Hamilton Trust Research Project

Solitons in Self-Dual Yang-Mills

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Contents

1 Self-Dual Yang-Mills Theory

With the aim of this research being to study soliton solutions of the self-dual Yang-Mills (SDYM) equations, we begin by briefly introducing Yang-Mills theory in its most general form. As we proceed, it is important to note that a general gauge theory is a theory in which the interactions are introduced by promoting global symmetries to local gauge symmetries. Yang-Mills represents one such class of these theories built upon the principle of invariance under a set of gauge transformations belonging to some compact, semi-simple Lie group $\mathcal{G}$ called the gauge group. In the construction of Yang-Mills theory, the principle of local gauge invariance necessarily leads to the introduction of a gauge field one-form A which plays the role of a connection in the gauge covariant derivative $D = d + A$. Certain choices of the gauge group lead to quantised theories which are used to describe the strong, weak and electromagnetic field interactions, and so, although the following discussion of the Yang-Mills equation is purely classical, the range of applications of this research can in principle extend beyond this realm into the area of non-perturbative quantum field theory.

1.1 Yang-Mills Theory

Yang-Mills theory is a special example of a gauge theory with a compact, semi-simple gauge group $\mathcal{G}$. The generators t^a of the corresponding Lie algebra $\mathfrak{g}$ are taken to satisfy the following algebraic and normalisation conditions

$$[t^a, t^b] = i f^{abc} t^c, \qquad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}; \tag{1.1}$$

where $a, b, c = 1, 2, \dots, \dim(\mathcal{G})$ are Lie-algebra indices in the adjoint representation. The gauge group $\mathcal{G}$ is non-abelian whenever the *structure constants* $f^{abc} \neq 0$. Any of the fields relevant to Yang-Mills theory possess a representation in terms of components or in the Lie algebra $\mathfrak{g}$ itself, they are related as

$$\Phi = \Phi^a t^a, \qquad \Phi^a = 2\mathrm{tr}(\Phi t^a). \tag{1.2}$$

In order to construct a gauge invariant Lagrangian, we must introduce gauge fields $A_\mu = A^a_\mu t^a$, the interaction of which is governed by their role as a connection in the gauge covariant derivative

$$D_\mu = \partial_\mu + A_\mu. \tag{1.3}$$

The two-form field strength tensor is defined as the commutator of covariant derivatives

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.4}$$

Let $U \in \mathcal{G}$, then under finite local gauge transformations the gauge field A_μ transforms as a connection in the adjoint representation

$$A_\mu \to A'_\mu = UA_\mu U^{-1} - U(\partial_\mu)U^{-1} \qquad \Longrightarrow \qquad F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1}. \tag{1.5}$$

It is clear from these transformations that the field strength itself is not gauge invariant. However, there still exists gauge invariant combinations of the field strength to be used in the action. For example, the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{1.6}$$

is gauge invariant and provides a kinetic term for the gauge field. The action for this so-called "pure" Yang-Mills Lagrangian is

$$S = -\frac{1}{2}\int d^4x \ \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{1.7}$$

As is evident in the field strength (1.4), this action also contains higher order terms in the gauge fields which represents the fact that pure Yang-Mills is a *self-interacting* theory. A complex scalar field Φ can also be introduced by observing that the terms $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ and $\Phi^\dagger\Phi$ are gauge invariant.

By imposing the principle of least action $\delta S = 0$ we arrive at the equation

$$D_\mu F^{\mu\nu} = 0. \tag{1.8}$$

The dependent variable in this equation is the gauge field and for non-abelian gauge groups, the equation is non-linear. It turns out that in this unrestricted form, analytic solutions are very hard to come by and so we normally impose further constraints on the field strength separately in order to make the system integrable. In fact, the work we will do in this project is largely based on a claim made by Ward which is that, by reducing the system of Yang-Mills equations through various symmetry arguments and choices of $\mathcal{G}$, we can arrive at many of the known integrable models.

1.2 The Bianchi Identity

The Hodge star operator $*$ acts on tensors returning their Hodge dual. For the field strength tensor of Yang-Mills theory, the dual tensor $\star F$ is defined as

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{1.9}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally anti-symmetric Levi-Civita symbol. The contraction of the two anti-symmetric tensors leads us to quite an elegant mathematical result called the Bianchi identity which can be derived as follows. The gauge covariant derivative satisfies the Jacobi identity

$$[D_\mu, [D_\nu, D_\lambda]] + [D_\lambda, [D_\mu, D_\nu]] + [D_\nu, [D_\lambda, D_\mu]] = 0, \tag{1.10}$$

under the Lie bracket operation. Using (1.4) we can write

$$[D_\mu, [D_\nu, D_\lambda]]\Phi = [D_\mu, F_{\nu\lambda}]\Phi = D_\mu(F_{\nu\lambda}\Phi) - F_{\nu\lambda}D_\mu\Phi = D_\mu F_{\nu\lambda}\Phi.$$

This allows us to rewrite the Jacobi identity in terms of the field strength as

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = D_{[\mu}F_{\nu\lambda]} = 0. \tag{1.11}$$

But now we notice that the anti-symmetrisation of the indices in the above equation can be written as the contraction (under relabelling)

$$\epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = D_\mu(\star F)^{\mu\nu} = 0. \tag{1.12}$$

This equation is the Bianchi identity for Yang-Mills theory and together with the Euler Lagrange equation (1.8), they are known as the *Yang-Mills Equations*.

