# Control System Design Solutions

Matthew Stringer

## Chapter 2 Answers

**Answer** (2.5a)**.** For each cart, we know that acceleration is equal to the net Force divided by the mass. Thus,

$$\ddot{z}_i = \frac{F_N}{M}.$$

We let the positive direction be to the right. Thus,

$$\begin{aligned}\ddot{z}_1 &= \frac{1}{M}\left(F_{m1} - F_s\right)\\ \ddot{z}_2 &= \frac{1}{M}\left(F_{m2} + F_s\right),\end{aligned}$$

where $F_{mi}$ are the forces from each respective motor, and $F_s$ is the force from the intermediate spring. Now lets derive the equations for the force $F_{mi}$. This will follow from Example 2B of the textbook.

$$\begin{aligned}e_i - v_i &= Ri_i\\ e_i - K_2 w_i &= Ri\\ \tau_i &= K_1 i_i\\ e_i - K_2 w_i &= \frac{R\tau_i}{K_1}\\ \tau_i &= \frac{K_1}{R}e_i - \frac{K_1K_2}{R}w_i\\ F_{mi} &= \frac{\tau_i}{r} = \frac{K_1}{rR}e_i - \frac{K_1K_2}{rR}w_i\\ w_i r &= \dot{z}_i\\ F_{mi} &= \frac{K_1}{rR}e_i - \frac{K_1K_2}{r^2R}\dot{z}_i\end{aligned}$$

Now we will use Hooke's law to derive the force of the spring,

$$F_s = kd = k(z_2 - z_1).$$

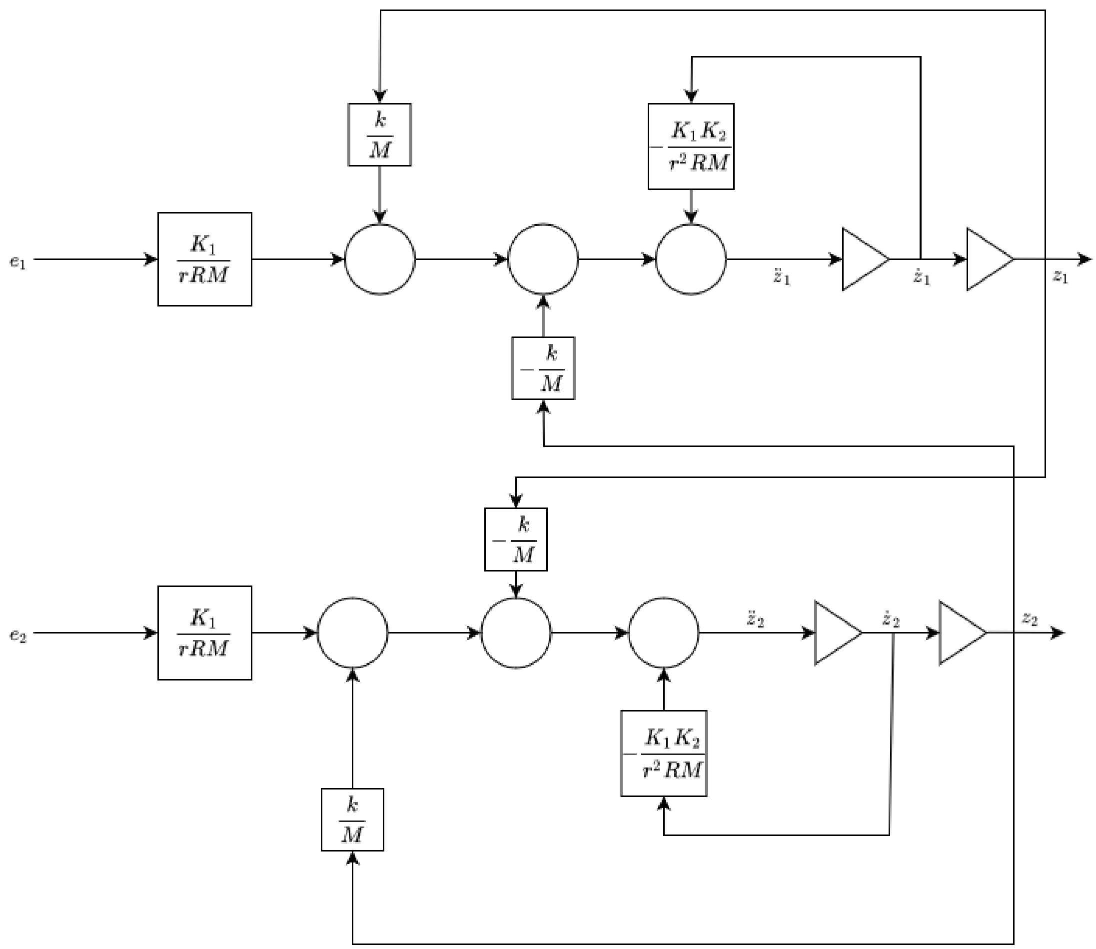
Thus, for our differential equations, we have

$$\begin{aligned}\ddot{z}_1 &= \frac{k}{M}z_1 - \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_1 + \frac{K_1}{rRM}e_1\\ \ddot{z}_2 &= -\frac{k}{M}z_1 + \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_2 + \frac{K_1}{rRM}e_2.\end{aligned}$$

