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Theoretical Physics Capstone Project Report

Parisi-Sourlas Quantisation of Constrained Systems

Matthew Dunne

Supervised by Chaolun Wu

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Abstract

In this project, a method to obtain covariant phase space path-integrals for constrained Hamiltonian systems was introduced and applied to a non-abelian gauge theory. For completeness, a pedagogical introduction to constrained Hamiltonian systems was presented in the language of Dirac [1]. The focus was then brought to systems with first class constraints and their associated gauge symmetry. This was supplemented with a discussion of the BRST formalism which is perhaps the most widely used approach for dealing with non-trivial first class constrained systems. The specific example of pure Yang-Mills theory was examined in both the Lagrangian and Hamiltonian formulation. The notably successful Lagrangian path-integral of Faddeev and Popov [2] was explained in the context of the BRST formalism. The Parisi-Sourlas formalism [3, 4] was then introduced as an alternative method to obtain path-integrals in the Hamiltonian formulation of constrained systems. The method was applied to pure Yang-Mills theory by extending a part of the phase space into a superspace and then constructing a representation of the unphysical sector in terms of superfields. Using this method, a covariant path-integral was derived in which the Gauss law constraints were implemented in the Coulomb gauge while respecting the pertinent Parisi-Sourlas supersymmetry.

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1 Constrained Hamiltonian Systems

To begin we shall introduce the Hamiltonian formulation of systems with constraints which is accredited independently to Dirac [1] and Bergmann [5]. In their independent endeavours they derived an algorithm for finding, classifying, and implementing constraints in the canonical formalism. In the context of this project, the focus will be on gauge theories which are perhaps the most important type of constrained Hamiltonian system. An abundance of physical theories enjoy the property of gauge symmetry and the quantisation of these theories has been a cumbersome undertaking in modern theoretical physics.

1.1 The Emergence of Constraints

For any classical system described by a Lagrangian $L(q, \dot{q})$ with generalised coordinates $q^i(t)$, $i = 1, \ldots, N$, the action can be written in the form

$$S[q, \dot{q}] = \int_{t_1}^{t_2} dt L(q, \dot{q}). \tag{1.1}$$

The canonical momenta conjugate to each q^i are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{1.2}$$

Hamilton's principle of least action, stating that any variation $\delta q^i(t)$ of classical trajectories that vanishes at the times t_1, t_2 leaves the action unchanged, can only be true if the N Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{1.3}$$

are satisfied. Any of these equations can be written in the equivalent form

$$\frac{d^2 q^j}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} - \frac{dq^j}{dt}\left(\frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\right). \tag{1.4}$$

From a dynamical viewpoint, this extended equation states that the accelerations are determined uniquely at all times by the positions and velocities if and only if the matrix on the left hand side is invertible, or equivalently:

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) \neq 0. \tag{1.5}$$

This matrix of second order partial derivatives of the Lagrangian with respect to the generalised coordinates is called the Hessian matrix. Here, and in what follows, the case where this determinant vanishes is the most relevant. In this case the accelerations are *not* uniquely determined by the positions and velocities. A Lagrangian with this property is said to be degenerate. The Euler-Lagrange equations for a degenerate system are not all independent, they can contain arbitrary functions of time indicating that there is some redundancy in our description of the dynamics. Furthermore it is easy to see that if the determinant (1.5) was to vanish, then the defining equations for the conjugate momenta would not be invertible so that locally we could not solve their defining equation (1.2) for the velocities $\dot{q}^i = \dot{q}^i(q,p)$. If the rank of the Hessian matrix is $N - M$, then computing the canonical momenta leads to a set of M *primary constraints*

$$G_a = G_a(q,p) = 0, \qquad a = 1, \ldots, M. \tag{1.6}$$

These constraints do not obstruct the Legendre transformation in passing to the canonical formalism however they do require specific treatment in order for the inverse transformation to be possible.

1.2 The Canonical Formalism

In the canonical formalism, the dynamics of the classical system are described by the generalised positions and momenta (q^i, p_i) of a $2N$-dimensional phase space. A Legendre transformation of the action in the Lagrangian formalism yields the canonical action

$$S[q,p] = \int_{t_1}^{t_2} dt \left(\sum_i p_i \dot{q}^i - H(q,p) \right). \tag{1.7}$$

Henceforth, the summation will be replaced by the usual summation convention of contracted indices. Under extremisation, the action remains stationary if Hamilton's equations are satisfied

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}. \tag{1.8}$$

The Poisson bracket of two phase space functions $f(q,p)$ and $g(q,p)$ is defined as

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{1.9}$$

From this definition, it is easy to see that the time evolution of any function on phase space is given by its Poisson bracket with the Hamiltonian

$$\dot{f} = \{f, H\}. \tag{1.10}$$

For this reason, it is typically said that the Hamiltonian *generates* the time evolution of the system. The Poisson bracket is clearly anti-symmetric and it satisfies the Leibniz rule

$$\{fg, h\} = \{f,h\}g + f\{g,h\}, \tag{1.11}$$

and the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \tag{1.12}$$

The full phase space spanned by the $2N$ variables (q^i, p_i) shall henceforth be referred to as the *ambient phase space*. Phase space is a symplectic manifold as is evident in the definition of the Poisson bracket and the algebraic structure of the canonical variables

$$\{q^i, p_j\} = \delta^i{}_j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0. \tag{1.13}$$

The ambient phase space is said to possess a *flat* symplectic structure, meaning that it has as its metric is the symplectic two-form

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}. \tag{1.14}$$

Upon performing the Legendre transformation to the canonical formalism, the constraints (1.6) are understood to define a $(2N - M')$-dimensional smoothly embedded submanifold of the ambient phase space where M' is the number of independent primary constraints among the full set. A point on this so-called *primary constraint surface* has a multi-valued inverse image when mapped to the $2N$-dimensional configuration space spanned by $(q, \dot{q})$. This motivates the introduction of a set of M' Lagrange multipliers $\lambda^a(q, p)$.

To see how Lagrange multipliers are introduced to restore the invertibility of the Legendre transformation, we note that a variation to the canonical Hamiltonian $H_0 = \dot{q}^i p_i - L$ results in the expression

$$\delta H_0 = \dot{q}^i \delta p_i + \delta \dot{q}^i p_i - \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - \delta q^i \frac{\partial L}{\partial q^i} = \dot{q}^i \delta p_i - \delta q^i \frac{\partial L}{\partial q^i}. \tag{1.15}$$

Expanding the left hand side and matching coefficients of δq^i and δp_i,

$$\left(\frac{\partial H_0}{\partial q^i} + \frac{\partial L}{\partial q^i}\right)\delta q^i + \left(\frac{\partial H_0}{\partial p_i} - \dot{q}^i\right)\delta p_i = 0. \tag{1.16}$$

In the case that the arbitrary variations are tangent to the primary constraint surface, the second term in brackets has the solution

$$\dot{q}^i(q, p) = \frac{\partial H_0}{\partial p_i} + \lambda^a \frac{\partial G_a}{\partial p_i}. \tag{1.17}$$

where $\lambda^a(q, p)$ are an additional set of M' variables. It is now clear that by introducing the λ^a the invertibility of the Legendre transformation is restored despite the vanishing of the determinant (1.5) at the cost of adding extra variables to the phase space. The variations δp_i themselves must satisfy the constraints (1.6) so it must be the case that the Hamiltonian is only well-defined on the primary constraint surface up to arbitrary extensions proportional to the constraints. Indeed, when one extends the original Hamiltonian H_0 in accordance with the variational equation (1.16), the action becomes

$$S[q^i, p_i, \lambda^a] = \int_{t_1}^{t_2} dt\ (p_i \dot{q}^i - H_0 - \lambda^a G_a) \tag{1.18}$$

with the equations of motion

$$\begin{aligned} \dot{p}_i &= \{p_i, H_0\} + \lambda^a \{p_i, G_a\}, \\ \dot{q}^i &= \{q^i, H_0\} + \lambda^a \{q^i, G_a\}, \\ G_a(q,p) &= 0. \end{aligned} \tag{1.19}$$

The additional variables introduced to restore the invertibility of the Legendre transformation now act as Lagrange multipliers which enforce the primary constraints. Now let's see what lies beyond primary constraints.

A further requirement that is imposed upon the primary constraints is that each of them be preserved throughout the time evolution of the system. Each $G_a(q,p)$ must then obey the following *consistency condition*

$$\frac{dG_a}{dt} = \{G_a, H_0\} + \lambda^b \{G_a, G_b\} = 0. \tag{1.20}$$

This equation can either lead to restrictions on the Lagrange multipliers or a new relation between the phase space variables. If, in the latter case, the relation is not already covered by the primary constraints then it defines a secondary constraint. In practice, a repeated application of this algorithm (adding each new constraint to the Hamiltonian before checking consistency) until all new relations have been obtained completes the full set of $K \geq M$ constraints on a Hamiltonian system:

$$G_a(q,p) = 0, \qquad a = 1, \ldots, K. \tag{1.21}$$

In the case that checking consistency restricts the value of a Lagrange multiplier then the constraint being checked is what will soon be introduced as a second class constraint. In a similar fashion to before, the complete set of constraints specify a $(2N - K)$-dimensional submanifold smoothly embedded in the ambient phase space on which the physical trajectories lie - the physical phase space. Equipped with a further $K - M'$ Lagrange multipliers, the *total Hamiltonian* is written as

$$H = H_0 + \lambda^a G_a, \qquad a = 1, \ldots, K \tag{1.22}$$

1.3 First and Second Class Constraints

For a constrained Hamiltonian system with a set of K constraints as in (1.21), it is emphasised that the relations are only constrained to vanish on the physical phase space by means of a weak equality symbol ($\approx$). This notation, attributed to the Dirac formalism [1], leads to the constraint equations being more appropriately written as

$$G_a(q,p) \approx 0. \tag{1.23}$$

Essentially this makes the idea of computing Poisson brackets of quantities which are equal to zero more comfortable, with the knowledge that they can be set to zero after the theory is set up. Constraints can be classified as first or second class based on the following definition. A function $f(q,p)$ defined on the ambient phase space is *first class* if

$$\{f, G_a\} \approx 0 \tag{1.24}$$

for all $a = 1, \ldots, K$. A function is *second class* if it is not first class, in other words, if it has a non-weakly vanishing Poisson bracket with any of the K constraint functions. Given any system one computes the constraint matrix $C_{ab} = \{G_a, G_b\}$, the rank R of this matrix on the physical subspace will be equal to the number of second class constraints meaning there are a further $K - R$ first class constraints present.

The second class constraints form a full set of functions with non-vanishing Poisson brackets such that their constraint matrix is non-singular

$$\{G_a, G_b\} \neq 0. \tag{1.25}$$

When Poisson brackets take this form they are said to define a structure. Since the constraint matrix does not vanish, there is no ambiguity in writing a set of R second class constraints as the strong equations

$$G_a = G_a(q,p) = 0, \qquad a = 1, \ldots, R \tag{1.26}$$

In most cases, the structure of the ambient phase space defined by the ordinary Poisson brackets is inconsistent with the structure defined by the second class constraints. That is to say, the second class constraints induce a symplectic two-form on the physical phase space that does not agree with the ambient version (1.14). In the Dirac formalism, one resolves this inconsistency by replacing the ordinary Poisson brackets with the *Dirac brackets*, defined by the equation

$$\{f, g\}_D = \{f, g\} - \{f, G_a\} C^{-1}_{ab} \{G_b, g\}. \tag{1.27}$$

From this definition, it is clear that the operation $\{*, *\}_D$ defines an induced symplectic two-form for which the unphysical (constraint) directions are projected out. It can easily shown that this bracket reproduces the correct time evolution of the system through the Hamiltonian. There are no more subtleties to do with the second class constraints, other than the fact that the constraint matrix, and hence the induced symplectic two-form, can be arbitrarily complicated. In practice, when both first and second class constraints emerge from an application of the Dirac-Bergmann algorithm, one initially ignores the first class constraints. They are then treated only after implementing the second class constraints using the Dirac bracket.

In this project we will be analysing systems for which the constraint matrix is made up of, for instance, derivative operators. It should be noted that in cases like this the Dirac bracket is usually not the most straightforward approach. Indeed, the methods we shall introduce bypass the notion of the Dirac bracket entirely.

1.4 Gauge Symmetry

Suppose the canonical formalism has been established for a constrained Hamiltonian system such that the full set of K constraints $G_a \approx 0$ are first class. In this case, the Poisson brackets between constraints take the general form

$$\{G_a, G_b\} = U_{ab}{}^{c} G_c \approx 0 \tag{1.28}$$

where $U_{ab}{}^{c}$ are called *structure functions*, possibly depending on the variables (q, p). From this equation we recognise that a set of first class constraints define an algebra rather than a structure. In addition, the consistency requirement takes the form

$$\{G_a, H\} = V_a{}^{b} G_b \approx 0. \tag{1.29}$$

This highlights the important result that, in a first class system, the consistency conditions do not restrict the Lagrange multipliers $\lambda^a(q, p)$. However, since these functions appear in the equations of motion (1.19) it follows that our description of the system's dynamics is left partly arbitrary. At any given time, a physical state is represented by a point on the constraint surface (these points are referred to as being on-shell). But when the equations of motion contain arbitrary functions of the phase space variables, a physical state is represented by an equivalence class of points on the constraint surface rather than a single unique point. These equivalence classes are more commonly referred to as gauge orbits and the associated local symmetry of a first class Hamiltonian system is called the gauge symmetry. The transformation which relates a pair of equivalent points on a gauge orbit is called a gauge transformation. Therefore, gauge transformations can be viewed as transformations which relate distinct values of the Lagrange multipliers. This is the viewpoint from which first class constrained systems are identified with gauge theories.

Where does this leave us then, with regards to describing physical dynamics? Firstly, the equations of motion should describe the same physics no matter what arbitrary value the Lagrange multipliers take on the gauge orbit. For this reason we say that the physical description of the theory is *gauge invariant*. This suggests that the responsibility of dealing with gauge symmetry is one that we have brought upon ourselves, it is a redundancy in the way we have formulated the theory leaving the equations of motion underdetermined. As we will see, this redundancy is by no means fatal and its treatment forms a rich theoretical subject.

The complete set $\{G_a\}$ of first class constraints can be thought of as the *generators* of gauge transformations, a result known as the Dirac conjecture. It remains a conjecture although there are no known counter-examples. This property of the constraints is a manifestation of the inverse Noether's theorem which states that, in a Hamiltonian system, constants of motion generate symmetries. A function $f(q, p)$ on the constraint surface can be identified as a physical observable if it is gauge invariant. In other words, if it is constrained to be on-shell and it's variation vanishes under the gauge transformations generated by $\{G_a\}$,

$$\delta f = \epsilon^a \{f, G_a\} = 0. \tag{1.30}$$

Where the *gauge parameters* ϵ^a generally depend on (q, p). Defining the gauge transformation of a function in this way then allows us to rewrite the equations of motion (1.19) as

$$\begin{aligned} \dot{p}_i &= \{p_i, H_0\} + \lambda^a \delta_a p_i, \\ \dot{q}^i &= \{q^i, H_0\} + \lambda^a \delta_a q^i. \end{aligned} \tag{1.31}$$

These equations are now more transparent than they were previously. Explicitly, they show that Hamilton's equations for gauge theories contain local symmetry transformations of the canonical variables.

There are a variety of known methods which eliminate the gauge freedom inherent in systems with the defining properties (1.28, 1.29) of a first class constrained system. One of these is the *reduced phase space* method in which the constraints are solved and the canonical formalism is set up only in terms of gauge invariant physical degrees of freedom [6]. In practice, methods such as this are obstructed by the loss of desirable physical properties such as Lorentz invariance or locality. For example, in Yang-Mills theory, the temporal component of the gauge field actually enters the phase space as a Lagrange multiplier, so its removal would sacrifice manifest covariance. In more dire circumstances the constraints may not have explicit solutions to begin with and so reduced phase space methods are only viable in simple cases.

Perhaps the most intuitive possibility, is the method of *canonical gauge fixing*. As the name suggests, this involves choosing a set of gauge fixing conditions $F^a(q, p) = 0$. A *good* choice of gauge fixing conditions defines a submanifold of the constraint surface that intersects each gauge orbit exactly once and satisfies

$$\det |\{F^a, G_b\}| \neq 0. \tag{1.32}$$

This equation states that the constraints and gauge fixing conditions mutually form a set of second class constraints.

