Vieta's Formula's and Newton's Sums

1) The Formulas:

1) Vieta's formula's: Let $f(x) = a_nx^n + ... + a_1x + a_0$ and let the roots of f be $r_1, r_2, ..., r_n$. If s_i denotes the sum of all possible products of some i roots of f (for example $s_2 = r_1r_2 + r_1r_3 + ... + r_1r_n + r_2r_3 + ... + r_2r_n + ... + r_{n-1}r_n$) then

$$s_i = (-1)^i \frac{a_{n-i}}{a_n} \tag{1}$$

Basic Proof Outline:
Since $f(x) = a_nx^n + ... + a_1x + a_0 = a_n(x - r_1)(x - r_2)...(x - r_n)$, the coefficient of x^i in $a_nx^n + ... + a_1x + a_0$ and $a_n(x - r_1)(x - r_2)...(x - r_n)$ have to be equal and equating the two values gives vieta's formula's.

2) Newton's sums: Let $f(x) = a_nx^n + ... + a_1x + a_0$ and let the roots of f be $r_1, r_2, ..., r_n$. Let $P_i = r_1^i + r_2^i + ... + r_n^i$. If $i \leq n$

$$P_ia_n + P_{i-1}a_{n-1} + ... + P_1a_{n-i+1} + ia_{n-i} = 0 \tag{2}$$

If $i > n$ then

$$a_nP_i + a_{n-1}P_{i-1} + ... + P_{i-n+1}a_1 + P_{i-n}a_0 = 0 \tag{3}$$

Basic Proof Outline:
Since $f(r_1) = f(r_2) = ... = f(r_n) = 0$, we have that $x^k(f(r_1) + f(r_2) + ... + f(r_n)) = 0$ and expanding this out based on each value of k (which can be negative) gives Newton's Sums.

2) Common Transformations:

For all of these we are transforming the polynomial f, which has roots $r_1, r_2, ..., r_n$ and degree n into the polynomial g.

1) Polynomial with roots $r_1 + a, r_2 + 2, ..., r_n + a$: $g(x) = f(x - a)$
2) Polynomial with roots $ar_1, ar_2, ..., ar_n$: $g(x) = f(\frac{x}{a})$
3) Polynomial with roots $\frac{1}{r_1}, \frac{1}{r_2}, ..., \frac{1}{r_n}$: $g(x) = x^nf(\frac{1}{x})$

3) Problems:

1) (AoPS) Find the sum of the 20^{th} powers of the roots of $z^{20} - 19z + 2$.

2) (Purple Comet) Suppose a, b, and c are real numbers that satisfy $a+b+c = 5$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{5}$. Find the greatest possible value of $a^3 + b^3 + c^3$.

3) (AIME) Let r, s, t be three roots of the equation

$$8x^3 + 1001x + 2008$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

4) (PUMaC) Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given α, β, and γ, find

$$\left(\frac{1}{\alpha - 2}\right)^2 + \left(\frac{1}{\beta - 2}\right)^2 + \left(\frac{1}{\gamma - 2}\right)^2$$

5) (HMMT) The polynomial $f(x) = x^3 - 3x^2 - 4x - 4$ has three real roots r_1, r_2, and r_3. Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1, s_2, and s_3, where $s_1 = r_1 + r_2 z + r_3 z^2$, $s_1 = r_1 z + r_2 z^2 + r_3$, $s_1 = r_1 z^2 + r_2 + r_3 z$, and $z = \frac{-1+i\sqrt{3}}{2}$. Find the real part of the sum of the coefficients of $g(x)$.

6) (HMMT) Let a and b be real numbers, and let r, s, and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

7) (HMMT) Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.

8) (OMO) Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

9) (HMMT) The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}$$

10) (HMMT) How many real triples (a,b,c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, $\tan 3y$ for some real y.

11) (Me) Let a, b, and c be the roots of the polynomial $x^3 - 7x^2 + 14x - 7$. Given that $\displaystyle\sum_{n=0}^{\infty} \left(\frac{a^n}{7^n(a-1)} + \frac{b^n}{7^n(b-1)} + \frac{c^n}{7^n(c-1)} \right) = -\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.