1.3 Self-Duality

The Yang-Mills equations have the property that they are linear whenever the gauge group is Abelian and non-linear whenever the gauge group is non-abelian . The non-linear form of the equations are almost impossible to solve analytically but the rich area of study we are exploring in this project is accessed by imposing the self-dual condition

$$\star F_{\mu\nu} = F_{\mu\nu} \tag{1.13}$$

on the field strength tensor. This greatly simplifies the Yang-Mills equations since when the self-duality condition is true, the Bianchi identity automatically guarantees that the Euler-Lagrange equation holds. In this sense, the Yang-Mills equations are fully determined by (1.13) which shall henceforth be called the *self-dual Yang-Mills* (SDYM) *equations*. This simplification of the equations also reduces Yang-Mills to an integrable model. For abstract systems like this one, it is hard to concretely define the notion of integrability but we will highlight this property wherever it manifests itself in what follows. Field equations like (1.13) essentially have an infinite number of degrees of freedom due to the fact that these fields are defined at every point in the space we are considering. Hence, that SDYM is integrable, implies that there must be an equally infinite number of conserved quantities and as we will see these quantities arise from so-called *hidden symmetries*.

1.4 Double Null Coordinates

In what has been discussed so far, we have implicitly treated Yang-Mills theory on some real 4-dimensional space. In the context of SDYM there are three important signatures which the metric on this space can possess, namely:

- The real Euclidean space $\mathbb{E}$ with signature $(++++)$
- The real Minkowski space $\mathbb{M}$ with signature $(+---)$
- The real ultrahyperbolic space $\mathbb{U}$ with signature $(++--)$

On each of these spaces, the SDYM equations take a slightly different form and so a convenient possibility is to formulate the model in the complex Minkowski space $\mathbb{CM}$ within which each of the above real spaces are embedded. The choice of metric leading to the most simple representation of the SDYM equations on $\mathbb{CM}$ is that of the *double-null coordinates*

$$ds^2 = 2(dzd\tilde{z} - dwd\tilde{w}) \qquad \text{with} \qquad z^m = (z, \tilde{z}, w, \tilde{w}). \tag{1.14}$$

The metric can be read off from this line element as

$$g_{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{1.15}$$

Defining each of the three real spaces in terms of these coordinates is then just a matter of choosing some holomorphic parametrisation of the complex coordinates together with a reality condition which maintains equivalence to the real space in question. In particular;

- $\mathbb{E}$ is given by the reality conditions $\tilde{z} = \bar{z}$, $\tilde{w} = -\bar{w}$ with the parametrisation

$$\begin{pmatrix} z & w \\ \tilde{w} & \tilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + ix^2 & x^3 + ix^4 \\ -(x^3 - ix^4) & x^1 - ix^2 \end{pmatrix}.$$

- $\mathbb{M}$ is given by the reality conditions $z, \tilde{z} \in \mathbb{R}$, $\tilde{w} = \bar{w}$ with the parametrisation

$$\begin{pmatrix} z & w \\ \tilde{w} & \tilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix}.$$

- $\mathbb{U}$ is given by the reality condition $z, \tilde{z}, w, \tilde{w} \in \mathbb{R}$ with the parametrisation

$$\begin{pmatrix} z & w \\ \tilde{w} & \tilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -(x^2 - x^4) & x^1 - x^3 \end{pmatrix}.$$

To this end we can unify the treatment of SDYM theory in terms of the complex double-null coordinates z^m. To begin we choose the gauge group to be a general complex matrix Lie group $\mathcal{G} = GL(N, \mathbb{C})$. Then the complexified version of the Yang-Mills field strength tensor is

$$F_{mn} = [D_m, D_n] = \partial_m A_n - \partial_n A_m + [A_m, A_n], \tag{1.16}$$

where the covariant derivative is defined in a similar way to before $D_m = \partial_m + A_m$. Enforcing the self-dual condition (1.13) on the complexified version of the theory, we find

$$F_{mn} = \frac{1}{2}\epsilon_{mnls}F^{ls}. \tag{1.17}$$

By exhausting the possible combinations of coordinate indices on the left hand side, we arrive at the following representation of the SDYM equations:

$$\begin{aligned} F_{zw} &= \partial_z A_w - \partial_w A_z + [A_z, A_w] = 0, \\ F_{\tilde{z}\tilde{w}} &= \partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0, \\ F_{z\tilde{z}} + F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_w A_{\tilde{w}} - \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] + [A_w, A_{\tilde{w}}] = 0. \end{aligned} \tag{1.18}$$

Equivalently, in terms of the gauge covariant derivative they can be written as

$$\begin{aligned} &[D_z, D_w] = 0, \\ &[D_{\tilde{z}}, D_{\tilde{w}}] = 0, \\ &[D_z, D_{\tilde{z}}] + [D_w, D_{\tilde{w}}] = 0. \end{aligned} \tag{1.19}$$

