Stochastic Processes Assignment 9

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1. Problem 4.45

Consider an irreducible finite Markov chain with states $0, 1, \ldots, N$.

(a) Starting in state i, what is the probability the process will ever visit state j? Explain!

As the state space is finite, all state spaces communicate with one another, thus are all recurrent. That makes the probability 1.

(b) Let $x_i = P\{visit\ state\ N\ before\ state\ 0|\ start\ in\ i\}$. Compute a set of linear equations that the x_i satisfy, $i = 0, 1, \ldots, N$.

Conditioning on the 1st move, we have

$$\begin{aligned} x_0 &= 0 \\ x_N &= 1 \\ x_i &= \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN} \quad for\ i = 1, \ldots, N-1 \end{aligned}$$

(c) If $\sum_j jP_{ij} = i$ for $i = 1, \ldots N-1$, show that $x_i = \frac{i}{N}$ is a solution to the equations in part (b).

First, $x_i = \frac{i}{N}$ works for x_0 and x_N.
Then,

$$\begin{aligned} \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN} &= \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN} \times x_0 + P_{i0} \times x_0 \\ &= \sum_{j=0}^{N} P_{ij} x_j \end{aligned}$$

If $x_i = \frac{i}{N}$,

$$\sum_{j=0}^{N} P_{ij} x_j = \sum_{j=0}^{N} P_{ij} \times \frac{j}{N} = \frac{i}{N}$$

Thus, $x_i = \frac{i}{N}$ works as a solution.

1. Problem 4.57

A particle moves among $n+1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.

Let C denote the event the quest is completed.
Then, by conditioning on the first step

$$
\begin{aligned}
P(C) &= P(C|moves\ clockwise) \times p + P(C|moves\ counterclockwise) \times q \\
&= \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^n} \times p + \frac{1-\frac{p}{q}}{1-(\frac{p}{q})^n} \times q
\end{aligned}
$$

1. Problem 4.58

In the gambler's ruin problem of Section 4.5.1, suppose the gambler's fortune is presently i, and suppose that we know that the gambler's fortune will eventually reach N (before it goes to 0).Given this information, show that the probability he wins the next gamble is

$$
\begin{cases} \frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, & \text{if } p \neq \frac{1}{2} \\ \frac{i+1}{2i}, & \text{if } p = \frac{1}{2} \end{cases}
$$

$$
P\{X_{n+1} = i+1 | X_n = i, \lim_{m\to\infty} x_m = N\} = \frac{P\{X_{n+1} = i+1, \lim_{m\to\infty} x_m = N | X_n = i\}}{P\{\lim_{m\to\infty} x_m = N | x_n = i\}}
$$

$$
P\{X_{n+1} = i+1 | x_n = i\} \times \frac{P\{\lim_{m\to\infty} x_m = N | x_n = i, x_{n+1} = i+1\}}{P\{lim_{m\to\infty} = N | X_n = 1\}} = \frac{p^P i + 1}{P_i}
$$

Thus,

$$
\text{Probability of winning next game} = \begin{cases} \frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, & \text{if } p \neq \frac{1}{2} \\ \frac{i+1}{2i}, & \text{if } p = \frac{1}{2} \end{cases}
$$

1. Problem 4.59

For the gambler's ruin model of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with i, $i = 0, 1, \ldots, N$. Show that M_i satisfies

$$
M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N-1
$$

Solve these equations to obtain

$$
M_i = \begin{cases} i(N-i), & \text{if } p = \frac{1}{2} \\ \frac{i}{q-p} - \frac{N}{q-p} \times \frac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2} \end{cases}
$$

1. Problem 4.60

The following is the transition probability matrix of a Markov chain with states 1, 2, 3, 4

$$\mathbf{P} = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .2 & .2 & .2 & .4 \\ .25 & .25 & .5 & 0 \\ .2 & .1 & .4 & .3 \end{pmatrix} \quad if \ X_0 = 1$$

(a) find the probability that state 3 is entered before state 4

Let p_i denote the probability that starting from state i, state 3 is entered before state 4.
Then, we have

$$\begin{cases} p_3 = 1 \\ p_4 = 0 \\ p_1 = 0.4p_1 + 0.3p_2 + 0.2p_3 + 0.1p_4 \\ p_2 = 0.2p_1 + 0.2p_2 + 0.2p_3 + 0.4p_4 \end{cases}$$

By solving the equations, $p_1 = \frac{11}{21}$

(b) find the mean number of transitions until either state 3 or state 4 is entered.

Let m_i denote the expected number of steps needed given that starting from state i, state 3 is entered before state 4.
We have

$$\begin{cases} m_3 = 0 \\ m_4 = 0 \\ m_1 = 0.4(1+m_1) + 0.3(1+m_2) + 0.2(1+m_3) + 0.1(1+m_4) \\ m_2 = 0.2(1+m_1) + 0.2(1+m_2) + 0.2(1+m_3) + 0.4(1+m_4) \end{cases}$$

which can be simplified to

$$\begin{cases} 6m_1 - 3m_2 = 10 \\ 8m_2 - 2m_1 = 10 \end{cases}$$

By solving the equations, $m_1 = \frac{55}{21}$