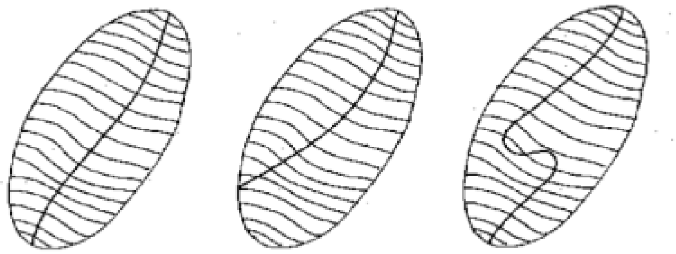

Figure 1.1: A good choice of gauge fixing conditions (left) and two bad ones [7].

The physical phase space is then the collection of points from the gauge orbits which satisfy the gauge fixing conditions and the Hamiltonian theory can be formulated in the ordinary unconstrained way. However, when the procedure is successfully carried through, one may still encounter the loss of desirable physical properties. In electrodynamics for example, canonical gauge fixing results in a description in terms of non-local differential operators. More complicated (and physically relevant) gauge theories thus need to be formulated in a different way in order for necessary physical properties to be maintained. The preservation of these properties is essential in the business of quantisation.

Gauge fixing conditions which globally achieve the requirements above can be difficult to validate on the constraint surface of more complicated theories. This problem is commonly referred to as the Gribov ambiguity [8]. The only alternative strategy for gauge fixing is to make use of the gauge symmetry rather than eliminating it. Following this idea naturally leads to working in a larger phase space where the gauge symmetry is promoted to an extended symmetry principle including some new variables.

2 The BRST Formalism

The BRST formalism [9, 10] was originally developed as an explanatory framework which rationalised the introduction of fermionic fields in the early work done on the quantisation of Yang-Mills theory. As we will see, the constraints of this particular theory cannot be explicitly solved, and moreover, the unphysical degrees of freedom can not be removed from the theory without compromising manifest covariance. As a resolution, it was found that unphysical degrees of freedom could be maintained along with the introduction of additional anti-commuting fields which provided precise cancellations of the amplitudes associated with unphysical states in perturbative calculations. It was later understood that these desirable cancellations were a consequence of the path integral being invariant under a set of so-called *BRST transformations* which emerge fundamentally from the BRST formalism. Remarkably, the BRST formalism provides a method to recast the gauge symmetry of *any* first class constrained system, not only Yang-Mills theory where it plays an indispensable role.

2.1 Anti-Commuting Variables

Anti-commuting variables can be understood as odd elements of a Grassmann algebra. The set of generators of a Grassmann algebra of finite dimension are made up of mutually anti-commuting elements ξ^A, $A = 1, \ldots, n$ and thus a general element can be written as

$$g = g_0 + g_A \xi^A + g_{AB} \xi^A \xi^B + \cdots . \tag{2.1}$$

When analysing a dynamical system which contains N commuting variables $q^i(t)$ and N' anti-commuting variables $\theta^\alpha(t)$ they are treated as even and odd elements of the *same* Grassmann algebra. The anti-commuting variables satisfy

$$(\theta^\alpha)^2 = 0, \qquad \theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha. \tag{2.2}$$

As expansions over the time-independent generators ξ^A, the variables (q^i, θ^α) can be formally written as

$$\begin{aligned} q^i(t) &= q_0^i(t) + q_{AB}^i(t) \xi^A \xi^B + \cdots , \\ \theta^\alpha(t) &= \theta_0^\alpha(t) + \theta_A^\alpha(t) \xi^A + \theta_{ABC}^\alpha(t) \xi^A \xi^B \xi^C + \cdots , \end{aligned} \tag{2.3}$$

where the even (bosonic) variables only contain Grassmann even combinations of the basis elements and the odd (fermionic) variables contain only Grassmann odd combinations of the basis elements. It is worth noting that, in applications, the basis elements ξ^A can be thought of as nothing more than a notational device which makes the distinction clear between commuting and anti-commuting variables. Functions of the variables (q^i, θ^α) are also elements of the same Grassmann algebra and can be written as an expansion of components over the basis ξ^A as in (2.3). A particularly useful subset of these functions are referred to as *superfunctions* [11]. They are functions of the variables (q^i, θ^α) with no explicit dependence on the components of the series in (2.3). A superfunction is generally written as

$$f(q, \theta) = f_0(q) + f_\alpha(q) \theta^\alpha + f_{\alpha\beta}(q) \theta^\beta \theta^\alpha + \cdots . \tag{2.4}$$

In the case that the system contains a finite number of anti-commuting variables θ^α, the superfunctions are guaranteed to have a finite number of terms since $(\theta^\alpha)^2 = 0$. A familiar example of a Grassmann algebra is that of differential forms over the space of smooth functions.

2.2 Canonical BRST Construction

Consider a Hamiltonian system for which the canonical formalism has been set up and a set of K first class constraints $G_a(p, q) \approx 0$ has been obtained. The BRST construction begins by extending the ambient phase space with the addition of K canonical pairs of anti-commuting variables (η^a, ζ_b). We refer to these variables as *ghosts*. In this $2(N + K)$-dimensional extended phase space, the generalised Poisson brackets of two functions

$f(q,p,\eta,\zeta)$, $g(q,p,\eta,\zeta)$ take the form

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} + (-1)^{\epsilon_f}\left(\frac{\partial f}{\partial \eta^a}\frac{\partial g}{\partial \zeta_a} + \frac{\partial f}{\partial \zeta_a}\frac{\partial g}{\partial \eta^a}\right). \tag{2.5}$$

The parameter ϵ_f used here is the Grassmann parity of f which is even/odd if f is Grassmann even/odd. According to this structure, the ghosts satisfy the canonical relations

$$\{\eta^a, \zeta_b\} = \{\zeta_b, \eta^a\} = -\delta^a_b. \tag{2.6}$$

Notice that the Poisson brackets for ghosts are symmetric, while the Poisson brackets for bosonic variables are anti-symmetric. By introducing ghosts into the ambient phase space of a first class constrained system, we result in an extended phase space which is subject to the so-called BRST symmetry. The conserved quantity associated with this symmetry is the BRST charge Ω which is defined in the most general case by

$$\Omega = \eta^a(G_a + W_a), \tag{2.7}$$

where $W_a = W_a(q,p,\eta,\zeta)$ is some Grassmann even function which is determined completely by the constraints and the subsequent properties of the BRST charge. Of the properties of the BRST charge, the most important is that the transformations generated by Ω are nilpotent, which is equivalent to the condition

$$\{\Omega,\Omega\} = 0. \tag{2.8}$$

The canonical BRST construction proceeds with the extension of the original Hamiltonian H_0 to include ghost terms. In particular, because H_0 is gauge invariant (1.29), it possesses a BRST invariant extension $H_0 \to H$ which satisfies

$$\delta_\Omega H = \{H,\Omega\} = 0. \tag{2.9}$$

The fact that the BRST charge commutes with the ghost-extended Hamiltonian means it can be used to generate BRST symmetry transformations of the ambient phase space variables and the newly introduced ghosts which are defined as

$$\delta_\Omega q^i = \{q^i,\Omega\} = \frac{\partial\Omega}{\partial p_i}, \qquad \delta_\Omega p_i = \{p_i,\Omega\} = \frac{\partial\Omega}{\partial q^i}; \tag{2.10}$$

$$\delta_\Omega \eta^a = \{\eta^a,\Omega\} = \frac{\partial\Omega}{\partial \zeta_a}, \qquad \delta_\Omega \zeta_a = \{\zeta_a,\Omega\} = \frac{\partial\Omega}{\partial \eta^a}. \tag{2.11}$$

Note that because of the presence of the constraints in the BRST charge, the transformations will take the form of the original gauge transformations generated by $\{G_a\}$ but now modified by ghost terms. In particular, the BRST transformations of (q^i, p_i) are exactly like the original gauge transformations (1.30) but now with the ghosts taking the place of the gauge parameters ϵ^a. A general feature, in fact, of this construction is that it promotes the local gauge symmetry of the original system to the global BRST symmetry of the extended one.

2.3 BRST Symmetry

Any gauge invariant physical quantities $f(q,p)$ that appear in the canonical formalism will satisfy

$$\{f,\eta^a\}=0, \qquad \{f,\zeta_a\}=0, \qquad \{f,G_a\}=0, \tag{2.12}$$

in the extended phase space where they remain defined only on the physical phase space. It follows that physical observables are BRST invariant quantities

$$\delta_\Omega f=\{f,\Omega\}=0. \tag{2.13}$$

More generally, any quantity which is gauge invariant (on or off shell) in the ambient phase space will be BRST invariant in the extended phase space. In addition to this, the nilpotent condition $\delta_\Omega^2=0$ ensures that there will always exist gauge invariant quantities that are trivial solutions $f=\delta_\Omega g(q,p,\eta,\zeta)$ to the above condition. Such quantities, which are BRST invariant purely by virtue of the nilpotence of the operator δ_Ω, are called *BRST exact*.

The time evolution of the system in the extended phase space is generated by the ghost-extended Hamiltonian H_{gh} which reproduces the original equations of motion for any previously gauge invariant functions of the theory. This is achieved of course by making sure that the extended Hamiltonian remains the same up to the addition of BRST exact terms. In some sense, this means that we have an abundance of options when it comes to choosing the extended Hamiltonian as long as it takes the form $H_{gh}=H_0+\{h,\Omega\}$. The elegance of this (purely mathematical) formalism is now revealed. In quantising a system with first class constraints, we want to factor out the infinitely many degrees of freedom associated with local gauge symmetry. By extending the phase space, we enlarge this symmetry to include ghosts which are not physical observables. The gauge symmetry, at its core, remains intact but now we have the freedom to alter the Hamiltonian in a number of ways. The trick, as it turns out, is to use this freedom to fix the gauge without explicitly reducing the number of degrees of freedom present during intermediate steps.

The ghost extended Hamiltonian not only generates the time evolution of BRST-invariant quantities but also that of non-BRST invariant ones. Indeed the BRST dependent ghosts fall within this category and their equations of motion read

$$\begin{aligned}\dot{\eta}^a&=\{\eta^a,H_{gh}\}=\eta^b V_b{}^a+\text{extensions},\\ \dot{\zeta}_a&=\{\zeta_a,H_{gh}\}=-V_a{}^b\zeta_b+\text{extensions},\end{aligned} \tag{2.14}$$

where the extensions are higher order terms in the ghosts. A significant fact now appears which is that if all canonical pairs of ghosts were to vanish initially then they would remain so throughout the time evolution of the system. In this case, the time evolution of (q, p) is completely determined by the restriction of the total Hamiltonian to $H_0 = H_0 + (\lambda^a = 0)G_a$ and all equations of motion are said to be gauge fixed. The problem remains that the constraints G_a need to be constrained to vanish separately in this case and so a slightly different approach is required for this to be practical. Instead, one defines the *gauge fixed* Hamiltonian

$$H_0 \to H_{gf} = H_0 + \{\Psi, \Omega\}. \tag{2.15}$$

The term that has been added is clearly BRST exact and the Hamiltonian remains a Grassmann even function by demanding that the so-called *gauge fermion* Ψ is Grassmann odd. This transformation of the Hamiltonian preserves the dynamics of BRST invariant functions but in general changes the dynamics of non-BRST invariant ones. In the BRST formalism this is achieved by choosing an appropriate gauge fermion. To see how gauge fixing is effectively achieved by this Hamiltonian, consider the choice of gauge fermion $\Psi = k^a \zeta_a$. Then at leading order in the BRST charge the Poisson bracket is

$$\{\Psi, \Omega\} = k^a G_b \{\zeta_a, \eta^b\} = -k^a G_a, \tag{2.16}$$

which mimics the "multiplier gauge" $\lambda^a = k^a$ while still treating the constraints in the appropriate way. The gauge fixed and BRST invariant action can be written as

$$S_{gf}[q, p, \eta, \zeta] = \int_{t_1}^{t_2} dt \; (\dot{q}^i p_i + \dot{\eta}^a \zeta_a - H - \{\Psi, \Omega\}). \tag{2.17}$$

It should come as no surprise that the BRST symmetry of the gauge fixed action is an intrinsic property of the extended phase space and does not depend on the choice of gauge fermion Ψ. The BRST construction can also be extended to quantum theories where the effect of the BRST operator δ_Ω on states in Hilbert space is well understood [11]. These cohomological discussions are beyond the scope of this project but an important aspect is the notion of ghost numbers. In the analysis presented so far, ghosts, where they have been introduced have had ghost number $+1$. It turns out that any physically relevant theory should have a Lagrangian with ghost number 0 and so it is necessary to amend what has been done up to this point by claiming that whenever a pair of ghosts (η^a, ζ_b) has been introduced, the variable ζ_b is actually a Grassmann odd *anti-ghost* with ghost number -1. In the succeeding sections we will encounter situations where more than one set of ghost, anti-ghost pairs are present in phase space and so to avoid confusion we will use the notation $(\eta^a, \bar{\eta}_a)$, $(\bar{\zeta}^a, \zeta_a)$. In phase space, this notation basically says that the ghost η is position-like so its conjugate $\bar{\eta}$ is a momentum-like anti-ghost, while the anti-ghost ζ is momentum-like and so its conjugate is the position-like ghost $\bar{\zeta}$.

3 Yang-Mills Theory

In this section the classical notions of gauge symmetry and first class constrained systems will be extended to a field theory, particularly pure Yang-Mills theory. A gauge theory is a type of field theory in which the Lagrangian is invariant under a set of gauge transformations belonging to a Lie group referred to as the *gauge group*. The Lagrangian of a physical theory typically involves derivative terms which do not transform covariantly under local gauge transformations. To resolve this, we say there is associated with each generator of the corresponding Lie algebra a gauge field one-form. The gauge fields then play the role of a connection in the gauge covariant derivative which replaces the ordinary derivative, allowing the gauge fields to couple to other fields in the theory. A pure gauge theory is one formulated purely in terms of the gauge fields in the absence of coupling to other fields. As we will see, this does not mean that the theory will be a non-interacting one.

3.1 The Pure Yang-Mills Lagrangian

Pure Yang-Mills theory is a special example of a gauge theory with a compact, semi-simple gauge group $\mathcal{G}$. The generators t^a of the corresponding Lie algebra are traceless, Hermitian, square matrices satisfying the following algebraic and normalisation conditions

$$[t^a, t^b] = i f^{abc} t^c, \qquad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}; \tag{3.1}$$

where $a, b, c = 1, 2, \ldots, \dim(\mathcal{G})$ are Lie-algebra indices in the adjoint representation. The gauge group $\mathcal{G}$ is non-abelian as is evident from the Lie brackets of the generators. It follows that the structure constants f^{abc} are totally antisymmetric and they satisfy the Jacobi identity

$$f^{ab}{}_d f^{cd}{}_e + f^{bc}{}_d f^{ad}{}_e + f^{ca}{}_d f^{bd}{}_e = \frac{1}{2} f^{[ab}{}_d f^{c]d}{}_e = 0. \tag{3.2}$$

For the purpose of calculations it is useful to define the operation of raising or lowering on the structure constants as $f^{cab} = \delta^{cd} f^{ab}{}_d$. Any of the Lie Algebra-valued fields ψ relevant to Yang-Mills theory can be written in matrix or component form related as $\psi = \psi^a t^a$. Correspondingly, the gauge fields of pure Yang-Mills theory are defined as

$$A_\mu(x) = A^a_\mu(x) t^a. \tag{3.3}$$

The interaction of gauge fields is governed by their role as a connection in the gauge covariant derivative

$$D_\mu = \partial_\mu - igA_\mu \tag{3.4}$$

with g being the coupling constant. The two-form field strength tensor is defined as the commutator of covariant derivatives

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{3.5}$$

This can be expressed in component form as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{3.6}$$

Let $U = e^{ig\vartheta} \in \mathcal{G}$, with $\vartheta = \vartheta^a t^a$ the gauge parameter, then under finite local gauge transformations the gauge potential A_μ transforms as a connection in the adjoint representation

$$A_\mu \to A'_\mu = U\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^{-1} \qquad \Longrightarrow \qquad F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1}. \tag{3.7}$$

It is clear from these transformations that the field strength itself is not gauge invariant. However, there still exists gauge invariant combinations of the field strength to be used in the action. In theories of fields we work in terms of the Lagrangian density $\mathcal{L}$ over the spatial dimensions. In this case, the Lagrangian is obtained by integrating $\mathcal{L}$ over $d^3\vec{x}$. In pure Yang-Mills theory the gauge invariant Lagrangian density is

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{3.8}$$

with the corresponding action

$$S_{YM} = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu}_a. \tag{3.9}$$

The gauge invariant combination of the field strengths in the Lagrangian contains a kinetic energy term (quadratic in derivatives) for the gauge fields A^a_μ which describe the gauge bosons of the quantised theory. However, as is evident in the field strength (3.6), it also contains higher order terms in the gauge fields which represents the fact that pure Yang-Mills is a *self-interacting* theory.

