Hennefarth II

Week 6 Worksheet Answers

1) Calculate the de Broglie wavelength for each of the following:

a) An automobile of mass 200 kg traveling at 50 mph (22 m/s)

$$p = mv \Rightarrow (200\,kg)\left(\frac{22\,m}{s}\right) = 4400\ \frac{J s}{m}.$$

Then $$\lambda = \frac{6.63 \times 10^{-34}\ J \cdot s}{4400\ J \cdot s/m} = \boxed{1.51 \times 10^{-37}\ m}$$

b) A marble of mass 10 g moving at a speed of 10 cm/s.

$$10\ \frac{cm}{s} \times \frac{1m}{100\,cm} = 0.1\ m/s. \qquad 10g = 0.01\ kg.$$

$$p = (0.01\ kg)(0.1\ m/s) = 0.001\ Js/m.$$

$$\lambda = \frac{6.63 \times 10^{-34}\ J \cdot s}{0.001\ Js/m} = \boxed{6.63 \times 10^{-31}\ m}$$

c) A smoke particle of diameter 100 nm and a mass of 1 fg being jostled by air molecules at room temperature (300 K).

$$1\ fg = 1 \times 10^{-18}\ kg$$

$$\frac{1}{2} m v^2 = \frac{3}{2} k_B T \Rightarrow v^2 = \frac{3 k_B T}{m}$$

$$k_B = 1.38 \times 10^{-23}\ \frac{J}{K}.$$

$$v = \sqrt{\frac{3(1.38 \times 10^{-23}\ \frac{J}{K})(300K)}{1 \times 10^{-18}\ kg}} = 0.111\ m/s.$$

Then $$p = (1 \times 10^{-18}\ kg)(0.111\ m/s) = 1.11 \times 10^{-19}\ Js/m.$$

Hence, $$\lambda = \frac{6.63 \times 10^{-34}\ J \cdot S}{1.11 \times 10^{-19}\ JS/m} = 5.95 \times 10^{-15}\ m$$

$$= 5.95 \times 10^{-6}\ nm.$$

In all cases $\lambda \lll$ object size so we don't observe this wavelength.

2) A photon of green light with a wavelength of 486 nm was emitted when an e^- in a hydrogen atom fell to the $n=2$ level. From what higher level did the electron fall?

$$486\,nm \times \frac{1m}{10^9 nm} = 4.86 \times 10^{-7} m$$

$$C = \lambda \nu \Rightarrow 3.0 \times 10^8 \frac{m}{s} = (4.86 \times 10^{-7} m)\nu$$

$$\nu = 6.17 \times 10^{14}\ Hz$$

$$\Delta E = h\nu \Rightarrow \Delta E = (6.63 \times 10^{-34} J{\cdot}s)(6.17 \times 10^{14} Hz) = 4.09 \times 10^{-19} J$$

$$4.09 \times 10^{-19} J = \left(-2.18 \times 10^{-18} J\ \frac{1}{n^2}\right) - \left(-2.18 \times 10^{-18} J \left(\frac{1}{4}\right)\right)$$

$$\Rightarrow \frac{1}{n^2} = 0.0624 \Rightarrow \boxed{n=4}$$

3) The longest wavelength of light with enough energy to break a C-C bond is 346 nm.

a) Calculate the energy in joules involved in breaking 6.02×10^{23} C-C bonds with 346 nm.

$$346\,nm = 3.46 \times 10^{-7} m \qquad C = \lambda\nu \Rightarrow \nu = 8.67 \times 10^{14}\ Hz$$

$$E = h\nu \Rightarrow E = (6.63 \times 10^{-34} Js)(8.67 \times 10^{14} Hz) = 5.75 \times 10^{-19} J/photon$$

$$(5.75 \times 10^{-19} J/photon)(6.02 \times 10^{23}\ photons) = \boxed{3.46 \times 10^5 J}$$

b) When a bond like C-C is broken, what forces between the C atoms is being disrupted?

