Series and Summations

1) Warm-Ups:

1) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$

2) Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

2) More Problems:

1) Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \dots$

2) Let F_n be the n-th term in the fibonacci sequence. Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{4^n}$

3) The sequence $\{a_n\}$ satisfies $a_0 = 201$, $a_1 = 2011$ and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$

Find $\frac{1}{S}$.

4) Evaluate $\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{m^n}$

5) Compute

$$\sum_{j>i\geq 0} \frac{1}{2^j 3^i}$$

6) Evaluate $\sum_{a=0}^{\infty} \sum_{b=a}^{\infty} \sum_{c=b}^{\infty} \frac{1}{2^{a+b+c}}$.

7) Compute

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$$

8*) Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n k!}$. Calculate $\sum_{n=2}^{\infty} f(n)$

9) Compute

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}}$$

10) Let $\mu(n)$ be the function such that $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for all $n > 1$.

Prove that if F, f are functions such that

$$F(n) = \sum_{d|n} f(d)$$

then the following holds

$$f(x) = \sum_{d|n} F(d)\mu(\frac{n}{d})$$

11) Compute $\displaystyle\sum_{n=1}^{\infty}\sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$

12) If

$$\sum_{n=1}^{\infty} \frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{\binom{n+100}{100}} = \frac{p}{q}$$

for relatively prime positive integers p, q, find $p+q$.

13) The real numbers $a_0, a_1, \ldots, a_{2013}$ and $b_0, b_1, \ldots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \ldots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013}(a_k b_{k-1} - a_{k-1}b_k)$$

14*) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}$

15) Evaluate

$$\left\lfloor \prod_{n=1}^{1992} \frac{3n+2}{3n+1} \right\rfloor$$

16) Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty} \frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}$$

17*) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{n}$

18*) Evaluate the infinite sum $\displaystyle\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}$

19) Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$ Assume that the sequence (b_j) is bounded. Evaluate

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

i) For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate: $\displaystyle\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$