Stochastic Processes Assignment 6

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1. Problem 5.24

There are two servers available to process n jobs. Initially, each server begins work on a job. Whenever a server completes work on a job, that job leaves the system and the server begins processing a new job (provided there are still jobs waiting to be processed). Let T denote the time until all jobs have been processed.If the time that it takes server i to process a job is exponentially distributed with rate $\mu_i,\ i = 1, 2$, find $E[T]$ and $Var(T)$.

Let T_i denote the time between the $(i-1)$th and ith job completion. The T_is are independent, while $T_i \sim exp(\mu_1 + \mu_2)$. Except for T_n, where $T_n \sim exp(\mu_2)$ if less time is spent at server 1 (i.e. with probability $\frac{\mu_1}{\mu_1+\mu_2}$) or $T_n \sim exp(\mu_1)$ if less time is spent at server 2 (i.e. with probability $\frac{\mu_2}{\mu_1+\mu_2}$).

$$\begin{aligned} E[T] &= \sum_{i=1}^{n-1} E[T_i] + E[T_n] \\ &= (n-1) \times \frac{1}{\mu_1+\mu_2} + \frac{1}{\mu_2} \times \frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1} \times \frac{\mu_2}{\mu_1+\mu_2} \end{aligned}$$

Then, $Var(T_i) = \frac{1}{(\mu_1+\mu_2)^2}$. Except for T_n again.

$$\begin{aligned} Var(T_n) &= E[T_n^2] - (E[T_n])^2 \\ &= [(\frac{1}{\mu_2})^2 + \frac{1}{\mu_2^2}] \times \frac{\mu_1}{\mu_1+\mu_2} + [(\frac{1}{\mu_1})^2 + \frac{1}{\mu_1^2}] \times \frac{\mu_2}{\mu_1+\mu_2} - (\frac{1}{\mu_2} \times \frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1} \times \frac{\mu_2}{\mu_1+\mu_2})^2 \end{aligned}$$

As a result,

$$Var[T] = (n-1)\frac{1}{(\mu_1+\mu_2)^2} + [(\frac{1}{\mu_2})^2 + \frac{1}{\mu_2^2}] \times \frac{\mu_1}{\mu_1+\mu_2} + [(\frac{1}{\mu_1})^2 + \frac{1}{\mu_1^2}] \times \frac{\mu_2}{\mu_1+\mu_2} - (\frac{1}{\mu_2} \times \frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1} \times \frac{\mu_2}{\mu_1+\mu_2})^2$$

2. Problem 5.34

Two individuals, A and B, both require kidney transplants. If she does not receive a new kidney, then A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the first kidney will go to A (or to B if B is alive and A is not at that time) and the next one to B (if still living).

(a) What is the probability that A obtains a new kidney?

Let T_A denote the time A dies of failing to receive a kidney transplant.
Let T_B denote the time B dies of failing to receive a kidney transplant.
Let T_1 denote the exact time where the first kidney arrives.
Let T_2 denote the time between the arrival of the first and second kidney.

Then we have
$P(A\ obtains\ a\ new\ kidney) = P(T_A > T_1) = \frac{\lambda}{\lambda+\mu_A}$

(b) What is the probability that B obtains a new kidney?

$$
\begin{aligned}
&P(B\ obtains\ a\ new\ kidney)\\
&\quad = P(B\ obtains\ a\ new\ kidney|T_1 = min(T_1,T_A,T_B)) \times P(T_1 = min(T_1,T_A,T_B))\\
&\quad\quad + P(B\ obtains\ a\ new\ kidney|T_A = min(T_1,T_A,T_B)) \times P(T_A = min(T_1,T_A,T_B))\\
&\quad\quad + P(B\ obtains\ a\ new\ kidney|T_B = min(T_1,T_A,T_B)) \times P(T_B = min(T_1,T_A,T_B))\\
&\quad = P(T_2 < T_B)P(T_1 = min(T_1,T_A,T_B)) + P(T_1 < T_B)P(T_A = min(T_1,T_A,T_B)) + 0\\
&\quad = \frac{\lambda}{\lambda+\mu_B} \times \frac{\lambda}{\lambda+\mu_A+\mu_B} + \frac{\lambda}{\lambda+\mu_B} \times \frac{\mu_A}{\lambda+\mu_A+\mu_B}\\
&\quad = \frac{\lambda}{\lambda+\mu_B} \times \frac{\lambda+\mu_A}{\lambda+\mu_A+\mu_B}
\end{aligned}
$$

3.

$X \sim exp(\lambda)$ and $Y \sim exp(\mu)$ are independent. Show that $E[Y|Y < X] = \frac{1}{\lambda+\mu}$.

Rewrite Y as $min(X,Y) + (Y - min(X,Y))^+$, then, using the proposition below,

$$
\begin{aligned}
E[Y|Y<X] &= E[min(X,Y) + (Y - min(X,Y))^+|Y<X]\\
&= E[min(X,Y)|Y<X] + E[(Y - min(X,Y))^+|Y<X]\\
&= E[min(X,Y)] + 0\\
&= \frac{1}{\lambda+\mu}
\end{aligned}
$$

Proposition If $Y_1, Y_2, \cdots Y_n$ are independent exponential random variables with respective rates $\mu_1, \mu_2, \cdots \mu_n$, then $min_i\ Y_i$ is exponential with rate $\sum_{i=1}^{n} \mu_i$ and that $min_i\ Y_i$ and the rank order of the variables $Y_1, Y_2, \cdots Y_n$ are independent.