Integer Polynomials

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1 Important Facts

1. If $P(x)$ is an integer polynomial, then for all distinct $a, b \in \mathbb{Z}$ we have $a - b | P(a) - P(b)$.
2. If $P(a) = 0$ then there exists a polynomial Q such that
$$P(x) = (x - a)Q(x)$$
3. For any polynomials P, S there exists polynomials Q, R where $\deg R < \deg S$ such that
$$P(x) = S(x)Q(x) + R(x)$$
4. (Rational Root Theorem) If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial with integer coefficients and a, b are relatively prime integers satisfying $P(\frac{a}{b}) = 0$, then $a|a_0$ and $b|a_n$.
5. (Eisenstein's Criterion) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients. If there exists a prime p such that $p \nmid a_n$, for $n - 1 \leq i \geq 0, p|a_i$ and $p^2 \nmid a_0$, then P is irreducible over the rationals.
6. If an integer polynomial P is reducible over the rationals, then it is also reducible over the integers.

2 Problems

1. (USAMO 1974) Let a, b, c be three distinct integers, and let $P(x)$ be a polynomial with integer coefficients. Show that we cannot have $P(a) = b, P(b) = c, P(c) = a$

2. Show that for all primes p the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over the rationals
3. Suppose an integer polynomial P takes on values ± 1 at three different integer points. Show that P has no integer roots.
4. (BAMO 2004) Find with proof all monic polynomials $f(x)$ with integer coefficients that satisfy
   - $f(0) = 2004$
   - If x is irrational, then $f(x)$ is also irrational.
5. Show that the polynomial $(x^2 + x)^{2^n} + 1$ is irreducible
6. If $a_1, a_2, \ldots, a_n$ are distinct integers, prove that the polynomial $P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$ is irreducible over the integers.
7. (IMO 2006) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.
8. Let $n > 1$ be an integer. Show that the polynomial $x^n + 5x^{n-1} + 3$ is irreducible over the integers.
9. Let $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be a non zero polynomial with integer coefficient such there $P(s) = P(r) = 0$ for some integers r, s with $0 < r < s$. Prove that there exists a k such that $a_k \leq -s$.
10. (BAMO 2012) Find all non-zero polynomials $P(x)$ that satisfy the following property: whenever a, b are relatively prime then $P(a), P(b)$ are also relatively prime.
11. (USA TST 2010) Let P be a polynomial with integer coefficients such that $P(0) = 0$ and
$$\gcd(P(0), P(1), P(2), \ldots) = 1.$$
Show there are infinitely many n such that
$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \ldots) = n.$$
12. (USAMO 2002) Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.