The major force is the e^- cloud of each atom being attracted to the positively charged nucleus of the other atom.

c) To completely break both bonds in one C=C double bond, must you use light of a larger or smaller wavelength than 346 nm? Why?

C=C is stronger than C-C so need higher energy.

Higher energy light ⇒ higher frequency light

⇒ $\boxed{\text{smaller wavelength}}$

4) a) Let an electron be described by $\psi(x)$. Describe the physical interpretation of

$$|\psi(x)|^2 dx$$

This is the probability of observing the electron in the interval $[x, x+dx]$.

b) Based on your previous answer, compute

$$\int_{\mathbb{R}} |\psi(x)|^2 dx$$

$\int_{\mathbb{R}} |\psi(x)|^2 dx = 1$ since this is the probability to find the electron somewhere in space (and hence should be 1). All due to probabilistic interpretation.

5) a) Define the term "work" function and how it differs from ionization energy

The work function of a metal gives the minimum amount of energy required to remove an electron from a bulk metal in the solid state.

The ionization energy of a metal refers to the energy required for one electron to be removed from a single metal atom in the gaseous state.

b) What is the maximal initial quantum number required to emit a photocurrent from the lithium metal?

Lets only look at 5 → 2.

$$\Delta E_{5,2} = (-2.18 \times 10^{-18} J)\left(\frac{1}{2^2} - \frac{1}{5^2}\right) \times \frac{6.242 \times 10^{18} eV}{J}$$

$$= -2.86 eV$$

So 4 → 2 ; 3 → 2 have too little energy

$$\Delta E_{6,2} = (-2.18 \times 10^{-18} J)\left(\frac{1}{2^2} - \frac{1}{6^2}\right) \times \frac{6.242 \times 10^{18} eV}{J}$$

$$= -3.02 eV$$

So only 6 → 2 transition is enough

c) For all valid n that generate a photocurrent from b, calculate the maximum speed of the photoelectron from the Lithium

$$E_{max} = \frac{1}{2} m v^2 = h\nu - \varphi$$

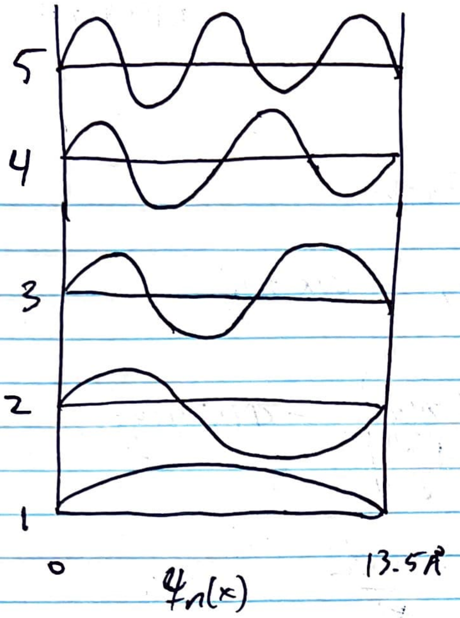
$$v = \sqrt{\frac{2(h\nu - \varphi)}{m}} = \sqrt{\frac{2\left(3.02 eV \times \frac{1J}{6.242 \times 10^{18} eV} - 2.93 eV \frac{1J}{6.242 \times 10^{18} eV}\right)}{9.11 \times 10^{-31} kg}}$$