In calculations we often prefer to treat the gauge transformations infinitesimally. To obtain the infinitesimal form of the gauge transformations (3.7), one expands

$$U(x) = \mathbb{1} + ig\vartheta^a(x)t^a + \cdots . \tag{3.10}$$

Substituting the infinitesimal form of $U(x)$ into the definitions of the finite gauge transformations (3.7) yields

$$\begin{aligned} \delta A_\mu &= D_\mu\vartheta = \partial_\mu\vartheta - ig[A_\mu, \vartheta], \\ \delta F_{\mu\nu} &= ig[\vartheta, F_{\mu\nu}]. \end{aligned} \tag{3.11}$$

Spelling out these expressions in component form

$$\begin{aligned} \delta A^a_\mu &= (D_\mu\vartheta)^a = \partial_\mu\vartheta^a + g f^{abc} A^b_\mu \vartheta^c, \\ \delta F^a_{\mu\nu} &= -g f^{abc}\vartheta^b F^c_{\mu\nu}. \end{aligned} \tag{3.12}$$

3.2 The Hamiltonian Formulation

In order to establish some notational familiarity, one can define the electric and magnetic components of the field strength as

$$(E^a)^i = F^a_{it}, \qquad (B^a)^i = \frac{1}{2}\epsilon^{ijk}F^a_{jk}. \tag{3.13}$$

To pass to the canonical formalism we compute the momenta conjugate to the gauge fields

$$\pi^\mu_a = \frac{\delta S}{\delta(\partial_t A^a_\mu)} = F^{t\mu}_a = \begin{cases} E^i_a & \text{if } \mu = i = 1,2,3; \\ 0 & \text{if } \mu = t. \end{cases} \tag{3.14}$$

The vanishing of the momenta conjugate to A^a_t is a primary constraint which indicates that the time component of the gauge field is an unphysical degree of freedom and this is most easily seen by writing down the canonical action of the theory. Firstly the equal-time Poisson brackets of the variables in the ambient phase space are

$$\{A^i_a(\vec{x},t), E^b_j(\vec{y},t)\} = \delta^i_j\delta^b_a\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.15}$$

The temporal components are, strictly speaking, extensions to the ambient phase space so we specify their Poisson bracket structure separately to be

$$\{\pi^t_a(\vec{x},t), A^b_t(\vec{y},t)\} = -\delta^b_a\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.16}$$

Then the Yang-Mills Hamiltonian is given by

$$H_{YM} = \int d^3x\mathcal{H}_{YM} = \int d^3x\bigg(\frac{1}{2}(\vec{E}^2_a + \vec{B}^2_a) + A^a_t(\vec{D}\cdot\vec{E})_a\bigg), \tag{3.17}$$

and the gauge invariant canonical action can be written as

$$S_{YM} = \int d^4x\bigg(\vec{E}_a\cdot(\partial_t\vec{A})^a - \frac{1}{2}(\vec{E}^2_a + \vec{B}^2_a) - A^a_t(\vec{D}\cdot\vec{E})_a\bigg). \tag{3.18}$$

In this expression the physical redundancy of the canonical pair (A^a_t, π^t_a) becomes evident. Appearing linearly, A^a_t plays the role of a Lagrange multiplier since its variation leads to the generalised Gauss law constraint

$$G_a(x) = (\vec{D}\cdot\vec{E})_a(x) = 0. \tag{3.19}$$

In this case, where the Lagrange multiplier is a coordinate of the ambient phase space, the constraints come in pairs. One is the Gauss law constraint which is secondary and restricts the physical degrees of freedom, the other is the vanishing of the momentum conjugate to the Lagrange multiplier which is primary and does not. This also means that we can quietly impose $\pi^t = 0$ from the outset without ambiguity since it will trivially be satisfied by any physical configurations. Over the set of Lie algebra indices, the Gauss law constraints form a full set and their Poisson bracket algebra can be computed as follows. The spatial part of the gauge covariant derivative is given by

$$\vec{D}^{ab} = \vec{\partial}\delta^{ab} + gf^{acb}\vec{A}^c. \tag{3.20}$$

Using this definition we can write

$$\{G_a(\vec{x}), G_b(\vec{y})\} = \{(\vec{D}\cdot\vec{E})_a(\vec{x}), (\vec{D}\cdot\vec{E})_b(\vec{y})\} = \{(\vec{\partial}\cdot\vec{E})_a + gf_{alm}\vec{A}_l\cdot\vec{E}_m, (\vec{\partial}\cdot\vec{E})_b + gf_{bed}\vec{A}_e\cdot\vec{E}_d\}$$

$$\begin{aligned} &= gf_{bed}\{\vec{\partial}\cdot\vec{E}_a, \vec{A}_e\cdot\vec{E}_d\} - gf_{alm}\{\vec{\partial}\cdot\vec{E}_b, \vec{A}_l\cdot\vec{E}_m\} \\ &+ g^2 f_{alm}f_{bed}\{\vec{A}_l\cdot\vec{E}_m, \vec{A}_e\cdot\vec{E}_d\}, \end{aligned}$$

where the position indices have been momentarily suppressed. Then applying the Leibniz rule twice gives

$$\begin{aligned} &= -gf_{bed}\vec{\partial}\{\vec{A}_e, \vec{E}_a\}\vec{E}_d + gf_{alm}\vec{\partial}\{\vec{A}_l, \vec{E}_b\}\vec{E}_m \\ &+ g^2 f_{alm}f_{bed}[-\vec{A}_e\{\vec{E}_d, \vec{A}_l\}\vec{E}_m - \vec{A}_l\{\vec{A}_e, \vec{E}_m\}\vec{E}_d], \end{aligned}$$

at which point the brackets are evaluated using (3.15) and after relabelling summation indices this reduces to

$$= g[f_{abc}\vec{\partial}\cdot\vec{E}_c + g(f_{ead}f_{bdm} + f_{bed}f_{adm})(\vec{A}_e\cdot\vec{E}_m)]\delta^{(3)}(\vec{x}-\vec{y}).$$

Then applying the Jacobi identity for the structure constants yields

$$\begin{aligned} \{G_a(\vec{x}), G_b(\vec{y})\} = \; & g[f_{abc}\vec{\partial}\cdot\vec{E}_c - gf_{abd}f_{edm}(\vec{A}_e\cdot\vec{E}_m)]\delta^{(3)}(\vec{x}-\vec{y}) \\ \overset{d\longleftrightarrow c}{=} \; & gf_{abc}(\vec{\partial}\cdot\vec{E}_c + gf_{cem}\vec{A}_e\cdot\vec{E}_m)\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned}$$

The resulting expression shows that the constraints locally form a Lie algebra under the Poisson bracket with the expected first class structure of a gauge theory

$$\{G_a(\vec{x}), G_b(\vec{y})\} = gf_{abc}G_c(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.21}$$

The defining equations (3.19) of the constraint surface are non-linear and cannot be solved explicitly. In other words one can not solve the constraint equations and formulate the theory purely in terms of unconstrained, gauge invariant degrees of freedom. This is the scenario we have previously discussed where reduced phase space methods are not viable. Historically, a quantised version of this theory was first obtained in the Lagrangian path-integral formalism by introducing Faddeev-Popov ghosts. This was the breakthrough that ultimately led to the BRST formalism in the first place. We will now discuss how this was achieved and how the result can be viewed from the standpoint of the BRST formalism.

3.3 Lagrangian Path-Integral Quantisation

The path-integral formalism can be applied to quantise a pure non-abelian gauge theory by applying the Faddeev-Popov method [2] to the corresponding Lagrangian system. The computation of any physical amplitude relies on defining the functional integral

$$\mathcal{Z} = \int \mathcal{D}A \exp\left[-\frac{i}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu}_a \right] \equiv \int \mathcal{D}A e^{iS_{YM}[A]}. \tag{3.22}$$

A gauge fixing condition $F(A) = 0$ is then required to factor out the infinitely many equivalent paths associated with each gauge direction. The generalised Lorenz gauge condition

$$F(A) = \partial^\mu A_\mu(x) - \omega(x),$$

where $\omega(x)$ can be any scalar function, is a good choice of gauge in this case. To impose the condition $F(A) = 0$ on the field configurations, it can be introduced into the functional integral in the form of the following identity in terms of the image $(A^\vartheta)_\mu$ of the gauge field under a gauge transformation parametrised by ϑ:

$$1 = \int \mathcal{D}\vartheta \delta(F(A^\vartheta)) \det\left(\frac{\delta F(A^\vartheta)}{\delta\vartheta}\right). \tag{3.23}$$

Using the infinitesimal expression for $(A')^a_\mu \equiv (A^\vartheta)^a_\mu$ defined by (3.12) the functional derivative in this identity can be computed as

$$\frac{\delta F(A^\vartheta)}{\delta\vartheta} = \partial^\mu D_\mu(x) \tag{3.24}$$

which is independent of the gauge parameter. Inserting this identity into the functional integral, one finds

$$\mathcal{Z} = \int \mathcal{D}A e^{iS_{YM}[A]} = \int \mathcal{D}\vartheta \int \mathcal{D}A e^{iS_{YM}[A]} \delta(F(A)) \det\left(\frac{\delta F(A^\vartheta)}{\delta\vartheta}\right). \tag{3.25}$$

Here, the integral over the gauge parameter has been factored out into an overall normalisation which cancels in the computation of physical amplitudes. The functional integral over $\omega(x)$ can be performed by a trick involving the introduction of a Gaussian weighting function centred on $\omega = 0$. This produces an additional term in the exponent, leading to the expression

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}\omega \exp\left[-i\int d^4x \frac{\omega^2}{2\xi} \right] \int \mathcal{D}\vartheta \int \mathcal{D}A e^{iS_{YM}[A]} \delta(\partial^\mu A_\mu - \omega(x)) \det(\partial^\mu D_\mu) \\ &= \int \mathcal{D}\vartheta \int \mathcal{D}\mathcal{A} e^{iS_{YM}[A]} \exp\left[i\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \right] \det(\partial^\mu D_\mu).\end{aligned} \tag{3.26}$$

The parameter ξ in the denominator is an arbitrary real number. The key feature of this method is that the determinant of $\partial^\mu D_\mu(x)$ can be written as a functional integral over a set of Faddeev-Popov ghosts $(\eta^a, \bar{\eta}_a)$ using the fermionic functional integral identity

$$\det(\partial^\mu D_\mu) = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp\left[-i \int d^4x \bar{\eta}_a (\partial^\mu D_\mu)\eta^a\right]. \tag{3.27}$$

To this end, the pure Yang-Mills theory is now described by the gauge fixed Faddeev-Popov effective Lagrangian

$$\mathcal{L}_{FP} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar{\eta}_a(\partial^\mu D_\mu)\eta^a \tag{3.28}$$

and the functional integral

$$\mathcal{Z} = \int \mathcal{D}\vartheta \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar{\eta} \exp\left[i \int d^4x \mathcal{L}_{FP}\right]. \tag{3.29}$$

One shortcoming of this method of quantisation is that it implicitly relies on the cancellations of emission probabilities for timelike and longitudinal gauge bosons which do not occur in the non-abelian theory. Fortunately, the BRST symmetry provides a layer of security by construction. Indeed, the gauge fixed action (3.34) can be used to define a functional integral modified to include the auxiliary field B_a (not the magnetic field)

$$\mathcal{Z} \sim \int \mathcal{D}A\mathcal{D}\eta\mathcal{D}\bar{\eta}\mathcal{D}B \exp\left\{\int d^4x \left[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + B_a(\partial^\mu A^a_\mu) - \frac{\xi}{2}(B_a)^2 - \bar{\eta}_a(\partial^\mu D_\mu)\eta^a\right]\right\}. \tag{3.30}$$

Using the Jacobi identity it can be verified by calculation that this form of the functional integral is BRST invariant under the transformations

$$\begin{aligned} \delta_\Omega A^a_\mu &= (D_\mu \eta)^a, \\ \delta_\Omega \eta^a &= -\frac{1}{2} g f^{bca} \eta^b \eta^c, \\ \delta_\Omega \bar{\eta}_a &= B_a, \\ \delta_\Omega B_a &= 0. \end{aligned} \tag{3.31}$$

One may freely integrate out B_a by completing the square which reduces the expression to (3.29). To see how the necessary cancellations are achieved through BRST symmetry [12], one notes that at a quantum level the nilpotent operator δ_Ω commutes with the Hamiltonian and thus divides the eigenstates into three subspaces. Among these subspaces lies the physical Hilbert space which consists of states which are BRST invariant (but not exact) and represent transversely polarised gauge bosons as needed. The unphysical eigenstates lie in subspaces associated with ghosts, anti-ghosts and gauge bosons of longitudinal polarisation.

3.4 Pure Yang-Mills Theory in the BRST Formalism

Now we can reproduce this result from scratch in the Lagrangian version of the BRST formalism. Starting from the pure Yang-Mills action (3.9), for each gauge degree of freedom a pair of ghost fields $(\eta^a, \bar{\eta}_a)$ are introduced to the configuration space. Then, the BRST transformations for the fields are introduced precisely as in (3.31). In doing so, we define the auxiliary field B^a which has no independent dynamics and exists solely for the purpose of BRST symmetry. The generalised Lorenz gauge is then selected by making an appropriate choice of the gauge fermion

$$\Psi = \bar{\eta}_a\left(\partial^\mu A^a_\mu - \frac{\xi}{2}B^a\right). \tag{3.32}$$

This modifies the Lagrangian by the addition of the BRST exact term

$$\delta_\Omega \Psi = B_a(\partial^\mu A^a_\mu) - \frac{\xi}{2}(B_a)^2 - \bar{\eta}_a(\partial^\mu D_\mu)\eta^a, \tag{3.33}$$

where the parameter ξ is identical to that which appeared in the Faddeev-Popov method. The full gauge fixed action S_{gf} for this choice of gauge fermion is identical to the one in the path-integral (3.30), in particular

$$S_{gf} = \int d^4x\left[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + B_a(\partial^\mu A^a_\mu) - \frac{\xi}{2}(B_a)^2 - \bar{\eta}_a(\partial^\mu D_\mu)\eta^a\right]. \tag{3.34}$$

Since the action is BRST invariant, it can be used to derive the BRST charge Ω explicitly by means of Noether's theorem. At the quantum level, this construction succeeds in formulating the physical field configurations as those which obey the constraints and gauge fixing conditions

$$G_a = (\vec{D}\cdot\vec{E})_a = 0, \quad \text{and} \quad \partial^\mu A^a_\mu = 0. \tag{3.35}$$

as needed.

4 Parisi-Sourlas Supersymmetry

The Lagrangian path-integral quantisation presented in the previous section for the pure Yang-Mills Lagrangian has proven to be an undeniably powerful way to obtain transition amplitudes [2]. However, the BRST-formalism and the path-integral still suffer from some important limitations. Firstly, many formal devices were required to obtain a functional integral which correctly implemented the gauge fixing conditions and constraints in an ad-hoc fashion. Furthermore the relevant BRST symmetry was not implemented fundamentally, but rather as a property of the gauge fixed Lagrangian that is observed after applying the Faddeev-Popov method. Indeed (much to their credit), the original practitioners of the method itself worked in the absence of any such formalism. Another glaringly obvious problem is that the BRST formalism can not be applied to quantise second class constraints unless they are first converted to a set of effective first class constraints. In practice, Lagrangian path-integrals are computed by discretising spacetime thereby converting the integration measure into an ordinary one. The spacetime is then brought back to continuum after integration by means of a limiting procedure, but the result has been shown to be dependent on the choice of discretisation making a consistent definition of the measure in the path-integral impossible [13]. Path-integrals derived in the canonical formalism however, do not suffer from this ambiguity. In the following discussion a method to obtain covariant phase space path-integrals in the canonical formalism will be presented and applied to pure Yang-Mills theory.

