Solitons in Self-Dual Yang-Mills Theory
Supervised by Tristan McLoughlin

Matthew Dunne

Trinity College Dublin

May 15, 2025

Outline

Gauge Theory

- A gauge theory is one in which the interactions are introduced by promoting global symmetries to local (gauge) symmetries

- For Yang-Mills theory, we take the set of transformations which leave the theory invariant to belong to some compact, semi-simple gauge group $\mathcal{G}$

- Quantities relevant for describing physics are fields and their derivatives, but derivatives cause a problem

Gauge Theory

- A gauge theory is one in which the interactions are introduced by promoting global symmetries to local (gauge) symmetries
- For Yang-Mills theory, we take the set of transformations which leave the theory invariant to belong to some compact, semi-simple gauge group $\mathcal{G}$
- Quantities relevant for describing physics are fields and their derivatives, but derivatives cause a problem

Gauge Theory

- A gauge theory is one in which the interactions are introduced by promoting global symmetries to local (gauge) symmetries
- For Yang-Mills theory, we take the set of transformations which leave the theory invariant to belong to some compact, semi-simple gauge group $\mathcal{G}$
- Quantities relevant for describing physics are fields and their derivatives, but derivatives cause a problem

Covariant Derivative

- We want everything to transform covariantly under local gauge transformations but the ordinary derivative ∂_μ does not

- Introduce a new field A_μ which plays the role of a connection in the gauge covariant derivative

$$D_\mu = \partial_\mu + A_\mu$$

- The degree to which these covariant derivatives fail to commute is measured by the field strength tensor

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Covariant Derivative

- We want everything to transform covariantly under local gauge transformations but the ordinary derivative ∂_μ does not

- Introduce a new field A_μ which plays the role of a connection in the gauge covariant derivative

$$D_\mu = \partial_\mu + A_\mu$$

- The degree to which these covariant derivatives fail to commute is measured by the field strength tensor

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Covariant Derivative

- We want everything to transform covariantly under local gauge transformations but the ordinary derivative ∂_μ does not

- Introduce a new field A_μ which plays the role of a connection in the gauge covariant derivative

$$D_\mu = \partial_\mu + A_\mu$$

- The degree to which these covariant derivatives fail to commute is measured by the field strength tensor

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Yang-Mills Theory

- The pure, gauge invariant Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\text{tr}(F^{\mu\nu}F_{\mu\nu})$$

- By employing Hamilton's principle $\delta S = 0$ we arrive at the Euler Lagrange equation

$$D_\mu F^{\mu\nu} = 0$$

Yang-Mills Theory

- The pure, gauge invariant Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\text{tr}(F^{\mu\nu}F_{\mu\nu})$$

- By employing Hamilton's principle $\delta S = 0$ we arrive at the Euler Lagrange equation

$$D_\mu F^{\mu\nu} = 0$$

Yang-Mills Equations

■ We can also define the Hodge dual of the field strength as

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

■ Using the Jacobi identity for D_μ, one can derive the Bianchi identity

$$D_\mu(\star F)^{\mu\nu} = 0$$

■ Collectively, this identity and the field equation $D_\mu F^{\mu\nu} = 0$ are known as Yang-Mills equations

Yang-Mills Equations

- We can also define the Hodge dual of the field strength as

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

- Using the Jacobi identity for D_μ, one can derive the Bianchi identity

$$D_\mu(\star F)^{\mu\nu} = 0$$

- Collectively, this identity and the field equation $D_\mu F^{\mu\nu} = 0$ are known as Yang-Mills equations

Yang-Mills Equations

- We can also define the Hodge dual of the field strength as

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

- Using the Jacobi identity for D_μ, one can derive the Bianchi identity

$$D_\mu(\star F)^{\mu\nu} = 0$$

- Collectively, this identity and the field equation $D_\mu F^{\mu\nu} = 0$ are known as Yang-Mills equations

Self-Duality

- Solutions of the Yang-Mills equations are almost impossible to find

- One way to turn this into an integrable model is to impose the self-duality condition on the field strength tensor

$$F^{\mu\nu} = \star F^{\mu\nu}$$

- When true, this guarantees both equations hold by the Bianchi identity and so it is sufficient to consider only solutions of the self-dual condition

- G Abelian / non-abelian $\implies$ Linear / Non-Linear self-dual Yang-Mills (SDYM) equations

Self-Duality

- Solutions of the Yang-Mills equations are almost impossible to find
- One way to turn this into an integrable model is to impose the self-duality condition on the field strength tensor

$$F^{\mu\nu} = \star F^{\mu\nu}$$

- When true, this guarantees both equations hold by the Bianchi identity and so it is sufficient to consider only solutions of the self-dual condition