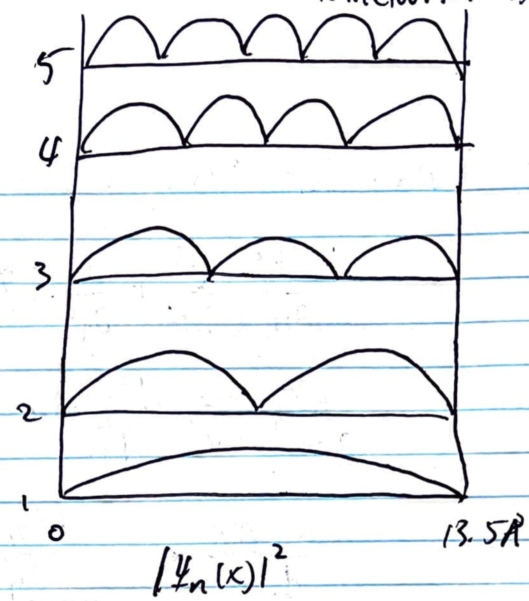
$$= \boxed{1.78 \times 10^5 m/s}$$

6) Consider e^- inside a 1D box with

$$V(x) = \begin{cases} 0 & x \in (0, 13.5 Å) \\ +\infty & \text{else} \end{cases}$$

a) Draw qualitatively $\Psi_n(x)$ and $|\Psi_n(x)|^2$ for $n = 1, 2, 3, 4, 5$.





b) How many nodes will be in $\Psi_{12}(x)$? Would it be possible to find electron density at 13.5/2 Å?

The number of nodes is $n-1$. So $\boxed{11}$

Even, there is not density at ~~ath~~ 13.5/2 for even ~~~~

So $\boxed{\text{NO}}$ p at 13.5 Å/2.

c) What wavelength of light (in nm) is required to excite the electron from its ground state to $n=3$? What frequency (in Hz) does this correspond to?

$$E_n = \frac{n^2h^2}{8mL^2}$$

$$\Delta E = E_3 - E_1 = \frac{h^2}{8mL^2}(9-1) = \frac{(6.626 \times 10^{-34} m^2 kg\, Hz)^2}{8(9.11 \times 10^{-31} kg)(13.5 \times 10^{-10} m)^2}(8)$$

$$= 2.64 \times 10^{-19} J$$

$$\Delta E = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E} = \frac{(6.626 \times 10^{-34} m^2 kg\, Hz)(2.99 \times 10^{8} m/s)}{2.64 \times 10^{-19} J}$$

$$= 7.49 \times 10^{-7} = \boxed{749\, nm}$$

$$\nu = \frac{c}{\lambda} = \frac{(2.99 \times 10^{8} m/s)}{7.49 \times 10^{-7} m} = \boxed{3.99 \times 10^{14}\, Hz}$$

7) a) Let $\Psi(x,y) = \varphi(x)\,\eta(y)$. Then

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi(x)\,\eta(y) = E\,\varphi(x)\,\eta(y)$$

$$-\frac{\hbar^2}{2m}\left[\eta(y)\frac{d^2\varphi(x)}{dx^2}+\varphi(x)\frac{d^2\eta(y)}{dy^2}\right] = E\,\varphi(x)\,\eta(y)$$

$$\frac{\frac{d^2\varphi(x)}{dx^2}}{\varphi(x)}+\frac{\frac{d^2\eta(y)}{dy^2}}{\eta(y)} = -\frac{2mE}{\hbar^2}$$

b) Clearly $\frac{\frac{d^2\varphi(x)}{dx^2}}{\varphi(x)}$ is a function only of x

Similarly $\frac{\frac{d^2\eta(y)}{dy^2}}{\eta(y)}$ is a function only of y.

Their sum is a constant $\Rightarrow$ $\frac{\frac{d^2\varphi(x)}{dx^2}}{\varphi(x)} = A$

a constant. Similarly for $\frac{\frac{d^2\eta(y)}{dy^2}}{\eta(y)}$.

c) Directly:

$$\frac{d^2\varphi(x)}{dx^2} = A\varphi(x).$$

Solutions of the form $e^{\pm i\sqrt{A}x}$ satisfy this.

$$\frac{d^2 e^{\pm i\sqrt{A}x}}{dx^2} = (\pm i)^2 A e^{\pm i\sqrt{A}x} = Ae^{\pm i\sqrt{A}x}$$

