ON PICARD GROUPS AND JACOBIANS OF DIRECTED GRAPHS

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ABSTRACT. The Picard group of a graph is a finitely generated abelian group, and the Jacobian is the torsion subgroup of the Picard group. These groups can be computed by using the Smith normal form of the Laplacian matrix of the graph or by using chip-firing games associated with the graph. One may consider its generalization to directed graphs by using Laplacian matrices. In this paper, we compute Picard groups and Jacobians for several classes of directed graphs (trees, cycles, and wheel graphs).

1. Introduction

In this paper, we compute Picard groups and Jacobians for several classes of directed graphs (trees, cycles, wheel graphs), which include undirected graphs by seeing them as all edges being bidirectional. Our proof is purely algebraic, based on explicit computations of Smith normal forms of Laplacian matrices of directed graphs. Briefly, we prove the following:

(1) For a tree T, the Jacobian $\mathrm{Jac}(T)$ is trivial for any orientation although the rank of $\mathrm{Pic}(T)$ does not have to be one.
(2) For cycles C_n with n vertices ($n \geq 3$) and for any $0 \leq k \leq n$, we construct an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_k$.
(3) For wheel graphs W_n with n vertices, we compute two specific orientations (not all edges are bidirectional), generalizing the undirected case in [Big99].

We further provide some computations results and conjectures for directed multipartite graphs.

A chip-firing game is a combinatorial game that one can play on finite graphs. To play a chip-firing game, one first puts chips (possibly negative as "debt") at each vertex of a graph G. At each turn, a vertex can borrow / lend chips from / to all neighbors simultaneously. The goal of the game is to find a finite sequence of borrowing / lending moves so that every vertex is debt-free. Hence, a natural question to ask is whether or not one can determine there is a winning moves for a given configuration of chips.

To study the game more systemically, one can write any chip configuration as an element of the free abelian group generated by the set of vertices $V(G)$ of a graph G which is denoted by $\mathrm{Div}(G)$. Each element of $\mathrm{Div}(G)$ is called a *divisor*, and a divisor D is *effective* if $D = \sum_{v \in V(G)} a_v v$, with $a_v \geq 0$. Then, one defines an equivalence relation on $\mathrm{Div}(G)$ as follows: $D \sim D'$ if and only if D' can be obtained from D by a finite sequence of borrowing / lending moves. This defines a group

$$\mathrm{Pic}(G) := \mathrm{Div}(G)/\sim,$$

called the *Picard group* of G. The torsion subgroup of $\mathrm{Pic}(G)$, denoted by $\mathrm{Jac}(G)$, is called the *Jacobian* of G.[1]

The combinatorial theory of chip-firing games has connections to various areas in mathematics. For instance, from a perspective motivated by algebraic geometry, one may view finite graphs as a discrete model for Riemann surfaces. In this case, chip configurations play the role of divisors on

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[1] Depending on literature, $\mathrm{Jac}(G)$ is also called as a critical group.

curves. We refer the reader to [**?**] for divisor theory on graphs. Briefly speaking, Baker and Norine [BN07] formulated and proved a version of Riemann-Roch theorem for finite graphs as follows:

$$r(D) - r(K-D) = \deg(D) - g + 1, \tag{1}$$

where

$$g = |E(G)| - |V(G)| + 1, \quad K = \sum_{v \in V(G)} (\deg(v) - 2)v \in \mathrm{Div}(G)$$

and $\deg(D)$ for D in $\mathrm{Pic}(G)$ is the total number of chips, and the rank $r(D)$ is one less than the minimum number of chips which need to be removed so that D is no longer equivalent to an effective divisor. When G is a dual graph of a strongly semistable model for an algebraic curve X (smooth, proper, geometrically connected) over a valued field, one also has a (degree-preserving) group homomorphism $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$.

For a graph G with the vertices $\{v_1, v_2, \ldots, v_n\}$, one defines an $n \times n$ matrix L_G, called the Laplacian matrix of G, as follows:

$$(L_G)_{ij} = \begin{cases} \text{valency of } v_i, & i = j \\ -(\text{\# of edges between } v_j \text{ and } v_j), & i \neq j. \end{cases}$$

To compute $\mathrm{Pic}(G)$, one can use the Laplacian matrix L_G of a graph G. One can easily check that any configuration of chips can be considered as a vector $v \in \mathbb{Z}^{V(G)}$, and the chip configuration v' by making a lending move (resp. borrowing move) at a vertex i from v is the following:

$$v' = v - (L_G^t)e_i \quad (\text{resp. } v' = v + (L_G^t)e_i). \tag{2}$$

With this, by considering $L_G^t : \mathbb{Z}^{V(G)} \to \mathbb{Z}^{V(G)}$, one can see that

$$\mathrm{Pic}(G) = \mathrm{coker}(L_G^t) \text{ and } \mathrm{Pic}(G) = \mathbb{Z} \times \mathrm{Jac}(G).$$

From (1), one is naturally led to think that one can play a chip-firing game with a given matrix M (replacing L_G), namely firing at a site i is:

$$v' = v - (M^t)e_i. \tag{3}$$

In fact, this has been studied. See [Kli18, Section 6] for details.

In this paper, we explore the case when a matrix M is the Laplacian matrix of a directed graph (Definition 2.3). Chip-firing games on directed graphs were first considered by A. Björner and L. Lovász in [**?**], and later further studied by D. Wagner [Wag00]. We also note that S. Backman [**?**] proved a directed version of Riemann-Roch formula for graphs (**??**).

In what follows by a proper orientation of an undirected graph we mean an orientation such that at least one of the edges is not bidirectional. To avoid any ambiguity we let $\vec{G}$ be a directed graph whose underlying undirected graph is G. The following is a natural question to ask.

Question. Let G be an undirected graph. Are there a proper orientation of G so that

$$\mathrm{Jac}(\vec{G}) = \mathrm{Jac}(G) \tag{4}$$

If so, can we explicitly find an orientation of G so that (3) holds? Finally, can we find asymptotic behavior of the number of orientations of G satisfying (3) for specific classes of graphs?