1.5 The Lax Representation

In the complex representation, the integrability of SDYM theory lies in the fact that (1.19) is equivalent to the condition that the Lax pair of operators

$$\begin{aligned} L &= D_w - \zeta D_{\tilde{z}}, \\ M &= D_z + \zeta D_{\tilde{w}}, \end{aligned} \tag{1.20}$$

exist and should commute for every value of the spectral parameter ζ. Indeed, to prove this we simply compute

$$\begin{aligned} [L, M] &= [D_w - \zeta D_{\tilde{z}}, D_z + \zeta D_{\tilde{w}}] \\ &= [D_w, D_z] + \zeta([D_w, D_{\tilde{w}}] + [D_z, D_{\tilde{z}}]) - \zeta^2 [D_{\tilde{z}}, D_{\tilde{w}}] \\ &= F_{wz} + \zeta(F_{w\tilde{w}} + F_{z\tilde{z}}) - \zeta^2 F_{\tilde{z}\tilde{w}}. \end{aligned}$$

Taking ζ to be arbitrary (but non-zero), the vanishing of this equation gives us the same SDYM equations from before;

$$F_{zw} = 0, \qquad F_{\tilde{z}\tilde{w}} = 0, \qquad F_{z\tilde{z}} + F_{w\tilde{w}} = 0. \tag{1.21}$$

This powerful method of encoding our system in a pair of differential operators most notably allows us to reduce the SDYM equations to a linear system. In particular, for some Lie algebra valued field ϕ, the linear system of equations $L\phi = M\phi = 0$ is invariant under the gauge transformation

$$L \to ULU^{-1}, \qquad M \to UMU^{-1}, \qquad \phi \to U\phi,$$

for some $U \in \mathcal{G}$. In accordance with the procedure introduced by Nimmo-Gilson-Ohta, we can then generalise this gauge invariant system so that the equations are formulated in terms of matrix valued operators and fields just as in Yang-Mills theory. To do this we first let the spectral parameter be an $N \times N$ matrix of complex constants ζ_{ij}. We also require that ζ acts from the right on the matrix valued field ϕ, such that L, M are no longer conventional operators. Surprisingly, this construction still works as we can now write down the linear system

$$\begin{aligned} L(\phi) &\equiv D_w\phi - (D_{\tilde{z}}\phi)\zeta = (\partial_w + A_w)\phi - ((\partial_{\tilde{z}} + A_{\tilde{z}})\phi))\zeta, \\ M(\phi) &\equiv D_z\phi + (D_{\tilde{w}}\phi)\zeta = (\partial_z + A_z)\phi + ((\partial_{\tilde{w}} + A_{\tilde{w}})\phi))\zeta. \end{aligned} \tag{1.22}$$

It is easy to see that (1.21) are recovered by imposing the condition $L(M(\phi)) = M(L(\phi))$ for all $\zeta \neq 0$, and furthermore the system is invariant under gauge transformations (1.5) of the connection.

1.6 Yang's Equation

Before we move on to study soliton solutions of the SDYM equations, we must introduce one more formulation of the equations that make the solutions yet more tractable. By looking at the equations as presented in (1.18), we see that the first two equations are actually local integrability conditions for the existence of two invertible matrices h and $\tilde{h}$ such that

$$\begin{aligned} D_z h &= \partial_z h + A_z h = 0, \\ D_w h &= \partial_w h + A_w h = 0, \\ D_{\tilde{z}}\tilde{h} &= \partial_{\tilde{z}}\tilde{h} + A_{\tilde{z}}\tilde{h} = 0, \\ D_{\tilde{w}}\tilde{h} &= \partial_{\tilde{w}}\tilde{h} + A_{\tilde{w}}\tilde{h} = 0. \end{aligned} \tag{1.23}$$

This means we can express the gauge fields as

$$\begin{aligned} A_z &= -(\partial_z h)h^{-1}, \qquad & A_w &= -(\partial_w h)h^{-1}, \\ A_{\tilde{z}} &= -(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}, \qquad & A_{\tilde{w}} &= -(\partial_{\tilde{w}}\tilde{h})\tilde{h}^{-1}. \end{aligned} \tag{1.24}$$

A very elegant result now follows, if we define *Yang's J-matrix* as $J = h^{-1}\tilde{h}$, then the third SDYM equation is equivalent to *Yang's equation*

$$\partial_{\tilde{z}}(J^{-1}\partial_z J) + \partial_{\tilde{w}}(J^{-1}\partial_w J) = 0. \tag{1.25}$$

Remarkably, this equation is gauge independent by virtue of the J matrix being a gauge invariant quantity. The correspondence between this equation and (1.18) can be verified by a fairly straightforward computation. Indeed, consider the first term which can first be written as

$$J^{-1}\partial_z J = \tilde{h}^{-1}h\big[(\partial_z h^{-1})\tilde{h} + h^{-1}\partial_z\tilde{h}\big].$$

Then using the matrix calculus identity

$$\partial_x M^{-1} = -M^{-1}(\partial_x M)M^{-1}, \tag{1.26}$$

we obtain

$$J^{-1}\partial_z J = \tilde{h}^{-1}h\big[-h^{-1}(\partial_z h)h^{-1}\tilde{h} + h^{-1}\partial_z\tilde{h}\big] = \tilde{h}^{-1}[A_z\tilde{h} + \partial_z\tilde{h}],$$

