Parisi-Sourlas Quantisation of Constrained Systems

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Introduction

In this project, a method to obtain covariant phase space path-integrals for constrained Hamiltonian systems was introduced and applied to pure Yang-Mills theory. In particular, it was shown that the Parisi-Sourlas (PS) formalism could be applied to quantise the Gauss' law constraints in the Coulomb gauge by introducing additional fermionic ghost fields. The resulting path-integral respected the PS supersymmetry of the extended phase space by construction. It was emphasised that this formalism generalises the well-known formalism of BRST quantisation.

Parisi-Sourlas Supersymmetry

Consider the $2(N+K)$-dimensional extended phase space of a system with K canonical pairs of Lagrange multipliers (λ^a, ϖ_a) corresponding to the first class constraints $G_a(q,p)$ and gauge fixing conditions $F^a(q,p)$. In order to establish equivalence to the $2(N-K)$-dimensional physical phase space, $4K$ Grassmann odd ghosts $(\eta^a, \bar\eta_a)$, $(\bar\zeta^a, \zeta_a)$ are introduced with the canonical relations

$$\{\eta^a, \bar\eta_b\} = \{\bar\zeta^a, \zeta_b\} = -\delta^a_{\ b}. \tag{1}$$

The unphysical sector of the extended phase space is spanned by the position variables $(\tilde F^a, \lambda^a, \eta^a, \bar\zeta^a)$ and momentum variables $(\tilde G_a, \varpi_a, \zeta_a, \bar\eta_a)$, where a change of variables is employed to map (G_a, F^a) into the *Darboux chart* $(\tilde G_a, \tilde F^a)$ such that

$$\{\tilde G_a, \tilde G_b\} = \{\tilde F^a, \tilde F^b\} = 0, \qquad \text{and} \qquad \{\tilde G_a, \tilde F^b\} = -\delta^b_a. \tag{2}$$

Using these variables, a 4-dimensional PS superspace can be introduced by locally attaching two fermionic coordinates $(\theta, \bar\theta)$ to the space. The corresponding path-integral on this superspace can always be made invariant under the group of orthosymplectic superrotations $Osp(1,1|2)$. This so-called *Parisi-Sourlas supersymmetry* in the unphysical sector also generalises the BRST supersymmetry. The BRST charge can be identified with a particular nilpotent generator Q_B of the superrotations. The freedom to add PS exact terms of the form $\{\Psi, Q_B\}$ into the canonical action can be used to impose the constraints and gauge fixing conditions in the path-integral without breaking Lorentz invariance.

$$\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\cdots\mathcal{D}\bar\zeta \exp\left[i\int dt(\dot q^i p_i - \dot\lambda^a\varpi_a + \dot\eta^a\bar\eta_a + \dot{\bar\zeta}^a\zeta_a - \{\Psi, Q_B\})\right] \tag{3}$$

Yang-Mills Theory

The gauge invariant pure Yang-Mills action is

$$S_{YM} = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu}_a \tag{4}$$

In the Hamiltonian formulation, the ambient phase space is parametrised by $(\vec A^a, \vec E^a)$. The temporal components $A^a_t(x)$ of the gauge field act as Lagrange multipliers for the first class Gauss' law constraints

$$G_a(\vec x) = (\vec D\cdot\vec E)_a(\vec x), \qquad \{G_a(\vec x), G_b(\vec y)\} = g f_{ab}^{\ \ c} G_c(\vec x)\delta^3(\vec x - \vec y). \tag{5}$$

These constraints are the generators of local gauge transformations. To eliminate the gauge freedom from our description we impose the Coulomb gauge condition $F^a(\vec x) = (\vec\partial\cdot\vec A)^a(\vec x)$. The conditions (F^a, G_a) are mutually second class constraints

$$\{G_a(\vec x), F^b(\vec y)\} = -(\vec\partial\cdot\vec D)^b_a(\vec x)\delta^{(3)}(\vec x - \vec y). \tag{6}$$

The physical phase space consists of the gauge fixed fields $(\vec A^a_{gf}, \vec E^a_{gf})$.

References

[1] A. J. Niemi, *Pedagogical Introduction to BRST*, Phys. Rept. **184:147–165**, (1989)
[2] G. Parisi and N. Sourlas, *Random Magnetic Fields, Supersymmetry, and Negative Dimensions*, Phys. Rev. Lett., **43:744–745**, (1979)
[3] S. Hwang. *Abelianization of gauge algebras in the Hamiltonian formalism*, Nucl. Phys. B, **351:425–440**, (1991).

Gauss' Law in the Darboux Chart

To apply the PS quantisation method to Yang-Mills theory, a change of variables was made so that the constraint algebra (5), (6) could be converted to the flat structure (2) of the Darboux chart. The gauge conditions were mapped identically to $\tilde F^a(\vec x) = (\vec\partial\cdot\vec A)^a(\vec x)$ while the solution for the Gauss' law constraints was shown to be given by

$$\tilde G_a(\vec x) = [(\vec\partial\cdot\vec D)^{-1}]^b_{\ a} G_b(\vec x) \tag{7}$$

The appearance of a non-local operator was deemed to be acceptable during intermediate stages of calculation after which the transformation could be inverted to restore locality.

