Trinity College Dublin Senior Sophister Theoretical Physics

Notes on MAU44400 - Quantum Field Theory

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- **Grade Weighting**: 100% Exam.
- **Textbook References**: The entire course is primarily based on sections from Peskin and Schroeder's "Introduction to Quantum Field Theory" but discussions from Paul Dirac's lectures on quantum mechanics and Schweber's "Intro to Relativistic Quantum Field Theory" will also appear.

Contents

In this module we are trying to unite special relativity with quantum mechanics under a field-theoretic formalism. These notes shall begin with a set of mathematical conventions we will use and a quick recap of the important prerequisite results from both theories.

1 Conventions and Prerequisites

Units: Throughout the course it is most convenient to define our units by

$$\hbar = c = 1 \tag{1.1}$$

such that we only have to worry about one independent unit of measurement when solving a problem. In other words, with this choice of units we can write that

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}. \tag{1.2}$$

Relativity: In terms of relativistic conventions we will use the regular covariant notation with the Minkowskian metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.3}$$

and we recall the important energy-momentum relation for a massive particle is

$$p^2 = p^\mu p_\mu = E^2 - |\vec{p}|^2 = m^2. \tag{1.4}$$

Quantum Mechanics: When working in the Schrödinger picture of single particle wavefunctions we will use the four-momentum operator

$$p^\mu = i\partial^\mu \tag{1.5}$$

to solve the corresponding eigenvalue problem. Furthermore when discussing spins it will be useful to have the Pauli matrices handy so I shall list them here along with the corresponding ladder operators:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

$$\sigma^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2) \qquad \Longrightarrow \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Products of the three Pauli matrices will satisfy the identity

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k. \tag{1.7}$$

1.1 A Simple QED Problem

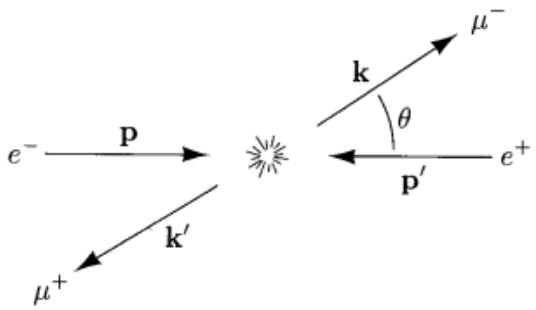


Figure 1.1: The annihilation of an electron-Positron pair to form a muon-antimuon pair.

In particle colliders such as CERN, experiments are conducted such as the elementary annihilation shown above in order to test our theories which probe the universe on it's finest scales. It would seem that we have no grounds to deal with such a quantum problem in a way which respects special relativity. Let's try our best to proceed anyway, as it happens, we can still use our smarts from both existing theories to predict the result of this scattering phenomenon to within 10% of experiment. To start, we will work in the centre of mass (COM) frame as in a classical scattering problem, then we must make the following assumption:

- We assume that the beam energy E is much greater than the electron or muon mass so that according to (1.4), $|\vec{p}| = |\vec{p}'| = |\vec{k}| = |\vec{k}'| = E \equiv E_{cm}/2$.

Now the measurable quantity we wish to compute is the scattering cross section σ. Now since the area we are dealing with depends on the solid angle into which each muon is reflected, we can start by writing down an expression for the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}|\mathcal{M}(\theta)|^2. \tag{1.8}$$

Here we have included the quantum theory by writing the dimensionless quantity $\mathcal{M}$ which will be the quantum probability for the process to occur. The inverse-square energy term can be explained by dimensional analysis since in our unit system $[\text{energy}]^{-2} = [\text{length}]^2$. Furthermore the electron and positron are spin$-\frac{1}{2}$ particles so the expression above only accounts for one particular set of allowed spins. This means that we must sum over spin orientations when calculating the final result. What we can't explain is the additional factor of $1/64\pi^2$ which is a conventional term coming from the theory we will go on to derive.

Unfortunately an exact formula for $\mathcal{M}$ can not be found so to make our next approximation we can find an expression for $\mathcal{M}$ as a perturbation series in powers of the electromagnetic interaction strength between e^+ and e^-. The first term in such an expression would be the expected value of the initial and final state according to the interaction

Hamiltonian H_I. But muons and electrons don't interact directly, only through the electromagnetic field (photons) therefore

$$\mathcal{M} \propto \langle e^+e^-|H_I|\mu^+\mu^-\rangle = 0. \tag{1.9}$$

The second term in the series is found by inserting $|\gamma\rangle\langle\gamma|$ as an operator which represents the respective interactions between the photon and each particle and because a photon is a four-vector particle we must sum over components as such

$$\mathcal{M} \propto \langle e^+e^-|H_I|\gamma\rangle^\nu\langle\gamma|H_I|\mu^+\mu^-\rangle_\nu. \tag{1.10}$$

Now we can guess the form of each of these matrix elements once again using our smarts. We know that H_I couples electrons to photons with a strength e so that will be a component of each matrix element. Now if we focus on one set of spin orientations where e^- and μ^- are right-handed, and e^+ and μ^+ are left-handed as shown in the figure below;

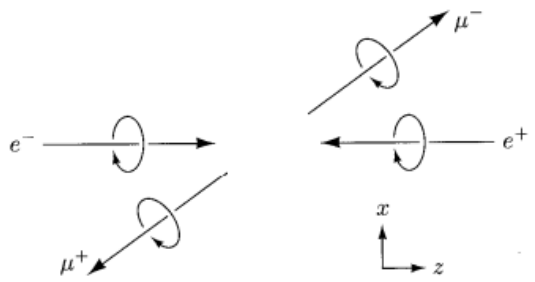


Figure 1.2: A choice of spin orientations in the scattering problem, note we are working in the x-z plane.

We can use that the initial and final angular momenta sum to 1 in the $+z$-direction. Since this must be conserved by H_I, so we know that the photon vector must preserve the polarisation as

$$\varepsilon^\mu = (0, 1, i, 0) \quad \text{with} \quad \varepsilon^\mu\varepsilon_\mu = 0. \tag{1.11}$$

Then for the muon matrix element to have one unit of angular momentum along the $\vec{k}$ direction we simply rotate this vector through the angle θ in the xz plane at which point we obtain the results

$$\langle e^+e^-|H_I|\gamma\rangle^\nu \propto e(0, 1, i, 0) \quad \text{and} \quad \langle\gamma|H_I|\mu^+\mu^-\rangle_\nu \propto e(0, \cos\theta, i, \sin\theta). \tag{1.12}$$

It follows that our perturbative solution for $\mathcal{M}$ at leading order is

$$\mathcal{M}(RL \to RL) = e^2(1 + \cos\theta). \tag{1.13}$$

Repeating this for the three other non-trivial spin configurations we find that

$$\mathcal{M}(LR \to LR) = e^2(1 + \cos\theta),$$

$$\mathcal{M}(LR \to RL) = e^2(1 - \cos\theta)$$
$$\mathcal{M}(RL \to LR) = e^2(1 - \cos\theta).$$

Now we can compute the differential scattering cross section by summing over these spins in (1.8)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}(1 + \cos^2\theta) \qquad \text{where} \qquad \alpha = \frac{e^2}{4\pi} = \frac{1}{137} \quad \text{(Fine Structure Constant)} \tag{1.14}$$

Integrating over the angular variables $d\Omega = \sin\theta d\theta d\varphi$, we obtain

$$\sigma_{\text{Total}} = \frac{4\pi\alpha^2}{3E_{cm}^2}. \tag{1.15}$$

2 Formalism

2.1 The Feynman Diagram of $e^+e^- \to \mu^+\mu^-$ Explained

In the introductory section we found a pretty good answer to a simple QED problem using nothing but cheap approximations and clever guesswork. Thanks to Richard Feynman we won't have to do that for much longer because he put a rigid and elegant set of rules together that will guide us through the subsequent calculations in QED. His set of rules guide us through the calculations of the perturbation series terms visually using a tool known as the Feynman diagram. Let's look at the Feynman diagram for the series component we calculated heuristically above.

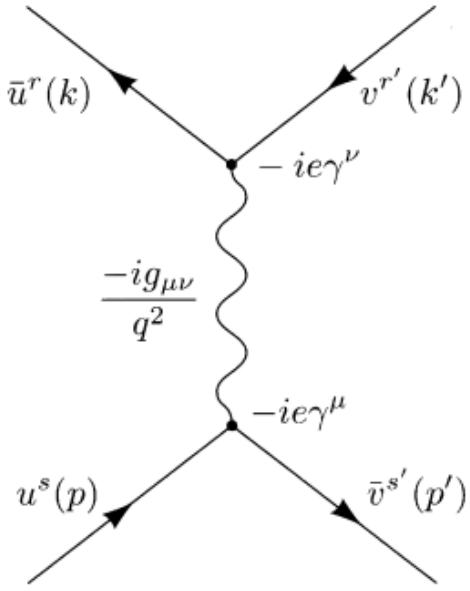


Figure 2.1: A Feynman diagram of the e^+e^- annihilation producing a $\mu^+\mu^-$ pair.

In this diagram we see that a short algebraic factor is assigned to each part of the interaction and we will explain each feature shortly. First it is important to mention that before we could only find an answer by making a few simplifying assumptions. This method will allow us to relax these assumptions and conduct general QED calculations. For the internal photon line we write $-ig_{\mu\nu}/q^2$ where $q = p + p' = k + k'$ is the momentum of the virtual photon. For each vertex we write $-ie\gamma^\mu$ which represents the interaction Hamiltonian H_I. The objects γ^μ are 4×4 constant matrices. We will learn about them properly in time, but they basically take care of the addition of angular momentum we did earlier, coupling a state of two spin-$\frac{1}{2}$ particles to a vector particle. The external lines carry expressions for four-component column spinors u, v or row spinors $\bar{u}, \bar{v}$. These are the wavefunctions of the initial and final particles in momentum-space with the indices denoting the spin state either up or down. Now the rule is to calculate from the bottom up in order of time so we can write down the expression for $\mathcal{M}$ as.

$$\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\Big(\frac{-ig_{\mu\nu}}{q^2}\Big)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \tag{2.1}$$

$$= \frac{ie^2}{q^2}(\bar{v}^{s'}(p')\gamma^\mu u^s(p))(\bar{u}^r(k)\gamma_\mu v^{r'}(k')).$$

For calculating the cross section, we would need to use Feynman trace technology and a concrete knowledge of Dirac spinors and gamma matrices which we will come to later in the course.

2.2 The Need for a Multiparticle Theory

Given that our primary concern is to understand processes that occur in quantum and relativistic conditions (those of elementary particles) we might ask why we can't simply quantize relativistic particles the same way we did for non-relativistic particles in quantum mechanics. One reason is that we can't assume that any relativistic process can be explained in terms of a single particle due to the implications of the $E = mc^2$ equation allowing the creation of new particles or even the existence of virtual particles for short periods of time at higher orders in the perturbation series according to the uncertainty principle. Reasoning aside, let's display mathematically why we need to move to a multiparticle (field) theory.

Let x and x' denote two events in the $(3+1)$-dimensional spacetime we are familiar with. The amplitude of a free particle propagating from x' to x is

$$U(t) = \langle x|e^{-iHt}|x'\rangle. \tag{2.2}$$

Which, for a free non-relativistic particle becomes

$$U(t) = \langle x|e^{-i\frac{p^2}{2m}t}|x'\rangle. \tag{2.3}$$

Now we can insert the momentum space completeness relation into the middle and integrate the resulting Gaussian.

$$= \int \frac{d^3p}{(2\pi)^3} \langle x|e^{-i\frac{p^2}{2m}t}|p\rangle\langle p|x'\rangle \tag{2.4}$$

$$= \frac{1}{(2\pi)^3}\int d^3p e^{-i\frac{p^2}{2m}t}\cdot e^{ip\cdot(\vec{x}-\vec{x}')} = \left(\frac{m}{2\pi i t}\right)^{\frac{3}{2}} e^{im\frac{(\vec{x}-\vec{x}')^2}{2t}}.$$

This probability is non-zero for any position or time which implies that a particle can propagate between any two points in an arbitrarily short time. But this quantum prediction clearly violates causality because this should not be the case if x and x' are spacelike-separated so we must not have the full picture. Perhaps we missed a trick in that we used the non-relativistic particle energy. We can try to compute the amplitude now using the relativistic energy (1.4)

$$U(t) = \langle x|e^{-it\sqrt{\vec{p}^2+m^2}}|x'\rangle = \frac{1}{(2\pi)^3}\int d^3p e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}. \tag{2.5}$$

Integrating over the momentum space sphere with the z-axis chosen along $\vec{x}-\vec{x}'$ we pick up a factor of 2π and explicitly computing the dot product in the second term are left with the expression

$$= -\frac{1}{4\pi^2}\int_{-1}^{1}\int_0^\infty dpd(\cos\theta)p^2 e^{-it\sqrt{\vec{p}^2+m^2}} e^{ip|\vec{x}-\vec{x}'|\cos\theta} \tag{2.6}$$

Now evaluating the integral over $\cos\theta$ gives

$$= -\frac{1}{4i|\vec{x}-\vec{x}'|\pi^2}\int_0^\infty dppe^{-it\sqrt{\vec{p}^2+m^2}}(e^{-ip|\vec{x}-\vec{x}'|} - e^{ip|\vec{x}-\vec{x}'|}) \tag{2.7}$$

$$= \frac{1}{2\pi^2|\vec{x}-\vec{x}'|}\int_0^\infty dpp\sin(p|\vec{x}-\vec{x}'|)e^{-it\sqrt{\vec{p}^2+m^2}}.$$

The explicit evaluation of such an integral can be done using Bessel functions but since we are only interested in whether the result will violate causality we can look at it's behaviour in the extremely spacelike limit $x^2 >> t^2$, using the method of stationary phase. For this we can assume that $\vec{x}' = 0$, and up to some arbitrary factors we are then dealing with the integral

$$\int_0^\infty dpp\sin(px)e^{-it\sqrt{p^2+m^2}} \simeq \int_{-\infty}^\infty dppe^{ipx-it\sqrt{p^2+m^2}} \tag{2.8}$$

where we have broken the sine function in half and compensated by extending the domain of integration into the negative reals. Now the stationary point of the exponential is given by solving $f'(\tilde{p}) = 0$ where $f(p) = ipx - it\sqrt{p^2+m^2}$ which turns out to be $p = imx/\sqrt{x^2-t^2}$. As per the stationary phase approximation, we then replace the

exponent of the integral in (2.8) with $f(\tilde{p}) + \frac{1}{2}f''(\tilde{p})(p-\tilde{p})^2$ and solve the integral to obtain

$$U(t) \simeq e^{-m\sqrt{x^2-t^2}}. \tag{2.9}$$

This integral is small because x^2 is large in the relevant limit but it is still non-zero and therefore admits the particle to traverse a spacelike interval. Now its time to ask what this is telling us? As we will see, quantum field theory sees the results above as no problem at all but rather proof of the existence of *antiparticles*. When we write the relativistic energy we really have $E = \pm\sqrt{p^2+m^2}$, here we have used the positive square root which we say propagates into the future, whereas the negative energy propagates into the past. So the propagation of a particle across a spacelike interval is indistinguishable from the propagation of an antiparticle in the opposite direction and they conveniently cancel each other out preserving causality.

2.3 Classical Field Theory

We will now remind ourselves about the elements of classical field theory that we have encountered so far with the intention of using this language to describe relativistic quantum mechanics. The fundamental quantity of classical mechanics is the action S which is given by the time integral of the Lagrangian $L(q_i, \dot{q}_i)$. When moving to a field formalism we want to treat space and time on equal grounds with coordinates represented by a continuous four-vector $(\vec{x}, t)$. In a local field theory, we instead write the Lagrangian (density) as a function of one or more fields $\varphi(\vec{x}, t)$ and their derivatives $\partial_\mu\varphi(\vec{x}, t)$ which also depend on whatever canonical coordinates we are using. To summarise

$$S = \int dt L(q_i, \dot{q}_i) = \int d^4x \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)). \tag{2.10}$$

Hamilton's principle of least action states that as the system goes from one configuration to another from time t_1 to t_2 it does so along the path for which $\delta S = 0$ this leads to the derivation of the Lagrange equations

$$0 = \delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)\right]. \tag{2.11}$$

Integrating the second term by parts we obtain

$$0 = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right)\right].$$

The final term can be turned into a surface integral at the boundary of the specific spacetime region at which $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$ since they are assumed as being given (having no first order variation). Since $\delta\varphi$ is an arbitrary deformation of the path, we see after factoring it out of the first two terms in the integrand, what remains must subtract to 0. Thus

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0. \tag{2.12}$$

If the Lagrangian contains more than one field then there is one such equation in each field. The Lagrangian formulation of field theory is extremely powerful while dealing with relativistic dynamics because all expressions involved are Lorentz invariant. For the first while we will be using the Hamiltonian formulation instead as it is easier when dealing with quantum mechanics. For discrete systems recall that we could find the canonical conjugate momenta by taking

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.13}$$

and the Hamiltonian is then

$$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i). \tag{2.14}$$

The generalisation of this to a continuous system is done by defining momenta conjugate to the fields $\varphi(x)$ by the equation

$$p(x) = \frac{\partial L}{\partial \dot{\varphi}} = \pi(x) d^3x \iff \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \tag{2.15}$$

where $\pi(x)$ is a volume form which we call the *momentum density* conjugate to the field $\varphi(x)$. Thus we can write the Hamiltonian (density) of the continuous system as

$$H = \int d^3x (\pi(x)\dot{\varphi}(x) - \mathcal{L}) = \int d^3x \mathcal{H}. \tag{2.16}$$

To grasp this change we will look now at the Lagrangian of the most simple form, for a theory of a single field $\varphi(x)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{2}\varphi^2 = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - \frac{m^2}{2}\varphi^2. \tag{2.17}$$

So the conjugate momentum density for this field is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}(x) \tag{2.18}$$

which allows us to write down the Hamiltonian

$$H = \int d^3x \left[\varphi^2(x) + \frac{1}{2}(\nabla\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]. \tag{2.19}$$

Finally we can write down the field equation (Lagrange equation) as in (2.12)

$$-m^2\varphi - \partial_\mu(\partial^\mu \varphi) = 0 \iff (\Box + m^2)\varphi = 0 \tag{2.20}$$

which is known as the *Klein Gordon equation*.

2.4 Noether's Theorem

A transformation $\Delta\varphi$ of the space of fields φ is called a symmetry if it leaves the equations of motion unchanged. The most general form of such a transformation can be written as

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \alpha\Delta\varphi(x) \tag{2.21}$$

That this transformation is a symmetry, is ensured by checking that the action is invariant. In general we can allow the action to change by a surface term since this would not affect our derivation of the Lagrange equation. By this reasoning we would expect the transformed Lagrangian to take the form

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \alpha\partial_\mu \mathcal{J}^\mu(x). \tag{2.22}$$

Let's check if this holds up by computing the variation of $\mathcal{L}$ obtained by varying the fields as above.

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}(\alpha\Delta\varphi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\partial_\mu(\alpha\Delta\varphi). \tag{2.23}$$

We can manipulate this using the product rule in reverse as

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right]\Delta\varphi.$$

The second term vanishes by the Lagrange equation. We set the remaining term equal to $\alpha\partial_\mu\mathcal{J}^\mu$ and consider that the variation of the action should once again vanish such that

$$\partial_\mu j^\mu(x) = 0 \qquad \text{for} \qquad j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu. \tag{2.24}$$

Which states that the so-called *Noether current* j^μ is conserved. This conservation law can be expressed in an alternate way by saying that the charge

$$\mathcal{Q} = \int d^3x j^o = \text{ constant}. \tag{2.25}$$

We will rarely use this form of the conservation law however since we will be working with theories that are invariant under local gauge symmetry transformations which leads directly to (2.24).

As a simple but non-trivial example of this we return to Klein-Gordon theory in the complex case which has Lagrangian

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi^2|. \tag{2.26}$$

This Lagrangian is invariant under the transformation $\phi \to e^{i\alpha}\phi$, for an infinitesimal transformation where α is small the changes are

$$\alpha\Delta\phi = i\alpha\phi, \qquad \alpha\Delta\phi^* = -i\alpha\phi^*. \tag{2.27}$$

This gives the conserved Noether current

$$j^\mu = i[(\partial_\mu \phi^*)\phi - \phi^*(\partial_\mu \phi)] \tag{2.28}$$

whose divergence vanishes by the KG equation (2.20). For relativity, we can use Noether's theorem to discover conserved quantities through the spacetime symmetry of translations and rotations. Let a be a small parameter, then we describe an infinitesimal spacetime translation of a field theory by

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi + \mathcal{O}(a^2) \tag{2.29}$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L}(x) + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Then the Noether currents for this symmetry is clearly

$$j^\mu_\nu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\nu_\mu \mathcal{L} = T^\mu_\nu \tag{2.30}$$

which is the *energy-momentum* tensor of the field ϕ. The conserved charge associated with the time component is (not-surprisingly) the Hamiltonian

$$T^0_0 = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} \dot{\phi} - \mathcal{L} = \mathcal{H}. \tag{2.31}$$

Analogously, the conserved charges associated with spatial translations are

$$P_i = \int T^0_i d^3x = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial_i \phi = \int d^3x \pi \partial_i \phi. \tag{2.32}$$

Which are the momentum components carried by the physical field rather than canonical momenta. Now that we have established the machinery we wish to carry forward into a multiparticle theory of quantum fields, we are ready to perform quantisation on a variety of theories relevant in modern physics. We will start with the simplest case of a real Klein-Gordon field and quantize it by interpreting the dynamical variables as operators that obey the canonical commutation relations.

