Control System Design Solutions

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Chapter 2 Answers

Answer (2.5a)**.** For each cart, we know that acceleration is equal to the net Force divided by the mass. Thus,

$$\ddot{z}_i = \frac{F_N}{M}.$$

We let the positive direction be to the right. Thus,

$$\begin{aligned}
\ddot{z}_1 &= \frac{1}{M}\left(F_{m1} - F_s\right) \\
\ddot{z}_2 &= \frac{1}{M}\left(F_{m2} + F_s\right),
\end{aligned}$$

where F_{mi} are the forces from each respective motor, and F_s is the force from the intermediate spring. Now lets derive the equations for the force F_{mi}. This will follow from Example 2B of the textbook.

$$\begin{aligned}
e_i - v_i &= Ri_i \\
e_i - K_2 w_i &= Ri \\
\tau_i &= K_1 i_i \\
e_i - K_2 w_i &= \frac{R\tau_i}{K_1} \\
\tau_i &= \frac{K_1}{R} e_i - \frac{K_1 K_2}{R} w_i \\
F_{mi} &= \frac{\tau_i}{r} = \frac{K_1}{rR} e_i - \frac{K_1 K_2}{rR} w_i \\
w_i r &= \dot{z}_i \\
F_{mi} &= \frac{K_1}{rR} e_i - \frac{K_1 K_2}{r^2 R} \dot{z}_i
\end{aligned}$$

Now we will use Hooke's law to derive the force of the spring,

$$F_s = kd = k(z_2 - z_1).$$

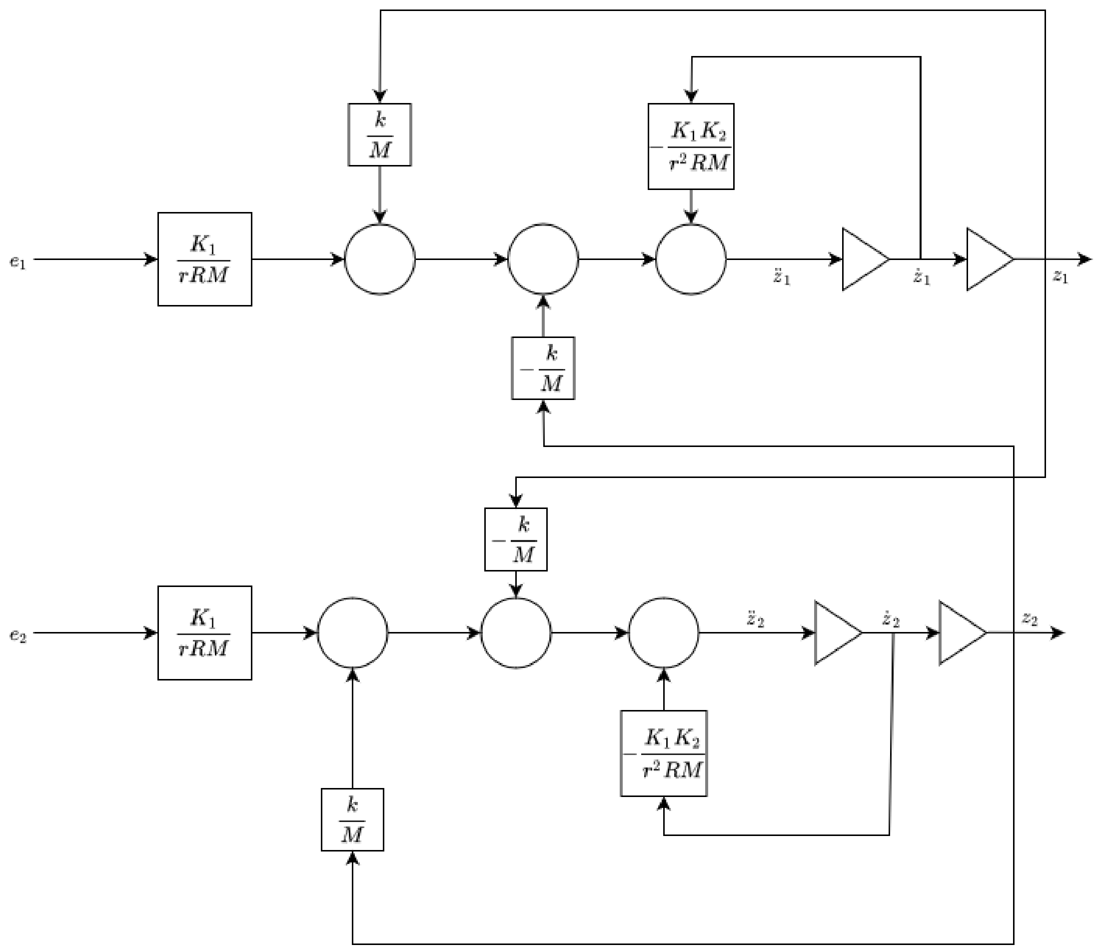
Thus, for our differential equations, we have