where (1.24) was used to recognise the gauge field A_z. Now

$$\begin{aligned} \partial_{\tilde{z}}(J^{-1}\partial_z J) &= (\partial_{\tilde{z}}\tilde{h}^{-1})A_z\tilde{h} + \tilde{h}^{-1}\big[(\partial_{\tilde{z}}A_z)\tilde{h} + A_z\partial_{\tilde{z}}\tilde{h}\big] + (\partial_{\tilde{z}}\tilde{h}^{-1})\partial_z\tilde{h} + \tilde{h}^{-1}\partial_{\tilde{z}}\partial_z\tilde{h} \\ &= -\tilde{h}^{-1}(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}A_z\tilde{h} + \tilde{h}^{-1}\big[\partial_{\tilde{z}}A_z - A_zA_{\tilde{z}}\big]\tilde{h} - \tilde{h}^{-1}(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}(\partial_z\tilde{h}) + \tilde{h}^{-1}\partial_z\partial_{\tilde{z}}\tilde{h} \\ &= \tilde{h}^{-1}\big[A_{\tilde{z}}A_z - A_zA_{\tilde{z}} + \partial_{\tilde{z}}A_z\big]\tilde{h} + \tilde{h}^{-1}\big[-(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}(\partial_z\tilde{h})\tilde{h}^{-1} + \partial_z(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}\big]\tilde{h} \\ &= \tilde{h}^{-1}\big[A_{\tilde{z}}A_z - A_zA_{\tilde{z}} + \partial_{\tilde{z}}A_z\big]\tilde{h} + \tilde{h}^{-1}\big[(\partial_{\tilde{z}}\tilde{h})(\partial_z\tilde{h}^{-1}) + \partial_z(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}\big]\tilde{h} \\ &= \tilde{h}^{-1}\big[A_{\tilde{z}}A_z - A_zA_{\tilde{z}} + \partial_{\tilde{z}}A_z + \partial_z\big((\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}\big)\big]\tilde{h} \\ &= \tilde{h}^{-1}\big[A_{\tilde{z}}A_z - A_zA_{\tilde{z}} + \partial_{\tilde{z}}A_z - \partial_zA_{\tilde{z}}\big]\tilde{h} = -\tilde{h}^{-1}F_{z\tilde{z}}\tilde{h}. \end{aligned}$$

When the same calculation is carried out for the second term, we similarly find that

$$\partial_{\tilde{w}}(J^{-1}\partial_w J) = -\tilde{h}^{-1}F_{w\tilde{w}}\tilde{h},$$

and hence, Yang's equation can be rewritten as

$$\partial_{\tilde{z}}(J^{-1}\partial_z J) + \partial_{\tilde{w}}(J^{-1}\partial_w J) = -\tilde{h}^{-1}(F_{z\tilde{z}} + F_{w\tilde{w}})\tilde{h} = 0.$$

With this formulation of the SDYM equations, we are free to study a broad range of interesting solutions depending on the gauge group and metric signature being considered. In this project we wish to explore a class of solutions known as *solitons* which we will encounter in the next section. In the interest of making our work as physically relevant as possible, we will focus on understanding (and visualising) SDYM solitons in some space with a well defined temporal dimension like the Minkowski space $\mathbb{M}$.

2 SDYM Solitons

2.1 Understanding Solitons

In the literature of mathematical physics, it is difficult to find a concrete definition of a soliton. Historically, solitons were first investigated by a Scottish naval engineer John Scott

Russell who observed water waves propagating in a narrow channel in the harbour of Edinburgh. He noticed that the wake of a ship coming to rest in the channel contained some waves which propagated seemingly indefinitely at a strongly stable velocity and amplitude. Following on horseback, he tracked the waves for a couple of miles until they were inevitably dissipated by the windings of the channel. Back at home, Scott Russell created some experiments to replicate the conditions from which he had seen these solitary waves originate and his research gave birth to the study of the soliton as a hydrodynamic phenomenon.

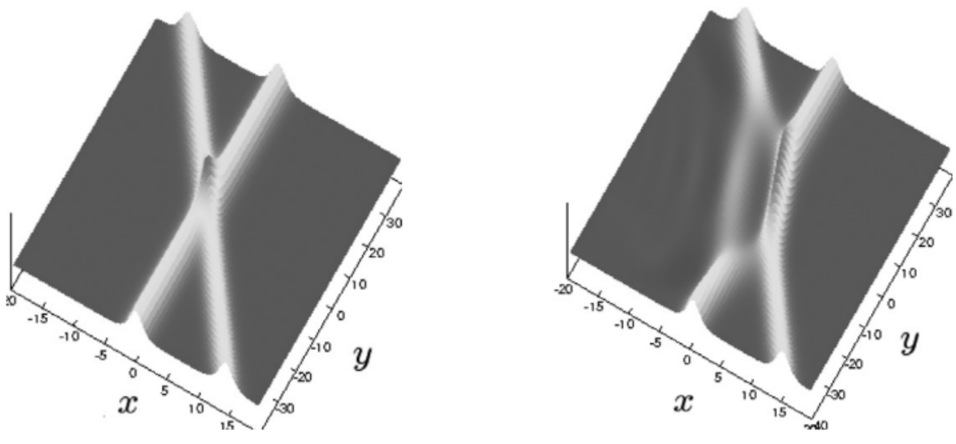
In the time since the work of Scott Russell, solitons have been theoretically classified much more generally (and rigorously) as types of solutions of non-linear partial differential equations which exhibit the defining properties of highly stable localised wavepackets. In particular, it has become apparent that many integrable models of varying dimension and complexity generate soliton solutions. The original problem of describing solitary water waves was solved with the introduction of the KdV equation in the late 19th century

$$4u_t - u_{xxx} - 6uu_x = 0. \tag{2.1}$$

This equation has solutions which successfully model the solitary water waves observed by Scott Russell, they are the soliton solutions given by

$$u(x,t) = 2\kappa^2 \mathrm{sech}^2(\kappa x + \kappa^3 t - \kappa x_0), \tag{2.2}$$

where κ and x_0 are constants which depend on the speed and amplitude of the wave. This solution to the $(1+1)$-dimensional system is able to retain its size and shape as t increases as the effects of the dispersive term and the non-linear term are in balance while regular waves are susceptible to topple or become unstable in the absence of such corrections. A snapshot of two such solutions can be observed in the figure below.



