# Stochastic Processes Assignment 3

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## 1.

Suppose we own a stock which is sensitive to the economy. There is a 70% chance the economy will grow next year; a 30% chance the economy will slow. If the economy grows, there is an 80% chance the stock will go up; otherwise 20% it will go down. On the other hand, if the economy slows, there is only a 30% chance the stock will go up; otherwise (1-30%) it will go down. If we know the stock went up, what is the probability that the economy grew?

Let $E$ denote the event the economy grows next year
Let $S$ denote the event the stock went up
Then: $P(E) = 0.7,\ P(E') = 0.3,\ P(S|E) = 0.8,\ P(S'|E) = 0.2,\ P(S|E') = 0.3,\ P(S|E') = 0.7$

$$P(E|S) = \frac{P(S \cap E)}{P(S)} = \frac{P(S|E)P(E)}{P(E)P(S|E) + P(E')P(S|E')} = \frac{56}{65}$$

## 2.

Suppose that $X$ and $Y$ are i.i.d. random variables following geometric distribution with parameter $p$. Show that $X + Y$ follows negative binomial distribution with parameters $n = 2$ and $p$. (a) prove by using moment generating function (mgf). (b) prove **without** using moment generating function (mgf).

### (a)

$M_X(t) = \frac{pe^t}{1-e^t(1-p)}$, $M_Y(t) = \frac{pe^t}{1-e^t(1-p)}$

$$M_{X+Y}(t) = E[e^{tX+tY}] = E[e^{tX}]E[e^{tY}] = \left[\frac{pe^t}{1-e^t(1-p)}\right]^2$$

While the mgf of a negative binomial distribution with parameters $(n, p)$ is $\left[\frac{pe^t}{1-e^t(1-p)}\right]^n$, by the uniqueness property of mgf,

$$X + Y \sim NB(2, p)$$

**(b)**

$$\begin{aligned}P(X+Y=k) &= \sum_{n=1}^{k-1} P(X=n,\ Y=k-n) \\ &= \sum_{n=1}^{k-1} P(X=n)P(Y=k-n) \\ &= \sum_{n=1}^{k-1} (1-p)^{n-1}p \cdot (1-p)^{k-1-n} \cdot p \\ &= \sum_{n=1}^{k-1} (1-p)^{k-2}p^2 = (k-1)p^2(1-p)^{k-2}\end{aligned}$$

Which is equivalent to $Z \sim NB(2, p)$ while $P(Z=k) = p^2(1-p)^{k-2}(k-1)$

## 3.

60 percent of the families in a certain community own their own car, 30% own their own home, and 20% own their own car and their own home. If a family is randomly chosen, what is the probability that this family owns a car or a house but not both.

Let C denote the event a family owns their own car.
Let H denote the event a family owns their own home.
Then, $P(C) = 0.6,\ P(H) = 0.3,\ P(C \cap H) = 0.2$

$$P(C \cap H') + P(C' \cap H) = P(C \cup H) - P(C \cap H) = 0.5$$

## 4.

If $X$ is a non-negative integer value random variable, show that $E[X] = \sum\limits_{n=1}^{\infty} P\{X \geq n\} = \sum\limits_{n=0}^{\infty} P\{X > n\}$

Let

$$I_n = \begin{cases} 1 & \text{if } X \geq n \\ 0, & \text{otherwise.} \end{cases}$$

and

$$J_n = \begin{cases} 1 & \text{if } X > n \\ 0, & \text{otherwise.} \end{cases}$$

Then, $X = \sum\limits_{n=1}^{\infty} I_n = \sum\limits_{n=0}^{\infty} J_n$. By taking expectations

$$E[X] = \sum_{n=1}^{\infty} E(I_k) = \sum_{n=0}^{\infty} E(J_k)$$

The result is shown since $E[I_n] = P(X \geq n)$ and $E[J_n] = P(X > n)$

## 5.

Suppose that $X$ and $Y$ are independent binomial random variables with parameters $(n, p)$ and$(m, p)$. Argue probabilistically (no computations necessary) that $X + Y$ is binomial with parameters $(n + m, p)$.

Since $X$ and $Y$ are independent, intuitively the trials they represent adds up without needing to consider their interaction. That way, the sum of outcomes of $n$ and $m$ trials should be equivalent to the outcome of $n + m$ trials.
In addition, considering the fact that binomial RVs with same probability of success can be partitioned into separate Bernoulli trials, we can also breakdown $X$ and $Y$ and then reassemble them back to a $(n + m)$ bernoulli trial.

## 6.

A coin, having probability $p$ of coming up heads, is successively flipped until two of the most recent three flips are heads. Let $N$ denote the number of flips. (Note that if the first two flips are heads, then $N = 2$.) Find $E[N]$.

