Symmetric Polynomials
(Boot camp)

1 Main Ideas

1. If f is a symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$, then it can be expressed as a sum of products of elementary symmetric polynomials in those variables.
2. (Newton's Sums) Let $x_1, x_2, \ldots, x_n$ be the roots of $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$.
   - If $r \leq n$, then $s_r + a_{n-1}s_{r-1} + \cdots a_{n-r+1}s_1 + na_{n-r} = 0$
   - If $r > n$, then $s_r + a_{n-1}s_{r-1} + \cdots a_0 s_{r-n} = 0$
3. Other Techniques: Factoring, Polynomial Transformations, Plugging values into polynomial

2 Lecture

1. Let a, b, c be the roots of $x^3 + ax^2 + bx + c$. Find $\sum_{\text{sym}} a^2$, $\sum_{\text{sym}} a^2 b$, and $\sum_{\text{sym}} \frac{a}{b}$.
2. Let $P(x) = x^3 + x^2 + x + 1$ have roots a, b, c. Find the polynomials with roots $\{a-1, b-1, c-1\}$, $\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}$, and $\{\frac{1}{a-1}, \frac{1}{b-1}, \frac{1}{c-1}\}$.
3. Find a constructive proof of Main Idea 1 and use it to find $\sum_{\text{sym}} a^2$, $\sum_{\text{sym}} a^2 b$
4. Let $P(x) = x^4 + 2x^2 + x + 1$. Find the sum of the squares and cubes of the roots of P.
5. Let $P(x) = x^2 + x + 3$. Find the sum of the 4th powers of the roots.
6. Given $a + b + c = 6$, $ab + bc + ca = 11$, $abc = 6$, find a, b, c.

3 Problems

1. Find the sums of the cubes of the roots of $x^3 + 2x^2 + x + 1$ in two ways.
2. Let $a + b + c = 12$, $a^2 + b^2 + c^2 = 50$, and $a^3 + b^3 + c^3 = 168$. Find a, b, c

3. Let a, b, c be the roots of $P(x) = x^3 + 2x^2 + 3x + 1$. Find

(a) $\displaystyle\sum_{\text{sym}} \frac{1}{b-1}$ (b) $\displaystyle\sum_{\text{sym}} a^4$ (c) $\displaystyle\sum_{\text{sym}} \frac{a}{b^2}$ (d) $\displaystyle\sum_{\text{sym}} \frac{a}{b-1}$

4. Find the sum of the 20th powers of the roots of $z^{20} - 19z - 1$.

5. Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

6. Find all possible triples of real numbers (x, y, z) satisfying

$$\begin{aligned} x^2y + y^2z + z^2x &= -1 \\ xy^2 + yz^2 + zx^2 &= 5 \\ xyz &= -2 \end{aligned}$$

7. The complex numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}$$