Polynomials: Intro

1 Main Ideas

1. (Conjugate Root Theorem) Suppose P is a polynomial with real coefficients. If $P(z) = 0$, then $P(\overline{z}) = 0$

2. (Rational Root Theorem) Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with integral coefficients, $a_n \neq 0$. If $P(x)$ has a rational root $r = \pm\frac{p}{q}$ with p, q relatively prime positive integers, p is a divisor of a_0 and q is a divisor of a_n.

3. (Vieta's Formula's) Let $f(x) = a_n x^n + ... + a_1 x + a_0$ and let the roots of f be $r_1, r_2, ..., r_n$. If s_i denotes the sum of all possible products of some i roots of f (for example $s_2 = r_1 r_2 + r_1 r_3 + ... + r_1 r_n + r_2 r_3 + ... + r_2 r_n + ... + r_{n-1} r_n$) then

$$s_i = (-1)^i \frac{a_{n-i}}{a_n}$$

2 Problems

1. Let a, b, c be the roots of $2x^3 - 3x^2 + 4x - 5$.
   (a) (Warm-up) Find $a + b + c$, $ab + bc + ca$, and abc
   (b) Find $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.
   (c) Find $a^2 + b^2 + c^2$
   (d) Find $a^2 b + b^2 a + b^2 c + c^2 b + c^2 a + a^2 c$
   (e) Find $a^3 + b^3 + c^3$
   (f) (Challenge) Find $\frac{1}{(a-2)^2} + \frac{1}{(b-2)^2} + \frac{1}{(c-2)^2}$

2. Determine all rational roots of $7x^4 - 8x^3 - 6x^2 - 13x + 2$.

3. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

4. Prove that $\sqrt{2}$ is irrational.

5. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients and satisfies $f(2i) = f(2 + i) = 1$. Find $a + b + c + d$.

6. Let r, s, t be three roots of the equation

$$8x^3 + 1001x + 2008$$

   Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

7. Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.

8. Let a, b, c be the roots of the cubic x^3+3x^2+5x+7. Given that P is a cubic polynomial such that $P(a) = b+c$, $P(b) = c+a$, $P(c) = a+b$, and $P(a+b+c) = -16$, find $P(0)$.

9. Prove that the sum

$$\sqrt{1000^2+1} + \sqrt{1001^2+1} + \cdots + \sqrt{2000^2+1}$$

is irrational

10. The complex numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}$$

11. Find all monic polynomials $f(x)$ with integer coefficients such that $f(0) = 2004$ and if x is irrational so is $f(x)$.