- $\mathcal{G}$ Abelian / non-abelian $\implies$ Linear / Non-Linear self-dual Yang-Mills (SDYM) equations

Self-Duality

- Solutions of the Yang-Mills equations are almost impossible to find
- One way to turn this into an integrable model is to impose the self-duality condition on the field strength tensor

$$F^{\mu\nu} = \star F^{\mu\nu}$$

- When true, this guarantees both equations hold by the Bianchi identity and so it is sufficient to consider only solutions of the self-dual condition

- $\mathcal{G}$ Abelian / non-abelian $\implies$ Linear / Non-Linear self-dual Yang-Mills (SDYM) equations

Self-Duality

- Solutions of the Yang-Mills equations are almost impossible to find
- One way to turn this into an integrable model is to impose the self-duality condition on the field strength tensor

$$F^{\mu\nu} = \star F^{\mu\nu}$$

- When true, this guarantees both equations hold by the Bianchi identity and so it is sufficient to consider only solutions of the self-dual condition
- $\mathcal{G}$ Abelian / non-abelian $\implies$ Linear / Non-Linear self-dual Yang-Mills (SDYM) equations

Significance

- What is so special about the SDYM equation?

- It has been shown that it can be reduced to a vast number of integrable models

- These reductions are achieved by choosing some particular gauge group $\mathcal{G}$ or imposing additional symmetries on the system

Significance

- What is so special about the SDYM equation?
- It has been shown that it can be reduced to a vast number of integrable models
- These reductions are achieved by choosing some particular gauge group $\mathcal{G}$ or imposing additional symmetries on the system

Significance

- What is so special about the SDYM equation?
- It has been shown that it can be reduced to a vast number of integrable models
- These reductions are achieved by choosing some particular gauge group $\mathcal{G}$ or imposing additional symmetries on the system

Examples

Some notable reductions of the SDYM equation are:

- The KdV equation
$$4u_t - u_{xxx} - 6uu_x = 0$$

- The sine-Gordon equation
$$\partial_t^2 \phi - \partial_x^2 \phi = \sin \phi$$

- The KP equation ($2 + 1$-dimensional)
$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

Where subscripts denote partial derivatives

Solitons

Solitons are wave-like solutions to PDEs with a few specific properties

- Without collision, an individual soliton is of permanent form with constant velocity and amplitude over time
- Localised within some region $\iff$ decaying to a constant at infinity
- After soliton collisions, the velocity, amplitude, and shape of each soliton are preserved.

Solitons

Solitons are wave-like solutions to PDEs with a few specific properties

- Without collision, an individual soliton is of permanent form with constant velocity and amplitude over time
- Localised within some region $\iff$ decaying to a constant at infinity
- After soliton collisions, the velocity, amplitude, and shape of each soliton are preserved.

Solitons

Solitons are wave-like solutions to PDEs with a few specific properties

- Without collision, an individual soliton is of permanent form with constant velocity and amplitude over time
- Localised within some region $\iff$ decaying to a constant at infinity
- After soliton collisions, the velocity, amplitude, and shape of each soliton are preserved.

Solitons

- As solutions to non-linear PDEs, the exact analytical form of solitons can be difficult to obtain

- Integrable models like SDYM allow explicit solutions to be found due to so-called *infinite hidden symmetries*

- For lower dimensional cases, these solutions are much more simple to obtain and visualise
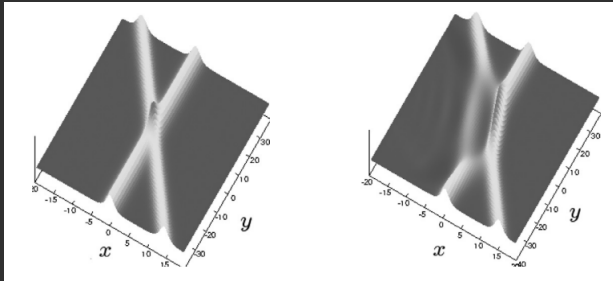
Solitons

- As solutions to non-linear PDEs, the exact analytical form of solitons can be difficult to obtain
- Integrable models like SDYM allow explicit solutions to be found due to so-called *infinite hidden symmetries*
- For lower dimensional cases, these solutions are much more simple to obtain and visualise

Solitons

- As solutions to non-linear PDEs, the exact analytical form of solitons can be difficult to obtain

- Integrable models like SDYM allow explicit solutions to be found due to so-called *infinite hidden symmetries*

- For lower dimensional cases, these solutions are much more simple to obtain and visualise

Solitons

A superposition of two KdV solitons

Developments

- For the case of the SDYM and KP equations we will consider recent work in this project which applies mathematical methods for constructing soliton solutions on both real and complex spaces.