$$\begin{aligned}
\ddot{z}_1 &= \frac{k}{M} z_1 - \frac{k}{M} z_2 - \frac{K_1 K_2}{r^2 RM} \dot{z}_1 + \frac{K_1}{rRM} e_1 \\
\ddot{z}_2 &= -\frac{k}{M} z_1 + \frac{k}{M} z_2 - \frac{K_1 K_2}{r^2 RM} \dot{z}_2 + \frac{K_1}{rRM} e_2.
\end{aligned}$$

Finally for our state-space representation of the system, we have,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & -\frac{k}{M} & -\frac{K_1 K_2}{r^2 RM} & 0 \\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1 K_2}{r^2 RM} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_1}{rRM} & 0 \\ 0 & \frac{K_1}{rRM} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Answer (2.5b)**.** Using the differential equations above, we can create the following blcok diagram.



Chapter 5 Answers

Answer (5.2a)**.** Removing the motor from the left most cart results in the following differential equations.

$$\ddot{z}_1 = \frac{1}{M}\left(-F_s\right)$$
$$\ddot{z}_2 = \frac{1}{M}\left(F_{m2} + F_s\right),$$

Thus, the differential equations become

$$\ddot{z}_1 = -\frac{k}{M}z_1 + \frac{k}{M}z_2$$
$$\ddot{z}_2 = -\frac{k}{M}z_1 + \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_2 + \frac{K_1}{rRM}e_2.$$

Thus, the state space system becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & -\frac{k}{M} & 0 & 0 \\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1K_2}{r^2RM} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_1}{rRM} \end{bmatrix} e_2.$$

Recall that in order for the system to be controllable, the matrix

$$Q = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

must have rank 4. Using matlab, we find that

$$Q = \begin{bmatrix} 0 & 0 & 0 & -40 \\ 0 & 1 & -100 & 10040 \\ 0 & 0 & -40 & 4000 \\ 1 & -100 & 10040 & -1008000 \end{bmatrix}.$$

Also using matlab, we find that the rank of this matrix is 4. Thus, the system is controllable.

Answer (5.2b)**.** Recall the state space system from problem 2.5

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & -\frac{k}{M} & -\frac{K_1K_2}{r^2RM} & 0 \\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1K_2}{r^2RM} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_1}{rRM} & 0 \\ 0 & \frac{K_1}{rRM} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In order for this system to be controllable, the rank of

$$Q = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

must be 4. By using matlab, we can find

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 & -100 & 0 & 10040 & -40 \\ 0 & 0 & 0 & 1 & 0 & -100 & -40 & 10040 \\ 1 & 0 & -100 & 0 & 10040 & -40 & -1008000 & 8000 \\ 0 & 1 & 0 & -100 & -40 & 10040 & 8000 & -1008000 \end{bmatrix}.$$

Also using matlab, we find that the rank of this matrix is 4.

Answer (5.2c)**.** Recall that the state vector is

$$x = \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}.$$

Thus, if you can only measure the position of the cart, our observation matrix must be

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Recall that in order the system is observable if and only if the matrix

$$N = \begin{bmatrix} C' & A'C' & (A')^2C' & (A')^3C' \end{bmatrix}$$

has rank 4. Using matlab we can find

$$N = \begin{bmatrix} 1 & 0 & 40 & -4000 \\ 0 & 0 & -40 & 4000 \\ 0 & 1 & -100 & 10040 \\ 0 & 0 & 0 & -40 \end{bmatrix}$$

Using the rank function in Matlab, we find that the rank of this matrix is always 4. Thus the system is observable.

Answer (5.2d)**.** If only the velocity of the first car is measurable, then

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

Using Matlab, we find

$$N = \begin{bmatrix} 0 & 0 & -40 & 4000 \\ 1 & 0 & 40 & -4000 \\ 0 & 0 & 0 & -40 \\ 0 & 1 & -100 & 10040 \end{bmatrix}.$$

and that the rank of this matrix is still 4.