Stochastic Analysis of the Horizontal Brownian Motion of a Foliation

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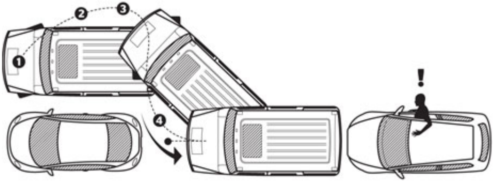
Sub-Riemannian manifolds

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $n+m$. We assume that $\mathbb{M}$ is equipped with a sub-bundle $\mathcal{H} \subset \mathbf{T}\mathbb{M}$ of dimension n and a fiberwise inner product $g_{\mathcal{H}}$ on that distribution.

- The distribution $\mathcal{H}$ is referred to as the set of *horizontal directions*.
- Sub-Riemannian geometry is the study of the geometry which is intrinsically associated to $(\mathcal{H}, g_{\mathcal{H}})$.

The geometry of parallel parking



How is it even possible to parallel park ?

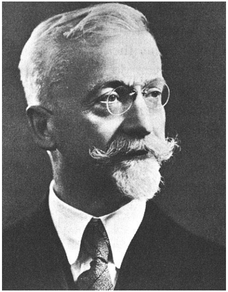
The geometry of parallel parking

We only have two degree of freedom. Either we steer the wheel, either we push the gas pedal in reverse or in displacement. This is an example of sub-Riemannian structure. The underlying group is the group of displacement of the plane which is generated by the rotations and the translations. There are two generators

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

X and Y Lie generate the roto-translation group and thus any two points can be joined by a horizontal curve.

Sub-Riemannian geometry



Sub-Riemannian geometry was internationally brought to the attention of mathematicians by E. Cartan at the Bologna International Congress of Mathematicians in 1928.

Riemannian foliations

In general, there is no canonical vertical complement of $\mathcal{H}$ in the tangent bundle $T\mathbb{M}$, but in many interesting cases $\mathcal{H}$ can be seen as the horizontal distribution of a Riemannian foliation $\mathcal{F}$.
In this talk, we will assume that the foliation $\mathcal{F}$ is totally geodesic with a bundle like metric g.

Examples:

- The Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ induces a sub-Riemannian structure on $\mathbb{S}^{2n+1}$ which comes from a totally geodesic foliation.
- The quaternionic Hopf fibration $\mathbf{SU}(2) \to \mathbb{S}^{4n+3} \to \mathbb{HP}^n$ induces a sub-Riemannian structure on $\mathbb{S}^{4n+3}$ which comes from a totally geodesic foliation.

More generally, totally geodesic Riemannian submersions, Sasakian and 3-Sasakian manifolds provide examples of sub-Riemannian structures associated with totally geodesic foliations.

Canonical variation of the metric

The metric g can be split as

$$g = g_{\mathcal{H}} \oplus g_{\mathcal{V}},$$

The one-parameter family of Riemannian metrics:

$$g_{\varepsilon} = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \varepsilon > 0,$$

is called the canonical variation of g. The sub-Riemannian limit is $\varepsilon \to 0$.

The Bott connection

There is a canonical connection on $\mathbb{M}$, the Bott connection, which is given as follows:

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{H}) \\ \pi_{\mathcal{H}}([X, Y]), X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}) \\ \pi_{\mathcal{V}}([X, Y]), X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}) \\ \pi_{\mathcal{V}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{V}) \end{cases}$$

where ∇^R is the Levi-Civita connection and $\pi_{\mathcal{H}}$ (resp. $\pi_{\mathcal{V}}$) the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). It is easy to check that for every $\varepsilon > 0$, this connection satisfies $\nabla g_\varepsilon = 0$.

The horizontal Laplacian

The horizontal Laplacian is the generator of the symmetric Dirichlet form

$$\mathcal{E}_{\mathcal{H}}(f,g) = -\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle_{\mathcal{H}} d\mu.$$

It is a diffusion operator L on $\mathbb{M}$ which is symmetric on $C_0^\infty(\mathbb{M})$ with respect to the volume measure μ. If $\mathcal{H}$ is bracket generating, then L is subelliptic.