As a higher dimensional generalisation, we also have the KP equation which is the $(2+1)$-dimensional system described by the equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0. \tag{2.3}$$

Once again this system is integrable and it admits exact soliton solutions which can be visualised and understood in a physical context. A remarkable fact is that both of these models (and many others) can be obtained by dimensionally reducing the equations of SDYM theory. In what follows, we will see how soliton solutions can be constructed in SDYM theory itself starting from Yang's equation (1.25). The goal of this project will ultimately be to find a way to understand SDYM solitons as well defined solitary waves propagating through some (possibly abstract) space.

Before proceeding, it will be useful to keep in mind the following set of properties which characterise solitons:

1. Without collision, individual solitons are of permanent form, moving with constant velocity and amplitude over time. That is, each soliton can be described mathematically by some function

$$u(x_i, t) = U(X) \quad \text{where} \quad X \equiv \kappa_i \cdot x_i - \omega t + \delta \tag{2.4}$$

   for some real constants $\kappa_i, \omega, \delta \in \mathbb{R}$.

2. Individual solitons are highly localised within a specific region of space such that their asymptotic behaviour is constant. That is, the soliton solution $U(X)$ satisfies the boundary condition

$$\frac{d^m}{dX^m} U(X) \to 0 \quad \text{as} \quad X \to \pm\infty. \tag{2.5}$$

3. Soliton collisions preserve the properties of each individual soliton involved up to a phase shift. More precisely, if we have a function describing the distribution of n individual solitons

$$u(x_i, t) = U(X_1, X_2, \ldots, X_n), \quad \text{where} \quad X_i = \kappa_i^{(j)} \cdot x_i - \omega^{(j)} t + \delta^{(j)},$$

   then for any given $I \in \{1, 2, \ldots, n\}$ such that

$$\begin{cases} X_I & \text{is a finite real number} \\ X_{i, i \neq I} & \to \pm\infty \end{cases} \quad \text{as} \quad t \to \pm\infty$$

   we have

$$U(X_1, X_2, \ldots, X_n) \overset{t \to \pm\infty}{\to} U_I(X_I + \Delta_I) \tag{2.6}$$

   where Δ_I is the phase shift of the I-th soliton which depends on the 2^{n-1} asymptotic regions of the $n-1$ other solitons.

2.2 Quasideterminants

In the lower dimensional systems of the KdV and KP equations, it is well known that multi-soliton solutions are described by functions containing Wronskian type determinants. Quasideterminants are a non-commutative version of determinants that we will need to be familiar with in order to discuss such solutions for SDYM theory and more generally any non-commutative integrable system. We will first give the definition and then develop some properties of the quasideterminant.

Let X be an $n \times n$ matrix over a noncommutative algebraic structure. Then the (i, j)-th quasideterminant of X is given by

$$|X|_{ij} \equiv x_{i,j} - R_{i,\hat{j}} \left(X_{\hat{i},\hat{j}}\right)^{-1} C_{\hat{i},j} \tag{2.7}$$

where $X_{\hat{i},\hat{j}}$ is the submatrix obtained from X by deleting the i-th row and the j-th column, $R_{i,\hat{j}}$ is the i-th row of X excluding the j-th element and $C_{\hat{i},j}$ is the j-th column of X excluding the i-th element. This definition of the (i, j)-th quasideterminant is invariant under the exchange of the i-th row and j-th column through $x_{i,j}$ with any other column-row combination. That is, the quasideterminant can be written in *canonical form* as

$$|X|_{ij} = \begin{vmatrix} X_{\hat{i},\hat{j}} & C_{\hat{i},j} \\ R_{i,\hat{j}} & \boxed{x_{i,j}} \end{vmatrix}. \tag{2.8}$$

The quasideterminant is also invariant under the operation in which the l-th row or column not containing the box element is added to any other row or column respectively. To prove this, we consider two cases, firstly when neither row (column) contains the box element. For simplicity we label the canonical form of the quasideterminant with block matrices A, B, C, d:

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = \begin{vmatrix} EA & EB \\ C & \boxed{d} \end{vmatrix} = d - C(EA)^{-1}EB = d - CA^{-1}B = \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}$$

where $E = I + M$, with M being the $(n-1) \times (n-1)$ matrix with 1 in the (k,l)-th position and zeroes elsewhere. One should be able to see how the action of E on the upper $n-1$ rows adds the k-th row to the l-th row of the matrix inside the quasideterminant. In the other case, we consider adding the l-th row to the n-th row containing the box element:

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = \begin{vmatrix} A & B \\ FA + C & \boxed{FB + d} \end{vmatrix} = FB + d - (FA + C)A^{-1}B = \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}$$

where F is the row vector with a 1 in the l-th position and zeroes elsewhere. One can check in exactly the same way that the same analysis carries through for the addition of columns rather than rows.

