Calculus 3 Notes

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Course Page

Contents

1 Chapter 12

1.1 Component form of a vector

The component form of $\vec{PQ}$ where both P and Q are defined is

$$Q - P$$

1.2 Vector projection

The projection of a vector u onto v is

$$\text{proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v$$

The component of the projection is

$$|u|\cos(\theta) = \frac{u \cdot v}{|v|}$$

1.3 Area of a Parallelogram/Triangle

Let a parallelogram or triangle is defined with 3 points, P,Q, and R. Also let A_p be the area of a parallelogram and A_t be the area of a triangle.

$$A_p = |\vec{PQ} \times \vec{PR}|$$

$$A_t = \frac{A_p}{2} = \frac{|\vec{PQ} \times \vec{PR}|}{2}$$

1.4 Distance from point to line

Let P be a reference point on the line, S be a point off of the line, and let v be a vector parallel to the line. Also let the distance between the line and S be d.

$$d = \frac{|\vec{PS} \times v|}{|v|}$$

1.5 Angle between 2 vectors

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$$

1.6 Angle between 2 planes

The vector normal to a plane is equivalent to the coefficients of x, y, and z. Then just find the angle between the two normal vectors

1.7 Work

The work done with force vector F and distance vector d is

$$W = F \cdot d$$

2 Section 13.2

2.1 Ideal Projectile Motion

If v_0 makes an angle α with the horizon, then

$$v_0 = (|v_0| \cos\alpha)\hat{\imath} + (|v_0| \sin\alpha)\hat{\jmath}$$

Also, assume $r_0 = 0\hat{\imath} + 0\hat{\jmath}$. With these formulas it can be derived that

$$r(t) = -\frac{1}{2}gt^2\hat{\jmath} + v_0 t$$

2.1.1 Range of a Projectile

$$R = \frac{v_0^2}{g}\sin(2\alpha)$$

Where R is the range, v_0 is the initial velocity, g is acceleration due to gravity, and α is the launch angle

2.1.2 Maximum Height of a Projectile

Lets say that y_{max} is the maximum height, then:

$$y_{max} = \frac{v_0 \sin\alpha}{2g}$$

3 Section 13.3

3.1 Finding Tangent Vector to a Path

If $r(t)$ is the function for the path, then $r'(t)$ represents the tangent vector for the path.

3.2 Length of Path

If $r(t)$ is the function for the path and D represents the length of the path from $t \in (t_0, t_1)$, then

$$D = \int_{t_0}^{t_1} |r'(t)| dt$$

4 Section 13.4

4.1 Curvature

If T is the tangent unit vector of a smooth curve, then the curvature of the curve is

$$\kappa = \left|\frac{dT}{ds}\right|$$

where s is arc length.

4.2 Formula for Calculating Curvature

If $s(t)$ is a smooth curve, then curvature is

$$\kappa = \left|\frac{\frac{dT}{dt}}{\frac{ds}{dt}}\right|$$

where

$$T = \frac{s'(t)}{|s'(t)|}$$

and $v = r'(t)$

4.3 Principal Unit Normal

At any point $\kappa \neq 0$, the principal unit normal vector for a curve is

$$N = \frac{1}{\kappa}\frac{dT}{ds}$$

4.4 Formula for Calculating N

If $r(t)$ is a smooth curve, then the principal unit normal is

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$$

5 Section 13.5

5.1 Binormal Vector

The binormal vector (B) is a vector that is orthogonal to both T and N.

$$B = T \times N$$

5.2 Acceleration Vector

If the acceleration vector is

$$a = a_T T + a_N N$$

then

$$a_T = s''(t) = \frac{d}{dt}|s'(t)| \text{ and } a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa |s'(t)|^2$$

also

$$a_N = \sqrt{|s''(t)|^2 - a_T^2}$$

5.3 Torsion

$$\frac{dB}{ds} = \frac{d(T \times N)}{ds} = \frac{dT}{ds} \times N + T \times \frac{dN}{ds}$$

Since N has the same direction as $\frac{dT}{ds}$, $N \times \frac{dT}{ds} = 0$

$$\frac{dB}{ds} = T \times \frac{dN}{ds}$$

Because $\frac{dB}{ds}$ is orthogonal to B and T, it must be parallel to N. Therefore, $\frac{dB}{ds}$ is a scalar multiple of N and

$$\frac{dB}{ds} = -\tau N$$

where τ is called the *torsion* of the curve. Note that

$$\frac{dB}{ds} \cdot N = -\tau N \cdot N = -\tau$$

5.4 Calculating Torsion

If $B = T \times N$, the torsion of a smooth curve is

$$\tau = -\frac{dB}{ds} \cdot N$$

also,

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$

6 Section 13.6

6.1 Motion in Polar and Cylindrical Coordinates

When a particle at $P(r, \theta)$ moves along a curve in the polar coordinate plane, we express its position in terms of moving unit vectors:

$$u_r = (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath} \quad u_\theta = -(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}$$

where u_r is tangent to the vector $\begin{bmatrix}\theta \\ r\end{bmatrix}$ and u_θ is normal to it. When we differentiate u_r and u_θ with respect to t, we can see how they change with time.

