Greatest Common Divisor

Matthew Lerner-Brecher

March 30, 2015

1 Warm-Ups

1. Evaluate $\gcd(1881, 1122)$.
2. Show that $\frac{21n+4}{14n+3}$ is irreducible for all n.
3. Show that $(a, b)[a, b] = ab$
4. Find the greatest common divisor of $x^2 + 2x - 3$ and $x^2 + 4x + 3$
5. Show that $\frac{12n+1}{30n+2}$ is irreducible for all n.

2 Problems

1. Find the largest positive n for which $n + 10$ divides $n^3 + 100$.
2. Find the number of ordered triples (a, b, c) of positive integers such that $[a, b] = 1000$ and $[b, c] = [c, a] = 2000$
3. Show that if
$$\gcd(m, n) + \operatorname{lcm}(m, n) = m + n$$
then one of m, n divides the other.
4. The numbers in the sequence 101, 104, 109, 116,... are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.
5. Show that
$$\frac{[a, b, c]^2}{[a, b][b, c][c, a]} = \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}$$

6. Find the greatest common divisor of the sequence $a_n = 16^n + 10n - 1$
7. Find a pair of integers x, y such that $57x + 33y = 1$.
8. Let a, b, c, d be positive integers such that $ab = cd$. Prove that
$$\gcd(a, c) \cdot \gcd(a, d) = a \cdot \gcd(a, b, c, d)$$
9. Find all sequences $a_1, a_2, \ldots, a_n$ that satisfy:
$$\gcd(a_i, a_j) = \gcd(i, j)$$
for all $i \neq j$.
10. Find all triples of positive integers (a, b, c) such that $a^3 + b^3 + c^3$ is divisible by a^2b, b^2c, c^2a.