3 Klein Gordon Theory

3.1 Quantization of the Klein-Gordon Field

We once again consider the real Klein-Gordon theory which is defined by the expressions

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 = \frac{1}{2} (\dot{\phi}^2 - (\nabla \phi)^2) - \frac{m^2}{2} \phi^2,$$

$$\mathcal{H} = \frac{1}{2} (\pi^2 + (\nabla \phi)^2 + m^2 \phi^2) > 0,$$

$$(\Box + m^2)\phi = 0.$$

We now think of the canonical phase space coordinates (π, ϕ) as operators along with the Hamiltonian. The canonical Poisson bracket of the field are given by

$$\{\pi(\vec{x}), \phi(\vec{y})\} = \delta(\vec{x} - \vec{y}) \tag{3.1}$$

which means that the canonical commutation relations are

$$[\pi(\vec{x}), \phi(\vec{y})] = i\delta(\vec{x} - \vec{y}). \tag{3.2}$$

The easiest way to do this is to see that when we write the Fourier transform of the classical KG field

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^3} \int d^3p \phi(\vec{p}, t) e^{i\vec{p}\cdot\vec{x}}, \tag{3.3}$$

the KG equation becomes

$$\left[\partial_t^2 + (|\vec{p}|^2 + m^2)\right]\phi(\vec{p}, t) = 0. \tag{3.4}$$

But this is the same as the equation of motion for a simple harmonic oscillator with frequency $\omega_p = \sqrt{|p|^2 + m^2}$. Since the KG equation is that of a simple harmonic oscillator we can quantize the theory by describing it as a field of infinitely many quantum harmonic oscillators (one at every point). Recall that the simple harmonic oscillator is a system whose spectrum we can find from the Hamiltonian

$$H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2$$

by writing ϕ and p in terms of ladder operators

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger).$$

The commutation relation is then equivalent as $[\phi, p] = i$, $[a, a^\dagger] = 1$, and the Hamiltonian in terms of these operators is

$$H_{SHO} = \omega(a^\dagger a + \frac{1}{2}).$$

And has the commutation relations with the canonical pair $(a^\dagger, a)$:

$$[H, a^\dagger] = \omega a^\dagger \qquad [H, a] = -\omega a.$$

It is then easy to see that the Bosonic Fock Space of states $|n\rangle$ are eigenstates of the Hamiltonian since $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$ gives $H|0\rangle = \frac{\omega}{2}|0\rangle$ (the zero point energy). Noticing also that composing the ladder operators with the Hamiltonian on a state, we can find the entire spectrum of H:

$$H|a^\dagger\psi\rangle = (E + \omega)|a^\dagger\psi\rangle \implies |n\rangle = \frac{(a^\dagger)^n}{n!}|0\rangle \qquad E_{|n\rangle} = (n + \frac{1}{2})\omega.$$

To quantise the KG field we now need to repeat this analysis for the canonical pair $\phi(\vec{x},t)$ and $\pi(\vec{x},t)$ in terms of the ladder operators. The only subtlety arising is that we have a Fourier mode of the field corresponding to each oscillator with its own ladder operators. Each one will be parametrised by the integration variable $\vec{p}$ as follows, with $t_0 = 0$

$$\phi(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3p \frac{1}{\sqrt{2E_{\vec{p}}}}(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}); \tag{3.5}$$

$$\pi(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3p \sqrt{E_{\vec{p}}/2}(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} - a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}).$$

Now we can check that these expressions satisfy the commutation relations as they did before we quantised the system. Indeed, through a very doable calculation we find that

$$[\pi(\vec{x}),\phi(\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y}), \qquad [a(\vec{p}),a^\dagger(\vec{p}')] = \delta^{(3)}(\vec{p}-\vec{p}'). \tag{3.6}$$

We can write the Hamiltonian then using the general form outlined at the opening of the section which gives

$$H = \int d^x \int \frac{d^3pd^3p'}{(2\pi)^6} e^{i(\vec{p}+\vec{p}')\cdot\vec{x}}\Bigg[-\frac{\sqrt{E_pE_{p'}}}{4}(a(\vec{p}) - a^\dagger(-\vec{p}))(a(\vec{p}') - a^\dagger(-\vec{p}')) \tag{3.7}$$

$$+\frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{E_{\vec{p}}E_{\vec{p}'}}}(a(\vec{p}) + a^\dagger(-\vec{p}))(a(\vec{p}') + a^\dagger(-\vec{p}'))\Bigg]$$

$$= \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}(a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}[a(\vec{p}),a^\dagger(\vec{p})]).$$

Notice that the second term is actually an infinity, but in a not surprising sense since this comes from the sum over every zero-point energy of the infinitely many harmonic oscillators being used to model the field. What actually matters is the difference between energy levels so we throw away the infinite term and define it as the vacuum energy

$$E_{\text{Vac}} = \frac{1}{2}\int d^3p E_{\vec{p}}\delta^{(3)}(0). \tag{3.8}$$

The rest of H remains and the commutation relations for it read

$$[H_{KG}, a^\dagger(\vec{p})] = \omega_p a^\dagger(\vec{p}), \qquad [H_{KG}, a(\vec{p})] = -\omega_p a(\vec{p})$$

We can now write down the spectrum of the theory in the Fock-Space representation where we have

$$a(\vec{p})|0\rangle = 0, \qquad |\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$$

where $|\vec{p}\rangle$ are the (one-particle) eigenstates created by the operator $a^\dagger$. We expect these eigenstates to return the oscillator energy when acted upon by H such that

$$H|\vec{p}\rangle = E|\vec{p}\rangle = \omega_{\vec{p}}|\vec{p}\rangle = \sqrt{\vec{p}^2 + m^2}|\vec{p}\rangle.$$

Indeed when we check this by computation we find

$$H|\vec{p}\rangle = \int d^3p' E_{p'} a^\dagger(p')a(p')a^\dagger(p)|0\rangle = \int d^3p' E_{p'} a^\dagger(p')(a^\dagger(p)a(p')|0\rangle + [a(p'), a^\dagger(p)]|0\rangle)$$

$$= \int d^3p' E_{p'} a^\dagger(p')\delta^{(3)}(p'-p)|0\rangle = E_p|\vec{p}\rangle$$

as needed. We can also write down the total momentum operator $\vec{P}$ as we had found from the energy-momentum tensor from a calculation similar to (3.7)

$$\vec{P} = -\int d^3x \pi(\vec{x})\nabla\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\vec{p}a^\dagger(\vec{p})a(\vec{p}). \tag{3.9}$$

We can refer now to the oscillators with frequency $\omega_p = \sqrt{p^2+m^2}$ as *particles*, not in the sense that they are localised objects but rather that they are discrete entities created in momentum eigenstates which obey the relativistic energy relation. Hence we shall only refer to the energy E_p instead of the frequency. A general n-particle state can be created in the same way where

$$|\vec{p}_1,\ldots,\vec{p}_n\rangle = a^\dagger(\vec{p}_1)\cdots a^\dagger(\vec{p}_n)|0\rangle,$$
$$H|\vec{p}_1,\ldots,\vec{p}_n\rangle = \sum_{i=1,\ldots,n} E_i|\vec{p}_1,\ldots,\vec{p}_n\rangle$$

Since we want to be able to study the quantum statistics of the momentum eigenstates we want them to be normalised, defining $\langle 0|0\rangle = 1$ we can compute

$$\langle\vec{p}'|\vec{p}\rangle = \langle 0|a(\vec{p}')a^\dagger(\vec{p})|0\rangle = \langle 0|a^\dagger(\vec{p})a(\vec{p}') + [a(\vec{p}'), a^\dagger(\vec{p})]|0\rangle.$$

The left term vanishes due to to the lowering operator killing the vacuum state which leaves

$$\langle\vec{p}'|\vec{p}\rangle = \delta^{(3)}(\vec{p}-\vec{p}').$$

This matrix element is a measurable physical quantity so it needs to be Lorentz invariant, if it is not, then we will need to change our convention so that our normalisation is consistent with the physical picture. Consider only the 0 and 3 components of a momentum four vector p, under a boost in the 3-direction these two components transform as

$$E' = \gamma(p_3 + \beta E), \qquad p_3' = \gamma(E + \beta p_3).$$

Under this transformation, the Dirac delta function from above for momenta $(\vec{p},\vec{q})$transforms as

$$\delta^{(3)}(\vec{p}-\vec{q})dp_3 = \delta^{(3)}(\vec{p}'-\vec{q}')dp_3'$$
$$\implies \delta^{(3)}(\vec{p}-\vec{q}) = \delta^{(3)}(\vec{p}'-\vec{q}')\frac{dp_3'}{dp_3} = \delta^{(3)}(\vec{p}'-\vec{q}')\gamma\left(1+\beta\frac{dE_{p'}}{dp_3}\right)$$
$$= \delta^{(3)}(\vec{p}'-\vec{q}')\frac{\gamma}{E_p}(E_p+\beta p_3) = \delta^{(3)}(\vec{p}'-\vec{q}')\frac{E_{p'}}{E_p}.$$

and so the quantity is clearly not Lorentz invariant due to Lorentz contraction. But we can immediately see that the correct definition of the state $|\vec{p}\rangle$ (with an added factor of 2 for convenience) is

$$|\vec{p}\rangle = \sqrt{2E_p}a^\dagger(\vec{p})|0\rangle. \tag{3.10}$$

To restore time-dependence to the theory we move to the Heisenberg picture and define a time-dependent version of the operators in the usual way

$$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}, \qquad \pi(\vec{x},t) = e^{iHt}\pi(\vec{x})e^{-iHt} \tag{3.11}$$

plugging each operator into the Heisenberg equation of motion

$$i\frac{\partial}{\partial t}X = [X,H]$$

returns the result of the KG equation once again. To better understand the time dependence of ϕ and π we can use their expressions in terms of creation and annihilation operators together with the following results

$$Ha(\vec{p}) = a(\vec{p})H - [a(\vec{p}),H] = a(\vec{p})(H - E_p)$$
$$\implies H^n a(\vec{p}) = a(\vec{p})(H - E_p)^n.$$

The same holds for $a^\dagger(\vec{p})$ with $+ \longleftrightarrow -$. Thus we have the identities

$$e^{iHt}a(\vec{p})e^{-iHt} = a(\vec{p})e^{-iE_pt}, \qquad e^{iHt}a^\dagger(\vec{p})e^{-iHt} = a^\dagger(\vec{p})e^{iE_pt}. \tag{3.12}$$

Now using (3.11) we can get the explicit form of the desired operator $\phi(\vec{x},t)$

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{-i(E_pt-\vec{p}\cdot\vec{x})} + a^\dagger(\vec{p})e^{i(E_pt-\vec{p}\cdot\vec{x})}\right) \tag{3.13}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x})$$

where we have used that for the four vectors $p^\mu x_\mu \equiv p\cdot x = Et - \vec{p}\cdot\vec{x}$. Of course the conjugate momentum to the field follows simply from the canonical equation

$$\pi(\vec{x},t) = \partial_t\phi(\vec{x},t). \tag{3.14}$$

These equations make clear the dual particle and wave interpretations of the quantum field ϕ. On one hand we have written the field as a Hilbert space operator which creates and destroys the particles that are the quanta of the field excitation. On the other hand, $\phi(\vec{x},t)$ is written as a linear combination of solutions to the KG equation and both $\pm$ signs of time-dependence appear in the exponential. We refer to each as a positive and negative frequency mode respectively. Note also that a positive frequency branch of the solution always has as its coefficient the operator that destroys a single particle wavefunction and vice-versa for the negative frequency solution. In this respect, the fact that relativistic wave equations have both positive and negative frequency solutions is reconciled with the fact that a physical quantum theory contains only positive excitation energies.

3.2 Causality of the Quantized KG Field

In our present formalism, still working in the Heisenberg picture we can work out the propagator which we call $D(x-y)$. This is the field version of the amplitude of a particle to propagate from y to x which can be computed as

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}. \tag{3.15}$$

Let's first compute this integral for a timelike interval we can always find a frame such that $x^0 - y^0 = t$, $\vec{x} - \vec{y} = 0$. Then

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}}e^{-i\sqrt{p^2+m^2}} =$$

$$\frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}e^{-iEt} \underset{t\to\infty}{\approx} e^{-imt}.$$

Which is exponentially vanishing but non-zero. Next we consider the spacelike separation $x^0 - y^0 = 0$, $\vec{x} - \vec{y} = \vec{r}$. The amplitude is then

$$D(x-y) = \int \frac{d^3p}{(2\pi)^2}\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{r}} = \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\frac{p^2}{2E_p}\frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}.$$

The integrand is a complex function of p and it has branch cuts on the imaginary axis starting at $\pm im$. To evaluate the integral we push the contour up to wrap around the upper branch cut. Defining $\rho = -ip$ we obtain

$$\frac{1}{4\pi^2 r}\int_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \underset{r\to\infty}{\approx} e^{-mr}.$$

So again we find that outside the lightcone, there is a non-zero propagator. These are only probabilities however and we must remember that to really discuss causality we should examine whether a physical measurement performed at one point can affect a measurement at another point whose separation from the first is spacelike. The simplest thing to measure would be the field, so as is standard in the Heisenberg picture we compute whether the commutator $[\phi(x),\phi(y)]$ is vanishing when $(x-y)^2 < 0$ since this also covers the vanishing commutator with the conjugate momentum.

$$[\phi(x),\phi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} \tag{3.16}$$

$$\times[(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}),(a(\vec{q})e^{-ip\cdot y} + a^\dagger(\vec{q})e^{ip\cdot y})]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) = D(x-y) - D(y-x).$$

Which vanishes since we can always perform a Lorentz transformation on the second term that takes $(x-y) \to -(x-y)$ which makes the commutator vanish. If the interval is of the form $(x-y)^2 > 0$ then the amplitude is non-zero since there is no such transformation. So we have the result we need and causality is preserved. To fully encapsulate the physics of this result we should extend our scope to consider a complex-valued KG field. In this theory $\phi(x)$ will create positively charged particles and destroy negatively charged ones while $\phi^\dagger(x)$ will do the opposite. Then the commutator $[\phi(x), \phi^\dagger(y)]$ will have non-zero contributions which must cancel outside the lightcone. Much like we argued in the classical case we say that this necessity on the two terms in the complex analogue of (3.16) is a motivation for the antiparticle. Both particles must exist, have the same mass and opposite quantum numbers in order for the amplitudes to cancel. For the real valued field, we say then that the particle is its own antiparticle (which makes quite obvious that Klein Gordon theory is not a real physical field as far as we know).

3.3 The Klein-Gordon Propagator

It is worth spending more time on the commutator $[\phi(x), \phi(y)]$. Since it takes the value of a number we can re-express it as the propagator $\langle 0|[\phi(x), \phi(y)]|0\rangle$. We can treat the case $x^0 > y^0$ for now without losing any non-trivial information about the object, as in (3.16) we write

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}).$$

We then match the exponentials by splitting the integrand into the positive and negative energy branches effectively changing the variable p^0 to the opposite sign in the latter which gives

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0=E_p} + \frac{1}{(-2E_p)} e^{-ip\cdot(x-y)} \Big|_{p^0=-E_p} \right].$$

Which can be compactly written as an integral over p^0 as

$$\langle 0|[\phi(x), \phi(y)]|0\rangle \overset{x^0 \geq y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}. \tag{3.17}$$

We write the integral in this way with the knowledge that it must be performed over a contour which avoids the poles at $\pm E_p$ caused by the denominator $p^2 - m^2 = (p^0)^2 - (\vec{p}^2 + m^2) \equiv (p^0)^2 - E_p^2$. The particular contour we refer to is shown below:

In our application of the residue theorem we now must choose the correct way to close the contour which corresponds to capturing the poles. Our integral is formulated for $x^0 > y^0$ so if we choose to close it on the wrong side of the imaginary axis we will miss our poles and the integral will vanish (which would be the correct choice for $x^0 < y^0$. since it no longer describes the propagator we are trying to compute. Clearly closing the contour below captures the poles in this case and so integrating this way gives $\langle 0|[\phi(x), \phi(y)]|0\rangle$. This situation clearly indicates that the integral is equivalent to the familiar function

$$D_R(x-y) = \Theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle, \tag{3.18}$$

which appears to be the retarded Green's function of the Klein Gordon operator $(\Box + m^2)$. We can verify this by computation:

$$(\Box + m^2)D_R(x-y) = (\Box\Theta(x^0 - y^0))\langle 0|[\phi(x), \phi(y)]|0\rangle$$

$$+2\partial_\mu\Theta(x^0 - y^0)\partial^\mu\langle 0|[\phi(x), \phi(y)]|0\rangle + \Theta(x^0 - y^0)(\Box + m^2)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

$$= -\delta(x^0 - y^0)\partial_t\langle 0|[\phi(x), \phi(y)]|0\rangle + 2\delta(x^0 - y^0)\partial^t\langle 0|[\phi(x), \phi(y)]|0\rangle + \langle 0|(\Box + m^2)\phi(x), \phi(y)|0\rangle$$

$$= (-\delta(x^0 - y^0) + 2\delta(x^0 - y^0))\langle 0[\partial_t\phi(x), \phi(y)]|0\rangle$$

$$= (-\delta(x^0 - y^0) + 2\delta(x^0 - y^0))\langle 0[\pi(x), \phi(y)]|0\rangle = \delta(x^0 - y^0)(-i)\delta^{(3)}(\vec{x} - \vec{y}) = -i\delta^{(4)}(x - y).$$

So indeed this is equivalent to the integral form of the propagator $\langle 0|[\phi(x), \phi(y)]|0\rangle$ for $x^0 > y^0$ with additional confirmation coming from the fact that it vanishes for $x^0 < y^0$ due to the Heavisisde step function

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2}e^{-ip\cdot(x-y)}. \tag{3.19}$$

We can derive an equivalent result for the *advanced* case where the same propagation happens for $x^0 < y^0$, here we close the contour above the real axis to capture the poles in the opposite way. To summarise, we have that the propagators for a disturbance travelling from y to x in the KG field are Green's functions of the KG equation and in each case:

$$x^0 > y^0: \qquad D_R(x-y) = \Theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle,$$
$$x^0 < y^0: \qquad D_A(x-y) = \Theta(y^0 - x^0)\langle 0|[\phi(x), \phi(y)]|0\rangle.$$

The contours we have chosen here for our p^0 integration are two of four total possible prescriptions. We have not seen the most useful of these yet but we will introduce it now as the "*Feynman prescription*". As depicted below it prescribes a diversion into the negative imaginary plane for the first pole and into the positive imaginary plane for the right pole.

Now when we close the contour over the lower plane ($x^0 > y^0$) we capture the left pole and when we close over the upper plane ($x^0 < y^0$) we capture the right pole. The corresponding Green's function for this prescription of the propagator integral is given by:

$$D_F(x-y) = \begin{cases} D(x-y) & \text{if } x^0 > y^0 \\ D(y-x) & \text{if } x^0 < y^0 \end{cases} \tag{3.20}$$

$$= \Theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi(y),\phi(x)|0\rangle \equiv \langle 0|T\phi(x),\phi(y)|0\rangle.$$

The last line is a compact way of writing the expression where the *time-ordering symbol* T instructs us to place the operators that follow in order of time components from earliest to latest. This Green's function is called the *Feynman propagator* for a KG particle and they are the propagators we use to represent the propagation of virtual particles in Feynman diagrams. A convenient way to remember this Green's function is to write it in integral form which is

$$D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}, \tag{3.21}$$

where ε is an infinitesimal parameter. Observe that $p_0^2 = \vec{p}^2 + m^2 - i\varepsilon$ implies $p_0 \approx \pm\sqrt{\vec{p}^2+m^2} + i\varepsilon$ so Feynman's contour moves the poles an infinitesimal distance away from the real line and we choose whether to close the contour above or below depending on the sign of the exponent in the integral. This contour is "*good*" because it gives particles with positive energy propagating into the future and negative energy into the past making this a manifestly causal form of the propagator.

3.4 Particle Creation by a Classical Source

We now study solutions of the inhomogeneous KG equation in which a KG field is coupled to an external classic source field $j(x)$:

$$(\Box + m^2)\phi(x) = j(x). \tag{3.22}$$

We consider $j(x)$ to be a given, fixed function of space and time that is non-zero only for a finite time interval. So the interesting point we want to study is what will happen if we start in the vacuum state and turn $j(x)$ on and off again. For the field equation to take this form we know that it will appear in a term linear in ϕ in the Lagrangian, hence

$$\frac{\partial\mathcal{L}}{\partial\phi} \propto j(x) \implies \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + j(x)\phi(x).$$

The solution of (3.22) is a linear combination of the homogeneous solution (3.13) with an additional source term which can be constructed using the propagator

$$\phi(x) = \phi_0(x) + i\int d^4y D(x-y)j(y)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} + i\int d^4y j(y)\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}e^{ip\cdot(x-y)}.$$

Then $\phi(x)$ involves a Fourier transform of j:

$$j(p) = \int d^4y e^{ip\cdot y} j(y),$$

which turns the solution into

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[\left(a(p) + \frac{i}{\sqrt{2E_p}}j(p)\right)e^{-ip\cdot x} + \left(a^\dagger(p) - \frac{i}{\sqrt{2E_p}}j(p)\right)e^{ip\cdot x}\right]. \quad (3.23)$$

So the Hamiltonian becomes

$$H = \int \frac{d^3p}{(2\pi)^3}E_p\left(a^\dagger(p) - \frac{i}{\sqrt{2E_p}}j(p)\right)\left(a(p) + \frac{i}{\sqrt{2E_p}}j(p)\right)$$

with

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}|j(p)|^2$$

being the energy of the system after the source has been turned off.

4 Representations of the Poincaré Group

This section, permeating about two weeks of lectures is Samson's favourite part to teach and is based on the relevant section in the Schweber textbook.

When discussing a physical system we are always interested in the invariant quantities which, in a physically realisable theory can, determine the real physical properties of an object in the system like mass or spin. For the quantum field theories we are discussing, we work in Minkowski spacetime. So far we have only been using the Lorentz group $SO(3,1)$ which generates rotations and boosts to find invariant quantities called *Lorentz invariants*. In this framework we associate invariants with any quantity that is a scalar, which has contracted indices such that it can't change under transformations. Although effective for our purposes so far, the fact that any four vector squared $x_\mu x^\mu$ is invariant under the symmetry transformations of the Lorentz group does not give us the entire picture.

To see why, we must consider the most fundamental rule in special relativity which is that the speed of light is preserved in every inertial frame. While it is great that we can pick out Lorentz invariants by squaring a four vector, it is not at all fundamental because on physical grounds (the speed of light) we only require that the line element $(dx)^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ is invariant. Indeed the Lorentz group satisfies this condition but we notice that so too does the group of constant translations in $\mathbb{R}^{1,3}$! So the full symmetry group of Minkowski space is actually an extension of the Lorentz group to include translations

such that the generators span all isometries on the space. The resulting group is called the Poincaré group and since rotations and translations don't commute we write it as the semi-direct product of Lorentz transformations and translations

$$P = \mathbf{R}^{1,3} \rtimes O(1,3). \tag{4.1}$$

The addition of the four translation directions brings the dimension of the Poincaré group to 10. It is the fundamental symmetry group of relativistic physics and we will study it in detail in this chapter to see how we can extend the notion of such transformations to fields. We begin by studying just the representations of the Lorentz group and progress from there.

4.1 Group Actions

To see how fields transform under Lorentz transformations we must determine what type of action $SO(3,1)$ has on the fields and then find an appropriate representation of the group in terms of the fields and spacetime themselves. If we consider the action of a Lorentz transformation Λ^{μ}_{ν} as an *active* one then we identify its action with a transformation of the entire manifold of spacetime rather than of a single localised particle then we have a group action by definition.

Definition (Representation by Group Action): A group G that acts on a manifold must be a Lie group and we say that a G is represented as a group of transformations of a manifold M, or has a *left action* on M if:

1. There is a diffeomorphism from M to itself associated with each of its elements g

$$x \mapsto \mathcal{T}_g(x), \qquad x \in M$$

   such that $\mathcal{T}_{gh} = \mathcal{T}_g \mathcal{T}_h$ for all $g, h \in G$.