Remark 1.1. With most orientations of G, we cannot expect to have $\mathrm{Pic}(\vec{G}) = \mathrm{Pic}(G)$, even for trees as the rank of $\mathrm{Pic}(\vec{G})$ is the number of the strong terminal components of $\vec{(G)}$; for a connected, undirected graph G, considered as all edges being bidirectional, there exists only one strong terminal component.

We partly answer the above question for directed trees, directed cycles, and directed wheel graphs. We further provide computational results for the asymptotic behavior of such orientations for cycles. Finally, based on experimental data, we conjecture structures of Picard groups for multipartite graphs.

We first consider directed trees. We prove this by mathematical induction by studying how Jacobians change when we add one edge e to a directed tree T. There are total three cases; (1) e is incoming, (2) e is outgoing, and (3) e is bidirectional. We prove that as in the case for undirected trees, the Jacobian of a directed tree is trivial.

Theorem A. *(Proposition 3.3) Let T be a tree with any orientation. Then* $\mathrm{Jac}(T) = \{0\}$.

Next, we consider cycles graphs C_n. For the undirected case, it is easy to check that $\mathrm{Jac}(C_n) = \mathbb{Z}_n$. For the directed case, we prove that for each $k \neq n$, one can construct an orientation of C_n so that $\mathrm{Jac}(C_n) = \mathbb{Z}_k$ (with orientation) as follows:

Theorem B. *(Theorem 4.1) Let C_n be a cycle graph with n vertices. For any $0 \leq m \leq n$, there exists an orientation of C_n such that $\mathrm{Jac}(C_n) = \mathbb{Z}_m$ with the orientation. Furthermore, we explicitly describe how to find an orientation of C_n to obtain $\mathbb{Z}_m$.*

Finally, we consider the wheel graphs W_n.[2] When W_n is undirected, i.e., equipped with all bidirectional edges, Biggs computed the Jacobian of W_n in [Big99]. For the directional cases, we compute two special cases. Let W_n be the wheel graph with all bidirectional edges, and W_n' with the edges of the rim are bidirectional and all its spoke edges point to the axel. Let W_n'' be the wheel graph where the edges of the rim are bidirectional and all its spoke edges point away from the axel. See, Example 5.2 for W_n W_n', and W_n''. We prove the following.

Theorem C. *(Propositions 5.3 and 5.4) With the same notation as above, we have the following.*

(1) The Laplacian matrices of W_n and W_n' are row equivalent. In particular, one has

$$\mathrm{Pic}(W_n) \simeq \mathrm{Pic}(W_n').$$

(2) The smith normal form of the Laplacian of W_n'' is of the form

$$\left[\begin{array}{c|ccc} I_{n-3} & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

where $(a,b) = \begin{cases} (n-1, n-1) & \textit{if } n \textit{ is even}; \\ (\frac{n-1}{2}, 2(n-1)) & \textit{if } n \textit{ is odd} \end{cases}.$

This paper is organized as follows. In Section 2, we review basic definitions and properties. In Section 3, we prove Theorem A on directed trees. In Section 4, we prove Theorem B on directed cycles. Finally, in Section 5, we prove Theorem C on directed wheel graphs. In Section 6, we provide our conjectures on multipartite graphs and some experimental data for asymptotic behaviors for the interested readers.

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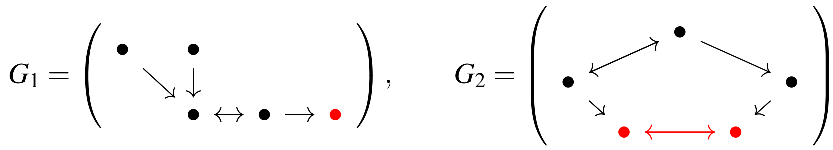
[2] Our notation W_n is the wheel graph with n-vertices.

2. Preliminaries

2.1. Directed graphs, Laplacian matrices, Picard groups, and Jacobians.

Definition 2.1 (Strong terminal component)**.** Let G be a directed graph. A *strong component* C of G is a subgraph of G such that for any two vertices v_i and v_j of C, there exist directed paths from v_i to v_j and from v_j to v_i, respectively. A strong component C of G is called *terminal* if there is no arrow from any vertex v of C to the set of vertices $V_G \setminus V_C$.

Example 2.2. Consider the following directed graphs:



In G_1 and G_2, the red subgraphs are strong terminal components.

Definition 2.3. For a graph G with the vertices $\{v_1, v_2, \dots, v_n\}$, one defines an $n \times n$ matrix L_G, called the Laplacian matrix of G, as follows:

$$(L_G)_{ij} = \begin{cases} \#\text{ of outgoing edges of } v_i, & i = j \\ -(\#\text{ of edges from } v_j \text{ to } v_j), & i \neq j. \end{cases}$$

Here are some examples of the Laplacian matrices of directed graphs.

Example 2.4.

$$T = \left(\begin{array}{ccccc} & & 1 & & \\ & & \downarrow & & \\ 2 & \to & 3 & \leftarrow & 4 \\ & & \uparrow & & \\ & & 5 & & \end{array}\right), \qquad L_T = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \tag{5}$$

$$T' = \left(\begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 2 & \leftarrow & 3 & \to & 4 \\ & & \downarrow & & \\ & & 5 & & \end{array}\right), \qquad L_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Let G be a directed graph and L_G be the Laplacian matrix of G, which defines a linear map

$$L_G^T : \mathbb{Z}^{|V(G)|} \to \mathbb{Z}^{|V(G)|}, \tag{7}$$

where L_G^T is the transpose of L_G.

Definition 2.5. Let G be a directed graph. The Picard group $\mathrm{Pic}(G)$ is the cokernel of L_G^T. The Jacobian $\mathrm{Jac}(G)$ of G is the torsion subgroup of $\mathrm{Pic}(G)$.

The following was proved by D. Wagner in [Wag00].

Theorem 2.6. [Wag00, Corollary 3.5] *For any directed graph G, the rank of* $\mathrm{Pic}(G)$ *is the number of terminal strong components of G.*

As in the undirected case, $\mathrm{Pic}(G)$ can be computed by using the Smith normal form of L_G. Since the Smith normal forms of L_G and L_G^T are same, we can just use L_G instead of L_G^T in (6).

