Quantisation of Constrained Systems

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March 27, 2025

Outline

Constrained Hamiltonian Systems

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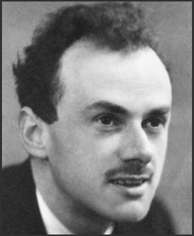

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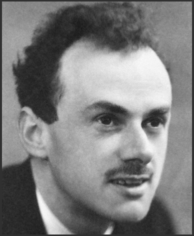
- "Lectures on Quantum Mechanics"[1] provides a recipe for dealing with general constraints



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Constrained Hamiltonian Systems

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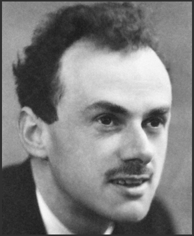
- "Lectures on Quantum Mechanics"[1] provides a recipe for dealing with general constraints
- Most importantly for this project, the concept of gauge freedom appears more naturally in the Hamiltonian formulation



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- When not all of the momentum variables are independent we also get a set of constraints

$$G_a(q, p) = 0 \qquad a = 1, \ldots, M$$

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- The physical trajectories must be consistent with the constraints so they are confined to the $(2N - M)$-dimensional subspace called the physical phase space
- In the remaining unphysical sector of phase space, the constraints do not necessarily vanish.
- For clarity one uses the weak equality $G_a \approx 0$ as a reminder that the constraints only vanish on the physical phase space.

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- Historically implemented using the Dirac bracket, a method which this project avoids

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- Dirac conjecture: All first class constraints generate gauge transformations

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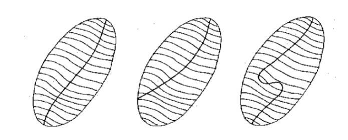
- Theories which we are very interested in quantising
- How do we resolve our redundant description of theories with gauge freedom?

Gauge Fixing

- Introduce gauge fixing conditions $F^a(q,p) = 0$



A good choice of gauge and two bad ones [2].

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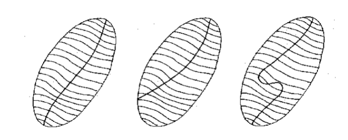

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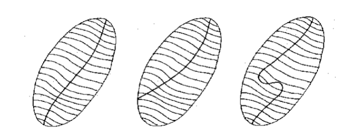
- Constraints and gauge fixing conditions together form a set of second class constraints!



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- It is difficult to verify that a gauge choice is a good one (Gribov ambiguity)
- Even when successful, the standard treatment of second class constraints involving the explicit removal of gauge degrees of freedom can cause essential properties like Lorentz invariance or locality to be lost
- We instead develop methods which make use of some extended symmetry of the gauge phase space in order to obtain a gauge invariant dynamical description

Dynamics over a Grassmann Algebra

- Anti-commuting variables θ^i are odd elements of a Grassmann algebra which satisfy

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- One may have a dynamical system with N commuting (bosonic) variables $q^i(t)$ and K anti-commuting (fermionic) variables $\theta^\alpha(t)$
- Both types of variables and functions of them can be taken as elements of the same Grassmann algebra

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- Ghosts have properties that can be exploited to perform gauge fixing without eliminating unphysical degrees of freedom at intermediate stages of analysis.
- Famously Fadeev-Popov ghosts were used to obtain a gauge fixed path-integral for Yang-Mills
- In my project I am exploring the application of one such method called Parisi-Sourlas quantisation[3]

Parisi-Sourlas Formalism

- Suppose that we want to formulate a path-integral for a theory of N dynamical variables with 2 constraints without explicitly reducing the action to a lower dimensional form. Then by analytic continuation we can imagine writing the reduced path integral in the equivalent form

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- The "negative dimensional coordinates" of the above integral can then be realised by anti-commuting variables

Ghosts as Negative Dimensional Coordinates: Parisi-Sourlas Supersymmetry

References

P.A.M. Dirac.
Lectures on Quantum Mechanics.
Belfer Graduate School of Science, monograph series. Dover Publications, 2001.

Bengtsson. I.
Constrained hamiltonian systems.
U. Stockholm, Unknown Year.

Antti J. Niemi.
Pedagogical Introduction to BRST.
Phys. Rept., 184:147–165, 1989.

Thank You!