$$\dot{u}_r = (-\dot{\theta}\sin\theta)\hat{\imath} + (\dot{\theta}\cos\theta)\hat{\jmath} \quad u_\theta = -(\dot{\theta}\sin\theta)\hat{\imath} - (\dot{\theta}\sin\theta)\hat{\jmath}$$

which equals

$$\dot{u}_r = \dot{\theta}u_\theta \quad u_\theta = -\dot{\theta}u_r$$

6.2 Polar Velocity Vector

$$v = \dot{r} = \frac{d}{dt}(ru_r) = \dot{r}u_r + r\dot{u}_r = \dot{r}u_r + r\dot{\theta}u_\theta$$

6.3 Polar Acceleration Vector

$$a = \dot{v} = (\ddot{r}u_r + \dot{r}\dot{u}_r) + (\dot{r}\dot{\theta}u_\theta + r\ddot{\theta}u_\theta + r\dot{\theta}\dot{u}_\theta)$$

which becomes

$$a = (\ddot{r} - r\dot{\theta}^2)u_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})u_\theta$$

6.4 Planet Movement

If r is the radius vector from the center of a sun of mass M to the center of a planet of mass m, then the force F of the gravitational attraction between the planet and the sun is:

$$F = -\frac{GmM}{|r|^2}\frac{r}{|r|}$$

where G is the universal gravitational constant.

$$G = 6.6726 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$$

By using Newton's second law, we can find

$$\ddot{r} = -\frac{GM}{|r|^2}\frac{r}{|r|}$$

Because $\ddot{r}$ is a scalar multiple of r, we know

$$r \times \ddot{r} = 0$$

so we know

$$\frac{d}{dt}(r \times \dot{r}) = \dot{r} \times \dot{r} + r \times \ddot{r} = r \times \ddot{r} = 0$$

meaning that the cross between r and $\dot{r}$ is constant

$$r \times \dot{r} = C$$

6.5 Kepler's First Law

The eccentricity of the ellipse that the planet follows is

$$e = \frac{r_0 v_0^2}{GM} - 1$$

and the polar equation is

$$r = \frac{(1+e)r_0}{1 + e\cos\theta}$$

where r_0 is the minimum distance from the sun. The sun's mass M is 1.99×10^{30}kg

6.6 Kepler's Third Law

The time T it takes a planet to go around its sun once is the planet's orbital period. Kepler's Third Law says that and the orbit's semimajor axis a are related by

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

7 Section 14.6

7.1 Standard Linear Approximation

For $f(x, y)$ at (x_0, y_0), the Standard Linear Approximation of $f(x, y)$ is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{7.1.1}$$

7.2 The Error in the Standard Linear Approximation

If f has continuous first and second partial derivative throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearizion satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2. \tag{7.2.1}$$

7.3 Tangent Plane

For $f(x, y)$ at (x_0, y_0), the tangent plane of $f(x, y)$ is:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{7.3.1}$$

If you have a surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ use

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \tag{7.3.2}$$

7.4 Normal Line

The normal line to $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ has the following equations:

$$\begin{aligned} x &= x_0 + f_x(P_0)t \\ y &= y_0 + f_y(P_0)t \\ z &= z_0 + f_z(P_0)t \end{aligned}$$

8 Section 14.7

8.1 Definitions of local maximums and minimums

If $f(x, y)$ is defined on a region R containing the point (a, b), then:

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

8.2 First Derivative Test for Local Extreme Values

If all partial derivatives are equal to zero or undefined, then they are critical points. In order to find local extrema, you must set all partial derivatives to zero and solve the system of equations.

8.3 Second Derivative Test for Local Extreme Values

Let D be the **discriminant** or **Hessian** of f so that

$$D = f_{xx} f_{yy} - f_{xy}^2$$

then

1. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $D > 0$ at (a, b).
2. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $D > 0$ at (a, b).
3. f has a **saddle point** at (a, b) if $D < 0$ at (a, b).
4. **the test is inconclusive** at (a, b) if $D = 0$ at (a, b) and another method must be in order to determine the behavior at (a, b).