2.3 The Darboux Transformation

In the context of integrable systems, it is often the case that one can use the invariance of a set of equations under some transformation to generate additional exact solutions from a *seed* solution. Indeed, for SDYM theory, such a technique has been devised starting from the linear system (1.22) in the Lax representation. This technique is called the Darboux transformation and it can be understood as follows. Firstly, since Yang's J matrix and hence Yang's equation (1.25) are gauge invariant, we are free to work in the convenient S-gauge which is defined as

$$h = J, \qquad \tilde{h} = I. \tag{2.9}$$

In this gauge, the gauge fields associated with the coordinates $\tilde{z}, \tilde{w}$ vanish while the other gauge fields can be expressed in terms of the J matrix

$$A_{\tilde{z}} = A_{\tilde{w}} = 0, \quad A_z = -(\partial_z J)J^{-1}, \quad A_w = -(\partial_w J)J^{-1}. \tag{2.10}$$

Then, in the S gauge, the linear system of the SDYM Lax representation becomes

$$\begin{aligned} L(\phi) &= (\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta = 0, \\ M(\phi) &= (\partial_z - (\partial_z J)J^{-1})\phi + (\partial_{\tilde{w}}\phi)\zeta = 0. \end{aligned} \tag{2.11}$$

Now if we let J be given and $\phi(\zeta)$ be the general solution of this linear system for arbitrary values of the spectral parameter, and let $\psi(\Lambda)$ be a particular solution of the system for $\zeta = \Lambda$, then the equations are form invariant under the following Darboux transformation:

$$\begin{aligned} \tilde{J} &= -\psi\Lambda\psi^{-1}J = \begin{vmatrix} \psi & 1 \\ \psi\Lambda & \boxed{0} \end{vmatrix} J, \\ \tilde{\phi} &= \phi\zeta - \psi\Lambda\psi^{-1}\phi = \begin{vmatrix} \psi & \phi \\ \psi\Lambda & \boxed{\phi\zeta} \end{vmatrix}. \end{aligned} \tag{2.12}$$

In other words, the Darboux transformed matrices $(\tilde{J}, \tilde{\phi})$ satisfy the equations

$$\begin{aligned}\tilde{L}(\tilde{\phi}) &= (\partial_w - (\partial_w \tilde{J})\tilde{J}^{-1})\tilde{\phi} - (\partial_{\tilde{z}}\tilde{\phi})\zeta = 0,\\ \tilde{M}(\tilde{\phi}) &= (\partial_z - (\partial_z \tilde{J})\tilde{J}^{-1})\tilde{\phi} + (\partial_{\tilde{w}}\tilde{\phi})\zeta = 0.\end{aligned} \tag{2.13}$$

This can be proven by direct calculation, indeed we show that this is true for $\tilde{L}(\tilde{\phi})$ as:

$$\tilde{L}(\tilde{\phi}) = (\partial_w - (\partial_w \tilde{J})\tilde{J}^{-1})\tilde{\phi} - (\partial_{\tilde{z}}\tilde{\phi})\zeta = (\partial_w - \partial_w(HJ)(HJ)^{-1})(\phi\zeta + H\phi) - \partial_{\tilde{z}}(\phi\zeta + H\phi)\zeta,$$

where we have defined the shorthand notation $H = -\psi\Lambda\psi^{-1}$ for convenience. Expanding out derivatives we get

$$\begin{aligned}\tilde{L}(\tilde{\phi}) = {}&(\partial_w\phi)\zeta - (\partial_w H)H^{-1}\phi\zeta - H(\partial_w J)J^{-1}H^{-1}\phi\zeta + (\partial_w H)\phi + H(\partial_w\phi)\\ &- (\partial_w H)\phi - H(\partial_w J)J^{-1}\phi - (\partial_{\tilde{z}}\phi)\zeta^2 - (\partial_{\tilde{z}}H)\phi\zeta - H(\partial_{\tilde{z}}\phi)\zeta.\end{aligned}$$

Two of these terms cancel, and then one may notice that

$$\begin{aligned}\partial_{\tilde{z}}H &= -\partial_{\tilde{z}}(\psi\Lambda\psi^{-1}) = -(\partial_{\tilde{z}}\psi)\Lambda\psi^{-1} + \psi\Lambda\psi^{-1}(\partial_{\tilde{z}}\psi)\psi^{-1}\\ &= -(\partial_w\psi - (\partial_w J)J^{-1}\psi)\psi^{-1} + \psi\Lambda\psi^{-1}(\partial_w\psi - (\partial_w J)J^{-1}\psi)\Lambda^{-1}\psi^{-1}\\ &= -(\partial_w\psi)\psi^{-1} + (\partial_w J)J^{-1} + \psi\Lambda\psi^{-1}\big((\partial_w\psi)\Lambda^{-1}\psi^{-1} - (\partial_w J)J^{-1}\psi\Lambda^{-1}\psi^{-1}\big)\\ &= (\partial_w J)J^{-1} - H(\partial_w J)J^{-1}H^{-1} - \big((\partial_w\psi)\Lambda\psi^{-1} - \psi\Lambda\psi^{-1}(\partial_w\psi)\psi^{-1}\big)\psi\Lambda^{-1}\psi^{-1}\\ &= (\partial_w J)J^{-1} - H(\partial_w J)J^{-1}H^{-1} - (\partial_w H)H^{-1}.\end{aligned}$$

