Solitons in Self-Dual Yang-Mills Theory

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INTRODUCTION

In this project, soliton solutions of the self-dual Yang-Mills (SDYM) equations were studied following recent work, showing that exact multi-soliton solutions can be generated using the method of *Darboux transformations*. In particular, for the gauge group $GL(2,\mathbb{C})$, exact 1 and 2-soliton solutions were derived on the 4-dimensional complex space $\mathbb{CM}$ and the corresponding Lagrangian density $\mathrm{tr}F^{\mu\nu}F_{\mu\nu}$ was computed. The solutions were then subjected to suitable reality conditions reducing the space to $\mathbb{M}$, such that a physical interpretation of their scattering and asymptotic behaviour could be developed.

SELF-DUAL YANG-MILLS THEORY

Yang-Mills is a theory of non-Abelian gauge fields A_μ which are one-forms taking values in some Lie algebra $\mathfrak{g}$, with the corresponding Lie group $\mathcal{G}$ being the *gauge group*. The action and equations of motion for the theory are written in terms of the two-form field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1}$$

Yang-Mills theory becomes an integrable model when the self-duality condition $F_{\mu\nu} = \star F_{\mu\nu}$ is imposed on the field strength, where $\star$ is the Hodge star operator. The SDYM equations are fully determined by this condition and their treatment on real 4-dimensional spaces can be unified in the complex Minkowski space $\mathbb{CM}$ with *double null coordinates* $(z,\tilde{z},w,\tilde{w})$ as

$$F_{zw} = F_{\tilde{z}\tilde{w}} = 0, \qquad F_{z\tilde{z}} + F_{w\tilde{w}} = 0. \tag{2}$$

The first two equations are equivalent to local conditions for the existence of two invertible matrices $h,\tilde{h}$ such that

$$\begin{aligned} A_z &= -(\partial_z h)h^{-1}, \qquad & A_w &= -(\partial_w h)h^{-1}, \\ A_{\tilde{z}} &= -(\partial_{\tilde{z}} \tilde{h})\tilde{h}^{-1}, \qquad & A_{\tilde{w}} &= -(\partial_{\tilde{w}} \tilde{h})\tilde{h}^{-1}. \end{aligned} \tag{3}$$

The SDYM equations then reduce to the gauge invariant Yang equation

$$\partial_{\tilde{z}}(J^{-1}\partial_z J) + \partial_{\tilde{w}}(J^{-1}\partial_w J) = 0, \tag{4}$$

where we define *Yang's J matrix* as $J = h^{-1}\tilde{h}$.

QUASIDETERMINANTS

In lower dimensional integrable models such as the KdV and KP equations, solutions describing more than one soliton are expressed in terms of Wronskian type determinants. In the case of SDYM, a non-commutative version of these determinants is used and this is called a quasideterminant. The quasideterminant of a matrix X with respect to the (i,j)-th element x_{ij} is given by

$$|X|_{ij} \equiv x_{i,j} - R_{i,\hat{j}}\left(X_{\hat{i},\hat{j}}\right)^{-1} C_{\hat{i},j}, \tag{5}$$

where $X_{\hat{i},\hat{j}}$ is the submatrix obtained from X by deleting the i-th row and the j-th column, $R_{i,\hat{j}}$ is the i-th row of X excluding the j-th element and $C_{\hat{i},j}$ is the j-th column of X excluding the i-th element. This definition is invariant under the exchange of the i-th row and j-th column with any other column-row combination. Using this property, we can always write the quasideterminant in *canonical form* as

$$|X|_{ij} = \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B, \qquad \text{where} \quad x_{ij} \equiv d. \tag{6}$$

THE LAX REPRESENTATION

The integrability of SDYM theory follows from the existence of a Lax pair of operators

$$\begin{aligned} L &= D_w - \zeta D_{\tilde{z}}, \\ M &= D_z + \zeta D_{\tilde{w}}, \end{aligned} \tag{7}$$

which satisfy $[L,M] = 0$ whenever the SDYM equations (2) hold. These operators define a linear system of equations for some matrix valued field ϕ as

$$\begin{aligned} L(\phi) &\equiv (\partial_w + A_w)\phi - ((\partial_{\tilde{z}} + A_{\tilde{z}})\phi))\zeta = 0, \\ M(\phi) &\equiv (\partial_z + A_z)\phi + ((\partial_{\tilde{w}} + A_{\tilde{w}})\phi))\zeta = 0. \end{aligned} \tag{8}$$