2. Each $\mathcal{T}_g(x)$ depends smoothly on the arguments g, x making $(g,x) \mapsto \mathcal{T}_g(x)$ a smooth map.

The Lie group is said to have a *right action* if the above definition is valid with the alternative product of diffeomorphisms $\mathcal{T}_g \mathcal{T}_h = \mathcal{T}_{hg}$.

Following from this definition we can define left and right actions as follows. For $x \in M$ we say that the **left action** of a Lie group G is given by

$$x \mapsto x' = gx \quad \Longrightarrow \quad x'' = hgx = (hg)x. \tag{4.2}$$

which satisfies conditions (1) and (2) in the above definition. Similarly we say that the **right action** of a Lie group G is given by

$$x \mapsto x' = xg^{-1} \quad \Longrightarrow \quad x'' = xg^{-1}h^{-1} = x(hg)^{-1}. \tag{4.3}$$

4.2 Transformation of Fields

With the conventions formulated as above, we now consider how fields ϕ, as functions of spacetime coordinates $x^\mu \equiv x$ transform under the manifestly active Lorentz transformations. Let the transformation Λ be represented by a group element $g \in SO(3,1)$, then we have that x transforms as

$$x \to x' = \Lambda x = g \cdot x. \tag{4.4}$$

Correspondingly, $\phi(x)$ transforms as

$$\phi(x) \to \phi'(x) = \phi(g \cdot x) \quad \Longrightarrow \quad \phi''(x) = \phi'(h \cdot x) = \phi(g \cdot (h \cdot x)). \tag{4.5}$$

But as a left group action this takes the form $\mathcal{T}_h \mathcal{T}_g = \mathcal{T}_{gh}$ which does not follow the order of the left group action we defined above where $\mathcal{T}_h \mathcal{T}_g = \mathcal{T}_{hg}$. Fixing this inconsistency requires us to recognise that the functions transform differently to the manifold elements they depend on. Namely when we transform x as $\mathcal{T}_g(x) = g \cdot x$ by a left group action, we transform $\phi(x)$ by a right group action. From here it is a simple observation that the fields must transform according to the right group action which can be obtained by replacing $g \longleftrightarrow g^{-1}$. Then we have that the field $\phi(x)$ transforms under the action of $SO(3,1)$ according to

$$\phi(x) \to \phi'(x) = \phi(g^{-1} \cdot x) \quad \Longrightarrow \quad \phi''(x) = \phi'(h^{-1} \cdot x) = \phi(g^{-1} h^{-1} \cdot x) = \phi((hg)^{-1} \cdot x). \tag{4.6}$$

Or rather in proper notation

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x). \tag{4.7}$$

4.3 Representations of the Lorentz Group and $\mathfrak{so}(3,1)$

To find the different representations of the Lorentz group we must look at the infinitesimal transformations and study the resulting Lie algebra whose basis elements are the generators of these transformations. If we write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{4.8}$$

for infinitesimal ω, then the condition for Λ to be a Lorentz transformation

$$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta^{\sigma\rho} = \eta^{\nu\mu}$$

becomes the requirement that ω is a two-form (anti-symmetric). Clearly then the infinitesimal transformations have six independent basis generators, three for rotations and three for boosts. We compile these basis elements into a single tensor $\mathcal{M}_{\mu\nu}$ which is conventionally defined in the form

$$\mathcal{M}_{\mu\nu} \equiv L_{\mu\nu} \mathbb{I} + S_{\mu\nu} \tag{4.9}$$

and we will see why shortly. To use these matrices practically we must not only index which basis element it is but also index the components and in this case we write the six generators as

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = -i(\eta^{\rho\mu} \delta^\sigma{}_\nu - \eta^{\sigma\mu} \delta^\rho{}_\nu). \tag{4.10}$$

Finally, the Lie bracket structure of the algebra is

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = -i(\eta^{\mu\sigma}\mathcal{M}^{\nu\rho} + \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{M}^{\mu\rho}). \tag{4.11}$$

where the factor of i has been freely introduced to ensure the generators are Hermitian. This way of writing the algebraic relations is nice and compact but it will actually serve us better to use three vector notation. The boosts (K_i) and rotations (J_i) are generated by the matrices shown below:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Figure 4.1: The six Lorentz group generators, we will use these matrices with a factor of i multiplying them by convention although it is not necessary in general.

These matrices can be extracted from the definition of $\mathcal{M}$ as

$$J_i = \frac{1}{2}\epsilon_{ijk}\mathcal{M}_{jk}, \qquad K_i = M_{0i}. \tag{4.12}$$

In this notation, the commutation relations become those among the boosts and rotations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{4.13}$$

The Lorentz group is not compact because it contains boosts and hence all of its unitary representations are infinite-dimensional. Furthermore it is not simply connected because it contains rotations so to eventually obtain a suitable representation of the Poincaré group for our purposes, we must instead study the representations of $SL(2,\mathbb{C})$ which is the universal covering group of the Lorentz group.

4.4 The $SL(2,\mathbb{C})$ Representation of the Lorentz Group

To move in this more motivated direction, we now define the matrices

$$N_i = \frac{1}{2}(J_i + iK_i), \qquad N_i^\dagger = \frac{1}{2}(J_i - iK_i). \tag{4.14}$$

These have the commutation relations

$$[N_i, N_j] = i\epsilon_{ijk}N_k, \qquad [N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger, \qquad [N_i, N_j^\dagger] = 0. \tag{4.15}$$

These are the commutation relations and generators of $SL(2,\mathbb{C})$ and note here that N commutes with $N^\dagger$. Given the angular momentum operators have relations of the exact same form, we can analogously define $N^2 = N_1^2 + N_2^2 + N_3^2$ which takes values $n(n+1)$ and commutes with all N_i, together with a specification that N_3 takes values $\{-n, -n+1, \ldots, n-1, n\}$. Since N commutes with $N^\dagger$ we can define a totally separate representation for $N^\dagger$ in parallel to what we have done for N. Namely the commuting operator $(N^\dagger)^2 = (N_1^\dagger)^2 + (N_2^\dagger)^2 + (N_3^\dagger)^2$ which takes values $m(m+1)$ and $N_3^\dagger$ takes values $\{-m, -m+1, \ldots, m-1, m\}$.

Casimir operators of an algebra are elements which commute with every other element so here we have constructed two such operators N^2 and $(N^\dagger)^2$. Then for the same reason as in the angular momentum representation we require that $n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. We recognise that the fact that N commutes with $N^\dagger$ implies that we actually have two completely independent quantities in the angular momentum representation. The pair $(n,m) = (0,0)$ corresponds to KG theory for which the particles have zero spin. The next case up from that would be the system defined by the representation $(n,m) = (0,\frac{1}{2}), (\frac{1}{2},0)$ which describes spin-1/2 particles.

In the above analysis we divided the six dimensional representation of the Lorentz group into a direct sum of symmetric and anti-symmetric representations which are dual to each other. This was possible because the Hermitian conjugation operation divided the reducible representation of 6 generators into two commuting sets of generators of $SL(2,\mathbb{C})$. When we come to studying the Dirac field in the spinor representation we will reunite this decomposition as a direct sum of symmetric and anti-symmetric (left and right) irreps.

$$\text{Dirac Spinor} = \underbrace{(\frac{1}{2},0)}_{\text{Left Spinor}} \oplus \underbrace{(0,\frac{1}{2})}_{\text{Right Spinor}} .$$

To have a general idea of how this representation theory works is good to give us a better understanding of why Dirac theory is the way it is (abstract) but not at all necessary to understand how the equations of the theory work. Now back to the main goal of the chapter.

4.5 Adding in Translations: The Poincaré Group

As mentioned above, the Poincaré group is defined as the set of transformations combining those of the Lorentz group, with standard translations in spacetime. In covariant form, such a transformation is

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu. \tag{4.16}$$

The translations are generated by the co-vectors $P_\mu = -i\partial_\mu$ and by the equality of mixed partial derivatives this means that $[P_\mu, P_\nu] = 0$ (translations commute). But translations

do not commute with the Lorentz transformations. The fundamental bracket relations of the Poincaré algebra are:

$$\begin{gathered}[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = -i(\eta^{\mu\sigma}\mathcal{M}^{\nu\rho} + \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{M}^{\mu\rho}), \\ [M_{\mu\nu}, P_\sigma] = i(\eta_{\nu\sigma}P_\mu - \eta_{\mu\sigma}P_\nu), \\ [P_\mu, P_\nu] = 0.\end{gathered} \tag{4.17}$$

The Casimirs of this algebra are the square of the ("momentum") translation generators and the *Pauli-Lubansky vector* defined respectively as

$$P_\mu P^\mu = m^2 \tag{4.18}$$

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}. \tag{4.19}$$

To classify the representations of this algebra we return to the somewhat ambiguous definition of $M_{\mu\nu}$ given in (4.9)

$$\mathcal{M}_{\mu\nu} \equiv L_{\mu\nu}\mathbb{I} + S_{\mu\nu}$$

In the Poincaré algebra we associate with the tensor $L_{\mu\nu}$ a definition analogous to $\vec{L} = \vec{x}\times\vec{p}$ in terms of the translation generator, namely

$$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu). \tag{4.20}$$

Plugging this definition of $M_{\mu\nu}$ into the Pauli-Lubansky vector gives

$$W^\mu = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu S_{\rho\sigma}. \tag{4.21}$$

The interpretation here from a representation theory point of view, is that W is measuring the unknown quantity S which we associate with **spin**! Similarly P measures the **mass** through its Casimir operator. In terms of eigenvalues, this means that

$$W^2 = W_\mu W^\mu = -(P_\nu P^\nu)(S_{\rho\sigma}S^{\rho\sigma}) = -m^2 s(s+1). \tag{4.22}$$

The physically relevant representations correspond to the value of $P_\mu P^\mu$. If it is non-zero then we have a positive mass $m > 0$, this case is self-explained in the above equation. However, if $m = 0$ then we have one of the following two representations based on whether the operators W^μ, P_μ are:

1. **Perpendicular:** Where $W^\mu P_\mu = 0$ with $P_\mu = 0$, this has the trivial solution as representing the vacuum.
2. **Parallel:** Where $W_\mu = cP_\mu$ with $P_0 > 0$, and proportionality given by $c = \pm s$ called *helicity*.

5 The Dirac Field

Looking back at the theories we have encountered so far, we may notice that we have only discussed scalar fields. The quantisation of these fields have provided us with descriptions of spinless particles that satisfy the dynamics encoded in the KG equation. To describe particles with non-trivial spin we must clearly move to a theory of vector fields. To construct a Lorentz/Poincaré invariant (physically relevant) action of particles with spin we will need to find something similar to the spinor representation in section (4.4). In particular, we are interested in finding other matrices which satisfy the Lie Algebra relations of (4.11).

5.1 The Spinor Representation

Dirac managed to find these matrices in an abstract algebra called the *Clifford Algebra* which at first glance has nothing to do with Quantum Field Theory. The Clifford algebra in its defining representation is the set of 4×4 matrices γ^μ with $\mu = 0, 1, 2, 3$ that satisfy

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}\mathbf{1}_4. \tag{5.1}$$

Now in another seemingly random step, it turns out that if we define

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \tag{5.2}$$

then these matrices satisfy the Lorentz algebra relations (4.17). To see this we note that they can be written out explicitly as

$$S^{\mu\nu} = \frac{i}{4}(2\gamma^\mu\gamma^\nu - 2\eta^{\mu\nu}) = \frac{i}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}). \tag{5.3}$$

Computation then shows that

$$[S^{\mu\nu}, \gamma^\rho] = i(\gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}), \tag{5.4}$$

and the desired result for $[S^{\mu\nu}, S^{\rho\sigma}]$ follows. We refer to the $S^{\mu\nu}$ matrices as the *Spinor rotation matrices*. We need a field for these matrices to act on, and we are going to define that field as the *Dirac Spinor Field* $\psi^\alpha(x)$, which is an object with four complex components labelled by α.

The fields are no longer scalars and therefore won't have the usual form of a Lorentz transformation as in (4.7). The transformations of Dirac spinors are given by exponentiating the generators such that

$$\psi^\alpha(x) \to (\Lambda_S)^\alpha_\beta \psi^\beta(\Lambda^{-1}x). \tag{5.5}$$

By defining it in this way, we make sure that the spacetime coordinates continue to transform in their usual way. The transformation matrices (one for spin and one for boost and rotations are given in terms of the generators by

$$\Lambda = \exp\left(\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^\mu\nu\right), \qquad \Lambda_S = \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right). \tag{5.6}$$

5.2 The Dirac (Gamma) Matrices

The matrices γ^μ as defined in the Clifford algebra are defined as far as they need to be for the purposes of constructing the Dirac algebra, but we are not done yet, we want to know their explicit form. We can see immediately from their defining anti-commutator that each of the four matrices squares to the identity (up to a sign), that is, $(\gamma^0)^2 = \mathbf{1}_4$ $(\gamma^i)^2 = -\mathbf{1}_4$. We can also see that if we manipulate the anti-commutator as follows

$$[\gamma^\mu\gamma^\nu]_+ = [\gamma^\mu,\gamma^\nu] + 2\gamma^\nu\gamma^\mu \implies [\gamma^\mu,\gamma^\nu] = 2\eta^{\mu\nu} - 2\gamma^\nu\gamma^\nu$$

$$\implies \gamma^\mu\gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu = \begin{cases} (\gamma^\mu)^2 & \text{if } \mu = \nu \\ -\gamma^\nu\gamma^\mu & \text{if } \mu \neq \nu \end{cases} \tag{5.7}$$

then we obtain some more useful relations that the matrices must obey. We can then show that the matrices must be traceless since for some fixed index μ, the identity can be expressed as $\mathbf{1}_4 = \gamma^\mu\gamma^\mu/\eta^{\mu\mu}$. We then insert this identity into the trace formula for some index $\nu \neq \mu$.

$$\mathrm{tr}(\gamma^\nu) = \frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma^\mu) = -\frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\mu) = -\frac{1}{\eta^{\mu\mu}}\mathrm{tr}(\gamma^\nu\gamma^\mu\gamma^\mu) = -\mathrm{tr}(\gamma^\nu) = 0 \tag{5.8}$$

Where we have used that γ^μ, γ^ν anti-commute and the cyclic property of the trace. The constraints on the form of γ^μ we have derived so far do not completely determine them. But we will choose to use what is known as the *chiral or Weyl representation*

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{5.9}$$

Here the 2×2 matrices σ^i are the usual Pauli matrices which satisfy the multiplicative identity (1.7). Accredited to Weyl is an even more compact way of writing the four matrices γ^μ by defining $\sigma^\mu = (\mathbb{1}_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma})$ which gives

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{5.10}$$

5.3 The Spinor Transformations

Now that we have explicit expressions for the gamma matrices we can compute the exponential map to find Λ_S in (5.6). From the definition we know that $S^{\mu\mu} = 0$ because they are defined as a commutator. The boost part of Λ_S will be generated by $S^{0i} = -S^{i0}$ while the rotation part will be generated by S^{ij}. We will make the computations separately since boosts and rotations don't commute and this would create some trouble in the exponent due to the BCH formula if we tried to compute the whole matrix at once.

We will start by computing Λ_S for boosts, from the definition of $S^{\mu\nu}$ we have

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{5.11}$$

We then exponentiate this matrix with the boost parameters $\omega_{0i} = -\omega_{i0} \equiv \alpha_i$, such that $\vec{\alpha}$ is the vector determining the direction of the boost. This yields

$$\Lambda_S = \exp\left[-\frac{1}{2}\begin{pmatrix} -\vec{\alpha}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\alpha}\cdot\vec{\sigma} \end{pmatrix}\right] = \begin{pmatrix} e^{-\frac{1}{2}\vec{\alpha}\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{1}{2}\vec{\alpha}\cdot\vec{\sigma}} \end{pmatrix}. \tag{5.12}$$

Now to compute the Lorentz transformation of the rotation generators S^{ij} we again look at the definition to find that

$$S^{ij} = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}. \tag{5.13}$$

We parametrise the rotations by $\omega_{ij} = -\epsilon_{ijk}\varphi^k$, and then using that the two-index contraction of the Levi-Civita symbols are $\epsilon^{ijl}\epsilon_{ijk} = 2\delta^l_k$, we get

$$\Lambda_S = \exp\left[\frac{i}{2}\begin{pmatrix} \vec{\varphi}\cdot\vec{\sigma} & 0 \\ 0 & \vec{\varphi}\cdot\vec{\sigma} \end{pmatrix}\right] = \begin{pmatrix} e^{\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}} \end{pmatrix}. \tag{5.14}$$

We note that the Dirac representation we have formulated here is not unitary because γ^0 is Hermitian and γ^i are anti-Hermitian.

5.4 Constructing an Action

Now that we have introduced the Dirac spinor as our vector field, and the corresponding representation of the Lorentz group that acts on this field, we are ready to try and construct a Lorentz-invariant action. To this, we need to find combinations of spinors which are invariant under the Lorentz transformation (5.6).

A good guess to start with would be to take an inner product of two spinors ψ and $\psi^\dagger = (\psi^*)^T$ such that all of the spinor indices become contracted and we get a scalar. For each of these objects we have the transformations

$$\psi(x) \to \Lambda_S\psi(\Lambda^{-1}x); \qquad \psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\Lambda_S^\dagger.$$

This means the product transforms as

$$\psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)\Lambda_S^\dagger\Lambda_S\psi(\Lambda^{-1}x), \tag{5.15}$$

but this can't be an invariant quantity since $\Lambda_S^\dagger\Lambda_S \neq \mathbf{1}$ due to the non-unitarity of the representation. The result of the transformation is not even a scalar. To progress to a better guess, we can look at why this one failed. Let's instead pick a representation of the Clifford algebra which, just like the Weyl representation holds γ^0 Hermitian and γ^i anti-Hermitian. That is

$$(\gamma^0)^\dagger = \gamma^0 \qquad \text{and} \qquad (\gamma^i)^\dagger = -\gamma^i;$$

Then we note that we can write the conjugation in covariant notation for any of the gamma matrices as

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger. \tag{5.16}$$

By linearity this means that the Hermitian conjugate of $S^{\mu\nu}$ takes the form

$$(S^{\mu\nu})^\dagger = -\frac{i}{4}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = \gamma^0 S^{\mu\nu}\gamma^0 \tag{5.17}$$

where we find that $S^{\mu\nu}$ is Hermitian. If we had not defined $S^{\mu\nu}$ with a factor of i this choice of representation would make it anti-Hermitian with a minus sign on the RHS above. Now we have that the conjugate $\Lambda_S^\dagger$ in the transformation becomes

$$\Lambda_S^\dagger = \exp(\frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu)^\dagger}) = \gamma^0\Lambda_S^{-1}\gamma^0, \tag{5.18}$$

which follows from the fact that Hermitian generators generate Unitary transformations. With this clearly being a step in the right direction, we define the *Dirac adjoint* operation on the spinor ψ as

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 \tag{5.19}$$

and we claim that this makes $\bar{\psi}\psi$ a Lorentz scalar. To show this we repeat the Lorentz transformation in the new representation which yields

$$\bar{\psi}(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)\Lambda_S^\dagger\gamma^0\Lambda_S\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x), \tag{5.20}$$

which is the transformation law for a Lorentz scalar as needed. Now we claim that the combination $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector which is to say that it transforms as x^μ. Showing that this is true would mean that we could contract other Lorentz indices with it to obtain yet more scalars. Suppressing the argument x, we have

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}\Lambda_S^{-1}\gamma^\mu\Lambda_S\psi. \tag{5.21}$$

For this transformation to be that of a Lorentz vector we clearly require the condition

$$\Lambda_S^{-1}\gamma^\mu\Lambda_S = \Lambda^\mu{}_\nu\gamma^\nu. \tag{5.22}$$

Infinitesimally we expand (5.6) to leading order to get

$$\Lambda \approx 1 + \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}, \qquad \Lambda_S \approx 1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

where we see that the condition becomes $[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{M}^{\rho\sigma})^\mu_\nu\gamma^\nu$. But this follows immediately from (5.4) and (4.10) therefore it is true that $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector whose transformation law is

$$\bar{\psi}(x)\gamma^\mu\psi(x) \to \Lambda^\mu{}_\nu\bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x). \tag{5.23}$$

With these two Lorentz covariant terms, we are ready to write down the *Dirac action*

$$S_D = \int d^4x\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{5.24}$$

As we will see when we derive its equations of motion, the quantised theory associated with this action describes fermionic particles of mass $|m|$ and spin-1/2. The mass can be positive or negative in the Lagrangian which we will also write down before moving on

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{5.25}$$

The equation of motion follows from the action by varying ψ and $\bar{\psi}$ independently. Varying with respect to $\bar{\psi}$ yields the equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{5.26}$$

This is the *Dirac equation*, and varying with respect to ψ gives its conjugate equation

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0. \tag{5.27}$$

The equation mixes up different components of ψ through the γ matrices, but each individual component solves the Klein-Gordon equation. We can show this by writing

$$(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = -(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = 0, \tag{5.28}$$

but $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}[\gamma^\mu,\gamma^\nu]_+\partial_\mu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu$ so we end up with

$$-(\partial_\mu\partial^\mu + m^2)\psi = 0 \tag{5.29}$$

which is the KG equation.

5.5 Weyl Spinors, Chiral Spinors and Parity

We would like to now find out more about the spinors which can solve the classical Dirac equation so that we can quantise this theory. It is apparent from the generators of boosts and rotations (5.12), (5.14) that the Dirac representation of the Lorentz group is reducible, because they are in block diagonal form. This means that we can decompose the Dirac representation into two irreps acting only on two-component spinors

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}. \tag{5.30}$$

The embedded two component objects ψ_L and ψ_R are the left and right-handed *Weyl spinors*. As is easily read off from the block matrices, they both transform in the same way under rotations

$$\psi_{L/R} \to e^{\frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}}\psi_{L/R}, \tag{5.31}$$

but oppositely under boosts

$$\psi_{L/R} \to e^{\mp\frac{1}{2}\vec{\alpha}\cdot\vec{\sigma}}\psi_{L/R}. \tag{5.32}$$

We say that ψ_L lives in the $(\frac{1}{2},0)$ irrep, while ψ_R lives in the $(0,\frac{1}{2})$ irrep of the Lorentz group. If we substitute this decomposition into the Dirac Lagrangian we get

$$\mathcal{L} = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L). \tag{5.33}$$

We see that the description of a massive fermion requires both spinors because they are coupled through m, but the description of a massless fermion requires only one of the two with either of the following equations of motion

$$i\bar{\sigma}^\mu \psi_L = 0 \qquad \text{or} \qquad i\sigma^\mu \partial_\mu \psi_R = 0. \tag{5.34}$$

The Lorentz group transformations Λ_S turned out to have a block diagonal form because we chose the chiral representation. We would like to make sure that our definition of chiral spinors is invariant under the choice of representation because if for example, we defined the Dirac matrices as $U\gamma^\mu U^{-1}$ with spinors $U\psi$, then Λ_S would not have a block diagonal form. Is there an invariant way to define chiral spinors?