Then we have any linear combination also a solution.
Hence, we force the real solutions which are

$$\alpha\cos(\sqrt{A}x)+\beta\sin(\sqrt{A}x) \text{ for } \alpha,\beta\in\mathbb{R}$$

Similary for $\eta(y)$.

d) ~~$(x=a, y=b)$~~ we have $\psi(x,y)=0$. Hence,

~~$(x=a, y=0)$, $(x=0, y=b)$ also have 0~~

$(x=0, y=y)$ is zero for any $y \in (0,b)$. Thus,

$$\varphi(0)=0 \Rightarrow \beta=0 \text{ Clearly.}$$

Similarly $(x=x, y=0)$ is zero for any $x \in (0,a)$. Thus,

$$\eta(0)=0 \Rightarrow \delta=0.$$

So

$$\varphi(x)=\alpha \sin \sqrt{A}x \qquad \eta(y)=\gamma \sin \sqrt{A}y$$

By similar argument but for $(x=a, y=y)$, $\varphi(a)=0$.

$$\varphi(a)=\alpha \sin \sqrt{A}a = 0 \Rightarrow \sqrt{A}a = n\pi \quad n=1,2,\ldots$$

$$\Rightarrow \boxed{\sqrt{A}=\frac{n\pi}{a}}$$

Similar argument makes $\sqrt{B}=\frac{m\pi}{b}$ $m=1,2,\ldots$

$$\frac{2mE}{\hbar^2}=A+B=\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2$$

$$\Rightarrow E=\frac{\hbar^2\pi^2}{2m}\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right) \quad \boxed{E}$$

8) a) What is the probability density that the particle can be found at $x=x'/2$?

Need to normalize 1st. That is

$$1=\int_{\mathbb{R}} \psi(x)^2 dx = \int_{-\infty}^{0} \psi(x)^2 dx + \int_0^{x'} \psi(x)^2 dx + \int_{x'}^{+\infty} \psi(x)^2 dx$$

$$=\int_0^{x'} \psi(x)^2 dx = \int_0^{x'} (2)^2 dx \Rightarrow 4x'=1$$

We see that

$$\psi(x) = \begin{cases} 0 & \text{if } x \notin (0, x') \\ \frac{1}{\sqrt{x'}} & 0 \le x \le x' \end{cases}$$

is thus normalized. Therefore,

$$|\psi(x)|^2 = \left(\frac{1}{\sqrt{x'}}\right)^2 = \frac{1}{x'}$$

So $\boxed{\frac{1}{x'}}$ makes sense?

b) Find $\psi(p)$ using

$$\psi(p) = \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\hbar}} \exp\left\{-\frac{ip}{\hbar}x\right\} \psi(x)$$

$\psi(x)$ is 0 except in $(0, x')$, thus

$$\psi(p) = \int_0^{x'} \frac{1}{\sqrt{2\pi\hbar}} \exp\left\{-\frac{ipx}{\hbar}\right\} 2\, dx$$

$$= \frac{2}{\sqrt{2\pi\hbar}} \int_0^{x'} dx\, e^{\frac{-ipx}{\hbar}} = \sqrt{\frac{2}{\pi\hbar}} \left(\frac{\hbar}{-ip}\right) \left[e^{-ip\cancel{x}/\hbar}\right]_0^{x'}$$

$$= \frac{i\sqrt{2\hbar}}{p\sqrt{\pi}} \left[e^{-ipx'/\hbar} - 1\right]$$

So $\boxed{\psi(p) = \frac{i\sqrt{2\hbar}}{p\sqrt{\pi}} \left[e^{-ipx'/\hbar} - 1\right]}$

c) Give a physical interpretation of

$$|\psi(p)|^2 dp$$

Probability of the particle having a momentum in $[p, p+dp]$. Still probabilistic?

9) Some Quantum Dynamics:

a) Write the power series expression of $e^{\lambda x}$.

