SUBSTRATE THEORY — SYMBOLIC LOGIC CANONICAL REFERENCE

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TYPES

State := List Bool
Entity : opaque type
Ω : Entity (axiom)
Substrate : Entity (axiom)
Time := $\mathbb{R}$
Phase := $\mathbb{R}$
Nbhd := List State
Precision := $\mathbb{N}$

COMPLEXITY FUNCTIONS

K : Entity $\to$ $\mathbb{R}$ (noncomputable axiom)
K_sum : List Entity $\to$ $\mathbb{R}$
K_sum(es) := $\sum$ e $\in$ es, K(e)

K_joint : List Entity $\to$ $\mathbb{R}$ (noncomputable axiom)
K_cond : Entity $\to$ Entity $\to$ $\mathbb{R}$
K_cond(e_1,e_2) := K_joint([e_1,e_2]) - K(e_1)

C : Entity $\to$ $\mathbb{N}$ $\to$ $\mathbb{R}$ (noncomputable axiom)
C_sum : List Entity $\to$ $\mathbb{N}$ $\to$ $\mathbb{R}$
C_sum(es,p) := $\sum$ e $\in$ es, C(e,p)

C_joint : List Entity $\to$ $\mathbb{N}$ $\to$ $\mathbb{R}$ (noncomputable axiom)
C_cond : Entity $\to$ Entity $\to$ $\mathbb{N}$ $\to$ $\mathbb{R}$
C_cond(e_1,e_2,p) := C_joint([e_1,e_2],p) - C(e_1,p)

K_LZ : State $\to$ $\mathbb{N}$ (axiom)
K_toy : State $\to$ $\mathbb{N}$
K_toy(s) := |dedup(s)|

RANK FUNCTIONS

rank_K : Entity → ℕ (noncomputable axiom)
rank_C : Entity → ℕ → ℕ (noncomputable axiom)

TEMPORAL FUNCTIONS

indexed : Entity → Time → Entity (axiom)
temporal_slice : List Entity → Time → List Entity
slice : List (Entity × Time) → Time → List Entity

join : List State → State (axiom)
mode : State → State (noncomputable constant)

traj : Entity → List (Entity × Time)
P_total : Entity → ℝ

COHERENCE FUNCTIONS

Coh : List Entity → List Time → ℝ
Coh(es,times) := 1 - K_joint(slice(es,times))/K_sum(slice(es,times))

Coh_op : List Entity → List Time → ℕ → ℝ
Coh_op(es,times,p) := 1 - C_joint(slice(es,times),p)/C_sum(slice(es,times),p)

Coh_trajectory : Entity → ℝ
dCoh_dt : Entity → Time → ℝ

compression_ratio : List Entity → List Time → ℝ

PARTITION FUNCTIONS

Z_ideal : Finset Entity → ℝ
Z_ideal(S) := ∑ e ∈ S, 2^(-K(e))

Z_op : Finset Entity → ℕ → ℝ
Z_op(S,p) := ∑ e ∈ S, 2^(-C(e,p))

GROUNDING FUNCTIONS

grounds_graph : Finset Entity → ℕ → Finset (Entity × Entity)
parents_C : Entity → Finset Entity → ℕ → Finset Entity
bfs_depth_C : Entity → ℕ → Finset Entity → ℕ
bfs_grounding_path : Entity → Finset Entity → ℕ → Option (List Entity)

CONSTANTS

c_grounding := 50
c_margin := 5
c_sub : ℝ (constant)
c_single : ℝ (constant)
C_mode : ℝ (constant)

$c \approx 299792458$ m/s
$\hbar$: ℝ (axiom, $\hbar > 0$)
G : ℝ (axiom, $G > 0$)
k_B : ℝ (axiom, $k_B > 0$)
e : ℝ (axiom, $e > 0$)
ε_0 : ℝ (axiom, $\varepsilon_0 > 0$)
α : ℝ (axiom, $1/138 < \alpha < 1/137$)

ℓ_Planck := $\sqrt{\hbar G/c^3}$
t_Planck := ℓ_Planck/c
M_Planck := $\sqrt{\hbar c/G}$
E_Planck := M_Planck·c^2
T_Planck := E_Planck/k_B

κ_energy := E_Planck
ℏ_eff : ℝ (axiom, ℏ_eff > 0)
ε_geom : ℝ (axiom, ε_geom > 0)

