Stochastic Processes Assignment 7

Matthew Yu Student ID: 0553501

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1. Problem 5.49

Events occur according to a Poisson process with rate λ.Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T, where $T > \frac{1}{\lambda}$. That is, if an event occurs at time $t,\ 0 \le t \le T$, and we decide to stop, then we win if there are no additional events by time T, and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T, then we lose. Also, if no events occur by time T, then we lose. Consider the strategy that stops at the first event to occur after some fixed time $s,\ 0 \le s \le T$.

(a) Using this strategy, what is the probability of winning?

$$\begin{aligned} P(win) &= P(N(T) - N(s) = 1) \\ &= \frac{[\lambda(T-s)]^1}{1!} e^{-\lambda(T-s)} \\ &= \lambda(T-s)e^{-\lambda(T-s)} \end{aligned}$$

(b) What value of s maximizes the probability of winning?

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\lambda(T-s)e^{-\lambda(T-s)}\right] = -\lambda e^{-\lambda(T-s)} + \lambda(T-s)\lambda e^{-\lambda(T-s)} \overset{\text{set}}{=} 0$$

By setting $k = -\lambda(T-s)$,

$$\begin{gathered} -\lambda e^k - k\lambda e^k = 0 \\ k = -\lambda(T-s) = -1 \\ \lambda(T-s) = 1 \\ s^* = T - \frac{1}{\lambda} \end{gathered}$$

(c) Show that one's probability of winning when using the preceding strategy with the value of s specified in part (b) is $\frac{1}{e}$

Given $s = 1$, according to (a) $P(win) = 1e^{-1} = \frac{1}{e}$

1. Problem 5.61

A system has a random number of flaws that we will suppose is Poisson distributed with mean c. Each of these flaws will, independently, cause the system to fail at a random time having distribution G. When a system failure occurs, suppose that the flaw causing the failure is immediately located and fixed.

(a) What is the distribution of the number of failures by time t?

Conditioning on the number of flaws in the system, we get

$$
\begin{aligned}
P[failures\ by\ time\ t = k] &= P[flaws\ detected\ by\ time\ t = k] \\
&= \sum_{i=0}^{\infty} P[flaws\ detected\ by\ time\ t = k | total\ flaws\ = i] \times P[total\ flaws\ = i]
\end{aligned}
$$

While

$$P[total\ flaws\ = i] = e^{-c}\frac{c^i}{i!}$$

$$P[flaws\ detected\ by\ time\ t = k | total\ flaws\ = i] = \begin{cases} 0 & \text{if } i < k \\ \frac{i!}{k!(i-k)!}[G(t)]^k[1-G(t)]^{i-k}, & i \geq k \end{cases}$$

Then, we have

$$
\begin{aligned}
P[failures\ by\ time\ t = k] &= \sum_{i=k}^{\infty} e^{-c}\frac{c^i}{i!}\frac{i!}{k!(i-k)!}[G(t)]^k[1-G(t)]^{i-k} \\
&= e^{-c}\frac{[cG(t)]^k}{k!}\sum_{i=k}^{\infty}\frac{[c(1-G(t))]^{i-k}}{(i-k)!} \\
&= e^{-c}\frac{[cG(t)]^k}{k!}\sum_{j=0}^{\infty}\frac{[c(1-G(t))]^{j}}{j!} \\
&= e^{-c}\frac{[cG(t)]^k}{k!}e^{c-cG(t)} \\
&= e^{-cG(t)}\frac{[cG(t)]^k}{k!} \sim Poisson(cG(t))
\end{aligned}
$$

(b) What is the distribution of the number of flaws that remain in the system at time t?

Similar to (a), we can derive that

$$P[flaws\ undetected\ by\ time\ t = k] \sim Poisson(1 - cG(t))$$

(c) Are the random variables in parts (a) and (b) dependent or independent?

Since we can first classify flaws into detected or undetected, with probability $G(t)$ and $1 - G(t)$ respectively, the two random variables are thereby independent.

1. Problem 5.74

The number of missing items in a certain location, call it X, is a Poisson random variable with mean λ.When searching the location, each item will independently be found after an exponentially distributed time with rate μ. A reward of R is received for each item found, and a searching cost of C per unit of search time is incurred. Suppose that you search for a fixed time t and then stop.

(a) Find your total expected return.

The probability of each item (independently) being found is $1 - e^{\mu t}$.
Thus, the expected return will be

$$R \times \lambda(1 - e^{\mu t}) - C \times t$$

(b) Find the value of t that maximizes the total expected return.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[R \times \lambda(1 - e^{\mu t}) - C \times t\right] = R\mu\lambda e^{-\mu t} - C \overset{\text{set}}{=} 0$$

$$e^{-\mu t} = \frac{c}{R\mu\lambda}$$

$$t^* = \frac{\ln R\mu\lambda - \ln C}{\mu}$$

(c) The policy of searching for a fixed time is a static policy. Would a dynamic policy, which allows the decision as to whether to stop at each time t, depend on the number already found by t be beneficial?

Independently, each of the number of items will be counted with probability $1 - e^{-\mu t}$ or not be counted with probability $e^{-\mu t}$, therefore the number of items not found yet by time t is independent of the number found. Thus, there is no guarantee that the dynamic policy works better.