Theorems 1.10 and 1.12 are quite striking and indicate the rough shape that a full solution to the Analyst's TSP in a general metric space might take. Recall that a metric space is *doubling* if every ball of radius r can be covered by at most D balls of radius $r/2$.

Theorem 1.12 (David and Schul [DS19, Theorem A]). *Let Σ be a connected, doubling metric space with doubling constant D and let $\mathscr{H}$ be a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$. For every $\varepsilon > 0$,*

$$(1.29) \qquad S_\infty^{\Sigma,\varepsilon}(\mathscr{H}) := \mathrm{diam}\, Q + \sum_{Q \in \mathscr{H}} \beta_\infty^\Sigma(Q)^{2+\varepsilon} \,\mathrm{diam}\, Q \lesssim_{\varepsilon,D,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

Remark 1.13. The doubling assumption in Theorem 1.12 allows the authors to simplify the overall proof of theorem. David and Schul conjecture (see [DS19, Remark 1.6]) that the doubling assumption can be dropped by implementing the techniques in [Sch07c].

David and Schul present several corollaries to Theorem 1.12 with alternative definitions of the metric beta numbers $\beta_\infty^\Sigma(Q)$ (see (1.27)). In particular, they obtain necessary conditions for the Analyst's TSP in ℓ_p, with $1 < p < \infty$, using traditional Jones' beta numbers $\beta_\Sigma(Q)$ (see (1.1)). Also see [DS19, Corollary D] for a more general statement on uniformly convex Banach spaces.

Corollary 1.14 (David and Schul [DS19]). *Let $1 < p < \infty$, let $\Sigma \subset \ell_p$ be a connected set with doubling constant D, and let $\mathscr{H}$ be a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$. For all $\varepsilon > 0$,*

$$(1.30) \qquad S_{E,\max(2,p)+\varepsilon}(\mathscr{H}) = \mathrm{diam}\, E + \sum_{Q \in \mathscr{H}} \beta_E(Q)^{\max(2,p)+\varepsilon} \,\mathrm{diam}\, Q \lesssim_{\varepsilon,D,p,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

In our main Theorem 1.6, we have removed the doubling assumption *and* the error ε from Corollary 1.14. This is accomplished by following the strategy in [Sch07c].

2. Modulus of smoothness and proof of the sufficient conditions

2.1. **Ordering flat sets in Banach spaces.** A simple, but important ingredient in all proofs of the Analyst's Traveling Salesman Theorem is that "almost flat" sets of points can be linearly ordered. To implement a generic Banach space version of the sufficient half of the Analyst's Traveling Salesman Theorem with universal constants (in the spirit of Hahlomaa [Hah05]), we first develop an instance of this principle. The following lemma is modeled after [BS17, Lemma 8.3].

Lemma 2.1 (flatness implies order). *Let $\mathbb{X}$ be a Banach space. Suppose that $V \subset \mathbb{X}$ is a δ-separated set with $\#V \geq 2$ and there exist lines L_1 and L_2 and a number $0 \leq \alpha < 1/6$ such that*

$$(2.1) \qquad \operatorname{dist}(v, L_i) \leq \alpha\delta \quad \text{for all } v \in V \text{ and } i = 1, 2.$$

Let $\pi_i : \mathbb{X} \to L_i$ denote a metric projection onto ℓ_i, i.e. any map satisfying

$$(2.2) \qquad \operatorname{dist}(x, \ell_i) = \operatorname{dist}(x, \pi_i(x)) \quad \text{for all } x \in \mathbb{X}.$$

There exist compatible identifications of L_1 and L_2 with $\mathbb{R}$ such that $\pi_1(v') \leq \pi_1(v'')$ if and only if $\pi_2(v') \leq \pi_2(v'')$ for all $v', v'' \in V$. If $v_1, v_2 \in V$, then

$$(2.3) \qquad (1 + 2\alpha)^{-1}|\pi_1(v_1) - \pi_2(v_2)| \leq |v_1 - v_2| \leq (1 + 3\alpha)|\pi_1(v_1) - \pi_1(v_2)|.$$

Proof. Without loss of generality, it suffices to assume $\delta = 1$. Let $V \subset \mathbb{X}$ be a 1-separated set with at least two points. Assume that there exist one-dimensional affine subspaces L_1 and L_2 in $\mathbb{X}$ and a number $0 \leq \alpha < 1/6$ such that

$$\operatorname{dist}(v, L_i) \leq \alpha \quad \text{for all } v \in V \text{ and } i = 1, 2.$$

Let π_i denote a metric projection onto L_i. For any distinct pair of points $v_1, v_2 \in V$,

$$1 \leq |v_1 - v_2| \leq |\pi_i(v_1) - \pi_i(v_2)| + 2\alpha,$$

because V is 1-separated and the distance of points in V to L_i is bounded by α. Hence

$$(2.4) \qquad |\pi_i(v_1) - \pi_i(v_2)| \geq |v_1 - v_2| - 2\alpha \geq 1 - 2\alpha > 2/3.$$

In particular, $2\alpha \leq 3\alpha|\pi_i(v_1) - \pi_i(v_2)|$, and it follows that

$$|v_1 - v_2| \leq (1 + 3\alpha)|\pi_i(v_1) - \pi_i(v_2)|.$$

This establishes the right half of (2.3). Similarly,

$$|\pi_i(v_1) - \pi_i(v_2)| \leq |v_1 - v_2| + 2\alpha \leq (1 + 2\alpha)|v_1 - v_2|,$$

which yields the left half of (2.3). In particular, note that

$$(2.5) \qquad |v_1 - v_2| \geq |\pi_i(v_1) - \pi_i(v_2)| - 2\alpha.$$

Suppose for contradiction that there are identifications of L_1 and L_2 with $\mathbb{R}$ and distinct points $v, v', v'' \in V$ such that $\pi_1(v) < \pi_1(v') < \pi_1(v'')$, but $\pi_2(v') < \pi_2(v) < \pi_2(v'')$. Set

$$x := |v - v'|, \qquad\qquad y := |v'' - v'|, \qquad\qquad z := |v'' - v|,$$
$$x_1 := |\pi_1(v) - \pi_1(v')|, \qquad y_1 := |\pi_1(v'') - \pi_1(v')|, \qquad z_1 := |\pi_1(v'') - \pi_1(v)|,$$
$$x_2 := |\pi_2(v) - \pi_2(v')|, \qquad y_2 := |\pi_2(v'') - \pi_2(v')|, \qquad z_2 := |\pi_2(v'') - \pi_2(v)|.$$

Heuristically, since $\pi_1(v) < \pi_1(v') < \pi_1(v'')$, we have $z \approx x + y$, and since $\pi_2(v') < \pi_2(v) < \pi_2(v'')$, we have $y \approx x + z$. Hence $z \approx z + 2x$, which yields a contradiction if α is sufficiently small. More precisely, by repeated application of (2.4) and (2.5),

$$z \geq z_1 - 2\alpha = x_1 + y_1 - 2\alpha \geq x_1 + y - 4\alpha \geq x_1 + y_2 - 6\alpha$$
$$= x_1 + x_2 + z_2 - 6\alpha \geq x_1 + x_2 + z - 8\alpha.$$

Rearranging, we obtain $4/3 < x_1 + x_2 \leq 8\alpha < 8/6$, which is absurd. Therefore, under any choice of identifications of L_1 and L_2 with $\mathbb{R}$, either $\pi_1(v) \leq \pi_1(v')$ if and only if $\pi_2(v) \leq \pi_2(v')$ for all $v, v' \in V$, or $\pi_1(v) \leq \pi_1(v')$ if and only if $\pi_2(v) \geq \pi_2(v')$ for all $v, v' \in V$. Thus, we can choose compatible identifications of L_1 and L_2 with $\mathbb{R}$ such that $\pi_1(v') \leq \pi_1(v'')$ if and only if $\pi_2(v') \leq \pi_2(v'')$ for all $v', v'' \in V$. $\qquad\square$

Corollary 2.2 (cf. [BNV19, Lemma 2.2]). *Let $\mathbb{X}$ be a Banach space. Suppose that $V \subset \mathbb{X}$ is a δ-separated set with $\#V \geq 2$ and there exists a line L and a number $0 \leq \alpha < 1/6$ such that*

$$(2.6) \qquad\qquad \mathrm{dist}(v, L) \leq \alpha\delta \quad \text{for all } v \in V.$$

Enumerate $V = \{v_1, \dots, v_n\}$ so that v_{i+1} lies to the right of v_i for all $1 \leq i \leq n-1$. Then

$$(2.7) \qquad\qquad \sum_{i=1}^{n-1} |v_{i+1} - v_i|^s \leq (1 + 3\alpha)^{2s}|v_1 - v_n|^s \quad \text{for all } s \in [1, \infty).$$

Proof. Let π denote a metric projection onto L. For all $1 \leq i \leq n$, set $x_i := \pi(v_i)$. Then

$$(1 + 3\alpha)^{-1}|x_{i+1} - x_i| \leq |v_{i+1} - v_i| \leq (1 + 3\alpha)|x_{i+1} - x_i| \quad \text{for all } 1 \leq i \leq n-1$$

by Lemma 2.1. Assume $s \in [1, \infty)$ and $\#V \geq 2$. Then

$$\sum_{i=1}^{n-1} \frac{|v_{i+1} - v_i|^s}{(1 + 3\alpha)^s} \leq \sum_{i=1}^{n-1} |x_{i+1} - x_i|^s$$

$$\leq \left(\sum_{i=1}^{n-1} |x_{i+1} - x_i| \right)^s = |x_1 - x_n|^s \leq (1 + 3\alpha)^s |v_1 - v_n|^s,$$

because $s \geq 1$ and $x_1, \dots, x_n$ appear in the given order on the line L by Lemma 2.1. This proves (2.7). $\qquad\square$

Remark 2.3. In a general Banach space, the metric projection is not unique and may be norm-increasing. For example, in $\ell^2_\infty = (\mathbb{R}^2, |\cdot|_\infty)$, consider the horizontal line L through the origin ("the x-axis") and the point $v = (1, \alpha)$ for some $0 < \alpha \leq 1$. Then $|v|_\infty = 1$, $\mathrm{dist}(v, L) = \alpha$, and a metric projection from v to L can be any point on the line segment $[1-\alpha, 1+\alpha] \times \{0\}$. In particular, if $\pi_L(v) = (1+\alpha, 0)$, then $|\pi_L(v)|_\infty = (1+\alpha)|v|_\infty > |v|_\infty$. This shows that in Lemma 2.1 for an arbitrary Banach space, we cannot expect to replace the lower bound in (2.3) with a 1-Lipschitz bound.

2.2. Lipschitz and Hölder continuous Traveling Salesman parameterizations in Banach spaces. Throughout this section, let $\mathbb{X}$ denote an arbitrary Banach space.

Definition 2.4 (doubling scales). Let $0 < \xi_1 \leq \xi_2 < 1$. A (ξ_1, ξ_2)-*doubling sequence of scales* is a sequence $(\rho_k)_{k=0}^\infty$ of positive numbers such that $\rho_0 = 1$ and for all $k \geq 0$, $\xi_1\rho_k \leq \rho_{k+1} \leq \xi_2\rho_k$. In the special case when $\xi_1 = \xi_2$, we may call $(\rho_k)_{k=0}^\infty$ a *geometric sequence of scales*.

Following [BNV19], let $\mathcal{V} = (V_k, \rho_k)_{k=0}^\infty$ be a sequence consisting of nonempty finite sets V_k in $\mathbb{X}$ and positive numbers ρ_k. Assume that there exist $x_0 \in \mathbb{X}$, $r_0 > 0$, $C^* \geq 1$, and $0 < \xi_1 \leq \xi_2 < 1$ such that $\mathcal{V}$ has the following properties.

(V0) The numbers $(\rho_k)_{k=0}^\infty$ are a (ξ_1, ξ_2)-doubling sequence of scales.

(V1) When $k = 0$, we have $V_0 \subset B(x_0, C^*r_0)$.

(V2) For all $k \geq 0$, we have $V_k \subset V_{k+1}$.

(V3) For all $k \geq 0$ and all distinct $v, v' \in V_k$, we have $|v - v'| \geq \rho_k r_0$.

(V4) For all $k \geq 0$ and all $v \in V_{k+1}$, there exists $v' \in V_k$ such that $|v - v_k| < C^*\rho_{k+1}r_0$.

With C^* and ξ_2 given, define the associated constant

$$(2.8) \qquad A^* := \frac{C^*}{1 - \xi_2}.$$

In addition to (V0)–(V4), assume that for each $k \geq 0$ and $v \in V_k$, we are given a number $\alpha_{k,v} \geq 0$ and a line $L_{k,v}$ in $\mathbb{X}$ such that

$$(V5) \qquad \sup_{x \in V_{k+1}B(v,30A^*\rho_k r_0)} \operatorname{dist}(x, L_{k,v}) \leq \alpha_{k,v}\rho_{k+1}r_0.$$

Definition 2.5 (flat pairs, see [BNV19]). Fix a parameter $\alpha_0 \in (0, 1/6)$. For all $k \geq 0$, define $\mathsf{Flat}(k)$ to be the set of pairs $(v, v') \in V_k \times V_k$ such that

(F1) $\rho_k r_0 \leq |v - v'| < 14A^*\rho_k r_0$, and

(F2) $\alpha_{k,v} < \alpha_0$ and v' is the first point in $V_k \cap B(v, 14A^*\rho_k r_0)$ to the left or to the right of v with respect to the ordering induced by $L_{k,v}$.

Given a pair $(v, v') \in \mathsf{Flat}(k)$, let $V_{k+1}(v, v')$ denote the set of all points $x \in V_{k+1} \cap B(v, 14A^*\rho_k r_0)$ such that x lies between v and v' (inclusive) with respect to the ordering induced by $L_{k,v}$.

Definition 2.6 (variation excess, see [BNV19]). For all $s \in [1, \infty)$, for all $k \geq 0$, and for all $(v, v') \in \mathsf{Flat}(k)$, define the *s-variation excess* $\tau_s(k, v, v')$ by

$$(2.9) \qquad \tau_s(k, v, v')|v - v'|^s = \max\left\{\left(\sum_{i=1}^{n-1} |v_{i+1} - v_i|^s\right) - |v - v'|^s, 0\right\},$$

where $V_{k+1}(v, v') = \{v_1, \ldots, v_n\}$ is enumerated so that $v_1 = v$ and for all $1 \leq i \leq n - 1$, $v_{i+1} \in V_{k+1}(v, v')$ is the first point after v_i in the direction from v to v' with respect to the ordering induced by $L_{k,v}$ (hence $v_n = v'$).

The following theorem extends Badger, Naples, and Vellis; see [BNV19, Theorem 5.1]. In its original form, the theorem was stated for $\mathbb{X} = \ell_2$ with a weaker restriction on α_1, achieved through targeted use of the Pythagorean theorem.

Theorem 2.7 (Hölder Traveling Salesman parameterizations for nets in Banach spaces). *Let $\mathbb{X}$ be a Banach space. In addition to (V0)–(V5), assume that*

$$(2.10) \qquad \alpha_0 \leq \alpha_1 := \frac{\xi_1(1 - \xi_2)}{87C^*}.$$

If the sum

$$(2.11) \qquad S_{\mathscr{V}}^s := \sum_{k=0}^{\infty} \sum_{(v,v') \in \mathsf{Flat}(k)} \tau_s(k,v,v') \rho_k^s + \sum_{k=0}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_0}} \rho_k^s < \infty,$$

then there exists a $(1/s)$-Hölder continuous map $f : [0,1] \to \mathbb{X}$ such that $f([0,1])$ contains $\bigcup_{k \geq 0} V_k$ and the $(1/s)$-Hölder constant of f satisfies $H \lesssim_{s,C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}^s)$.

Proof. Repeat the proof of [BNV19, Theorem 5.1] (see [BNV19, §§2–5]) *mutatis mutandis.* Use Lemma 2.1 and Corollary 2.2 above in place of [BNV19, Lemmas 2.1 and 2.2]. □

Remark 2.8 (outline of key modifications)**.** We specify some details to aid the reader with the proof of Theorem 2.7. The reader is first urged to read through [BNV19, §2], followed by item (C0) in [BNV19, §3.10], which indicates how key parameters are chosen.

In Hilbert space, the initial upper bound $1/16$ on the size of α_0 is made so that

$$1 + 3\alpha_0^2 \leq 1 + 3(1/16)^2 < 1.1.$$

See [BNV19, Lemma 2.3]. In generic Banach space, let us initially require $\alpha_0 \leq 1/31$ so that $1 + 3\alpha_0 \leq 1 + 3(1/31) < 1.1$. Then using Lemma 2.1 instead of [BNV19, Lemma 2.1], the rest of the proof and conclusion of [BNV19, Lemma 2.3] goes through as written.

The algorithm presented in [BNV19, §3] requires no changes in the Banach setting.

The principal estimates in the proof of the theorem occur in [BNV19, §4]. No changes are required until we reach the proof of [BNV19, Lemma 4.6], where we need to use Corollary 2.2 instead of [BNV19, Lemma 2.2]. This time, we require that $\alpha_0 \leq 1/62$ so that we can replace the original estimate $1 + 3\alpha_0^2 < 1.1$ with the estimate $(1 + 3\alpha_0)^2 < 1.1$. The next required change occurs at the end of the proof of [BNV19, Lemma 4.9]. Using Corollary 2.2 once again, we see the original requirement $1 + 3\alpha_0^2 - \xi_1/14A^* \leq 1$ becomes $(1 + 3\alpha_0)^2 - \xi_1/14A^* \leq 1$, or equivalently $6\alpha_0 + 9\alpha_0^2 \leq \xi_1/14A^*$. With our *a priori* bound $\alpha_0 \leq 1/62$, this certainly holds provided $(6 + 9/62)\alpha_0 \leq \xi_1/14A^*$. Thus, after noting that $(6 + 9/62)14 = 86.03...$, it suffices to take

$$\alpha_1 \leq \frac{\xi_1}{87A^*} = \frac{\xi_1(1 - \xi_2)}{87C^*}.$$

Note that

$$\min\left\{\frac{1}{31}, \frac{1}{62}, \frac{\xi_1(1 - \xi_2)}{87C^*}\right\} = \frac{\xi_1(1 - \xi_2)}{87C^*},$$

because $0 < \xi_1 \leq \xi_2 < 1$ and $C^* \geq 1$. There are two final uses of Lemma 2.1 instead of [BNV19, Lemma 2.1] to estimate the separation of points after projection onto an approximating line $\ell_{k,v}$, once in the proof of [BNV19, Proposition 4.11] and once in the proof of [BNV19, (4.3)]. This change affects the value of the implicit constant in [BNV19, Proposition 4.11], but not dependencies of the constant.

To finish the proof of Theorem 2.7, repeat the argument in [BNV19, §5.1] verbatim.

