Arithmetic Functions

0) Multiplicative Functions:

Definition: f is a multiplicative if for any relatively prime positive integers a, b we have $f(ab) = f(a)f(b)$.

1) Important Multiplicative Functions[1]:

For all of the following it is assumed that the prime factorization of n is $p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$

1. $\sigma(n)$=sum of the divisors of n
   (a) $\sigma(n) = \frac{p_1^{e_1+1}-1}{p_1-1} \frac{p_2^{e_2+1}-1}{p_2-1} \cdots \frac{p_n^{e_n+1}-1}{p_n-1}$
2. $\tau(n) = d(n)$=the number of factors of n
   (a) $\tau(n) = (e_1+1)(e_2+1)\cdots(e_n+1)$
3. $\phi(n)$= number of integers less than n which are relatively prime to n
   (a) $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})\cdots(1 - \frac{1}{p_n})$

2) Problems:

1) The only prime factors of an integer n are 2 and 3. If $\sigma(n) = 1815$, find n.

2) Find all n such that $\sigma(n) = n + 6$.

3) Prove the formula given for $\tau(n)$

4) How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers (To spare you the calculations: $2004 = 2 \cdot 3 \cdot 167$).

5) Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \ldots + \tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find a and b.

6) Find all positive integers n such that $\tau(n)^2 = 2n$

7) Prove that if $n > 6$ then $\phi(n) \geq \sqrt{n}$. (Challenge: Prove that for all $\epsilon < 1$, there exists some M such that for all $n \geq M$ we have $\phi(n) \geq n^{\epsilon}$)

[1] Examples 1a and 3a will be presented in class

8) Prove that if f is a multiplicative function then so is $F(n) = \sum_{d|n} f(d)$

9) The Möbius function $\mu(n)$ is defined as follows [2]

$$\mu(x) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2|n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \dots p_r \text{ for distinct } p_i \end{cases}$$

Prove that if $n > 1$

$$\sum_{d|n} \mu(d) = 0$$

10) Find all odd integers $n \leq 5000$ such that $n|\phi(n)\tau(n)$.

11) Find the largest integer k such that $\phi(\sigma(2^k)) = 2^k$ (Hint: $641|2^{32}+1$).

12) Prove that if $\sigma(N) = 2N + 1$, then N is the square of an odd integer

13) Show that $\phi(n) + \sigma(n) \geq 2n$

14) Determine all positive integers m for which there exists a positive integer n such that $\dfrac{\tau(n^2)}{\tau(n)} = m$.

[2] If you want to read more about the Möbius function and dirichlet convolutions you should go to these links: `www.artofproblemsolving.com/Resources/Papers/SatoNT.pdf` and `http://www.math.upenn.edu/~wilf/gfology2.pdf` . If you know some group theory you should also check out this link: `https://www.math.hmc.edu/~benjamin/papers/SuryLetter.pdf`