For $Z \in \mathcal{V}$, we consider the unique skew-symmetric map J_Z defined on the horizontal bundle $\mathcal{H}$ such that for every horizontal vector fields X and Y,

$$g_{\mathcal{H}}(J_Z(X), Y) = g_{\mathcal{V}}(Z, T(X,Y)).$$

The transverse Bochner-Weitzenböck formulas

Theorem (B., Kim, Wang 2014)

Let

$$\square_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) + \frac{1}{\varepsilon} J^* J - \mathfrak{Ric}_{\mathcal{H}}.$$

Then, for every smooth function f on $\mathbb{M}$,

$$dLf = \square_\varepsilon df,$$

and for any smooth one-form η,

$$\frac{1}{2} L \|\eta\|_\varepsilon^2 - \langle \square_\varepsilon \eta, \eta \rangle_\varepsilon = \|\nabla_{\mathcal{H}} \eta - \mathfrak{T}^\varepsilon_{\mathcal{H}} \eta\|_\varepsilon^2 + \left\langle \mathfrak{Ric}_{\mathcal{H}}(\eta) - \frac{1}{\varepsilon} J^* J(\eta), \eta \right\rangle_{\mathcal{H}}.$$

The horizontal Brownian motion, Gradient formula

The horizontal Brownian motion $(X_t)_{t\geq 0}$ is the Markov diffusion process with generator $\frac{1}{2}L$. Consider the process $\tau_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_{X_0}\mathbb{M}$ which is the solution of the following covariant Stratonovitch stochastic differential equation:

$$\begin{aligned} & d[\tau_t^\varepsilon \alpha(X_t)] \\ =& \tau_t^\varepsilon \left(\nabla_{\circ dX_t} - \mathfrak{T}^\varepsilon_{\circ dX_t} - \frac{1}{2}\left(\frac{1}{\varepsilon}\mathbf{J}^2 + \mathfrak{Ric}_{\mathcal{H}} \right) dt \right) \alpha(X_t), \end{aligned}$$

where α is any smooth one-form. τ_t^ε is the damped parallel transport for the metric connection $\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}$.

The horizontal Brownian motion, Gradient formula

Theorem (Baudoin, 2015)

Let $\varepsilon > 0$. Let $f \in C_0^\infty(\mathbb{M})$. Then for every $t \geq 0$, and $x \in \mathbb{M}$,

$$dP_t f(x) = \mathbb{E}_x(\tau_t^\varepsilon df(X_t)).$$

Gradient bound for the semigroup

Form now on, we assume

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} \geq -K \|\eta\|_{\mathcal{H}}^2, \quad \langle J^* J\eta, \eta \rangle_{\mathcal{H}} \leq \kappa \|\eta\|_{\mathcal{H}}^2,$$

with $K, \kappa \geq 0$.

Lemma

For every $t \geq 0$, we have almost surely,

$$\|\tau_t^\varepsilon \alpha(X_t)\|_\varepsilon \leq e^{\frac{1}{2}\left(K + \frac{\kappa}{\varepsilon}\right)t} \|\alpha(X_t)\|_\varepsilon.$$

Corollary

For every $f \in C_0^\infty(\mathbb{M})$, $\varepsilon > 0$, $t \geq 0$,

$$\|dP_t f\|_\varepsilon \leq e^{\frac{1}{2}\left(K + \frac{\kappa}{\varepsilon}\right)t} P_t \|df\|_\varepsilon.$$

Horizontal Malliavin calculus

The anti-development of $(X_t)_{t\geq 0}$,

$$B_t = \int_0^t \|_{0,s}^{-1} \circ dX_s,$$

is a Brownian motion in the horizontal space $\mathcal{H}_{x}.$. For $F = f(X_{t_1}, \cdots, X_{t_n})$, $f \in C_0^\infty(\mathbb{M}^n)$, we define the **Damped gradient:**

$$\tilde{D}_t^\varepsilon F = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)(\tau_t^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(X_{t_1}, \cdots, X_{t_n}), \quad 0 \leq t \leq T.$$

Integration by parts formula

Theorem (B.-Feng, 2015)

Let $\varepsilon > 0$. Let $F = f(X_{t_1}, \cdots, X_{t_n})$, $f \in C_0^\infty(\mathbb{M}^n)$. For any C^1 and adapted process $\gamma : [0, T] \to \mathcal{H}_x$ such that $\mathbb{E}_x(\int_0^T \|\gamma'(s)\|_{\mathcal{H}}^2 ds) < \infty$,

$$\mathbb{E}_x \left(F \int_0^T \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \right) = \mathbb{E}_x \left(\int_0^T \langle \tilde{D}_s^\varepsilon F, \|_{0,s} \gamma'(s) \rangle ds \right).$$

Clark-Ocone formula

Theorem (B.-Feng, 2015)

Let $\varepsilon > 0$. Let $F = f(X_{t_1}, \cdots, X_{t_n})$, $f \in C_0^\infty(\mathbb{M}^n)$. Then

$$F = \mathbb{E}_x(F) + \int_0^T \langle \mathbb{E}_x(\tilde{D}_s^\epsilon F | \mathcal{F}_s), //_{0,s} dB_s \rangle.$$