Corollary 2.9. *Let $\mathbb{X}$ be a Banach space. Assume $\mathscr{V} = (V_k, \rho_k)_{k=0}^\infty$ satisfies (V0)–(V5) above. If the sum*

$$(2.12) \qquad S_{\mathscr{V}} := \sum_{k=0}^\infty \sum_{v \in V_k} \alpha_{k,v} \rho_k < \infty,$$

then there exists a rectifiable curve Γ containing $\bigcup_{k \geq 0} V_k$ such that

$$(2.13) \qquad \mathcal{H}^1(\Gamma) \lesssim_{C^*, \xi_1, \xi_2} r_0 (1 + S_{\mathscr{V}}).$$

Proof. Set $\alpha_0 = \alpha_1$, which depends only on ξ_1, ξ_2, and C^*. By Corollary 2.2 with $s = 1$, we have $\tau_1(k, v, v') \leq 6\alpha_{k,v} + 9\alpha_{k,v}^2 \leq 7\alpha_{k,v}$ for every flat pair $(v, v') \in \mathsf{Flat}(k)$. Thus,

$$S_{\mathscr{V}}^1 = \sum_{k=0}^\infty \sum_{(v,v') \in \mathsf{Flat}(k)} \tau_1(k, v, v') \rho_k + \sum_{k=0}^\infty \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_1}} \rho_k$$

$$\leq \sum_{k=0}^\infty \sum_{(v,v') \in \mathsf{Flat}(k)} 7\alpha_{k,v} \rho_k + \alpha_1^{-1} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_1}} \alpha_{k,v} \rho_k \leq \alpha_1^{-1} S_{\mathscr{V}} < \infty.$$

By Theorem 2.7, there exists a Lipschitz map $f : [0,1] \to \mathbb{X}$ such that $\Gamma := f([0,1])$ contains $\bigcup_{k=0}^\infty V_k$ and $\mathcal{H}^1(\Gamma) \leq \mathrm{Lip}(f) \lesssim_{C^*, \xi_1, \xi_2} r_0(1 + S_{\mathscr{V}}^1) \lesssim_{C^*, \xi_1, \xi_2} r_0(1 + S_{\mathscr{V}})$. $\qquad \square$

For completeness, we show how to use Corollary 2.9 to derive a beta number criterion for a set in a Banach space to be contained in a rectifiable curve. The following theorem is best viewed as the Banach space analogue of Theorem 1.10, expressed with the geometric Jones' beta numbers (1.1) instead of metric beta numbers (1.27). To recall the definition of the sum $S_{E,1}(\mathscr{G})$ of beta numbers over a multiresolution family $\mathscr{G}$ for E, see (1.15).

Theorem 2.10 (sufficient half of Schul's theorem in arbitrary Banach spaces)**.** *Let $\mathbb{X}$ be a Banach space. If $E \subset \mathbb{X}$ and $S_{E,1}(\mathscr{G}) < \infty$ for some multiresolution family $\mathscr{G}$ for E with inflation factor $A_{\mathscr{G}} \geq 240$, then E is contained in a rectifiable curve Γ in $\mathbb{X}$ with*

$$(2.14) \qquad \mathcal{H}^1(\Gamma) \lesssim_{A_{\mathscr{G}}} S_{E,1}(\mathscr{G}).$$

Proof. Let $\mathbb{X}$ be a Banach space, let $E \subset \mathbb{X}$, let $\mathscr{G}$ be a multiresolution family for E with inflation factor $A_{\mathscr{G}} \geq 240$, and assume that $S_{E,1}(\mathscr{G}) < \infty$. Then E is bounded and there exists a unique integer $k_0 \in \mathbb{Z}$ such that

$$(2.15) \qquad 2^{-k_0} \leq \mathrm{diam}\, E < 2 \cdot 2^{-k_0}.$$

For all $k \geq 0$, define $\rho_k = 2^{-k}$ and $V_k = X_{k_0+k}$, where $(X_j)_{j \in \mathbb{Z}}$ are the 2^{-j}-nets for E used to define $\mathscr{G}$. Set parameters $C^* = 2$, $\xi_1 = \xi_2 = \frac{1}{2}$, and $r_0 = 2^{-k_0}$, and choose any $x_0 \in V_0 = X_{k_0}$. Then the sequence $\mathscr{V} = (V_k, \rho_k)_{k=0}^\infty$ satisfies properties (V0)–(V4) above. Note that $A^* = 4C^* = 8$ and $30A^* = 240$, since $\xi_1 = \xi_2 = \frac{1}{2}$. For each $k \geq 0$ and $v \in V_k$, set $\alpha_{k,v} = 8A_{\mathscr{G}}\beta_E(B(v, A_{\mathscr{G}} 2^{-(k_0+k)}))$ and choose $L_{k,v}$ to be any line such that

$$(2.16) \qquad \sup_{x \in E \cap B(v, A_{\mathscr{G}} 2^{-(k_0+k)})} \mathrm{dist}(x, L_{k,v}) \leq 2\beta_E(B(v, A_{\mathscr{G}} 2^{-(k_0+k)})) \, \mathrm{diam}\, B(v, A_{\mathscr{G}} 2^{-(k_0+k)}).$$

Because $A_{\mathscr{G}} \geq 240 = 30A^*$, it follows that for all $k \geq 0$ and $v \in V_k$,

$$\sup_{x \in V_{k+1} \cap B(v, 30A^* \rho_k r_0)} \mathrm{dist}(x, L_{k,v}) \leq 2\beta_E(B(v, A_{\mathscr{G}} 2^{-(k_0+k)})) \, \mathrm{diam}\, B(v, A_{\mathscr{G}} 2^{-(k_0+k)})$$

$$(2.17) \qquad\qquad\qquad\qquad \leq 8A_{\mathscr{G}}\beta_E(B(v, A_{\mathscr{G}} 2^{-(k_0+k)}))2^{-(k_0+k+1)}$$

$$= \alpha_{k,v}\rho_{k+1}r_0.$$

Thus, property (V5) is satisfied, as well. To proceed, observe that

$$(2.18) \quad S_{\mathscr{V}} = \sum_{k=0}^{\infty}\sum_{v \in V_k} \alpha_{k,v}\rho_k = \sum_{k=0}^{\infty}\sum_{x \in V_k} 8A_{\mathscr{G}}\beta_E(B(v, A_{\mathscr{G}} 2^{-(k_0+k)}))2^{-k} \leq 8A_{\mathscr{G}} 2^{k_0} S_{E,1}(\mathscr{G}).$$

Since $S_{\mathscr{V}} \leq 8A_{\mathscr{G}} S_{E,1}(\mathscr{G}) < \infty$, there is a rectifiable curve Γ containing $\bigcup_{k=0}^{\infty} V_k$ such that

$$(2.19) \qquad\qquad \mathcal{H}^1(E) \lesssim_{C^*,\xi_1,\xi_2} r_0(1 + S_{\mathscr{V}}) \lesssim r_0(1 + 8A_{\mathscr{G}} r_0^{-1} S_{E,1}) \lesssim_{A_{\mathscr{G}}} S_{E,1}$$

by Corollary 2.9 and (2.15). Finally, note that since $(V_k)_{k=0}^{\infty}$ is a sequence of $2^{-(k_0+k)}$-nets for E and Γ is closed, Γ contains the set $\overline{\bigcup_{k=0}^{\infty} V_k} \supset E$, as well. $\qquad\square$

Remark 2.11. The constant 240 in Theorem 2.10 has not been optimized and can be at least somewhat reduced at the cost of growing the implicit constant in (2.14). In the future event that a smaller constant is needed, the reader should first consult [BS17, §3.10]

2.3. **Triangle inequality excess in uniformly smooth Banach spaces.** Our goal in this section is to prove that in a uniformly smooth Banach space of power type $p \in (1, 2]$, the exponent 1 in Corollary 2.9 and Theorem 2.10 may be replaced with the exponent p. In the process, we will verify (1.17) and (1.19) in Theorem 1.6. The essential step is to improve the exponent in the bound (2.3) in Lemma 2.1. To achieve this, we follow the strategy used by Edelen, Naber, and Valtorta [ENV19] in their proof of one-dimensional Reifenberg-type theorems in uniformly smooth Banach spaces. The approach utilizes a special projection operator, which is available in uniformly smooth Banach spaces.

Definition 2.12. Let $\mathbb{X}$ be a Banach space. The *modulus of smoothness* $\rho_{\mathbb{X}}$ of $\mathbb{X}$ is the function $\rho_{\mathbb{X}} : [0, \infty) \to [0, \infty)$ defined by

$$(2.20) \qquad\qquad \rho_{\mathbb{X}}(t) := \sup_{|x|=1,|y|=t} \frac{1}{2}(|x + y| + |x - y|) - 1 \quad \text{for all } t \in [0, \infty).$$

Definition 2.13. We say that $\mathbb{X}$ is *uniformly smooth* if $\rho_{\mathbb{X}}(t)/t = o(t)$ as $t \to 0$. In this case, we say that $\mathbb{X}$ is *smoothness power type* $p \in [1, 2]$ if $\rho_{\mathbb{X}}(t)/t^p = O(t)$ as $t \to 0$.

Remark 2.14 (essential facts)**.** For general background on the modulus of smoothness and uniformly smooth spaces, including the following inequalities, see e.g. [Die75, Chapter Three] or [LT79, Chapter 1, §e]. On any Banach space $\mathbb{X}$, the modulus of smoothness $\rho_{\mathbb{X}}$ is a non-decreasing convex function such that $\rho_{\mathbb{X}}(0) = 0$ and

$$(2.21) \qquad\qquad \sqrt{1 + t^2} - 1 = \rho_{\ell_2}(t) \leq \rho_{\mathbb{X}}(t) \leq t \quad \text{for all } t \geq 0.$$

Since $\rho_{\mathbb{X}}$ is convex and $\rho_{\mathbb{X}}(0) = 0$,

$$(2.22) \qquad \frac{\rho_{\mathbb{X}}(t_1)}{t_1} \leq \frac{\rho_{\mathbb{X}}(t_2)}{t_2} \quad \text{for all } 0 < t_1 \leq t_2.$$

Furthermore, there exists a constant $1 < L_0 < 3.18$ (see Figiel [Fig76, Proposition 10]) such that

$$(2.23) \qquad \frac{\rho_{\mathbb{X}}(t_2)}{t_2^2} \leq L_0 \frac{\rho_{\mathbb{X}}(t_1)}{t_1^2} \quad \text{for all } 0 < t_1 \leq t_2,$$

The modulus of smoothness $\rho_{\mathbb{X}}$ and modulus of convexity $\delta_{\mathbb{X}}$ (see §3.2) are related by

$$(2.24) \qquad \rho_{\mathbb{X}^*}(t) = \sup\left\{ \frac{1}{2}t\epsilon - \delta_{\mathbb{X}}(\epsilon) : 0 < \epsilon \leq 2 \right\} \quad \text{for all } t > 0.$$

Hence the dual $\mathbb{X}^*$ of a uniformly convex Banach space $\mathbb{X}$ (see §3.2) is uniformly smooth. Finally, every uniformly smooth Banach space is reflexive.

Example 2.15. By Hanner's inequalities [Han56], $\rho_{\ell_p}(t) = p^{-1}t^p + o(t^p)$ when $1 < p \leq 2$; and $\rho_{\ell_p}(t) = \frac{1}{2}(p-1)t^2 + o(t^2)$ when $2 \leq p < \infty$. In particular, the ℓ_p spaces are uniformly smooth with power type $\min(p, 2)$ when $1 < p < \infty$.

We now recall the definition of Alber's generalized projection operator [Alb96a, §7] in the special case of projection onto a line. Given a real Banach space $\mathbb{X}$, let $\mathbb{X}^*$ denote the dual of $\mathbb{X}$ and let $J : \mathbb{X} \to \mathbb{X}^*$ denote a *normalized duality mapping*, i.e. a map satisfying

$$(2.25) \qquad |J(x)|_{\mathbb{X}^*} = |x| \quad \text{and} \quad \langle J(x), x \rangle = |x|^2 \quad \text{for all } x \in \mathbb{X},$$

where $\langle f, x \rangle \equiv f(x) \in \mathbb{R}$ denotes the natural pairing of $f \in \mathbb{X}^*$ and $x \in \mathbb{X}$. Alternatively, J is a subgradient of the convex function $x \in \mathbb{X} \mapsto \frac{1}{2}|x|^2$ (see [Asp67, Kie02]). The norm on any (uniformly) smooth Banach space $\mathbb{X}$ is Gateaux (uniformly Fréchet) differentiable, and thus, J is uniquely determined (see e.g. [Die75, Chapter Two]) when $\mathbb{X}$ is smooth. For example, when $\mathbb{X} = \ell_p$ with $1 < p < \infty$,

$$J(x) = |x|_{\ell_p}^{2-p} y \in \ell_p^* = \ell_{p'},$$

where $y = (|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, \dots)$ and p' is the conjugate exponent to p.

Definition 2.16 ([Alb96a, §7]; [ENV19, Definition 3.31]). Let $\mathbb{X}$ be a Banach space and let L be a one-dimensional linear subspace of $\mathbb{X}$. Define the *J-projection* Π_L onto L by

$$(2.26) \qquad \Pi_L(x) := \langle J(v), x \rangle v \quad \text{for all } x \in \mathbb{X},$$

where J is a normalized dual mapping and v is a point in L with $|v| = 1$. When L is a one-dimensional affine subspace of $\mathbb{X}$, define $\Pi_L \equiv p + \Pi_{L-p}(\cdot - p)$ for any choice of $p \in L$. Clearly, Π_L maps $\mathbb{X}$ onto L. For all lines L, we also define $\Pi_L^{\perp} \equiv \text{Id}_{\mathbb{X}} - \Pi_L$.

Let us record some elementary properties of J-projection.

Lemma 2.17. *Let $\mathbb{X}$ be a Banach space and let L be a one-dimensional linear subspace of $\mathbb{X}$. The J-projection Π_L satisfies each of the following properties.*

(1) For all $x \in L$, we have $\Pi_L(x) = x$.
(2) For all $x \in L^\perp \equiv \Pi_L^\perp(\mathbb{X})$, we have $\langle Jv, x \rangle = 0$ and $\Pi_L(x) = 0$.
(3) For all $x \in \mathbb{X}$, we have $|\Pi_L(x)| \leq |x|$.
(4) For all $x \in \mathbb{X}$, we have $\mathrm{dist}(x, L) \leq |\Pi_L^\perp(x)| \leq 2\,\mathrm{dist}(x, L)$.

Proof. Let $v \in L$ be the unit vector in the definition of Π_L. If $x \in L$, say $x = cv$, then

$$\Pi_L(x) = \langle Jv, cv \rangle v = c\langle Jv, v \rangle v = c|v|^2 v = cv = x.$$

This gives the first point. To see the second point, for any $x \in L^\perp$, say $x = \Pi_L^\perp(y)$,

$$\langle Jv, x \rangle = \langle Jv, y - \langle Jv, y \rangle v \rangle = \langle Jv, y \rangle - \langle Jv, y \rangle \langle Jv, v \rangle$$
$$= \langle Jv, y \rangle - \langle Jv, y \rangle |v|^2 = \langle Jv, y \rangle - \langle Jv, y \rangle = 0.$$

Hence $\Pi_L(x) = \langle Jv, x \rangle v = 0v = 0$. For the third point, observe that for any $x \in \mathbb{X}$,

$$|\Pi_L(x)| = |\langle Jv, x \rangle v| \leq |Jv|_{\mathbb{X}^*} |x||v| = |v|^2 |x| = |x|.$$

To see the last point, suppose that $x \in \mathbb{X}$. Clearly, $|\Pi_L^\perp(x)| = |\Pi_L(x) - x| \geq \mathrm{dist}(x, L)$. Choose $y \in L$ such that $|x - y| = \mathrm{dist}(x, L)$. Then

$$|\Pi_L(x) - x| \leq |\Pi_L(x) - y| + |x - y|$$
$$= |\Pi_L(x) - \Pi_L(y)| + |x - y| \leq 2|x - y| = 2\,\mathrm{dist}(x, L). \qquad \square$$

We now check that in any Banach space, the J-projection Π_L induces a well-defined order on a sufficiently flat, separated set of points, by checking compatibility with the order induced by a metric projection π_L. The importance of this fact for us is that in the definition of flat pairs in the traveling salesman algorithm (see Definition 2.5), it does not matter whether we order points by π_L or Π_L.

Lemma 2.18 (order compatibility for Π_L and π_L). *Let $\mathbb{X}$ be a Banach space. Suppose that V is a δ-separated set with $\#V \geq 2$ and there exists a line L and a number $0 \leq \alpha < 1/8$ such that $\mathrm{dist}(x, L) \leq \alpha\delta$ for all $x \in V$. Then the J-projection Π_L induces an order on V compatible with the order induced by a metric projection π_L onto L.*

Proof. Let $V \subset \mathbb{X}$ be a δ-separated set with $\#V \geq 2$, let L be a line in $\mathbb{X}$, let $0 \leq \alpha < 1/6$, and assume that $\mathrm{dist}(x, L) \leq \alpha\delta$ for all $x \in V$. Without loss of generality, we may assume that L is a linear subspace of $\mathbb{X}$. Fix any metric projection π_L onto L. The restriction on α ensures that π_L induces a unique order on V by Lemma 2.1. If $x, y \in V$ are distinct, then

$$(2.27) \quad |x - y| \geq |\Pi_L(x) - \Pi_L(y)| \geq |x - y| - |\Pi_L^\perp(x)| - |\Pi_L^\perp(y)| \geq (1 - 4\alpha)\delta > \frac{1}{3}\delta > 0$$

by the triangle inequality and Lemma 2.17. We may now check that the J-projection Π_L induces an order on V that is compatible with the order induced by π_L. To that end, suppose that $x, y, z \in V$ are distinct points, write $x' = \pi_L(x)$, $y' = \pi_L(y)$, $z' = \pi_L(z)$ and $x'' = \Pi_L(x)$, $y'' = \Pi_L(y)$, $z'' = \Pi_L(z)$, and suppose to get a contradiction that there
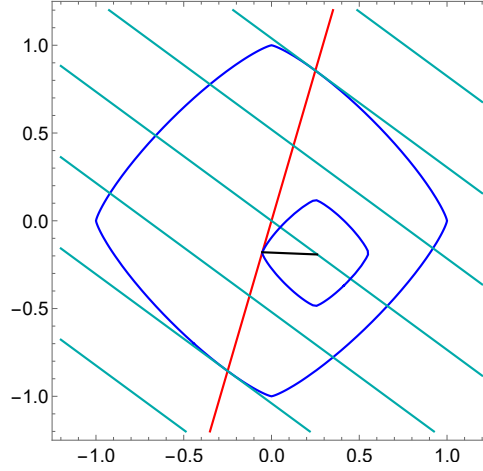
FIGURE 3. In cyan, the fibers of the J-projection Π_L of $\mathbb{X} = \ell_{5/4}^2$ onto the (red) line $L = \text{span}(v)$, where $v = (1/4, y_0)$, $(1/4)^{5/4} + y_0^{5/4} = 1$. In black, the metric projection $\pi_L(w)$ onto L of $w = (1/4, y_1)$ with $\Pi_L(w) = (0,0)$.

exists identifications of L with $\mathbb{R}$ such that $x' < y' < z'$ and $y'' < x'' < z''$. Heuristically, because of the order of the triples on the line L,

$$|x - z| \approx |x - y| + |y - z| \approx |x - y| + \underbrace{|x - y| + |x - z|},$$

which is impossible if α is sufficient small. More precisely, on one hand, by (2.4) and (2.5) from the proof of Lemma 2.1 (recalling that there we normalized $\delta = 1$),

$$|x - z| \geq |x' - z'| - 2\alpha\delta = |x' - y'| + |y' - z'| - 2\alpha\delta \geq |x - y| + |y - z| - 6\alpha\delta,$$

since $x' < y' < z'$. On the other hand, by (2.27),

$$|y - z| \geq |y'' - z''| = |y'' - x''| + |x'' - z''| \geq |y - x| + |x - z| - 8\alpha\delta,$$

since $y'' < x'' < z''$. Combining the previous two displayed equations and recalling V is δ-separated, we obtain

$$|x - z| \geq 2|x - y| + |x - z| - 14\alpha\delta \geq |x - z| + 2\delta - 14\alpha\delta,$$

or equivalently $(2 - 14\alpha)\delta \leq 0$. This is a contradiction, because $\alpha < 1/8$ implies that $2 - 14\alpha > 2 - 14/8 = 1/4 > 0$. It readily follows that there is an identification of L with $\mathbb{R}$ such that $\pi_L(x) \leq \pi_L(y)$ if and only if $\Pi_L(x) \leq \Pi_L(y)$ for all $x, y \in V$. $\qquad\square$

Remark 2.19 (geometric interpretation in smooth spaces). Assume that $\mathbb{X}$ is smooth. For a line L, spanned by a unit vector $v \in \mathbb{X}$, the J-projection Π_L admits the following geometric description. Let $T_v \partial B(0,1)$ denote the tangent hyperplane to $\partial B(0,1)$ at the point v (which exists because $\mathbb{X}$ is smooth). By Lemma 2.17(2), a point $x \in \mathbb{X}$ satisfies $\Pi_L(x) = cv$ if and only if $x \in (cv + T_v \partial B(0,1))$. See Figure 3.