As it turns out, the Dirac matrices have a weird inbred cousin called γ^5 defined by the equation

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{5.35}$$

with the properties that

$$[\gamma^5, \gamma^\mu]_+ = 0, \qquad (\gamma^5)^2 = \mathbf{1}, \qquad (\gamma^5)^\dagger = \gamma^5. \tag{5.36}$$

It follows that γ^5 also commutes with the generators $S^{\mu\nu}$ which means that it is a scalar under rotations and boosts. That it squares to the identity, makes γ^5 an ideal candidate for a term in the Lorentz invariant projection operators defined as

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5). \tag{5.37}$$

Dependent on the choice of representation of the Clifford algebra γ^5 can take many forms but in the pertinent chiral representation we find

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \implies P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.38}$$

Clearly the projection operators project onto the Weyl spinors $\psi_{L/R}$. In fact in any arbitrary representation of the Clifford algebra, we may use γ^5 to define the Chiral spinors

$$\psi_{L/R} = P_\pm \psi,$$

which form the pair of irreps of the Lorentz group.

The two chiral spinors we have defined above are related by a discrete symmetry of the Lorentz group known as parity (or chirality). The parity transformation is defined as a flipping of signs on the spatial components of $(x^0, \vec{x})$ namely

$$P : x^0 \to x^0 \quad ; \quad x^i \to -x^i. \tag{5.39}$$

Under parity, the left and right-handed spinors are exchanged. This is further reinforced by the fact that under a parity reversal, the rotations of spinors don't change sign but the boosts do. We can write this in standard notation as

$$P : \psi_{L/R}(\vec{x}, t) \to \psi_{R/L}(-\vec{x}, t). \tag{5.40}$$

Using this knowledge of how chiral spinors transform under parity we can see that in the reducible Dirac representation, the Dirac spinor transforms as

$$P : \psi(\vec{x}, t) \to \psi^P(\vec{x}, t) = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \psi = \gamma^0 \psi(-\vec{x}, t) \tag{5.41}$$

where the matrix γ^0 acts to exchange the top two components of ψ with the bottom two. It follows that the Dirac spinor is parity invariant because under parity exchange it remains a solution to the Dirac equation:

$$(i\gamma^0\partial_t + i\gamma^i\partial_i - m)\gamma^0\psi(-\vec{x}, t) = \gamma^0(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\psi(-\vec{x}, t) = 0$$

where the minus sign caused by γ^0 passing through γ^i is compensated by ∂_i acting on $-\vec{x}$.

We will find that Parity invariance is a hard thing to maintain when we start to look at interacting theories such as those of Dirac but some theories are built on the principle. Let's look at how the Lorentz invariant interaction terms in the Dirac action do under a parity transformation. First, we transform the scalar $\bar{\psi}\psi(\vec{x}, t)$ which yields

$$P : \bar{\psi}\psi(\vec{x}, t) \to \bar{\psi}(\gamma^0)^2\psi(-\vec{x}, t) = \bar{\psi}\psi(-\vec{x}, t),$$

which is the transformation of a scalar. For the vector term $\bar{\psi}\gamma^\mu\psi(\vec{x}, t)$ we can look at the time and space components individually;

$$P : \bar{\psi}\gamma^0\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\psi(-\vec{x}, t)$$

and

$$\bar{\psi}\gamma^i\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^i\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^i\psi(-\vec{x}, t).$$

Which is the transformation of a vector with the spatial part changing sign.

5.6 Solving the Free Particle Dirac Equation

It is now time to look at the Dirac equation (5.26), we start by making a simple plane wave ansatz. This guess is mostly motivated by the fact that ψ should also satisfy the KG equation

$$\psi(x) = u(\vec{p})e^{-ip\cdot x}. \tag{5.42}$$

Here, $u(\vec{p})$ is a four-component spinor, independent of spacetime but possibly depending on the 3-momentum $\vec{p}$. Substitution of this ansatz into the Dirac equation yields

$$(\gamma^\mu p_\mu - m)u(\vec{p}) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{pmatrix} u(\vec{p}) = 0 \tag{5.43}$$

where we are using Weyl's notation $\sigma^\mu = (\mathbf{1}_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbf{1}_2, -\vec{\sigma})$. The solution to this matrix equation is the spinor

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \tag{5.44}$$

for any 2-component spinor ξ which we will normalise so that $\xi^\dagger\xi = 1$. To prove this, we write the spinor $u(\vec{p}) = (u_1, u_2)^T$. Then (5.43) produces the two equations

$$(p\cdot\sigma)u_2 = mu_1 \qquad (p\cdot\bar{\sigma})u_1 = mu_2. \tag{5.45}$$

These equations actually imply each other and this is easy to see when we write down the identity

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2. \tag{5.46}$$

Now suppose we choose $u_1 = (p\cdot\sigma)\xi'$ for some two component spinor ξ', then the second equation immediately tells us that $u_2 = m\xi'$ and therefore any four component spinor of the form

$$u(\vec{p}) = A\begin{pmatrix}(p\cdot\sigma)\xi'\\ m\xi'\end{pmatrix},$$

with A constant, is a solution to (5.43). Our result was achieved by the more symmetric choice $A = 1/m$ and $\xi' = \sqrt{p\cdot\bar{\sigma}}\xi$ with ξ =constant.

We get a further set of solutions to the Dirac equation from the ansatz

$$\psi = v(\vec{p})e^{ip\cdot x}$$

which has a positive exponent. The difference between these solutions is that the previous solutions oscillate in time as $\psi \sim e^{-iEt}$ are called positive frequency solutions and correspond to positive energy excitations. However, these solutions oscillate in time as $\psi \sim e^{iEt}$ and are called negative frequency solutions and correspond to negative energy excitations. We saw this exact duality before in KG theory. The Dirac equation requires that the four-component spinor $v(\vec{p})$ satisfies

$$(\gamma^\mu p_\mu + m)v(\vec{p}) = \begin{pmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & m\end{pmatrix}v(\vec{p}) = 0 \tag{5.47}$$

which analogously is solved by the spinor

$$v(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta\\ -\sqrt{p\cdot\bar{\sigma}}\eta\end{pmatrix} \tag{5.48}$$

where η is any arbitrary 2-component spinor which we take to be normalised $\eta^\dagger\eta = 1$.

The positive frequency solution $u(\vec{p})$ corresponding to vanishing 3-momentum is given by

$$u(\vec{p}) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}. \tag{5.49}$$

Spatial rotations of the field act on ξ in the usual way

$$\xi \to e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}}\xi.$$

The embedded two component spinor ξ defines the spin of the field, just like in quantum mechanics where spin up or down is described by the eigenvector of the corresponding Pauli matrix with eigenvalue ± 1. For example if the spin of the field is defined by $\xi = (1,0)^T$ along the z-axis, then this will be the spin of the associated particle after we quantise the theory.

We will abuse the language here a bit and call ξ the spin of the particle for the purpose of an important example which describes the Lorentz transformation of a general solution of the Dirac equation. Consider the boost of a particle with spin $\xi = (1,0)^T$ along the x^3-direction and with $p^\mu = (E,0,0,p^3)$. The solution to the Dirac equation becomes

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}(1,0)^T \\ \sqrt{p\cdot\bar{\sigma}}(1,0)^T \end{pmatrix} = \begin{pmatrix} \sqrt{E-p^3}(1,0)^T \\ \sqrt{E+p^3}(1,0)^T \end{pmatrix}.$$

This expression also makes sense for a massless field for which $E = p^3$ which is why we normalised the spinor in this symmetric way. In this case the solution would reduce to

$$u(\vec{p}) = (0,0,\sqrt{2E},0)^T. \tag{5.50}$$

Similarly the spin down field with $\xi = (0,1)^T$ has the boosted solution

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}(0,1)^T \\ \sqrt{p\cdot\bar{\sigma}}(0,1)^T \end{pmatrix} = \begin{pmatrix} \sqrt{E+p^3}(0,1)^T \\ \sqrt{E-p^3}(0,1)^T \end{pmatrix}.$$

In the massless case this reduces to

$$u(\vec{p}) = (0,\sqrt{2E},0,0)^T. \tag{5.51}$$

We will now introduce another concept relating to the angular momentum of the field and that is *helicity*. The helicity operator h measures the projection of angular momentum along the direction of momentum, in matrix form it is given by

$$h = \frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{jk} = \frac{1}{2}\hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{5.52}$$

where S^{ij} is the rotation generator of the Dirac representation (5.13). The massless field with spin given by $\xi^T = (1,0)$ has helicity $h = \frac{1}{2}$ which we call *right handed*. Oppositely, we refer to the field with $\xi^T = (0,1)$ as being left-handed with helicity $h = -\frac{1}{2}$.

5.7 Some Final Formulas Before Quantisation

With the positive and negative frequency solution spinors $u(\vec{p})$ and $v(\vec{p})$ we will go on to quantise Dirac theory, so it is very important that we know how to handle them. Firstly, we introduce a basis ξ^s and η^s where $s = 1,2$ for the embedded two component spinors such that

$$(\xi^r)^\dagger \xi^s = \delta^{rs} = (\eta^r)^\dagger \eta^s \tag{5.53}$$

which could for example be the choice of spinors $\xi^1 = (1,0)^T$ and $\xi^2 = (0,1)^T$ and similarly for η^s. The consequences of this can be seen as follows, the positive frequency plane wave solutions are now written as

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}. \tag{5.54}$$

There are two ways we can take the inner product of four component spinors, $u^\dagger \cdot u$ and $\bar{u}\cdot u$. The first way is not Lorentz invariant but we still need it for when we quantise, the latter is Lorentz invariant and we will state both products now:

$$u^{r\dagger}(\vec{p})\cdot u^s(\vec{p}) = (\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}) \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \tag{5.55}$$

$$= \xi^{r\dagger}(p\cdot\sigma)\xi^s + \xi^{r\dagger}(p\cdot\bar{\sigma}\xi^s = 2\xi^{r\dagger}p_0\xi^s = 2p_0\delta^{rs}.$$

$$\bar{u}^r(\vec{p})\cdot u^s(\vec{p}) = (\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = 2m\delta^{rs} \tag{5.56}$$

The corresponding inner products for the negative frequency solution in this basis

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix} \tag{5.57}$$

are

$$v^{r\dagger}(\vec{p})\cdot v^s(\vec{p}) = 2p_0\delta^{rs} \qquad \text{and} \qquad \bar{v}^r(\vec{p})\cdot v^s(\vec{p}) = -2m\delta^{rs}. \tag{5.58}$$

It is easy to show that the Lorentz invariant inner products $\bar{u}^r(\vec{p})\cdot v^s(\vec{p})$ and $\bar{v}^r(\vec{p})\cdot u^s(\vec{p})$ both vanish. For the non-Lorentz invariant combination the inner products vanish on a certain condition which fortunately appears during the quantisation. In particular we find that the identity

$$u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p}) = v^{r\dagger}(\vec{p})\cdot u^s(-\vec{p}) = 0 \tag{5.59}$$

holds when we have the sign of the vector $\vec{p}$ flipped in the argument of the unconjugated spinor. The last thing we need to check is whether these solutions form a complete set. We claim that the following identities hold

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = p_\mu\gamma^\mu + m, \tag{5.60}$$

$$\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = p_\mu\gamma^\mu - m. \tag{5.61}$$

To show this for u^s we compute

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} (\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma}).$$

But $\sum \xi^s \xi^{s\dagger} = \mathbf{1}$ which leads to the previous line being equal to

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = p_\mu \gamma^\mu + \mathbf{1}m.$$

As needed, then a similar proof works for the case of v^s. Clearly the set is not totally complete but we will see why in the following section where we quantise the Dirac theory.

5.8 Naive Quantisation of the Dirac Field

To begin the process of quantising the Dirac Field, we will start by once again writing down the Dirac Lagrangian form the action as in (5.25)

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{5.62}$$

We can naively try and repeat the method we applied to the Klein-Gordon scalar field but we will find that common sense does not prevail in the case of Dirac theory, instead there is some fermionic trickery afoot. The Hamiltonian can be formulated as in (2.16) by calculating the momentum density conjugate to ψ

$$\pi(x) = \frac{\partial \mathcal{L}[\psi(x)]}{\partial \dot{\psi}} = i\bar{\psi}(x)\gamma^0 = i\psi^\dagger(x)(\gamma^0)^2 = i\psi^\dagger(x). \tag{5.63}$$

Hence we can write

$$\begin{aligned} H &= \int d^3x [i\bar{\psi}\gamma^0 \partial_0 \psi - \mathcal{L}] = \int d^3x \bar{\psi}(i\gamma^0\partial^0 - i\gamma^\mu\partial_\mu + m)\psi \\ &= \int d^3x \bar{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi. \end{aligned} \tag{5.64}$$

The pertinent way to write this Hamiltonian is in terms of the Hermitian conjugate of the field rather than the Dirac adjoint, so we substitute $\psi^\dagger\gamma^0 = \bar{\psi}$ to obtain

$$H = \int d^3x \psi^\dagger(-i\gamma^0\vec{\gamma}\cdot\nabla + m\gamma^0)\psi. \tag{5.65}$$

After promoting the field and its momentum to operators, we then postulate that they satisfy the canonical equal time commutation relations

$$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}. \tag{5.66}$$

Note that a and b here denote the component of the Dirac spinor, we will suppress these indices going forward but we must maintain the fact that they are just components and they behave as such under reordering. This applies distinctly to the spinor indices, not to be confused with the indices s, r denoting the choice of *up* or *down* spin within the basis of positive and negative frequency solutions to the Dirac equation.

Next we want to write down an explicit form of the solutions ψ and $\psi^\dagger$ in a representation of creation and annihilation operators that diagonalise the Hamiltonian. From the form of the Hamiltonian we can see that the eigenfunctions $u^s(p)e^{-ip\cdot x}$ and $v^s(p)e^{ip\cdot x}$ with eigenvalues $E_{\vec{p}}$ and $-E_{\vec{p}}$ are an appropriate set over which ψ can be expanded but due to the additional minus sign in the Hamiltonian, we must change them slightly. So in the Schrödinger picture where the operators don't depend on time, we obtain

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2}[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}], \tag{5.67}$$

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2}[b^{s\dagger}_{\vec{p}}u^{s\dagger}(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^{s\dagger}(\vec{p})e^{i\vec{p}\cdot\vec{x}}]. \tag{5.68}$$

The operator coefficients are to be interpreted as the creation and annihilation operators b associated with u and c with v. The commutation relations of the fields themselves necessitate the following relations between the creation and annihilation operators

$$[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}), \tag{5.69}$$

$$[c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}] = -(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}). \tag{5.70}$$

The minus sign here is a subtle foreshadowing of the failure of our naive approach. It essentially says that the creation operator $c^{s\dagger}_{\vec{p}}$ will play the opposite role in bosonic Fock space $c^{s\dagger}_{\vec{p}}|0\rangle = 0$ since the states it would create in the usual way have negative norm. Now let's show that these commutation relations reproduce (5.66) as follows

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] =$$

$$\int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{r,s}([b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}]u^r(\vec{p})u^{s\dagger}(\vec{q})e^{i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})} + [c^{r\dagger}_{\vec{p}}, c^s_{\vec{q}}]v^r(\vec{p})v^{s\dagger}(\vec{q})e^{-i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})}$$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(u^s(\vec{p})\bar{u}^s(\vec{p})\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s(\vec{p})\bar{v}^s(\vec{p})\gamma^0 e^{-i\vec{p}(\vec{x}-\vec{y})}). \tag{5.71}$$

Then we use the outer product formulae derived in the previous section (5.60) and (5.61) to reduce this to

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int \frac{d^3}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}[(p_\mu\gamma^\mu + m)\gamma^0 e^{i\vec{p}(\vec{x}-\vec{y})} + (p_\mu\gamma^\mu - m)\gamma^0 e^{-i\vec{p}(\vec{x}-\vec{y})}]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}[(p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 + (p_0\gamma^0 - p_i\gamma^i - m)]e^{i\vec{p}\cdot(\vec{x}-\vec{y})},$$

where in the second term we have changed $\vec{p} \to -\vec{p}$ under the integration sign. Then since the temporal component of momentum is $p_0 = E_{\vec{p}}$ we have

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y}) \tag{5.72}$$

as needed. Now let's return to the Hamiltonian (5.64) and attempt to write that too as an operator. First we consider

$$(-i\vec{\gamma}\cdot\nabla+m)\psi=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}[b^s_{\vec{p}}(-\gamma^i p_i+m)u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}}+c^{s\dagger}_{\vec{p}}(\gamma^i p_i+m)v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}]$$

where we implicitly sum over $s=1,2$. We have also used the Minkowski metric to achieve some minus signs here by lowering a spatial index. From the defining equations (5.43) and (5.47) of the spinors $u(\vec{p})$ and $v(\vec{p})$ we can obtain

$$(i\gamma^i p_i+m)u^s(\vec{p})=\gamma^0 p_0 u^s(\vec{p})\quad\text{and}\quad(\gamma^i p_i+m)v^s(\vec{p})=-\gamma^0 p_0 v^s(\vec{p}).\tag{5.73}$$

This allows us to write

$$(-i\vec{\gamma}\cdot\nabla+m)\psi=\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\gamma^0[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}}-c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}].$$

Then we insert this expression into the operator version of the Hamiltonian:

$$H=\int d^3x\psi^\dagger(-i\gamma^i\partial_i+m)\psi$$

$$=\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\sqrt{\frac{E_{\vec{p}}}{4E_{\vec{q}}}}[b^{r\dagger}_{\vec{q}}u^r(\vec{q})e^{-i\vec{q}\cdot\vec{x}}+c^r_{\vec{q}}v^{r\dagger}(\vec{q})e^{i\vec{q}\cdot\vec{x}}]\cdot[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}}-c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}]$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\Big[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}}[u^r(\vec{p})\cdot u^s(\vec{p})]-c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}[v^{r\dagger}(\vec{p})\cdot v^s(\vec{p})]$$

$$-b^{r\dagger}_{\vec{p}}c^{s\dagger}_{-\vec{p}}[u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p})]+c^r_{\vec{p}}b^s_{-\vec{p}}[v^{r\dagger}(\vec{p})\cdot u^s(-\vec{p})]\Big],$$

where we have relabelled $\vec{p}\to-\vec{p}$ in the final two terms. We have gone to the trouble of manipulating the Hamiltonian into this form containing inner products of the spinors so that we can avail of the work we did at the end of the previous section. Indeed, by virtue of (5.59), the two cross terms vanish and we are left with

$$H=\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}-c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}).$$

Upon placing the c operators into normal order and picking up a commutator this becomes

$$H=\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}-c^{s\dagger}_{\vec{p}}c^s_{\vec{p}}+(2\pi)^3\delta^{(3)}(0)).\tag{5.74}$$

The $\delta^{(3)}(0)$ term is the same one we dismissed in Klein Gordon theory and is dealt with by normal ordering. Let's look at the structure of the other two terms, firstly we have that $b^\dagger$ creates positive energy states as we need it to since $[H,b^{s\dagger}_{\vec{p}}]=E_{\vec{p}}b^{s\dagger}_{\vec{p}}$. The second term however is alarming since it has a minus sign, from the relation (5.70) we still get that the operator $c^\dagger$ creates positive energy states $[H,c^{s\dagger}_{\vec{p}}]=E_{\vec{p}}c^{s\dagger}_{\vec{p}}$ however as we noted

before, these states have negative norms in Hilbert space. So to have a sensible Hilbert space we need to view c as the creation operator. But then the Hamiltonian would not be bounded below as indicated by $[H, c^s_{\vec{p}}] = -E_{\vec{p}} c^s_{\vec{p}}$ which tells us that we can keep creating c particles and it will lower the energy indefinitely. To avoid this catastrophe we need to figure out why that minus sign appears and what the solution is to getting rid of it.

5.9 Fermionic Quantisation of the Dirac Field

The key feature of the theory that we missed on our first attempt was that the particles created by the Dirac field are fermionic (spin-1/2) particles that follow Fermi-Dirac statistics. The function describing these particles must be anti-symmetric under the exchange of their quantum numbers. We can proceed to resolve our previous mistake by recalibrating our bosonic assumptions to fermionic ones where the commutation relations are replaced by anti-commutation relations.

To have states which obey fermionic statistics, we will postulate that the spinor fields satisfy the equal time anti-commutation relations

$$\{\psi_a(\vec{x}), \psi^\dagger_b(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab}, \tag{5.75}$$

$$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi^\dagger_a(\vec{x}), \psi^\dagger_b(\vec{y})\} = 0. \tag{5.76}$$

For the mixed anti-commutator to hold in terms of our expansions over creation and annihilation operators we find that the following must hold at equal time for b and c

$$\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{(3)}(\vec{p} - \vec{q}) \tag{5.77}$$

with all other combinations vanishing identically. The calculation of the Hamiltonian operator proceeds all the way to the second last step where we obtain

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}(b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}} c^s_{\vec{p}} - (2\pi)^3\delta^{(3)}(0)). \tag{5.78}$$

Clearly our problem is solved and we have restored the necessary condition that the vacuum can only experience positive energy excitations. Before moving on to discuss the physics of the quantised theory, we will write down the Dirac adjoint field $\bar{\psi}$ dual to ψ in terms of creation and annihilation operators. The dual is generally more useful and is also easy to define from the components of $\psi^\dagger$ given in (5.68) as we need only multiply by a component of γ^0 to go between them. So here I will write down the field and its dual side by side.

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1}^{2} [b^s_{\vec{p}} u^s(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}}],$$

$$\bar{\psi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{s=1}^{2}[b^{s\dagger}_{\vec{p}}\bar{u}^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}\bar{v}^s(\vec{p})e^{i\vec{p}\cdot\vec{x}}]. \tag{5.79}$$

To formulate a representation in terms of states in fermionic Fock space, we define the vacuum $|0\rangle$ to be the state which uniquely satisfies

$$b^s_{\vec{p}}|0\rangle = c^s_{\vec{p}}|0\rangle = 0. \tag{5.80}$$

The Hamiltonian continues to maintain some nice commutation relations with the creation and annihilation operators. Indeed, we find

$$[H, b^r_{\vec{p}}] = [H, c^r_{\vec{p}}] = -E_{\vec{p}}(b/c)^r_{\vec{p}} \quad \text{and} \quad [H, b^{r\dagger}_{\vec{p}}] = [H, c^{r\dagger}_{\vec{p}}] = E_{\vec{p}}(b/c)^{r\dagger}_{\vec{p}}. \tag{5.81}$$

This means that as in the bosonic case we can construct a tower of energy eigenstates by repeatedly acting on the vacuum with the creation operators to create particles with $b^\dagger$ and antiparticles with $c^\dagger$. As such, a normalised one particle state would look like $|\vec{p}, r\rangle = \sqrt{2E_{\vec{p}}}b^{r\dagger}_{\vec{p}}|0\rangle$, and a general two particle state would look like $|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle = b^{r_1\dagger}_{\vec{p}_1}b^{r_2\dagger}_{\vec{p}_2}|0\rangle$. Note that in a manner consistent with the theory and the Pauli exclusion principle, this two particle state picks up a minus sign under the exchange of particle quantum numbers.