Substituting this expression for the $(\partial_{\tilde{z}}H)$ term in our expression for $\tilde{L}(\tilde{\phi})$ above yields

$$\begin{aligned}\tilde{L}(\tilde{\phi}) &= (\partial_w\phi)\zeta - (\partial_w H)H^{-1}\phi\zeta - H(\partial_w J)J^{-1}H^{-1}\phi\zeta + H(\partial_w\phi) - H(\partial_w J)J^{-1}\phi\\ &\quad - (\partial_{\tilde{z}}\phi)\zeta^2 - \big((\partial_w J)J^{-1} - H(\partial_w J)J^{-1}H^{-1} - (\partial_w H)H^{-1}\big)\phi\zeta - H(\partial_{\tilde{z}}\phi)\zeta\\ &= \big[(\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta\big]\zeta + H\big[(\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta\big]\\ &= L(\phi)\zeta - \psi\Lambda\psi^{-1}L(\phi) = \begin{vmatrix}\psi & L(\phi)\\ \psi\Lambda & \boxed{L(\phi)\zeta}\end{vmatrix}.\end{aligned}$$

An almost identical calculation holds for $\tilde{M}(\tilde{\phi})$. With this result, we see that for every particular solution $\psi_i(\Lambda_i)$ we have of the linear system (2.11), we can generate more solutions by applying the Darboux transformation to (J, ϕ). The solution generated by i iterations of this transformation is expressed in terms of a quasideterminant of order $i-1$.

2.4 Generating Solutions of the SDYM Equations

To apply the method of Darboux transformations to generate solutions of the SDYM equations, we first note that in the S-gauge (2.9), Yang's equation takes the form

$$\partial_{\tilde{z}}\big((\partial_z J)J^{-1}\big) + \partial_{\tilde{w}}\big((\partial_w J)J^{-1}\big) = 0. \tag{2.14}$$

To generate a solution using one iteration of the Darboux transformation, let J_1 be an initial J matrix which defines the initial linear system

$$\begin{aligned}L_1(\phi) &= (\partial_w - (\partial_w J_1)J_1^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta = 0,\\ M_1(\phi) &= (\partial_z - (\partial_z J_1)J_1^{-1})\phi + (\partial_{\tilde{w}}\phi)\zeta = 0.\end{aligned} \tag{2.15}$$

In this case, we call J_1 a seed solution. After solving this system for a number of specified values of the spectral parameter Λ_i, we are left with a set of particular solutions $\psi_i(\Lambda_i)$ where $i = 1, 2, \ldots, n$. In what follows we will use Φ_i to denote the general solution of the linear system with respect to the matrix J_i, and Ψ_i to denote the particular solution of that system which is by definition given by

$$\Psi_i = \Phi_i\big|_{(\phi,\zeta)\to(\psi_i,\Lambda_i)}. \tag{2.16}$$

Now we apply the Darboux transformation to the seed solution of the initial linear system which yields

$$\begin{aligned} J_2 = \tilde{J}_1 &= -\Psi_1\Lambda_1\Psi_1^{-1}J_1 = \begin{vmatrix} \psi_1 & 1 \\ \psi_1\Lambda_1 & \boxed{0} \end{vmatrix}, \\ \Phi_2 = \tilde{\Phi}_1 &= \Phi_1\zeta - \Psi_1\Lambda_1\Psi_1^{-1}\Phi_1 = \begin{vmatrix} \psi_1 & \phi \\ \psi_1\Lambda_1 & \boxed{\phi\zeta} \end{vmatrix}. \end{aligned} \tag{2.17}$$

The matrices (J_2, Φ_2) satisfy the linear system of equations

$$\begin{aligned} L_2(\Phi_2) &= (\partial_w - (\partial_w J_2)J_2^{-1})\Phi_2 - (\partial_{\tilde{z}}\Phi_2)\zeta = 0, \\ M_2(\Phi_2) &= (\partial_z - (\partial_z J_2)J_2^{-1})\Phi_2 + (\partial_{\tilde{w}}\Phi_2)\zeta = 0. \end{aligned} \tag{2.18}$$

To perform a second iteration, we must first use the definition (2.16) of Ψ_i to obtain a particular solution to the linear system (2.18), this gives

$$\Psi_2 = \Phi_2\big|_{(\phi,\zeta)\to(\psi_2,\Lambda_2)} = \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix}. \tag{2.19}$$

Now we apply the Darboux transformation once more to the second linear system, we have

$$\begin{aligned} J_3 &= \tilde{J}_2 = -\Psi_2\Lambda_2\Psi_2^{-1}J_2 \\ &= -\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix} \Lambda_2 \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix}^{-1} J_2 = \begin{vmatrix} \psi_1 & \psi_2 & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & 0 \\ \psi_1\Lambda_1^2 & \psi_2\Lambda_2^2 & \boxed{0} \end{vmatrix} J_1, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \Phi_3 &= \tilde{\Phi_2} = \Phi_2\zeta - \Psi_2\Lambda_2\Psi_2^{-1}\Phi_2 \\ &= \begin{vmatrix} \psi_1 & \phi \\ \psi_1\Lambda_1 & \boxed{\phi\zeta} \end{vmatrix}\zeta - \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix}\Lambda_2 \begin{vmatrix} \psi_1 & \psi_2 \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix}^{-1} \begin{vmatrix} \psi_1 & \phi \\ \psi_1\Lambda_1 & \boxed{\psi_2\Lambda_2} \end{vmatrix} \\ &= \begin{vmatrix} \psi_1 & \psi_2 & \phi \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \psi\zeta \\ \psi_1\Lambda_1^2 & \psi_2\Lambda_2^2 & \boxed{\phi\zeta^2} \end{vmatrix}. \end{aligned} \tag{2.21}$$