Example 2.7. Let T and T' be as in Example 2.4. One can easily check the Smith normal forms as follows:

$$\mathrm{SNF}(L_T) = \left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0 \end{array}\right], \qquad \mathrm{SNF}(L_{T'}) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0_{4\times 4} \end{array}\right]$$

In particular, $\mathrm{Pic}(T) = \mathbb{Z}$ and $\mathrm{Pic}(T') = \mathbb{Z}^4$. This shows that different from undirected graphs, the rank of $\mathrm{Pic}(G)$ may be greater than one for directed graphs G. Since T has one terminal strong component and T' has four terminal strong component, Theorem 2.6 also provide the ranks of $\mathrm{Pic}(T)$ and $\mathrm{Pic}(T')$. Combined with Proposition 3.3 (stating that $\mathrm{Jac}(T) = \{0\}$ for any directed tree T), one obtains the desired Picard groups without computing the Smith normal forms.

In the following, we review some basic properties of Smith normal forms which will be used throughout the paper.

2.2. Smith normal forms.

Definition 2.8. We say m by n matrices M and N are *equivalent* if there exist invertible matrices P of size m and Q of size n such that $M = PNQ$.

Definition 2.9 (Smith normal form)**.** Suppose $M \in \mathrm{Mat}_{m\times n}(R)$, where R is a commutative ring and $m \leq n$. The Smith normal form of M denoted by $\mathrm{SNF}(M) = (d_{ij})$ is an m by n matrices with entries in R such that

(1) $\mathrm{SNF}(M)$ is equivalent to M,
(2) $d_{i,i} | d_{i+1,i+1}$ for $i = 1, \dots, m-1$, and
(3) $d_{ij} = 0$ if $i \neq j$.

Remark 2.10. Let $M \in M_{m\times n}(\mathbb{Z})$, and let $I_k(M)$ denote the ideal generated by $k \times k$ minors of M, where $I_k(M) = 0$ if $k > \min\{m,n\}$ and $I_k = \langle 1 \rangle$ if $k \leq 0$. For a matrix $N = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right]$ and for any k, $I_k(M) = I_{k+1}(N)$, and the cokernels of M and N are isomorphic.

The following is well-known. For instance, wee [Sta16, Theorem 2.4].

Theorem 2.11. *Let R be a unique factorization domain such that any two elements have a greatest common divisor (gcd). Suppose that $M \in \mathrm{Mat}_{m\times n}(R)$ has a Smith normal form $L = (x_1, \dots, x_m)$. Then, for $1 \leq k \leq m$, the product $x_1 \cdots x_k$ is equal to the gcd of all $k \times k$ minors of M, with the convention that if all $k \times k$ minors are 0 then their gcd is 0.*

3. Picard groups and Jacobians of directed trees

Lemma 3.1. *Let G be a directed graph. If we attach either an incoming arrow or a two-sided arrow to create another directed graph G', then $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.*

Proof. Let α be an arrow which is glued to G. Let $|V(G)| = n$. We label the vertexes of G as $1, 2, \dots, n$. Suppose first that α is incoming and α is glued at the vertex n. Let $L_G = (a_{ij})$ (resp. $L_{G'}$) be the Laplacian matrix of G (resp. G'). Then the matrix $L_{G'}$ is of the following form.

$$L_{G'} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right] \tag{8}$$

By a column operation between the last two columns, we obtain the following matrix:

$$\left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right]. \tag{9}$$

This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$.

Next, suppose that α is a two-sided arrow. Then similar to the above, we obtain the following Laplacian matrix for G':

$$L_{G'} = \left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n+1 & -1 \\ \hline 0 & 0 & \cdots & -1 & 1 \end{array}\right]. \tag{10}$$

By a column operation, the matrix (9) becomes the matrix (8). This shows that $\mathrm{Pic}(G) = \mathrm{Pic}(G')$. $\square$

Remark 3.2. For the undirected case, when one glues two graphs G_1 and G_2 along one vertex to obtain G, then $\mathrm{Pic}(G) = \mathrm{Pic}(G_1) \times \mathrm{Pic}(G_2)$. But, this is no longer true for directed graphs. For instance, the tree T in (**??**) can be considered as a directed graph obtained by gluing the following two directed graphs G_1 and G_2 along the vertices 3 and 3'"

$$G_1 = \left(\begin{array}{cc} & 1 \\ & \downarrow \\ 2 \rightarrow & 3 \end{array}\right), \quad G_2 = \left(\begin{array}{cc} 3' & \leftarrow 4 \\ \uparrow & \\ 5 & \end{array}\right) \tag{11}$$

But, we have

$$L_{G_1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathrm{Pic}(G_1) = \mathbb{Z}, \quad L_{G_2} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Longrightarrow \mathrm{Pic}(G_2) = \mathbb{Z}$$

It follows that $\mathrm{Pic}(T) \neq \mathrm{Pic}(G_1) \times \mathrm{Pic}(G_2)$.

For the undirected trees T, $\mathrm{Pic}(T) = \mathbb{Z}$. But, for directed trees, the rank of $\mathrm{Pic}(T)$ can be arbitrarily large depending the number of strong terminal component of T. Nonetheless, we prove that $\mathrm{Jac}(T) = \{0\}$ in the following.

Proposition 3.3. *Let T be a tree with any orientation. Then* $\mathrm{Jac}(T) = \{0\}$*, i.e.,* $\mathrm{Pic}(T)$ *is torsion-free.*

Proof. We inductively prove this. When T_0 is a tree with one arrow, one can easily check that $\mathrm{Pic}(T_0) = \mathbb{Z}$ or $\{0\}$ (depending on the number of strong terminal components).

Suppose that T_k is an oriented tree with k arrows. When we add one arrow α to T_k to construct T_{k+1}, there are three cases; α is (1) incoming, (2) outgoing, and (3) two-sided. When α is either incoming or two-sided arrow, then it follows from Lemma 3.1 that $\mathrm{Pic}(T_k) = \mathrm{Pic}(T_{k+1}) = \mathbb{Z}^r$, where r is the number of the terminal strong components of T_k and T_{k+1}, since in this case it does not increase the number of the terminal strong components.