DARBOUX TRANSFORMATION

Since Yang's J matrix and hence Yang's equation (4) are gauge invariant, we are free to work in the convenient S-gauge which is defined as

$$h = J, \qquad \tilde{h} = I. \tag{9}$$

In this gauge, the gauge fields associated with the coordinates $\tilde{z},\tilde{w}$ vanish while the other gauge fields can be expressed in terms of the J matrix

$$A_{\tilde{z}} = A_{\tilde{w}} = 0, \quad A_z = -(\partial_z J)J^{-1}, \quad A_w = -(\partial_w J)J^{-1}, \tag{10}$$

while the linear system of the SDYM Lax representation becomes

$$\begin{aligned} L(\phi) &= (\partial_w - (\partial_w J)J^{-1})\phi - (\partial_{\tilde{z}}\phi)\zeta = 0, \\ M(\phi) &= (\partial_z - (\partial_z J)J^{-1})\phi + (\partial_{\tilde{w}}\phi)\zeta = 0. \end{aligned} \tag{11}$$

If one has a particular solution of this system $\psi(\Lambda)$ for $\zeta = \Lambda$, then the equations are invariant under the Darboux transformation

$$\begin{aligned} \tilde{J} &= -\psi\Lambda\psi^{-1}J = \begin{vmatrix} \psi & 1 \\ \psi\Lambda & \boxed{0} \end{vmatrix} J, \\ \tilde{\phi} &= \phi\zeta - \psi\Lambda\psi^{-1}\phi = \begin{vmatrix} \psi & \phi \\ \psi\Lambda & \boxed{\phi\zeta} \end{vmatrix}. \end{aligned} \tag{12}$$

This enables one to take some seed solution J of the SDYM equations and generate new solutions iteratively.

SDYM SOLITONS

For the gauge group $GL(2,\mathbb{C})$, the seed solution $J_1 = I$ generates a SDYM 1-soliton under one iteration of the Darboux transformation with the following ansatz:

$$\begin{aligned} &\psi = \begin{pmatrix} a_1e^L + a_2e^{-L} & c_1e^M + c_2e^{-M} \\ b_1e^L + b_2e^{-L} & d_1e^M + d_2e^{-M} \end{pmatrix}, \qquad \Lambda = \mathrm{diag}(\lambda,\mu), \\ &L = (\lambda\alpha)z + \beta\tilde{z} + (\lambda\beta)w + \alpha\tilde{w}, \qquad M = (\mu\gamma)z + \delta\tilde{z} + (\mu\delta)w + \gamma\tilde{w}, \end{aligned} \tag{13}$$

where $a_i, b_i, c_i, d_i, \alpha, \beta, \gamma, \delta, \mu, \lambda$ are complex constants. The Lagrangian density for this solution takes the form of a codimension-1 soliton

$$\mathrm{tr}F^{\mu\nu}F_{\mu\nu} \propto 2\mathrm{sech}^2X - 3\mathrm{sech}^4X \tag{14}$$

in 4-dimensional space. One can define n particular solutions $\psi_i(\Lambda_i)$ taking the above form (12) with different parameters. An n-soliton solution can be generated by applying the Darboux transformation n times to the seed solution which gives

$$J_{n+1} = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1\Lambda_1 & \psi_2\Lambda_2 & \cdots & \psi_n\Lambda_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1\Lambda_1^n & \psi_2\Lambda_2^n & \cdots & \psi_n\Lambda_n^n & \boxed{0} \end{vmatrix} \tag{15}$$

The scattering process of the n-soliton solution can then be analysed by taking the limit of the Lagrangian density at spatial infinity. It has been verified that the above solution induces a Lagrangian density which decays into n isolated domain wall solitons in the asymptotic region up to a phase shift of the individual solitons. This behaviour was explored explicitly in the 2-soliton case.

REFERENCES

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