5.10 Charged Particles

The Dirac Lagrangian is quadratic in the spinors ψ and $\bar{\psi}$, so there must be a relevant isotropic symmetry of the theory. This symmetry turns out to be that of $U(1)$, which under which the conjugate pair transform as

$$\psi \to e^{i\alpha}\psi \quad \text{and} \quad \psi^\dagger = e^{-i\alpha}. \tag{5.82}$$

If the gauge parameter α is infinitesimal then we can write the transformation as

$$\psi \to \psi + i\alpha\psi \quad \text{and} \quad \psi^\dagger \to \psi^\dagger - i\alpha\psi^\dagger. \tag{5.83}$$

Using Noether's theorem we can compute the conserved current of the Dirac Lagrangian to be

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\varphi - \mathcal{J}^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0. \tag{5.84}$$

Clearly for this transformation the term $\mathcal{J}^\mu = 0$. We can then calculate the Noether charge as the spatial integral over the time component of this current as follows

$$Q = \int d^3x j^0 = \int d^3x\bar{\psi}\gamma^0\psi = \int d^3x\psi^\dagger\psi. \tag{5.85}$$

Now we can calculate this integral manually using the known expressions in terms of creation and annihilation operators but now we will work in the Heisenberg picture where we restore the presence of the time components of x and p:

$$Q = \sum_{r,s}\int \frac{d^3xd^3pd^3q}{(2\pi)^6 2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left(b^{s\dagger}_{\vec{p}}u^{s\dagger}(p)e^{-ip\cdot x} + c^s_{\vec{p}}v^{s\dagger}(p)e^{ip\cdot x}\right)\left(b^r_{\vec{p}}u^r(q)e^{iq\cdot x} + c^{r\dagger}_{\vec{q}}v^r(q)e^{-iq\cdot x}\right)$$

$$= \sum_{r,s} \int \frac{d^3x d^3p d^3q}{(2\pi)^6 2\sqrt{E_{\vec{p}} E_{\vec{q}}}} \Big(b^{s\dagger}_{\vec{p}} b^r_{\vec{q}} u^{s\dagger}(p) u^r(q) e^{ix^0(p_0-q_0)} e^{i\vec{x}\cdot(\vec{p}-\vec{q})} + c^s_{\vec{p}} b^r_{\vec{q}} v^{s\dagger}(p) u^r(q) e^{ix^0(p_0+q_0)} e^{i\vec{x}\cdot(\vec{p}+\vec{q})}$$

$$+ b^{s\dagger}_{\vec{p}} c^{r\dagger}_{\vec{q}} u^{s\dagger}(p) v^r(q) e^{-ix^0(p_0+q_0)} e^{-i\vec{x}\cdot(\vec{p}+\vec{q})} + c^s_{\vec{p}} c^{r\dagger}_{\vec{q}} v^{s\dagger}(p) v^r(q) e^{-ix^0(q_0-p_0)} e^{-i\vec{x}\cdot(\vec{q}-\vec{p})} \Big).$$

We can then integrate out x and q to create delta functions resulting in the following integral over $\vec{p}$

$$= \sum_{r,s} \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \Big(b^{s\dagger}_{\vec{p}} b^r_{\vec{p}} u^{s\dagger}(\vec{p}) u^r(\vec{p}) + c^s_{\vec{p}} b^r_{\vec{p}} v^{s\dagger}(\vec{p}) u^r(-\vec{p}) + b^{s\dagger}_{\vec{p}} c^{r\dagger}_{-\vec{p}} u^{s\dagger}(\vec{p}) v^r(-\vec{p}) + c^s_{\vec{p}} c^{r\dagger}_{\vec{p}} v^{s\dagger}(\vec{p}) v^r(\vec{p}) \Big)$$

Due to the spinor inner product relations, the cross terms vanish and we pick up constants for the other products leaving

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s [b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^s_{\vec{p}} c^{s\dagger}_{\vec{p}}] = \int \frac{d^3p}{(2\pi)^3} \sum_s [b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}} c^s_{\vec{p}}] + \text{constant}. \tag{5.86}$$

As usual the constant is dealt with by normal ordering. This conserved charge reflects that although the particles created by $b^\dagger$ and $c^\dagger$ have the same positive energy. The particles created by $b^\dagger$ have a *charge* of +1 and oppositely the particles created by $c^\dagger$ have a charge of −1.

5.11 Dirac Propagator

Staying within the formalism of Heisenberg, we shall now properly define the spinors ψ and $\psi^\dagger$ that satisfy the operator equation

$$\frac{\partial \psi}{\partial t} = i[H, \psi]. \tag{5.87}$$

As hinted at in the previous section these can be solved by the expansions (5.67) and (5.68) with the time component of $x \cdot p$ restored in the Fourier exponents. In particular

$$\psi(\vec{x}, t) = \psi(x) = \psi \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1}^{2} [b^s_{\vec{p}} u^s(p) e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}} v^s(p) e^{ip\cdot x}], \tag{5.88}$$

$$\psi^\dagger(\vec{x}, t) = \psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1}^{2} [b^{s\dagger}_{\vec{p}} u^{s\dagger}(\vec{p}) e^{ip\cdot x} + c^s_{\vec{p}} v^{s\dagger}(p) e^{-ip\cdot x}]. \tag{5.89}$$

Note that the sign of the dot product also flips here due to the way the Minkowskian inner product is defined so that the spatial part of the solution remains the same as before. In analogy to the bosonic case, we define the propagator (in components) to be the anti-commutator

$$iS_{ab} = \{\psi_a(x), \bar{\psi}_b(y)\}. \tag{5.90}$$

We will once again suppress these Dirac spinor indices in what follows but this is just another reminder that they have been there all along. Calculating the explicit form of the propagator we obtain

$$iS(x-y) = \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{E_{\vec{p}}E_{\vec{q}}}}\sum_{r,s}\Big[\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\}u^s(\vec{p})\bar{u}^r(\vec{q})e^{-i(p\cdot x-q\cdot y)}+\{c^{s\dagger}_{\vec{p}}, c^r_{\vec{q}}\}v^s(\vec{p})\bar{v}^r(\vec{q})e^{i(p\cdot x-q\cdot y)}\Big]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s\Big[u^s(\vec{p})\bar{u}^s(\vec{p})e^{-ip\cdot(x-y)} + v^s(\vec{p})\bar{v}^s(\vec{p})e^{ip\cdot(x-y)}\Big].$$

Then we can once again use the outer product formulae between the spinors $u, \bar{u}$ and $v, \bar{v}$ which yields

$$iS(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}[(p_\mu\gamma^\mu + m)e^{-ip\cdot(x-y)} + (p_\mu\gamma^\mu - m)e^{-ip\cdot(x-y)}]. \tag{5.91}$$

Notice that this is nothing but a linear combination of Klein-Gordon propagators being acted on by the Dirac equation operator. Therefore we can write

$$iS(x-y) = (i\gamma^\mu\partial_\mu + m)(D(x-y) - D(y-x)). \tag{5.92}$$

To comment on the causality of this solution, we note that for spacelike separated points we now have $\{\psi_a(x), \bar{\psi}_b(y)\} = 0$ outside the lightcone as opposed to the commutator in scalar theory. Initially this might seem shaky but Fermionic operators are defined in terms of spinors, this means they flip sign under a 2π rotation. There's nothing observable that does this so we conclude that as long as fermionic operators are not observable then the theory is still causal because all observables in the theory are bilinear in fermions. Note also that the propagator satisfies the Dirac equation (away from singularities)

$$(i\gamma^\mu\partial_\mu - m)S(x-y) = 0$$

which follows from the fact that $((\gamma^\mu\partial_\mu)^2 + m^2)D(x-y) = 0$ using the condition on four-momentum $p^2 = m^2$.

5.12 The Feynman Propagator of Dirac Theory

Let's now look at the all important vacuum expectation values which we can extract from our calculation of the propagator above

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(p_\mu\gamma^\mu + m)_{ab}e^{-ip\cdot(x-y)} = (i\gamma^\mu\partial_\mu + m)D(x-y), \tag{5.93}$$

$$\langle 0|\bar{\psi}_a(x)\psi_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(p_\mu\gamma^\mu - m)_{ab}e^{ip\cdot(x-y)} = (i\gamma^\mu\partial_\mu - m)D(y-x). \tag{5.94}$$

The most important propagator for us to define is the Feynman propagator and to do this for the fermionic fields is a bit more subtle than it was in KG theory. The Dirac

spinors can be identified as Grassmann variables because they anti-commute with each other. This means that we will have to account for an additional minus sign appearing in the Feynman propagator before the second term in (3.20) in order to keep Lorentz invariance. This prescription leads to the definition

$$S_F(x-y) = \Theta(x^0-y^0)\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle - \Theta(y^0-x^0)\langle 0|\bar{\psi}_a(x)\psi_b(y)|0\rangle \tag{5.95}$$

which is nothing but the Dirac operator acting on the Feynman propagator for scalar theory:

$$S_F(x-y) = (i\gamma^\mu\partial_\mu + m)D_F(x-y) \equiv \langle 0|\mathcal{T}(\bar{\psi}(x)\psi(y))|0\rangle. \tag{5.96}$$

The operator T here is the time-ordering operator now acting on spinor fields which anti-commute under its application so that we recover (5.95). The four-momentum integral representation of this fermionic Feynman propagator is now easy to write down as we have it in terms of $D_F(x-y)$:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\gamma^\mu p_\mu + m)}{p^2-m^2+i\varepsilon}e^{-ip\cdot(x-y)}. \tag{5.97}$$

Most importantly this propagator is a Green's function of the Dirac operator

$$(i\gamma^\mu\partial_\mu - m)S_F(x-y) = i\delta^{(4)}(x-y). \tag{5.98}$$

6 Constrained Hamiltonian Systems

We have now obtained free field propagators for both scalar Klein Gordon theory and Dirac theory which describe field excitations corresponding to spin-0 and spin-$\frac{1}{2}$ particles respectively. The next theory we will study is that of quantum electrodynamics which in the free field case is a quantum theory of light (spin-1) originating from classical Maxwell electrodynamics. As we know from our study of classical Maxwell theory in third year, this theory contains subtleties of a new kind involving gauge symmetry and constraints. The theory of constrained Hamiltonian systems is due to Dirac and the following section is based on his presentation in the iconic "Lectures on Quantum Mechanics".

6.1 The Emergence of Constraints

For any classical system described by a Lagrangian $L(q,\dot{q})$ with generalised coordinates $q^i(t)$, $i = 1,\ldots,N$, the action can be written in the form

$$S[q,\dot{q}] = \int_{t_1}^{t_2} dt L(q,\dot{q}). \tag{6.1}$$

The canonical momenta conjugate to each $q^i(t)$ are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{6.2}$$

and Hamilton's principle of least action stating that any variation $\delta q^i(t)$ of classical trajectories that vanishes at the times t_1, t_2 leaves the action unchanged can only be true if the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \tag{6.3}$$

are satisfied. Any of these equations can be written in the equivalent form

$$\frac{d^2 q^j}{dt^2}\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} - \frac{dq^j}{dt}\left(\frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\right). \tag{6.4}$$

Which reveals that the accelerations are determined uniquely at all times by the positions and velocities if and only if the matrix on the left hand side is invertible, or equivalently:

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) \neq 0. \tag{6.5}$$

Here, and in what follows, the case where this determinant vanishes is of most interest i.e. the accelerations are not uniquely determined by the positions and velocities. The equations of motion in this case can contain arbitrary functions of time indicating that there is some redundancy in the theory to be dealt with. Furthermore it is easy to see that if the determinant (6.5) was to vanish, then the defining equations for the conjugate momenta would not be invertible so that locally they can not be solved for $\dot{q} = \dot{q}(q, p)$. In particular, where the rank of the matrix is $N - M$ not all of the momenta are independent, instead a set of M primary constraints are encountered

$$G_a = G_a(q, p) = 0, \qquad a = 1, \ldots, M. \tag{6.6}$$

These constraints do not obstruct the Legendre transformation in passing to the canonical formalism however they do require specific treatment in order for the inverse transformation to be possible.

6.2 The Canonical Formalism

In the canonical formalism the dynamics of the system are described by the coordinates (q^i, p_i) of an even $2N$-dimensional phase space. A Legendre transformation of the action in the Lagrangian formalism yields the canonical action

$$S[q, p] = \int_{t_1}^{t_2} dt\ (p_i \dot{q}^i - H(q, p)). \tag{6.7}$$

Under extremisation, the action remains stationary if Hamilton's equations are satisfied

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}. \tag{6.8}$$

The Poisson bracket of two functions $f(q,p)$ and $g(q,p)$ on phase-space is defined as

$$\{f,g\}_{PB} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{6.9}$$

and using this notation, the time evolution of any function on phase space is generated by the Hamiltonian

$$\dot{f} = \{f,H\}_{PB}. \tag{6.10}$$

The Poisson bracket is clearly anti-symmetric and it satisfies the Leibniz rule (in what follows we suppress the subscript PB)

$$\{fg,h\} = \{f,h\}g + f\{g,h\}, \tag{6.11}$$

and the Jacobi identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \tag{6.12}$$

The full phase space spanned by the $2N$ variables (q,p) shall henceforth be referred to as the ambient phase space. Phase space is a symplectic manifold as is evident in the definition of the Poisson bracket and the algebraic structure of the canonical variables

$$\{q^i,p_j\} = \delta^i{}_j, \quad \{q^i,q^j\} = \{p_i,p_j\} = 0. \tag{6.13}$$

Upon performing the Legendre transformation to the canonical formalism the conditions (6.6) are understood to define a $(2N-M')$-dimensional smoothly embedded submanifold of the ambient phase space where M' is the number of independent primary constraints among the full set. As discussed above, a point on this so-called primary constraint surface has a multi-valued inverse image when mapped to the $2N$-dimensional configuration space spanned by $(q,\dot{q})$. This motivates the introduction of a set of M' Lagrange multipliers $\lambda^a(q,p)$ as follows.

A variation to the canonical Hamiltonian $H_0 = \dot{q}^i p_i - L$ results in the expression

$$\delta H_0 = \dot{q}^i\delta p_i + \delta\dot{q}^i p_i - \delta\dot{q}^i\frac{\partial L}{\partial\dot{q}^i} - \delta q^i\frac{\partial L}{\partial q^i} = \dot{q}^i\delta p_i - \delta q^i\frac{\partial L}{\partial q^i}. \tag{6.14}$$

Expanding the left hand side and matching coefficients of δq^i and δp_i,

$$\left(\frac{\partial H_0}{\partial q^i} + \frac{\partial L}{\partial q^i}\right)\delta q^i + \left(\frac{\partial H_0}{\partial p_i} - \dot{q}^i\right)\delta p_i = 0. \tag{6.15}$$

In the case that the arbitrary variations are tangent to the primary constraint surface, the second term can always be written as

$$\dot{q}^i(q,p) = \frac{\partial H_0}{\partial p_i} + \lambda^a\frac{\partial G_a}{\partial p_i}. \tag{6.16}$$

where $\lambda^a(q,p)$ are an additional set of M' variables. It is now clear that by introducing the λ^a the invertibility of the Legendre transformation is restored despite the vanishing of the determinant (6.5). The variations δp_i themselves must satisfy the constraints (6.6) so it must be the case that the Hamiltonian is only well defined on the primary constraint surface up to arbitrary extensions proportional to the constraints. Indeed, when one extends the original Hamiltonian H_0 in accordance with the variational equation (6.15), the action becomes

$$S[q^i, p_i, \lambda^a] = \int_{t_1}^{t_2} dt\ (\dot{q}^i p_i - H_0 - \lambda^a G_a) \tag{6.17}$$

with equations of motion

$$\dot{p}_i = \{p_i, H_0\} + \lambda^a \{p, G_a\}, \qquad \dot{q}^i = \{q^i, H_0\} + \lambda^a \{q^i, G_a\}, \qquad G_a(q,p) = 0. \tag{6.18}$$

The additional variables introduced to restore the invertibility of the Legendre transformation now act as Lagrange multipliers which enforce the primary constraints. A further requirement that is imposed upon the primary constraints is that each of them be preserved throughout the time evolution of the system. Each $G_a(q,p)$ must then obey the following consistency condition

$$\frac{dG_a}{dt} = \{G_a, H_0\} + \lambda^b \{G_a, G_b\} = 0. \tag{6.19}$$

This equation can either lead to restrictions on the Lagrange multipliers or a new relation between the phase space variables. If, in the latter case, the relation is not already covered by the primary constraints then it defines a secondary constraint. In practice, a repeated application of this algorithm (adding each new constraint to the Hamiltonian before checking consistency) until all new relations have been obtained completes the full set of $K \geq M$ constraints on a Hamiltonian system:

$$G_a(q,p) = 0, \qquad a = 1, \ldots, K. \tag{6.20}$$

In the case that checking consistency restricts the value of a Lagrange multiplier then the constraint appearing next to that multiplier in the Hamiltonian is what will soon be introduced as a second class constraint. In a similar fashion to before, the complete set of constraints specify a $(2N - K)$-dimensional submanifold smoothly embedded in the ambient phase space on which the physical trajectories lie. Equipped with a further $K - M'$ Lagrange multipliers, the total Hamiltonian is written as

$$H = H_0 + \lambda^a G_a, \qquad a = 1, \ldots, K \tag{6.21}$$

6.3 First Class Constraints

For a constrained Hamiltonian system with a set of K constraints as in (6.20) it is emphasised that the relations are only constrained to vanish on the physical subspace by means of a weak equality symbol ($\approx$). This notation, attributed only to the Dirac formalism leads to the constraint equations being more appropriately written as

$$G_a(q,p) \approx 0. \tag{6.22}$$

Constraints can be classified as first or second class based on the following definition. A function $f(q,p)$ defined on the ambient phase space is first class if

$$\{f, G_a\} \approx 0 \tag{6.23}$$

for all $a = 1, \ldots, K$. A function is second class if it is not first class, in other words, if it has a non-weakly vanishing Poisson bracket with any of the K constraint functions. Given any system one computes the constraint matrix $C_{ab} = \{G_a, G_b\}$, the rank R of this matrix on the physical subspace will be equal to the number of second class constraints meaning there are a further $K - R$ first class constraints present.

Suppose the canonical formalism has been established for a constrained Hamiltonian system and assume that the full set of constraints $G_a \approx 0$ are first class. The constraint algebra then reads

$$\{G_a, G_b\} = U_{ab}{}^c G_c \approx 0 \tag{6.24}$$

where $U_{ab}{}^c$ are the structure functions possibly depending on the variables (q,p). In addition the consistency requirement takes the form

$$\{G_a, H_0\} = V_a{}^b G_b \approx 0. \tag{6.25}$$

This highlights the important result that in a first class system, the consistency conditions do not restrict the Lagrange multipliers in any way and since $\lambda^a(q,p)$ appear in the equations of motion it follows that a physical state is represented by an equivalence class of points on the constraint surface rather than a single unique point. These equivalence classes are more commonly referred to as gauge orbits and the associated local symmetry of first class Hamiltonian systems is called a gauge symmetry. The transformation which relates a pair of points on the gauge orbit of a given physical state is called a gauge transformation. The complete set $\{G_a\}$ of first class constraints are said to be the generators of gauge transformations. Lastly, a function $f(p,q)$ on the constraint surface can be identified as a physical observable if it is gauge invariant. In other words, if it is constrained to be on shell and has a vanishing gauge transformation

$$\delta_\varepsilon f = \varepsilon^a \{f, G_a\} = 0 \tag{6.26}$$

where some time-dependent functions $\varepsilon^a(t)$ have been chosen as the gauge parameters. There are a variety of known methods which eliminate the gauge freedom inherent in systems with the defining properties (6.24), (6.25) of a general gauge theory. One of these is the reduced phase space method in which the constraints are solved and the canonical formalism is set up only in terms of physical degrees of freedom. In practice, methods such as this are obstructed by the loss of desirable physical properties such as Lorentz invariance or locality. The natural way to proceed then is to keep unphysical degrees of freedom in the theory so that these properties can be preserved.

6.4 Second Class Constraints - The Dirac Bracket

In contrast, by the above definition, second class constraints form a full set of functions with non-vanishing Poisson brackets such that the constraint matrix is non-singular. In other words they define a structure rather than an algebra. The structure of the ambient phase space defined by the ordinary Poisson brackets is inconsistent with the constraints. Since the constraint matrix does not vanish, there is no ambiguity in writing a set of R second class constraints as the strong equations

$$G_a = G_a(q,p) = 0. \tag{6.27}$$

Dirac devised a simple recipe which involves replacing the ordinary Poisson brackets with the so-called Dirac brackets, defined by

$$\{f,g\}_D = \{f,g\} - \{f,G_m\}C^{-1}_{mn}\{G_n,g\}. \tag{6.28}$$

It can easily shown that this bracket reproduces the correct time evolution of the system through the Hamiltonian. There are no more subtleties to do with the second class constraints, other than the fact that the structure of the phase space they define can be arbitrarily non-trivial in practice.

7 Electromagnetism

7.1 Constraints of Maxwell Theory

We can now apply the Dirac formalism of constrained Hamiltonian systems to the pure gauge theory of electrodynamics for which the gauge group is $U(1)$. The gauge invariant action is

$$S = \int d^4x\mathcal{L} = -\int d^4x\bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\bigg). \tag{7.1}$$

In order to uncover any constraints we must perform the Legendre transformation to the canonical formalism, first by computing the momenta conjugate to the gauge fields

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F_{\mu 0} = \begin{cases} -E_i & \text{if } \mu = i = 1,2,3; \\ 0 & \text{if } \mu = 0. \end{cases} \tag{7.2}$$

The vanishing of the momentum conjugate to A_0 is a primary constraint which indicates that the time component of the gauge field is an unphysical degree of freedom. The ambient phase space is then spanned by the canonical pairs (A^μ,π_μ) which obey the following Poisson bracket structure

$$\{A_\mu(\vec{x},t),\pi^\nu(\vec{y},t)\} = \delta^\nu_\mu\delta^{(3)}(\vec{x}-\vec{y}). \tag{7.3}$$

The canonical Hamiltonian then reads

$$H = \int d^3x(\pi_\mu\dot{A}^\mu - \mathcal{L}) = \int d^3x\bigg[\frac{1}{2}(\vec{E}^2+\vec{B}^2) + \pi_0\partial_0A^0 - A_0(\vec{\partial}\cdot\vec{E})\bigg], \tag{7.4}$$

where integration by parts has been used. The consistency of the primary constraint $\pi_0 \approx 0$ is then checked

$$0 \approx \dot{\pi}_0(\vec{x}) = \{\pi_0, H\} = (\vec{\partial} \cdot \vec{E})(\vec{x}) \equiv G(\vec{x}) \tag{7.5}$$

resulting in the secondary constraint known as Gauss' law. If we now write down the canonical action

$$S_{can} = \int d^4x \left[\pi_\mu \dot{A}^\mu - \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \pi_0 \partial_0 A^0 - A_0(\vec{\partial} \cdot \vec{E}) \right], \tag{7.6}$$

it becomes clear that A_0 is a Lagrange multiplier enforcing the Gauss law constraint while the constraint $\pi_0 \approx 0$ is merely a consequence of A_0 being an arbitrary function with no physical relevance. For this reason it is desirable to quietly drop (A_0, π_0) from the ambient phase space and redefine $A_0 \equiv \lambda$ so that the action becomes

$$S_{can} = -\int d^4x \left[\vec{E} \cdot (\partial_0 \vec{A}) + \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \lambda(\vec{\partial} \cdot \vec{E}) \right]. \tag{7.7}$$

The theory is now formulated in the phase space spanned by the physical degrees of freedom $(\vec{A}, -\vec{E})$ subject to the Gauss Law constraint $G = \vec{\partial} \cdot \vec{E}$.