Next, suppose that α is an outgoing arrow. Let's label the vertexes of G as $v_1, v_2, \dots, v_n$. Suppose that the arrow α is attached to the vertex v_j. Let $L_k = (a_{ij})$ be the Laplacian matrix of T_k. Then we have the following:

$$L_{k+1} = \left[\begin{array}{cccccc|c|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jj}+1 & \cdots & a_{jn} & -1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{array}\right] \tag{12}$$

To compute the Smith normal form, by relabeling vertices, we may assume L_{k+1} is the following matrix:

$$\left[\begin{array}{ccc|c|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \vdots & \vdots & \cdots & a_n & 1 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] \tag{13}$$

Since $\mathrm{Pic}(T_k) = \mathbb{Z}^r$, there exist $P, Q \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$PL_kQ = \left[\begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & 0_r \end{array}\right] \tag{14}$$

where 0_r is an $r \times r$ zero matrix. Consider the following block matrices of size $(n+1)\times(n+1)$:

$$P' = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right], \quad Q' = \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right] \tag{15}$$

Then, we have

$$P'L_{k+1}Q' = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} L_k & e_n \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array}\right] \tag{16}$$

We first consider the case when v_n is a sink. In particular, T_{k+1} and T_k have the same number of terminal strong components. In this case, the n^{th} row of L_{k+1} in (12) is the zero row. In particular, we can take P so that

$$Pe_n = e_n \tag{17}$$

Therefore, we have

$$P'L_{k+1}Q' = \left[\begin{array}{c|c} PL_kQ & Pe_n \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} PL_kQ & e_n \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c|c|c} I_{n-r} & & 0 & 0 \\ \hline & 0_{r-1} & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \tag{18}$$

It follows that $\mathrm{Pic}(T_{k+1}) = \mathrm{Pic}(T_k)$.

Now, suppose that v_n is not a sink. Let $Pe_n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. There are two cases:

Case 1: Suppose that $x_{n-r+1} = x_{n-r+2} = \cdots = x_n = 0$. In this case, one can easily observe that after some column operations, $P'L_{k+1}Q'$ becomes the Smith normal form of L_{k+1}. In particular, $\mathrm{Jac}(T_{k+1}) = \mathrm{Jac}(T_k)$, and hence $\mathrm{Pic}(T_{k+1}) = \mathbb{Z} \times \mathrm{Pic}(T_k)$.

Case 2: Suppose that at least one of $x_{n-r+1}, x_{n-r+2}, \cdots, x_n$ is not equal to zero. Then, the Smith normal form of L_{k+1} becomes the last matrix in (17). In particular, $\mathrm{Jac}(T_{k+1}) = \mathrm{Jac}(T_k)$, and hence $\mathrm{Pic}(T_{k+1}) = \mathbb{Z} \times \mathrm{Pic}(T_k)$. □

Example 3.4. Consider the following oriented tree:

$$T = \left(\begin{array}{ccccc} 1 & & 2 & & 3 \\ & \nwarrow & \downarrow & & \updownarrow \\ & & 4 & \leftrightarrow & 5 & \rightarrow & 6 \end{array} \right)$$

The Laplacian matrix of T is the following:

$$L_T = D_T - A_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of L_T is the following 6×6 matrix:

$$\left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_2 \end{array} \right]$$

This shows that $\mathrm{Pic}(T) = \mathbb{Z}^2$.

Example 3.5. Consider the following oriented tree obtain from Example 3.4 by gluing an outgoing arrow α:

$$T' = \left(\begin{array}{ccccccc} 1 & & 2 & & 3 \\ & \nwarrow & \downarrow & & \updownarrow \\ 7 & \xleftarrow[\alpha]{} & 4 & \leftrightarrow & 5 & \rightarrow & 6 \end{array} \right)$$

Now, we have

$$L_{T'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of $L_{T'}$ is the following 7×7 matrix:

$$\left[\begin{array}{c|c} I_4 & 0 \\ \hline 0 & 0_3 \end{array} \right]$$

This shows that $\mathrm{Pic}(T) = \mathbb{Z}^3$.

Example 3.6. Consider the following oriented tree obtain from Example 3.4 by gluing an outgoing arrow α:

$$T'' = \left(\begin{array}{ccccccc} 1 & & 2 & & 3 & & 7 \\ & \nwarrow & \downarrow & & \updownarrow & & \uparrow\alpha \\ & & 4 & \leftrightarrow & 5 & \rightarrow & 6 \end{array} \right)$$

Now, we have

$$L_{T''} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Smith normal form of $L_{T''}$ is the following 7×7 matrix:

$$\left[\begin{array}{c|c} I_5 & 0 \\ \hline 0 & 0_2 \end{array}\right]$$

This shows that $\mathrm{Pic}(T'') = \mathbb{Z}^2$.

4. Picard groups of directed cycles

4.1. Directed cycles with cyclic Jacobians. In this subsection, we prove the following theorem. Our idea is to find an isomorphism between $\mathrm{Pic}(C_n)$ and $\mathrm{Pic}(C_{n+1})$ by properly choosing an orientation of C_{n+1} from a given orientation of C_n.

Theorem 4.1. *Let $n \geq 3$. For each $k \leq n$, there exists an orientation of C_n such that $\mathrm{Jac}(C_n)$ (with that orientation) is $\mathbb{Z}_k$.*

Example 4.2. With the following orientations of C_3

$$G_1 = \left(\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl] \\ \bullet \arrow[u] & \end{tikzcd}\right), \quad G_2 = \left(\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[dl] \\ \bullet \arrow[u] & \end{tikzcd}\right), \quad G_3 = \left(\begin{tikzcd} \bullet \arrow[r, leftrightarrow] & \bullet \\ \bullet \arrow[u, leftrightarrow] \arrow[ur, leftrightarrow] & \end{tikzcd}\right)$$

we have $\mathrm{Jac}(G_1) = 0$, $\mathrm{Jac}(G_2) = \mathbb{Z}_2$, and $\mathrm{Jac}(G_3) = \mathbb{Z}_3$.

Definition 4.3. For a directed graph G, we let $e_{i \to j}$ be a directed edge whose source (resp. target) is i (resp. j). We let $e_{i \leftrightarrow j}$ be the bidirectional edge between $i, j \in V(G)$.

Definition 4.4. Let C_n be a directed cycle.