Lemma 2.20. *If* $\mathbb{X}$ *is a smooth Banach space, then*

$$\frac{d}{dt}|x + ty|^2 = 2\langle J(x+ty), y\rangle$$

for all $x, y \in \mathbb{X}$ *and* $t \in \mathbb{R}$ *with* $x + ty \neq 0$ *and* $y \neq 0$.

Proof. Since $2J$ is the subgradient of $x \in \mathbb{X} \mapsto |x|^2$, we have

$$|z|^2 - |x+ty|^2 \geq \langle 2J(x+ty), z - (x+ty)\rangle \quad \text{for all } z \in \mathbb{X}.$$

The claim follows by applying the inequality to the difference quotient

$$\frac{|x + (t+h)y|^2 - |x+ty|^2}{h}$$

along values $h \to 0^+$ and for $h \to 0^-$, where the limit exists by Gateaux differentiability of the norm. $\qquad \square$

The following estimate by Alber is crucial for our application below.

Lemma 2.21 (see Alber [Alb96a, Remark 7.3] or [Alb96b, (2.13)]). *Let* $\mathbb{X}$ *be a uniformly smooth Banach space and write* $h_{\mathbb{X}}(t) \equiv \rho_{\mathbb{X}}(t)/t$. *Then*

$$(2.28) \qquad |J(x) - J(y)|_{\mathbb{X}^*} \leq 4C_0 h_{\mathbb{X}}(8C_0 L_0 |x-y|) \quad \text{for all } x, y \in \mathbb{X},$$

where $C_0 = 2\max\left(1, \sqrt{\frac{1}{2}(|x|^2 + |y|^2)}\right)$ *and* L_0 *is Figiel's constant* (2.23).

The next estimate is similar to [ENV19, Lemma 3.27]. Unfortunately, the proof of the lemma given in [ENV19] is partially based on [ENV19, Lemma 3.26], which appears to misstate Lemma 2.21. Thus, we supply a proof of the estimate.

Lemma 2.22. *Let* $\mathbb{X}$ *be a uniformly smooth Banach space and let* L *be a one-dimensional linear subspace of* $\mathbb{X}$. *If* $x \in \mathbb{X}$, $|x| \leq 1$, *and* $\text{dist}(x, L) \leq \alpha|x|$, *then*

$$(2.29) \qquad |x| \leq |\Pi_L(x)| + \frac{8\alpha}{1 - 4\alpha} h_{\mathbb{X}}(51\alpha|x|),$$

where $h_{\mathbb{X}}(t) \equiv \rho_{\mathbb{X}}(t)/t$.

Proof. For brevity, write $y = \Pi_L(x)$ and $z = \Pi_L^{\perp}(x) = x - y$. If $\text{dist}(x, L) \leq \alpha|x|$, then $|y| \leq |x|$ and $|z| \leq 2\alpha|x|$ by Lemma 2.17. Thus, by Lemma 2.20 and the fundamental theorem of calculus,

$$|x| - |y| = \int_0^1 \frac{d}{dt}|y + tz|\, dt = \int_0^1 \frac{1}{2}|y+tz|^{-1}\frac{d}{dt}|y+tz|^2\, dt$$

$$= \int_0^1 |y+tz|^{-1}\langle J(y+tz), z\rangle\, dt \leq \frac{1}{(1-4\alpha)|x|}\int_0^1 \langle J(y+tz), z\rangle\, dt,$$

where we used the rough estimate $|y + tz| \geq |y| - |z| \geq |x| - 2|z| \geq (1 - 4\alpha)|x|$.

Now, by Lemma 2.17, $\langle J(y), z \rangle = 0$. Therefore, by Alber's inequality (2.28),

$$\int_0^1 \langle J(y+tz), z \rangle \, dt = \int_0^1 \langle J(y+tz) - J(y), z \rangle \, dt \leq \int_0^1 |J(y+tz) - J(y)|_{\mathbb{X}^*} |z| \, dt$$

$$\leq 4C_0 |z| \int_0^1 h_{\mathbb{X}}(8C_0 L_0 |tz|) \, dt \leq 4C_0 |z| h_{\mathbb{X}}(8C_0 L_0 |z|),$$

where $C_0 = \sup_{0 \leq t \leq 1} \max\{1, \sqrt{\frac{1}{2}(|y|^2 + |y+tz|^2)}\} \leq 1$ (since $|x| \leq 1$) and $1 < L_0 < 3.18$. Recall that $h_{\mathbb{X}}(t)$ is non-decreasing (see Remark 2.14) and $|z| \leq 2\alpha |x|$. Thus,

$$\int_0^1 \langle J(y+tz), z \rangle \, dt \leq 8\alpha |x| h_{\mathbb{X}}(51\alpha |x|),$$

as $16L_0 < 50.88$. Combining the displayed estimates yields (2.29). $\qquad\square$

We may now improve Lemma 2.1 in uniformly smooth Banach spaces.

Lemma 2.23 (cf. Lemma 2.1). *Let $\mathbb{X}$ be a uniformly smooth Banach space. Suppose that V is a δ-separated set with $\#V \geq 2$ and there exists a line L and a number $0 \leq \alpha \leq 43/1224 = 0.0351...$ such that $\mathrm{dist}(x, L) \leq \alpha\delta$ for all $x \in V$. If $v_1, v_2 \in V$, then*

$$(2.30) \qquad |\Pi_L(v_1) - \Pi_L(v_2)| \leq |v_1 - v_2| \leq (1 + \rho_{\mathbb{X}}(102\alpha))|\Pi_L(v_1) - \Pi_L(v_2)|.$$

Proof. Let $V \subset \mathbb{X}$ be a δ-separated set with $\#V \geq 2$, let L be a line in $\mathbb{X}$, let $\alpha \geq 0$, and assume that $\mathrm{dist}(x, L) \leq \alpha\delta$ for all $x \in V$. Fix any pair of distinct points $v_1, v_2 \in V$. By applying two translations and invoking the triangle inequality, we may assume that L is a linear subspace of $\mathbb{X}$, $0 = v_1 \in L$, and $v := v_2$ satisfies $|v| \geq \delta$ and $\mathrm{dist}(v, L) \leq 2\alpha\delta$. Applying a dilation, we may further assume that $|v| = 1$. Then $\mathrm{dist}(v, L) \leq 2\alpha\delta \leq 2\alpha|v|$ and by Lemma 2.22,

$$|v| \leq |\Pi(v)| + \frac{16\alpha}{1 - 8\alpha} h_{\mathbb{X}}(102\alpha) = |\Pi(v)| + \frac{8}{51(1 - 8\alpha)} \rho_{\mathbb{X}}(102\alpha).$$

Recall that $\rho_{\mathbb{X}}(t) \leq t$ in any Banach space; see (2.21). Hence

$$|\Pi(v)| \geq |v| - \frac{8}{51(1 - 8\alpha)} \rho_{\mathbb{X}}(102\alpha) \geq 1 - \frac{16\alpha}{1 - 8\alpha} = \frac{1 - 24\alpha}{1 - 8\alpha}.$$

We now require that

$$\frac{8}{51(1 - 8\alpha)} \leq \frac{1 - 24\alpha}{1 - 8\alpha},$$

or equivalently, $\alpha \leq 43/1224$. Then combining the displayed equations yields the right hand side of (2.30). The left hand side of (2.30) follows immediately from Lemma 2.17. $\quad\square$

Corollary 2.24 (cf. Corollary 2.2). *Let $\mathbb{X}$ be a uniformly smooth Banach space. Suppose that $V \subset \mathbb{X}$ is a δ-separated set with $\#V \geq 2$ and there exists a line L and a number $0 \leq \alpha \leq 43/1224$ such that $\mathrm{dist}(x, L) \leq \alpha\delta$ for all $x \in V$. Enumerate $V = \{v_1, \ldots, v_n\}$*

so that v_{i+1} lies to the right of v_i for all $1 \leq i \leq n-1$, relative to the ordering induced by the metric projection π_L or the J-projection Π_L (see Lemma 2.18). Then

$$(2.31) \qquad \sum_{i=1}^{n-1} |v_{i+1} - v_i|^s \leq (1 + \rho_{\mathbb{X}}(102\alpha))^s |v_1 - v_n|^s \quad \text{for all } s \in [1, \infty).$$

Proof. Repeat the proof of Corollary 2.2 *mutatis mutandis.* Use Lemma 2.23 instead of Lemma 2.1. □

Theorem 2.25 (Analyst's Traveling Salesman parameterizations for nets in uniformly smooth Banach spaces, cf. Corollary 2.9)**.** *Let $\mathbb{X}$ be a uniformly smooth Banach space and let $\rho_{\mathbb{X}}$ denote its modulus of smoothness (see Definition 2.12). Assume that $\mathscr{V} = (V_k, \rho_k)_{k=0}^{\infty}$ satisfies (V0)–(V5) in §2.2. If the sum*

$$(2.32) \qquad S_{\rho_{\mathbb{X}}}(\mathscr{V}) := \sum_{k=0}^{\infty} \sum_{v \in V_k} \rho_{\mathbb{X}}(102\alpha_{k,v})\rho_k < \infty,$$

then there exists a rectifiable curve Γ containing $\bigcup_{k=0}^{\infty} V_k$ such that

$$(2.33) \qquad \mathcal{H}^1(\Gamma) \lesssim_{C^*, \xi_1, \xi_2} r_0(1 + S_{\rho_{\mathbb{X}}}(\mathscr{V})).$$

Proof. As in the proof of Corollary 2.9, set $\alpha_0 = \alpha_1$, which only depends on C^*, ξ_1, and ξ_2. Note that $\alpha_1 \leq 1/384 < 43/1224$. Thus, by Corollary 2.24 with $s = 1$ and the bound $\rho_{\mathbb{X}}(102\alpha_1) \geq \rho_{\ell_2}(102\alpha_1) \simeq \alpha_1^2 \simeq_{C^*, \xi_1, \xi_2} 1$ (see (2.21)),

$$S_{\mathscr{V}}^1 = \sum_{k=0}^{\infty} \sum_{(v,v') \in \mathsf{Flat}(k)} \tau_1(k, v, v')\rho_k + \sum_{k=0}^{\infty} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_1}} \rho_k$$

$$\leq \sum_{k=0}^{\infty} \sum_{(v,v') \in \mathsf{Flat}(k)} \rho_{\mathbb{X}}(102\alpha_{k,v})\rho_k + \rho_{\ell_2}(102\alpha_1)^{-1} \sum_{\substack{v \in V_k \\ \alpha_{k,v} \geq \alpha_1}} \rho_{\mathbb{X}}(102\alpha_{k,v})\rho_k$$

$$\lesssim_{C^*, \xi_1, \xi_2} S_{\rho_{\mathbb{X}}}(\mathscr{V}) < \infty.$$

By Theorem 2.7, there exists a Lipschitz map $f : [0,1] \to \mathbb{X}$ such that $\Gamma := f([0,1])$ contains $\bigcup_{k=0}^{\infty} V_k$ and $\mathcal{H}^1(\Gamma) \leq \mathrm{Lip}(f) \lesssim_{C^*, \xi_1, \xi_2} r_0(1 + S_{\mathscr{V}}^1) \lesssim_{C^*, \xi_1, \xi_2} r_0(1 + S_{\rho_{\mathbb{X}}}(\mathscr{V}))$. □

Theorem 2.26 (sufficient half of Schul's theorem in uniformly smooth Banach spaces)**.** *Let $\mathbb{X}$ be a uniformly smooth Banach space of power type $p \in (1, 2]$. If $E \subset \mathbb{X}$ and $S_{E,p}(\mathscr{G}) < \infty$ for some multiresolution family $\mathscr{G}$ for E with inflation factor $A_{\mathscr{G}} \geq 240$, then E is contained in a rectifiable curve Γ in $\mathbb{X}$ with*

$$(2.34) \qquad \mathcal{H}^1(\Gamma) \lesssim_{\rho_{\mathbb{X}}, A_{\mathscr{G}}} S_{E,p}(\mathscr{G}).$$

Proof. Repeat the proof of Theorem 2.10 *mutatis mutandis.* Use Theorem 2.25 in lieu of Corollary 2.9. □

Because the Banach space ℓ_p is uniformly smooth of power type $\min\{p, 2\}$ when $1 < p < \infty$, the sufficient conditions (1.17) and (1.19) in Theorem 1.6 follow immediately from Theorem 2.26.

3. Modulus of convexity and proof of the necessary conditions

3.1. **Canonical parameterization of finite length continua and beta numbers associated to a parameterization.** At the heart of the proof of necessary conditions in Analyst's Traveling Salesman Theorems is the existence of parameterizations of finite length continuum by Lipschitz curves. An excellent source for the essential background is the recent paper [AO17] by Alberti and Ottolini.

Given a continuous map $f : [0, 1] \to \mathbb{X}$ into a metric space and a closed, non-degenerate interval $I \subset [0, 1]$, the *variation* $\mathrm{var}(f, I)$ of f over I is defined by

$$(3.1) \qquad \mathrm{var}(f, I) := \sup_{a_1 \le \cdots \le a_{n+1}} \sum_{i=1}^{n} |f(a_{i+1}) - f(a_i)| \in [0, \infty],$$

where the supremum ranges over all finite increasing sequences in I. Associated to f, define the *multiplicity function* $m(f, \cdot) : X \to [0, \infty]$, $m(f, x) = \#f^{-1}(x)$ for all $x \in \mathbb{X}$. The following proposition records the well-known connection between the variation of f (intrinsic length) and the Hausdorff measure of the image of f (extrinsic length).

Proposition 3.1 ([AO17, Proposition 3.5]). *Let $f : [0, 1] \to \mathbb{X}$ be a continuous map into a metric space and let $I \subset [0, 1]$ be a closed, non-degenerate interval. If $\mathrm{var}(f, I) < \infty$, then $m(f|_I, \cdot)$ is a Borel function and*

$$(3.2) \qquad \mathrm{var}(f, I) = \int_{\mathbb{X}} m(f|_I, x) \, d\mathcal{H}^1(x).$$

A theorem of Ważewski [Waż27] asserts that every connected, compact metric space Σ with finite one-dimensional Hausdorff measure $\mathcal{H}^1$ admits a Lipschitz parameterization by the interval $[0, 1]$ with Lipschitz constant $\mathrm{Lip}(f) = \sup_{x \ne y} |f(x) - f(y)|/|x - y|$ at most $2\mathcal{H}^1(\Sigma)$.[3] Alberti and Ottolini have recently proved the following refinement of Ważewski's theorem (in particular, that f has degree zero). Property (2) says that the parameterization f of Σ is *essentially 2-to-1*.

Theorem 3.2 ([AO17, Theorem 4.4]). *Let Σ be a connected, compact metric space with $\mathcal{H}^1(\Sigma) < \infty$. Then there exists a continuous function $f : [0, 1] \to \Sigma$ such that*

(1) f is closed, Lipschitz, surjective, and has degree zero (see [AO17]);
(2) $m(f, x) = 2$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$, and $\mathrm{var}(f, [0, 1]) = 2\mathcal{H}^1(\Sigma)$; and,
(3) f has constant speed equal to $2\mathcal{H}^1(\Sigma)$.

In any Banach space $\mathbb{X}$, a connected set $\Sigma \subset \mathbb{X}$ has the property that $\mathcal{H}^1(\Sigma) = \mathcal{H}^1(\overline{\Sigma})$, where $\overline{\Sigma}$ denotes the closure of Σ in $\mathbb{X}$. Moreover, if $\Sigma \subset \mathbb{X}$ is closed, connected, and $\mathcal{H}^1(\Sigma) < \infty$, then Σ is compact. The proofs of these facts are simple exercises with the definitions, using convexity of $\mathbb{X}$; see e.g. [Sch07c, §5] (although stated there for $\mathbb{X} = \ell_2$, the proofs there hold in any Banach space).

Corollary 3.3. *Let $\mathbb{X}$ be a Banach space. If $\Sigma \subset \mathbb{X}$ is connected and $\mathcal{H}^1(\Sigma) < \infty$, then there exists an essentially 2-to-1 Lipschitz surjection $f : [0, 1] \to \overline{\Sigma}$ with $\mathrm{Lip}(f) = 2\mathcal{H}^1(\Sigma)$.*

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[3]This fails dramatically for higher-dimensional curves, see e.g. the "Cantor ladders" in [BNV19, §9.2].

For the remainder of §3, we fix a connected set Σ in a Banach space $\mathbb{X}$ with $\mathcal{H}^1(\Sigma) < \infty$ and we fix a parameterization $f : [0,1] \to \overline{\Sigma}$ given by Corollary 3.3. Following [Oki92, Sch07c], we refer to subcurves of f as "arcs"; note that we do not require arcs be 1-to-1.

Definition 3.4 (arcs and associated quantities). An *arc*, $\tau = f|_{[a,b]}$, of $\overline{\Sigma}$ is the restriction of f to some interval $[a,b] \subset [0,1]$. Given an arc $\tau : [a,b] \to \overline{\Sigma}$, define

- Domain$(\tau) = [a,b]$, Image$(\tau) = \tau([a,b]) = f([a,b])$, Diam$(\tau) = $ diam Image(τ),
- Start$(\tau) = \tau(a) = f(a)$, End$(\tau) = \tau(b) = f(b)$, and
- Edge$(\tau) = [f(a), f(b)]$, i.e Edge(τ) is the line segment in $\mathbb{X}$ from $f(a)$ to $f(b)$.

Definition 3.5 ([Oki92, Sch07c]). Given an arc τ of $\overline{\Sigma}$, we define the *arc beta number*

$$(3.3) \qquad \tilde{\beta}(\tau) := \sup_{x \in \mathsf{Image}(\tau)} \frac{\mathrm{dist}\left(x, \mathsf{Edge}(\tau)\right)}{\mathsf{Diam}(\tau)} \in [0,1].$$

Let us now recall a key element in the proof of the necessary conditions in Theorems 1.2 and 1.5, first introduced by Okikiolu and later formalized by Schul.

