Functional Equations

Contents

1 List of Tactics

General things to look for / questions to ask yourself when solving functional equations (we'll go into greater depth on many of these in the sheet):

- Can you find any functions which satisfy the equation? Common functions to try are constants, $\pm x$, x^2.
- What is the value of f at $x = 0, 1$?
- Where is the function 0,1?
- What are the domain and range of the function?
- Is the function injective, surjective, or bijective?
- Does the function have any fixed points? If so, where?
- Is the function additive or multiplicative? Is it in any way related to the Cauchy equation?
- Is the function periodic?
- Does iterating the function give any nice results?
- Is the function strictly increasing or decreasing?
- Can you come up with any bounds on the function?
- If you're trying to prove a fact and the function maps integers to integers than can you prove what you want by induction?

2 Plugging Stuff in

2.1 Main Idea

The problems in this section can either be solved by coming up with some sort of cycle or just repeatedly plugging in numbers (or functions) and trying to find the value of different points on the function.

2.2 Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $xf(x) + f(1 - x) = x^3 - x$. Find all possible functions f.

2. Find all functions satisfying for all $x, y \in \mathbb{R}$:

$$xf(y) + yf(x) = (x + y)f(x)f(y)$$

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following holds:

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

3 Substitutions

3.1 Main Idea

This is a small section is devoted to problems that can be solved making some substitution in terms of f (i.e. defining some $h(x) = g(f(x))$ which is easier to solve for).

3.2 Problems

1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that $xf(y) = yf(x)$.
2. Find all $f : \mathbb{R} \to \mathbb{R}$ that satisfy $(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$ for all $x, y \in \mathbb{R}$

4 Injectivity and Surjectivity

4.1 Main Idea

Injectivity:

<u>Defintion:</u> A function $f : A \to B$ is considered to be injective if for all $a, b \in A$, $f(a) = f(b)$ if and only if $a = b$.

<u>How to Prove It:</u> Injectivity is usually proved via a proof by contradiction; we assume that there exists a, b such that $a \neq b$ and $f(a) = f(b)$ and then try to arrive at some sort of contradiction. For example, take the functional equation $f(f(x) + y) = x + y$ and suppose a, b satisfy the conditions we just mentioned. Then we have $a+y = f(f(a)+y) = f(f(b)+y) = b+y$. This cannot b=possibly be true since $a \neq b$ so we have a contradiction and the function must be injective.

<u>How to Use It:</u> The main thing that injectivity allows us to do is that if we have an equation of the form $f(g(x)) = f(h(x))$ and we know that f is injective then we must have $g(x) = f(x)$.

Surjectivity:

<u>Definition:</u> A function $f : A \to B$ is said to be surjective if for all $y \in B$ there exists some $x \in A$ such that $f(x) = y$.

<u>How to Prove It:</u> Surjectivity is often a little bit harder to prove than injectivity. The best way to prove injectivity is to manipulate the equation into something of the form $f(g(x)) = h(x)$ where $h(x)$ is some function that you know spans all the numbers in B. For example, if you can reduce a functional equation to $f(f(x) + y) = x^3 + y$ this would imply that f is surjective since for all m there exists x, y such that $x^3 + y = m$.

How to Use It: Surjectivity allows us to make nice substitutions into functional equations. For example, suppose we have the equation $f(f(x)) = f(x)$. Without surjectivity all this equation tells us is that $f(z) = z$ if z is in the range of f. With surjectivity, this implies that $f(z) = z$ for all z. Essentially, surjectivity allows us to make substitutions like $z = f(x)$ without having to worry about what specific values z can take on. Adding on, using surjectivity we can make substitutions along the lines of "let x be the value such that $f(x) = 1$".

4.2 Problems

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + f(x + f(y)) = 2x + y$.
2. Find all functions f from reals to reals such that $f(f(x) + y) = 2x + f(f(y) - x)$.
3. Find all pairs of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x + g(y)) = x \cdot f(y) - y \cdot f(x) + g(x)$ for all real x, y.

5 Monotonic Functions

A monotonic function is any function which is either nondecreasing or nonincreasing. Like injectivity, monotonicity is generally proved using a proof by contradiction; we assume there exists some $a > b$ such that $f(a) < f(b)$ (or alternateively $f(a) > f(b)$ if we are trying to prove that it's nonincreasing) and then try to arrive at some sort of contradiction. The nice thing about monotonicity is that it allows us to compare the values of f for different a, b and bound f above or below. Also, as you'll see in the next section, the fact that a function is monotone is useful if $f(x)$ is additive.

5.1 Main Idea

5.2 Problems

1. Find all $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ that satisfy $f(f(n)) = 3n$ and $f(n+1) > f(n)$. What is the value of $f(10)$?
2. Find all $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(n+1) > f(f(n))$.

6 The Cauchy Equation

6.1 Main Idea

A Cauchy equation is any functional equation of the form $f(x+y) = f(x)+f(y)$. The nice thing about the Cauchy equation is that if any of the following three conditions are true, then $f(x) = cx$ for some constant c.

- $f : \mathbb{Q} \to \mathbb{Q}$
- f is continuous at some point
- f is monotonic on some interval
- f is bounded on some interval

The idea behind the following problems is reducing the equation to a Cauchy equation or deriving some Cauchy equation from the given one.

