Diophantine Equations

1) Tips and Tricks:

1) Look at the parity of the equations

2) Look at the equations under different mods (The fewer residues each side can have the better)

3) Look at the gcd of different groups/pairs of numbers

4) If there are multiple equations, try adding and subtracting them

5) Factor, rearrange, and simplify the equations

6) Infinite descent

7) Compare magnitudes of the different sides of an equation.

2) Problems:

1) Find all triples of positive integers (x, y, z) such that $x^2 + y - z = 100$ and $x + y^2 - z = 124$.

2) Find all (x, y, m, n) such that $(x^2 + y^2)^n = (xy)^m$

3)

*4) How many ordered pairs of integers (x, y) satisfy

$$8(x^3 + x^2y + y^2x + y^3) = 15(x^2 + y^2 + xy + 1)$$

*6) Find all positive integer solutions of $(x^2 + 2)(y^2 + 3)(z^2 + 4) = 60xyz$

*7) (Putnam 2001) Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a + 1)^n = 2001$$

*8) Find all pairs (x, y) of positive integers that satisfy the equation

$$y^2 = x^3 + 16$$

9) Find the number of ordered pairs (a, b) of positive integers that are solutions of the following equation

$$a^2 + b^2 = ab(a + b)$$

*10) (PUMaC 2011) Let a and b be positive integers such that $a + bz = x^3 + y^4$ has no solutions for any integers x, y, z with b as small as possible, and a as small as possible for the minimum b. Find a and b.

11) Prove that the system

$$x^6 + x^3 + x^3y + y = 147^{157}$$
$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, z.

12) Prove the system of equations:

$$x^2 + 6y^2 = z^2$$
$$6x^2 + y^2 = t^2$$