Definition 3.6 ([Oki92, Sch07c]). A *filtration* $\mathscr{F} = \bigcup_{n=n_0}^{\infty} \mathscr{F}_n$ is a family of arcs in $\overline{\Sigma}$ with the following properties.

(1) *Tree structure:* If $\tau' \in \mathscr{F}_{n+1}$, then there exists a unique arc $\tau \in \mathscr{F}_n$ such that Domain$(\tau) \subset$ Domain(τ').
(2) *Geometric diameters:* For every $\tau \in \mathscr{F}_n$, $\rho^{-n} \leq$ Diam$(\tau) \leq A\rho^{-n}$ for some constants $\rho > 1$ and $A > 1$ independent of τ.
(3) *Trivial overlaps:* For all $\tau, \tau' \in \mathscr{F}_n$, either $\tau = \tau'$, or Domain(τ) and Domain(τ') intersect in at most one point.
(4) *Partitioning:* $\bigcup_{\tau \in \mathscr{F}_n}$ Domain$(\tau) = \bigcup_{\tau \in \mathscr{F}_{n_0}}$ Domain(τ) for every $n \geq n_0$.

Lemma 3.7 ([Oki92, Sch07c]). *Suppose that $\mathbb{X}$ is a Hilbert space. If $\mathscr{F} = \bigcup_{n=n_0}^{\infty} \mathscr{F}_{n_0}$ is a filtration, then*

$$(3.4) \qquad \sum_{\tau \in \mathscr{F}} \tilde{\beta}(\tau)^2 \, \mathsf{Diam}(\tau) \lesssim_{A,\rho} \mathcal{H}^1 \left(\bigcup_{\tau \in \mathscr{F}_{n_0}} \mathsf{Image}(\tau) \right).$$

The exponent 2 in (3.4) is a consequence of the Pythagorean theorem or parallelogram law in Hilbert space. With Lemma 3.7 in hand, the remainder of the proof of necessary conditions in the Analyst's Traveling Salesman Theorem in $\mathbb{R}^n$ or ℓ_2 is essentially metric, with a strong harmonic analysis flavor. We outline the last steps in more detail in §3.3.

3.2. **Okikiolu's filtration lemma in uniformly convex spaces.** We now develop an analogue of Lemma 3.7 in uniformly convex spaces by following the proof in ℓ_2 from [Sch07c] (which is based on [Oki92]) and replacing the parallelogram law in Hilbert space with a suitable inequality in uniformly convex spaces from [DS19].

Definition 3.8. Let $\mathbb{X}$ be a Banach space. The *modulus of convexity* $\delta_{\mathbb{X}}$ of $\mathbb{X}$ is the function $\delta_{\mathbb{X}} : [0, 2] \to [0, 1]$ defined by

$$(3.5) \qquad \delta_{\mathbb{X}}(\epsilon) := \inf\left\{1 - \frac{|x+y|}{2} : |x| = |y| = 1 \text{ and } |x - y| \geq \epsilon\right\}.$$

Definition 3.9. A Banach space is called *uniformly convex* if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. In this case, we say that $\mathbb{X}$ is *convexity power type* $p \in [2, \infty)$ if there exists $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$.

Remark 3.10 (essential facts)**.** For general background on the modulus of convexity and uniformly convex spaces, we again refer the reader to [Die75, Chapter Three] or [LT79, Chapter 1, §e]. On any Banach space $\mathbb{X}$, the modulus of convexity satisfies the inequality

$$(3.6) \qquad \delta_{\mathbb{X}}(\epsilon) \leq \delta_{\ell_2}(\epsilon) = 1 - \sqrt{1 - \epsilon^2/4}.$$

Moreover, for all $\epsilon \in (0, 2]$,

$$(3.7) \qquad \delta_{\mathbb{X}}(\epsilon) = \inf\left\{1 - \frac{|x+y|}{2} : |x| \leq 1, \ |y| \leq 1, \text{ and } |x - y| \geq \epsilon\right\};$$

see e.g. [LT79, p. 60]. In addition,

$$(3.8) \qquad \frac{\delta_{\mathbb{X}}(\epsilon_1)}{\epsilon_1} \leq \frac{\delta_{\mathbb{X}}(\epsilon_2)}{\epsilon_2} \quad \text{for all } 0 < \epsilon_1 \leq \epsilon_2 \leq 2;$$

see e.g. [LT79, Lemma 1.e.8]. In contrast with the modulus of smoothness $\rho_{\mathbb{X}}$ (see §2.3), the modulus of convexity $\delta_{\mathbb{X}}$ is not necessarily a convex function. The dual $\mathbb{X}^*$ of a uniformly smooth Banach space $\mathbb{X}$ is uniformly convex and every uniformly convex Banach space is reflexive.

Example 3.11. By Hanner's inequalities [Han56], $\delta_{\ell_p}(\epsilon) = \frac{1}{8}(p-1)\epsilon^2 + o(\epsilon^2)$ when $1 < p \leq 2$; and $\delta_{\ell_p}(\epsilon) = p^{-1}2^{-p}\epsilon^p + o(\epsilon^p)$ when $2 \leq p < \infty$. In particular, the ℓ_p spaces are uniformly convex with power type $\max(2, p)$ when $1 < p < \infty$.

In [DS19], David and Schul observed that the modulus of convexity on a uniformly smooth space of power type p can be used to control the triangle inequality excess from below. Because of its centrality to the proof of (1.18) and (1.20), we include a proof of their estimate here for reference. Actually, we provide a slightly stronger statement. There is a large literature on related substitutes for the Pythagorean theorem and parallelogram law in Banach spaces; see e.g. [BD72, Byn76, CR15, CMR18].

Lemma 3.12 (see [DS19, Lemma 8.2])**.** *Suppose $\mathbb{X}$ is a uniformly convex Banach space. If $x, y, z \in \mathbb{X}$ are distinct points and L is the line containing x and z, then*

$$(3.9) \qquad |x - y| + |y - z| - |x - z| \geq 2r\,\delta_{\mathbb{X}}\left(\frac{\mathrm{dist}(y, L)}{r}\right) \quad \text{for all } r \geq \mathrm{diam}\{x, y, z\}.$$

Proof. If $y \in L$, then (3.9) holds trivially. Thus, we may assume $\mathrm{dist}(y, L) > 0$.

Because the function $g(w) = |w - x|/|x - z|$ is continuous along $[x, z]$ with $g(x) = 0$ and $g(z) = 1$, there exists $y_0 \in [x, z] \subset L$ such that

$$(3.10) \qquad \frac{|y_0 - x|}{|x - z|} = \frac{|y - x|}{|x - y| + |y - z|}.$$

Because $y_0 \in [x, z]$, we have $|x - y_0| + |y_0 - z| = |x - z|$ and it follows that

$$(3.11) \qquad \frac{|y_0 - z|}{|x - z|} = \frac{|y - z|}{|x - y| + |y - z|}.$$

Rearranging (3.10), we see that

$$|y_0 - x| = |y - x| \frac{|x - z|}{|x - y| + |y - z|} \leq |y - x|$$

by the triangle inequality. By a parallel argument, starting from (3.11), $|y_0 - z| \leq |y - z|$. Therefore, $y, y_0 \in B(x, |y - x|) \cap B(z, |y - z|)$. We will bound $\operatorname{dist}(y, L)$ by $|y - y_0|$.

Now, let $y' = (y + y_0)/2$ and $h = |y - y_0| = 2|y - y'|$. Invoking (3.7) on a scaled and translated copy of $B(x, |y - x|)$ and similarly on $B(z, |y - z|)$, we obtain

$$\delta_{\mathbb{X}} \left(\frac{h}{|x - y|} \right) \leq 1 - \frac{|y' - x|}{|y - x|} \quad \text{and} \quad \delta_{\mathbb{X}} \left(\frac{h}{|z - y|} \right) \leq 1 - \frac{|y' - z|}{|y - z|}.$$

Therefore, noting that $|x - y| \leq r$ and $|y - z| \leq r$,

$$|x - y| + |y - z| - |x - z| \geq |x - y| + |y - z| - |x - y'| - |z - y'|$$
$$\geq |x - y| \, \delta_{\mathbb{X}} \left(\frac{h}{|x - y|} \right) + |z - y| \, \delta_{\mathbb{X}} \left(\frac{h}{|z - y|} \right)$$
$$\geq 2r \delta_{\mathbb{X}}(h/r)$$

by (3.8). Since $y_0 \in L$, we have $\operatorname{dist}(y, L) \leq |y - y_0| = h$ and (3.9) follows. $\qquad \square$

Lemma 3.13 (filtration lemma, cf. Lemma 3.7). *Let $\mathbb{X}$ be a uniformly convex Banach space of power type $p \in [2, \infty)$, say $\delta_{\mathbb{X}}(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. If $\mathscr{F} = \bigcup_{n=n_0}^{\infty} \mathscr{F}_n$ is a filtration in the sense of Definition 3.6, then*

$$(3.12) \qquad \sum_{\tau \in \mathscr{F}} \tilde{\beta}(\tau)^p \operatorname{Diam}(\tau) \lesssim_{c,p,A,\rho} \sum_{\tau \in \mathscr{F}_{n_0}} \operatorname{var}(f, \operatorname{Domain}(\tau)) - \sum_{\tau \in \mathscr{F}_{n_0}} \mathcal{H}^1(\operatorname{Edge}(\tau))$$
$$\lesssim_{c,p,A,\rho} \mathcal{H}^1 \left(\bigcup_{\tau \in \mathscr{F}_{n_0}} \operatorname{Image}(\tau) \right)$$

The implicit constant is of the form $c^{-1} A^{p-1} C(\rho, p)$ and blows up as either $c \downarrow 0$ or $p \uparrow \infty$; see (3.19).

Proof. We mimic the proof of Lemma 3.7 in [Sch07c], invoking Lemma 3.12 at a critical juncture to replace estimates depending on Hilbert space geometry. For every $\tau \in \mathscr{F}_n$ and $k \in \mathbb{N}$, we let $\mathscr{F}_{\tau,k}$ denote the kth generation descendants of τ,

$$(3.13) \qquad \mathscr{F}_{\tau,k} := \{ \tau' \in \mathscr{F}_{n+k} : \tau' \subset \tau \}.$$

Also, for every $\tau \in \mathscr{F}$, define

$$(3.14) \qquad \Delta(\tau) := \left(\sum_{\tau' \in \mathscr{F}_{\tau,1}} |\,\mathsf{Start}(\tau') - \mathsf{End}(\tau')| \right) - |\,\mathsf{Start}(\tau) - \mathsf{End}(\tau)| \in [0, \infty).$$

We immediately see that for each $n_1 \geq n_0$,

$$\sum_{\tau \in \mathscr{F}_{n_0}} |\,\mathsf{Start}(\tau) - \mathsf{End}(\tau)| + \sum_{n=n_0}^{n_1} \sum_{\tau \in \mathscr{F}_n} \Delta(\tau) = \sum_{\tau' \in \mathscr{F}_{n_1+1}} |\,\mathsf{Start}(\tau') - \mathsf{End}(\tau')|$$

$$\leq \sum_{\tau \in \mathscr{F}_{n_0}} \mathrm{var}(f, \mathsf{Domain}(\tau)).$$

Thus, because $|\,\mathsf{Start}(\tau) - \mathsf{End}(\tau)| = \mathcal{H}^1(\mathsf{Edge}(\tau))$ for each arc and f is essentially 2-to-1, recalling (3.2), we obtain,

$$(3.15) \qquad \begin{aligned} \sum_{\tau \in \mathscr{F}} \Delta(\tau) &\leq \sum_{\tau \in \mathscr{F}_{n_0}} \mathrm{var}(f, \mathsf{Domain}(\tau)) - \sum_{\tau \in \mathscr{F}_{n_0}} \mathcal{H}^1(\mathsf{Edge}(\tau)) \\ &\leq \sum_{\tau \in \mathscr{F}_{n_0}} \int_{\mathbb{X}} m(\tau, x) \, d\mathcal{H}^1(x) \leq 2\mathcal{H}^1 \left(\bigcup_{\tau \in \mathscr{F}_{n_0}} \mathsf{Domain}(\tau) \right). \end{aligned}$$

Next, for every $\tau \in \mathscr{F}$, we define the *discretized distance* d_τ by

$$(3.16) \qquad d_\tau := \sup_{\tau' \in \mathscr{F}_{\tau,1}} \sup_{x \in \mathsf{Edge}(\tau')} \mathrm{dist}(x, \mathsf{Edge}(\tau)).$$

In addition, for every arc τ and $k \in \mathbb{N}$, choose an arc $\tau_k \in \mathscr{F}_{\tau,k}$ such that d_{τ_k} is maximal among all arcs in $\mathscr{F}_{\tau,k}$. We also write $\tau_0 \equiv \tau$. We claim that

$$(3.17) \qquad \tilde{\beta}(\tau) \,\mathsf{Diam}(\tau) \leq \sum_{k=0}^{\infty} d_{\tau_k}.$$

To see this, choose an auxiliary sequence $\tau^k \in \mathscr{F}_{\tau,k}$ inductively so that $\tau^0 \equiv \tau$ and $\tilde{\beta}(\tau^{k+1}) \,\mathsf{Diam}(\tau^{k+1})$ is maximal over all arcs in $\mathscr{F}_{\tau^k,1}$. Then

$$\tilde{\beta}(\tau) \,\mathsf{Diam}(\tau) = \sum_{k=0}^{\infty} \left(\tilde{\beta}(\tau^k) \,\mathsf{Diam}(\tau^k) - \tilde{\beta}(\tau^{k+1}) \,\mathsf{Diam}(\tau^{k+1}) \right),$$

because the series is telescoping and absolutely convergent by our assumption that the arcs have geometrically decaying diameters. Moreover,

$$\tilde{\beta}(\tau^k) \,\mathsf{Diam}(\tau^k) - \tilde{\beta}(\tau^{k+1}) \,\mathsf{Diam}(\tau^{k+1}) \leq d_{\tau^k}$$

by the triangle inequality and definition of the sequence τ^k. Hence

$$\tilde{\beta}(\tau) \,\mathsf{Diam}(\tau) \leq \sum_{k=0}^{\infty} d_{\tau^k} \leq \sum_{k=0}^{\infty} d_{\tau_k}$$

by maximality of the distance d_{τ_k} among all arcs in $\mathscr{F}_{\tau,k}$. This verifies (3.17).

The proof up until this point is valid in any Banach space. To continue, we now suppose that $\delta_{\mathbb{X}}$ is convexity power type $p \in [2, \infty)$, say $\delta_{\mathbb{X}}(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. By Lemma 3.12 and the triangle inequality,

$$(3.18) \quad 2c\,\frac{d_\tau^p}{\mathsf{Diam}(\tau)^{p-1}} \leq \left(\sum_{\tau' \in \mathscr{F}_{\tau,1}} |\mathsf{Start}(\tau') - \mathsf{End}(\tau')| \right) - |\mathsf{Start}(\tau) - \mathsf{End}(\tau)| = \Delta(\tau)$$

for any arc $\tau \in \mathscr{F}$. Now, by (3.17) and Minkowski's inequality for ℓ_p,

$$\underbrace{\left(\sum_{\tau \in \mathscr{F}} \tilde{\beta}(\tau)^p \, \mathsf{Diam}(\tau) \right)^{\frac{1}{p}}}_{I} \leq \left(\sum_{\tau \in \mathscr{F}} \left(\sum_{k=0}^{\infty} d_{\tau_k} \right)^p \mathsf{Diam}(\tau)^{1-p} \right)^{\frac{1}{p}} \leq \sum_{k=0}^{\infty} \left(\sum_{\tau \in \mathscr{F}} \frac{d_{\tau_k}^p}{\mathsf{Diam}(\tau)^{p-1}} \right)^{1/p}.$$

If $\tau \in \mathscr{F}$, say $\tau \in \mathscr{F}_n$, and $k \in \mathbb{N}$, then

$$\mathsf{Diam}(\tau_k) \leq A\,\rho^{-(n+k)} = A\,\rho^{-k}\rho^{-n} \leq A\,\rho^{-k}\,\mathsf{Diam}(\tau).$$

Hence

$$I \leq \sum_{k=0}^{\infty} \left(A\rho^{-k} \right)^{\frac{p-1}{p}} \left(\sum_{\tau \in \mathscr{F}} \frac{(d_{\tau_k})^p}{\mathsf{Diam}(\tau_k)^{p-1}} \right)^{\frac{1}{p}}.$$

Thus, by (3.18),

$$(3.19)
\begin{aligned}
I &\leq \sum_{k=0}^{\infty} (2c)^{-\frac{1}{p}} \left(A\rho^{-k} \right)^{\frac{p-1}{p}} \left(\sum_{\tau \in \mathscr{F}} \Delta(\tau_k) \right)^{\frac{1}{p}} \\
&\leq \sum_{k=0}^{\infty} (2c)^{-\frac{1}{p}} \left(A\rho^{-k} \right)^{\frac{p-1}{p}} \left(\sum_{\tau \in \mathscr{F}} \Delta(\tau) \right)^{\frac{1}{p}} = \frac{(2c)^{-\frac{1}{p}} A^{\frac{p-1}{p}}}{1 - \rho^{\frac{1-p}{p}}} \left(\sum_{\tau \in \mathscr{F}} \Delta(\tau) \right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, combining (3.15) and (3.19), we obtain (3.12). $\qquad\square$

3.3. Filtration design and Schul's martingale argument: bounding sums of $\beta_\Sigma(Q)$ from above with sums of $\tilde{\beta}(\tau)$.

Because the ℓ_p spaces are uniformly convex of power type $\max\{2, p\}$ when $1 < p < \infty$, the necessary conditions (1.18) and (1.20) in Theorem 1.6 are immediate consequences of the following theorem.

Theorem 3.14 (necessary half of Schul's theorem in uniformly convex Banach spaces). *Let $\mathbb{X}$ be a uniformly convex Banach space of power type $p \in [2, \infty)$. If $\Sigma \subset \mathbb{X}$ is connected and $\mathscr{H}$ is a multiresolution family for Σ with inflation factor $A_{\mathscr{H}} > 1$, then*

$$(3.20) \qquad\qquad S_{\Sigma,p}(\mathscr{H}) \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

Proof. Repeat the proof of [Sch07c, Theorem 1.1] (see [Sch07c, §3]) *mutatis mutandis.* Use Lemma 3.13 in place of [Sch07c, Lemma 3.11]. $\qquad\square$

To end this section, we outline the steps in the proof of Theorem 3.14 in more detail. Because (3.20) is trivial when $\mathcal{H}^1(\Sigma) = \infty$, we may continue to assume that $\mathcal{H}^1(\Sigma) < \infty$ and work with the Lipschitz parameterization $f : [0, 1] \to \overline{\Sigma}$ fixed above in §3.1.

Definition 3.15 ([Oki92, Sch07c])**.** For any closed ball $Q = B(x, r)$ in $\mathbb{X}$, let

(3.21) $\qquad \Lambda(Q) := \{f|_{[a,b]} : [a, b] \text{ is a connected component of } f^{-1}(\overline{\Sigma} \cap Q)\}.$

The elements of $\Lambda(Q)$ are called *arcs* of $\overline{\Sigma}$ in Q. An arc $\tau \in \Lambda(Q)$ is called *almost flat* if

(3.22) $\qquad\qquad\qquad\qquad \tilde{\beta}(\tau) \leq \epsilon_2 \beta_\Sigma(Q),$

where $0 < \epsilon_2 \ll 1$ is a constant depending on at most the inflation factor $A_{\mathscr{H}}$ of $\mathscr{H}$ to be specified below. Let $S(Q)$ denote the set of almost flat arcs in $\Lambda(Q)$.

