Due In Class: Thursday, October 1

Reading: Finish reading Chapter 2.

Do the following problems.

Problem A: Let X be a metric space. Let $\mathcal{P} = \mathcal{P}(X)$ denote the collection of subsets of X; let $\mathcal{S} = \mathcal{S}(X)$ denote the collection of nonempty subsets of $\mathbb{R}^2$. Define a function $\mathrm{ex} : \mathcal{P} \times \mathcal{S} \to [0, \infty]$ by the rule

$$\mathrm{ex}(A, B) = \begin{cases} \sup_{a \in A} \left(\inf_{b \in B} \mathrm{dist}(a, b) \right) & \text{if } A \in \mathcal{S} \\ 0 & \text{if } A = \emptyset. \end{cases}$$

(Here we formally write $\sup E = \infty$ for a set $E \subseteq \mathbb{R}$ if E is not bounded above. By convention $x + \infty = \infty + \infty = \infty$ for all $x \in \mathbb{R}$.) The quantity $\mathrm{ex}(A, B)$ is called the *excess of A over B*.

Prove that excess satisfies the triangle inequality: $\mathrm{ex}(A, C) \le \mathrm{ex}(A, B) + \mathrm{ex}(B, C)$ for all sets $A, B, C \in \mathcal{P}$ such that $\mathrm{ex}(A, B)$, $\mathrm{ex}(A, C)$ and $\mathrm{ex}(B, C)$ are all defined.

Problem B: Let X be a metric space. Let $\mathcal{CB} = \mathcal{CB}(X)$ denote the collection of closed, bounded subsets of X. Define $\mathrm{HD} : \mathcal{CB} \times \mathcal{CB} \to [0, \infty)$ by the rule

$$\mathrm{HD}(A, B) = \max \{\mathrm{ex}(A, B), \mathrm{ex}(B, A)\} \quad \text{for all } A, B \in \mathcal{CB}.$$

The quantity $\mathrm{HD}(A, B)$ is called the *Hausdorff distance between A and B*.

(a) Prove that $(\mathcal{CB}, \mathrm{HD})$ is a metric space.

(b) Describe in words and/or pictures the open ball $B([0, 1] \times \{0\}, \frac{1}{4})$ in this metric space $\mathcal{CB}(\mathbb{R}^2)$.

Problem C: Exercise 2.12.

Problem D: Exercise 2.13. *You must prove your assertions.*

Problem E: Exercise 2.15.

Problem F: Exercise 2.22.