Concentration inequality

Theorem

Let $\varepsilon > 0$. We have for every $T > 0$ and $r \geq 0$

$$\mathbb{P}_x\left(\sup_{0\leq t\leq T} d_\epsilon(X_t, x) \geq \mathbb{E}_x\left[\sup_{0\leq t\leq T} d_\epsilon(X_t, x)\right] + r\right)$$
$$\leq \exp\left(-\frac{r^2}{2Te^{\left(K+\frac{\kappa}{\epsilon}\right)T}}\right).$$

Curvature dimension inequality

Using the Bochner-Weitzenböck formulas, we can also quickly recover the generalized curvature dimension inequality first discovered by B.-Garofalo (2009) in a less general framework by using Γ-calculus If $f \in C^\infty(\mathbb{M})$, we denote

$$\Gamma_2(f) = \frac{1}{2} L \|\nabla_{\mathcal{H}} f\|^2 - \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} Lf \rangle_{\mathcal{H}}$$

and

$$\Gamma_2^Z(f) = \frac{1}{2} L \|\nabla_{\mathcal{V}} f\|^2 - \langle \nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} Lf \rangle_{\mathcal{V}}.$$

Curvature dimension inequality

Theorem (B., Kim, Wang 2014)

Assume that for every smooth horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} \geq \rho_1 \|\eta\|^2_{\mathcal{H}}, \quad \langle J^* J(\eta), \eta \rangle_{\mathcal{H}} \leq \kappa \|\eta\|^2_{\mathcal{H}},$$

and that for every $Z \in \mathcal{V}$,

$$\mathbf{Tr}(J_Z^* J_Z) \geq \rho_2 \|Z\|^2_{\mathcal{V}},$$

with $\rho_1 \in \mathbb{R}, \rho_2 > 0$ and $\kappa \geq 0$. Then for every $\nu > 0$,

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \|\nabla_{\mathcal{H}} f\|^2 + \frac{\rho_2}{4} \|\nabla_{\mathcal{V}} f\|^2$$

Bonnet-Myers theorem

As proved in B.-Garofalo, a notable consequence of the generalized curvature dimension inequality is the Bonnet-Myers result.

Theorem

Assume that for every smooth horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} \ge \rho_1 \|\eta\|^2_{\mathcal{H}}, \quad \langle J^* J(\eta), \eta \rangle_{\mathcal{H}} \le \kappa \|\eta\|^2_{\mathcal{H}},$$

and that for every $Z \in \mathcal{V}$,

$$\frac{1}{4}\mathbf{Tr}(J_Z^* J_Z) \ge \rho_2 \|Z\|^2_{\mathcal{V}},$$

with $\rho_1, \rho_2 > 0$ and $\kappa \ge 0$. Then the manifold $\mathbb{M}$ is compact and the following diameter bound for the CC distance holds:

$$\mathbf{diam}(\mathbb{M}) \le 2\sqrt{3}\pi \sqrt{\frac{\kappa + \rho_2}{\rho_1 \rho_2}\left(1 + \frac{3\kappa}{2\rho_2}\right) n}.$$

Bonnet-Myers theorem

To put things in perspective, we point out that

$$\mathbf{Ricci}_{\varepsilon}(Z,Z) = \mathbf{Ricci}_{\mathcal{V}}(Z,Z) + \frac{1}{4\varepsilon^2}\mathbf{Tr}(J_Z^* J_Z)$$

$$\mathbf{Ricci}_{\varepsilon}(X,Z) = 0$$

$$\mathbf{Ricci}_{\varepsilon}(X,X) = \mathbf{Ricci}_{\mathcal{H}}(X,X) - \frac{1}{2\varepsilon}\|JX\|^2$$

Volume doubling property, Poincaré inequality on balls

Theorem

Assume that for every smooth horizontal one-form η,

$$\langle \mathfrak{Ric}_{\mathcal{H}}(\eta), \eta \rangle_{\mathcal{H}} \geq \rho_1 \|\eta\|_{\mathcal{H}}^2, \quad \langle J^* J(\eta), \eta \rangle_{\mathcal{H}} \leq \kappa \|\eta\|_{\mathcal{H}}^2,$$

and that for every $Z \in \mathcal{V}$,

$$\mathbf{Tr}(J_Z^* J_Z) \geq \rho_2 \|Z\|_{\mathcal{V}}^2,$$

with $\rho_1 \geq 0, \rho_2 > 0$ and $\kappa \geq 0$. Then, there exist constants $C_d, C_p > 0$, for which one has for every $x \in \mathbb{M}$ and every $r > 0$:

$$\mu(B(x, 2r)) \leq C_d \, \mu(B(x, r));$$

$$\int_{B(x,r)} |f - f_B|^2 d\mu \leq C_p r^2 \int_{B(x,r)} \|\nabla_{\mathcal{H}} f\|^2 d\mu,$$

for every $f \in C^1(\overline{B}(x, r))$.