To proceed, we categorize the balls Q in the multiresolution family $\mathscr{H}$ according the behavior of their associated arcs $\Lambda(Q)$. First, let $\mathscr{H}_0$ denote the collection of all balls $Q \in \mathscr{H}$ such that

(3.23) $\qquad\qquad\qquad\qquad \Sigma \setminus 4Q = \emptyset,$

where $\lambda Q = B(x, \lambda r)$ denotes the concentric dilate of the ball $Q = B(x, r)$ by $\lambda > 0$. Next, define the collection

(3.24) $\qquad\qquad\qquad \widehat{\mathscr{H}} := \{Q, 2Q, 4Q : Q \in \mathscr{H} \setminus \mathscr{H}_0\}$

and for all balls $R \in \widehat{\mathscr{H}}$, choose an arc $\tau_R \in \Lambda(R)$ such that $\mathsf{Image}(\tau_R)$ contains the center of the ball R. Starting from the coarsest scale in $\widehat{\mathscr{H}}$ and working inductively, we can require that if $R_1, R_2 \in \widehat{\mathscr{H}}$ and $R_1 = (1/2)R_2$, then $\mathsf{Domain}(\tau_{R_1}) \subset \mathsf{Domain}(\tau_{R_2})$. Beyond this compatibility restriction, we may choose the arc τ_R associated to each $R \in \widehat{\mathscr{H}}$ in an arbitrary fashion.

Continuing to follow [Sch07c] (see page 345), for each $R \in \widehat{\mathscr{H}}$, we write $\beta_{S(R)}(R)$ as short hand for $\beta_{\bigcup\{\mathsf{Image}(\tau):\tau \in S(R)\}}(R)$, where we interpret $\beta_{S(R)}(R) = 0$ if the set $S(R)$ of almost flat arcs is empty. Also fix a universal constant $0 < \epsilon_1 \ll 1$ to be specified below. Then, for each scaling factor $\lambda \in \{1, 2, 4\}$, we define three families $\mathscr{H}_1^\lambda$, $\mathscr{H}_2^\lambda$, and $\mathscr{H}_3^\lambda$ of balls in $\mathscr{H} \setminus \mathscr{H}_0$,

(3.25) $\qquad \mathscr{H}_1^\lambda := \{Q \in \mathscr{H} \setminus \mathscr{H}_0 : \tilde{\beta}(\tau_{\lambda Q}) > \epsilon_2 \beta_\Sigma(\lambda Q)\},$

(3.26) $\qquad \mathscr{H}_2^\lambda := \{Q \in \mathscr{H} \setminus \mathscr{H}_0 : \tilde{\beta}(\tau_{\lambda Q}) \leq \epsilon_2 \beta_\Sigma(\lambda Q) \text{ and } \beta_{S(\lambda Q)}(\lambda Q) > \epsilon_1 \beta_\Sigma(Q)\},$

(3.27) $\qquad \mathscr{H}_3^\lambda := \{Q \in \mathscr{H} \setminus \mathscr{H}_0 : \tilde{\beta}(\tau_{\lambda Q}) \leq \epsilon_2 \beta_\Sigma(\lambda Q) \text{ and } \beta_{S(\lambda Q)}(\lambda Q) \leq \epsilon_1 \beta_\Sigma(Q)\}.$

Informally, $\mathscr{H}_1^\lambda$ consists of balls in the multiresolution family whose distinguished arc $\tau_{\lambda Q}$ is "wiggly" or "far from flat", and thus, the arc beta number $\tilde{\beta}(\tau_{\lambda Q})$ controls the flatness $\beta_\Sigma(\lambda Q)$ of Σ in λQ. By contrast, the distinguished arc $\tau_{\lambda Q}$ for balls in $\mathscr{H}_2^\lambda$ or $\mathscr{H}_3^\lambda$ is much flatter than the set Σ is in λQ. In this case, we distinguish between balls in $\mathscr{H}_2^\lambda$, for which the almost flat arcs in λQ collectively dominate the flatness of Σ in Q, and balls in $\mathscr{H}_3^\lambda$, for which the almost flat arcs in λQ are collectively much flatter than Σ in Q. See Figure 4 on page 346 of [Sch07c] for an illustration of the different families of balls. Even though the distinguished arc τ_Q is almost flat for all $Q \in \mathscr{H}_3^1$, the requirement $\beta_{S(Q)}(Q) \leq \epsilon_1 \beta_\Sigma(Q)$ ensures there exists at least one arc in $\Lambda(Q)$ that is not almost flat.

With the families $\mathscr{H}_0$, $\mathscr{H}_1^\lambda$, $\mathscr{H}_2^\lambda$, and $\mathscr{H}_3^\lambda$ now defined, the proof of Theorem 3.14 reduces to establishing an estimate like (3.20) for each category. We refer the reader to [Sch07c] for details.

Lemma 3.16 (counting). *If $\mathbb{X}$ is an arbitrary Banach space, then for all $s > 1$,*

$$(3.28) \qquad \sum_{Q \in \mathscr{H}_0} \beta_\Sigma(Q)^s \operatorname{diam}(Q) \lesssim_{s, A_{\mathscr{H}}} \mathcal{H}^1(\Sigma).$$

Proof. Modify the proof of [Sch07c, Lemma 3.9]. Any ball $Q \in \mathscr{H}_0$ has $\Sigma \subset 4Q$, and the lemma follows from counting the maximum number of balls at all higher scales and estimating of the beta number $\beta_\Sigma(Q)$ for balls $Q \in \mathscr{H}$ with $\operatorname{diam} Q \geq (1/4) \operatorname{diam} \Sigma$. The requirement that $s > 1$ ensures a certain series converges geometrically. $\qquad \square$

Lemma 3.17 (filtrations I). *If $\mathbb{X}$ is uniformly convex of power type $p \in [2, \infty)$, then*

$$(3.29) \qquad \sum_{Q \in \mathscr{H}_1^\lambda} \beta_\Sigma(Q)^p \operatorname{diam} Q \lesssim_{p, \delta_{\mathbb{X}}, A_{\mathscr{H}}, \epsilon_2} \mathcal{H}^1(\Sigma) \quad \text{for each } \lambda \in \{1, 2, 4\}.$$

Proof. Modify the proof of [Sch07c, Lemma 3.14] using Lemma 3.13 from above. In more detail, to control sums of $\beta_\Sigma(Q)^p \operatorname{diam} Q$ over $\mathscr{H}_1^\lambda$, one can build $O(A_{\mathscr{H}})$ filtrations out of non-almost-flat arcs associated to Q and then invoke Lemma 3.13. The construction of the filtrations, carried out in [Sch07c, Lemma 3.13], works in any Banach space, and only the final step, invoking Lemma 3.13, requires uniform convexity of the norm of power type p. $\qquad \square$

Lemma 3.18 (filtrations II). *Assume ϵ_1 is sufficiently small. If $\mathbb{X}$ is uniformly convex of power type $p \in [2, \infty)$, then*

$$(3.30) \qquad \sum_{Q \in \mathscr{H}_3^1 \cap \mathscr{H}_3^2 \cap \mathscr{H}_3^4} \beta_\Sigma(Q)^p \operatorname{diam} Q \lesssim_{p, \delta_{\mathbb{X}}, A_{\mathscr{H}}, \epsilon_2} \mathcal{H}^1(\Sigma).$$

Proof. Modify the proof of [Sch07c, Lemma 3.16] using Lemma 3.13 from above, like in the proof of the previous lemma. The requirement on ϵ_1 and the restriction to balls Q in $\mathscr{H}_3^1 \cap \mathscr{H}_3^2 \cap \mathscr{H}_3^4$ (i.e. the intersection of the three families $\mathscr{H}_3^1$, $\mathscr{H}_3^2$, and $\mathscr{H}_3^4$) are imposed to bound the overlap in certain families of arcs by $O(A_{\mathscr{H}})$. $\qquad \square$

Lemma 3.19 (Schul's martingale argument). *Assume ϵ_2 is sufficiently small depending on $A_{\mathscr{H}}$ (it suffices to take $\epsilon_2 \simeq A_{\mathscr{H}}^{-1}$). If $\mathbb{X}$ is uniformly convex of power type $p \in [2, \infty)$, then*

$$(3.31) \qquad \sum_{Q \in \mathscr{H}_2^\lambda} \beta_\Sigma(Q)^1 \operatorname{diam} Q \lesssim_{p, \delta_{\mathbb{X}}, A_{\mathscr{H}}, \epsilon_1, \epsilon_2} \mathcal{H}^1(\Sigma) \quad \text{for each } \lambda \in \{1, 2, 4\}.$$

Proof. Because $A_{\mathscr{H}} > 1$ is arbitrary, it suffices to establish the case $\lambda = 1$. Modify the proof of [Sch07c, Proposition 3.21] using Lemma 3.13 from above. There are three steps. In similar fashion to the proofs of Lemmas 3.17 and 3.18 above, the uniform convexity

assumption is used in the proof of [Sch07c, Lemma 3.24] to initially prove that

$$\sum_{Q \in \Delta_{2.2}} \beta_{S(Q)}(Q)^p \operatorname{diam}(Q) \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}},\epsilon_2} \mathcal{H}^1(\Sigma)$$

for a certain subfamily $\Delta_{2.2}$ of $\mathscr{H}_2^1$. (In Schul's notation, $\mathscr{H}_2^1$ is denoted by $\mathscr{G}_2$.) However, on this subfamily, $\beta_{S(Q)}(Q) \gtrsim_{A_{\mathscr{H}}} 1$, whence

$$\sum_{Q \in \Delta_{2.2}} \beta_{S(Q)}(Q)^1 \operatorname{diam}(Q) \lesssim_{p,A_{\mathscr{H}}} \sum_{Q \in \Delta_{2.2}} \beta_{S(Q)}(Q)^p \operatorname{diam} Q \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}},\epsilon_2} \mathcal{H}^1(\Sigma).$$

In the second and third steps in the proof of the proposition (see [Sch07c, §3.3.1, §3.3.2]), Schul constructs "geometric martingales" to directly control the sum of $\beta_{S(Q)}(Q)^1 \operatorname{diam} Q$ over additional subfamilies Δ_1 and $\Delta_{2.1}$ of $\mathscr{H}_2^1$; these steps carry through in an arbitrary Banach space, as written, without modification. To finish the proof for $\mathscr{H}_2^1$, simply note $\beta_\Sigma(Q) \leq \epsilon_1^{-1}\beta_{S(Q)}(Q)$ by definition of the family. $\qquad\square$

Putting it all together, on a uniformly convex space of power type $p \in [2,\infty)$, choose parameters $0 < \epsilon_1 \ll 1$ (a universal constant) and $0 < \epsilon_2 \ll 1$ with $\epsilon_2 \simeq A_{\mathscr{H}}^{-1}$ so that Lemmas 3.18 and 3.19 are valid. Then

$$\sum_{Q \in \mathscr{H}} \beta_\Sigma(Q)^p \operatorname{diam} Q \leq I + II + III + IV \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}}} \mathcal{H}^1(\Sigma),$$

where

$$I := \sum_{Q \in \mathscr{H}_0} \beta_\Sigma(Q)^p \operatorname{diam} Q \lesssim_{p,A_{\mathscr{H}}} \mathcal{H}^1(\Sigma) \qquad \text{by Lemma 3.16,}$$

$$II := \sum_{\lambda \in \{1,2,4\}} \sum_{Q \in \mathscr{H}_1^\lambda} \beta_\Sigma(Q)^p \operatorname{diam} Q \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}},\epsilon_2} \mathcal{H}^1(\Sigma) \qquad \text{by Lemma 3.17,}$$

$$III := \sum_{\lambda \in \{1,2,4\}} \sum_{Q \in \mathscr{H}_2^\lambda} \beta_\Sigma(Q)^p \operatorname{diam} Q \leq \sum_{\lambda \in \{1,2,4\}} \sum_{Q \in \mathscr{H}_2^\lambda} \beta_\Sigma(Q)^1 \operatorname{diam} Q \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}},\epsilon_1,\epsilon_2} \mathcal{H}^1(\Sigma)$$

$$\text{by Lemma 3.19,}$$

$$\text{and} \quad IV := \sum_{Q \in \mathscr{H}_3^1 \cap \mathscr{H}_3^2 \cap \mathscr{H}_3^4} \beta_\Sigma(Q)^p \operatorname{diam} Q \lesssim_{p,\delta_{\mathbb{X}},A_{\mathscr{H}},\epsilon_2} \mathcal{H}^1(\Sigma) \qquad \text{by Lemma 3.18.}$$

4. Sharpness of the exponents via examples

Our goal in this section is to verify the sharpness of the exponents on beta numbers in Theorem 1.6. To do so, we build Koch-snowflake-like curves Γ, for which we can estimate beta number sums over *arbitrary* multiresolution families. This type of construction is not new, see e.g. [BJ94, Roh01, BNV19, ENV19] for motivating examples, but the details are subtle.

The organization is as follows. In §4.1, we verify sharpness of the exponent 2 in (1.18) and (1.19) by building curves in the Euclidean plane. In §4.2, we verify sharpness of the exponent p in (1.17) and (1.20) by building curves in infinite-dimensional Banach spaces. Finally, in §4.3, we carry out additional estimates to record a proof of Proposition 1.1.
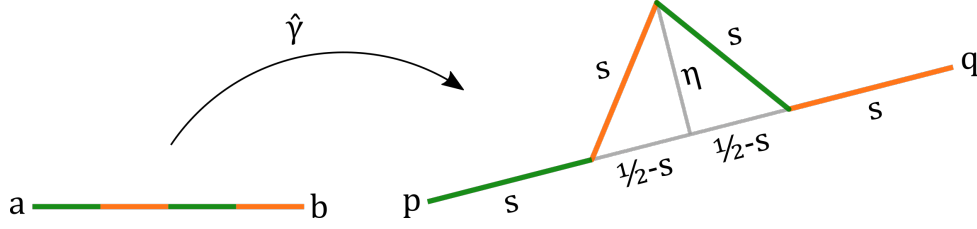
FIGURE 4. The snowflake map $\hat{\gamma}$, displayed with $|\vec{v}| = 1$, $\eta = 1/4$, and $s = 5/16$.

4.1. **Examples with critical exponent 2.** The main results of this subsection are the following two propositions. Recall that ℓ_p^n denotes $(\mathbb{R}^n, |\cdot|_p)$.

Proposition 4.1. *There exists a curve Γ in ℓ_2^2 such that $\mathcal{H}^1(\Gamma) = \infty$ and $S_{\Gamma, 2+\varepsilon}(\mathscr{G}) < \infty$ for every multiresolution family $\mathscr{G}$ for Γ and every $\varepsilon > 0$. In particular, the exponent 2 in (1.19) is sharp.*

Proposition 4.2. *There exists a curve Γ in ℓ_2^2 such that $\mathcal{H}^1(\Gamma) < \infty$ and $S_{\Gamma, 2-\varepsilon}(\mathscr{G}) = \infty$ for every multiresolution family $\mathscr{G}$ for Γ and every $\varepsilon > 0$. In particular, the exponent 2 in (1.18) is sharp.*

Sharpness of (1.18) and (1.19) in ℓ_p when $p \neq 2$ follows from the case $p = 2$, because ℓ_p contains a subspace isomorphic to ℓ_p^2, which in turn is bi-Lipschitz equivalent to ℓ_2^2. Therefore, in this section, we focus on $\ell_2^2 \equiv \mathbb{R}^2$, the standard Euclidean plane. We wish to emphasize that in both statements the curve is independent of ε. We build the curves using the following procedure.

Algorithm 4.3 (snowflake-like curves in $\mathbb{R}^2$). Suppose that $\vec{p}, \vec{q} \in \mathbb{R}^2$ and $\gamma : [a, b] \to \mathbb{R}^2$ is a constant speed parameterization of $I = [\vec{p}, \vec{q}]$ from $\gamma(a) = \vec{p}$ to $\gamma(b) = \vec{q}$. That is,

$$\gamma(t) = \vec{p} + \frac{t - a}{b - a}(\vec{q} - \vec{p}) \quad \text{for all } t \in [a, b].$$

Write $\vec{v} := \vec{q} - \vec{p}$ and let $\vec{y} \in \mathbb{R}^2$ denote the unique unit vector with $\vec{y} \perp \vec{v}$ such that $\vec{y}$ points to the left of the oriented line segment from $\vec{p}$ to $\vec{q}$. Given a *relative height* $0 \leq \eta \leq 1/\sqrt{12}$, we define a piecewise linear path $\hat{\gamma} : [a, b] \to \mathbb{R}^2$, as follows. Set $s := 1/4 + \eta^2 \in [1/4, 1/3]$. Divide $[a, b]$ into quarters. For all $a \leq t \leq (3/4)a + (1/4)b$,

$$\hat{\gamma}(t) = \vec{p} + 4\left(\frac{t - a}{b - a}\right)s\vec{v}$$

(see Figure 4). For all $(3/4)a + (1/4)b \leq t \leq (1/2)a + (1/2)b$,

$$\hat{\gamma}(t) = \vec{p} + s\vec{v} + 4\left(\frac{t - (3/4)a - (1/4)b}{b - a}\right)\left((1/2 - s)\vec{v} + \eta|\vec{v}|\vec{y}\right).$$

For all $(1/2)a + (1/2)b \leq t \leq (1/4)a + (3/4)b$,

$$\hat{\gamma}(t) = \vec{p} + (1/2)\vec{v} + \eta|\vec{v}|\vec{y} + 4\left(\frac{t - (1/2)a - (1/2)b}{b - a}\right)\left((1/2 - s)\vec{v} - \eta|\vec{v}|\vec{y}\right).$$

Finally, for all $(1/4)a + (3/4)b \leq t \leq b$,

$$\hat{\gamma}(t) = \vec{p} + (1-s)\vec{v} + 4\left(\frac{t - (1/4)a - (3/4)b}{b-a}\right)s\vec{v}.$$

We say that $\hat{\gamma}$ is obtained from γ by *adding a bump of relative height η (on the left side)*. On each quarter of $[a, b]$, $\hat{\gamma}(t)$ traces a line segment of length $s|\vec{v}|$ at constant speed. Thus, $\hat{\gamma}$ has Lipschitz constant $4s|\vec{v}|/(b-a) = (1 + 4\eta^2)|\vec{v}|/(b-a)$. Additionally, we have $\|\gamma - \hat{\gamma}\|_\infty = |\gamma((a+b)/2) - \hat{\gamma}((a+b)/2)| = \eta|\vec{v}|$.