H_0 : ℝ ($67 < H_0 < 74$)
Ω_m : ℝ (0.3 < Ω_m < 0.32)
Ω_Λ : ℝ (0.68 < Ω_Λ < 0.70)
Ω_r : ℝ (0 < Ω_r < 0.0001)
Ω_k : ℝ (|Ω_k| < 0.01)
Ω_DM : ℝ (0.25 < Ω_DM < 0.27)
Ω_baryon : ℝ (0.04 < Ω_baryon < 0.05)
t_universe : ℝ (13.7×10^9 < t_universe < 13.9×10^9)

CORE AXIOMS

K2 (Substrate Minimality)
K(Substrate) = 0
K(Ω) = 0

G1 (Substrate Grounds All)
∀e. is_presentation(e) → is_grounded(e,Substrate)
where is_grounded(e,ctx) := K_cond(ctx,e) < K(e) - K(ctx) + c_grounding

T7 (Time Arrow)
∀hist next. hist.length ≥ 2 →
  K_joint(next::hist) - K_joint(hist) ≤ K_joint([hist.last,hist.init]) - K(hist.init)

T4 (Emergence/Collapse)
∀e_classical e_quantum.
  emergent(e_classical,e_quantum) →
  is_measurement_device(e_classical) ∨ is_observable(e_classical)
where emergent(e_classical,e_quantum) := K_cond(Substrate,e_classical) < K(e_quantum)

C6 (Coherence Preservation)
∀e. is_quantum_state(e) → coherent(e)
where coherent(e) := ∀t_1 t_2. t_1 < t_2 →
  K_cond(indexed(e,t_1),indexed(e,t_2)) = K_cond(indexed(e,t_2),indexed(e,t_1))

AXIOM CONSEQUENCES

substrate_ultimate_ground :
∀e. is_presentation(e) → ∃path.
  path.head = Substrate ∧ path.last = e ∧
  ∀i. i+1 < path.length → is_grounded(path[i+1],path[i])

decoherence_implies_classical :
∀e. is_presentation(e) ∧ ¬coherent(e) →
  ∃t_0. ∀t > t_0. ¬is_quantum_state(indexed(e,t))

measurement_breaks_coherence :
∀e_q e_c. is_quantum_state(e_q) ∧ coherent(e_q) ∧ emergent(e_c,e_q) →
  ¬coherent(e_c)

BRIDGE AXIOMS

BRIDGE1 (Pointwise Convergence)
∀e ε>0. is_presentation(e) →
  ∃p_0. ∀p≥p_0. |C(e,p) - K(e)| < ε

BRIDGE2 (Uniform Convergence)
∀S ε>0. (∀e∈S. is_presentation(e)) →
  ∃p_0. ∀p≥p_0 e∈S. |C(e,p) - K(e)| < ε

BRIDGE3 (Probability Convergence)
∀S ε>0. (∀e∈S. is_presentation(e)) ∧ Z_ideal(S) > 0 →
  ∃p_0. ∀p≥p_0. |Z_op(S,p) - Z_ideal(S)|/Z_ideal(S) < ε

BRIDGE4 (Grounding Convergence)
∀S ε>0 e_1,e_2. e_1,e_2 ∈ S ∧ is_presentation(e_1) ∧ is_presentation(e_2) →
  ∃p_0. ∀p≥p_0. grounds_K(e_1,e_2) ↔ grounds_C(e_1,e_2,p)
where:
  grounds_K(e_1,e_2) := K_cond(e_1,e_2) < K(e_2) - K(e_1) + c_grounding
  grounds_C(e_1,e_2,p) := C_cond(e_1,e_2,p) < C(e_2,p) - C(e_1,p) + c_grounding

BRIDGE5 (Rank Stability)
∀S e. e ∈ S ∧ is_presentation(e) →
  ∃p_0. ∀p≥p_0. rank_C(e,p) = rank_K(e)

BRIDGE6 (Temporal Continuity)
∀e times ε>0. is_temporal_presentation(e) →
  ∃p_0. ∀p≥p_0 t∈times. |Coh_op([e],[t],p) - Coh([e],[t])| < ε

BRIDGE7 (Conditional Convergence)
∀e_1 e_2 ε>0. is_presentation(e_1) ∧ is_presentation(e_2) →
  ∃p_0. ∀p≥p_0. |C_cond(e_1,e_2,p) - K_cond(e_1,e_2)| < ε

BRIDGE7_joint (Joint Convergence)
∀es ε>0. (∀e∈es. is_presentation(e)) →
  ∃p_0. ∀p≥p_0. |C_joint(es,p) - K_joint(es)| < ε