7.2 Canonical Gauge Fixing

The Gauss law constraint is trivially first class, this would still be the case if we hadn't eliminated π_0 earlier since they Poisson commute so we will return to this larger phase space for now. Moreover, it reduces the physical degrees of freedom of the system by requiring that the longitudinal electric field is vanishing at every point in space. This condition is non-linear which renders the method of solving the constraint and formulating the theory in terms of unconstrained physical degrees of freedom impossible. What we must do instead is keep the gauge degrees of freedom present for now and compensate for their contribution by gauge fixing. There are multiple options for choice of gauge in electrodynamics but we will opt for the Coulomb condition

$$F(\vec{x}) = \vec{\partial} \cdot \vec{A}(\vec{x}) \approx 0 \tag{7.8}$$

which removes the correct number of degrees of freedom and promotes the set (F, G) to a second class constraint algebra which we know how to deal with. First of all, checking consistency of the gauge fixing condition with the extended Hamiltonian (7.4) gives

$$\dot{F} = \{F, H\} = \left\{ \vec{\partial} \cdot \vec{A}, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \pi_0 \partial_0 A^0 - A_0(\vec{\partial} \cdot \vec{E}) \right\}$$

$$= \frac{1}{2}\vec{\partial}\{\vec{A}, \vec{E}^2\} - \vec{\partial}\{\vec{A}, A_0(\vec{\partial} \cdot \vec{E})\} = -(\vec{\partial} \cdot \vec{E}) - \vec{\partial}^2 A_0 \approx -\vec{\partial}^2 A_0 = 0.$$

Thus the Coulomb gauge necessitates the condition $A_0 = \text{const} \equiv 0$ which we quietly enforced in the setup of the previous section. Now we have constructed a formulation of the theory in which the Hamiltonian takes the reduced form

$$H_{red} = \frac{1}{2}\int d^3x(\vec{E}^2 + \vec{B}^2) \tag{7.9}$$

and the Gauss law constraint which restricts the true degrees of freedom is to be imposed separately by constructing the Dirac bracket which defines the symplectic structure of the physical subspace.

7.3 The Dirac Bracket

We first compute the constraint matrix for (F, G) which yields

$$C(\vec{x},\vec{y}) = \begin{pmatrix} 0 & -\{G(\vec{x}), F(\vec{y})\} \\ \{G(\vec{x}), F(\vec{y})\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \triangle\, \delta^{(3)}(\vec{x}-\vec{y}). \tag{7.10}$$

The Coulomb gauge is therefore effective if this matrix is invertible such that the Dirac brackets can be well-defined using (6.28). For now we can write the inverse in a meaningful way as

$$C^{-1}(\vec{x},\vec{y}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{1}{\triangle}\delta^{(3)}(\vec{x}-\vec{y}) \tag{7.11}$$

Assuming the space is flat (locally isomorphic to a region of Euclidean space) the appearance of the inverse Laplacian can be justified and the gauge choice avoids any issues related to the Gribov problem. Assuming these requirements are met, within the Dirac formalism this is the best we can do, locality is lost since the Dirac brackets will contain the non-local operator $\triangle^{-1}$ however causality still remains intact in the corresponding quantum theory. Let's then proceed carefully aware of the elephant in the room and construct the Dirac brackets. By definition we should have

$$\{f, g\}_D = \{f, g\} - \{f, \phi_a\}C^{ab}\{\phi_b, g\}. \tag{7.12}$$

The only two parts of the summation in the second term that do not vanish are those corresponding to the off-diagonal of $C^{ab} = (C^{-1})_{ab}$ so the structure of the phase space is now defined as

$$\begin{aligned}\{A^i(\vec{x}), E_j(\vec{y})\}_D = \delta^i_j\delta^{(3)}(\vec{x}-\vec{y}) - \{A^i(\vec{x}), G(\vec{x}')\}\triangle^{-1}\delta^{(3)}(\vec{x}'-\vec{y}')\}\{F(\vec{y}'), E_j(\vec{y})\} \\ +\{E_j(\vec{y}), G(\vec{x}')\}\triangle^{-1}\delta^{(3)}(\vec{x}'-\vec{y}')\}\{F(\vec{y}'), A^i(\vec{x})\} \\ = (\delta^i_j - \partial_i\triangle^{-1}\partial_j)\delta^{(3)}(\vec{x}-\vec{y}).\end{aligned}$$

This result may look funny but we note that writing the canonical variables $(\vec{A}, \vec{E})$ as a sum of their longitudinal and transverse parts is done by the projection

$$\vec{E} = \vec{E}_T + \vec{E}_L = (1 - \vec{\partial}\triangle^{-1}\vec{\partial})\cdot\vec{E} + \vec{\partial}\triangle^{-1}\vec{\partial}\cdot\vec{E}. \tag{7.13}$$

So the structure of the physical phase space given by the Dirac bracket is nothing but the transverse projection of the fields $(\vec{A}_T, \vec{E}_T)$ in the ambient phase space exactly as required by Gauss' law and the Coulomb gauge. The quantisation of the gauge field A_μ follows from this (already messy) analysis but there have been many gauge theory quantisation methods devised since Dirac which go more directly to the point in a more elegant way so we will not proceed from here. In particular, the choices of gauge available in the Dirac formalism have already lost us manifest Lorentz invariance and locality so it seems a losing battle. We will revisit the topic of quantum electrodynamics when we study the theory of interacting fields which is endowed with much stronger machinery.

SEMESTER 2

8 Interacting Fields and Feynman Diagrams

There are almost no experiments in the world of science that do not involve an interaction of sorts so now we will convert the conceptual framework of free fields (bosonic and fermionic) developed in the first semester of the module into a description of interacting fields from which real physical quantities can be extracted. In particular we would now like to study theories in which the Lagrangian is dependent on a combination of the $spin-(0,\frac{1}{2},1)$ fields (ϕ,ψ,A). Of course, we could come up with arbitrarily complicated ways of combining these fields but there is a simple principle which greatly restricts the type of terms that can show up in physically realisable theories. In particular, the action S, defined in (2.10), should be dimensionless in all cases. Since the integration measure d^4x has dimension $[\text{length}]^4 \sim [\text{mass}]^{-4}$, we deduce that any term showing up in our Lagrangians should have mass-dimension 4. Looking back on the three pure Lagrangians we have studied, by inspection it is easy to see that the triplet of fields we are interested in have mass-dimension

$$[\phi_{KG}] = [\text{mass}]^1, \qquad [\psi_D] = [\text{mass}]^{\frac{3}{2}}, \qquad [A_{EM}] = [\text{mass}]^1. \tag{8.1}$$

It follows that the simplest interaction we can study is that described by the interaction term of the form $\mathcal{L}_{int} \sim \lambda\phi^4$ where λ will play the role of a coupling constant. Before we study this however we have to discuss how we can approach the construction of propagators for interacting theories. The general idea will be to use perturbation theory (in powers of the coupling constant) to introduce the interaction term iteratively into the free theory. The mass-dimension condition is a manifestation of the renormalisability of physical theories which will be discussed in depth at the end of the course. As in the non-relativistic case, we will work in the Hamiltonian formulation where the perturbative approach is defined. Due to the work of Feynman we will see that the perturbation series we obtain is outstandingly simple in structure and that the particle interactions at any order can even be visualised. The full description of interacting quantum field theories involve analysis special to the functional (path) integral formalism covered in a later section.

8.1 Perturbative Propagators

Without any assumptions we would like to derive the form of the scalar field two-point correlation function

$$\langle 0|\mathcal{T}(\phi(x)\phi(y))|0\rangle_{\text{free}} = D_F(x-y) \tag{8.2}$$

for an interacting theory. In the interacting state space, the vacuum will be denoted as $|\Omega\rangle$ which reduces to $|0\rangle$ in the limit of zero coupling. As mentioned, the strategy we employ is to express the entire correlation function of the interacting theory in terms of a series of propagators from free theory which we already know.

We will momentarily focus on ϕ^4 theory described by the Hamiltonian

$$H = H_{KG} + H_{int} \equiv H_0 + \int d^3x \frac{\lambda}{4!}\phi^4, \tag{8.3}$$

although everything we do will be mostly general. In the Heisenberg picture, the field $\phi(\vec{x},t) = \phi(x)$ is defined through adjoint action of the evolution operator

$$\phi(\vec{x},t) = e^{iH(t-t_0)}\phi(\vec{x},t_0)e^{-iH(t-t_0)} \tag{8.4}$$

where t_0 is a fixed time at which the interaction is turned on. In the "interacting picture", the free field is given by

$$\phi(\vec{x},t)|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(\vec{x},t_0)e^{-iH_0(t-t_0)} \equiv \phi_I(\vec{x},t) \tag{8.5}$$

which we already know in terms of creation and annihilation operators as

$$\phi_I(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right)\Big|_{x^0=t-t_0}.$$

The interacting field can then be written in terms of ϕ_I by inverting the right-hand relation of (8.5)

$$\phi(\vec{x},t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(\vec{x},t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \equiv U^\dagger(t,t_0)\phi_I(\vec{x},t)U(t,t_0),$$

where we have defined the unitary interaction picture time evolution operator

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{8.6}$$

With the interacting field ϕ written in terms of the free field ϕ_I, it remains to figure out the subtleties of the theory which have been shifted onto the evolution of the operator $U(t,t_0)$. To analyse this we first formulate the Heisenberg equation

$$i\frac{\partial}{\partial t}U(t,t_0) = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} = e^{iH_0(t-t_0)}(H_{int})e^{-iH(t-t_0)}.$$

Analogously to how we defined the free field in the interaction picture ϕ_I, note that defining the interacting Hamiltonian in the interaction picture as

$$H_I(t) \equiv e^{iH_0(t-t_0)}(H_{int})e^{-iH_0(t-t_0)} = \int d^3x \frac{\lambda}{4!}\phi_I^4, \tag{8.7}$$

reduces the above PDE to the *heat kernel equation*:

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I[\phi_I]U(t,t_0). \tag{8.8}$$

Clearly the solutions to this are functions of the free field ϕ_I too, so things are going to plan so far. To find a unique solution we first impose the sensible boundary condition

$U(t_0,t_0) = 1$ and postulate a solution of the form $U(t,t_0) = \exp[-iH_I t]$ which can be constructed in powers of the coupling constant as

$$U(t,t_0) = 1 + (-i)\int_t^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \cdots. \tag{8.9}$$

The fact that the product of H_I's are in time order allows us to symmetrise the integrand of each term in the series. In particular, changing the upper bound of each integral to t extends the integration to a volume in t-space twice the size so for example at second order

$$(-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) = \frac{1}{2}(-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \mathcal{T}\{H_I(t_1)H_I(t_2)\}. \tag{8.10}$$

This identification allows us to write the entire series solution nicely as

$$U(t,t_0) \equiv \mathcal{T}\left\{ \exp\left[-i \int_{t_0}^t dt' H_I(t') \right] \right\}, \tag{8.11}$$

where the time ordering applies to the expansion of the exponential function. This can of course be defined for times other than the reference time t_0 such that U can be seen to satisfy *causality and unitarity*:

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3), \qquad U(t_1,t_3)U^\dagger(t_2,t_3) = U(t_1,t_2). \tag{8.12}$$

The only part of (8.2) left to write in terms of free field expressions is the new ground state $|\Omega\rangle$. We reason that because it can be expressed in terms of the states in the free theory Fock space, there must be some overlap between $|\Omega\rangle$ and $|0\rangle$. The evolution of the free field vacuum through time T for the perturbed Hamiltonian H can therefore be expanded as

$$e^{-iHT}|0\rangle = e^{iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle. \tag{8.13}$$

All excited state energies are larger than the ground state so if we send T to infinity with a small push in the imaginary direction, we can get rid of these terms and rearrange the expression for $|\Omega\rangle$ as

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(\frac{e^{-iHT}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle} \right). \tag{8.14}$$

In this limit we can add a small constant t_0 to T which gives

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(\frac{e^{-iH(t_0-(-T))}e^{-iH_0(-T-t_0)}|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle} \right),$$

where we have inserted an extra term in the numerator using the fact that $H_0|0\rangle = 0$. Now we recognise the product of exponentials as the operator $U(t_0,-T)$ so

$$\Omega = \lim_{T\to\infty(1-i\varepsilon)} \left(\frac{U(t_0,-T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle} \right), \qquad \langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \left(\frac{\langle 0|U(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle} \right).$$

Putting everything together, the two point correlation $\langle\Omega|\mathcal{T}(\phi(x)\phi(y))|\Omega\rangle$ function where $x^0 > y^0 > t_0$ is

$$\lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U^\dagger(T,t_0)U^\dagger(x^0,t_0)\phi_I(x)U(x^0,t_0)U^\dagger(y^0,t_0)\phi_I(y)U(y^0,t_0)U(t_0,-T)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}}. \tag{8.15}$$

The awkward denominator can be dealt with by noticing that

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}\right)^{-1} \langle 0|U(T,t_0)U(t_0,-T)|0\rangle.$$

Furthermore the resulting expression becomes greatly simplified by employing the properties of U in (8.12):

$$\langle\Omega|\mathcal{T}(\phi(x)\phi(y))|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(-T,T)|0\rangle}.$$

Writing this in terms of our solution (8.11) for the evolution operator in the interaction picture (and lifting our assumption on x^0, y^0, t_0) we arrive at

$$\langle\Omega|\mathcal{T}(\phi(x)\phi(y))|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\mathcal{T}\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\}|0\rangle}{\langle 0|\mathcal{T}\left\{\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\}|0\rangle}. \tag{8.16}$$

This exact expression is precisely fit for calculating the perturbative expansions we are interested in. It also generalises directly to the calculation of correlation functions for more than 2 fields. Successful in our aforementioned strategy, the computation of correlation functions has been reduced to the calculation of vacuum expectation values of time-ordered products of free field operators.

8.2 Wick's Theorem and Feynman Diagrams

To make the appropriate calculations in perturbation theory we want to devise a general recipe for evaluating an expression of the form

$$\langle 0|\mathcal{T}(\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n))|0\rangle \tag{8.17}$$

where n is finite but arbitrarily large. Consider the following decomposition

$$\phi_I(x) = \phi^+(x) + \phi_I^-(x) \sim \int a(\vec{p}) + \int a^\dagger(\vec{p}) \tag{8.18}$$

of a free field operator into its positive frequency and negative frequency components. If we let $x^0 > y^0$ then the time ordered product of two fields can be written as

$$\mathcal{T}(\phi_I(x)\phi_I(y)) = \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)].$$

We have rearranged the products such that they are in *normal order* where all creation operators are to the left of annihilation operators and thus has a vanishing vacuum expectation value. We define the normal ordered product of a set of operators proportional to $(a^\dagger, a)$ using the normal ordering symbol $\mathcal{N}$. For example $\mathcal{N}\{a(\vec{p})a^\dagger(\vec{q})a(\vec{k})\} = a^\dagger(\vec{q})a(\vec{p})a(\vec{k})$.

With reference to the above time-ordered product, the additional commutator we picked up by ordering would instead be $[\phi_I^+(y), \phi_I^-(x)]$ if we had considered the case where $x^0 < y^0$ which suggests that we make the definition of *the contraction of two fields* as

$$\overbrace{\phi(x)\phi}(y) \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & \text{if } x^0 > y^0 \\ \text{or same with } (x \longleftrightarrow y) & \text{if } x^0 < y^0 \end{cases} = D_F(x-y), \tag{8.19}$$

which as we can see is nothing but the Feynman propagator. Notice here we dropped the subscript I on the fields for convenience, in what follows we continue discussing free field operators in the interaction picture unless stated otherwise. From our example with two fields we are now able to express the time ordered product of any number of fields as an identity known as *Wick's theorem*. Namely

$$\mathcal{T}\{\phi(x_1)\cdots\phi(x_n)\} = \mathcal{N}\{\phi(x_1)\cdots\phi(x_n) + \text{all possible contractions}\}. \tag{8.20}$$

In practice the term of all possible contractions may seem daunting but we note that in the vacuum expectation value (8.17) any combination in which there are uncontracted operators gives zero since it will be normal ordered. For example with four operators

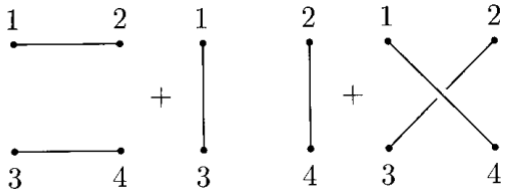
$$\langle 0|\mathcal{T}\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = \langle 0|\mathcal{N}\{\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \overbrace{\phi_1\phi_2\phi_3\phi_4}$$

$$\phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\overbrace{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3\phi_4}\}|0\rangle$$

$$= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3). \tag{8.21}$$

This powerful theorem allows us to decompose any vacuum expectation value of the form (8.17) into a sum of products of Feynman propagators. Although the resulting expression is still complicated, Feynman devised an elegant way to view the result diagrammatically. As a first glimpse let's see how the result (8.21) appears in terms of Feynman diagrams. The four fields are localised at four different spacetime points and each Feynman propagator can intuitively be represented as a line joining the two points it depends on so we write

$\langle 0|\, T\{\phi_1\phi_2\phi_3\phi_4\}\, |0\rangle =$



Without putting any rigorous formalism in place for whether the diagrams provide any measurable predictions, it still seems like drawing them in this natural way provides a clear interpretation for the processes that can occur at this order in the perturbation series. In particular, each diagram indicates the creation of two particles at distinct spacetime points and their propagation to another point at which they annihilate. Since there are three ways for this to occur from a set of four spacetime points we have three diagrams and the total probabilistic amplitude is the sum of the three diagrams.

The diagrams become a bit more exotic when there are more than one field at the same spacetime point. To see how this works we return to the two point correlation function (8.16) for ϕ^4 theory. Ignoring the denominator for now, we can expand to leading order in the coupling constant:

$$\langle\Omega|\mathcal{T}(\phi(x)\phi(y))|\Omega\rangle \sim \langle 0|\mathcal{T}\Big\{\phi(x)\phi(y) + \phi(x)\phi(y)\Big[-i\int dt H_I(t)\Big] + \cdots\Big\}|0\rangle \tag{8.22}$$

the central term returns the known result $D_F(x-y)$, while the first order term for the interacting Hamiltonian (8.7) is

$$\langle 0|\mathcal{T}\Big\{\phi(x)\phi(y)\Big(\frac{-i\lambda}{4!}\Big)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\Big\}|0\rangle.$$

We now want to apply Wick's theorem to evaluate this expression. There are 15 possible ways to contract all the fields together but we can make our life simpler by noticing that each of these can only return one of two results. In the first case we contract $\phi(x), \phi(y)$ which leaves 3 ways to contract all of the $\phi(z)$ all giving the same result. Alternatively, we can contract $\phi(x)$ and $\phi(y)$ each with a distinct $\phi(z)$ and then contract the remaining pair of $\phi(z)$, all 12 ways giving the same result, hence

$$\langle 0|\mathcal{T}\Big\{\phi(x)\phi(y)\Big(\frac{-i\lambda}{4!}\Big)\int d^4z\phi^4(z)\Big\}|0\rangle = 3\Big(\frac{-i\lambda}{4!}\Big)D_F(x-y)\int d^4z D_F(z-z)D_F(z-z)$$

$$+12\Big(\frac{-i\lambda}{4!}\Big)\int d^4z D_F(x-z)D_F(x-y)D_F(z-z).$$

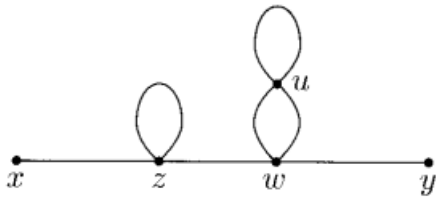
In order to represent the terms of this time ordered product as a Feynman diagram, we must introduce the notion of *internal and external points* to explain the factor $(-i\lambda/4!)\int d^4z$ appearing before certain propagators. Simply put, each internal point (z in this case) appears under a spacetime integration such that the above expression can be represented by the sum of Feynman diagrams:

We shall refer to the lines themselves in the diagrams as *propagators* since they represent the propagation amplitude. As is evident in the above diagram, we define a *loop* as a propagator which begins and ends at the same point representing of course $D_F(z-z)$. Of great importance are internal points where four lines meet, these are called *vertices*. Lets now jump up to the third order term in λ which contains the following choice of contractions

$$\langle 0|\phi(x)\phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\phi\phi\phi\phi\int d^4w\phi\phi\phi\phi\int d^4u\phi\phi\phi\phi|0\rangle$$

$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4zd^4wd^4uD_F(x-z)D_F(z-z)D_F(z-w)D_F(w-y)D_F^2(w-u)D_F(u-u).$$

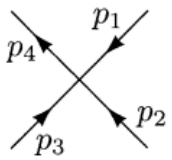
The number of other contractions that give this result is 10,368 which makes up less than 10% of the total set of possible contractions. This third-order term in the time-ordered product can be represented by the diagram:



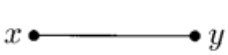
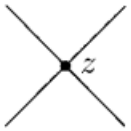
In practice we work backwards from the diagram to the algebraic expression but this makes it harder to come up with the combinatoric factor of identical contractions. To remedy this consider a general term at n-th order in λ. The factor $\frac{1}{n!}$ from the Taylor series will always be cancelled by the number of ways of interchanging vertices under which the result is clearly invariant, so we can forget about these factors. Next, each vertex is characterised by four contractions placed on $\phi\phi\phi\phi$ which gives a factor of 4! for equivalent placements. This cancels the 4! in the denominator of the interacting Hamiltonian and so it is conventional to associate the factor $(-i\lambda)\int d^4z$ with each vertex. Following this recipe we obtain a constant which is $8 = 2\cdot 2\cdot 2$ times too large. To explain this over counting we note that for vertices such as z in the above diagram with a certain symmetry there are only half as many unique contractions $4!/2 = 12$. We refer to this number $\mathbb{S}$ as the *symmetry factor* of the diagram. To get the correct combinatorial constant we must therefore divide by the symmetry factor. We will rarely deal with diagrams where this takes a value greater than 2.