Starting from the arc length parameterization $\gamma_0 : [0, 1] \to \mathbb{R}^2$ of the line segment $I_0 = [\vec{0}, \vec{e}_1]$ from $\gamma_0(0) = \vec{0}$ to $\gamma_0(1) = \vec{e}_1$, we now define a sequence of piecewise linear maps $\gamma_i : [0, 1] \to \mathbb{R}^2$ by iteratively adding bumps of relative height η_i. Suppose that γ_i has been defined for some $i \geq 0$ so that $\gamma_i|_{J_{i,k}}$ is a constant speed parameterization of a line segment on each interval $J_{i,k} = [a_{i,k}, b_{i,k}]$ of the form

$$J_{i,k} = \left[a + \frac{(k-1)}{4^i}(b-a), a + \frac{k}{4^i}(b-a)\right] \qquad (1 \leq k \leq 4^i, \ k \in \mathbb{Z}).$$

For each index $1 \leq k \leq 4^i$, define $\gamma_{i+1}|_{J_{i,k}} = \widehat{\gamma_i|_{J_{i,k}}}$ by adding a bump of relative height η_{i+1}. This defines a map γ_{i+1}. By induction, we obtain the full sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$; the image of each map γ_n is composed of 4^n line segments of length $r_n := (\text{Lip}\,\gamma_n)4^{-n}$. Evidently, for all $n = 0, 1, 2, \ldots$

$$(4.1) \qquad \text{Lip}(\gamma_n) = \prod_{i=1}^{n}(1 + 4\eta_i^2) \leq (4/3)^n,$$

$$(4.2) \qquad r_{n+1} \leq (1/3)r_n,$$

$$(4.3) \qquad \|\gamma_n - \gamma_{n+1}\|_\infty \leq \eta_{n+1}r_n \leq 3^{-n}\eta_{n+1} < 3^{-n}.$$

Therefore, $\gamma : [0, 1] \to \mathbb{R}^2$, which is defined pointwise by

$$(4.4) \qquad \gamma(t) = \gamma_0(t) + \sum_{n=0}^{\infty}(\gamma_{n+1}(t) - \gamma_n(t)) \quad \text{for all } t \in [0, 1],$$

is continuous as the uniform limit of the maps γ_n by (4.3). We denote $\gamma([0, 1])$ by Γ, and for each n, we denote $\gamma_n([0, 1])$ by Γ_n. For all integers $n, k \geq 0$ with $0 \leq k \leq 4^n$, we call the point $\gamma_n(k/4^n)$ a *vertex* of Γ_n.

Remark 4.4 (modulus of continuity). In fact, (4.1) and (4.3) imply that the maps γ_n and γ are uniformly $\log_4(3)$-Hölder continuous (see e.g. [BNV19, Appendix B]). This is the optimal modulus of continuity of the von Koch snowflake curve, which corresponds to the choice of relative heights $\eta_n = 1/\sqrt{12}$ for all n. Furthermore, if $\sum_{n=1}^{\infty} \eta_n^2 < \infty$, then (4.1) implies that γ_n and γ are uniformly Lipschitz with Lipschitz constant at most $\prod_{n=1}^{\infty}(1 + 4\eta_n^2) \simeq \exp(\sum_{n=1}^{\infty} \eta_n^2)$.

Remark 4.5 (vertices). For each integer $n \geq 0$, let V_n denote the set of vertices in Γ_n and let $\widetilde{V}_n := V_n \setminus V_{n-1}$ denote the set of "new vertices" in Γ_n with the convention that

$\widetilde{V}_0 = V_0$. For later use, we observe that

$$(4.5) \qquad\qquad\qquad V_0 \subset V_1 \subset V_2 \subset \cdots,$$

$$(4.6) \qquad V_n = \widetilde{V}_0 \cup \widetilde{V}_1 \cup \cdots \cup \widetilde{V}_{n-1} \cup \widetilde{V}_n \qquad (\widetilde{V}_i \cap \widetilde{V}_j = \emptyset \text{ when } i \neq j)$$

$$(4.7) \qquad\qquad \#\widetilde{V}_0 = 2, \quad \#\widetilde{V}_n = 3 \cdot 4^{n-1} \ (n \geq 1), \quad \#V_n = 1 + 4^n$$

Remark 4.6 (injectivity)**.** The restriction $\eta_i \leq 1\sqrt{12}$ on the relative heights ensures that the parameterization γ of Γ constructed by Algorithm 4.3 is injective. This can be shown by a geometric argument similar to the proof that the standard von Koch snowflake curve is the attractor of an iterated function system that satisfies the open set condition (i.e. draw equilateral triangles on the left side of each segment in Γ_n). We leave details to the dedicated reader. For other models of generalized von Koch curves, the question of injectivity of a parameterization is quite subtle, see e.g. [KP10].

Lemma 4.7. *If Γ is constructed by Algorithm 4.3, then*

$$(4.8) \qquad \exp\left(3.3\overline{3} \sum_{n=1}^{\infty} \eta_n^2\right) \leq \mathcal{H}^1(\Gamma) = \prod_{n=1}^{\infty}(1 + 4\eta_n^2) \leq \exp\left(4 \sum_{n=1}^{\infty} \eta_n^2\right).$$

In particular, Γ is a rectifiable curve if and only if $\sum_{n=1}^{\infty} \eta_n^2 < \infty$. Moreover, in that case, Γ is Ahlfors regular with constants depending only on $\mathcal{H}^1(\Gamma)$.

Proof. The one-dimensional Hausdorff measure $\mathcal{H}^1$ enjoys the bound $\mathcal{H}^1(K) \geq \operatorname{diam} K$ for every connected set $K \subset \mathbb{R}^2$ (see e.g. [AO17, Lemma 2.11]). Thus, for each $n \geq 1$,

$$\mathcal{H}^1(\Gamma) = \sum_{k=1}^{4^n} \mathcal{H}^1(\gamma([(k-1)4^{-n}, k4^{-n}])) \geq \sum_{k=1}^{4^n} \operatorname{diam} \gamma([(k-1)4^{-n}, k4^{-n}])$$

$$\geq \sum_{k=1}^{4^n} \operatorname{diam} \gamma_n([(k-1)4^{-n}, k4^{-n}]) = \sum_{k=1}^{4^n} \prod_{i=1}^{n}(1/4 + \eta_i^2) = \prod_{i=1}^{n}(1 + 4\eta_i^2),$$

where the initial equality holds by Remark 4.6. Hence $\mathcal{H}^1(\Gamma) \geq \prod_{n=1}^{\infty}(1+4\eta_n^2)$. Conversely, $\mathcal{H}^1(\Gamma) \leq \operatorname{Lip} \gamma \leq \prod_{n=1}^{\infty}(1 + 4\eta_n^2)$. Therefore,

$$\mathcal{H}^1(\Gamma) = \prod_{n=1}^{\infty}(1 + 4\eta_n^2) = \exp\left(\sum_{n=1}^{\infty} \log(1 + 4\eta_n^2)\right).$$

Finally, since $0 \leq 4\eta_n^2 \leq 1/3$, we may use Taylor's theorem to bound $\log(1 + 4\eta_n^2) \leq 4\eta_n^2$ and $\log(1 + 4\eta_n^2) \geq 4\eta_n^2 - \frac{1}{2}(4\eta_n^2)^2 \geq (10/3)\eta_n^2$.

Suppose that $\mathcal{H}^1(\Gamma) < \infty$. By (4.2), (4.3), and the bound $\eta_n \leq 1/\sqrt{12}$ for all $n \geq 1$,

$$\|\gamma_n - \gamma\|_\infty \leq \sum_{j=n}^{\infty} \eta_{j+1} r_j \leq \frac{1}{\sqrt{12}} \sum_{k=0}^{\infty} r_{n+k} \leq \frac{1}{\sqrt{12}} r_n \sum_{k=0}^{\infty} 3^{-k} = \frac{\sqrt{3}}{4} r_n < 0.45 r_n.$$

Thus, given $x \in \Gamma$ and $n \geq 1$, we may pick $y \in \Gamma_n$ such that $|x - y| \leq (\sqrt{3}/4)r_n$. Next, let $v \in V_n$ be an endpoint of the segment in Γ_n containing y that is closest to y so that $|v - y| \leq (1/2)r_n$. Hence $B(x, 0.05r_n) \subset B(v, r_n)$ and we may use (4.8) to estimate

$$(4.9) \quad \mathcal{H}^1(\Gamma \cap B(x, 0.05r_n)) \leq \mathcal{H}^1(\Gamma \cap B(v, r_n)) \leq 2 \exp\left(\sum_{i=n}^{\infty} 4\eta_i^2\right) r_n \leq 2e^{6/5}\mathcal{H}^1(\Gamma)r_n,$$

because $\Gamma_n \cap B(v, r_n)$ consists of one or two line segments of length r_n. From (4.9), one easily deduces that $\mathcal{H}^1(\Gamma \cap B(x, r)) \simeq_{\mathcal{H}^1(\Gamma)} r$ for every $x \in \Gamma$ and every $0 < r \leq 1 = \operatorname{diam}\Gamma$. Therefore, Γ is Ahlfors regular with constants depending only on $\mathcal{H}^1(\Gamma)$. $\qquad\square$

Lemma 4.8. *Assume that Γ is constructed by Algorithm 4.3. For all $i \geq 0$ and $j \geq i$,*

$$(4.10) \qquad\qquad \frac{\sqrt{3}}{4}\eta_i \leq \beta_{\Gamma_j}(B(v, r_j)) \leq 2\eta_i \quad \text{for all } v \in \widetilde{V}_i,$$

where $\widetilde{V}_i$ denote the "new vertices" in Γ_i (see Remark 4.5) and $\eta_0 = 0$. Furthermore,

$$(4.11) \qquad\qquad \frac{\sqrt{3}}{4}\eta_i \leq \beta_\Gamma(\overline{B(v, r_i)}) \quad \text{for all } i \geq 0 \text{ and } v \in \widetilde{V}_i.$$

Proof. Note that when $v \in \widetilde{V}_0$ is an endpoint of Γ_j, the set $\Gamma_j \cap B(v, r_j)$ is a line segment. Thus, $\beta_{\Gamma_j}(B(v, r_j)) = 0$ for all $v \in \widetilde{V}_0$ and $j \geq 0$. Next, suppose that $i \geq 1$ and $v \in \widetilde{V}_i$. Using the line containing the segment in $\Gamma_{i-1} \cap B(v, r_i)$ to approximate the beta number, we obtain the estimate

$$\beta_{\Gamma_i}(B(v, r_i)) \leq \frac{\eta_i}{2(\frac{1}{4} + \eta_i^2)} \leq 2\eta_i.$$

When $j \geq i$, the set $\Gamma_j \cap B(v, r_j)$ agrees up to a dilation centered at v with $\Gamma_i \cap B(v, r_i)$. Thus, $\beta_{\Gamma_j}(B(v, r_j)) \leq 2\eta_i$ for all $v \in \widetilde{V}_i$ and $j \geq i$, as well.

To establish the lower bound, one can use symmetry to find the best fitting line for each of the two triangles formed by consecutive segments of side length s in Figure 4. The best fitting lines are parallel to the third side of each triangle and the altitudes of the triangles are η and $\sqrt{s}\eta = (\frac{1}{4} + \eta^2)^{1/2}\eta$. It follows that for all $v \in \widetilde{V}_i$ and $j \geq i$,

$$\beta_{\Gamma_j}(B(v, r_j)) = \beta_{\Gamma_i}(B(v, r_i)) \geq \frac{\frac{1}{2}(\frac{1}{4} + \eta_i^2)^{1/2}\eta_i}{2(\frac{1}{4} + \eta_i^2)} = \frac{\eta_i}{4(\frac{1}{4} + \eta_i^2)^{1/2}} \geq \frac{\sqrt{3}}{4}\eta_i,$$

since $\eta_i \leq 1/\sqrt{12}$. Finally, because $\Gamma \cap \overline{B(v, r_i)}$ contains the vertices a, b, c of one of the two triangles, we find that $\beta_\Gamma(\overline{B(v, r_i)}) \geq \beta_{\{a,b,c\}}(\overline{B(v, r_i)}) \geq (\sqrt{3}/4)\eta_i$, as well. $\qquad\square$

4.1.1. *Proof of Proposition 4.1.* Build Γ using Algorithm 4.3 with relative heights $4\eta_i^2 := 1/(i + 15)\log(i + 15)$ for all $i \geq 1$. See Remark 4.9 below for an explanation of the logarithmic factor in the relative heights. Note that $\eta_{i+1} \leq \eta_i < 1/8$ for all i. Because $\sum_{i=1}^{\infty} \eta_i^2 = \infty$, we have $\mathcal{H}^1(\Gamma) = \infty$ by Lemma 4.7. To proceed, let $(X_k)_{k \in \mathbb{Z}}$ be any family of nested 2^{-k}-nets for Γ, and let $\mathscr{G} = \{B(x, A2^{-k}) : x \in X_k, k \in \mathbb{Z}\}$ be the associated

multiresolution family with inflation factor $A > 1$. Since Γ is bounded, to prove that $S_{\Gamma,2+\varepsilon}(\mathscr{G}) < \infty$, it suffices to show that (cf. Lemma 3.16)

$$S_{\Gamma,2+\varepsilon}(\mathscr{G}') := \sum_{B \in \mathscr{G}'} \beta_\Gamma(B)^{2+\varepsilon} \operatorname{diam} B < \infty,$$

where $\mathscr{G}'$ is the subfamily of all balls starting from some initial generation k_0.

Recall that each intermediate curve Γ_n consists of 4^n line segments of length

$$r_n = 4^{-n} \prod_{i=1}^{n}(1 + 4\eta_i^2) = 4^{-n} \exp\left(\sum_{i=1}^{n} \log(1 + 4\eta_i^2)\right).$$

By Taylor's theorem, we have

$$4\eta_i^2 - \frac{1}{2}(4\eta_i^2)^2 \leq \log(1 + 4\eta_i^2) \leq 4\eta_i^2 \quad \text{for all } i \geq 1.$$

Combined with the elementary bounds

$$\sum_{i=1}^{n} \frac{1}{(i+15)\log(i+15)} \leq \frac{1}{16\log(16)} + \int_{16}^{n+15} \frac{1}{x\log(x)} \, dx$$

$$< \frac{1}{16} + \log(\log(n+15))$$

and

$$\sum_{i=1}^{n} \frac{1}{(i+15)\log(i+15)} - \frac{1}{2(i+15)^2\log(i+15)^2}$$

$$\geq \int_{16}^{n+16} \left(\frac{1}{x\log(x)} - \frac{1}{2x^2\log(x)^2}\right) dx > \log(\log(n+16)) - 1.2$$

we obtain the rough estimate (for the record, $0.3 < e^{-1.2}$ and $e^{1/16} < 1.1$):

(4.12) $0.3 \log(n+16)4^{-n} < r_n < 1.1 \log(n+15)4^{-n} \quad \text{for all } n \geq 1.$

To continue, let $k \geq k_0$ (with k_0 sufficiently large depending on A to be specified below) and let $x \in X_k$. Our immediate goal is to estimate $\beta_\Gamma(B(x, A2^{-k}))$ from above in terms of $\beta_{\Gamma_m}(B(v, r_m))$ for some suitably chosen generation $m = m(A, k) \geq 1$ and vertex $v \in \Gamma_m$. Write $c := \log_4 A > 0$ so that $A2^{-k} = 4^{-(\frac{1}{2}k-c)}$. We now require $\frac{1}{2}k_0 - c \geq 3$, which ensures

$$A2^{-k_0} \leq 4^{-3} < 0.03 \log(16)4^{-1}.$$

Since $k \geq k_0$, there exists a unique integer $m \geq 1$ with

(4.13) $0.03 \log(m+16)4^{-(m+1)} < A2^{-k} \leq 0.03 \log(m+15)4^{-m}.$

By (4.12), it follows that

(4.14) $r_m \lesssim A2^{-k} < \frac{1}{10}r_m.$

Next, by (4.2), (4.3), and fact that $\eta_{i+1} \leq \eta_i < 1/8$ for all i,

$$\|\gamma_m - \gamma\|_\infty \leq \sum_{j=m}^\infty \eta_{j+1} r_j \leq \eta_{m+1} \sum_{l=0}^\infty r_{m+l} \leq \eta_{m+1} r_m \sum_{k=0}^\infty 3^{-k} = (3/2)\eta_{m+1} r_m < (3/16) r_m.$$

In particular, we can find $y \in \Gamma_m$ with $|x - y| < (3/16) r_m$ and then choose a vertex v in Γ_m such that $|y - v| \leq (1/2) r_m$ (i.e. v is an endpoint of the segment containing y). Since

$$\frac{1}{10} + \frac{3}{16} + \frac{1}{2} < \frac{3}{4},$$

$B(x, A2^{-k}) \subset B(v, (3/4) r_m)$. Invoking the bound $\|\gamma - \gamma_m\|_\infty \leq (3/2)\eta_{m+1} r_m < (3/16) r_m$ again, we conclude that

$$E := \mathrm{excess}(\Gamma \cap B(x, A2^{-k}), \Gamma_m \cap B(v, r_m)) \leq (3/2)\eta_{m+1} r_m,$$

where $\mathrm{excess}(S, T) = \sup_{s \in S} \inf_{t \in T} |s - t|$ denotes the *excess of S over T*. Hence

$$(4.15) \qquad \beta_\Gamma(B(x, A2^{-k})) \leq \frac{E}{2A2^{-k}} + \frac{r_m}{A2^{-k}} \cdot \beta_{\Gamma_m}(B(v, r_m)) \lesssim \eta_{m+1} + \beta_{\Gamma_m}(B(v, r_m))$$

by the triangle inequality and (4.14). Note that by Lemma 4.8,

$$(4.16) \qquad \eta_{m+1} + \beta_{\Gamma_m}(B(v, r_m)) \leq \eta_{m+1} + 2\eta_i \quad \text{when } v \in \widetilde{V}_i \subset V_m,$$

where $\eta_0 = 0$. In particular, $\beta_\Gamma(B(x, A2^{-k})) \lesssim \eta_{m+1}$ when $v \in \widetilde{V}_0$ and $\beta_\Gamma(B(x, A2^{-k})) \lesssim \eta_i$ when $v \in \widetilde{V}_i$ for some $1 \leq i \leq m$.