4.1 Reduced Phase Space Path-Integrals

We are interested in deriving a path integral for a constrained Hamiltonian system which is formally equivalent to that which would be obtained using the reduced phase space approach [6]. To illustrate the way in which path-integrals are constructed when the reduced phase space method is possible, consider the $2N$-dimensional ambient phase space of a system described by canonical coordinates (q^i, p_i). Suppose this system also contains a set of K first class constraints $G_a(q,p)$ which *can* be solved for the unphysical coordinates q^a where $a = 1, \ldots, K$. The reduced phase space is then the $2(N-K)$-dimensional subspace acquired by eliminating the unphysical coordinates and their canonical momenta (q^a, p_a). Eliminating the unphysical momenta requires the introduction of a set of gauge fixing conditions $F^a(q,p)$ subject to the condition (1.32) Assuming the gauge fixing conditions are chosen such that they have mutually vanishing Poisson brackets, a canonical transformation can be used to identify F^a with the unphysical momenta p_a and the reduced phase space Γ^* is then given by the defining equations $G_a = F^a = 0$. The relevant path integral is then defined by the restriction of the measure and the action to the reduced phase space

$$\mathcal{Z} \sim \int \mathcal{D}q^* \mathcal{D}p_* \exp\left[i \int dt \; (\dot{q}^* p_* - H_0(q^*, p_*)) \right]. \tag{4.1}$$

This can be expressed in the equivalent unrestricted form where (F^a, G_a) are yet to be imposed explicitly

$$\mathcal{Z} \sim \int \mathcal{D}q \mathcal{D}p \delta(G_a) \delta(F^a) \det[\{G_a, F^b\}] \exp\left[i \int dt \; (\dot{q}^i p_i - H_0) \right]. \tag{4.2}$$

In contrast to (3.25), this path integral is now expressed in terms of constraints as well as gauge fixing conditions in the canonical formalism. Following the Faddeev-Popov method in this case involves introducing Lagrange multipliers (λ^a, Π_a) and ghosts $(\eta^a, \bar{\eta}_a)$ to write the reduced phase space integral in the equivalent extended form

$$\mathcal{Z} \sim \int \mathcal{D}q \mathcal{D}p \mathcal{D}\lambda \mathcal{D}\varpi \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp\left[i \int dt \; (\dot{q}^i p_i - H_0 - \lambda^a G_a - \varpi_a F^a - \eta^a \{G_a, F^b\} \bar{\eta}_b) \right]. \tag{4.3}$$

In the case of Yang-Mills theory the reduced phase space method fails to be practical as manifest covariance is immediately lost in the reduced phase space. Notwithstanding this fact, when covariant quantisation methods are applied the resulting quantum theory should maintain equivalence to the corresponding reduced phase space quantum theory. In what follows we shall introduce a method which, at its core, is built upon this idea.

4.2 The Parisi-Sourlas Formalism

The Parisi-Sourlas (PS) formalism [3] refers to a particular treatment of constrained Hamiltonian systems in an extended phase space which preserves manifest Lorentz invariance and reproduces a path integral equivalent to (4.1) without any restriction to lower dimensional manifolds. Conceptually, the principle of PS supersymmetry can be understood as follows [4]. The path-integral for an N-dimensional constrained system with 2 unphysical degrees of freedom has a physical, reduced space path-integral which is an infinite-dimensional version of the ordinary integral

$$\int d^{N-2}x f(x^2). \tag{4.4}$$

Where $f(x^2)$ is an $SO(N-2)$ rotation invariant function. As in (4.2), this integral can be written in an unrestricted form where the 2 unphysical variables are not explicitly eliminated from the integrand. In this case the $SO(N-2)$ rotation invariance is extended to an $SO(N)$ rotation invariance of both the function $f(x^2)$ and the measure $d^{N-2}x$. In a manner similar to the BRST formalism, the PS formalism involves the introduction of anti-commuting coordinates $(\theta, \bar{\theta})$ such that a representation of the $(N-2)$-dimensional integral above can be constructed over an $(N+2)$-dimensional superspace parametrised by N bosonic and 2 fermionic coordinates $(x, \theta, \bar{\theta})$. In this superspace, denoted by Σ, the $SO(N)$ rotation invariance is extended further to an $(N+2)$-dimensional superrotation invariance. A general coordinate on Σ is expressed as $y^\alpha = (y^\mu, y^\theta, y^{\bar{\theta}}) = (x^\mu, \theta, \bar{\theta})$ and the relevant superrotation group is determined by the set of transformations that leave

$$y^2 = g_{\alpha\beta} y^\alpha y^\beta = x^2 + \bar{\theta}\theta \tag{4.5}$$

unchanged, where the non-zero elements of the superspace metric tensor are

$$g_{\mu\nu} = \eta_{\mu\nu}, \qquad g_{\bar{\theta}\theta} = -g_{\theta\bar{\theta}} = \frac{1}{2}. \tag{4.6}$$

This symmetry of the superspace is in fact generated by the graded Lie group $Osp(N|2)$ of orthosymplectic superrotations composed of bosonic $SO(N)$ rotations and fermionic $Sp(2)$ rotations. The generators $R^{\alpha\beta}$ of this symmetry group are elements of the $\mathfrak{osp}(N|2)$ graded Lie algebra with the Lie bracket structure given by

$$[R^{\alpha\beta}, R^{\gamma\delta}] = g^{\beta\gamma} R^{\alpha\delta} - (-1)^{\gamma\delta} g^{\beta\delta} R^{\alpha\gamma} - (-1)^{\alpha\beta} g^{\alpha\gamma} R^{\beta\delta} + (-1)^{\alpha\beta}(-1)^{\gamma\delta} g^{\alpha\delta} R^{\beta\gamma} \tag{4.7}$$

The grading $(-1)^{\alpha\beta}$ only takes the value (-1) if both indices are associated with fermionic degrees of freedom $(\theta, \bar{\theta})$. One may also express the generators in the differential representation

$$R^{\alpha\beta} = (-1)^{\alpha\beta} y^\alpha \partial^\beta - y^\beta \partial^\alpha, \tag{4.8}$$

where the derivative operator is defined in the usual way $\partial_\alpha = g_{\alpha\beta}\partial^\beta = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$. In the PS formalism, the restricted integral (4.4) can then be formulated as an integral over the superspace with $f(x, \theta, \bar{\theta})$ depending on its variables in an $Osp(N|2)$ invariant fashion. In particular, under the assumption that f takes the form of a Gaussian function, it turns out that the following equation holds

$$\int d^N x d\bar{\theta} d\theta f(x^2 + \bar{\theta}\theta) = \int d^{N-2} x f(x^2). \tag{4.9}$$

It is worth noting here that Grassmann integration and left-differentiation are related very closely through the identity

$$\int d\theta_1 \cdots d\theta_n f(\theta_1, \ldots, \theta_n) = \frac{\partial}{\partial\theta_1} \cdots \frac{\partial}{\partial\theta_n} f(\theta_1, \ldots, \theta_n), \tag{4.10}$$

for θ_i $i = 1, \ldots, n$ odd elements of a Grassmann algebra. The equation (4.9) represents explicitly the idea of this method; that fermionic degrees of freedom can be introduced in order to cancel the contributions of unphysical degrees of freedom without eliminating them explicitly from the theory. In constrained Hamiltonian systems, the superrotation invariance of the function $f(x^2 + \bar{\theta}\theta)$ is not always guaranteed but the subspace of unphysical variables can always be brought to this form by a change of variables. We conclude that the subspace spanned by the physical degrees of freedom does not require any further treatment in the PS formalism. With the goal of deriving path-integrals in the canonical formalism, we must look at how this logic can be applied to the unphysical sector of an extended phase space.

Consider the extended phase space of a constrained system defined by the coordinates (q^i, p_i) and the canonical pair of Lagrange multipliers (λ^a, Π_a) which correspond to the first class constraints G_a and gauge fixing conditions F^a respectively. Due to Darboux, there is a theorem [14] which confirms the existence of a local change of variables $(G_a, F^a) \to (\tilde{G}_a, \tilde{F}^a)$ such that

$$\{\tilde{G}_a, \tilde{G}_b\} = \{\tilde{F}^a, \tilde{F}^b\} = 0, \qquad \text{and} \qquad \{\tilde{G}_a, \tilde{F}^b\} = -\delta_a^b. \tag{4.11}$$

In what follows it is assumed that such a change of variables has been employed such that $(\tilde{F}^a, \tilde{G}_a)$ are identifiable with the unphysical positions and momenta (q^a, p_a) through a canonical transformation. In order to establish equivalence to the $2(N-K)$-dimensional physical phase space, $2K$ ghost, anti-ghost pairs $(\eta^a, \bar{\eta}_a)$ and $(\bar{\zeta}^a, \zeta_a)$ are introduced resulting in a phase space which includes a further $4K$ anti-commuting variables. The ghosts and anti-ghosts satisfy the canonical relations (2.6) under the generalised Poisson bracket (2.5). Associated with each constraint index a there is an 8-dimensional unphysical phase space composed of four canonically conjugate pairs of position and momentum variables with an equal number of bosonic and fermionic variables. The corresponding PS superspace is introduced as a 4-dimensional superspace by attaching the fermionic coordinates $(\theta, \bar{\theta})$ to the space. To illustrate how the rest of the construction proceeds, we can for instance parametrise the superspace using light-cone position and momentum variables

$$\begin{aligned} (q^a)^\alpha &= (\hat{x}^a, x^a, \theta^a, \bar{\theta}^a) \equiv (\tilde{F}^a, -\lambda^a, \bar{\zeta}^a, \eta^a), \\ (p_a)_\alpha &= (p_a, \hat{p}_\alpha, \bar{\gamma}_a, \gamma_a) = (\tilde{G}_a, \Pi_a, \zeta_a, \bar{\eta}_a). \end{aligned} \tag{4.12}$$

The relevant PS supersymmetry group is then $Osp(1,1|2)$, and in the representation of light-cone coordinates the generators in the corresponding Lie algebra $\mathfrak{osp}(1,1|2)$ are given by

$$(R_a)^{\alpha\beta} = (q^a)^\alpha (p_a)^\beta - (-1)^{\alpha\beta}(q^a)^\beta (p_a)^\alpha. \tag{4.13}$$

A generalisation of the BRST symmetry in the PS formalism can then be seen by computing the nilpotent generator

$$R^{-\theta} = \sum_a (R_a)^{-\theta} = \bar{\zeta}^a \tilde{G}_a + \bar{\eta}_a \lambda^a \tag{4.14}$$

which satisfies the general definition of the standard BRST charge Ω in (2.7). For completeness we can also compute the nilpotent generator of anti-BRST transformations which appears in BRST supersymmetry

$$R^{-\bar{\theta}} = \sum_a (R_a)^{-\bar{\theta}} = \eta^a \tilde{G}_a - \zeta_a \lambda^a. \tag{4.15}$$

It follows that the superrotations of a superspace function f corresponding to these generators can be written as

$$\delta_{-\theta} f = \{f, R^{-\theta}\}, \qquad \delta_{-\bar{\theta}} f = \{f, R^{-\bar{\theta}}\}; \tag{4.16}$$

which precisely mimic the form of the transformations (2.13) generated by the BRST (or anti-BRST) charge. It should now be quite evident that the PS supersymmetry is a generalisation of the BRST supersymmetry. The quantum theory in this representation of the unphysical sector is then described by the phase space path-integral version of (4.9) which can be written as

$$\mathcal{Z}_{\text{unphysical}} = \int \mathcal{D}(q^a)\mathcal{D}(p_a) \exp\left[i \int dt((\dot{q}^a)^\alpha (p_a)_\alpha - \{\Psi, R^{-\theta}\})\right]. \tag{4.17}$$

The function Ψ here is the PS version of the gauge fermion, in general it is a linear combination of terms such that $\{\Psi, R^{-\theta}\}$ is PS *exact*. As we will see in our application of this method, the PS exact term being added to the path-integral can be written in an even more symmetric form $\{\{\Psi, R^{-\bar{\theta}}\}, R^{-\theta}\}$ which makes use of multiple nilpotent generators. In practice the quantisation procedure is not yet complete, it remains to write this path-integral locally in terms of the original variables on the phase space which can be achieved by a canonical transformation. Once the appropriate canonical transformation has been made the relevant path-integral over the entire phase space is

$$\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\mathcal{D}\varpi\mathcal{D}\eta\mathcal{D}\bar{\eta}D\zeta\mathcal{D}\bar{\zeta} \exp\left[i \int dt(\dot{q}^i p_i - \dot{\lambda}^a \Pi_a + \dot{\eta}^a \bar{\eta}_a + \dot{\bar{\zeta}}^a \zeta_a - \{\Psi, \Omega\})\right] \tag{4.18}$$

with the generalised gauge fermion and BRST charge being written in terms of the original localised variables. Of utmost importance is the fact that the resulting quantum theory obtained by treating first class constraints in the PS formalism results in canonical path integrals which are formally equivalent to those obtained in the reduced phase space quantisation procedure as desired.

4.3 Abelianisation of the Gauss Law Constraints

We now return to our study of pure Yang-Mills theory in the canonical formalism as presented in Section 3.2. Applying the PS formalism in this case is much more subtle. Firstly, the PS formalism presented above has to be carefully generalised to a discussion of gauge fields rather than discretised coordinates. The ambient phase space is determined by the gauge fields and their conjugate momenta $(\vec{A}^a, \vec{E}_a)$ with extensions provided by the Lagrange multipliers A^a_t and their conjugate momenta π^t_a. As before the physical states are those obeying the Gauss law constraints G_a and the trivial primary constraint $\pi^0_a = 0$. To eliminate the gauge freedom inherent in this phase space, we impose the transversality condition $\vec{\partial} \cdot \vec{A}^a = 0$ which is also known as the Coulomb gauge. The constraints and gauge fixing conditions together read

$$G_a(\vec{x}) = (\vec{D} \cdot \vec{E})_a(\vec{x}), \qquad F^a(\vec{x}) = (\vec{\partial} \cdot \vec{A})^a(\vec{x}), \qquad a = 1, \ldots, K. \tag{4.19}$$

In this picture, the physical phase space consists of the gauge fixed fields $(\vec{A}^a_{gf}, \vec{E}^a_{gf})$ which are solutions of the above equations. On the other hand, the non-transversely polarised fields become unphysical degrees of freedom which are gauge-dependent. The gauge fixed fields are related to their original form $(\vec{A}^a, \vec{E}_a)$ defined on the ambient phase space by a gauge transformation of the form (3.7), in other words the gauge we have chosen is said to be *accessible*. This is important to remember because we ultimately want to express the quantum theory in terms of the original variables and their supersymmetric extensions picked up in the PS formalism. The second, and most immediate detail we must address is that our constraints and gauge fixing conditions do not induce the required structure on the unphysical subspace. In particular, the constraints satisfy the non-trivial Poisson bracket algebra (3.21). The construction of the PS superspace in the unphysical sector relies on (F^a, G_a) being identifiable with position and momentum-like variables respectively, thereby satisfying the algebraic structure (4.11). In the non-abelian case, Darboux's theorem must be applied so that the formalism can initially be set up in terms of the transformed variables $(\tilde{F}^a, \tilde{G}_a)$ of a local coordinate system called the *Darboux chart* which has the required *flat* symplectic structure (4.11).