(1) Suppose that $e_{(n-1)\to n}, e_{1\to n} \in E(C_n)$. We let C'_{n+1} be a directed cycle whose directed edges are given as follows:

$$E(C_{n+1}) = E(C_n) - \{e_{1\to n}\} \cup \{e_{1\to(n+1)}\}$$

Pictorially, we have the following

$$C_n = \Big(\cdots \ v_{n-1} \ \to v_n \ \leftarrow v_1 \ \cdots\Big) \implies C'_{n+1} = \Big(\cdots \ v_{n-1} \ \to v_n \ \to v_{n+1} \ \leftarrow v_1 \ \cdots\Big).$$

(2) Suppose that $e_{(n-1)\to n}, e_{n\to 1} \in E(C_n)$. We let C''_{n+1} be a directed cycle whose directed edges are given as follows:

$$E(C_{n+1}) = E(C_n) - \{e_{n\to 1}\} \cup \{e_{n\to(n+1)}, e_{(n+1)\to 1}\}$$

Pictorially, we have the following

$$C_n = \Big(\cdots \ v_{n-1} \ \to v_n \ \to v_1 \ \cdots\Big) \implies C_{n+1} = \Big(\cdots \ v_{n-1} \ \to v_n \ \to v_{n+1} \ \to v_1 \ \cdots\Big).$$

Definition **??** will be used to find an orientation of C_{n+1} from a directed C_n so that $\mathrm{Pic}(C_n)$ is isomorphic to $\mathrm{Pic}(C_{n+1})$. Here are some examples of Definition **??**, where we color the new vertex and edges in red.

Example 4.5 ($C_5 \implies C'_6$)**.**

$$C_5 = \left(\begin{tikzcd}[column sep=small] & & 5 & & \\ 4 \arrow[urr] \arrow[dr] & & & & 1 \arrow[ull] \arrow[dl] \\ & 3 \arrow[rr, leftrightarrow] & & 2 & \end{tikzcd}\right) \implies C'_6 = \left(\begin{tikzcd}[column sep=small] & 5 \arrow[r, red] & {\color{red}6} & \\ 4 \arrow[ur] \arrow[dr] & & & 1 \arrow[ul, red] \arrow[dl] \\ & 3 \arrow[r, leftrightarrow] & 2 & \end{tikzcd}\right)$$

Example 4.6 ($C_5 \implies C_6''$).

$$C_5 = \left(\begin{array}{ccccc} & & 5 & & \\ 4 & \nearrow & & \searrow & 1 \\ \searrow & & & & \swarrow \\ & 3 & \longleftrightarrow & 2 & \end{array} \right) \implies C_6'' = \left(\begin{array}{cccccc} & & 5 & \longrightarrow & 6 & \\ & \nearrow & & & & \searrow \\ 4 & & & & & 1 \\ & \searrow & & & & \swarrow \\ & & 3 & \longleftrightarrow & 2 & \end{array} \right)$$

The following is our key lemma for cycles.

Lemma 4.7. *Fix an orientation of C_n. If not every edge of C_n (with a fixed orientation) is bidirectional, there is an orientation of C_{n+1} such that* $\mathrm{Pic}(C_n) \simeq \mathrm{Pic}(C_{n+1})$.

Proof. Let $V(C_n) = \{v_1, \ldots, v_n\}$ and $D_{C_n} = (d_{ij})$ be the diagonal matrix of C_n (with a given orientation). Since not every edge is bidirectional, there exists i such that $d_{ii} = 0$ or 1. Without loss of generality, we may assume $i = n$ so that the adjacent vertices are v_{n-1} and v_1.

<u>Case 1:</u> Suppose that $d_{nn} = 0$. In this case, the Laplacian matrix of C_n is of the following form.

$$L_{C_n} = \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

where $l, k \in \{0, 1\}$. Now, the Laplacian matrix $L_{C'_{n+1}}$ is of the following form:

$$L_{C'_{n+1}} = \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $r_1, \ldots, r_{n+1}$ (resp. $c_1, \ldots, c_{n+1}$) be the rows (resp. columns) of $L_{C'_{n+1}}$. Replace r_1 with $r_1 = r_n$, then c_n with $c_n + c_{n+1}$, and then switch r_n and r_{n+1} (also multiplying -1), we have the following:

$$\begin{bmatrix} l+1 & -l & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} L_{C_n} & 0 \\ \hline 0 & 1 \end{array} \right].$$

Now our claim follows from Remark 2.10.

<u>Case 2:</u> Now suppose $d_{nn} = 1$. Without loss of generality, we may assume the vertex v_n has one outgoing edge to v_1. There are two cases, depending on the existence of $e_{1 \to n}$. The Laplacian of these two graphs are equivalent as follows:

$$L_{C_n} = \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \to r_1 - r_n} \begin{bmatrix} l+1 & -l & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & -k & k+1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

where $l,k \in \{0,1\}$. Therefore, we may assume that L_{C_n} is the first matrix in (19). Now, the Laplacian matrix $L_{C''_{n+1}}$ is of the following form:

$$L_{C''_{n+1}} = \begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -1 \\ -1 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $r_1,\dots,r_{n+1}$ (resp. $c_1,\dots,c_{n+1}$) be the rows (resp. columns) of $L_{C''_{n+1}}$. Switch r_{n+1} and r_n, then replace r_n with r_n+r_{n+1}, then replace c_n with $c_n - c_{n+1}$ r_n, and then multiply -1 to r_{n+1}, we have the following:

$$\begin{bmatrix} l & -l & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -k & k+1 & -1 & 0 \\ -1 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} L_{C_n} & 0 \\ \hline 0 & 1 \end{array}\right].$$

Our claim follows from Remark 2.10. $\square$

Lemma 4.8. *For any $n \geq 4$ and any $m \in \{n-1,n\}$, there exists an orientation of C_n such that* $\mathrm{Jac}(C_n) = \mathbb{Z}_m$.

Proof. Case 1: If $m = n$, then we can simply that the orientation of C_n so that all edges are bidirectional. In this case, we have $\mathrm{Pic}(C_n) \simeq \mathbb{Z}_n$.