Next, we bound the number of times a scale m and vertex $v \in V_m$ are associated to a point $x \in X_k$. On one hand, $\#X_k \cap B(v, r_m) \lesssim A^2$ for each vertex $v \in V_m$ by (4.14), since X_k is 2^{-k}-separated and our construction takes place in $\mathbb{R}^2$. (We could remove the dimension dependence by using Lemma 2.1, but do not require a sharp upper bound.) On the other hand, the scale m associated to $k \geq k_0$ satisfies

$$0.03 \log(m + 16) 4^{-(m+1)} < 4^{-(\frac{1}{2}k - c)} \leq 0.03 \log(m + 15) 4^{-m},$$

where $c = \log_4 A$. Taking logarithms and rearranging, we have

$$2(m + c - \log_4(0.03 \log(m + 15))) \leq k < 2(m + 1 + c - \log_4(0.03 \log(m + 16))).$$

Thus, the number of integers k associated to a given integer m is at most

$$2(m + 1 + c - \log_4(0.03 \log(m + 16))) - 2(m + c - \log_4(0.03 \log(m + 15))) \leq 2.$$

To finish, fix a parameter $\varepsilon > 0$. In view the previous paragraph, (4.13), and (4.15), we see that

$$(4.17) \qquad \begin{aligned} S_{\Gamma, 2+\varepsilon}(\mathscr{G}') &= \sum_{k=k_0}^\infty 2A2^{-k} \sum_{x \in X_k} \beta_\Gamma(B(x, A2^{-k}))^{2+\varepsilon} \\ &\lesssim_{A,\varepsilon} \sum_{m=1}^\infty \log(m + 15) 4^{-m} \sum_{v \in V_m} (\eta_{m+1} + \beta_{\Gamma_m}(B(v, r_m)))^{2+\varepsilon}. \end{aligned}$$

Decomposing $V_m = \widetilde{V}_0 \cup \cdots \cup \widetilde{V}_m$ (see Remark 4.5) and invoking (4.16),

$$(4.18) \quad \sum_{v \in V_m} (\eta_{m+1} + \beta_{\Gamma_m}(B(v, r_m)))^{2+\varepsilon} \lesssim_\varepsilon \sum_{v \in \widetilde{V}_0} \eta_{m+1}^{2+\varepsilon} + \sum_{i=1}^m \sum_{v \in \widetilde{V}_i} \eta_i^{2+\varepsilon} \lesssim_\varepsilon 1 + \sum_{i=1}^m 4^{i-1} \eta_i^{2+\varepsilon}.$$

Combining the previous two displayed equations, it follows that

$$(4.19) \quad S_{\Gamma, 2+\varepsilon}(\mathscr{G}') \lesssim_{A,\varepsilon} \underbrace{\sum_{m=1}^\infty \log(m+15) 4^{-m}}_{I} + \underbrace{\sum_{m=1}^\infty \log(m+15) 4^{-m} \sum_{i=1}^m 4^{i-1} \eta_i^{2+\varepsilon}}_{II}.$$

It is apparent that $I \lesssim 1$. To bound II, exchange the order of summation:

$$(4.20) \quad \begin{aligned} II &= \sum_{i=1}^\infty 4^{i-1} \eta_i^{2+\varepsilon} \sum_{m=i}^\infty \log(m+15) 4^{-m} \lesssim \sum_{i=1}^\infty 4^{i-1} \eta_i^{2+\varepsilon} \log(i+15) 4^{-i} \\ &\lesssim \sum_{i=1}^\infty \log(i+15) \eta_i^{2+\varepsilon} = \sum_{i=1}^\infty \log(i+15) \left(\frac{1}{4(i+15)\log(i+15)} \right)^{1+\frac{1}{2}\varepsilon} \lesssim_\varepsilon 1. \end{aligned}$$

We conclude that $S_{\Gamma, 2+\varepsilon}(\mathscr{G}') \lesssim_{A,\varepsilon} 1$. Therefore, by our initial discussion, $S_{\Gamma, 2+\varepsilon}(\mathscr{G}) < \infty$ for every $\varepsilon > 0$ and every multiresolution family $\mathscr{G}$ for Γ. This completes the proof of Proposition 4.1.

Remark 4.9 (importance of the logarithmic factor). Seeking out examples verifying the sharpness of (1.19), it is natural to first look at snowflake curves Γ built with relative heights $\eta_i^2 \simeq 1/i$. By carrying out the outline above with relative heights $4\eta_i^2 = \delta/(i+i_0)$ with parameters $\delta > 0$ and $i_0 \geq 1$, one obtains

$$II \lesssim \sum_{i=1}^\infty (i+i_0)^\delta \left(\frac{1}{4(i+i_0)} \right)^{1+\frac{1}{2}\varepsilon},$$

where the latter expression is finite precisely when $\delta < (1/2)\varepsilon$. Thus, for every $\varepsilon > 0$, we could build a curve Γ_ε with $\mathcal{H}^1(\Gamma_\varepsilon) = \infty$ and $S_{\Gamma_\varepsilon, 2+\varepsilon}(\mathscr{G}) < \infty$ by selecting $\delta = \delta(\varepsilon)$ sufficiently small. The logarithmic correction used in the proof of Proposition 4.1 allows us to find a single curve Γ such that $\mathcal{H}^1(\Gamma) = \infty$ and $S_{\Gamma, 2+\varepsilon}(\mathscr{G}) < \infty$ for all $\varepsilon > 0$.

4.1.2. *Proof of Proposition 4.2.* Construct Γ using Algorithm 4.3 with relative heights $4\eta_i^2 = 1/(i+2)\log(i+2)^2$ for all $i \geq 1$. Then $\eta_{i+1} \leq \eta_i < 1/\sqrt{12}$ for all $i \geq 1$ and

$$(4.21) \quad \sum_{i=1}^\infty 4\eta_i^2 = \sum_{n=3}^\infty \frac{1}{n\log(n)^2} = 1.069... < \infty$$

Hence $\mathcal{H}^1(\Gamma) \leq \exp(\sum_{i=1}^\infty 4\eta_i^2) < \exp(1.07) < 3$ by Lemma 4.7. Similarly, we have that the intermediate curves Γ_n consist of 4^n segments of length r_n, where

$$(4.22) \quad 4^{-n} < r_n \leq \exp\left(\sum_{i=1}^n 4\eta_i^2 \right) 4^{-n} < 3 \cdot 4^{-n} \quad \text{for all } n \geq 1.$$

Let $(X_k)_{k \in \mathbb{Z}}$ be an arbitrary family of nested 2^{-k}-nets for Γ, let $A > 1$, and let $\mathscr{G} = \{B(x, A2^{-k}) : x \in X_k, k \in \mathbb{Z}\}$ be the associated multiresolution family for Γ. We wish to show that $S_{\Gamma, 2-\varepsilon}(\Gamma) = \infty$ for all $\varepsilon > 0$. By Lemma 4.7, Γ is Ahlfors regular with constants determined by $\mathcal{H}^1(\Gamma)$. In particular, we know that

$$\#X_k \simeq 2^k \quad \text{for every } k \geq 0.$$

Thus, writing $\beta(k) := \inf_{x \in X_k} \beta_\Gamma(B(x, A2^{-k}))$, we have

$$(4.23) \qquad S_{\Gamma, 2-\varepsilon}(\mathscr{G}) \geq \sum_{k=k_1}^{\infty} \sum_{x \in X_k} \beta_\Gamma(B(x, A2^{-k}))^{2-\varepsilon} 2A2^{-k} \gtrsim_A \sum_{k=k_1}^{\infty} \beta(k)^{2-\varepsilon}.$$

To proceed, we will bound $\beta(k)$ from below in terms of η_m for sufficiently large k.

Choose k_0 sufficiently large such that $A2^{-k_0} < 6$. Suppose that $k \geq k_0$. Let $m(k) \geq 1$ be the unique integer such that $6 \cdot 4^{-m} \leq A2^{-k} < 6 \cdot 4^{-(m-1)}$. By (4.22), we have

$$(4.24) \qquad\qquad\qquad 2r_m \leq A2^{-k} \lesssim r_m.$$

Given $x \in X_k$, choose $v \in V_m$ such that $|x - v| < r_m$ (cf. proof of Lemma 4.7). Then $B(v, r_m) \subset B(x, 2r_m) \subset B(x, A2^{-k})$. Hence

$$\beta_\Gamma(B(x, A2^{-k})) \geq \frac{r_m}{A2^{-k}} \beta_\Gamma(B(v, r_m)) \gtrsim \eta_m$$

by Lemma 4.8. As $x \in X_k$ was arbitrary, $\beta(k) \gtrsim \eta_m$. Now, $m \leq \frac{1}{2}k + \log_4(6/A) + 1$. Choose $k_1 \geq k_0$ sufficiently large such that $\log_4(6/A) + 3 \leq \frac{1}{2}k_1$. Then, for every $k \geq k_1$, we have $m + 2 \leq k$ and

$$\beta(k) \gtrsim \eta_m \gtrsim \frac{1}{(m+2)^{1/2} \log(m+2)} \gtrsim \frac{1}{k^{1/2} \log(k)}.$$

Therefore, for every $\varepsilon > 0$,

$$S_{\Gamma, 2-\varepsilon}(\mathscr{G}) \gtrsim_A \sum_{k=k_1}^{\infty} \beta(k)^{2-\varepsilon} \gtrsim_A \sum_{k=k_1}^{\infty} k^{\frac{1}{2}\varepsilon - 1} \log(k)^{\varepsilon - 2} = \infty.$$

This completes the proof of Proposition 4.2.

4.2. **Examples with critical exponent $p \neq 2$.** To complete the proof of Theorem 1.6, we return to the infinite-dimensional setting. Our goal is to establish:

Proposition 4.10. *For all $1 < p < \infty$, there is a curve Γ in ℓ_p such that $\mathcal{H}^1(\Gamma) = \infty$ and $S_{\Gamma, p+\varepsilon}(\mathscr{G}) < \infty$ for every multiresolution family $\mathscr{G}$ for Γ and every $\varepsilon > 0$. In particular, the exponent p in (1.17) is sharp.*

Proposition 4.11. *For all $1 < p < \infty$, there is a curve Γ in ℓ_p such that $\mathcal{H}^1(\Gamma) < \infty$ and $S_{\Gamma, p-\varepsilon}(\mathscr{G}) = \infty$ for every multiresolution family $\mathscr{G}$ for Γ and every $\varepsilon > 0$. In particular, the exponent p in (1.20) is sharp.*

We construct the curves in both propositions using the following algorithm, which is inspired by examples of Edelen, Naber, and Valtorta [ENV19, §5.2] in $L^p([0,1])$. The key point is that because we are working in an infinite dimensional space, we may build each intermediate iteration of the snowflake by adding bumps in a new coordinate direction.

Algorithm 4.12 (snowflake-like curves in ℓ_p with bumps along coordinate directions). Let $\{e_i\}_{i=1}^{\infty}$ denote the standard basis in ℓ_p, i.e. $e_i(j) = \delta_{ij}$. Suppose $x, y \in \operatorname{span}\{e_1, \ldots, e_k\}$ and $\gamma : [a,b] \to \operatorname{span}\{e_1, \ldots, e_k\} \cong \ell_p^k$ is a constant speed parameterization of $I = [x, y]$ from $\gamma(a) = x$ to $\gamma(b) = y$. That is,

$$\gamma(t) = x + \frac{t-a}{b-a}(y-x) \quad \text{for all } t \in [a,b].$$

Write $v := y - x$. Given a *relative height* $0 \leq \eta < 1/2$, we define a piecewise linear path $\hat{\gamma} : [a,b] \to \operatorname{span}\{e_1, \ldots, e_{k+1}\} \cong \ell_p^{k+1}$, as follows. Define $s \in [1/4, 1/2)$ to be the unique solution of $s^p = (\frac{1}{2} - s)^p + \eta^p$. Divide $[a,b]$ into quarters. For all $a \leq t \leq (3/4)a + (1/4)b$,

$$\hat{\gamma}(t) = x + 4\left(\frac{t-a}{b-a}\right)sv$$

(cf. Algorithm 4.3). For all $(3/4)a + (1/4)b \leq t \leq (1/2)a + (1/2)b$,

$$\hat{\gamma}(t) = x + sv + 4\left(\frac{t - (3/4)a - (1/4)b}{b-a}\right)\left((1/2 - s)v + \eta|v|_p e_{k+1}\right).$$

For all $(1/2)a + (1/2)b \leq t \leq (1/4)a + (3/4)b$,

$$\hat{\gamma}(t) = x + (1/2)v + \eta|v|e_{k+1} + 4\left(\frac{t - (1/2)a - (1/2)b}{b-a}\right)\left((1/2 - s)v - \eta|v|_p e_{k+1}\right).$$

Finally, for all $(1/4)a + (3/4)b \leq t \leq b$,

$$\hat{\gamma}(t) = x + (1 - s)v + 4\left(\frac{t - (1/4)a - (3/4)b}{b-a}\right)sv.$$

We say that $\hat{\gamma}$ is obtained from γ by *adding a bump of relative height η in the direction* e_{k+1}. On each quarter of $[a,b]$, $\hat{\gamma}(t)$ traces a line segment in ℓ_p of length $s|v|_p$ at constant speed. Thus, $\hat{\gamma}$ has Lipschitz constant $4s|v|_p/(b-a)$. Additionally, we have $\|\gamma - \hat{\gamma}\|_\infty = |\gamma((a+b)/2) - \hat{\gamma}((a+b)/2)|_p = \eta|v|_p$.

Starting from the arc length parameterization $\gamma_0 : [0,1] \to \operatorname{span}\{e_1\}$ of the line segment $I_0 = [0, e_1]$ from $\gamma_0(0) = 0$ to $\gamma_0(1) = e_1$, we now define a sequence of piecewise linear maps $\gamma_i : [0,1] \to \operatorname{span}\{e_1, \ldots, e_{i+1}\}$ by iteratively adding bumps of relative height η_i in the direction e_{i+1}. Suppose that γ_i has been defined for some $i \geq 0$ so that $\gamma_i|_{J_{i,k}}$ is a constant speed parameterization of a line segment in $\operatorname{span}\{e_1, \ldots, e_{i+1}\}$ on each interval $J_{i,k} = [a_{i,k}, b_{i,k}]$ of the form

$$J_{i,k} = \left[a + \frac{(k-1)}{4^i}(b-a), a + \frac{k}{4^i}(b-a)\right] \qquad (1 \leq k \leq 4^i, \ k \in \mathbb{Z}).$$

For each index $1 \leq k \leq 4^i$, define $\gamma_{i+1}|_{J_{i,k}} = \widehat{\gamma_i|_{J_{i,k}}}$ by adding a bump of relative height η_{i+1} in the direction e_{i+2}. This defines a map γ_{i+1}. By induction, we obtain the full

sequence $\gamma_0, \gamma_1, \gamma_2, \ldots$; the image of each map γ_n is composed of 4^n line segments in $\text{span}\{e_1, \ldots, e_{n+1}\}$ of equal length $r_n = (\text{Lip}\,\gamma_n)4^{-n} = s_1 \cdots s_n$, where $s_i \in [1/4, 1/2)$ denotes the solution to $s_i^p = (\frac{1}{2} - s_i)^p + \eta_i^p$. Moreover,

$$(4.25) \qquad \text{Lip}(\gamma_n) = \prod_{i=1}^{n} 4 s_i < 2^n,$$

$$(4.26) \qquad r_{n+1} \leq (1/2) r_n,$$

$$(4.27) \qquad \|\gamma_n - \gamma_{n+1}\|_\infty \leq \eta_{n+1} r_n \leq 2^{-n} \eta_{n+1} < 2^{-n}.$$

Therefore, $\gamma : [0,1] \to \ell_p$, which is defined pointwise by

$$(4.28) \qquad \gamma(t) = \gamma_0(t) + \sum_{n=0}^{\infty} (\gamma_{n+1}(t) - \gamma_n(t)) \quad \text{for all } t \in [0,1],$$

is continuous as the uniform limit of the maps γ_n by (4.27). We denote $\gamma([0,1])$ by Γ, and for each n, we denote $\gamma_n([0,1])$ by Γ_n. For all integers $n, k \geq 0$ with $0 \leq k \leq 4^n$, we call the point $\gamma_n(k/4^n)$ a *vertex* of Γ_n. Remark 4.5 holds in this setting, as well.

Remark 4.13 (modulus of continuity and injectivity). In fact, (4.25) and (4.27) imply that the maps γ_n and γ are uniformly $(1/2)$-Hölder continuous. The parameterization γ is injective and this can be verified by showing that $\pi_1 \circ \gamma$ is injective, where $\pi_1 : \ell_p \to \mathbb{R}$ is projection onto the first coordinate. We leave details for the reader.

Remark 4.14 (improved bounds on r_n). Let Γ in ℓ_p $(1 < p < \infty)$ be constructed by Algorithm 4.12 with relative heights $0 \leq \eta_i \leq \overline{\eta}$ for all i, for some universal constant $\overline{\eta}$ to be specified below. Let $s_i \in [1/4, 1/2)$ be defined by $s_i^p = (\frac{1}{2} - s_i)^p + \eta_i^p$. Then

$$s_i \leq ((1/4)^p + \eta_i^p)^{1/p} = \frac{1}{4}\left(1 + (4\eta_i)^p\right)^{1/p}.$$

Since $p > 1$, Taylor's theorem with remainder gives

$$1 + \frac{1}{p}\delta - \frac{p-1}{2p^2}\delta^2 \leq (1+\delta)^{1/p} \leq 1 + \frac{1}{p}\delta \quad \text{for all } 0 \leq \delta < 1.$$

On the one hand, assume that $\overline{\eta} < 1/4$. Then $(4\eta_i)^p < 4\overline{\eta} < 1$ and we obtain

$$s_i \leq \frac{1}{4}\left(1 + \frac{1}{p}(4\eta_i)^p\right) = \frac{1}{4} + \frac{1}{4p}(4\eta_i)^p.$$

On the other hand, assume that $\overline{\eta} < 1/8$. Then $s_i \leq 1/4 + (1/4p)2^{-p} < 3/8$. Hence $1/8 < \frac{1}{2} - s_i \leq 1/4$ and using the Taylor bound we can write

$$s_i = \left(\frac{1}{2} - s_i\right)\left(1 + \left(\frac{1}{2} - s_i\right)^{-p}\eta_i^p\right)^{1/p}$$

$$\geq \frac{1}{2} - s_i + \frac{1}{p}\left(\frac{1}{2} - s_i\right)^{1-p}\eta_i^p - \frac{p-1}{2p^2}\left(\frac{1}{2} - s_i\right)^{1-2p}\eta_i^{2p}$$

$$\geq \frac{1}{2} - s_i + \frac{1}{4p}(4\eta_i)^p - \frac{p-1}{16p^2}(8\eta_i)^{2p}.$$

Rearranging the inequality, we obtain

$$s_i \geq \frac{1}{4} + \frac{1}{8p}(4\eta_i)^p - \frac{p-1}{32p^2}(8\eta_i)^{2p} = \frac{1}{4} + \left[\frac{1}{8p} - \frac{p-1}{32p^2}(16\eta_i)^p\right](4\eta_i)^p.$$

We now specify that $\bar{\eta} = 1/16$ so that

$$\frac{1}{8p} - \frac{p-1}{32p^2}(16\eta_i)^p \geq \frac{1}{8p} - \frac{p-1}{32p^2} \geq \frac{3}{32p}.$$

Therefore,

(4.29) $$\frac{1}{4} + \frac{3}{32p}(4\eta_i)^p \leq s_i \leq \frac{1}{4} + \frac{1}{4p}(4\eta_i)^p \quad \text{whenever } \eta_i \leq \frac{1}{16}.$$

Thus, if $\eta_i \leq 1/16$ for all i, then the length $r_n = \prod_{i=1}^{n} s_i$ of each segment in Γ_n satisfies

(4.30) $$4^{-n}\prod_{i=1}^{n}\left(1 + \frac{3}{8p}(4\eta_i)^p\right) \leq r_n \leq 4^{-n}\prod_{i=1}^{n}\left(1 + \frac{1}{p}(4\eta_i)^p\right).$$

The reader may verify that when $p = 2$, the bound (4.30) is compatible with (4.1).

Lemma 4.15. *If Γ is constructed by Algorithm 4.12 with relative heights $\eta_i \leq 1/16$, then*

(4.31) $$\exp\left(\frac{1}{4p}\sum_{n=1}^{\infty}(4\eta_n)^p\right) \leq \mathcal{H}^1(\Gamma) \leq \exp\left(\frac{1}{p}\sum_{n=1}^{\infty}(4\eta_n)^p\right).$$

In particular, Γ is a rectifiable curve if and only if $\sum_{n=1}^{\infty}\eta_n^p < \infty$. Moreover, in that case, Γ is Ahlfors regular with constants depending only on $\mathcal{H}^1(\Gamma)$.