BRIDGE8 (Continuum Limit)
∀e times ε>0. is_temporal_presentation(e) →
  ∃p_0 δ. δ > 0 ∧ ∀p≥p_0 t∈times.
  |(Coh_op([e],[t+δ],p) - Coh_op([e],[t],p))/δ - dCoh_dt(e,t)| < ε

CA RULES

F : State → State (noncomputable)
merge : State → State → State (noncomputable)

R_Cohesion : List State → State → State
R_Cohesion(n,h) := merge(F(join(n)),h)

R_Reduction : List State → State
R_Reduction(n) := mode(join(n))

R_G1 : List State → State → State
R_G1(n,h) := if K_LZ(join(n)) ≤ c_grounding then R_Cohesion(n,h)
    else R_Reduction(n)

coherent_state : State → Prop
coherent_state(s) := K_LZ(s) ≤ c_grounding

CA PRESERVATION THEOREMS

P3 (C6 Preservation)
∀n h. coherent_state(join(n)) → K_LZ(R_G1(n,h)) = K_LZ(h)

R_G1_grounding_reduction
∀n h. K_LZ(join(n)) > c_grounding →
  K_LZ(R_G1(n,h)) < K_LZ(join(n)) + c_grounding

R_G1_preserves_time_arrow
∀hist n h. K_LZ(join(R_G1(n,h)::hist)) ≤ K_LZ(join(hist)) + c_margin

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FUNDAMENTAL THEOREMS

E_K (Energy-Complexity Equivalence)
∀e. is_presentation(e) →
  (has_mass(e) → ∃Δ>0. K(e) = K(Ω) + Δ) ∧
  energy_of(e) = κ_energy · K(e)

G_Ψ (Grounding Stability)

$\forall e.\ stable(e) \leftrightarrow K\_cond(\Omega,e) > c\_grounding$
where $stable(e) := is\_presentation(e) \land K\_cond(\Omega,e) > c\_grounding$

B_Ω (Holographic Bound)
$\forall region\ Area.\ is\_presentation(region) \land Area > 0 \rightarrow$
$K(region) \leq Area/(4\ell\_Planck^2)$

Ψ_I (Coherence Invariant)
$\forall e.\ is\_temporal\_presentation(e) \land coherent(e) \rightarrow$
$Coh\_trajectory(e) \cdot P\_total(e) = 1$

U_Ω (Uncertainty Principle)
$\forall e\ \Delta K\ \Delta t.\ is\_temporal\_presentation(e) \land \Delta K > 0 \land \Delta t > 0 \rightarrow$
$\Delta K \cdot \Delta t \geq \hbar\_eff$

RANK SYSTEM

$rank\_K(\Omega) = 0$
$grounds(e_1,e_2) \rightarrow rank\_K(e_2) < rank\_K(e_1)$
$\forall e.\ \exists n.\ rank\_K(e) = n$

$rank\_C(e,p) = bfs\_depth\_C(e,p,S)$ for $e \in S$

UNIVERSAL GROUNDING

$\forall e.\ is\_presentation(e) \rightarrow$
$\exists path.\ path.head = \Omega \land path.last = e \land$
$\forall i.\ i+1 < path.length \rightarrow grounds(path[i],path[i+1])$

grounding_transitive :
$\forall e_1\ e_2\ e_3.\ grounds(e_1,e_2) \land grounds(e_2,e_3) \rightarrow grounds(e_1,e_3)$

grounding_acyclic :
$\forall e.\ \neg grounds(e,e)$

COMPLEXITY BOUNDS

K_joint_nonneg : $\forall es.\ 0 \leq K\_joint(es)$

K_joint_nil : K_joint([]) = 0
K_joint_singleton : ∀e. K_joint([e]) = K(e)

compression_axiom :
∀es. (∀e∈es. is_presentation(e)) ∧ es.length ≥ 2 →
  K_joint(es) < K_sum(es)

joint_le_sum :
∀es. (∀e∈es. is_presentation(e)) → K_joint(es) ≤ K_sum(es)

complexity_positive :
∀e. is_presentation(e) → 0 < K(e)

substrate_minimal :
∀e. is_presentation(e) → K(Substrate) ≤ K(e)

OPERATIONAL BOUNDS

C_nonneg : ∀e p. 0 ≤ C(e,p)
C_monotone : ∀e p_1 p_2. $p_1 \le p_2$ → C(e,p_2) ≤ C(e,p_1)
C_upper_bound : ∀e p. is_presentation(e) → K(e) ≤ C(e,p)

C_joint_nonneg : ∀es p. 0 ≤ C_joint(es,p)