From the definition of $F^a(\vec{x})$, it is clear that the gauge fixing conditions are abelian, i.e. $\{F^a(\vec{x}), F^b(\vec{y})\} = 0$. This is convenient since the identity mapping $F^a(\vec{x}) \to \tilde{F}^a(\vec{x})$ is a valid choice for which $\tilde{F}^a$ can be identified with the unphysical position-like variables. The ease of this choice naturally shifts all of the subtleties of the change of variables onto the Gauss law constraints. We have already seen the Faddeev-Popov matrix (3.23) in the Lorenz gauge. This is the analogue of the constraint matrix $\{G_a(\vec{x}), F^b(\vec{y})\}$ in the Lagrangian formulation. Now treating Yang-Mills theory as a Hamiltonian system in the Coulomb gauge, we are interested in the constraint matrix

$$\{G_a(\vec{x}), F^b(\vec{y})\} = -(\vec{\partial} \cdot \vec{D})^b_a(\vec{x})\delta^{(3)}(\vec{x} - \vec{y}). \tag{4.20}$$

which shows that the constraints and gauge fixing conditions mutually form a set of second class constraints as required. Abelianisation is a method which can be applied to convert the Gauss law constraints into momentum-like variables [15]. Let $\tilde{G}_a(\vec{x})$ be the desired functionals in the Darboux chart. Following the abelianisation procedure, this change of variables are given by the transformation

$$\tilde{G}_a(\vec{x}) = \int (M^{-1})_a{}^b(\vec{x},\vec{x}')G_b(\vec{x}')d^3\vec{x}', \tag{4.21}$$

where M^{-1} is an invertible matrix. It is straightforward to define the inverse transformation where the Gauss law constraints are expressed as

$$G_a(\vec{x}) = \int M_a{}^b(\vec{x},\vec{x}')\tilde{G}_b(\vec{x}')d^3\vec{x}. \tag{4.22}$$

A solution for this matrix is found in [16] which prescribes;

$$M_a{}^b(\vec{x},\vec{x}') = -\{G_a(\vec{x}), F^b(\vec{x}')\} = (\vec{\partial}\cdot\vec{D})_a{}^b(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}'). \tag{4.23}$$

This matrix is quite complicated and it's inverse is clearly going to be non-local. Despite this we can still proceed to derive the explicit transformation $G_a \to \tilde{G}_a$. To begin, we make the transformation $G_a(\vec{x}) \to \tilde{G}_a(\vec{x})$ more explicit by inserting the solution (4.21) into the known relation (3.21) and simplifying the result as follows:

$$\{G_a(\vec{x}), G_b(\vec{y})\} = \left\{ \int (\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\tilde{G}_c(\vec{x}')d^3\vec{x}', \int (\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\tilde{G}_d(\vec{y}')d^3\vec{y}' \right\}$$

expanding the Poisson bracket using the Leibniz rule gives

$$\begin{aligned} = -\iint \Big[&\{(\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}'), (\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\}\tilde{G}_c(\vec{x}')\tilde{G}_d(\vec{y}') \\ &+ (\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\{\tilde{G}_d(\vec{y}'), (\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\}\tilde{G}_c(\vec{x}') \\ &+ (\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\{(\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}'), \tilde{G}_c(\vec{x})\}\tilde{G}_d(\vec{y}')\Big] d^3\vec{x}'d^3\vec{y}'. \end{aligned}$$

We notice immediately that the Poisson bracket vanishes in the first term due to it taking the form $\{\vec{\partial}\cdot\vec{A}+\vec{\partial}^2, \vec{\partial}\cdot\vec{A}+\vec{\partial}^2\} = 0$ and expanding inside the remaining brackets yields

$$\begin{aligned} = -\iint \Big[&(\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\{\tilde{G}_d(\vec{y}'), (\vec{\partial}^2\delta_a^c + gf_a{}^{ec}\vec{\partial}\cdot\vec{A}_e(\vec{x}))\delta^{(3)}(\vec{x}-\vec{x}')\}\tilde{G}_c(\vec{x}') \\ &+ (\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\{(\vec{\partial}^2\delta_b{}^d + gf_b{}^{hd}\vec{\partial}\cdot\vec{A}_h(\vec{y}))\delta^{(3)}(\vec{y}-\vec{y}'), \tilde{G}_c(\vec{x}')\}\tilde{G}_d(\vec{y}')\Big] d^3\vec{x}'d^3\vec{y}'. \end{aligned}$$

Once again the contributions of the form $\{\vec{\partial}^2\delta^{(3)}(\cdots), \star\}$ will vanish and using the definition of the transformed gauge conditions $\tilde{F}^a(\vec{x}) = (\vec{\partial}\cdot\vec{A})^a(\vec{x})$ the remaining terms reduce to

$$= -g \iint \Big[f_a{}^{ec}(\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\{\tilde{G}_d(\vec{y}'), \tilde{F}_e(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\}\tilde{G}_c(\vec{x}')$$
$$+ f_b{}^{hd}(\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\{\tilde{F}_h(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}'), \tilde{G}_c(\vec{x}')\}\tilde{G}_d(\vec{y}')\Big] d^3\vec{x}' d^3\vec{y}'.$$

Now by the postulated algebra (4.11) of the new coordinates the Poisson brackets can be evaluated within the integral:

$$= g \iint [f_a{}^{ec}(\vec{\partial}\cdot\vec{D})_b{}^d(\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\delta_{de}\delta^{(3)}(\vec{y}'-\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\tilde{G}_c(\vec{x}')$$
$$- f_b{}^{hd}(\vec{\partial}\cdot\vec{D})_a{}^c(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\delta_{ch}\delta^{(3)}(\vec{x}'-\vec{y})\delta^{(3)}(\vec{y}-\vec{y}')\tilde{G}_d(\vec{y}')] d^3\vec{x}' d^3\vec{y}'.$$

One delta function can be eliminated from each term by integrating out $\vec{x}'$ in the first term and $\vec{y}'$ in the second term, then relabelling the integration variables such that only $\vec{x}'$ remains yields

$$= g \int [f_a{}^{ec}(\vec{\partial}\cdot\vec{D})_{be}(\vec{y})\delta^{(3)}(\vec{y}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x})\tilde{G}_c(\vec{x})$$
$$+ f_b{}^{dh}(\vec{\partial}\cdot\vec{D})_{ah}(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\tilde{G}_d(\vec{y})] d^3\vec{x}'$$

Equivalently, the integrand can be written in a much more symmetric fashion

$$= g \int [f_a{}^{ec}(\vec{\partial}\cdot\vec{D})_{be}(\vec{y})\delta^{(3)}(\vec{y}-\vec{x}')\tilde{G}_c(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x})$$
$$- f_b{}^{hd}(\vec{\partial}\cdot\vec{D})_{ah}(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\tilde{G}_d(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})] d^3\vec{x}'$$

In the first term, we integrate by parts moving $\vec{\partial}_y$ first to $\vec{x}'$ and then through $\tilde{G}_c(\vec{x})$ onto the last delta function. Similarly, in the second term we move $\vec{\partial}_x$ to $\vec{x}'$ and then through $\tilde{G}_d(\vec{x}')$ onto the last delta function leaving

$$= -g \int [f_a{}^{ec}\vec{D}_{be}(\vec{y})\delta^{(3)}(\vec{y}-\vec{x}')\tilde{G}_c(\vec{x}')\ \vec{\partial}_{x'}\delta^{(3)}(\vec{x}'-\vec{x})$$
$$- f_b{}^{hd}\vec{D}_{ah}(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')\tilde{G}_d(\vec{x}')\ \vec{\partial}_{x'}\delta^{(3)}(\vec{x}'-\vec{y})] d^3\vec{x}'.$$

Under integration by parts $\vec{D}_{ab}$ becomes $-\vec{D}_{ba}$, after repeatedly applying this in both terms such that the spatial covariant derivative now acts on $\tilde{G}$, applying a further integration by parts to the spatial derivative gives

$$\{G_a(\vec{x}), G_b(\vec{y})\} = g \int [f_a{}^{ec}\delta^{(3)}(\vec{y}-\vec{x}')(\vec{\partial}\cdot\vec{D})_{eb}(\vec{x}')\tilde{G}_c(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{x})$$
$$- f_b{}^{hd}\delta^{(3)}(\vec{x}-\vec{x}')(\vec{\partial}\cdot\vec{D})_{ha}(\vec{x}')\tilde{G}_d(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})] d^3\vec{x}'$$

$$
\begin{aligned}
&= g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_a{}^{ec}\vec{D}_{eb}(\vec{x}')\tilde{G}_c(\vec{x}')-f_b{}^{hd}\vec{D}_{ha}(\vec{x}')\tilde{G}_d(\vec{x}')]\\
&\overset{d\longleftrightarrow c}{=} g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_a{}^{ec}\vec{D}_{eb}(\vec{x}')-f_b{}^{hc}\vec{D}_{ha}(\vec{x}')]\tilde{G}_c(\vec{x}')\\
&= g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_a{}^{ec}(\vec{\partial}_{x'}\delta_{eb}+gf_{elb}\vec{A}^l(\vec{x}'))\\
&\quad -f_b{}^{hc}(\vec{\partial}_{x'}\delta_{ha}+gf_{hla}\vec{A}^l(\vec{x}'))]\tilde{G}_c(\vec{x}')\\
&= g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_{ab}{}^c\vec{\partial}_{x'}+g(f_a{}^{ec}f_{elb}-f_b{}^{hc}f_{hla})\vec{A}^l(\vec{x}')]\tilde{G}_c(\vec{x}')
\end{aligned}
$$

Relabelling $h\longleftrightarrow e$ and using the totally anti-symmetric property of the structure constants this can be rewritten as

$$
= g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_{ab}{}^c\vec{\partial}_{x'}+g(f_a{}^{ce}f_{bel}+f_{cb}{}^ef_{ael})\vec{A}^l(\vec{x}')]\tilde{G}_c(\vec{x}').
$$

The Jacobi identity for this set of indices reads

$$
f_a{}^{ce}f_{bel}+f_{cb}{}^ef_{ael}+f_{ba}{}^ef_{cel}=0, \tag{4.24}
$$

which can be applied to the integrand above to reduce the bracket to the form

$$
\begin{aligned}
\{G_a(\vec{x}),G_b(\vec{y})\} &= g\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})\vec{\partial}_{x'}\cdot[f_{ab}{}^e\delta_e^c\vec{\partial}_{x'}+gf_{ab}{}^ef_{el}^{\ c}\vec{A}^l(\vec{x}')]\tilde{G}_c(\vec{x}')\\
&= gf_{ab}^{\ e}\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})(\vec{\partial}\cdot\vec{D})_e{}^c(\vec{x}')\tilde{G}_c(\vec{x}').
\end{aligned}
$$

Now we compare this to our known result in terms of Gauss law generators which can be expanded in an identical way

$$
\{G_a(\vec{x}),G_b(\vec{y})\} = gf_{ab}^{\ c}G_c(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) = gf_{ab}^{\ c}\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')G_c(\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y}).
$$

Indeed, by comparing this to the expression we have just derived using the definition (4.22) we see that we can equate the following two expressions

$$
\begin{aligned}
&f_{ab}^{\ c}\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})G_c(\vec{x}')\\
=&f_{ab}^{\ e}\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})(\vec{\partial}\cdot\vec{D})_e{}^c(\vec{x})\tilde{G}_c(\vec{x}').
\end{aligned}
$$

To further simplify this, we write $f_{ab}^{\ c}=f_{ab}^{\ e}\delta_e^c$ above to cancel the structure constants

$$
\int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})G_e(\vec{x}') = \int d^3\vec{x}'\delta^{(3)}(\vec{x}-\vec{x}')\delta^{(3)}(\vec{x}'-\vec{y})(\vec{\partial}\cdot\vec{D})_e{}^c(\vec{x}')\tilde{G}_c(\vec{x}').
$$

Finally, performing the integration over $\vec{x}'$ on both sides yields

$$
G_e(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) = (\vec{\partial}\cdot\vec{D})_e{}^c(\vec{x})\tilde{G}_c(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) = M_e{}^c(\vec{x},\vec{y})\tilde{G}_c(\vec{x}). \tag{4.25}
$$

Despite the obvious non-locality this gives a way to write the result of abelianisation in a meaningful way as

$$\tilde{G}_a(\vec{x}) = (M^{-1})^b{}_a(\vec{x},\vec{y})G_a(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) = [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a G_b(\vec{x}). \tag{4.26}$$

For our purposes, an expression like this is acceptable to work with for a number of reasons. Firstly, these functionals form an abelian constraint algebra and this can be seen to follow from the fact that $\tilde{F}^a$ and $M_a{}^b(\vec{x},\vec{y})$ depend only on $\vec{A}^a$ and not $\vec{E}_a$ [16]. Secondly, after we introduce the PS supersymmetry in the unphysical sector of the phase space, quantities such as the BRST charge will be related their counterparts in the original coordinates by a canonical transformation [17] which can be used to transform back to a fully local description after the formalism is set up. This reinforces the intention to use the Darboux coordinates $(\tilde{F}^a, \tilde{G}_a)$.

It remains to show that the abelian gauge fixing conditions and constraints $(\tilde{F}^a, \tilde{G}_a)$ form a pair of canonically conjugate variables in the sense of (4.11). Indeed, by computing the Poisson bracket

$$\{\tilde{G}_a(\vec{x}), \tilde{F}^b(\vec{y})\} = \{[(\vec{\partial}\cdot\vec{D})^{-1}]^c{}_a G_c(\vec{x}), F^b(\vec{y})\}$$

$$= [(\vec{\partial}\cdot\vec{D})^{-1}]^c{}_a\{G_c(\vec{x}), F^b(\vec{y})\} = -[(\vec{\partial}\cdot\vec{D})^{-1}]^c{}_a(\vec{\partial}\cdot\vec{D})_c{}^b(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}) = -\delta_a{}^b\delta^{(3)}(\vec{x}-\vec{y})$$

we verify that this is the case.

4.4 The Superfield Representation

The unphysical sector of the phase space is locally converted into a PS superspace by first identifying the Darboux coordinates $(\tilde{F}^a, \tilde{G}_a)$ as the unphysical (non-gauge fixed) parts of the ambient phase space variables $(\vec{A}^a, \vec{E}_a)$. For each existing pair $(\tilde{F}^a, \tilde{G}_a)$ we extend the phase space by introducing a canonically conjugate pair (B^a, ω_a) of auxiliary fields. These variables play a role analogous to that of the Lagrange multipliers introduced before except now as part of a field theory and in the new coordinates defined by the abelianisation procedure. Note that we are constructing the theory fundamentally from the ambient phase space which does not include the temporal gauge field as previously discussed. The auxiliary fields can also be taken to be Grassmann even and they satisfy

$$\{\omega_a(\vec{x}), B^b(\vec{y})\} = -\delta_a{}^b\delta^{(3)}(\vec{x}-\vec{y}) \tag{4.27}$$

Finally, canonical pairs of ghost fields $(\eta^a, \bar{\eta}_a)$, $(\bar{\zeta}^a, \zeta_a)$ are introduced for each unphysical degree of freedom, they satisfy the continuous generalisation of the fermionic Poisson bracket structure (2.6);

$$\{\eta^a(\vec{x}), \bar{\eta}_b(\vec{y})\} = \{\bar{\zeta}^a(\vec{x}), \zeta_b(\vec{y})\} = -\delta^a_b\delta^{(3)}(\vec{x}-\vec{y}). \tag{4.28}$$

Using the position-like variables $(\tilde{F}^a, B^a, \eta^a, \bar{\zeta}^a)$ and momentum-like variables $(\tilde{G}_a, \omega_a, \zeta_a, \bar{\eta}_a)$, the task is now to form a representation of the unphysical sector as a PS superspace over the Grassmann coordinates $(x,\theta,\bar{\theta})$ introduced previously.