Proof. The outline is similar to the proof of Lemma 4.7. For each $n \geq 1$,

$$\mathcal{H}^1(\Gamma) \overset{\text{Rk 4.13}}{=} \sum_{k=1}^{4^n} \mathcal{H}^1(\gamma([(k-1)4^{-n}, k4^{-n}])) \geq \sum_{k=1}^{4^n} \text{diam }\gamma([(k-1)4^{-n}, k4^{-n}])$$

$$\overset{(4.30)}{\geq} \sum_{k=1}^{4^n} 4^{-n}\prod_{i=1}^{n}\left(1 + \frac{3}{8}p^{-1}(4\eta_i)^p\right) = \prod_{i=1}^{n}\left(1 + \frac{3}{8}p^{-1}(4\eta_i)^p\right).$$

We conclude that

$$\prod_{n=1}^{\infty}\left(1 + \frac{3}{8p}(4\eta_n)^p\right) \leq \mathcal{H}^1(\Gamma) \leq \text{Lip }\gamma \leq \liminf_{m\to\infty}\text{Lip }\gamma_m \overset{(4.30)}{\leq} \prod_{n=1}^{\infty}\left(1 + \frac{1}{p}(4\eta_n)^p\right).$$

To derive (4.31), rewrite each infinite product as the exponential of an infinite sum and use Taylor's theorem bounds for $\log(1+x)$ with $0 \leq x \leq 1/4$.

Suppose that $\mathcal{H}^1(\Gamma) < \infty$. By (4.26), (4.27), and assumption $\eta_n \leq 1/16$ for all $n \geq 1$,

$$\|\gamma_n - \gamma\|_\infty \leq \sum_{j=n}^{\infty}\eta_{j+1}r_j \leq \frac{1}{16}\sum_{k=0}^{\infty}r_{n+k} \leq \frac{1}{16}r_n\sum_{k=0}^{\infty}2^{-k} = \frac{1}{8}r_n.$$

Thus, given $x \in \Gamma$ and $n \geq 1$, we may pick $y \in \Gamma_n$ such that $|x - y| \leq (1/8)r_n$. Next, let $v \in V_n$ be an endpoint of the segment in Γ_n containing y that is closest to y so that $|v - y| \leq (1/2)r_n$. Hence $B(x, (3/8)r_n) \subset B(v, r_n)$ and we may use (4.8) to estimate

$$(4.32) \quad \mathcal{H}^1(\Gamma \cap B(x, \tfrac{3}{8}r_n)) \leq \mathcal{H}^1(\Gamma \cap B(v, r_n)) \leq 2 \exp\left(\sum_{i=n}^{\infty} \frac{1}{p}(4\eta_i)^p\right) r_n \leq 2e^4 \mathcal{H}^1(\Gamma)r_n,$$

because $\Gamma_n \cap B(v, r_n)$ consists of one or two line segments of length r_n. It follows that Γ is Ahlfors regular with constants depending only on $\mathcal{H}^1(\Gamma)$. $\qquad\square$

Lemma 4.16. *Assume that Γ is constructed by Algorithm 4.12 with $\eta_i < 1/8$ for all i. For all $i \geq 0$ and $j \geq i$,*

$$(4.33) \qquad\qquad \frac{1}{4}\eta_i \leq \beta_{\Gamma_j}(B(v, r_j)) \leq 2\eta_i \quad \text{for all } v \in \widetilde{V}_i,$$

where $\widetilde{V}_i$ denote the "new vertices" in Γ_i (see Remark 4.5) and $\eta_0 = 0$. Furthermore,

$$(4.34) \qquad\qquad \frac{1}{4}\eta_i \leq \beta_\Gamma(\overline{B(v, r_i)}) \quad \text{for all } i \geq 0 \text{ and } v \in \widetilde{V}_i.$$

Proof. The upper bound agrees with the case $p = 2$ above. Note that when $v \in \widetilde{V}_0$ is an endpoint of Γ_j, the set $\Gamma_j \cap B(v, r_j)$ is a line segment. Thus, $\beta_{\Gamma_j}(B(v, r_j)) = 0$ for all $v \in \widetilde{V}_0$ and $j \geq 0$. Next, suppose that $i \geq 1$ and $v \in \widetilde{V}_i$. Using the line containing the segment in $\Gamma_{i-1} \cap B(v, r_i)$ to approximate the beta number, we obtain the estimate

$$\beta_{\Gamma_i}(B(v, r_i)) \leq \frac{\eta_i r_{i-1}}{2r_i} = \frac{\eta_i}{2s_i} \leq 2\eta_i,$$

where $\sup_{x \in \Gamma_i} \text{dist}(x, \ell) \leq \eta_i r_{i-1}$, because at stage i in the construction we added the bump in the direction e_{i+1} and we recall that $s_i \geq \frac{1}{4}$. When $j \geq i$, the set $\Gamma_j \cap B(v, r_j)$ agrees up to a dilation centered at v with $\Gamma_i \cap B(v, r_i)$. Thus, $\beta_{\Gamma_j}(B(v, r_j)) \leq 2\eta_i$ for all $v \in \widetilde{V}_i$ and $j \geq i$, as well.

Before we determine the lower bounds in (4.33) and (4.34), we recall two basic properties of the beta numbers. First, for any set $E \subset B(v, r)$, we have $\beta_E(B(v, r)) = \beta_{\overline{E}}(\overline{B(v, r)})$. (This may fail if $E \not\subset B(v, r)$!) Second, $\beta_E(B(v, r))$ is increasing in E. Because $V_i \subset \Gamma_j$ for all $j \geq i$ and $V_i \subset \Gamma$, for any $v_k \in V_i$ the vertices v_{k-1} and v_{k+1} that are adjacent with respect to the global parametrization satisfy

$$\{v_{k-1}, v_k, v_{k+1}\} \subset \overline{\Gamma_j \cap B(v_k, r_i)} \quad \text{for any } j \geq i$$

and the inclusion also holds with Γ in place of Γ_j. Therefore, up to a translation and dilation, there are two relevant configurations of vertices:

$$E_v(\eta) := \{0\} \cup \{sv\} \cup \{\tfrac{1}{2}v + \eta e_{n+1}\}, \quad F_v(\eta) := \{sv\} \cup \{\tfrac{1}{2}v + \eta e_{n+1}\} \cup \{(1-s)v\},$$

where $v \in \text{span}\{e_1, \ldots, e_n\}$ is an arbitrary vector with $|v|_p = 1$ and $s^p = (\tfrac{1}{2} - s)^p + \eta^p$ with $\eta < 1/8$. By Remark 4.14, $1/4 \leq s < 3/8$. The optimal lower bound in (4.33) and (4.34) is given by $\beta(\eta_i)$, where

$$\beta(\eta) := \min\left\{\beta_{E_v(\eta)}(\overline{B(sv, s)}), \beta_{F_v(\eta)}(\overline{B(\tfrac{1}{2}v + \eta e_{n+1}, s)})\right\}.$$

We work separately for each configuration, beginning with $E_v(\eta)$. Let L_1 be the line containing $[0, \frac{1}{2}v + \eta e_{n+1}]$. In the ve_{n+1}-plane, the line L_1 is the locus of points (x, y) satisfying $y = 2\eta x$. When $x \geq \frac{1}{4} - \frac{1}{2}\eta$,

$$y \geq 2\eta \left(\frac{1}{4} - \frac{1}{2}\eta \right) \geq \frac{1}{2}\eta - \frac{1}{16}\eta = \frac{7}{16}\eta.$$

Since $s \geq \frac{1}{4}$, it follows that $B(sv, (7/16)\eta) \cap L_1 = \emptyset$. Hence

(4.35) $\mathrm{dist}(sv, L_1) \geq (7/16)\eta.$

We now argue that $\beta_{E_v(\eta)}(\overline{B(sv, s)}) \geq (3/32s)\eta$ by way of contradiction. Assume that we can find a line L such that $E_v(\eta) \subset B_{(3/16)\eta}(L)$. By convexity, $L_1 \cap B(sv, s) \subset B_{(3/16)\eta}(L)$, as well. Let $x_1 \in L \cap B(0, (3/16)\eta)$ and $x_2 \in L \cap B(\frac{1}{2}v + \eta e_{n+1}, (3/16)\eta)$. Once again, by convexity, the segment of L that falls between x_1 and x_2 is contained in $B_{(3/16)\eta}(L_1)$. Since $\mathrm{dist}(x, sv) \geq \frac{1}{2}s \geq (3/16)\eta$ for all $x \in L \setminus B_{(3/16)\eta}(L_1)$, the points $x \in L$ such that $\mathrm{dist}(x, sv) \leq (3/16)\eta$ are contained in $B_{(3/16)\eta}(L_1)$. Thus, by the triangle inequality,

(4.36) $E_v(\eta) \subset B_{(6/16)\eta}(L_1).$

Since (4.35) and (4.36) are incompatible, we have reached a contradiction. Therefore,

$$\beta_{E_v(\eta)}(\overline{B(sv, s)}) \geq \frac{3}{32s}\eta \geq \frac{1}{4}\eta.$$

Using symmetry, one sees the best fitting line for $F_v(\eta)$ is $\frac{1}{2}\eta e_{n+1} + \mathrm{span}(v)$, whence

$$\beta_{F_v(\eta)}(\overline{B(\tfrac{1}{2}v + \eta e_{n+1}, s)}) \geq \frac{\eta}{4s} \geq \frac{2}{3}\eta.$$

Thus, the extremal configuration is given by $E_v(\eta)$ and $\beta(\eta) \geq (1/4)\eta$. $\square$

4.2.1. *Proof of Proposition 4.10.* Let $1 < p < \infty$. Let Γ be the curve constructed by Algorithm 4.12 with relative heights $\eta_i^p = \delta/(i + i_0) \log(i + i_0)$ for all $i \geq 1$, where $\delta > 0$ and i_0 are chosen so that $\eta_1 \leq 1/16$. Then $\mathcal{H}^1(\Gamma) = \infty$ in ℓ_p by Lemma 4.15. To prove that $S_{\Gamma, p+\varepsilon}(\mathscr{G}) < \infty$ for every multiresolution family $\mathscr{G}$ for Γ and $\varepsilon > 0$, repeat the proof of Proposition 4.1, *mutatis mutandis*, using Lemma 4.16 *in lieu* of Lemma 4.8.

4.2.2. *Proof of Proposition 4.11.* Let $1 < p < \infty$. Let Γ be the curve constructed by Algorithm 4.12 with relative heights $\eta_i^p = \delta/(i + i_0) \log(i + i_0)^2$ for all $i \geq 1$, where $\delta > 0$ and i_0 are chosen so that $\eta_1 \leq 1/16$. Then $\mathcal{H}^1(\Gamma) < \infty$ in ℓ_p and Γ is Ahlfors regular by Lemma 4.15. To prove that $S_{\Gamma, p-\varepsilon}(\mathscr{G}) = \infty$ for every multiresolution family $\mathscr{G}$ for Γ and $\varepsilon > 0$, repeat the proof of Proposition 4.2, *mutatis mutandis*, again substituting Lemma 4.16 for Lemma 4.8.

4.3. **Proof of Proposition 1.1.** Let $1 < p < q < \infty$. To begin, we verify that if $\Gamma \subset \ell_p$ is a rectifiable curve, then Γ is also rectifiable when viewed as a curve embedded in ℓ_q. As is well known, $\ell_p \subset \ell_q$ and $|v|_q \leq |v|_p$ for every $v \in \ell_p$. Hence the diameter of a set in ℓ_p does not increase when embedded into ℓ_q. Thus, $\mathcal{H}^s_{\ell_q}(E) \leq \mathcal{H}^s_{\ell_p}(E)$ for every $s > 0$ and $E \subset \ell_p$, where $\mathcal{H}^s_{\ell_r}$ denotes the s-dimensional Hausdorff measure in ℓ_r. In particular, every rectifiable curve in ℓ_p is also a rectifiable curve in ℓ_q, possibly with shorter length.

We now construct a curve Γ such that $\mathcal{H}^1_{\ell_p}(\Gamma) = \infty$ and $\mathcal{H}^1_{\ell_q}(\Gamma) < \infty$ for every $q > p$. Build Γ in ℓ_p using Algorithm 4.12 with relative heights

$$\eta_i^p = \frac{\delta}{i \log(i + i_0)} \quad \text{for all } i \geq 1,$$

where $\delta > 0$ and $i_0 \in \mathbb{N}$ are chosen so that $0 < \eta_1 \leq 1/16$. Note that $\sum_{i=1}^\infty \eta_i^p = \infty$. Therefore, $\mathcal{H}^1_{\ell_p}(\Gamma) = \infty$ by Lemma 4.15.

Fix an exponent q with $p < q < \infty$. We break the proof that $\mathcal{H}^1_{\ell_q}(\Gamma) < \infty$ into a series of lemmata. First, we calculate the $\mathcal{H}^1_{\ell_q}$-growth of a line segment under the snowflaking procedure $\gamma \mapsto \hat{\gamma}$ used in Algorithm 4.12. We emphasize that the estimate in the following lemma is independent of n.

Lemma 4.17. *Given $v \in \mathrm{span}\{e_1, ..., e_n\}$ with $|v|_p = 1$, let $\gamma : [0, 1] \to \mathrm{span}\{e_1, ..., e_n\}$ be the unit speed parameterization of the line segment $I = [0, v]$ in ℓ_p. Let $\hat{\gamma}$ be obtained from γ by adding a bump of relative height η in the direction e_{n+1} (see Algorithm 4.12). If $\eta \leq 1/16$, then $\mathcal{H}^1_{\ell_q}(I) = |v|_q$ and*

$$(4.37) \qquad \mathcal{H}^1_{\ell_q}(\hat{\gamma}([0,1])) - \mathcal{H}^1_{\ell_q}(I) \lesssim_q \left(\frac{\eta}{|v|_q}\right)^q |v|_q.$$

Proof. The estimate is nearly identical to the calculation in Remark 4.14. Since $\eta \leq 1/16$, each of the four edges of $\hat{\gamma}(I)$ in ℓ_p have length $s \in [1/4, 3/8)$ given implicitly by the equation $s^p = (\frac{1}{2} - s)^p + \eta^p$. In ℓ_q, $\mathcal{H}^1_{\ell_q}(I) = |v|_q$ (since I is a segment) and the four edges of $\hat{\gamma}(I)$ have total length

$$\mathcal{H}^1_{\ell_q}(\hat{\gamma}([0,1])) = 2s|v|_q + 2\left((\tfrac{1}{2} - s)^q|v|_q^q + \eta^q\right)^{1/q}$$

$$= 2s|v|_q + (1 - 2s)|v|_q\left(1 + (\tfrac{1}{2} - s)^{-q}|v|_q^{-q}\eta^q\right)^{1/q}$$

$$\leq |v|_q\left(1 + (\tfrac{1}{2} - s)^{-q}|v|_q^{-q}\eta^q\right)^{1/q}$$

Applying $(1 + \delta)^{1/q} \leq 1 + \frac{1}{q}\delta$ for all $\delta \geq 0$ with $\delta = (\frac{1}{2} - s)^{-q}|v|_q^{-q}\eta^q$, we conclude

$$\mathcal{H}^1_{\ell_q}(\hat{\gamma}([0,1])) \leq |v|_q + \frac{1}{q}\left(\frac{\eta}{(\frac{1}{2} - s)|v|_q}\right)^q |v|_q \leq |v|_q + \frac{1}{q}\left(\frac{8\eta}{|v|_q}\right)^q |v|_q. \qquad \square$$

The next estimate is elementary and left as an exercise for the reader.

Lemma 4.18. *For all $n \in \mathbb{N}$ and $v \in \ell_p^n$, $|v|_q \geq n^{\frac{1}{q} - \frac{1}{p}}|v|_p$.*

We may now give a uniform estimate on the length of the intermediate curves Γ_n approximating Γ in ℓ_q.

Lemma 4.19. *With the exponents and the parameter $\eta_i^p = \delta/(i \log(i + i_0))$ fixed as above, the intermediate curves produced by Algorithm 4.12 satisfy*

$$\mathcal{H}^1_{\ell_q}(\Gamma_n) \leq C(p, q) < \infty \quad \text{for all } n \geq 1,$$

where $C(p, q)$ is a constant depending on p and q with $C(p, q) \uparrow \infty$ as $q \downarrow p$.

Proof. The initial segment $\Gamma_0 = [0, e_1]$ satisfies $\mathcal{H}^1_{\ell_q}(\Gamma_0) = 1$. Let $n \geq 0$, let $1 \leq k \leq 4^n$, and let $J = \gamma_n([(k-1)4^{-n}, k4^{-n}])$ be an edge in Γ_n. Assign $J^+ := \gamma_{n+1}([(k-1)4^{-n}, k4^{-n}])$. By Lemma 4.17 and Lemma 4.18,

$$\mathcal{H}^1_{\ell_q}(J^+) - \mathcal{H}^1_{\ell_q}(J) \lesssim_q \left(\frac{\eta_{n+1} \mathcal{H}^1_{\ell_p}(J)}{\mathcal{H}^1_{\ell_q}(J)} \right)^q \mathcal{H}^1_{\ell_q}(J) \lesssim_q \left(\frac{\eta_{n+1}}{(n+1)^{\frac{1}{q} - \frac{1}{p}}} \right)^q \mathcal{H}^1_{\ell_q}(J).$$

Summing over the 4^n edges J in Γ_n, it follows that

$$\mathcal{H}^1_{\ell_q}(\Gamma_{n+1}) \leq \left(1 + C_q \left(\frac{\eta_{n+1}}{(n+1)^{\frac{1}{q} - \frac{1}{p}}} \right)^q \right) \mathcal{H}^1_{\ell_q}(\Gamma_n),$$

where C_q is a positive constant depending only on q. Therefore,

$$\mathcal{H}^1_{\ell_q}(\Gamma_n) \leq \prod_{i=1}^n \left(1 + C_q \left(\frac{\eta_i}{i^{\frac{1}{q} - \frac{1}{p}}} \right)^q \right) \leq \exp \left(C_q \sum_{i=1}^n \left(\frac{\eta_i}{i^{\frac{1}{q} - \frac{1}{p}}} \right)^q \right).$$

Finally, recalling our choice of η_i,

$$\sum_{i=1}^n \left(\frac{\eta_i}{i^{\frac{1}{q} - \frac{1}{p}}} \right)^q \leq \sum_{i=1}^\infty \left(\frac{\delta^{1/p} i^{-1/p} \log(i + i_0)^{-1/p}}{i^{\frac{1}{q} - \frac{1}{p}}} \right)^q = \sum_{i=1}^\infty \frac{\delta^{q/p}}{i \log(i + i_0)^{q/p}} < \infty,$$

because $q/p > 1$. All together, $\mathcal{H}^1_{\ell_q}(\Gamma_n) \leq C(p, q) < \infty$, where $C(p, q) \uparrow \infty$ when $q \downarrow p$. $\quad\square$

By (4.27), γ_n converges uniformly to γ in ℓ_p, and thus, in ℓ_q. Hence Γ_n converges to Γ in the Hausdorff metric on compact subsets of ℓ_q. Therefore, by Gołąb's semicontinuity theorem (see e.g. [AO17, Theorem 2.9]), $\mathcal{H}^1_{\ell_q}(\Gamma) \leq \liminf_{n \to \infty} \mathcal{H}^1_{\ell_q}(\Gamma_n) \leq C(p, q) < \infty$ for all $q > p$. This completes the proof of Proposition 1.1.

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