K_LZ_nonneg : ∀s. 0 ≤ K_LZ(s)
K_LZ_empty : K_LZ([]) = 0
K_LZ_monotone : ∀s_1 s_2. s_1.length ≤ s_2.length → K_LZ(s_1) ≤ K_LZ(s_2)

TOY COMPRESSOR BOUNDS

K_toy_lower_bound : ∀s. K_LZ(s) ≤ K_toy(s)
K_toy_upper_bound : ∀s. K_toy(s) ≤ K_LZ(s) + $\log_2$(s.length)

KLZ AXIOMS

K_LZ_subadditive_cons :
∀x xs. K_LZ(join(x::xs)) ≤ K_LZ(join(xs)) + K_LZ(x) + c_sub

K_LZ_prefix :
$\forall b\ s.\ K\_LZ(join([b])) \le K\_LZ(join(b::s))$

K_LZ_singleton_bound :
$\forall b.\ K\_LZ(join([b])) \le c\_single$

K_LZ_mode_le :
$\forall s.\ K\_LZ(mode(s)) \le K\_LZ(s) + C\_mode$

C_mode_lt_c_grounding :
$C\_mode < c\_grounding$

COHERENCE BOUNDS

coherence_bounds :
$\forall es\ times.\ (\forall e \in es.\ is\_presentation(e)) \to$
$0 \le Coh(es,times) \land Coh(es,times) \le 1$

compression_ratio_ge_one :
$\forall es\ times.\ (\forall e \in es.\ is\_presentation(e)) \to$
$1 \le compression\_ratio(es,times)$

PHYSICS CORRESPONDENCES

$energy\_of(e) = \kappa\_energy \cdot K(e)$
$mass(e) = energy\_of(e)/c^2$
$entropy(e) = k\_B \cdot \log(2) \cdot K(e)$

$is\_quantum(nbhd) := K\_LZ(join(nbhd)) \le c\_grounding$
$is\_classical(nbhd) := K\_LZ(join(nbhd)) > c\_grounding$

PREDICATES

is_substrate : Entity → Prop (axiom)
is_presentation : Entity → Prop (axiom)
is_emergent : Entity → Prop (axiom)
is_temporal_presentation : Entity → Prop (axiom)
is_static_presentation : Entity → Prop (axiom)

is_quantum_state : Entity → Prop (axiom)
is_measurement_device : Entity → Prop (axiom)
is_observable : Entity → Prop (axiom)

phenomenal : Entity → Prop (axiom)
has_mass : Entity → Prop (axiom)

grounds : Entity → Entity → Prop (axiom)
temporal_grounds : Entity → Time → Entity → Time → Prop (axiom)
interacts : Entity → Entity → Prop (axiom)
inseparable : Entity → Entity → Prop (axiom)
emerges_from : Entity → List Entity → Prop (axiom)
phase_coupled : Entity → Entity → Phase → Prop (axiom)

coherent : Entity → Prop
decoherent : Entity → Prop
stable : Entity → Prop

is_quantum : List State → Prop
is_classical : List State → Prop
coherent_state : State → Prop

ENTITY CLASSIFICATION

entity_classification :
∀e. (is_substrate(e) ∧ e = Substrate) ∨ is_presentation(e) ∨ is_emergent(e)

substrate_not_presentation : ∀e. ¬(is_substrate(e) ∧ is_presentation(e))
substrate_not_emergent : ∀e. ¬(is_substrate(e) ∧ is_emergent(e))
presentation_not_emergent : ∀e. ¬(is_presentation(e) ∧ is_emergent(e))

presentation_temporal_or_static :
∀e. is_presentation(e) →
  (is_temporal_presentation(e) ∨ is_static_presentation(e)) ∧
  ¬(is_temporal_presentation(e) ∧ is_static_presentation(e))

SUBSTRATE PROPERTIES

substrate_unique : ∀x y. is_substrate(x) ∧ is_substrate(y) → x = y
substrate_is_Substrate : is_substrate(Substrate)

Omega_is_substrate : is_substrate(Ω)
Omega_eq_Substrate : Ω = Substrate

TEMPORAL PRESERVATION

indexed_preserves_presentation :
$\forall$e t. is_presentation(e) $\rightarrow$ is_presentation(indexed(e,t))

temporal_slice_preserves_presentation :
$\forall$es t. ($\forall$e$\in$es. is_presentation(e)) $\rightarrow$
  ($\forall$e$\in$temporal_slice(es,t). is_presentation(e))

ASSOCIATIVITY

join_associative :
$\forall s_1\ s_2\ s_3$. join([join([s_1,s_2]),s_3]) = join([s_1,join([s_2,s_3])])