The Superfield Representation

To generalise the discussion of PS supersymmetry to a theory of fields, the Yang-Mills phase space was extended by introducing canonically conjugate auxiliary fields (B^a, ω_a) and ghost fields $(\eta^a, \bar\eta_a)$, $(\bar\zeta^a, \zeta_a)$. Together with the Darboux coordinates $(\tilde F^a, \tilde G_a)$, each unphysical subspace was then converted into a PS superspace by defining the superfield position and momentum variables

$$\begin{aligned} F^{(\mathrm{SF})a}(x,\theta,\bar\theta) &\equiv \tilde F^a + \bar\zeta^a\theta + \eta^a\bar\theta + B^a\bar\theta\theta; \\ G^{(\mathrm{SF})}_a(x,\theta,\bar\theta) &\equiv \omega_a + \theta\bar\eta_a + \bar\theta\zeta_a + \tilde G_a\bar\theta\theta. \end{aligned} \tag{8}$$

In this representation, it was shown that the BRST transformations were given by the action of the operator Q_{PS} associated with translations in the θ direction. In particular, the equations $Q_{\mathrm{PS}}F^{(\mathrm{SF})a} = \partial_\theta F^{(\mathrm{SF})a}$ and $Q_{\mathrm{PS}}G^{(\mathrm{SF})}_a = \partial_\theta G^{(\mathrm{SF})}_a$ were used to arrive at the PS transformations

$$\begin{aligned} &Q_{\mathrm{PS}}\tilde F^a = -\bar\zeta^a \quad Q_{\mathrm{PS}}\bar\zeta^a = 0 \quad Q_{\mathrm{PS}}\eta^a = -B^a \quad Q_{\mathrm{PS}}B^a = 0 \\ &Q_{\mathrm{PS}}\omega_a = \bar\eta_a \quad Q_{\mathrm{PS}}\bar\eta_a = 0 \quad Q_{\mathrm{PS}}\zeta_a = \tilde G_a \quad Q_{\mathrm{PS}}\tilde G_a = 0 \end{aligned} \tag{9}$$

It was then shown that the combinations $(F^{(\mathrm{SF})a}G^{(\mathrm{SF})}_a)$, $(F^{(\mathrm{SF})a})^2$ and $(G^{(\mathrm{SF})}_a)^2$ of the superfields were invariant under these transformations. After adding these terms to the action, the gauge fermion Ψ and BRST supercharge Q_B were found to be

$$\begin{aligned} \Psi &= (\tilde F^a\zeta_a - \omega_a\eta^a) - 2(\tilde F^a\eta^a) + 2(\omega_a\zeta_a), \\ Q_B &= \int d^3\vec y[\bar\zeta^b(\vec y)\tilde G_b(\vec y) + B^b(\vec y)\bar\eta_b(\vec y)]. \end{aligned} \tag{10}$$

The unphysical phase space could then be described by the non-local, gauge fixed, and PS invariant action

$$S_{\mathrm{unphysical}} = \int d^4x\left[\int d\theta d\bar\theta\partial_t(F^{(\mathrm{SF})a})G^{(\mathrm{SF})}_a + \{Q_B, \Psi\}\right]. \tag{11}$$

Localising the Action

To bring the path-integral over the entire phase space to a local form, a canonical transformation was performed. To establish the necessary equivalence between the old and new position variables, the original components of the gauge field $\vec A^a$ were taken into a larger phase space with the auxiliary fields $(\mathcal{B}^a, \mathcal{P}_a)$ and ghost fields $(c^a, \bar c_a)$, $(\bar b^a, b_a)$ by means of a supersymmetric gauge transformation

$$\vec A^a(\vec x) \to \vec{\mathcal{A}}^a(\vec x) = \vec A^a(\vec x) + \vec D^a_{\ b}\bar b^b(\vec x)\theta + \vec D^a_{\ b}c^b(\vec x)\bar\theta + \vec D^a_{\ b}\mathcal{B}^a(\vec x)\bar\theta\theta. \tag{12}$$

The transformation was then defined by the type 2 generating functional

$$F_2[\vec{\mathcal{A}}^a, G^{(\mathrm{SF})}_a] = \int d^3\vec y d\theta d\bar\theta[G^{(\mathrm{SF})}_a(\vec y)(\vec\partial\cdot\vec{\mathcal{A}})^a(\vec y)]. \tag{13}$$

This functional was used to relate the ghosts and auxiliary fields in each set of variables explicitly. The remaining fields $(\vec A^a_{gf}, \tilde F^a, \vec E^a_{gf}, \tilde G_a)$ did not require such treatment since they collectively span the ambient phase space, and therefore, after transforming back to the original variables their contribution to the action must be equal to S_{YM} as given in (4). Once all of the fields were localised, it was concluded that the resulting path-integral was equivalent to the known result in terms of the Fadeev-Popov Lagrangian.