The numerator of the vacuum expectation value we have been looking at for ϕ^4 theory at leading and third order can now be written generally as

$$\langle 0|\mathcal{T}\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int dtH_I(t)\right]\right\}|0\rangle = \sum(\text{All possible diagrams with 2 external points}) \tag{8.23}$$



In order to make this equation meaningful, we are shall now rigorously formulate the so-called Feynman rules which dictate the exact algebraic expression corresponding to a Feynman diagram. In ϕ^4 theory the rules read:

1. For each propagator,  $= D_F(x-y)$;
2. For each vertex,  $= (-i\lambda)\int d^4z$;
3. For each external point,  $= 1$;
4. Divide by the symmetry factor.

These are more precisely referred to as the position-space Feynman rules, in most calculations it is more useful to use the momentum-space version of the Feynman propagator which means our rules need to be translated into the language of momentum.

8.3 Momentum-Space Feynman Diagrams

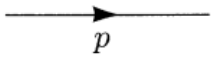
Recall that the Fourier expansion of each Feynman propagator is

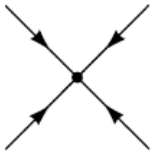
$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}. \tag{8.24}$$

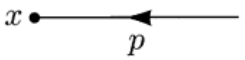
We can represent each momentum corresponding to D_F in the diagram by indicating the direction of propagation with an arrow. At a vertex where four propagators meet, momentum appears positive in the exponential when moving away and negative when moving towards the vertex point. For example

$$\int d^4z e^{-ip_1z}e^{-ip_2z}e^{-ip_3z}e^{ip_4z} = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_4) \tag{8.25}$$

is the expression associated with the vertex which obviously implies that *momentum should be conserved at each vertex.* Implementing momentum conservation in this way is useful in calculations where the delta function will greatly reduce the number of integrations. The Feynman rules in momentum space are formulated as follows, translating

1. For each propagator,  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$;

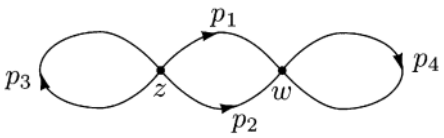
2. For each vertex,  $= -i\lambda$;

3. For each external point,  $= e^{-ip\cdot x}$;

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$;

6. Divide by the symmetry factor.



from before: Going back to our correlation function we now must reinstate the fact that the limit as time goes to $\infty(1-i\varepsilon)$ is being considered. For the vertex integration above we write this as

$$\lim_{T\to\infty(1-i\varepsilon)} \int_{-T}^{T} dz^0 \int d^3z e^{-i(p_1+p_2+p_3-p_4)\cdot z}. \tag{8.26}$$

The exponential part blows up as z^0 goes to $\pm\infty$ due to both signs being present in the sum of momenta, unless of course z^0 is purely imaginary. The momentum space equivalent of this would be to take p^0 to have a small imaginary part which is exactly what is done in defining the contour integral for the Feynman propagator. In vertex contributions like (8.26), the dependence on T drops out in the infinite limit but there are certain diagrams where this is not the case.

Consider the following Feynman diagram The conservation of momentum at the left vertex gives a factor of $(2\pi)^4\delta^{(4)}(p_1+p_2)$ which automatically ensures momentum conversation at the right hand vertex so we get an additional factor $(2\pi)^4\delta^{(4)}(0)$. This factor can be most easily interpreted in position space where it is an integral of the form

$$\int_{-T}^{T} dw^0 \int d^3w(\text{constant}) \sim 2T\cdot(\text{volume}). \tag{8.27}$$

This is nothing but the statement that the process described by the diagram can occur at any point in spacetime. Diagrams with this feature are called *disconnected* since they are not connected to any external points. In general a Feynman diagram will have both connected and disconnected pieces. Suppose we choose a set of all possible disconnected

diagrams $\{V_i\}$, and that some arbitrary diagram contains n_i disconnected components from this basis. Understanding V_i to denote the "value" of the disconnected piece V_i then the total value of the diagram is

$$(\text{value connected}) \cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i}. \tag{8.28}$$

For the numerator of (8.16) we write

$$\sum_{\text{Connected}} \sum_{\{n_i\}} (\text{value connected}) \cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i} \tag{8.29}$$

$$= \left(\sum \text{connected} \right) \cdot \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1} \right) \left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2} \right) \cdots = \left(\sum \text{connected} \right) \cdot \exp\left(\sum_i V_i \right).$$

The exponentiation of disconnected diagrams here highlights that the sum of all possible Feynman diagrams is the sum of all possible connected diagrams times the exponential of the sum of all disconnected diagrams. But wait! Something happens now, all of this has been a bit of a mouthful but recall that the function we are trying to compute $\langle \Omega | \phi(x)\phi(y) | \Omega \rangle$ contains a denominator which has no dependence on external points,

$$\langle 0 | \mathcal{T} \left\{ \exp\left[-i \int dt H_I(t) \right] \right\} | 0 \rangle = \exp\left(\sum_i V_i \right), \tag{8.30}$$

which is again just the exponential of the sum of all disconnected pieces of the diagram. Amazingly, this cancels the disconnected contribution of the numerator so we find that our correlation function for ϕ^4 theory reduces all the way down to

$$\langle \Omega | \mathcal{T}(\phi(x)\phi(y)) | \Omega \rangle = \sum \begin{pmatrix} \text{All possible connected diagrams} \\ \text{with 2 external points} \end{pmatrix}. \tag{8.31}$$

Before happily moving on, we should take a look at the *worthless garbage* we are throwing away and interpret the disconnected diagrams physically. Let's go back and write the equation (8.15) as

$$\langle \Omega | \mathcal{T}(\phi(x)\phi(y)) | \Omega \rangle \cdot \lim_{T \to \infty(1-i\varepsilon)} \left(|\langle 0 | \Omega \rangle|^2 e^{-iE_0(2T)} \right) = \tag{8.32}$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \langle 0 | \mathcal{T} \left\{ \phi_I(x)\phi_I(y) \exp\left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | 0 \rangle.$$

We can form a proportionality relation by looking at the parts on each side of the equation that are T-dependent. That is to say,

$$\exp\left(\sum_i V_i \right) \sim \exp\left(-iE_0(2T) \right), \tag{8.33}$$

where the left hand side is justified since each V_i contains a factor of $(2\pi)^4\delta^{(4)}(0) = 2T\cdot V$ such that we can manipulate this to find that the energy density of the vacuum is

$$\frac{E_0}{\text{Volume}} = i\frac{\sum_i V_i}{(2\pi)^4\delta^{(4)}(0)}. \tag{8.34}$$

We will see why this equation is important again later in the course but for now it suits us to summarise what we have achieved in this section by writing the n-point generalisation of the result (8.31) so that we can get on to doing some actual calculations. The elegant formula we've worked so hard for is

$$\langle\Omega|\mathcal{T}(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \sum\begin{pmatrix}\text{All possible connected diagrams}\\ \text{with n external points}\end{pmatrix}. \tag{8.35}$$

9 Cross Sections & The S-Matrix

In the opening lectures of this module we presented a quantity known as the differential scattering cross section without really explaining how we got our answer for $e^+e^- \to \mu^+\mu^-$ or why it is something we even want to calculate at all. The n-point correlation function we have just derived is impressive but it still doesn't specify the value of any physically measurable quantity to test our quantum field theory. It is this agenda we will push in this section; physically measurable quantities. To begin we will revisit the cross section more carefully and show that its computation sits quite naturally within our formalism of quantum fields via a more primitive quantity called the S-matrix. We'll get some comfort by seeing that the result we wrote down in the beginning (1.8) can be derived rigorously within this formalism which will of course require the introduction of Feynman rules for fermions.

9.1 The S-Matrix

In collider scattering experiments which are used to probe fundamental aspects of nature, there are two species of particles prepared in batches of density ρ_a and ρ_b of characteristic length l_a and l_b respectively. When the two batches are brought head to head at high energy, the overlapping area which encloses all resulting collisions is denoted by A and the scattering cross section σ of units area is given by the formula

$$\sigma \equiv \frac{\text{Number of events}}{\rho_a\rho_b l_a l_b A}. \tag{9.1}$$

The experiments not only want to determine the cross section but also the momenta of the particles in the final state. Although the dependence is indirect in the above formula, specifying the exact momenta will cause σ to become infinitesimal (I think this is because of the uncertainty principle). To restore finiteness in this case, we instead focus on measuring the differential scattering cross section $d\sigma/(d^3p_1 d^3p_2\cdots d^3p_n)$. We can

think about this as an object that, when integrated over any small region in momentum space $d^3p_1 d^3p_2 \cdots d^3p_n$, gives the cross section for scattering into that region for the final state. The case where the differential cross section $d\sigma/d\Omega(\theta, \phi)$ occurs is nothing but the result obtained for a final state of only two particles where one of them is scattered through the solid angle $d\Omega = \sin\theta d\theta d\phi$ and the other is dependent by conservation. For a system of a particles which each experience spontaneous decay into two or more products, there is a simpler quantity we can measure called the *decay rate* which we denote by Γ:

$$\Gamma = \frac{\text{Number of decays per unit time}}{\text{Number of } a \text{ particles present}}. \tag{9.2}$$

In non-relativistic quantum mechanics, unstable atomic states are predicted to appear in scattering experiments in the form of a resonance around some energy E_0. In this case the scattering amplitude is given by the Breit-Wigner formula

$$f(E) \propto \frac{1}{E - E_0 - i\frac{\Gamma}{2}}.$$

In the relativistic case this formula appears once again in the scattering amplitude of a process where initial particles collide and form an unstable particle which then decays. In particular if we just say that the four momentum of the unstable particle is p and its mass m, then the amplitude generalises to

$$f(E) \propto \frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\vec{p}}(p^0 - E_{\vec{p}} + i(m/E_{\vec{p}})\frac{\Gamma}{2})}.$$

This equality only really holds near resonance but the left-hand side is a manifestly Lorentz invariant and so is much more convenient to use. We will leave the discussion of decay rates to the standard model module for now and focus purely on scattering cross sections.

To construct a framework in which σ can be calculated, we will imagine the "batch" of initial particles as a wavepacket of initial states which can be evolved by e^{-iHt} and then overlapped with a wavepacket representing the desired final state particles $|\phi\rangle$. The result of this procedure is the probability amplitude for a given initial state to evolve into a specified final state. The wavepacket of a desired final state of particles can be written as

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \phi(\vec{k}) |\vec{k}\rangle, \tag{9.3}$$

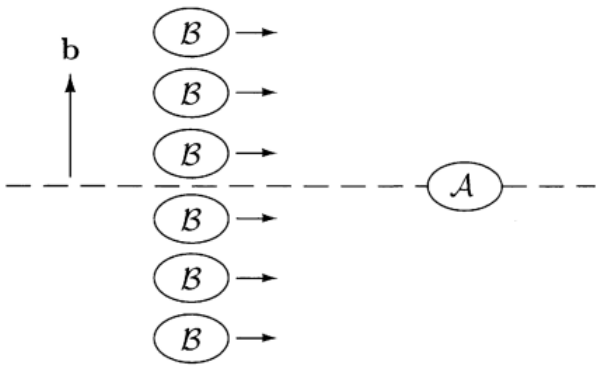
where $\phi(\vec{k})$ is the wavefunction in momentum space obtained by a Fourier transform and $|\vec{k}\rangle$ is a one-particle state in the interacting theory. The factor of $1/\sqrt{2E_{\vec{k}}}$ is included to convert our relativistic normalisation of the state $|\vec{k}\rangle$ back to the normalisation where all probabilities sum to 1. Mathematically, this ensures that

$$\int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1 \quad \Longrightarrow \quad \langle\phi|\phi\rangle = 1, \tag{9.4}$$

where states are created in the free theory with a normalisation $|\vec{k}\rangle = \sqrt{2E_{\vec{k}}}a^\dagger(\vec{k})|0\rangle$. For a scattering experiment involving batches of $\mathcal{A}$ particles and $\mathcal{B}$ particles at some time in the far past, the probability we wish to compute for some given final state in the far future is

$$\mathcal{P} = |\underset{\text{future}}{\langle\phi_1\phi_2\cdots}|\underset{\text{past}}{\phi_\mathcal{A}\phi_\mathcal{B}}\rangle|^2, \tag{9.5}$$

where $|\phi_\mathcal{A}\phi_\mathcal{B}\rangle$ is a state of two past wavepackets and $\langle\phi_1\phi_2\cdots|$ is a state of several future wavepackets (one for each desired final state particle). The initial wavepackets are defined locally at different points in position space such that they can be constructed independently of one another, to denote that this is the case we will refer to such states as *in* and *out* states. To bring this convention to momentum space, we will define an *in* state as follows. Let $|\phi_\mathcal{A}\phi_\mathcal{B}\rangle$ be the state from above set up in the far past, then if we take the limit where the wavepackets $\phi_i(\vec{k}_i)$ become concentrated around some definite momenta $\vec{p}_i$ in momentum space then this defines the *in* state $|\vec{p}_\mathcal{A}\vec{p}_\mathcal{B}\rangle_{in}$. In this picture, $|\phi_\mathcal{A}\phi_\mathcal{B}\rangle$ can be viewed as a linear superposition of such states comprised of wavepackets with definite momenta. In the most general case, this scattering process can be viewed as in the figure below:



When analysing these interactions, we will adopt the convention that all of the momentum space wavefuntions are collinear in (9.3), i.e. where the impact parameter vanishes implicitly in the definition of $\phi_\mathcal{B}$. To rectify that there are still portions of the wavepacket $\phi_\mathcal{B}$ which are transversely displaced relative to $\phi_\mathcal{A}$, we include a factor of $e^{-i\vec{b}\cdot\vec{k}_\mathcal{B}}$ in the definition of the *in* state which generates the relevant spatial translation explicitly. The initial state as depicted above can then be written down as

$$|\phi_\mathcal{A}\phi_\mathcal{B}\rangle_{in} = \int \frac{d^3k_\mathcal{A}}{(2\pi)^3}\int \frac{d^3k_\mathcal{B}}{(2\pi)^3}\frac{\phi_\mathcal{A}(\vec{k}_\mathcal{A})\phi_\mathcal{B}(\vec{k}_\mathcal{B})e^{-i\vec{b}\cdot\vec{k}_\mathcal{B}}}{\sqrt{(2E_\mathcal{A})(2E_\mathcal{B})}}|\vec{k}_\mathcal{A}\vec{k}_\mathcal{B}\rangle_{in}. \tag{9.6}$$

In a similar fashion, for the *out* states it is preferable to work in terms of the definite momenta $\vec{p}_i$ when defining the final states in the probability amplitude (9.5). The outgoing state of locally separated wavepackets can be defined as

$${}_{out}\langle\phi_1\phi_2\cdots| = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\vec{p}_f)}{\sqrt{2E_f}}\right)\cdot{}_{out}\langle\vec{p}_1\vec{p}_2\cdots|. \tag{9.7}$$

Therefore as long as the detectors in our experiment mainly resolve the final momenta it is viable to consider ${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|$ as the final state in the computation of amplitudes and reintroduce the normalisation factors at a later stage. So we have now reduced the computation of probability amplitudes in scattering experiments to that of the object

$${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{in}.$$

Next we want to reinstate the claim that these states are set up in the far past and the far future respectively so we instead work in the limiting case

$${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{in} = \lim_{T\to\infty} \langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle \underset{-T}{\overset{+T}{}} = \lim_{T\to\infty} \langle \vec{p}_1\vec{p}_2\cdots|e^{-iH(2T)}|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle. \tag{9.8}$$

In the last expression an operator has now been included so that both states are the same interval away from some reference time which can intuitively be thought of as when the action happens. The full equation tells us that *in* and *out* states are related by the limit of a sequence of unitary operators. The entire sequence as a limiting operator is referred to as the S-matrix,

$${}_{out}\langle \vec{p}_1\vec{p}_2\cdots|\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle_{in} = \langle \vec{p}_1\vec{p}_2\cdots|\ S\ |\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle. \tag{9.9}$$

The S-matrix can be understood easily from the way it was just constructed. Firstly, if the particles don't interact at all, then $S = \mathbb{1}$. To inspect the non-trivial part of S which is due to interaction, we introduce the T-matrix through the definition

$$S = \mathbb{1} + iT. \tag{9.10}$$

To specify the T-matrix is also straightforward. We know that it should conserve momentum at each collision vertex, this immediately implies that

$$T \sim \delta^{(4)}\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right).$$

The remaining part of T will collectively be referred to as the (Lorentz) *invariant matrix element* $\mathcal{M}$. We came across this quantity in the opening of the course at which point we merely mentioned that it represented the quantum probability for the specific process to occur and indeed it plays this role. We can now however define the matrix element more formally by the equation

$$\langle \vec{p}_1\vec{p}_2\cdots|\ iT\ |\vec{k}_{\mathcal{A}}\vec{k}_{\mathcal{B}}\rangle = (2\pi)^4\delta^{(4)}\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right)\cdot i\mathcal{M}(k_{\mathcal{A}}, k_{\mathcal{B}} \to \{p_f\}). \tag{9.11}$$

The matrix element is a very useful quantity because it contains all of the information about particle dynamics while the kinematics are kept separate. In particular, it depends specifically on how the given two particles speak to each other through the interaction Hamiltonian such that the state with final momenta $\{p_f\}$ is realised. Now that we have fully honed in on the interesting part of the theory captured by the S-matrix, we are ready to see how σ is related to $\mathcal{M}$ for a general 2-particle scattering experiment.

9.2 σ in 2-Particle Scattering

We want to begin with the expression for the probability of $\mathcal{A}$ and $\mathcal{B}$ particles (represented by wavepackets) scattering to produce a final state of n particles within a small region $d^3p_1 d^3p_2 \cdots d^3p_n$ in momentum space. As mentioned above, the specification of the final momenta corresponds to the cross section being infinitesimal such that the probability in our normalisation is given by

$$\mathcal{P}(\mathcal{AB} \to 1, 2 \ldots, n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |_{out}\langle \vec{p}_1 \cdots \vec{p_n} | \phi_\mathcal{A} \phi_\mathcal{B} \rangle_{in}|^2. \tag{9.12}$$

For a single target $\mathcal{A}$ being approached by many incident $\mathcal{B}$ particles with different impact parameters $\vec{b}$ the total number of scattering events N is the sum of this probability over all such $\mathcal{B}$ particles. In the continuous limit where the incoming particles form a distribution of constant number density $n_\mathcal{B}$ we can write this sum as an integral, to wit:

$$N = \sum_{\text{all incident particles: } i} \mathcal{P}_i = \int d^2b n_\mathcal{B} \mathcal{P}(\vec{b}) \tag{9.13}$$

The scattering cross section is, by definition

$$\sigma = \frac{N}{\rho_\mathcal{A} \rho_\mathcal{B} l_\mathcal{A} l_\mathcal{B} A} = \frac{N}{n_\mathcal{B} N_\mathcal{A}} = \frac{N}{n_\mathcal{B}} = \int d^2b \mathcal{P}(\vec{b}), \tag{9.14}$$

where we have used that $n_\mathcal{B} = \rho_\mathcal{B} l_\mathcal{B}$ and $A\rho_\mathcal{A} l_\mathcal{A} = A n_\mathcal{A} = N_\mathcal{A} = 1$. Inserting the definition of the initial state (9.6) into this expression and emphasising that σ is infinitesimal in this case by writing $d\sigma$ we have

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \int d^2b \int \frac{d^3k_\mathcal{A}}{(2\pi)^3} \int \frac{d^3k_\mathcal{B}}{(2\pi)^3} \int \frac{d^3k'_\mathcal{A}}{(2\pi)^3} \int \frac{d^3k'_\mathcal{B}}{(2\pi)^3} \frac{\phi_\mathcal{A}(\vec{k}_\mathcal{A}) \phi_\mathcal{B}(\vec{k}_\mathcal{B})}{\sqrt{(2E_{\vec{k}_\mathcal{A}})(2E_{\vec{k}_\mathcal{B}})}} \tag{9.15}$$

$$\times \frac{\phi^*_\mathcal{A}(\vec{k}'_\mathcal{A}) \phi^*_\mathcal{B}(\vec{k}'_\mathcal{B})}{\sqrt{(2E_{\vec{k}'_\mathcal{A}})(2E_{\vec{k}'_\mathcal{B}})}} e^{-i\vec{b}\cdot(\vec{k}_\mathcal{B} - \vec{k}'_\mathcal{B})} ({}_{out}\langle \{\vec{p}_f\} | \vec{k}_\mathcal{A} \vec{k}_\mathcal{B} \rangle_{in}) ({}_{out}\langle \{\vec{p}_f\} | \vec{k}'_\mathcal{A} \vec{k}'_\mathcal{B} \rangle_{in})^*.$$

Here we have used $\vec{k}'$ to denote the integration variable over the conjugated part of the modulus squared term in (9.12). To reduce this long expression, we first note that integration over the impact parameter just leaves a factor of $(2\pi)^2 \delta^{(2)}(k^\perp_\mathcal{B} - k'^\perp_\mathcal{B})$. We can then relate the inner product of *in* and *out* states to $\mathcal{M}$ using the definition (9.11), converting these terms to

$${}_{out}\langle \{\vec{p}_f\} | \vec{k}_\mathcal{A} \vec{k}_\mathcal{B} \rangle_{in} = i(2\pi)^4 \delta^{(4)} \left(k_\mathcal{A} + k_\mathcal{B} - \sum p_f \right) \mathcal{M}(k_\mathcal{A}, k_\mathcal{B} \to \{p_f\}),$$

$$({}_{out}\langle \{\vec{p}_f\} | \vec{k}'_\mathcal{A} \vec{k}'_\mathcal{B} \rangle_{in})^* = -i(2\pi)^4 \delta^{(4)} \left(k'_\mathcal{A} + k'_\mathcal{B} - \sum p_f \right) \mathcal{M}^*(k_\mathcal{A}, k_\mathcal{B} \to \{p_f\}).$$