For a non-abelian gauge theory, we can not use the light-cone position and momentum variables (4.12). There is an alternative and more direct way to introduce and parametrise each four-dimensional superspace corresponding to each constraint index "a" in this case. Recalling that commuting and anti-commuting variables of the same dynamical system are also elements of the same Grassmann algebra. We can use the fact that superfunctions of these variables are also elements of the same Grassmann algebra to define superspace coordinates which are expansions of these variables over the $(\theta,\bar{\theta})$ directions. Superfunctions of fields are appropriately named *superfield multiplets*, or simply *superfields*, a topic discussed in depth for example in [18]. In this case, there are only two distinct anti-commuting coordinate directions so the highest order term in the expansion will be proportional to $\bar{\theta}\theta$. Accordingly, for each subspace identified by a constraint index a, we introduce a 4-dimensional PS superspace with superfield (SF) position and momentum variables defined as

$$F^{(\mathrm{SF})a}(x,\theta,\bar{\theta}) \equiv \tilde{F}^a + \bar{\zeta}^a\theta + \eta^a\bar{\theta} + B^a\bar{\theta}\theta; \tag{4.29}$$

$$G_a^{(\mathrm{SF})}(x,\theta,\bar{\theta}) \equiv \omega_a + \theta\bar{\eta}_a + \bar{\theta}\zeta_a + \tilde{G}_a\bar{\theta}\theta. \tag{4.30}$$

Here, the dependence on the spacetime coordinate x is implicit in the constituent fields. To see how this representation of the superspace gives a more direct description of the desired PS supersymmetry, we note that the Grassmann odd supercharges $R^{-\theta}, R^{-\bar{\theta}}$ are now associated with the operators $Q_{\mathrm{PS}}, \bar{Q}_{\mathrm{PS}}$ which generate *translations* in the θ and $\bar{\theta}$ directions respectively. Hence, the transformation rules for the individual fields in each multiplet can be derived from the equations

$$Q_{\mathrm{PS}}F^{(\mathrm{SF})} = \partial_\theta F^{(\mathrm{SF})}, \qquad \bar{Q}_{\mathrm{PS}}F^{(\mathrm{SF})} = \partial_{\bar{\theta}} F^{(\mathrm{SF})}; \tag{4.31}$$

$$Q_{\mathrm{PS}}G^{(\mathrm{SF})} = \partial_\theta G^{(\mathrm{SF})}, \qquad \bar{Q}_{\mathrm{PS}}G^{(\mathrm{SF})} = \partial_{\bar{\theta}} G^{(\mathrm{SF})}. \tag{4.32}$$

Where the constraint index has been suppressed for notational convenience. To evaluate derivatives of the superfields with respect to the fermionic coordinates $(\theta,\bar{\theta})$, we adopt the convention of left differentiation. In addition, the fact that derivative operators of Grassmann odd coordinates are also anti-commuting will be used in what follows. Suppressing the constraint indices momentarily, for the θ-direction, we have

$$\begin{aligned} Q_{\mathrm{PS}}F^{(\mathrm{SF})} &= (Q_{\mathrm{PS}}\tilde{F}) + (Q_{\mathrm{PS}}\bar{\zeta})\theta + (Q_{\mathrm{PS}}\eta)\bar{\theta} + (Q_{\mathrm{PS}}B)\bar{\theta}\theta = \partial_\theta F^{(\mathrm{SF})} = -\bar{\zeta} - B\bar{\theta}, \\ Q_{\mathrm{PS}}G^{(\mathrm{SF})} &= (Q_{\mathrm{PS}}\omega) - \theta(Q_{\mathrm{PS}}\bar{\eta}) - \bar{\theta}(Q_{\mathrm{PS}}\zeta) + (Q_{\mathrm{PS}}\tilde{G})\bar{\theta}\theta = \partial_\theta F^{(\mathrm{SF})} = \bar{\eta} - \bar{\theta}\tilde{G}. \end{aligned} \tag{4.33}$$

By matching the left and right hand side we find that the *PS transformations* are

$$\begin{aligned} &Q_{\text{PS}}\tilde{F} = -\bar{\zeta} && Q_{\text{PS}}\omega = \bar{\eta} \\ &Q_{\text{PS}}\bar{\zeta} \ = 0 && Q_{\text{PS}}\bar{\eta} = 0 \\ &Q_{\text{PS}}\eta \ = -B && Q_{\text{PS}}\zeta = \tilde{G} \\ &Q_{\text{PS}}B = 0 && Q_{\text{PS}}\tilde{G} = 0 \end{aligned} \tag{4.34}$$

Similarly, the equations for the $\bar{\theta}$-direction are

$$\begin{aligned} \bar{Q}_{\text{PS}}F^{(\text{SF})} &= (\bar{Q}_{\text{PS}}\tilde{F}) + (\bar{Q}_{\text{PS}}\bar{\zeta})\theta + (\bar{Q}_{\text{PS}}\eta)\bar{\theta} + (\bar{Q}_{\text{PS}}B)\bar{\theta}\theta = \partial_{\bar{\theta}}F^{(\text{SF})} = -\eta + B\theta, \\ \bar{Q}_{\text{PS}}G^{(\text{SF})} &= (\bar{Q}_{\text{PS}}\omega) - \theta(\bar{Q}_{\text{PS}}\bar{\eta}) - \bar{\theta}(\bar{Q}_{\text{PS}}\zeta) + (\bar{Q}_{\text{PS}}\tilde{G})\bar{\theta}\theta = \partial_{\bar{\theta}}F^{(\text{SF})} = \zeta + \theta\tilde{G}. \end{aligned} \tag{4.35}$$

Leading to the *anti-PS transformations*

$$\begin{aligned} &\bar{Q}_{\text{PS}}\tilde{F} = -\eta && \bar{Q}_{\text{PS}}\omega = \zeta \\ &\bar{Q}_{\text{PS}}\bar{\zeta} \ = B && \bar{Q}_{\text{PS}}\bar{\eta} = -\tilde{G} \\ &\bar{Q}_{\text{PS}}\eta \ = 0 && \bar{Q}_{\text{PS}}\zeta = 0 \\ &\bar{Q}_{\text{PS}}B = 0 && \bar{Q}_{\text{PS}}\tilde{G} = 0 \end{aligned} \tag{4.36}$$

In accordance with the PS formalism, the nilpotent transformations (4.34, 4.36) of the fields under the action of the operators $Q_{\text{PS}}, \bar{Q}_{\text{PS}}$ allow us to find PS exact combinations of the superspace variables $(F^{(\text{SF})a}, G_a^{(\text{SF})})$ that can be freely added to the action.

5 Parisi-Sourlas Quantisation of Yang-Mills Theory

Starting from the Hamiltonian formulation of pure Yang-Mills theory, we have now taken the necessary steps to proceed with quantisation in the Parisi-Sourlas formalism. In particular, the change of variables $(F^a, G_a) \to (\tilde{F}^a, \tilde{G}_a)$ has been made to the Darboux chart in which the constraints are abelian and the superfield representation of the PS supersymmetry group has been adopted. We can now construct a PS invariant path-integral in a coordinate system where all of the fields are localised.

5.1 Constructing an Action

To determine the allowed form of the action such that the PS supersymmetry is respected in the unphysical sector, recall that Grassmann integration and differentiation are essentially equivalent operations as described in (4.10). It follows that when we write down the canonical action of the superspace, only terms proportional to $\bar{\theta}\theta$ will contribute, in particular

$$S_{\text{unphysical}} = \int d^4x d\theta d\bar{\theta}(\cdots) = \int d^4x \partial_\theta \partial_{\bar{\theta}}(\cdots).$$

To find candidate terms to be added to such an action on top of the canonical 1-form $(G_a^{(\mathrm{SF})}\partial_t F^{(\mathrm{SF})a})$, consider the combinations $(F^{(\mathrm{SF})a}G_a^{(\mathrm{SF})})$, $(F^{(\mathrm{SF})a})^2$ and $(G_a^{(\mathrm{SF})})^2$ of the superfields. The decision to analyse these three terms is made based on the fact that the superrotations of the PS supersymmetry group should preserve the inner product on the superspace. Quadratic terms and contractions of the superfields are therefore of great interest. Using the definitions (4.29, 4.30), and the fact that $\theta^2 = \bar{\theta}^2 = 0$ one finds

$$
\begin{aligned}
F^{(\mathrm{SF})a}G_a^{(\mathrm{SF})} &= \tilde{F}^a(\omega_a + \theta\bar{\eta}_a + \bar{\theta}\zeta_a + \bar{\theta}\theta\tilde{G}_a) + \bar{\zeta}^a\theta(\omega_a + \bar{\theta}\zeta_a) + \eta^a\bar{\theta}(\omega_a + \theta\bar{\eta}_a) + B^a\omega_a\bar{\theta}\theta \\
&= \tilde{F}^a\omega_a + \theta(\tilde{F}^a\bar{\eta}_a - \bar{\zeta}^a\omega_a) + \bar{\theta}(\tilde{F}^a\zeta_a - \eta^a\omega_a) + \bar{\theta}\theta(B^a\omega_a - \bar{\zeta}^a\zeta_a + \eta^a\bar{\eta}_a + \tilde{F}^a\tilde{G}_a).
\end{aligned}
$$

The second combination is

$$
\begin{aligned}
(F^{(\mathrm{SF})a})^2 &= (\tilde{F}^a)^2 + \tilde{F}^a(\bar{\zeta}^a\theta + \eta^a\bar{\theta} + B^a\bar{\theta}\theta) + \bar{\zeta}^a\theta\eta^a\bar{\theta} + \eta^a\bar{\theta}\bar{\zeta}^a\theta + \tilde{F}^a(\eta^a\bar{\theta} + \bar{\zeta}^a\theta + B^a\bar{\theta}\theta) \\
&= (\tilde{F}^a)^2 + 2\tilde{F}^a(\bar{\zeta}^a\theta + \eta^a\bar{\theta}) + 2(\bar{\zeta}^a\eta^a + \tilde{F}^a B^a)\bar{\theta}\theta.
\end{aligned}
$$

And finally

$$
\begin{aligned}
(G_a^{(\mathrm{SF})})^2 &= (\omega_a)^2 + \omega_a(\theta\bar{\eta}_a + \bar{\theta}\zeta_a + \tilde{G}_a\bar{\theta}\theta) + \omega_a(\theta\bar{\eta}_a + \bar{\theta}\zeta_a) + \theta\bar{\eta}_a\bar{\theta}\zeta_a + \bar{\theta}\zeta_a\theta\bar{\eta}_a + \omega_a\tilde{G}_a\bar{\theta}\theta \\
&= (\omega_a)^2 + 2\omega_a(\theta\bar{\eta}_a + \bar{\theta}\zeta_a) + 2(\tilde{G}_a\omega_a + \bar{\eta}_a\zeta_a)\bar{\theta}\theta.
\end{aligned}
$$

As we are only interested in terms proportional to $\bar{\theta}\theta$ we reduce these expressions by dropping all the other terms leaving only

$$
\begin{aligned}
F^{(\mathrm{SF})a}G_a^{(\mathrm{SF})} &= \bar{\theta}\theta(B^a\omega_a - \bar{\zeta}^a\zeta_a + \eta^a\bar{\eta}_a + \tilde{F}^a\tilde{G}_a) + \cdots, \\
(F^{(\mathrm{SF})a})^2 &= 2\bar{\theta}\theta(\bar{\zeta}^a\eta^a + \tilde{F}^a B^a) + \cdots, \\
(G_a^{(\mathrm{SF})})^2 &= 2\bar{\theta}\theta(\tilde{G}_a\omega_a + \bar{\eta}_a\zeta_a) + \cdots.
\end{aligned}
\tag{5.1}
$$

Before adding these terms to the action, it must be verified that they are PS exact so that the action possesses the relevant PS supersymmetry. Indeed, we simply use the transformation rules (4.34, 4.36) for our fields which reveals that

$$
\begin{aligned}
Q_{\mathrm{PS}}\int d\theta d\bar{\theta}(F^{(\mathrm{SF})a}G_a^{(\mathrm{SF})}) &= (Q_{\mathrm{PS}}B^a)\omega_a + B^a(Q_{\mathrm{PS}}\omega_a) - (Q_{\mathrm{PS}}\bar{\zeta}^a)\zeta_a + \bar{\zeta}^a(Q_{\mathrm{PS}}\zeta_a) \\
+ (Q_{\mathrm{PS}}\eta^a)\bar{\eta}_a - \eta^a(Q_{\mathrm{PS}}\bar{\eta}_a) + (Q_{\mathrm{PS}}\tilde{F}^a)\tilde{G}_a &+ \tilde{F}^a(Q_{\mathrm{PS}}\tilde{G}_a) = B^a\bar{\eta}_a + \bar{\zeta}^a\tilde{G}_a - B^a\bar{\eta}_a - \bar{\zeta}^a\tilde{G}_a = 0,
\end{aligned}
$$

for the cross term and

$$
\begin{aligned}
Q_{\mathrm{PS}}\int d\theta d\bar{\theta}(F^{(\mathrm{SF})a})^2 &= 2[(Q_{\mathrm{PS}}\bar{\zeta}^a)\eta^a - \bar{\zeta}^a(Q_{\mathrm{PS}}\eta^a) + (Q_{\mathrm{PS}}\tilde{F}^a)B^a + \tilde{F}^a(Q_{\mathrm{PS}}B^a)] \\
&= 2[\bar{\zeta}^a B^a - \bar{\zeta}^a B^a] = 0,
\end{aligned}
$$

$$
\begin{aligned}
Q_{\mathrm{PS}}\int d\theta d\bar{\theta}(G_a^{(\mathrm{SF})})^2 &= 2[(Q_{\mathrm{PS}}\tilde{G}_a)\omega_a + \tilde{G}_a(Q_{\mathrm{PS}}\omega_a) + (Q_{\mathrm{PS}}\bar{\eta}_a)\zeta_a - \bar{\eta}_a(Q_{\mathrm{PS}}\zeta_a)] \\
&= 2[\tilde{G}_a\bar{\eta}_a - \tilde{G}_a\bar{\eta}_a] = 0,
\end{aligned}
$$

for the quadratic terms. It should be noted that the vanishing of these terms is also a consequence of the operator $Q_{\rm PS}$ being identifiable with the ∂_θ in the differential representation. By virtue of Grassmann differentiation this means we are essentially taking two integrals over the fermionic coordinate θ which is guaranteed to vanish. Now we can write the action on the PS superspace as

$$S_{\rm unphysical} = \int d^4x d\theta d\bar{\theta}[(\partial_t + 1)(F^{({\rm SF})a})G_a^{({\rm SF})} + (F^{({\rm SF})a})^2 + (G_a^{({\rm SF})})^2]. \tag{5.2}$$

This can be written in terms of the fields using (5.1), after performing a trivial integration of $\bar{\theta}\theta$ the action is

$$\begin{aligned} S_{\rm unphysical} = \int d^4x[(\partial_t + 1)(\tilde{F}^a)\tilde{G}_a - (\partial_t + 1)(\bar{\zeta}^a)\zeta_a + (\partial_t + 1)(\eta^a)\bar{\eta}_a \\ + (\partial_t + 1)(B^a)\omega_a + 2(\bar{\zeta}^a\eta^a + \tilde{F}^a B^a) + 2(\tilde{G}_a\omega_a + \bar{\eta}_a\zeta_a)]. \end{aligned} \tag{5.3}$$

The path integral associated with this action takes the form of (4.17) but for the current variables we are working with this would be a highly non-local solution to the problem posed by the constraints of the theory so we will postpone such an expression until these undesirable properties have been addressed.

5.2 The Gauge Fermion

To make the relationship between the BRST and PS supersymmetry more explicit in the context of pure Yang-Mills theory, we will show that it is also possible to derive the analogue of the BRST charge Ω in the superfield representation. In the lightcone coordinate representation used to illustrate the PS formalism, the BRST charge was identified with the generator $R^{-\theta}$, that acts according to (4.16). In the superfield representation, the gauge fermion is first derived from the action (5.2) by requiring that

$$\int d\theta d\bar{\theta}[F^{({\rm SF})a}G_a^{({\rm SF})} + (F^{({\rm SF})a})^2 + (G_a^{({\rm SF})})^2] = Q_{\rm PS}\Psi = \{Q_B, \Psi\}, \tag{5.4}$$

where Q_B is the supercharge which generates the PS transformations (4.34). In other words, the PS transformations we have derived can equivalently be written as

$$Q_{\rm PS}\chi = \{Q_B, \chi\},$$

for any field χ on the superspace. The gauge fermion is then obtained by writing the LHS integrand of (5.4) as the operator $Q_{\rm PS}$ acting on a smaller set of terms:

$$\begin{aligned} &\int d\theta d\bar{\theta}[F^{({\rm SF})a}G_a^{({\rm SF})} + (F^{({\rm SF})a})^2 + (G_a^{({\rm SF})})^2] = B^a\omega_a - \bar{\zeta}^a\zeta_a + \eta^a\bar{\eta}_a + \tilde{F}^a\tilde{G}_a \\ &+ 2(\bar{\zeta}^a\eta^a + \tilde{F}^a B^a) + 2(\tilde{G}_a\omega_a + \bar{\eta}_a\zeta_a) = Q_{\rm PS}[(\tilde{F}^a\zeta_a - \omega_a\eta^a) - 2(\tilde{F}^a\eta^a) + 2(\omega_a\zeta_a)], \end{aligned}$$

such that the gauge fermion can be read off as

$$\Psi = (\tilde{F}^a\zeta_a - \omega_a\eta^a) - 2(\tilde{F}^a\eta^a) + 2(\omega_a\zeta_a). \tag{5.5}$$

