

# Differential Equations

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# 1 Prerequisite Information

## 1.1 Vector Spaces

A vector space is a set along with two operations an addition and scalar multiplication over some field  $\mathbb{F}$  that satisfies the following:

- Closure under +
- Closure under scalar multiplication
- Associativity over +
- Commutativity over +
- Additive identity (referred to as 0)
- Additive inverses
- An identity for scalar multiplication (referred to as 1)
- Mixed associativity:  $\beta(\alpha\mathbf{v}) = (\beta\alpha)\mathbf{v}$
- Mixed distributive I:  $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$
- Mixed Distributive II:  $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

We now give a few Definitions and Theorems about vector spaces.

**Definition** (Linear map).  $L : \mathbb{V} \rightarrow \mathbb{W}$  where  $\mathbb{V}, \mathbb{W}$  are vector spaces is a Linear map if for all  $\mathbf{u}, \mathbf{v} \in \mathbb{V}$  we have  $f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  and  $a, b \in \mathbb{R}$

**Definition** (Null Space). The Null space of  $L$  is defined as the set  $\{\mathbf{u} \in \mathbb{V} | L(\mathbf{u}) = 0\}$

**Theorem 1.1.** *The Null space of any linear map  $L$  is a subspace of  $\mathbb{V}$ .*

*Proof.* We just need to show that the set is closed under + and scalar multiplication, since the fact that it is a subset of  $L$  over the same operations implies that it satisfies the other axioms. For addition:

$$\begin{aligned} L[\mathbf{u} + \mathbf{v}] &= L[\mathbf{u}] + L[\mathbf{v}] \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

For scalar multiplication:

$$\begin{aligned} L[\alpha \mathbf{u}] &= \alpha L[\mathbf{u}] \\ &= \alpha(0) \\ &= 0 \end{aligned}$$

□

**Definition** (Linearly Independent). A set of vectors  $\{v_1, v_2, \dots, v_n\}$  is considered to be linearly independent if the only tuple of scalars  $(a_1, a_2, \dots, a_n)$  that satisfies

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

is the tuple consisting solely of 0s

**Definition.** (Span) A set of vectors  $V$  spans a vector space  $\mathbb{W}$  if for all  $w \in \mathbb{W}$  there exists scalars  $a_1, a_2, \dots, a_n$  such that

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = w$$

**Definition.** (Basis) A set of vectors is a basis of  $\mathbb{W}$  if they are linearly independent and span  $\mathbb{W}$ .

**Theorem 1.2.** *All bases of a vector space have the same size.*

*Proof.* Suppose that there are multiple bases of different sizes for some vector space  $\mathbb{V}$ . Let  $(v_1, \dots, v_n)$  be the basis of the smallest size and let  $(w_1, \dots, w_k)$  be a basis of a larger size. Since  $(v_1, \dots, v_n)$  spans  $\mathbb{V}$  all elements of  $(w_1, \dots, w_k)$  can be represented as sums of scalar multiples of  $(v_1, \dots, v_n)$ . However, since there are fewer elements in  $(v_1, \dots, v_n)$ , the expression

$$a_1 w_1 + a_2 w_2 + \dots + a_n v_n$$

has more than 1 solution for  $(a_1, a_2, \dots, a_n)$ , which is a contradiction on the fact that  $(w_1, \dots, w_k)$  is linearly independent. □

**Definition.** (Dimension) The dimension of a vector space  $\mathbb{V}$  denoted  $\dim \mathbb{V}$  is the size of the bases of  $\mathbb{V}$

## 1.2 Differentiation over Complex Numbers

Let  $h = f + gi$  be function that maps reals to complex numbers where  $f, g$  are real differentiable functions. We define the derivative over complex numbers as the same way as defined for real numbers:

$$\begin{aligned}(f + gi)' &= \lim_{h \rightarrow 0} \frac{(f + gi)(x + h) - (f + gi)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + i \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \\ &= f' + g'i\end{aligned}$$

We now will also show that if  $r \in \mathbb{C}$  then,

$$\frac{d}{dt} e^{rt} = r e^{rt}$$

Let  $r = \alpha + i\beta$ . Thus

$$e^{rt} = e^{\alpha t} \cos \beta t + i e^{\alpha t} \sin \beta t$$

Differentiating using the above rule for complex differentiation gives.

$$\begin{aligned}(e^{rt})' &= (e^{\alpha t} \cos \beta t)' + i(e^{\alpha t} \sin \beta t)' \\ &= [\alpha \cos \beta t - \beta \sin \beta t] e^{\alpha t} + i[\alpha \sin \beta t + \beta \cos \beta t] e^{\alpha t} \\ &= e^{\alpha t} [(\alpha + i\beta \cos \beta t) + (-\beta + i\alpha) \sin \beta t] \\ &= e^{\alpha t} [(\alpha + i\beta) \cos \beta t + (\alpha + i\beta)i \sin \beta t] \\ &= r e^{rt}\end{aligned}$$

## 1.3 Generalized Product Rule

**Theorem 1.3.** (Generalized Product Rule)  $\left(\frac{d}{dt}\right)^{\circ l} (fg) = \sum_{j=0}^l \binom{l}{j} f^j g^{l-j}$

where  $\circ$  denotes function composition

*Proof.* Simple induction argument □

## 2 Basics of Analysis

### 2.1 Uniform Norms and Types of Convergence

**Definition.** (Uniform Norm)  $\|\phi\| := \sup_{t \in I} |\phi(t)|$  where  $\phi(t)$  is a real-valued function on the interval  $I$ .

This exists whenever  $\phi$  is bounded on  $I : \exists B \in \mathbb{R}$  s.t.  $|\phi(t)| \leq B$  for all  $t \in I$ . Then  $\|\phi\|$  is the least upper bound of  $\{\phi(t) : t \in I\} \subseteq \mathbb{R}$  (the existence of least upper bounds or suprema is an axiom - called The Completeness Axiom). If  $\phi$  is continuous and  $I$  is a closed interval, then

$$\|\phi\| := \max_{t \in I} |\phi(t)| = |\phi(t^*)|$$

where  $t^* \in I$

**Definition.** (Uniform Convergence) Given a sequence of bounded functions

$$\phi_0, \phi_1, \dots, \phi_n, \dots$$

all defined on the same interval,  $I$  and given bounded function  $\phi$  on  $I$ , we say  $\phi_n \rightarrow^U \phi$  if  $\|\phi_n - \phi\| \rightarrow 0$  as  $n \rightarrow \infty$

**Definition.** (Pointwise Convergence)  $\phi_n \rightarrow^P \phi$  means:  $\forall t \in I, \phi_n(t) \rightarrow \phi(t)$  as  $n \rightarrow \infty$

**Theorem 2.1.** *Uniform Convergence implies Pointwise Convergence but not the converse*

*Proof.* The forward statement is obvious. The fact that the converse is not true can be illustrated with  $I = [0, 1]$  and  $\phi_n = t^n$  as it converges pointwise to  $\phi(t) \equiv 0$  but not uniformly.  $\square$

**Theorem 2.2.** *If  $\phi$  is bounded and is the uniform limit of a sequence of continuous functions  $\phi_n$  on closed interval  $I$ , then  $\phi$  must be continuous.*

*Proof.* Pick any  $\epsilon > 0$ : we want to show that  $\exists \delta > 0$  s.t.  $|\phi(t) - \phi(t^*)| < \epsilon$  if  $|t - t^*| < \delta$

$$\begin{aligned} |\phi(t) - \phi(t^*)| &= |\phi(t) - \phi_n(t) + \phi_n(t) - \phi_n(t^*) + \phi_n(t^*) - \phi(t^*)| \\ &\leq |\phi(t) - \phi_n(t)| + |\phi_n(t) - \phi_n(t^*)| + |\phi_n(t^*) - \phi(t^*)| < \epsilon \end{aligned}$$

pick  $N$  big enough so that  $\|\phi_n - \phi\| < \frac{\epsilon}{3}$  for all  $n \geq N$ . Since  $\|\phi_n - \phi\| < \frac{\epsilon}{3}$  is the supremum, we must have the above equation.  $\square$

## 2.2 Metric Spaces and Cauchy Sequences

**Definition.** (Metric Space) A nonempty set  $X$  together with a distance function (metric)  $d : X \times X \rightarrow \mathbb{R}$  satisfying the axioms

- $d(x, y) = d(y, x)$  for all  $x, y \in \mathbb{R}$

- $d(x, y) \geq 0$  and  $d(x, y) = 0$  iff  $x = y$  for all  $x, y \in X$ .
- $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ .

**Definition.** (Cauchy Sequence) In a metric space, an infinite sequence  $x_n$  is said to be cauchy if for any  $\epsilon > 0$   $\exists N \in \mathbb{N}$  s.t. for any  $n, m \geq N$  we have  $d(x_n, x_m) < \epsilon$ .

**Definition.** In a metric space a sequence converges to a limit  $x \in X$  if for any  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,  $d(x_n, x) < \epsilon$ .

**Theorem 2.3.** *Every convergent sequence in a metric space must be cauchy.*

*Proof.* Basic argument using  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ .  $\square$

**Definition.** A metric space  $(X, d)$  is complete if every cauchy sequence in  $X$  converges to a limit in  $X$ .

**Example 2.2.1.** The reals under the standard metric  $|x - d|$  is complete, but the rationals under the same metric is not.

### 2.3 Convergence

**Lemma 2.4.** *Every sequence of real numbers has at least one monotone subsequence.*

**Lemma 2.5.** *Every cauchy sequence of the reals is automatically bounded.*

*Proof.* Direct result of definition of cauchy sequence.  $\square$

**Lemma 2.6.** *Every sequence of reals that is bounded has a convergent subsequence.*

**Lemma 2.7.** *If any subsequence of a cauchy sequence converges to  $a \in \mathbb{R}$ , then the whole sequence converges to  $a$ .*

**Theorem 2.8.** *The space of continuous functions  $C^0[a, b]$  on an interval is a complete metric space under the uniform norm  $d(x, y) = \|x - y\| = \sup_{t \in [a, b]} |x(t) - y(t)|$ .*

*Proof.* Take a cauchy sequence  $\phi_n$  in  $C^0[a, b]$ . Pick any  $t \in [a, b]$  and look at the sequence  $\phi_n(t)$ . We claim this sequence is cauchy. For any  $\epsilon > 0$ , there is value  $N$  s.t. for any  $n, m \geq N$  we have  $|\phi_n(t) - \phi_m(t)| < \epsilon$  since  $|\phi_n(t) - \phi_m(t)| \leq \|\phi_n - \phi_m\| < \epsilon$ . Since every cauchy sequence of reals converges to a real limit define  $\phi(t) := \lim_{n \rightarrow \infty} \phi_n(t) \in \mathbb{R}$ . We have a new

function  $\phi : [a, b] \rightarrow \mathbb{R}$ . We see by the definition of  $\phi$  that  $\phi_n$  converges pointwise to  $\phi$  we need to show that it converges universally. Once we have this, by an earleir theorem  $\phi$  is continuous. Now fix  $\epsilon > 0$ . By cauchyness,  $\exists N$  s.t.  $\|\phi_n - \phi_N\| < \frac{\epsilon}{2}$  for all  $n \geq N$ . We must have  $|\phi(t) - \phi_N(t)| \leq \frac{\epsilon}{2}$  (Why?). This implies that  $\sup_{t \in [a, b]} |\phi(t) - \phi_N(t)| \leq \frac{\epsilon}{2}$ . By the triange inequality we now have

$$\begin{aligned} \|\phi_n - \phi_N\| &\leq \|\phi_n - \phi_N\| + \|\phi_N - \phi\| \\ &< \epsilon \end{aligned}$$

Thus  $\phi_n$  converges uniformly.  $\square$

### 3 Existence and Uniqueness

In this section we'll be focusing on the Picard Theorem, which is the main theorem regarding the existence and uniqueness of differential equations.

#### 3.1 Basic Form of Picard Theorem

**Theorem 3.1.** (*Picard Theorem*) Let  $F(t, x)$  be a function defined in some rectangle  $R = [a, b] \times [c, d]$ . Suppose  $F$  is continuous, and  $F_x = \frac{\partial}{\partial x} F$  is continuous throughout  $R$ . Then for any  $(t_0, x_0)$  with  $(t_0, x_0) \in \text{int}R = (a, b) \times (c, d)$ , there exists exactly one function  $x(t)$  on a maximal domain  $[t_0 - h, t_0 + h] \subset [a, b]$  s.t.  $x'(t) \equiv F(t, x)$  for all  $t \in [t_0 - h, t_0 + h]$  and  $x(t_0) = x_0$ .

*Proof.* Essentially over some interval and for a given  $F$  we have to have the I.V.P.:

$$\begin{aligned} x' &= F(t, x) \\ x(t_0) &= x_0 \end{aligned}$$

These equations imply that  $x(t)$  is differentiable and therefore continuous. This then implies that  $F(t, x(t))$  is integratable. Integrating it gives,

$$x(t) - x(t_0) = \int_{t_0}^t F(\tau, x(\tau))d\tau$$

Thus the I.V.P. imply that  $x$  is continuous and the following equation equa-tion:

$$x(t) = x_0 + \int_{t_0}^t F(\tau, x(\tau))d\tau \quad \forall t \in [t_0 - h, t_0 + h]$$

We can then use this new set of information to arrive back at the I.V.P. by differentiating it. Thus the new set of information is equivalent to the I.V.P.  
Now define the operator

$$T[x](t) := x_0 + \int_{t_0}^t F(\tau, x(\tau)) d\tau$$

Let  $M = \max_{(t,x) \in \mathbb{R}} |F(t, x)| \geq 0$  and let  $K = \max_{(t,x) \in \mathbb{R}} |\frac{\partial}{\partial x} F(t, x)| \geq 0$ . By the extreme value theorem these exist. Now define  $\phi_0(t) := x_0$  for all  $t \in \mathbb{R}$  and  $\phi_{n+1}(t) := T[\phi_n](t)$  for all  $t \in [t_0 - h, t_0 + h]$  where  $h$  is to be determined. Furthermore let  $2q$  be the  $x$  length of  $R$  we are working in and let  $2p$  be the  $y$  length. We have

$$\begin{aligned} |\phi_{n+1}(t) - x_0| &= \left| \int_{t_0}^t F(\tau, \phi_n(\tau)) d\tau \right| \\ &\leq \left| \int_{t_0}^t |F(\tau, \phi_n(\tau))| d\tau \right| \\ &\leq M \int_{t_0}^t 1 d\tau \\ &\leq Mh \end{aligned}$$

We know that  $|\phi_{n+1}(t) - x_0| \leq q$  by induction so to bound  $h$  suppose  $h \leq \frac{q}{M}$ . We also clearly have  $h \leq p$ . We now want the following to be true

$$||T[\phi] - T[\psi]|| \leq k ||\phi - \psi||$$

For some  $k \in (0, 1)$  where  $\psi \in C^0[t_0 - h, t_0 + h]$  We have

$$\begin{aligned} |T[\phi](t) - T[\psi](t)| &= \left| \int_{t_0}^t F(\tau, \phi(\tau)) - F(\tau, \psi(\tau)) d\tau \right| \\ &\leq \left| \int_{t_0}^t |F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))| d\tau \right| \end{aligned}$$

By the mean value theorem we get

$$\left| \int_{t_0}^t |F(\tau, \phi(\tau)) - F(\tau, \psi(\tau))| d\tau \right| = \int_{t_0}^t \left| \frac{\partial}{\partial x} F(\tau, \theta) \right| |\phi(\tau) - \psi(\tau)| d\tau$$

and we want the RHS of the above is less than or equal to  $k ||\phi - \psi|| h$  which will give us  $h < \frac{1}{K}$ . So let  $h \min(p, \frac{q}{M}, \frac{1}{2k})$ .  $\square$

### 3.2 Vector Form of Picard Theorem

**Theorem 3.2.** (*Picard Theorem*) Let  $\mathbf{F}(t, \mathbf{x}) = F(t, x_1, x_2, \dots, x_n)$  be continuous and have continuous partial derivatives with respect to  $x_1, x_2, \dots, x_n$  in some box  $B = [a, b] \times [c_1, d_1] \times \dots \times [c_n, d_n] \subset \mathbb{R}^{n+1}$ . Then for any  $(t_0, \mathbf{x}_0) = (t_0, x_{1|0}, x_{2|0}, \dots, x_{n|0}) \in \text{int}B$ , there is exactly one function  $\mathbf{x}(t)$  defined on a maximal domain  $[t_0 - h, t_0 + h]$  s.t.  $\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$  for all  $t \in [t_0 - h, t_0 + h]$  and  $\mathbf{x}(t_0) = \mathbf{x}_0 = \langle x_{1|0}, x_{2|0}, \dots, x_{n|0} \rangle$ .

*Proof.* We are looking at the Initial Value Problem:

$$\begin{aligned}\mathbf{x}' &= \mathbf{F}(t, \mathbf{x}) \\ \mathbf{x}(t_0) &= \mathbf{x}_0\end{aligned}$$

For all  $t$  in some interval  $I$  define

$$\begin{aligned}\phi(t) &= \langle \phi_1(t), \dots, \phi_n(t) \rangle \\ \|\phi(t)\| &= \sqrt{\phi_1(t)^2 + \dots + \phi_n(t)^2} \\ \|\phi\|_I &= \sup_{t \in I} \|\phi(t)\| < \infty\end{aligned}$$

$\phi$  is bounded (on  $I$ ) if all its component functions  $\phi_1, \dots, \phi_n$  are bounded on  $I$  (Assume this). Let

$$\begin{aligned}\phi_0(t) &\equiv x_0 \\ \phi_{n+1}(t) &\equiv x_0 + \int_{t_0}^t \mathbf{F}(\tau, \phi_n(\tau)) d\tau\end{aligned}$$

Assume  $\mathbf{F}$  is continuous on a box  $B$  in  $\mathbb{R}^{n+1}$  centered at  $(t_0, \mathbf{x}_0)$ .

$$\max_{(t, \mathbf{x}) \in B} \|\mathbf{F}\| = M \geq 0.$$

Let the box  $B$  have dimensions  $2p \times 2q \times 2q \times \dots \times 2q$ . Assume  $|\phi_{k|j} - \phi_{j|0}| \leq q$  for all  $j \in \{1, 2, \dots, n\}$ . This gives  $\|\phi_k(t) - \mathbf{x}_0\| \leq \sqrt{n}q$ . We now want to get the same fact for further cases via induction. We have,

$$\begin{aligned}\|\phi_{k+1}(t) - \mathbf{x}_0\| &= \left\| \int_{t_0}^t \mathbf{F}(\tau, \phi_k(\tau)) d\tau \right\| \\ &\leq \left| \int_{t_0}^t \|\mathbf{F}(\tau, \phi_k(\tau))\| d\tau \right| \\ &\leq Mh \\ &\leq q\end{aligned}$$

(In the last step we choose  $h \leq \frac{q}{M}$ ). This now implies  $|\phi_{k+1|j}(t) - x_{0|j}| \leq ||x_{k+1}(t) - \mathbf{x}_0|| \leq q$ . Now consider  $\frac{\partial}{\partial \mathbf{x}} \mathbf{F}$  (the Jacobian Matrix). We'll set up our recursive equations as follows:

$$\begin{aligned}\phi_0(t) &\equiv x_0 \\ \phi_{n+1}(t) &\equiv x_0 + \int_{t_0}^t F(\tau, \phi_n(\tau)) d\tau\end{aligned}$$

We pick  $h = \min(p, \frac{q}{M}, \frac{1}{2n^2K}) > 0$ . We want to show by induction that  $|\phi_{k|j} - x_{0|j}| \leq q$ . The base case comes from our choice of  $h$  and for the inductive step:

$$\begin{aligned}|\phi_{k+1|j} - x_{0|j}| &= \left| \int_{t_0}^t F_j(\tau, \phi_k(\tau)) d\tau \right| \\ &\leq Mh \\ &\leq q\end{aligned}$$

...

□

## 4 Linear Equations with Constant Coefficients

### 4.1 Definition

In this section we will focus on solving Linear Equations with Constant Coefficients. That is equations of the form:

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_2x'' + a_1x' + a_0x = f(t)$$

for real constants  $a_{n-1}, \dots, a_0$ . For the first 2 sections we will focus on solving homogeneous equations (i.e. those with  $f(t) = 0$ ).

### 4.2 Homogeneous of order 2

We will begin by defining some notation to be used later on.

**Definition** (Operator).  $f$  is a function that maps ordinary functions to other functions.

We are dealing with equations of the form  $x'' + ax' + bx = 0$  where  $(a, b \in \mathbb{R})$ . Let  $L[x] = x'' + ax' + b$  be a linear differential operator. We can view  $L$  as a linear map on the vector space  $\mathbb{D}^2(I)$ : the set of all twice-differentiable ordinary functions with domain  $I$ . We want to find the null

space of  $L$ . By the Picard Theorem, the dimension of this vector space is 2, so if we can find two linearly independent solutions, we can easily find all solutions.

Furthermore let  $t \rightarrow x(t) \in \mathbb{D}^2$ . The latter equation implies

$$\begin{aligned} L[x] &= \frac{d^2}{dt^2}x(t) + a\frac{d}{dt}x(t) + bx(t) \\ L &= \frac{d^2}{dt^2} + a\frac{d}{dt} + b(Id) \end{aligned}$$

Last class we took  $L[e^{rt}] = p_L(r)e^{rt}$  and got  $p_L(r) = r^2 + ar + b$ . We will try to use the roots of  $p_L(r)$  to find two solutions to the desired equation. Choose  $r = r_1$  a root of  $p_L$  then  $x_1 = e^{r_1 t}$  is a solution to  $L[x_1] = p_L(r_1)e^{r_1 t} \equiv 0$ . This gives  $x_2(t) = e^{r_2 t}$  where  $p_L(r_2) = 0$ .

Now our methods have only given us real solutions. Now we attempt to find all solutions:

- If  $(a^2 > 4b)$ , The roots are real and distinct, say  $r_1 > r_2$

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

We get two linearly independent solutions,  $x_1(t) = e^{r_1 t}$ ,  $x_2(t) = e^{r_2 t}$ . Since the solutions of  $L[x] = 0$  forms a vector space of dimension 2, all solutions must be of the form:  $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$  for all  $(c_1, c_2 \in \mathbb{R})$ .

- If  $(a^2 = 4b)$ , so  $r_1 = r_2 = \frac{-a}{2} \in \mathbb{R}$ . Now we only have one solution, so we need a second root to find all solutions.

*Fact.* If  $r_1$  is a repeated root of  $p_L(r)$ , then  $r_1$  is also a root of  $\frac{d}{dr}p_L(r)$ .

We will now use this fact to find another solutions. We have

$$L[e^{rt}] \equiv_{r,t} p_L e^{rt}$$

Taking the partial derivative with respect to  $r$

$$\begin{aligned} \frac{\partial}{\partial r} L[e^{rt}] &\equiv \frac{\partial}{\partial r} [p_L(r)e^{rt}] \\ \frac{\partial}{\partial r} \circ L &= \frac{\partial}{\partial r} \circ \left[ \frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} + b(Id) \right] \\ &= \frac{\partial}{\partial r} \circ \left( \frac{\partial}{\partial t} \circ \frac{\partial}{\partial t} \right) + a \left[ \frac{\partial}{\partial r} \circ \frac{\partial}{\partial t} \right] + b \left[ \frac{\partial}{\partial r} \circ Id \right] \end{aligned}$$

where  $\circ$  denotes function composition and  $Id$  denotes the identity function. Since function composition is associative and we can switch the order in how we partial differentiate, this is equivalent to

$$L \circ \frac{\partial}{\partial r} = \left( \frac{\partial}{\partial t} \circ \frac{\partial}{\partial t} \right) \circ \frac{\partial}{\partial r} + a \left[ \frac{\partial}{\partial t} \circ \frac{\partial}{\partial r} \right] + b \left[ Id \circ \frac{\partial}{\partial r} \right]$$

Now going back to  $L[e^{rt}] \equiv_{r,t} p_L e^{rt}$  and applying this idea gives us

$$L[te^{rt}] \equiv \left( \frac{\partial}{\partial r} p_L(r) + tp_L(r) \right) e^{rt}$$

Based on the above fact,  $r_1$  is a root of both  $\frac{\partial}{\partial r} p_L(r), p_L(r)$ . Thus taking  $r = r_1$  gives us  $L[te^{r_1 t}] \equiv 0$ . Now we have two linearly independent solutions, so based upon the above give all solutions are of the form

$$x(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

for all  $(c_1, c_2 \in \mathbb{R})$

- If  $a^2 < 4b$ , then

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where  $\alpha = \frac{-a}{2} \in R$ . We also have

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

We know

$$\begin{aligned} z(t) &= e^{r_1 t} \\ &= e^{\alpha t} (\cos \beta t + i \sin \beta t) \end{aligned}$$

Is a complex root of  $p_L(r)$  and therefore a complex solution to the equation. Now let

$$\begin{aligned} x(t) &= e^{\alpha t} (\cos \beta t) \\ y(t) &= e^{\alpha t} (\sin \beta t) \end{aligned}$$

. Now we have

$$\begin{aligned} L[z(t)] &\equiv 0 \\ L[x(t) + iy(t)] &\equiv 0 \\ L[x] + iL[y] &\equiv 0 \end{aligned}$$

Since both  $x, y$  are real-valued, both  $L[x], L[y]$  are real valued. This implies that

$$L[x] \equiv L[y] \equiv 0$$

Thus both  $e^{\alpha t}(\sin \beta t)$  and  $e^{\alpha t}(\cos \beta t)$  are solutions which implies the general solution is  $L \equiv 0$  of the form  $c_1 e^{\alpha t}(\sin \beta t) + c_2 e^{\alpha t}(\cos \beta t)$

### 4.3 Homogeneous of order $\geq 3$

We will now give a basic idea of how to find  $n$  fundamental real-valued solutions of a linear n-th order homogeneous equation with constant coefficients. Let  $L : \mathbb{D}^n(I) \rightarrow \mathbb{F}(I)$  where  $I$  is a  $t$ -interval be defined as:

$$L[x] = x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x$$

We want to find all solutions to  $L[x] \equiv 0$ , since  $L$  is a linear map this is equivalent to finding  $\text{null } L$ . By the Picard theorem, the dimension of  $\text{null } L$  is  $n$ . As a result we if we can find a set  $W$  of  $n$  linearly independent solutions to the equation, then the set  $\text{span } W$ . We will now proceed to find  $n$  such solutions. Let  $p_L(r)$  be defined such that  $L[e^{rt}] = p_L(r)e^{rt}$ . As before this gives us:

$$\begin{aligned} p_L(r) &= r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \\ &= (r - \rho_1)^{m_1} \cdots (r - \rho_R)^{m_R} ((r - \alpha_1)^2 + \beta_1)^{\mu_1} \cdots ((r - \alpha_c)^2 + \beta_c)^{\mu_c} \end{aligned}$$

where the  $\rho_k$  is a real root and  $\alpha_k + i\beta_k$  is a complex root. For each real root  $\rho_j$  contributes exactly  $m_j$  fundamental solutions which will be:

$$e^{\rho_j t}, \quad te^{\rho_j t}, \quad t^2 e^{\rho_j t}, \dots t^{m_j} e^{\rho_j t}$$

Each pair of conjugate roots will contribute exactly  $2\mu_j$  fundamental solutions which will be:

$$\begin{aligned} e^{\alpha_j t} \cos \beta_j t, \quad te^{\alpha_j t} \cos \beta_j t, \dots, t^{\mu_j} e^{\alpha_j t} \cos \beta_j t \\ e^{\alpha_j t} \sin \beta_j t, \quad te^{\alpha_j t} \sin \beta_j t, \dots, t^{\mu_j} e^{\alpha_j t} \sin \beta_j t \end{aligned}$$

We can show that the first solution in each set can be obtained using the same logic from section 3.2 and then the others can be obtained by repeatedly differentiating  $L[t^k e^{rt}]$  and using the fact that a root of multiplicity  $m$  is the root of  $p_L(r)$  and its first  $m - 1$  derivatives.

Now in order to show that these span the set of all solutions, we need to show that all these solutions are linearly independent. Suppose there was a nontrivial linear relation between the fundamental solutions given above. The equation

$$c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) = 0$$

then has a solution where not all of the  $c_i$  are 0. Based on what we had as the roots, the expression is equivalent to

$$\sum_{j=1}^R p_j(t) e^{\rho_j t} + \sum_{j=1}^C q_j(t) e^{(\alpha+i\beta_j)t} \equiv 0$$

For some polynomials  $p_i, q_i$  for all  $i$  with fixed degree. Now let

$$\delta := \max\{\rho_1, \dots, \rho_R, \alpha_1, \dots, \alpha_C\}$$

and divide all the terms by  $e^{\delta t}$ :

$$\begin{aligned} \sum_{j=1}^R p_j(t) e^{(\rho_j - \delta)t} + \sum_{j=1}^C q_j(t) e^{(\alpha+i\beta_j - \delta)t} &\equiv 0 \\ P(t) + \sum_{j:a_j=\delta} q_j(t) e^{i\beta_j t} &\equiv - \sum_{j:\rho_j<\delta} e^{(\rho_j - \delta)t} \end{aligned}$$

The RHS will go to 0 as  $t$  goes to infinity, while the LHS will not. Therefore we have a contradiction as desired. The given solutions then span the set of all solutions.

#### 4.4 General Solution

#### 4.5 Problems

1. Find the general solution of  $x'' + 3x' - 4x = 0$
2. Find the general solution of  $x'' + 3x' + 4x = 0$
3. Find the general solution of  $x'' - 6x' + 9x = 0$

## 5 Answers To Problems

### 5.1 Section 3

1.  $c_1 e^{-4rt} + c_2 e^t$  for constants  $c_1, c_2$
2.  $e^{-\frac{3t}{2}} (c_1 \cos \frac{\sqrt{7}t}{2} + c_2 \sin \frac{\sqrt{7}t}{2})$  for constants  $c_1, c_2$
3.  $e^{3t}(c_1 + c_2 t)$  for constants  $c_1, c_2$