Case 2: Suppose that $m = n-1$. Consider the orientation of C_n as follows:

$$E(C_n) = \{e_{n\to 1}, e_{2\to 1}, e_{3\to 2}\} \cup \{e_{i\leftrightarrow j} \mid i \neq j \text{ and } \{i,j\} \not\subseteq \{n,1,2,3\}\}. \tag{21}$$

Pictorially, we have

$$C_n = \Big(\cdots \ v_n \longrightarrow v_1 \longleftarrow v_2 \longleftarrow v_3 \ \cdots\Big)$$

and all other edges are bidirectional. We claim that with the orientation (20), we have

$$\mathrm{Pic}(C_n) = \mathbb{Z}_{n-1}.$$

In fact, in this case, the Laplacian matrix of C_n is the following:

$$L_{C_n} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Let $r_1,\dots,r_n$ (resp. $c_1,\dots,c_n$) be the rows (resp. columns) of L_{C_n}. Replace r_n with $r_n - r_2$, and then c_2 with $c_2 + c_1$. Then, switch r_1 and r_2, and then multiply -1 to r_1. As a result, we obtain the following matrix:

$$M_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

Now, it follows from Remark 4.9, one can observe that M_n is equivalent to the following matrix:

$$M_n' = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L_{C_{n-1}} \end{array}\right].$$

Now the $(n-1)\times(n-1)$ bottom right submatrix is that of C_{n-1} with all bidirectional edges and Remark 2.10 proves the claim.

$\square$

Remark 4.9. For $n \geq 3$, the Laplacian of C_n each of its edge is bidirectional, the Laplacian of C_n is of the form

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}$$

Since $[1\cdots 1]L = 0$, the first row is a $\mathbb{Z}$-linear combination of the next $(n-1)$ rows. This justifies the last equivalence in the proof above.

Proof of Theorem 4.1. We induct on the number of edges. The base case is when $n = 3$ which is Example 4.2. For $n > 3$, by Lemma 4.8, we have orientations of C_n with $\mathrm{Jac}(C_n) = \mathbb{Z}_n$ and $\mathbb{Z}_{n-1}$. Now the rest follows from Lemma 4.7. $\square$

4.2. Construction of cyclic Jacobians of C_n.

Definition 4.10. By a *path* of C_n, we mean a connected subgraph of C_n in which all arrows are oriented in a single direction or are bidirectional.

Example 4.11. Consider the following orientation of C_5:

$$C_5 = \left(\begin{tikzcd} & 1 \arrow[dl, leftrightarrow] \arrow[dr] & \\ 5 \arrow[d] & & 2 \arrow[d] \\ 4 \arrow[rr, leftrightarrow] & & 3 \end{tikzcd} \right)$$

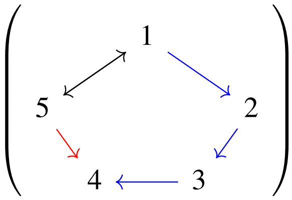
The following are some paths of C_5:

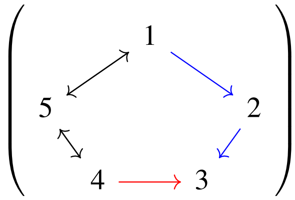
$$P_1 = \left(\, 5 \longleftrightarrow 1 \, \right), \qquad P_2 = \left(\, 1 \to 2 \to 3 \, \right), \qquad P_3 = \left(\, 4 \longleftrightarrow 3 \, \right)$$

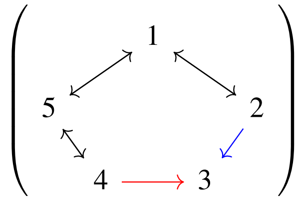
In this subsection, we prove the following.

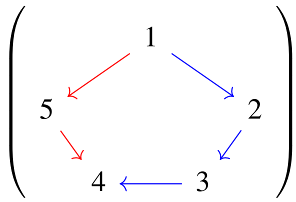
Theorem 4.12. *Let C_n be a cycle graph with a fixed orientation. Then, we have $\mathrm{Jac}(G) = \mathbb{Z}_{k+2}$, where k is the number of bidirectional edges clockwise of the counter-clockwise path and counter-clockwise of the clockwise path.*

Example 4.13. Consider the following orientations of C_5 with two paths. The counter-clockwise path is in red and the clockwise path is in blue. The bidirectional edges are shown in black.

$G_1 =$ 

, $G_2 =$ 

, $G_3 =$ 

$G_4 =$ 

Then, we have $\mathrm{Jac}(G_1) = \mathbb{Z}_3$, $\mathrm{Jac}(G_2) = \mathbb{Z}_4$, $\mathrm{Jac}(G_3) = \mathbb{Z}_5$, and $\mathrm{Jac}(G_4) = \{0\}$.

We will need the following.

Lemma 4.14. *For $n \geq 2$, let M_n be an $n \times n$ matrix whose diagonal entries are 2 and sub-diagonals are -1 as follows:*

$$M_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{22}$$

Then $\det(M_n) = n+1$.

Proof. We prove this by mathematical induction. When $n = 2$, we have

$$M_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Hence, we have $\det(M_2) = 3$.

Now, suppose the statement is true for all $k \leq n-1$. To compute $\det(M_n)$, we use the Laplace expansion along the first row, i.e.,

$$\det(M_n) = 2\det(M_{n-1}) + \det(N),$$

where N is an $(n-1)\times(n-1)$ matrix of the following form:

$$N = \left[\begin{array}{c|cccc} -1 & -1 & 0 & \cdots & 0 \\ \hline 0 & & M_{n-2} & & \end{array}\right]. \tag{23}$$

In particular, $\det(N) = -\det(M_{n-2})$, and hence by inductive assumption, we have

$$\det(M_n) = 2\det(M_{n-1}) + \det(N) = 2((n-1)+1) + (-1)((n-2)+1) = n+1.$$

$\square$

Remark 4.15. Note that the $(n-1)\times(n-1)$ minor of M_n after deleting the last row and last column is $(-1)^{n-1}$. Hence the Smith normal form of M_n is

$$\left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & n+1 \end{array}\right], \tag{24}$$

where I_{n-1} is the identity matrix of size $n-1$.