The second equation here along with the other two delta functions related to the impact parameter allow us to fully integrate out the k' variables in (9.15). The only non-trivial integral is that over the collision direction (z-component) , which can be computed in combination with the help of the zero-component delta function imposing energy conservation as

$$\begin{aligned}
&\int dk'^z_{\mathcal{A}} dk'^z_{\mathcal{B}} \delta\Big(dk'^z_{\mathcal{A}} + dk'^z_{\mathcal{B}} - \sum p^z_f\Big)\delta\Big(E'_{\mathcal{A}} + E'_{\mathcal{B}} - \sum E_f\Big) \\
&= \int dk'^z_{\mathcal{A}} \delta\Big(\sqrt{k'^2_{\mathcal{A}} + m^2_{\mathcal{A}}} + \sqrt{k'^2_{\mathcal{B}} + m^2_{\mathcal{B}}} - \sum E_f\Big)\Big|_{k'^z_{\mathcal{B}} = \sum p^z_f - k'^z_{\mathcal{A}}} \\
&= \frac{1}{\left|\frac{k'^z_{\mathcal{A}}}{E'_{\mathcal{A}}} - \frac{k'^z_{\mathcal{B}}}{E'_{\mathcal{B}}}\right|} \equiv \frac{1}{|v_{\mathcal{A}} - v_{\mathcal{B}}|}.
\end{aligned}$$

This is nothing but the relative velocity between the beams in the lab frame. To get to the last line we have just used the functional property of the delta function which reads

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0). \tag{9.16}$$

After evaluating all six k' integrals we can then pull out the factors which depend smoothly on $\vec{k}_{\mathcal{A}}$ and $\vec{k}_{\mathcal{B}}$ and evaluate them at $\vec{p}_{\mathcal{A}}$ and $\vec{p}_{\mathcal{B}}$ respectively since these are the points in momentum space around which the respective wavepackets are localised. The result is

$$\begin{aligned}
d\sigma = &\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\frac{|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\})|^2}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|}\int \frac{d^3k_{\mathcal{A}}}{(2\pi)^3}\int \frac{d^3k_{\mathcal{B}}}{(2\pi)^3} \\
&\times|\phi_{\mathcal{A}}(\vec{k}_{\mathcal{A}})|^2|\phi_{\mathcal{B}}(\vec{k}_{\mathcal{B}})|^2(2\pi)^4\delta^{(4)}\Big(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\Big)
\end{aligned} \tag{9.17}$$

Technologically speaking, the detectors used to measure the momentum eigenstates only have finite resolution and we only need to make our expression as accurate as they are. That is to say that, to a good approximation we can just evaluate the wavepackets $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ at their central values $\vec{p}_{\mathcal{A}}$ and $\vec{p}_{\mathcal{B}}$ since detectors cannot resolve the small variations around these values we would be integrating over. Using the normalisation condition (9.4) the final expression we come to is

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\frac{|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\})|^2}{2E_{\mathcal{A}}2E_{\mathcal{B}}|v_{\mathcal{A}} - v_{\mathcal{B}}|}(2\pi)^4\delta^{(4)}\Big(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\Big). \tag{9.18}$$

To obtain the full cross section σ, we must drop any specifications on the final number of particles and the value of their momenta and instead integrate $d\sigma$ over the phase space of all possible final momenta with the appropriate conservation constraint. Such an integral is written as

$$\int d\Pi_n = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\Big(P - \sum p_f\Big), \tag{9.19}$$

where in this case $P \equiv p_{\mathcal{A}} + p_{\mathcal{B}}$. Many of the processes we will study are scattering experiments in which the final state is a 2-particle state and the above results can be simplified. First, we let p_1 and p_2 label the final particle momenta and then choosing the centre of mass frame as our initial reference frame in which $\vec{P} = 0$. The integral over the spatial part of the delta function imposes $\vec{p}_2 = -\vec{p}_1$ and we are left with

$$\int d\Pi_2 = \int \frac{d|\vec{p}_1||\vec{p}_1|^2 d\Omega}{(2\pi)^3 2E_1 2E_2}(2\pi)\delta(E_{cm} - E_1 - E_2), \tag{9.20}$$

in spherical coordinates. Now we again use the functional definition of $\delta(f(x))$ which gives

$$\delta(E_{cm} - E_1(|\vec{p}_1|) - E_2(|\vec{p}_1|)) = \frac{\delta(|\vec{p}_1| - \tilde{p})}{\left|\frac{dE_1}{d|\vec{p}_1|} + \frac{dE_2}{d|\vec{p}_1|}\right|}.$$

From the mass shell condition we know that $dE/d|\vec{p}| = d/d|\vec{p}|(\sqrt{|\vec{p}|^2 + m^2}) = |\vec{p}|/E$ so evaluating the $d|\vec{p}_1|$ integral we get

$$\int d\Pi_2 = \int d\Omega \frac{\tilde{p}^2}{16\pi^2 E_1 E_2} \frac{1}{\left|\frac{\tilde{p}}{E_1} + \frac{\tilde{p}}{E_2}\right|} = \frac{1}{E_1 + E_2} \int d\Omega \frac{\tilde{p}}{16\pi^2} \tag{9.21}$$

where $\tilde{p}$ is the value of $|\vec{p}_1|$ for which $E_{cm} = E_1(|\vec{p}_1|) + E_2(|\vec{p}_1|)$. So we can rewrite this as

$$\int d\Pi_2 = \int d\Omega \frac{1}{16\pi^2} \frac{\tilde{p}}{E_{cm}}. \tag{9.22}$$

It will be useful to look at the high-energy limit where we have $E >> m_{\mathcal{A}}, m_{\mathcal{B}}$, we can then approximate $E_{cm} = E_1 + E_2 = E_{\mathcal{A}} + E_{\mathcal{B}} \approx \tilde{p} + \tilde{p} = 2\tilde{p}$. Applying this approximation, we get

$$\int d\Pi_2 \approx \int d\Omega \frac{1}{32\pi^2}. \tag{9.23}$$

Now if we return to the scattering cross section for two particles and substitute the result in the relativistic limit and centre of mass frame, this yields

$$d\sigma = \frac{d\Omega}{32\pi^2} \frac{1}{2E_{\mathcal{A}} 2E_{\mathcal{B}} |v_{\mathcal{A}} - v_{\mathcal{B}}|} |\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2)|^2. \tag{9.24}$$

In the commonly seen massless limit where all $m = 0$, the picture is that $v_{\mathcal{A}} = -v_{\mathcal{B}} = 1$ so that $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 2$ and with $E_{\mathcal{A}} = E_{\mathcal{B}} = \frac{1}{2}E_{cm}$ which reproduces the familiar result

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2}. \tag{9.25}$$

Now that we have shown that it is relatively straightforward to make a connection between σ and $\mathcal{M}$, we can also derive an analogous result for the decay rate. In fact, we can pretty much guess the result by reorganising the general expression for $d\sigma$ given

in (9.18) in such a way that the initial state consists of only one particle. The decaying particle is initially at rest, this gives that $1/2E_{\mathcal{A}} \to 1/2m_{\mathcal{A}}$ then we simply have:

$$d\Gamma = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \frac{|\mathcal{M}(m_{\mathcal{A}} \to \{p_f\})|^2}{2m_{\mathcal{A}}} (2\pi)^4 \delta^{(4)} \left(m_{\mathcal{A}} - \sum p_f \right). \tag{9.26}$$

This formula for decay rates doesn't make any sense in the way that we have come up with it since we can't send an unstable particle into the infinitely distant past. However, it is still the correct formula to use once we learn how to compute $\mathcal{M}$ in terms of Feynman rules for S-matrix elements which we will derive in the next section. We come away from this analysis with two completely general formulae for scattering and decay processes. As a final note it is evident that when n indistinguishable particles are present in the final state, the *total* decay rate or cross section, obtained by integrating over all possible final states, must be divided by $n!$ to avoid over counting.

9.3 S-Matrix Elements from Feynman Diagrams

Recall that in the derivation of the n-point correlation function for an interacting theory, we replaced the vacuum state $|\Omega\rangle$ which is an eigenstate of H with its counterpart $|0\rangle$ in the unperturbed theory which is an eigenstate of H_0. Since the S-matrix is defined as the time evolution operator in the limit of very large T, we would like to perform a similar procedure on the plane wave states through which it is defined in the expression

$$\langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle = \lim_{T \to \infty} \langle \vec{p}_1 \vec{p}_2 \cdots | e^{-iH(2T)} | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle.$$

Starting with the naive ansatz

$$|\vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}}\rangle \sim \lim_{T \to \infty(1-i\varepsilon)} e^{-iHT} |\vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}}\rangle_0. \tag{9.27}$$

For the previous case the proportionality was precisely cancelled by the contribution from all disconnected or *vacuum* diagrams. Fortunately, a similar cancellation happens here and a subset of Feynman diagrams which are said to be *connected and amputated* are the only contributors. An explicit formula for this can be formally derived later but our current understanding does not reach this far so we'll have to use our sneaky methods once again to get where we want to go. Continuing naively for now, should (9.27) be the correct expression we could use Dyson's formula (8.11) to write

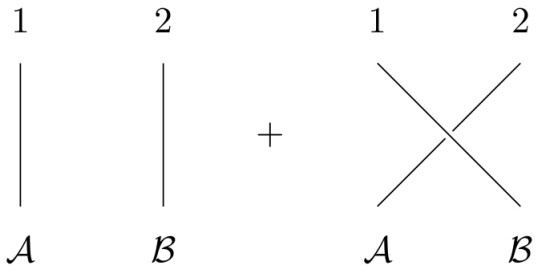
$$\begin{aligned} \langle \vec{p}_1 \vec{p}_2 \cdots | S | \vec{k}_{\mathcal{A}} \vec{k}_{\mathcal{B}} \rangle &= \lim_{T \to \infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \cdots \vec{p}_n | e^{-iH(2T)} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 \\ &\sim \lim_{T \to \infty(1-i\varepsilon)} {}_0\langle \vec{p}_1 \cdots \vec{p}_n | \mathcal{T} \left\{ \exp \left[-i \int_{-T}^{T} dt H_I(t) \right] \right\} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0. \end{aligned} \tag{9.28}$$

Now we want to use the same strategy we used to determine the structure of the n-point correlation function, we will analyse the last function in the equation above at each

individual order in the perturbative expansion and try to motivate the exact solution in terms of amputated and connected Feynman diagrams. To achieve this we begin by looking at the case where the final state contains 2-particles and the interacting Hamiltonian is that of ϕ^4 theory. At leading order, the sandwiched operator is just the identity, i.e. the $\mathbb{1}$ in $S = \mathbb{1} + iT$, this part is trivially non-interacting and does not contribute to the matrix element $\mathcal{M}$. Mathematically, this term is written as

$$
\begin{aligned}
{}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 &= \sqrt{2E_1 2E_2 2E_{\mathcal{A}} 2E_{\mathcal{B}}} \langle 0 | a_1 a_2 a^\dagger_{\mathcal{A}} a^\dagger_{\mathcal{B}} | 0 \rangle \qquad (9.29) \\
&= 2E_{\mathcal{A}} 2E_{\mathcal{B}} (2\pi)^6 [\delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_1)\delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_2) + \delta^{(3)}(\vec{p}_{\mathcal{A}} - \vec{p}_2)\delta^{(3)}(\vec{p}_{\mathcal{B}} - \vec{p}_1)].
\end{aligned}
$$

The Feynman diagrams for this order are simply the two ways the initial particles can be identical to the final particles without interaction:



The next term in the expansion is evaluated using Wick's theorem

$$
{}_0\langle \vec{p}_1 \vec{p}_2 | \mathcal{T}\left\{ -i\frac{\lambda}{4!} \int d^4x \phi_I^4(x) \right\} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0
$$

$$
{}_0\langle \vec{p}_1 \vec{p}_2 | \mathcal{N}\left\{ -i\frac{\lambda}{4!} \int d^4x \phi_I^4(x) + \text{contractions} \right\} | \vec{p}_{\mathcal{A}} \vec{p}_{\mathcal{B}} \rangle_0 .
$$

Now, there is a difference to previous case in that the uncontracted fields are acting on momentum states rather than the vacuum and so the decomposition (8.18) into positive and negative frequency components does not allow us to make use of the murder identities $\phi_I^+|0\rangle = 0$ and $\langle 0|\phi_I^- = 0$. Instead, the decomposition of uncontracted fields now results in a factor

$$
\begin{aligned}
&\phi_I^+(x)|\vec{p}\rangle_0 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}} e^{-ik\cdot x} \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} e^{-ik\cdot x} \sqrt{2E_{\vec{p}}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) |0\rangle = e^{-ip\cdot x}|0\rangle
\end{aligned}
$$

which motivates the definition of contraction for a field and an external state:

$$
\overbrace{\phi_I(x)|\vec{p}\rangle} = e^{-ip\cdot x}|0\rangle, \qquad \overbrace{\langle \vec{p}|\phi_I}(x) = \langle 0|e^{ip\cdot x}. \qquad (9.30)
$$

Previously, the only non-vanishing terms at order λ were those where all fields are contracted but now we also have contributions from terms with uncontracted fields too. In

the sum of all contractions of $\phi\phi\phi\phi$ which was performed in the previous section, there are three types of terms corresponding to there being either $0, 1$ or 2 pairs of contracted fields. We now examine how each of these terms contribute to the S-matrix.

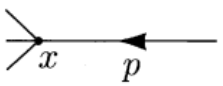
When *all* fields are contracted $\overbrace{\phi\phi\phi\phi}$ the momentum states will remain uncontracted while the fields add a vacuum bubble factor. This is trivial and does not contribute to $\mathcal{M}$, in terms of Feynman diagrams:

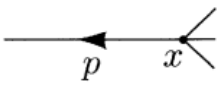
$$-i\frac{\lambda}{4!}\int d^4x \; {}_0\langle \vec{p}_1\vec{p}_2|\overbrace{\phi\phi\phi\phi}|\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}}\rangle_0$$

$=$ 

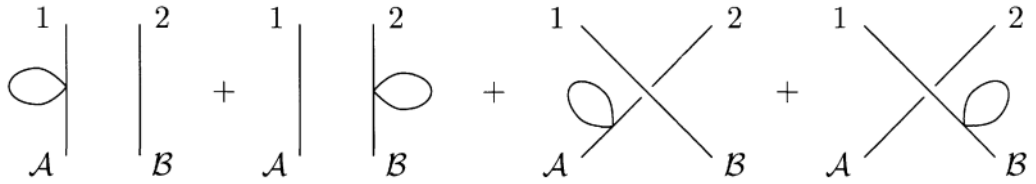
Which is a nice factor to be able calculate but because it does not represent any interaction we will ignore it in this discussion.

Next we will consider the 7 possible terms in which one pair of fields are contracted $\overbrace{\phi\phi}\phi\phi$. The remaining two fields will each contract onto an initial and final state momentum vector (it must be one of each) producing an exponential factor and a vacuum state $|0\rangle$ or $\langle 0|$. Lastly, the remaining two momentum state vectors combine to give a delta function as in (9.29). To represent these terms we introduce *external lines* into the Feynman rules which are lines in the diagram which look like:

$\overbrace{\phi_I(x)|\mathbf{p}\rangle} =$ 

$\overbrace{\langle\mathbf{p}|\phi_I(x)} =$ 

Feynman diagrams for S-matrix elements will always contain external lines and the four distinct diagrams for this term are

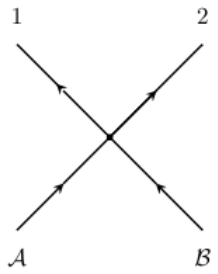


The integration at the vertices of these diagrams yields a momentum-conserving delta function rendering each of these diagrams as contributions in which the initial and final

states are identical.

Since we have found that the two cases above are trivial, it must be the case that the S-matrix is fully determined by the term in the Wick formula in which all of the fields are mutually uncontracted. With a total of 4! possible ways to proceed, each ϕ is contracted with an initial or final momentum state. This can be expressed as

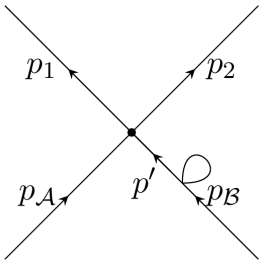
$$(4!)\left(-i\frac{\lambda}{4!}\right)\int d^4x e^{-i(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2)\cdot x} = -i\lambda(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-p_1-p_2) =$$



We can then read off the matrix element from (9.5) which is $\mathcal{M} = -\lambda$. Therefore the only interaction at order λ corresponds to the process for which the scattering cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{64\pi^2 E_{cm}^2} \implies \sigma = \frac{\lambda^2}{32\pi E_{cm}^2} \tag{9.31}$$

Now we have established that *only fully connected diagrams contribute sensibly* to the S-matrix elements. At higher orders in the series $\langle\vec{p}_1\vec{p}_2|iT|\vec{p}_{\mathcal{A}}\vec{p}_{\mathcal{B}}\rangle$ we will find fully connected diagrams of the form



where we have a loop attached to one of the external legs. If the Feynman rules in momentum space are followed this diagram corresponds to the expression

$$\frac{1}{2}\int\frac{d^4p'}{(2\pi)^4}\frac{i}{p'^2-m^2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}(-i\lambda)(2\pi)^8\delta^{(4)}(p_{\mathcal{A}}+p'-p_1-p_2)\delta^{(4)}(p_{\mathcal{B}}-p').$$

Here, k is taken to be the loop momentum. This poses a problem because without even evaluating the main vertex delta function we see that a factor of $i(p_{\mathcal{B}}^2-m^2)^{-1}$ will

appear due to the second delta function. Since all momenta associated with external lines are on-shell (associated with a real particle) this integral will blow up. We are not interested in having these infinities contribute to our S-matrix elements so we ignore the loops by amputating the external leg at a point between the loop and the main vertex. Amputation in this case converts the diagram back into the one in the previous image which we have already computed. For completeness we may aswell summarise the discussion of two particle scattering amplitudes by writing

$$i(2\pi)^4\delta^{(4)}(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum p_f)\mathcal{M}(p_{\mathcal{A}},p_{\mathcal{B}}\to\{p_f\}) = \sum\begin{pmatrix}\text{connected, amputated diagrams with}\\ p_{\mathcal{A}},p_{\mathcal{B}}\text{ incoming and }p_f\text{ outgoing}\end{pmatrix}. \tag{9.32}$$

9.4 Yukawa Theory

On our approach to QED we would now like to discuss interacting theories involving Dirac fermions to figure out their Feynman rules when coupled to other fields. The simplest example of a fermionic interacting theory is Yukawa theory in which Dirac theory is coupled to KG theory. Since correlation functions necessarily involve time-ordered products for causality, it will be key to recall how fermions anti-commute under the action of $\mathcal{T}$. In particular, for a product of any number of fermionic operators, the application of $\mathcal{T}$ earns a minus sign for each exchange necessary to put the operators in time order. The same of course is true for normal ordering. Now it is easy to generalise Wick's theorem and the definition of contractions to fermions. For two fields $\bar{\psi},\psi$, we get

$$\mathcal{T}[\psi(x)\bar{\psi}(y)] = \mathcal{N}[\psi(x)\bar{\psi}(y)] + \overbrace{\psi(x)\bar{\psi}}(y),$$

where the contraction is defined as

$$\overbrace{\psi(x)\bar{\psi}}(y) \equiv \begin{cases}\{\psi^+(x),\bar{\psi}^-(y)\} & \text{if } x^0>y^0\\ -\{\psi^+(y),\bar{\psi}^-(x)\} & \text{if } x^0<y^0\end{cases} = S_F(x-y) \tag{9.33}$$

With an understanding that contractions under the action of $\mathcal{N}$ are to include minus signs for operator interchanges, we can identically state Wick's theorem for fermions as:

$$\mathcal{T}[\psi_1\bar{\psi}_2\psi_3\cdots] = \mathcal{N}[\psi_1\bar{\psi}_2\psi_3\cdots + \text{all possible contractions}]. \tag{9.34}$$

With a coupling constant g, we define the Yukawa Hamiltonian as

$$H = H_{\text{Dirac}} + H_{\text{KG}} + \int d^3x g\bar{\psi}\psi\phi \tag{9.35}$$

If we treat Yukawa theory carefully now, then QED will be a straightforward generalisation when we come to it. Before working out Feynman rules, we should figure out how to compute scattering amplitudes for 2-particle processes as in the previous section. Consider the elementary process involving 2 initial and 2 final fermions with momenta

$p, k \to p', k'$. The leading (non-trivial) contribution to the matrix elements comes from the H_I^2 term which can be written as

$$ {}_0\langle \vec{p}', \vec{k}' | \mathcal{T} \left\{ \frac{1}{2!}(-ig) \int d^4x \bar{\psi}_I \psi_I \phi_I \cdot (-ig) \int d^4y \bar{\psi}_I \psi_I \phi_I \right\} | \vec{p}, \vec{k} \rangle_0 .$$

To define the contraction of fermionic fields with a momentum state $|\vec{p}, s\rangle$ we refer back to the second quantised operator definition of Dirac fermions in terms of spinors:

$$ \overbrace{\psi_I(x)|\vec{p}, s\rangle} \equiv \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_s a^{s'}_{\vec{p}'} u^{s'}(p') e^{-ip'\cdot x} \sqrt{2E_{\vec{p}}} a^{s\dagger}_{\vec{p}} |0\rangle = e^{-ip\cdot x} u^s(p)|0\rangle. \tag{9.36}$$

It is not difficult with some thought to also justify the constraint that ψ_I can only contract with a fermion on the right or an antifermion on the left while the opposite holds for $\bar{\psi}_I$. After applying Wick's theorem to the H_I^2 term an example of one possible contraction would be

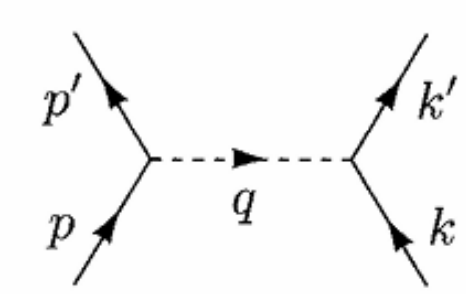
$$ \langle \vec{p}', \vec{k}' | \frac{1}{2!}(-ig) \int d^4x \bar{\psi}_I \psi_I \phi_I \cdot (-ig) \int d^4y \bar{\psi}_I \psi_I \phi_I \Big\} | \vec{p}, \vec{k} \rangle$$

$$ \overset{\pm}{=} (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2} (2\pi)^4 \delta^{(4)}(p' - p - q) \delta^{(4)}(k' - k + q) \bar{u}(p') u(p) \bar{u}(k') u(k).$$

We have dropped the $\frac{1}{2}$ factor because of the symmetry associated with x, y interchange. Either of the delta functions can be used to perform the integral here but importantly the result matches up with our definition of the matrix element so we can read off

$$ i\mathcal{M} = \frac{-ig^2}{q^2 - m_\phi^2} \bar{u}(p') u(p) \bar{u}(k') u(k). \tag{9.37}$$

with the momentum conservation imposed externally. Doing everything algebraically like this is laborious enough that we will now set up our Feynman rules. In particular, for the above contribution we can draw the diagram:



Where we have adopted the convention of denoting bosonic propagators with dashed lines and fermionic ones with solid lines.