Inequalities: AM-GM and Cauchy-Schwarz

1) Important Facts:

1) The AM-GM inequality: For any set of non-negative reals the following holds:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$$

2) The Cauchy-Schwarz inequality: For any two sequences of real numbers: $a_1, \ldots a_n$ and $b_1 \ldots b_n$ the following holds:

$$(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

3) Weighted AM-GM: Let $w_1, \ldots, w_n$ be positive reals such that $w_1 + \ldots + w_n = 1$. If $a_1, \ldots, a_n$ is any set of non-negative reals, then the following holds:

$$w_1 a_1 + w_2 a_2 + \ldots + w_n a_n \geq a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n}$$

2) Problems:

1) Let a be a positive integer. Find the minimum possible value of $a + \frac{1}{a}$.

2) Let a, b, c be reals. Find the maximum possible value of

$$\frac{ab + bc + ca}{a^2 + b^2 + c^2}$$

3) Let a, b, c, d be reals such that $a^2 + b^2 + c^2 + d^2 = 1$. Find the maximum possible value of $a + 2b + 3c + 4d$.

4) (AIME) Find the minimum value of $\dfrac{9x^2 \sin^2 x + 4}{x \sin x}$ for $0 < x < \pi$.

5) Let m and n be positive integers. Find the minimum value of $x^m + 1/x^n$

6) Let $P(x)$ be a polynomials with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$

7) (SUMO) Find the maximum possible value of

$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$

8) (SUMO) There are 2011 positive numbers with both their sum and the sum of their reciprocals equal to 2012. Let x be one of these numbers. Find the maximum possible value of $x + \frac{1}{x}$

9) (HMMT) Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

10) (PFTB) Prove that if $x, y, z > 0$ satisfy $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, then

$$\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1} \leq \sqrt{x+y+z}$$

11) (IMO) Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \ldots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \ldots (1+a_n)^n > n^n$$

12) Let x, y, z be positive integers such that $xyz = 1$. Prove

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$