Polynomials: Roots

1) Important Facts:

1) If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

2) Vieta's formula's: Let $f(x) = a_n x^n + ... + a_1 x + a_0$ and let the roots of f be $r_1, r_2, ..., r_n$. If s_i denotes the sum of all possible products of some i roots of f (for example $s_2 = r_1r_2 + r_1r_3 + ... + r_1r_n + r_2r_3 + ... + r_2r_n + ... + r_{n-1}r_n$) then

$$s_i = (-1)^i \frac{a_{n-i}}{a_n} \tag{1}$$

Example:
If $P(x) = ax^3 + bx^2 + cx + d$ is a polynomial with roots r_1, r_2, r_3 then $r_1 + r_2 + r_3 = -\frac{b}{a}$, $r_1r_2 + r_2r_3 + r_3r_1 = \frac{c}{a}$, and $r_1r_2r_3 = -\frac{d}{a}$

3) Intermediate Value Theorem: If P is a polynomial (or any continuous function) then for any $y \in [P(a), P(b)]$ there exists some $x \in [a, b]$ such that $P(x) = y$.

2) Problems:

1) Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

2) Find the sum of the roots, real and non-real, of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$ given that there are non multiple roots.

3) Let r, s, t be the roots of the equation $x^3 + 2x^2 + 3x + 1$. Find $\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}$.

4) Let r, s, t be three roots of the equation

$$8x^3 + 1001x + 2008$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

5) $f(x)$ is a monic quartic polynomial such that $f(-1) = -1$, $f(2) = -4$, $f(-3) = -9$, and $f(4) = -16$. Find $f(1)$.

6) Let $P(x)$ be a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

Find $P(-1)$.

7) Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

8) Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

9) A polynomial $P(x)$ of degree n satisfies $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$. Find $P(n + 1)$

10) The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}$$

11) Prove that for any monic polynomial P of degree n, there exists monic polynomials R, S also of degree n such that $P(x) = \frac{R(x)+S(x)}{2}$ and both R, S have exactly n real roots.

12) Find all pairs of polynomials p, q such that both have real coefficients and

$$p(x)q(x + 1) - p(x + 1)q(x) = 1$$