Finally for our state-space representation of the system, we have,

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \ddot{z}_1\\ \ddot{z}_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{k}{M} & -\frac{k}{M} & -\frac{K_1K_2}{r^2RM} & 0\\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1K_2}{r^2RM}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ \dot{z}_1\\ \dot{z}_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{K_1}{rRM} & 0\\ 0 & \frac{K_1}{rRM}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix}.$$

**Answer** (2.5b)**.** Using the differential equations above, we can create the following blcok diagram.




# Chapter 5 Answers

**Answer** (5.2a)**.** Removing the motor from the left most cart results in the following differential equations.

$$\ddot{z}_1 = \frac{1}{M}\left(-F_s\right)$$
$$\ddot{z}_2 = \frac{1}{M}\left(F_{m2} + F_s\right),$$

Thus, the differential equations become

$$\ddot{z}_1 = -\frac{k}{M}z_1 + \frac{k}{M}z_2$$
$$\ddot{z}_2 = -\frac{k}{M}z_1 + \frac{k}{M}z_2 - \frac{K_1K_2}{r^2RM}\dot{z}_2 + \frac{K_1}{rRM}e_2.$$

Thus, the state space system becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & -\frac{k}{M} & 0 & 0 \\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1K_2}{r^2RM} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_1}{rRM} \end{bmatrix} e_2.$$

Recall that in order for the system to be controllable, the matrix

$$Q = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

must have rank 4. Using matlab, we find that

$$Q = \begin{pmatrix} 0 & 0 & 0 & \sigma_2 \\ 0 & \sigma_3 & \sigma_1 & \sigma_4 \\ 0 & 0 & \sigma_2 & \frac{K_1{}^2\, K_2\, k}{M^3\, R^2\, r^3} \\ \sigma_3 & \sigma_1 & \sigma_4 & -\frac{K_1\left(\frac{K_1\, K_2\, k}{M^2\, R\, r^2} + \frac{K_1\, K_2\, \sigma_5}{M\, R\, r^2}\right)}{M\, R\, r} \end{pmatrix}$$

where

$$\sigma_1 = -\frac{K_1{}^2\, K_2}{M^2\, R^2\, r^3}$$

$$\sigma_2 = -\frac{K_1\, k}{M^2\, R\, r}$$

$$\sigma_3 = \frac{K_1}{M\, R\, r}$$

$$\sigma_4 = \frac{K_1\, \sigma_5}{M\, R\, r}$$

$$\sigma_5 = \frac{k}{M} + \frac{K_1{}^2\, K_2{}^2}{M^2\, R^2\, r^4}$$

Also using matlab, we find that the rank of this matrix is 4. Thus, the system is controllable.

**Answer** (5.2b)**.** Recall the state space system from problem 2.5

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M} & -\frac{k}{M} & -\frac{K_1 K_2}{r^2 RM} & 0 \\ -\frac{k}{M} & \frac{k}{M} & 0 & -\frac{K_1 K_2}{r^2 RM} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_1}{rRM} & 0 \\ 0 & \frac{K_1}{rRM} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In order for this system to be controllable, the rank of

$$Q = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

must be 4. By using matlab, we can find

$$Q = \begin{pmatrix} 0 & 0 & \sigma_4 & 0 & \sigma_1 & 0 & \sigma_5 & \sigma_2 \\ 0 & 0 & 0 & \sigma_4 & 0 & \sigma_1 & \sigma_2 & \sigma_5 \\ \sigma_4 & 0 & \sigma_1 & 0 & \sigma_5 & \sigma_2 & \sigma_6 & \sigma_3 \\ 0 & \sigma_4 & 0 & \sigma_1 & \sigma_2 & \sigma_5 & \sigma_3 & \sigma_6 \end{pmatrix}$$

where

$$\sigma_1 = -\frac{K_1{}^2\, K_2}{M^2\, R^2\, r^3}$$

$$\sigma_2 = -\frac{K_1\, k}{M^2\, R\, r}$$

$$\sigma_3 = \frac{2\, K_1{}^2\, K_2\, k}{M^3\, R^2\, r^3}$$

$$\sigma_4 = \frac{K_1}{M\, R\, r}$$

$$\sigma_5 = \frac{K_1\, \sigma_7}{M\, R\, r}$$

$$\sigma_6 = -\frac{K_1\left(\frac{K_1\, K_2\, k}{M^2\, R\, r^2} + \frac{K_1\, K_2\, \sigma_7}{M\, R\, r^2}\right)}{M\, R\, r}$$

$$\sigma_7 = \frac{k}{M} + \frac{K_1{}^2\, K_2{}^2}{M^2\, R^2\, r^4}$$

Also using matlab, we find that the rank of this matrix is always 4.