6.2 Problems

1. Let f be a function which is either monotonic or continuous. Find all possible f given that for all $x, y > 0$, $f(x+y) = f(x)f(y)$.
2. Let f be a function which is either monotonic or continuous. Find all possible f given that for all $x, y > 0$, $f(xy) = f(x)f(y)$.
3. Find all functions which are both additive and multiplicative.
4. Find all $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(x^2 - y^2) = xf(x) - yf(y)$ for all $x, y \in \mathbb{R}$.
5. Find all functions f from the reals to the reals such that for all reals x, y, z, t

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

7 Iteration

7.1 Main Idea

The idea behind this section is iterating functions to get some expression which can be expressed in multiple ways using the functional equation.

7.2 Problems

1. Find all $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(f(m)) = m + 1$.
2. Find all $f : \mathbb{N} \to \mathbb{N}$ that satisfy $f(f(a) + f(b)) = a + b$.

8 General Problems

1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y)+f(y+z)+f(z+x) \geq 3f(x+2y+3z)$ for all $x, y, z \in \mathbb{R}$.
2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy

$$f((x+z)(y+z)) = (f(x) + f(z))(f(y) + f(z))$$

for all $x, y, z \in \mathbb{R}$

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy $f(m + f(n)) = f(m) + n$ for all $m, n \in \mathbb{Z}$
4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy $f(1 + xf(y)) = yf(x + y)$ for all $x, y \in \mathbb{R}$.
5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xf(x) + f(y)) = (f(x))^2 + y$ for all $x, y \in \mathbb{R}$.
6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y^2) = f(x) + |yf(y)|$ for all $x, y \in \mathbb{R}$.

9 Hints

I did my best to balance out the hints but some of these are still big giveaways.

2.1 Try plugging $1 - x$ into the equation.

2.2 A single substitution will give you $f(x)$ directly. Note the fact that the two sides are of different degrees in terms of $f(x)$. Try taking advantage of this.

2.3 Try making x, y different combinations of 0,1.

3.1 Divide both sides by x.

3.2 Divide both sides by $x^2 - y^2$

4.1 Prove injectivity or surjectivity. Either one alone is enough to solve the equation.

4.2 Take $y = -f(x)$. What does this tell you about f?

4.3 The function is surjective, but you can't arrive at that fact simply based on the method I described. A bigger hint is in white:

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5.1 Try finding the values of other specific values of f.

5.2 Try to prove by induction that $f(x)$ is monotonically increasing.

6.1 Take the log of both sides.

6.2 Do the same thing as in 6.1 just make a second substitution in addition.

6.3 Try proving that f is bounded by showing that $f(x) \geq 0$ on some interval

6.4 Once you've gotten the equation in the form of a Cauchy try finding some $g(x)$ such that $f(g(x))$ can be expressed in terms of $f(x)$ in two ways.

6.5 Try setting different combinations of the variables to 0. What information do you get about $f(0)$? Do you get anything that resembles a cauchy equation? Try proving that f is monotonically increasing so you can use the solutions of the cauchy equation.

7.1 In what ways can you express $f(f(f(x)))$?

7.2 In what ways can you express $f(f(f(a)+f(b))+f(f(c)+f(d)))$

8.1 This problem only revolves around plugging stuff in.

8.2 Is f odd or even?

8.3 Injectivity, surjectivity, cauchy equation, iteration

8.4 Try proving Injectivity, surjectivity.

8.5 Prove $f(x)$ is surjective then try to get to the fact that $f(x)^2 = x^2$. Once you have that fact you're not done. You need to try and prove that $f(x) = x$ everywhere or $f(x) = -x$ everywhere. All that equation implies is that at each point it's one of the two.

8.6 Cauchy, bounding.

10 Answers

2.1 $f(x) = x^2 - x$

2.2 $f(x) = 0$ everywhere, $f(x) = 1$ everywhere, or $f(x) = 1$ everywhere except at 0 and $f(0) = 0$. Note: If you get something along the lines of $f(x) = f(x)^2$, it does NOT mean that $f(x) = 0$ everywhere or $f(x) = 1$ everywhere; it means any given point is either 0 or 1. You need to prove separately which points take on which value.

2.3 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1935849&sid=4bbd4fb43a17c848f14fe2d12cd29ef1#p1935849`

3.1 $f(x) = cx$ for a constant c.

3.2 $f(x) = x^3 + cx$ for a constant c.

4.1 Only $f(x) = x$ works.

4.2 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=118698&sid=4bbd4fb43a17c848f14fe2d12cd29ef1#p118698`

4.3 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=837232&sid=4bbd4fb43a17c848f14fe2d12cd29ef1#p837232`

5.1 `http://math.stackexchange.com/questions/337208/function-olympiad-problem-define-fn-such`

5.2 Only $f(x) = x$ works.

6.1 $f(x) = e^{cx}$ for some c or $f(x) = 0$.

6.2 $f(x) = x^c$ for some c or $f(x) = 0$.

6.3 Only $f(x) = 0$, $f(x) = x$ work.

6.4 `http://www.artofproblemsolving.com/Wiki/index.php/2002_USAMO_Problems/Problem_4`

6.5 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=118703&sid=4bbd4fb43a17c848f14fe2d12cd29ef1#p118703`

7.1 There are no such functions.

7.2 Only $f(x) = x$ works.

8.1 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=1414400&sid=eb22ff946c51162f8deb21708d21e5c1#p1414400`

8.2 `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=325471`

8.3 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=343342&sid=432856b45171e229fbca216dc050c9de#p343342`

8.4 `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=323174`

8.5 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=363615&sid=eb22ff946c51162f8deb21708d21e5c1#p363615`

8.6 `http://www.artofproblemsolving.com/Forum/viewtopic.php?p=3195788&sid=eb22ff946c51162f8deb21708d21e5c1#p3195788`