The supercharge Q_B is then the combination of fields on the superspace whose Poisson bracket with the gauge fermion is precisely $Q_{\mathrm{PS}}\Psi$. We will now show that this is given by the ansatz

$$Q_B = \int d^3\vec{y}\Big(\bar{\zeta}^b(\vec{y})\tilde{G}_b(\vec{y}) + B^b(\vec{y})\bar{\eta}_b(\vec{y})\Big), \tag{5.6}$$

by computing the bracket directly;

$$\begin{aligned}
\{Q_B, \Psi\} &= \int d^3\vec{y}\{\bar{\zeta}^b(\vec{y})\tilde{G}_b(\vec{y}) + B^b(\vec{y})\bar{\eta}_b(\vec{y})\ ,\ \tilde{F}^a(\vec{x})\zeta_a(\vec{x}) - \omega_a(\vec{x})\eta^a(\vec{x})\} \\
&+ \int d^3\vec{y}\{\bar{\zeta}^b(\vec{y})\tilde{G}_b(\vec{y}) + B^b(\vec{y})\bar{\eta}_b(\vec{y}), -2\tilde{F}^a(\vec{x})\eta^a(\vec{x}) + 2\omega_a(\vec{x})\zeta_a(\vec{x})\} \\
&= \int d^3\vec{y}\{\bar{\zeta}^b(\vec{y}), \tilde{F}^a(\vec{x})\zeta_a(\vec{x}) + 2\omega_a(\vec{x})\zeta_a(\vec{x})\}\tilde{G}_b(\vec{y}) \\
&+ \int d^3\vec{y}\bar{\zeta}^b(\vec{y})\{\tilde{G}_b(\vec{y}), \tilde{F}^a(\vec{x})\zeta_a(\vec{x}) - 2\tilde{F}^a(\vec{x})\eta^a(\vec{x})\} \\
&+ \int d^3\vec{y}\{\bar{\eta}_b(\vec{y}), -\omega_a(\vec{x})\eta^a(\vec{x}) - 2\tilde{F}^a(\vec{x})\eta^a(\vec{x})\}B^b(\vec{y}) \\
&+ \int d^3\vec{y}\bar{\eta}_b(\vec{y})\{B^b(\vec{y}), -\omega_a(\vec{x})\eta^a(\vec{x}) + 2\omega_a(\vec{x})\zeta_a(\vec{x})\}.
\end{aligned}$$

Once again applying the Leibniz rule yields

$$\begin{aligned}
= \int d^3\vec{y}\big[&- \tilde{F}^a(\vec{x})\{\zeta_a(\vec{x}), \bar{\zeta}^b(\vec{y})\}\tilde{G}_b(\vec{y}) - 2\omega_a(\vec{x})\{\zeta_a(\vec{x}), \bar{\zeta}^b(\vec{y})\}\tilde{G}_b(\vec{y}) \\
&- \bar{\zeta}^b(\vec{y})\{\tilde{F}^a(\vec{x}), \tilde{G}_b(\vec{y})\}\zeta_a(\vec{x}) + 2\bar{\zeta}^b(\vec{y})\{\tilde{F}^a(\vec{x}), \tilde{G}_b(\vec{y})\}\eta^a(\vec{x}) \\
&- \omega_a(\vec{x})\{\eta^a(\vec{x}), \bar{\eta}_b(\vec{y})\}B^b(\vec{y}) - 2\tilde{F}^a(\vec{x})\{\eta^a(\vec{x}), \bar{\eta}_b(\vec{y})\}B^b(\vec{y}) \\
&+ \bar{\eta}_b(\vec{y})\{\omega_a(\vec{x}), B^b(\vec{y})\}\eta^a(\vec{x}) - 2\bar{\eta}^b(\vec{y})\{\omega_a(\vec{x}), B^b(\vec{y})\}\zeta_a(\vec{x})\big],
\end{aligned}$$

which can be evaluated directly using the known Poisson brackets of the fields

$$\begin{aligned}
= \int d^3\vec{y}[&\tilde{F}^a(\vec{x})\tilde{G}_a(\vec{y}) + 2\tilde{G}_a(\vec{y})\omega_a(\vec{x}) - \bar{\zeta}^a(\vec{y})\zeta_a(\vec{x}) + 2\bar{\zeta}^a(\vec{y})\eta^a(\vec{x}) + B^a(\vec{y})\omega_a(\vec{x}) \\
&+ 2\tilde{F}^a(\vec{x})B^a(\vec{y}) + \eta^a(\vec{x})\bar{\eta}_a(\vec{y}) + 2\bar{\eta}_a(\vec{y})\zeta_a(\vec{x})]\delta^{(3)}(\vec{x} - \vec{y}).
\end{aligned}$$

Integrating out the delta function simply replaces all instances of $\vec{y}$ with $\vec{x}$ in the integrand and our final expression is the PS exact term from the action as needed

$$\begin{aligned}
\{Q_B, \Psi\} = &\tilde{F}^a(\vec{x})\tilde{G}_a(\vec{x}) - \bar{\zeta}^a(\vec{x})\zeta_a(\vec{x}) + \eta^a(\vec{x})\bar{\eta}_a(\vec{x}) + B^a(\vec{x})\omega_a(\vec{x}) \\
&+ 2\big[\bar{\zeta}^a(\vec{x})\eta^a(\vec{x}) + \tilde{F}^a(\vec{x})B^a(\vec{x}) + \tilde{G}_a(\vec{x})\omega_a(\vec{x}) + \bar{\eta}_a(\vec{x})\zeta_a(\vec{x})\big].
\end{aligned} \tag{5.7}$$

For completeness, we can then write down the canonical action on the superspace in terms of the supercharge Q_B;

$$S_{\text{unphysical}} = \int d^4x \bigg[\int d\theta d\bar{\theta} \partial_t (F^{(\text{SF})a}) G_a^{(\text{SF})} + \{Q_B, \Psi\} \bigg] \tag{5.8}$$

which is identical to (5.3). This analogy can be extended even further by noticing that the gauge fermion itself can be written as the anti-PS operator $\bar{Q}_{\text{PS}}$ acting on an even smaller set of terms on the superspace. To obtain this highly symmetric form of the action, we use the anti-PS transformations (4.36) to verify that the gauge fermion can be written as

$$\Psi = \bar{Q}_{\text{PS}}[(\tilde{F}^a + \omega_a)^2 - \tilde{F}^a \omega_a] = \bar{Q}_{\text{PS}}[(\tilde{F}^a)^2 + (\omega_a)^2 + \tilde{F}^a \omega_a] \equiv \bar{Q}_{\text{PS}} \Phi \tag{5.9}$$

where Φ is now a bosonic combination of the fields. Indeed we can compute

$$\begin{aligned}
\bar{Q}_{\text{PS}} \Phi &= \bar{Q}_{\text{PS}}(\tilde{F}^a) \tilde{F}^a + \tilde{F}^a \bar{Q}_{\text{PS}}(\tilde{F}^a) + \bar{Q}_{\text{PS}}(\omega_a) \omega_a \\
&\quad + \omega_a \bar{Q}_{\text{PS}}(\omega_a) + \bar{Q}_{\text{PS}}(\tilde{F}^a) \omega_a + \tilde{F}^a \bar{Q}_{\text{PS}}(\omega_a) \\
&= \tilde{F}^a \zeta_a - \omega_a \eta^a - 2(\tilde{F}^a \eta^a) + 2(\omega_a \zeta_a) = \Psi.
\end{aligned}$$

In this sense, we can now introduce the analogue of the anti BRST charge as the supercharge $\bar{Q}_B$ which is defined as the combination of fields on the superspace such that

$$\Psi = \bar{Q}_{\text{PS}} \Phi = \{\bar{Q}_B, \Phi\}. \tag{5.10}$$

Since the gauge fermion must be Grassmann odd we recognise that the supercharge $\bar{Q}_B$ must also be Grassmann odd. We will now show that it is given by the ansatz

$$\bar{Q}_B = \int d^3\vec{y} \bigg(B^b(\vec{y}) \zeta_b(\vec{y}) + \tilde{G}_b(\vec{y}) \eta^b(\vec{y}) \bigg), \tag{5.11}$$

by computing the Poisson bracket (5.10) directly:

$$\begin{aligned}
\{\bar{Q}_B, \Phi\} &= \int d^3\vec{y} \{B^b(\vec{y}) \zeta_b(\vec{y}) + \tilde{G}_b(\vec{y}) \eta^b(\vec{y}), (\tilde{F}^a(\vec{x}))^2 + (\omega_a(\vec{x}))^2 + \tilde{F}^a(\vec{x}) \omega_a(\vec{x})\} \\
&= \int d^3\vec{y} \{B^b(\vec{y}) \zeta_b(\vec{y}), \omega_a(\vec{x}) \omega_a(\vec{x}) + \tilde{F}^a(\vec{x}) \omega_a(\vec{x})\} \\
&+ \int d^3\vec{y} \{\tilde{G}_b(\vec{y}) \eta^b(\vec{y}), \tilde{F}^a(\vec{x}) \tilde{F}^a(\vec{x}) + \tilde{F}^a(\vec{x}) \omega_a(\vec{x})\} \\
&= \int d^3\vec{y} \bigg(\{B^b(\vec{y}), \omega_a(\vec{x}) \omega_a(\vec{x})\} \zeta_b(\vec{y}) + \{B^b(\vec{y}), \tilde{F}^a(\vec{x}) \omega_a(\vec{x})\} \zeta_b(\vec{y}) \bigg) \\
&+ \int d^3\vec{y} \bigg(\{\tilde{G}_b(\vec{y}), \tilde{F}^a(\vec{x}) \tilde{F}^a(\vec{x})\} \eta^b(\vec{y}) + \{\tilde{G}_b(\vec{y}), \tilde{F}^a(\vec{x}) \omega_a(\vec{x})\} \eta^b(\vec{y}) \bigg) \\
&= \int d^3\vec{y} \bigg(-2\omega_a(\vec{x}) \{\omega_a(\vec{x}), B^b(\vec{y})\} \zeta_b(\vec{y}) - \tilde{F}^a(\vec{x}) \{\omega_a(\vec{x}), B^b(\vec{y})\} \zeta_b(\vec{y}) \bigg) \\
&+ \int d^3\vec{y} \bigg(-2\tilde{F}^a(\vec{x}) \{\tilde{F}^a(\vec{x}), \tilde{G}_b(\vec{y})\} \eta^b(\vec{y}) - \omega_a(\vec{x}) \{\tilde{F}^a(\vec{x}), \tilde{G}_b(\vec{y}\} \eta^b(\vec{y}) \bigg).
\end{aligned}$$

Now using the known Poisson bracket relations between the fields this reduces to

$$
\begin{aligned}
\{\bar{Q}_B, \Phi\} &= \int d^3\vec{y}\big(2\omega_a(\vec{x})\zeta_a(\vec{y}) + \tilde{F}^a(\vec{x})\zeta_a(\vec{y}) - 2\tilde{F}^a(\vec{x})\eta^a(\vec{y}) - \omega_a(\vec{x})\eta^a(\vec{y})\big)\delta^{(3)}(\vec{x}-\vec{y}) \\
&= \tilde{F}^a(\vec{x})\zeta_a(\vec{x}) - \omega_a(\vec{x})\eta^a(\vec{x}) - 2\tilde{F}^a(\vec{x})\eta^a(\vec{x}) + 2\omega_a(\vec{x})\zeta_a(\vec{x}) = \Psi.
\end{aligned}
$$

Now that we have made this association explicit, the PS exact term in the superspace action can be written as a composition of BRST commutators in terms of the supercharges $Q_B, \bar{Q}_B$ and the bosonic contribution ϕ:

$$
S_{\text{unphysical}} = \int d^4x \left[\int d\theta d\bar{\theta} \partial_t (F^{(\text{SF})a}) G_a^{(\text{SF})} + \{Q_B, \{\bar{Q}_B, \Phi\}\} \right]. \tag{5.12}
$$

This can be written in an even more elegant form if we consider the differential representation of the Parisi-Sourlas transformations (4.31, 4.32), in this case we write

$$
\{Q_B, \{\bar{Q}_B, \Phi\}\} = Q_{\text{PS}} \bar{Q}_{\text{PS}} \Phi \equiv \partial_\theta \partial_{\bar{\theta}} \Phi = \int d\theta d\bar{\theta} \Phi. \tag{5.13}
$$

Now the superspace action for the unphysical sector of the theory can be written in the compact form

$$
S_{\text{unphysical}} = \int d^4x d\theta d\bar{\theta} \Big(\partial_t (F^{(\text{SF})a}) G_a^{(\text{SF})} + \Phi \Big). \tag{5.14}
$$

The equivalence of the action, as presented in (5.14), to the result obtained using the standard supersymmetric BRST quantisation method is made clear by the relation (5.13). The remaining step to be taken in our analysis is to recast this integral in a local form. The supercharges $Q_B, \bar{Q}_B$ are related to their counterparts in supersymmetric BRST quantisation by a canonical transformation as we have stated previously. This provides a convenient possibility for writing the fields we have introduced in the PS formalism in terms of their local counterparts on the original phase space.

5.3 Canonical Transformation

The action (5.14), which can be written out explicitly as in (5.3), is the desired action for the unphysical sector of the theory but in order for the path-integral to be expressed fully in terms of local fields, we must perform a coordinate transformation. The best way to go about such a coordinate transformation, without disturbing the canonical structure of the theory, is to perform a canonical transformation. Determining the correct method to use however, is non-trivial in this case.

Firstly, we know that the phase space we have constructed is essentially divided into the physical sector and the unphysical sector. The position and momentum variables of the former are the gauge fixed fields $(\vec{A}^a_{gf}, \vec{E}^a_{gf})$ which are nothing but the original fields on the ambient phase space projected onto their transverse components $(\vec{A}^a)^\perp, (\vec{E}_a)^\perp$. The unphysical sector has as its position variables the gauge fixing conditions $\tilde{F}^a = (\vec{\partial} \cdot \vec{A})^a$ along with the auxiliary and ghost fields introduced in the PS formalism. The unphysical momenta are the Gauss law constraints in the Darboux chart $\tilde{G}_a$ with similar extensions. If we follow the generating functional approach, the full set $((\vec{A}^a)^\perp, F^{(\mathrm{SF})a}, (\vec{E}^a)^\perp, G_a^{(\mathrm{SF})})$ of coordinates on the entire phase space define the "new" variables. Note that we are using the (SF) notation to compactly represent the four fields within each multiplet. A type 2 generating functional will depend on the "old" position variables but the gauge field $\vec{A}^a$ on the ambient phase space does not extend to the superspace parametrised by $(\theta, \bar{\theta})$ in our current construction. A canonical transformation only makes sense if the old and new coordinates span the same phase space so it follows that the ambient phase space must be extended beforehand.

In order to achieve this, we apply a local supersymmetric gauge transformation to $\vec{A}^a$ which extends its domain into the superspace. This transformation will take exactly the same form as (3.7) except the gauge parameter ϑ^a is now replaced by the superfield multiplet,

$$\Theta^a(x) = \bar{b}^a(x)\theta + c^a(x)\bar{\theta} + \mathcal{B}^a(x)\bar{\theta}\theta, \tag{5.15}$$

which does not include the central bosonic term ϑ^a already built into the gauge field. This choice of gauge parameter leaves the original part of $\vec{A}$ invariant but introduces additional extensions to it which are functions of the position-like ghost fields $(c^a, \bar{b}^a)$ and the auxiliary field $\mathcal{B}^a$. We denote the result of this transformation as the gauge superfield $\vec{\mathcal{A}}$ which can be computed in the infinitesimal limit where we expand

$$U(x) = \mathbb{1} + ig\Theta(x) + \frac{(ig)^2}{2}\Theta^2 + \cdots . \tag{5.16}$$

Note that the gauge multiplet Θ has a contribution at order Θ^2 in this case. In particular, computing Θ^2 in matrix notation, we find

$$\Theta^2 = \bar{b}c\bar{\theta}\theta - c\bar{b}\bar{\theta}\theta = [\bar{b}, c]\bar{\theta}\theta = i\bar{b}^a c^b f_{ab}{}^c t_c \bar{\theta}\theta$$

Then

$$\begin{aligned}
\vec{\mathcal{A}}(x) &= U(x)\left(\vec{A}(x) + \frac{i}{g}\vec{\partial}_x\right)U^{-1}(x) \\
&= \left(\mathbb{1} + ig\Theta(x)\frac{(ig)^2}{2}\Theta^2(x)\right)\left(\vec{A}(x) + \frac{i}{g}\vec{\partial}_x\right)\left(\mathbb{1} - ig\Theta(x) + \frac{(ig)^2}{2}\Theta^2(x)\right) \\
&= \vec{A}(x) + \vec{D}\left(\Theta(x) - \frac{ig}{2}\Theta^2(x)\right).
\end{aligned}$$

Writing this out explicitly in terms of components, the gauge superfield is

$$\vec{\mathcal{A}}^a(x) = \vec{A}^a(x) + (\vec{D}\bar{b})^a(x)\theta + (\vec{D}c)^a(x)\bar{\theta} + \left((\vec{D}\mathcal{B})^a(x) + \frac{g}{2}\vec{D}^{ab}(f_{bcd}\bar{b}^c c^d)\right)\bar{\theta}\theta. \tag{5.17}$$