Lemma 4.16. *Fix an orientation of C_n. Let $V(C_n) = \{v_1, \dots, v_n\}$. Suppose that the vertex v_1 does not have any outgoing edge, and all other vertices have two outgoing edges. Then* $\mathrm{Jac}(C_n) = \mathbb{Z}_n$.

Proof. The Laplacian matrix of C_n is the matrix obtained by replacing the first row of M_n (as in (**??**)) with the zero row as follows:

$$L_{C_n} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{25}$$

By the same argument as in Remark 4.9, this matrix is equivalent to

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & M_{n-1} \end{array}\right], \tag{26}$$

where M_{n-1} is the matrix in Lemma **??**. Thus, by Remark **??**, we have

$$\mathrm{Jac}(C_n) = \mathbb{Z}_n.$$

Note that one can also use the proof of Lemma 4.8 with Lemma **??** to see this. $\square$

Proof of Theorem 4.12. The general case follows from Lemma **??** combined with the first and the second parts of the proof of Lemma 4.8 for $\mathbb{Z}_{n-1}$. In particular, the construction stated in Theorem 4.12 corresponds to the reduction in the second case of $d_{n,n} = 1$. $\square$

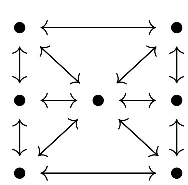
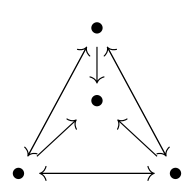
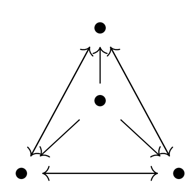
5. Picard groups of directed wheel graphs

In this section, we use the following notation for orientations of wheel graphs.

Definition 5.1 (Wheel graph)**.** By a wheel graph W_n, we mean a graph obtained by connecting a single universal vertex to all vertices of a cycle C_{n-1}, which we consider as a directed graph with all bidirectional edges.

(1) By W_n', we mean a wheel with an oriented as follows: the edges of the rim are bidirectional and all its spoke edges point to the axel.
(2) By W_n'', we mean a wheel with an oriented as follows: the edges of the rim are bidirectional and all its spoke edges point away from the axel.

Example 5.2.

$W_7 =$  $W_4' =$  $W_4'' =$ 

Proposition 5.3. *Let L_{W_n} (resp. $L_{W_n'}$) be the Laplacian matrix of W_n (resp. W_n'). Then, L_{W_n} and $L_{W_n'}$ are row equivalent. In particular, one has* $\mathrm{Pic}(W_n) \simeq \mathrm{Pic}(W_n')$.

Proof. We use the labeling convention that v_1 is the axel and the vertices on the rim are $v_2, \dots, v_n$. One can directly see that the Laplacian matrices of W_n and W_n' are row equivalent. To be precise, from the Laplacian $L_{W_n'}$, one can obtain the Laplacian L_{W_n} by subtracting all other rows from the first row by using the fact that $\mathbf{1_n} \cdot L_{w_n} = 0$:

$$L_{W_n'} = \left[\begin{array}{c|cccccccc} 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hline -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & -1 & 3 & -1 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & \cdots & 0 \\ -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & \cdots & 0 & -1 & 3 \end{array}\right], \quad L_{W_n} = \left[\begin{array}{c|cccccccc} n-1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ \hline -1 & 3 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & -1 & 3 & -1 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & \cdots & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & \cdots & 0 \\ -1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ -1 & -1 & 0 & 0 & \cdots & 0 & -1 & 3 \end{array}\right]$$

□

Proposition 5.4. *The Smith normal form of the Laplacian of W_n'' is of the following form*

$$\left[\begin{array}{c|ccc} I_{n-3} & 0 & 0 & 0 \\ \hline 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \textit{where } (a,b) = \begin{cases} (n-1, n-1) & \textit{if } n \textit{ is even}; \\ (\frac{n-1}{2}, 2(n-1)) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We use the labeling convention that v_1 is the axel and the vertices on the rim are $v_2, \ldots, v_n$. Let L_{W_n} be the Laplacian matrix of W_n. It is clear that $\det(L_{W_n}) = 0$. Hence, from Theorem 2.11, it suffices to show that

$$I_{n-1}(L_{W_n}) = \langle (n-1)^2 \rangle \text{ and } I_{n-2}(L_{W_n}) = \begin{cases} \langle n-1 \rangle & \text{if } n \text{ is even} \\ \langle (n-1)/2 \rangle & \text{if } n \text{ is odd.} \end{cases}$$

We reduce the Laplacian matrix L_{W_n}. Let $r_1, \ldots, r_n$ (resp. $c_1, \ldots, c_n$) be the rows (resp. columns) of L_{W_n}. First, by replacing c_1 with $c_1 + \cdots + c_n$ and then r_n with $r_2 + \cdots + r_n$, we obtain the following matrix:

$$\left[\begin{array}{c|cccccccc} 0 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right] \tag{27}$$

Finally, by replacing c_n with $c_2 + \cdots + c_n$, we obtain the following matrix:

$$\left[\begin{array}{c|cccccccc} 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 & n-1 \\ \hline 0 & 2 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right] \tag{28}$$

Thus, to compute the Smith normal form of L_{W_n}, it suffices to find the Smith normal form of the following matrix.

$$N = \left[\begin{array}{c|c} \mathbf{1}_{n-2} & n-1 \\ \hline M_{n-2} & \mathbf{0}_{n-1} \end{array}\right],$$

where M_{n-2} is the matrix in Lemma **??**. Note that $\det(M_{n-2}) = n-1$ and this shows that

$$I_{n-1}(L_{W_n}) = I_{n-1}(N) = \langle \det(N) \rangle = \langle (n-1)^2 \rangle.$$

To compute the $(n-2)$ minors, first note that if the last column is a part of a minor, then it is divisible by $(n-1)$. Since $\det(M_{n-2}) = (n-1) \in I_{n-2}(N)$, we only need to consider the $(n-2)$ minors of the first $(n-2)$ columns of N. In other words, $I_{n-2}(L_{W_n})$ is generated by $(n-1)$ and the $(n-2)$-determinants of the matrix which is obtained from M_{n-2} by replacing the i^{th} row by $\mathbf{1}_{n-2}$.