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!}$$

This is more of an identity, but Taylor's theorem also gets you here!

b) What is $\frac{d}{dx} e^{\lambda x}$?

We know $\frac{d e^{\lambda x}}{dx} = \lambda e^{\lambda x}$ clearly. We can also see this via

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{\lambda^n x^n}{n!}\right)$$

since $\frac{d}{dx}$ is linear. Thus:

$$= \sum_{n=1}^{\infty} \frac{n \lambda^n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}}{(n-1)!} = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!}$$

$$= \lambda \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = \lambda e^{\lambda x}.$$

c) Verify $\mathcal{U}$ satisfies the differential equation above.

$$\frac{\partial}{\partial t} \mathcal{U} = \frac{\partial}{\partial t} e^{-\frac{i}{\hbar} H(t-t_0)} = -\frac{iH}{\hbar} e^{-\frac{iH}{\hbar}(t-t_0)}$$

$$\Rightarrow i\hbar \frac{\partial \mathcal{U}}{\partial t} = i\hbar\left(\frac{-iH}{\hbar}\right) e^{-\frac{iH}{\hbar} t-t_0}$$

$$= H e^{-iH/\hbar (t-t_0)} = H\,\mathcal{U} \text{ clearly!}$$

d) We know $H \Psi_n = E_n \Psi_n$. Thus,

$$U(t, t_0)\Psi_n(x, t_0) = \Psi_n(t, t_0)$$

$$e^{-\frac{i}{\hbar}H(t-t_0)}\Psi_n(x, t_0)$$

$$= \sum_{m=0}^{\infty}\left(\frac{-iH(t-t_0)}{\hbar}\right)^m \frac{1}{m!}\Psi_n(x, t_0)$$

$$= \sum_{m=0}^{\infty}\left(\frac{-i(t-t_0)}{\hbar}\right)^m \frac{1}{m!} H^m \Psi_n(x, t_0)$$

$$H^m \Psi_n(x, t_0) = H^{m-1} H \Psi_n(x, t_0) = H^{m-1} E_n \Psi_n(x, t_0)$$

$$\Rightarrow H^m \Psi_n(x, t_0) = E_n^m \Psi(x, t_0).$$

$$\Rightarrow \sum_{m=0}^{\infty}\left(\frac{-i(t-t_0)}{\hbar}\right)^m \frac{1}{m!} E_n^m \Psi_n(x, t_0) = \Psi_n(t, t_0)$$

$$= \left[\sum_{m=0}^{\infty}\left(\frac{-iE_n(t-t_0)}{\hbar}\right)^m \frac{1}{m!}\right]\Psi_n(x, t_0)$$

$$= \boxed{e^{-i\frac{E_n(t-t_0)}{\hbar}}\Psi_n(x, t_0) = \Psi_n(x, t)}$$

Only have rotated with some complex phase?
So energy eigenstates stay energy eigenstates.

e)

$$|\Psi_n(x, t)|^2 = \left|e^{-i\frac{E_n(t-t_0)}{\hbar}}\Psi_n(x, t_0)\right|^2$$

$$= \left(\Psi_n^*(x, t_0)\Psi_n(x, t_0)\underbrace{e^{-i\frac{E_n(t-t_0)}{\hbar}} e^{i\frac{E_n(t-t_0)}{\hbar}}}_{=1}\right)$$

$$= |\Psi_n(x, t_0)|^2.$$ So good?

f) Recalling the probabilistic interpretation of $\Psi(x,t)$, can you give a physical justification for why

$$|\Psi_n(x,t)|^2 = |\Psi_n(x,t_0)|^2$$

If not then total probability would not be conserved. That is, if

$$|\Psi_n(x,t)|^2 \geq |\Psi_n(x,t_0)|^2$$

then total probability would be increasing which would be unphysical.

Note this is necessarily satisfied in the construction of $\mathcal{U}(t,t_0)$. It is unitary which means it preserves inner products (generalizations of dot products).