After this second iteration, the transformed matrices (J_3, Φ_3) satisfy the linear system

$$\begin{aligned} L_3(\Phi_3) &= (\partial_w - (\partial_w J_3)J_3^{-1})\Phi_3 - (\partial_{\tilde{z}}\Phi_3)\zeta = 0, \\ M_3(\Phi_3) &= (\partial_z - (\partial_z J_3)J_3^{-1})\Phi_3 + (\partial_{\tilde{w}}\Phi_3)\zeta = 0. \end{aligned} \tag{2.22}$$

One can continue to repeat this procedure for each particular solution $\psi_i(\Lambda_i)$ of the initial linear system corresponding to the seed solution J_1. The most general form of the solutions are captured by the following.

Let J_1 be a given seed solution of the linear system (2.15) for which $\psi_i(\Lambda_i)$, $i = 1, \ldots, n$ are particular solutions with respect to the spectral parameters $\zeta = \Lambda_i$. Then under the

Darboux transformation

$$
\begin{aligned}
J_{k+1} &\equiv -\Psi_k \Lambda_k \Psi_k^{-1} J_k = \begin{vmatrix} \Psi_k & 1 \\ \Psi_k \Lambda_k & \boxed{0} \end{vmatrix} J_k, \\
\Phi_{k+1} &\equiv \Phi_k \zeta - \Psi_k \Lambda_k \Psi_k^{-1} \Phi_k = \begin{vmatrix} \Psi_k & \Phi_k \\ \Psi_k \Lambda_k & \boxed{\Phi_k \zeta} \end{vmatrix},
\end{aligned}
\tag{2.23}
$$

where

$$
\Phi_1 \equiv \phi, \qquad \Psi_k \equiv \Phi_k \big|_{(\phi,\zeta)\to(\psi_k,\Lambda_k)}, \tag{2.24}
$$

the J matrix J_{n+1} and general solution Φ_{n+1} obtained after n iterations can be written as

$$
\begin{aligned}
J_{n+1} &= \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0} \end{vmatrix} J_1, \\
\Phi_{n+1} &= \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & \phi \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & \phi\zeta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{\phi\zeta^n} \end{vmatrix}.
\end{aligned}
\tag{2.25}
$$

The proof of this result follows from an inductive argument based on the iterations we have presented above. The procedure is now clear, we can take some simple seed solution J_1 and solve the resulting linear system to obtain a set of particular solutions $\psi_i(\Lambda_i)$ for values of the spectral parameter $\zeta = \Lambda_i$. Then we can apply the Darboux transformation iteratively to generate matrices $J_2, J_3, \ldots, J_{n+1}$ which are necessarily solutions of Yang's equation (2.14). In the following section we will see how this framework allows us to construct solutions of the SDYM equations which describe one or more solitons.

2.5 SDYM Solitons

Before we test whether soliton solutions can be generated, it is necessary to state the conditions which some J matrix must satisfy in order for it to be considered a soliton solution. In particular, we call a J matrix a SDYM 1-soliton solution if the resulting Lagrangian density $\mathrm{tr} F_{\mu\nu} F^{\mu\nu}$ takes the form of the function

$$
U(X) = \sum_k c_k \mathrm{sech}^k X, \tag{2.26}
$$

where X is a non-homogeneous function of the spacetime coordinates. Furthermore, let $\{J_1, J_2, \ldots, J_n\}$ be 1-soliton solutions each with a Lagrangian density taking the above form. We call some matrix $\tilde{J}$, a SDYM n-soliton if its Lagrangian density takes the form $U(X_1, X_2, \ldots, X_n)$ and satisfies the condition that for any given $I \in \{1, 2, \ldots, n\}$ such that

$$
\left\{ \begin{array}{ll} X_I & \text{is a finite real number} \\ X_{i, i\neq I} & \to \pm\infty \end{array} \right. \quad \text{as} \quad t \to \pm\infty
$$

we have

$$
U(X_1, X_2, \ldots, X_n) \overset{t\to\pm\infty}{\to} U_I(X_I + \Delta_I) \tag{2.27}
$$

where Δ_I is the phase shift of the I-th soliton which depends on the 2^{n-1} asymptotic regions of the $n-1$ other solitons.

Now let's construct some solutions of the SDYM equations which satisfy this criteria under the gauge group $\mathcal{G} = GL(2, \mathbb{C})$. First, consider the seed solution $J_1 = I$ in the S-gauge (2.9). For this choice of J_1, the linear system (2.15) reduces to a much simpler form:

$$\begin{aligned} \partial_w \phi - (\partial_{\tilde{z}} \phi)\zeta &= 0, \\ \partial_z \phi + (\partial_{\tilde{w}} \phi)\zeta &= 0. \end{aligned} \tag{2.28}$$

If we choose the spectral parameter to be the diagonal matrix $\zeta = \Lambda = \mathrm{diag}(\lambda, \mu)$ for some complex constants λ, μ, then the resulting system is solved immediately by the ansatz: