# Expanded Dynamic Games: Allowing for Simultaneous Moves

A defining feature of the kind of dynamic game we've studied so far, dynamic games of perfect information, is that players' actions are **never** taken simultaneously—all actions are sequential.

We are ready to graduate to a more general concept of a dynamic game, in which some actions by players **can** be chosen simultaneously.

Of course, not all actions are simultaneous, or else it would be a regular simultaneous-move game, not a dynamic game.

Allowing for simultaneous moves will require us to change some of the formal elements that define a dynamic game.

To review, a dynamic game of perfect information is defined by:

--the set of players

--the set of outcomes, defined as sequences of actions that take the game from its beginning to its completion

--a player function, which tells us which player is making the next choice at any proper subhistory

--players' payoffs at each outcome

Allowing simultaneous moves will not require changing the notion of a player or a payoff, but **will** require changing the notion of an outcome and a player function.

# Adjustment to outcomes

With no simultaneous actions, an outcome is interpreted as a set of actions that were taken in sequence.  For instance, if XYZX is an outcome of a dynamic game of perfect information, that means that one way for the game to be played is, “the player who goes first chooses action X, then a player chooses action Y, then a player chooses action Z, then a player chooses action X, at which point the game ends.”

# Adjustment to outcomes

With the possibility of simultaneous moves, outcomes are still lists of actions, and it is still true that if one action is taken before another, the first action appears before the second in the list.  However, some actions in the list may have been taken simultaneously.

In this case, we indicate this in our list by grouping actions taken simultaneously together, using some kind of notation.

# Adjustment to outcomes

For instance, we might group actions taken simultaneously in parentheses.

As an example, suppose an outcome of a dynamic game is X(XYX)(XZ). We would interpret that as meaning, "in this outcome, the player who went first chose action X, with no other player choosing anything simultaneously; after that, three players made choices simultaneously, with two of those players choosing action X and the third choosing action Y; then two players made choices simultaneously, with one of them choosing action X and the other choosing action Z; then the game ended."

# Adjustment to outcomes

Allowing these groupings of consecutive actions to show simultaneity is the only change we need to make to our notion of outcomes.

# Adjustments to the Player Function

We also need to change the structure of the player function when the dynamic game allows some simultaneous moves.

One reason is that the structure of histories of the game is different.

# Adjustments to the Player Function

The idea of a history of the game is that it is any record of actions that could be taken to start a play of the game, up to a certain point, but not necessarily to finish it.

In a dynamic game of perfect information, that is simply any sequence of actions that appears at the beginning of an outcome.

Going back to our prior example, if XYZX is an outcome of a dynamic game, then X, XY, and XYZ are all histories of the game (and $\emptyset$, of course).

# Adjustments to the Player Function

With simultaneous moves, a history can't start when only some of those moves have been made, and others haven't. Either the entire set of simultaneous actions have been chosen, or none of them has been chosen.

Going back to the example X(XYX)(XZ), X is a history (the player who moved first chose X, but nothing else has happened yet), and X(XYX) is a history (the player who moved first chose X, then the three players who moved simultaneously after that chose X, Y and X). But, for example, XXY (or X(XY)) is **not** history—there is no point in time at which the set of actions taken so far in the game consists of just two Xs and one Y.

# Adjustments to the Player Function

In terms of generating the set of histories from the set of outcomes, we can no longer identify histories just by "breaking off" the ends of outcomes, as we did for dynamic games of perfect information. We can break off immediately before or immediately after a set of grouped simultaneous actions, but not in the middle of that set.

# Adjustments to the Player Function

With this change, we continue to call any history of the game that is not an outcome a proper subhistory. At any proper subhistory, the game is still being played. In particular, someone is about to make the next choice.

In a dynamic game of perfect information, that someone is always a single player. But with the possibility of simultaneous moves, that someone might be **multiple** players, all moving simultaneously.

# Adjustments to the Player Function

Mathematically, this means that the output (or image) of the player function is not expressed as a single player, but as a **set** of players, defined as the players who simultaneously move next. Of course, this set might contain only one player, meaning that that player moves by herself.

Math note: since the output of this player function is a set, rather than a single player, it is formally not a function, but a correspondence. If you've never heard of that term, don't worry about it; I will continue to call it the "player function" despite this technicality.

# Adjustments to the Player Function

Returning to our outcome X(XYX)(XZ), we said that X and X(XYX) are proper subhistories, as well as the null history Ø. A possible specification of the player function of the game might include

P(Ø)={1}

P(X)={1,2,3}

P(X(XYX))={1,3}

We would interpret this as, "at the beginning of the game, player 1 moves by herself; at the history at which player 1 chose X, players 1, 2 and 3 all move simultaneously; at the history at which player 1 chose X, then players 1, 2 and 3 chose X, Y and X simultaneously, players 1 and 3 move simultaneously."

# Adjustments to the Player Function

One note: in a dynamic game of perfect information, actions have to be put in the order in which they are taken. But with actions that are taken simultaneously can be written in any order within their grouping.

At a history at which multiple players move simultaneously, we should be specific about the way we are ordering those players' actions.

For instance, in our X(XYX)(XZ) example, we might specify, "at history X, the three simultaneous actions that come next will be ordered with 1's action first, then 2's then 3's." Then we would know that (XYX) means that 1 and 3 choose X, and 2 chooses Y. Whereas, (YXX) would mean that 1 chooses Y, and 2 and 3 choose X.

# Finding SPNE of dynamic games with simultaneous moves

The definition of a subgame is the same in dynamic games with simultaneous moves as in dynamic games of perfect information: at any proper subhistory, there is a subgame that begins at that proper subhistory.

Our method of finding subgame perfect Nash equilibria is also similar: we perform backward induction by first analyzing subgames at the “end” of the game, then working our way back toward the beginning.

In a dynamic game of perfect information, a subgame at the “end” means a subgame in which the only thing that happens is that one player chooses one action, and then the game ends.

With the possibility of simultaneous moves, a subgame at the “end” means a subgame that consists only of single **round** of moves, after which the game ends.  However, a round might consist of **simultaneous actions by multiple players**.  That is, it is not a one-player subgame, but a multiplayer simultaneous-move subgame.

# Example

Here is a dynamic game with simultaneous moves, and two players. In this game, player 1 moves first, by herself, and chooses action X or action Y. If she chooses action X, the game ends. If she chooses action Y, then players 1 and 2 next choose actions simultaneously, with 1 choosing either action U or action D, and 2 choosing either action L or action R.

Payoffs will be specified later.

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What is the nature of that subgame?

It’s just a two-player simultaneous-move game, like we’re used to.

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Any way that can help us!

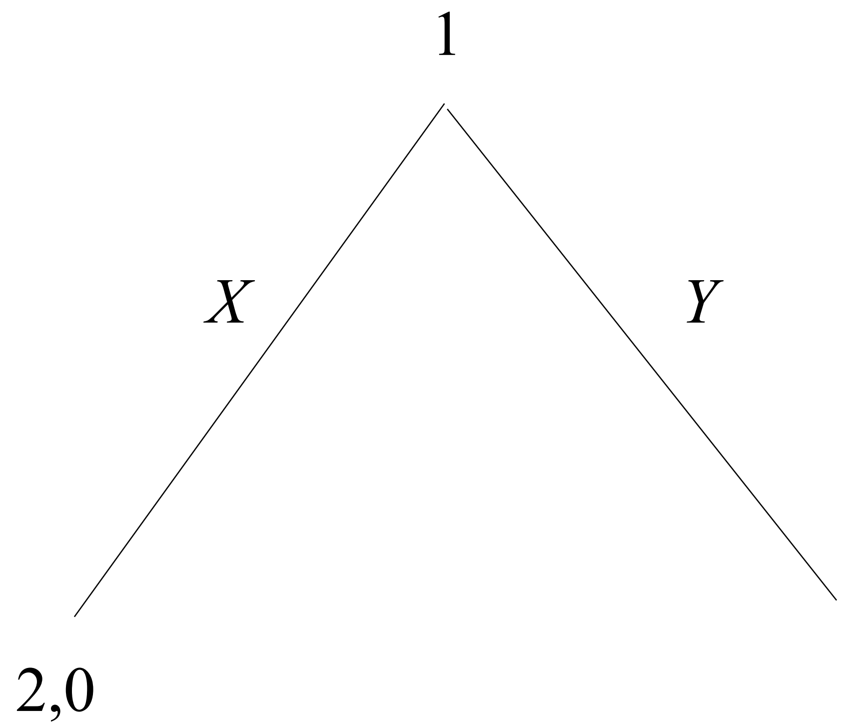
# In our example, we know that the proper subgame is a simultaneous-move game between two players:

Player 2

Player 1

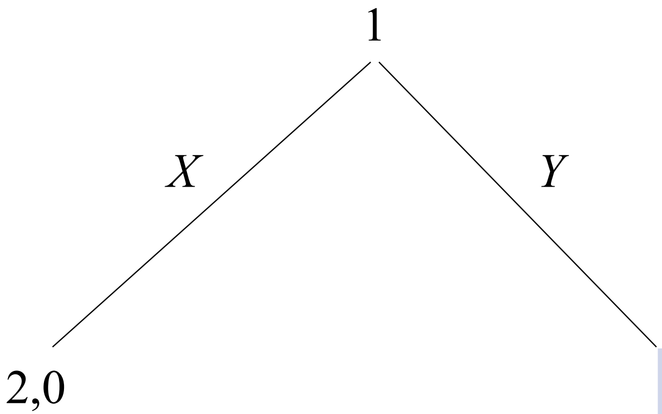
| | $L$ | $R$ |
|---|---|---|
| $U$ | 3,1 | 0,0 |
| $D$ | 0,0 | 1,3 |

# But, that game starts with one player making a choice:



# Can we combine them?

# Sure, why not

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1

| | $L$ | $R$ |
|---|---|---|
| $U$ | 3,1 | 0,0 |
| $D$ | 0,0 | 1,3 |

“Graphs, with some nodes consisting of matrices,” is not an everyday mathematical construction.

But, the only reason not to use some representation is if it doesn’t help us analyze the game. This one probably does.

To find SPNE using backward induction, we start with game's one proper subgame. What are the Nash equilibria?

Player 2

| | $L$ | $R$ |
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Player 1

To find SPNE using backward induction, we start with game's one proper subgame. What are the Nash equilibria?

In pure strategies, $(U,L)$ and $(D,R)$.

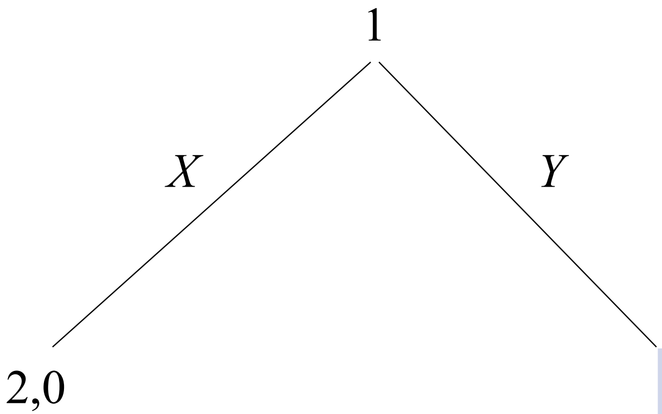
In mixed strategies, there is one in which player 1 plays $U$ with probability ¾, and player 2 plays $L$ with probability ¼. Player 1's expected payoff in this Nash equilibrium is ¾.

In dynamic games of perfect information, the only time we ran into the issue of more than one Nash equilibrium in a subgame was when a player was indifferent over multiple outcomes, and had multiple optimal choices.

But, once there can be simultaneous moves in a dynamic game, a subgame is liable to have more than one Nash equilibrium as a general matter, not as a special case.

Just like in games of perfect information, when this happens, we have to perform backward induction separately for **each** of the Nash equilibria of a subgame.

So, here, we'll start with an SPNE in which the players play ($U$,$L$) at the subgame that begins at $Y$. What will player 1 do at the beginning of the game?

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| | | *L* | *R* |
|---|---|---|---|
| 1 | *U* | 3,1 | 0,0 |
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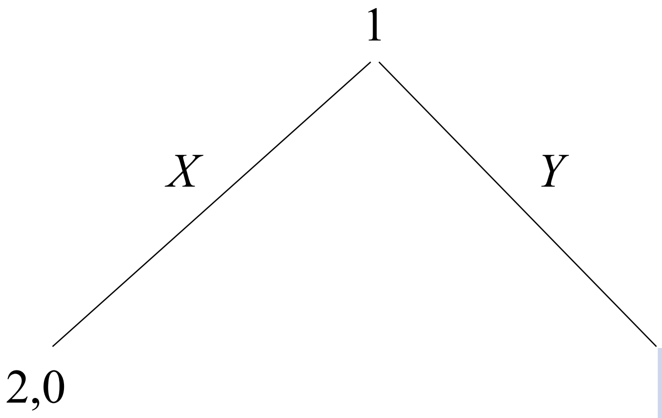
Player 1 uses strategy $YU$, player 2 uses strategy $L$.

Next, if the players play $(D,R)$ at the subgame that begins at $Y$, what will player 1 do at the beginning of the game?

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| 1 | | L | R |
|---|---|---|---|
| | U | 3,1 | 0,0 |
| | D | 0,0 | 1,3 |

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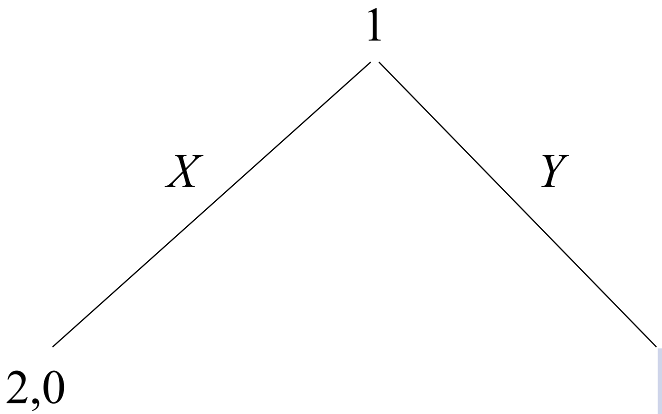
Player 1 uses strategy $XD$, player 2 uses strategy $R$.

Finally, if the players play $(\sigma_1(U) = \frac{3}{4}, \sigma_2(L) = \frac{1}{4})$ at the subgame that begins at $Y$, what will player 1 do at the beginning of the game?

Choose $X$. So, what is the SPNE?

Player 1 uses strategy $X; \sigma_1(U) = \frac{3}{4}$, player 2 uses strategy $\sigma_2(L) = \frac{1}{4}$.

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Choose *X*. So, what is the SPNE?

Player 1 uses strategy $X; \sigma_1(U) = \frac{3}{4}$, player 2 uses strategy $\sigma_2(L) = \frac{1}{4}$.