Control System Design Solutions

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Chapter 2 Answers

Answer (2.5a)**.** For each cart, we know that acceleration is equal to the net Force divided by the mass. Thus,

$$\ddot{z}_i = \frac{F_N}{M}.$$

We let the positive direction be to the right. Thus,

$$\begin{aligned}\ddot{z}_1 &= \frac{1}{M}\left(F_{m1} - F_s\right)\\ \ddot{z}_2 &= \frac{1}{M}\left(F_{m2} + F_s\right),\end{aligned}$$

where F_{mi} are the forces from each respective motor, and F_s is the force from the intermediate spring. Now lets derive the equations for the force F_{mi}. This will follow from Example 2B of the textbook.

$$\begin{aligned}e_i - v_i &= Ri_i\\ e_i - K_2 w_i &= Ri\\ \tau_i &= K_1 i_i\\ e_i - K_2 w_i &= \frac{R\tau_i}{K_1}\\ \tau_i &= \frac{K_1}{R}e_i - \frac{K_1K_2}{R}w_i\\ F_{mi} &= \frac{\tau_i}{r} = \frac{K_1}{rR}e_i - \frac{K_1K_2}{rR}w_i\\ w_i r &= \dot{z}_i\\ F_{mi} &= \frac{K_1}{rR}e_i - \frac{K_1K_2}{r^2R}\dot{z}_i\end{aligned}$$

Now we will use Hooke's law to derive the force of the spring,

$$F_s = kd = k(z_2 - z_1).$$

Thus, for our differential equations, we have

$$\begin{aligned}\ddot{z}_1 &= \frac{k}{M}z_1 - \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_1 + \frac{K_1}{rRM}e_1\\ \ddot{z}_2 &= -\frac{k}{M}z_1 + \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_2 + \frac{K_1}{rRM}e_2.\end{aligned}$$

Finally for our state-space representation of the system, we have,

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \ddot{z}_1\\ \ddot{z}_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{k}{M} & -\frac{k}{M} & -\frac{K_1K_2}{r^2RM} & 0\\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1K_2}{r^2RM}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \dot{z}_1\\ \dot{z}_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{K_1}{rRM} & 0\\ 0 & \frac{K_1}{rRM}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}.$$

Answer (2.5b). Using the differential equations above, we can create the following blcok diagram.