8.4 Finding Absolute Maxima and Minima on Closed Bounded Regions

In order to find absolute extrema for $f(x, y)$ on a closed and bounded region R,

1. *List the interior points of* R where f may have local maxima or minima and evaluate f at these points. These are critical points of f.
2. *List the boundary points of* R where f has local maxima and minima and evaluate f at these points. For every boundary, fix one or more of the variables in order to create a function of a single variable and find its local maxima and minima.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

9 Section 14.8

Using Lagrange multipliers, you can find extreme values of a function whose domain is constrained to lie within a subset of a plane.

9.1 Lagrange Multipliers

If you have two functions, $f(x, y)$ and $g(x, y) = c$, you can find extreme values of f on $g(x, y) = c$ by finding locations where $\nabla f = \lambda \nabla g$. λ is the Lagrange Multiplier.

10 Section 14.9

10.1 Local Linearization

The linearization of a function at the point (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If $\boldsymbol{x_0} = (x_0, y_0)$ and $\boldsymbol{x} = (x, y)$,

$$L(\boldsymbol{x}) = f(\boldsymbol{x_0}) + \nabla f \cdot (\boldsymbol{x} - \boldsymbol{x_0})$$

10.2 Quadratic Approximation

The approximation of a function at the point (x_0, y_0) is

$$\begin{aligned} Q(x, y) =& f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &+ \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2 \end{aligned}$$

10.2.1 Hessian Matrix

Let f be a function of (x, y)

$$\boldsymbol{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

10.2.2 Representing Quadratic Forms with vectors

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

You can also let

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

to get

$$ax^2 + 2bxy + cy^2 = \boldsymbol{X}^T \boldsymbol{A} \boldsymbol{X}$$

10.2.3 Vector form of Quadratic Approximation

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$$Q(X) = f(X_0) + \nabla f \cdot (X - X_0) + \frac{1}{2}(X - X_0)^T \boldsymbol{H}_{X_0}(X - X_0)$$

11 Chapter 15 Double Integral

11.1 Definition of the Double Integral

Double Integral = volume below the graph $z = f(x, y)$ over a region R in xy-plane.

$$\int\int_R f(x, y) dA$$

Definition: Cut R into small pieces of area ΔA Lets divide the region R into i x and y components The height of each rectangular prism is $f(x_i, y_i)$ and the area of the region underneath is ΔA_i

$$\sum_i f(x_i, y_i)\Delta A_i$$

Finally, you get take the limit as $\Delta A_i \to 0$ to get the double integral.

11.2 Calculating the Double Integral

To compute the double integral, take **slices**. Let $S(x)$ = area of slice by plane parallel to the yz-plane. Then,

$$\text{volume} = \int_{x_{min}}^{x_{max}} S(x) dx.$$

$$S(x) = \int_{y_{min}(x)}^{y_{max}(x)} f(x, y) dy$$

$$\text{Finally, } \int\int_{\mathbf{R}} \mathbf{f(x, y)dA} = \int_{\mathbf{x_{min}}}^{\mathbf{x_{max}}} \left[\int_{\mathbf{y_{min}(x)}}^{\mathbf{y_{max}(x)}} \mathbf{f(x, y)dy} \right] \mathbf{dx}$$

This is called an **Iterated Integral** because one integral is integrated over the next. BTW: dA becomes $dy \cdot dx$ because dA is the area of each infinitesimal rectangle which equals $dy \cdot dx$

11.3 Double Integral in Polar Coordinate

Slices of circles on the polar coordinate plane are approximately rectangles when they are very tiny. Therefore,

$$\Delta A \approx \Delta r \Delta \theta$$

and therefore,

$$dA = dr d\theta$$

If you have the function $f(\theta, r)$, then in order to find the volume under the curve, use the formula

$$V = \int\int_R r f(\theta, r) dA$$

11.4 Applications of Double Integrals

1. Find area of a region R

$$A = \int\int_R 1 dA$$

2. Finding mass of a flat object with density δ =mass per unit area

$$\Delta m = \delta \Delta A$$
$$m = \int\int_R \delta dA$$

3. Finding average value of f in region R

$$\bar{f} = \frac{1}{\text{Area}} \int\int_R f dA$$

Weighted average of f with density δ:

$$\frac{1}{\text{Mass}(R)} \int\int_R f \delta dA$$

4. Center of Mass of a (planar) object (with density δ)?
Center of mass is at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{\text{Mass}} \int\int_R x \delta dA$$
$$\bar{y} = \frac{1}{\text{Mass}} \int\int_R y \delta dA$$

Left off here