Note that M_{n-2} is symmetric. By Cramer's rule, these $(n-2)$-minors are the entries of the solution matrix $\mathbf{x}$ of the matrix equation

$$M_{n-2}\mathbf{x} = \det(M_{n-2}) \cdot \mathbf{1}_{n-2}^T$$

up to sign. This is done in the following lemma, and it completes the proof.

□

Lemma 5.5. *Consider the matrix equation*

$$M_n\mathbf{x} = \det(M_n)\mathbf{1_n}^T = (n+1)\mathbf{1_n}^T,$$

*where M_n is as in Lemma **??**. Let x_k denote the k^{th} entry of $\mathbf{x}$. Then $x_k = ak^2 + bk$, where*

$$a = -\frac{(n+1)}{2}, \quad b = \frac{(n+1)^2}{2}.$$

Furthermore, $\gcd(x_1, \dots, x_n)$ *is* $(n+1)$ *if n is even and* $(n+1)/2$ *if n is odd.*

Proof. Note that $\det(M_n) = n+1$ by Lemma **??**. Consider a sequence $f(k) = f_k = ak^2 + bk$. For any k, we have

$$-f_{k-1} + 2f_k - f_{k+1} = -2a.$$

First, we set $a = -(n+1)/2$. In addition, the equations $2x_1 - x_2 = n+1$ and $-x_{n-1} + 2x_n = n+1$ are equivalent to the conditions $f_0 = f_{n+1} = 0$. These conditions imply $b = (n+1)^2/2$ since $0, -b/a$ are the roots of $f(k)$.

We compute the gcd of $f_1, \dots, f_n$. First, we claim that $\gcd(f_1, \dots, f_n) = \gcd(f_1, n+1)$. Since $\gcd(f_1, \dots, f_n)$ divides $2f_1 - f_2 = n+1$, it suffices to show that $\gcd(f_1, n+1)$ divides $f_2, \dots, f_n$. This follows by induction since for $k = 2, \dots, n$, $f_k = 2f_{k-1} - f_{k-2} - (n+1)$. Here we use the fact that $f_0 = 0$. Since $f_1 = -(n+1)/2 + (n+1)^2/2 = \frac{n(n+1)}{2}$ is an integer, the expressions $f_k = -2f_{k-1} + f_{k-2}$ also proves that $f_1, \dots, f_n$ are integers.

Finally, we show that $\gcd(f_1, n+1)$ is $n+1$ when n is even and $(n+1)/2$ when n is odd. Recall that $f_1 = \frac{n(n+1)}{2}$. If n is even, then $n+1$ divides f_1. Thus, $\gcd(f_1, n+1) = n+1$. If n is odd, then

$$\gcd(f_1, n+1) = \gcd(\frac{n+1}{2} \cdot n, n+1) = \gcd(\frac{n+1}{2}, n+1) = \frac{n+1}{2}.$$

The 2^{nd} last equality follows from the fact that $\frac{n+1}{2}$ is an integer and for integers a, b, c, $\gcd(ab, c) = \gcd(a, c)$ if b, c are relatively prime.

Thus $\mathbf{x} = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T$ is a solution having the asserted gcd. This completes the proof. □

6. Conjectures and Future directions

6.1. Picard groups of directed multipartite graphs. By a directed multipartite graph[3], we mean a graph G such that there exists a partition of vertices $V(G) = V_1 \sqcup \cdots \sqcup V_r$ such that

(1) There is no edge between vertices in V_i for all $i = 1, \dots, r$,
(2) For each $i \in \{1, \dots, r\}$, and $u \in V_i$, $v \in V_{i+1}$, we have $e_{u \to v} \in E(G)$.

We were able to find notable patterns in both a *Perceptron* style model with two layers and a *Hidden Layer* model with three layers.

[3]The structure of the graphs that we investigate are intentionally designed to resemble artificial neural networks.

Conjecture 6.1 (Picard groups multipartite graphs for two layers, i.e, bipartite graphs)**.** Let G be the multipartite graph with two layers in the form of $f \to s$ where a and b are the number of vertices in the first and second layers, respectively. Then, one has

$$\mathrm{Pic}(G) = \mathbb{Z}_b^{a-1} \times \mathbb{Z}^b.$$

For three layers, the Picard group is significantly more complex. Based on our experimental results, we conjecture the following.

Conjecture 6.2 (Picard groups multipartite graphs for three layers)**.** Let G be the multipartite graph with three lays in the form of $a \to b \to c$ where a, b, and c are the number of vertices in the first, second, and third layers, respectively. Suppose that $a, b, c > 1$.

(1) Suppose that a, b, c satisfy one of the following two conditions:
  (a) b is odd, $b \nmid c$, and $a \leq b$, or
  (b) c is odd, b is even, and $a \leq b$.
  Then, one has

$$\mathrm{Pic}(G) = \mathbb{Z}_c^{b-a-1} \times \mathbb{Z}_{bc}^{a} \times \mathbb{Z}^c.$$

(2) Suppose that a, b, c satisfy one of the following two conditions:
  (a) b is odd, $b \nmid c$, and $a > b$, or
  (b) c is odd, b is even and $a > b$.
  Then, one has

$$\mathrm{Pic}(G) = \mathbb{Z}_b^{a-b+1} \times \mathbb{Z}_{bc}^{b-1} \times \mathbb{Z}^c.$$

(3) Suppose that a, b, c satisfy one of the following two conditions:
  (a) b is odd and $b \mid c$, or
  (b) c is even, b is even, and $b \mid c$.
  Then, one has

$$\mathrm{Pic}(G) = \mathbb{Z}_b^{c-1} \times \mathbb{Z}_c \times \mathbb{Z}_{bc} \times \mathbb{Z}^c$$

(4) Suppose that c is even, b is even, and $b \nmid c$, and $b > a$. Then, one has

$$\mathrm{Pic}(G) = \mathbb{Z}_2^{a-1} \times \mathbb{Z}_c^{b-a-1} \times \mathbb{Z}_{(bc)/2}^{a-1} \times \mathbb{Z}_{bc} \times \mathbb{Z}^c$$

6.2. Invariant factors and path numbers. Jaiung: Here we explain the invariant factors and path numbers. I do not know precisely which conjecture or data that we have.

Conjecture 6.3. content...

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