The term at order $\bar{\theta}\theta$ can always be broken up using the Leibniz rule for the gauge covariant derivative which gives

$$\left((\vec{D}\mathcal{B})^a(x) + \frac{g}{2}f^a_{\ bc}(\vec{D}\bar{b})^b(x)c^c(x) + \frac{g}{2}f^a_{\ bc}\bar{b}^b(x)(\vec{D}c)^c(x)\right)\bar{\theta}\theta \tag{5.18}$$

Now that we have established the form of the "old" coordinates, the choice is made to perform the canonical transformation through a type 2 generating functional $\mathcal{F}_2 = \mathcal{F}_2[\vec{\mathcal{A}}^a, (\vec{E}^a)^\perp, G_a^{(\mathrm{SF})}]$. We then propose that the generating functional takes the form of the following ansatz

$$\begin{aligned}
&\mathcal{F}_2[\vec{\mathcal{A}}^a, (\vec{E}^a)^\perp, G_a^{(\mathrm{SF})}] = \int d^3\vec{y}d\theta d\bar{\theta}[G_a^{(\mathrm{SF})}(\vec{y})(\vec{\partial}\cdot\vec{\mathcal{A}})^a(\vec{y})] \\
&= \int d^3\vec{y}\left[\omega_a(\vec{\partial}\cdot\vec{D})^a_{\ b}\left(\mathcal{B}^b + \frac{g}{2}f^b_{\ cd}\bar{b}^c c^d\right) - \bar{\eta}_a(\vec{\partial}\cdot\vec{D})^a_{\ b}c^b + \zeta_a(\vec{\partial}\cdot\vec{D})^a_{\ b}\bar{b}^b + \tilde{G}_a(\vec{\partial}\cdot\vec{A})^a\right],
\end{aligned} \tag{5.19}$$

where the dependence of the fields and derivatives on $\vec{y}$ is suppressed in the integrand. Before writing down the transformation equations for $\mathcal{F}_2$, it is noted that the variables $((\vec{A}^a)^\perp, \tilde{F}^a, (\vec{E}^a)^\perp, \tilde{G}_a)$ do not need to be related to the old variables explicitly since the action on their part of the phase space (the ambient phase space) is nothing but the spacetime integral over $-\frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}$ so we simply declare that the action takes this form when these variables are localised. For the remaining "new" variables, the type 2 transformation equations read

$$\begin{aligned}
\eta^a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\bar{\eta}_a(\vec{x})} = -(\vec{\partial}\cdot\vec{D})^a_{\ b}(\vec{x})c^b(\vec{x}), \\
\bar{\zeta}^a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\zeta_a(\vec{x})} = (\vec{\partial}\cdot\vec{D})^a_{\ b}(\vec{x})\bar{b}^b(\vec{x}), \\
B^a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\omega_a(\vec{x})} = (\vec{\partial}\cdot\vec{D})^a_{\ b}(\vec{x})\left(\mathcal{B}^b(\vec{x}) + \frac{g}{2}f^b_{\ cd}\bar{b}^c(\vec{x})c^d(\vec{x})\right).
\end{aligned} \tag{5.20}$$

To show how these transformations are derived from $\mathcal{F}_2$ we take for example the second equation which gives

$$\begin{aligned}
\bar{\zeta}^a(\vec{x}) = \frac{\delta\mathcal{F}_2}{\delta\zeta_a(\vec{x})} &= \frac{\delta}{\delta\zeta_a(\vec{x})}\int d^3\vec{y}\zeta_b(\vec{y})(\vec{\partial}\cdot\vec{D})^b_{\ c}(\vec{y})\bar{b}^c(\vec{y}) = \int d^3\vec{y}\left(\frac{\delta\zeta_b(\vec{y})}{\delta\zeta_a(\vec{x})}\right)(\vec{\partial}\cdot\vec{D})^b_{\ c}(\vec{y})\bar{b}^c(\vec{y}) \\
&= \int d^3\vec{y}(\vec{\partial}\cdot\vec{D})^b_{\ c}(\vec{y})\bar{b}^c(\vec{y})\delta^a_b\delta^{(3)}(\vec{x}-\vec{y}) = (\vec{\partial}\cdot\vec{D})^a_{\ c}(\vec{x})\bar{b}^c(\vec{x}),
\end{aligned}$$

where the result obtained above is simply given by relabelling the dummy index $c \longleftrightarrow b$. Similarly, the transformation equations for the remaining "old" variables denoted by $(b_a, \bar{c}_a, \varpi_a)$ are

$$\begin{aligned}
b_a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\bar{b}^a(\vec{x})} = (\vec{\partial}\cdot\vec{D})^b_{\ a}(\vec{x})\zeta_b(\vec{x}) + \frac{g}{2}f_{ab}^{\ \ d}c^b(\vec{x})(\vec{\partial}\cdot\vec{D})^c_{\ d}(\vec{x})\omega_c(\vec{x}) \\
\bar{c}_a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta c^a(\vec{x})} = -(\vec{\partial}\cdot\vec{D})^b_{\ a}(\vec{x})\bar{\eta}_b(\vec{x}) + \frac{g}{2}f^{ad}_{\ \ b}\bar{b}^b(\vec{x})(\vec{\partial}\cdot\vec{D})^c_{\ d}(\vec{x})\omega_c(\vec{x}) \\
\varpi_a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\mathcal{B}^a(\vec{x})} = (\vec{\partial}\cdot\vec{D})^b_{\ a}(\vec{x})\omega_b(\vec{x}).
\end{aligned} \tag{5.21}$$

The derivation of these transformations requires a bit more work, for example the third equation is derived using the properties of $(\vec{\partial}\cdot\vec{D})$ under integration by parts as

$$\begin{aligned}
\varpi_a(\vec{x}) = \frac{\delta\mathcal{F}_2}{\delta\mathcal{B}^a(\vec{x})} &= \frac{\delta}{\delta\mathcal{B}^a(\vec{x})}\int d^3\vec{y}\omega_b(\vec{y})(\vec{\partial}\cdot\vec{D})^b{}_c(\vec{y})\mathcal{B}^c(\vec{y}) = -\frac{\delta}{\delta\mathcal{B}^a(\vec{x})}\int d^3\vec{y}(\vec{\partial}\omega(\vec{y}))_b\cdot(\vec{D}\mathcal{B}(\vec{y}))^b \\
&= -\frac{\delta}{\delta\mathcal{B}^a(\vec{x})}\int d^3\vec{y}(\vec{\partial}\omega(\vec{y}))_b\cdot\left[(\vec{\partial}\mathcal{B})^b(\vec{y}) + gf^{blm}\vec{A}^l(\vec{y})\mathcal{B}^m(\vec{y})\right] \\
&= \frac{\delta}{\delta\mathcal{B}^a(\vec{x})}\int d^3\vec{y}\vec{\partial}^2\omega_b(\vec{y})\mathcal{B}^b(\vec{y}) - gf^{blm}\int d^3\vec{y}(\vec{\partial}\zeta(\vec{y}))_b\cdot\vec{A}^l(\vec{y})\mathcal{B}^m(\vec{y}) \\
&= \vec{\partial}^2\omega_a(\vec{x}) + gf^{blm}\int d^3\vec{y}(\vec{\partial}\cdot\vec{A})^l(\vec{y})\omega_b(\vec{y})\left(\frac{\delta\mathcal{B}^m(\vec{y})}{\delta\mathcal{B}^a(\vec{x})}\right) \\
&= \vec{\partial}^2\omega_a(\vec{x}) + gf^{blm}\int d^3\vec{y}(\vec{\partial}\cdot\vec{A})^l(\vec{y})\omega_b(\vec{y})\delta_a^m\delta^{(3)}(\vec{x}-\vec{y}) \\
&= \vec{\partial}^2\omega_a(\vec{x}) + gf^{bl}{}_a(\vec{\partial}\cdot\vec{A})^l(\vec{x})\omega_b(\vec{x}) = (\vec{\partial}\cdot\vec{D})^b{}_a\omega_b(\vec{x}).
\end{aligned}$$

For consistency, it is easy to check that this generating functional also reproduces the relations

$$\begin{aligned}
\tilde{F}^a(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta\tilde{G}_a(\vec{x})}, \\
E_a^i(\vec{x}) &= \frac{\delta\mathcal{F}_2}{\delta A_i^a(\vec{x})},
\end{aligned} \tag{5.22}$$

which it is required to satisfy in order for the canonical transformation to be well-defined. The first of these two relations is trivial since the term $\tilde{G}_a(\vec{y})\tilde{F}^a(\vec{y})$ appears alone in the integrand of $\mathcal{F}_2$. On the other hand, the second relation can be be derived as follows

$$\begin{aligned}
\frac{\delta\mathcal{F}_2}{\delta A_i^a(\vec{x})} &= \int d^3\vec{y}\tilde{G}_b(\vec{y})\frac{\delta(\partial^k A_k)^b(\vec{y})}{\delta A_i^a(\vec{x})} \\
&= -\int d^3\vec{y}\tilde{G}_b(\vec{y})\frac{\delta}{\delta A_i^a(\vec{x})}\int d^3\vec{x}' A_k^b(\vec{x}')\partial_{x'}^k\delta^{(3)}(\vec{y}-\vec{x}') \\
&= -\int d^3\vec{y}\tilde{G}_b(\vec{y})\int d^3\vec{x}'\frac{\delta A_k^b(\vec{x}')}{\delta A_i^a(\vec{x})}\partial_{x'}^k\delta^{(3)}(\vec{y}-\vec{x}') \\
&= \int d^3\vec{y}\partial_y^k\tilde{G}_b(\vec{y})\delta_k^i\delta_a^b\delta^{(3)}(\vec{x}-\vec{y}) = \partial_x^i\tilde{G}_a(\vec{x}) \\
&= \partial^i((\partial_j D^j)^{-1})^b{}_a D^l E_{lb}(\vec{x}) = \delta_j^i\delta_l^j\delta_a^b E_{lb}(\vec{x}) = E_a^i(\vec{x}).
\end{aligned}$$

The transformations (5.20, 5.21) can now be used to rewrite the superspace action (5.14) as a function over the phase space spanned by the position variables $(\vec{A}^a, c^a, \bar{b}^a, \mathcal{B}^a)$ and the momentum variables $(\vec{E}_a, b_a, \bar{c}_a, \varpi_a)$. We will refer to this phase space as the *original* one. Before we do this we must now invert the transformations (5.21) so that the $(\zeta,\bar{\eta})$ ghosts can be written as functions of the original momentum variables. The third transformation can actually be used to simplify the first two and in the end we obtain

$$\begin{aligned}
\zeta_a(\vec{x}) &= [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a b_b(\vec{x}) - \frac{g}{2}f_{ab}{}^c[(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_d c^d(\vec{x})\varpi_c(\vec{x}) \\
\bar{\eta}_a(\vec{x}) &= -[(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a\bar{c}_b(\vec{x}) + \frac{g}{2}f_{ab}{}^c[(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_d\bar{b}^d(\vec{x})\varpi_c(\vec{x}) \\
\omega_a(\vec{x}) &= [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a\varpi_b(\vec{x})
\end{aligned} \tag{5.23}$$

To conclude our analysis and reach a final result, we first note that the action on the original phase space is divided into the Yang Mills action on the ambient phase space

$$S_{YM} = -\frac{1}{4}\int d^4x F^a_{\mu\nu} F^{\mu\nu}_a,$$

and the superspace action (5.3) *without* the bare term involving the Darboux coordinates $(\partial_t + 1)(\tilde{F}^a)\tilde{G}_a$ since these variables are part of the ambient phase space. Then after performing applying the canonical transformation to the remaining terms we reach a gauge fixed action which is invariant under the BRST and anti-BRST transformations generated by Ω and $\bar{\Omega}$ respectively. These charges are given by the localised form of the supercharges $Q_B, \bar{Q}_B$ after the canonical transformation is performed. In particular

$$\begin{aligned}\Omega = &\int d^3\vec{y}\Big[(\vec{\partial}\cdot\vec{D})^a{}_b \bar{b}^b [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a G_b\Big] \\ &+ \int d^3\vec{y}\Big[(\vec{\partial}\cdot\vec{D})^a{}_b\Big(\mathcal{B}^b + \frac{g}{2}f^b{}_{cd}\bar{b}^c c^d\Big)\Big(- [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a \bar{c}_b + \frac{g}{2}f_{ab}{}^c[(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_d \bar{b}^d \varpi_c\Big)\Big],\end{aligned}$$

and applying integration by parts yields

$$\Omega = \int d^3\vec{y}\Big[\bar{b}^a G_a - \Big(\mathcal{B}^a + \frac{g}{2}f^a_{bc}\bar{b}^b c^c\Big)\bar{c}_a + \frac{g}{2}\Big(\mathcal{B}^a + \frac{g}{2}f^a_{bc}\bar{b}^b c^c\Big) f_{ad}{}^e \bar{b}^d \varpi_e\Big]. \tag{5.24}$$

For the anti-BRST charge we have

$$\begin{aligned}\bar{\Omega} = &\int d^3\vec{y}\Big[- [(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a G_b (\vec{\partial}\cdot\vec{D})^a{}_b c^b\Big] \\ &+ \int d^3\vec{y}\Big[(\vec{\partial}\cdot\vec{D})^a{}_b\Big(\mathcal{B}^b + \frac{g}{2}f^b{}_{cd}\bar{b}^c c^d\Big)\Big([(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_a b_b - \frac{g}{2}f_{ab}{}^c[(\vec{\partial}\cdot\vec{D})^{-1}]^b{}_d c^d \varpi_c\Big)\Big]\end{aligned}$$

which can similarly be written as

$$\bar{\Omega} = \int d^3\vec{y}\Big[- c^a G_a + \Big(\mathcal{B}^a + \frac{g}{2}f^a{}_{bc}\bar{b}^b c^c\Big) b_a - \frac{g}{2}\Big(\mathcal{B}^a + \frac{g}{2}f^a{}_{bc}\bar{b}^b c^c\Big) f_{ad}{}^e c^d \varpi_e\Big]. \tag{5.25}$$

These expressions for $\Omega, \bar{\Omega}$ (in our chosen notation) are analogous to the standard BRST and anti-BRST charges which appear as the conserved charges associated with the respective symmetries of the Yang-Mills BRST Lagrangian. In this case, there arises some additional terms due to the fact that the Parisi-Sourlas supersymmetry is an extension of the standard BRST symmetry. The full action for the theory after the canonical transformation is given by

$$S_{YM} + S_{\text{unphysical}} = -\frac{1}{4}\int d^4x\Big[F^a_{\mu\nu}F^{\mu\nu}_a + \int d\theta d\bar{\theta}\partial_t(F^{(\text{SF})a})G^{(\text{SF})}_a - (\partial_t + 1)(\tilde{F}^a)\tilde{G}_a + \{\Omega, \{\bar{\Omega}, \Phi\}\}\Big] \tag{5.26}$$

Explicitly, this action contains a term of the form $\varpi_a G_a$ from which the auxiliary field ϖ can be identified with the temporal component of the gauge field A_t as in (3.18). Furthermore we find that the auxiliary field $\mathcal{B}$ couples to the gauge fixing condition as $F^a\mathcal{B}^a$, thereby enforcing the requirement $F^a = 0$ on the allowed field configurations. We have subtracted the term $(\partial_t + 1)(\tilde{F}^a)\tilde{G}_a$ from the superspace action to compensate for the fact that it has now been absorbed into the ambient action S_{YM}. It follows that the Parisi-Sourlas phase space path-integral is given by

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}\mathcal{B}\mathcal{D}\varpi\mathcal{D}c\mathcal{D}b\mathcal{D}\bar{c}\mathcal{D}\bar{b}\exp\left[i(S_{YM} + S_{\text{unphysical}})\right] \tag{5.27}$$

We have shown that pure Yang-Mills theory can be quantised as a constrained Hamiltonian system in the Parisi-Sourlas formalism. The result is a gauge fixed, phase space path-integral which adheres to the pertinent Parisi-Sourlas supersymmetry, an extension of the BRST supersymmetry commonly used in the quantisation of non-abelian gauge theories. Remarkably, the method used to achieve this result can be applied to quantise systems with first *and* second class constraints while preserving properties such as causality, locality and Lorentz invariance. In the specific treatment of non-Abelian gauge theory we have presented, it is emphasised that the use of the superfield representation in combination with the Parisi-Sourlas formalism is a new technique and we speculate that it could have a wide range of applications as an alternative to BRST quantisation.

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