Let $H$ and $T$ denote the outcome being heads or tails respectively.
Based on the first to tosses, $E[N]$ can be rewritten as

$$\begin{aligned} E[N] &= 2 + E(N|HH)P(HH) + E(N|TT)P(TT) + E(N|TH)P(TH) + E(N|HT)P(HT) \\ &= 2 + E(N|HH)p^2 + E(N|TT)(1-p)^2 + E(N|TH)p(1-p) + E(N|HT)p(1-p) \end{aligned}$$

To proceed, each of the conditional expectations are computed, where:

$E(N|HH) = 0$ as requirements are met already in first two tosses
$E(N|TT) = E(N)$ for no heads appear implies a startover
$E(N|HT) = 1 + 0 \times P(H) + E(N|TT)P(T) = 1 + (1-p)E(N)$
$E(N|TH) = 1 + E(N|HH)P(H) + E(N|HT)P(T) = 1 + (1-p)[1 + (1-p)E(N)]$

Then, $E(N) = 2 + p(1-p)[1 + (1-p)E(N)] + p(1-p)[1 + (1-p) + (1-p)^2E(N)] + (1-p)^2E(N)$

$$\begin{aligned} E(N) &= \frac{2 + 2p(1-p) + p(1-p)^2}{1 - p(1-p)^2 - p(1-p)^3 - (1-p)^2} \\ &= \frac{1 + 2p - p^2}{p^2(2-p)} \end{aligned}$$

## 7. Problem 3.27

A coin that comes up heads with probability $p$ is continually flipped until the pattern T, T, H appears. (That is, you stop flipping when the most recent flip lands heads, and the two immediately proceeding it lands tails.) Let $X$ denote the number of flips made, and find $E[X]$.

Let $H$ and $T$ denote the outcome being heads or tails respectively.
By conditioning the first toss, $E(X) = [1 + E(X)]p + E(X|T)(1-p)$

Then, as

$$E(X|T) = E(X|TH)p + E(X|TT)(1-p) = [2 + E(X)]p + (2 + \frac{1}{p})(1-p)$$

$E(X)$ can be rewritten as:

$$\begin{aligned} E(X) &= [1 + E(X)]p + [2 + E(X)]p(1 = p) + (2 + \frac{1}{p})(1-p)^2 \\ &= \frac{1}{p(1-p)^2} \end{aligned}$$

## 8. Problem 3.26

You have two opponents with whom you alternate play. Whenever you play A, you win with probability $p_A$; whenever you play B, you win with probability $p_B$, where $p_B > p_A$. If your objective is to minimize the expected number of games you need to play to win two in a row, should you start with A or B? Hint: Let $E[N_i]$ denote the mean number of games needed if you initially play $i$. Derive an expression for $E[N_A]$ that involves $E[N_B]$; write down the equivalent expression for $E[N_B]$ and then subtract.

Let $w$ and $l$ denote the event of a win and loss, respectively.
Then, $E[N_A] = E[N_A|w]p_A + E[N_A|l](1 - p_A)$
By conditioning on the next game,

$$\begin{aligned} E[N_A|w] &= E[N_A|ww]p_B + E[N_A|wl](1 - p_B) \\ &= 2p_B + (2 + E[N_A])(1 - p_B) \\ &= 2 + (1 - p_B)E[N_A] \end{aligned}$$

And while $E[N_A|l] = 1 + E[N_A]$,

$$E[N_A] = \{2 + (1 - p_B)E[N_A]\}p_A + (1 + E[N_B])(1 - p_A)$$

following the same logic,

$$E[N_B] = \{2 + (1 - p_A)E[N_B]\}p_B + (1 + E[N_A])(1 - p_B)$$

After subtracting,

$$E[N_A] - E[N_B] = -(p_A - 1)(p_B - 1)E[N_A] - (p_A - 1)(p_B - 1)E[N_B] + (p_A - p_B)$$

$$E[N_A] - E[N_B] = \frac{(p_A - p_B)}{1 + (p_A - 1)(p_B - 1)}$$

Given that $1 > p_B > p_A > 0$, the result is negative. Therefore, one should start by playing A.

## 9. Problem 3.30

Let $X_i, i \geq 0$ be independent and identically distributed random variables with probability mass function

$$p(j) = P\{X_i = j\}, j = 1, \cdots m, \sum_{j=1}^{m} p(j) = 1$$

Find $E[N]$, where $N = min\{n > 0 : X_n = X_0\}$.

$$\begin{aligned} P(N > n) &= P\{X_1 \neq X_0, X_2 \neq X_0 \cdots X_n \neq X_0\} \\ &= \sum_{j=1}^{m} P\{X_1 \neq X_0, X_2 \neq X_0 \cdots X_n \neq X_0 | X_0 = j\} \cdot P(X_0 = j) \\ &= \sum_{j=1}^{m} [1 - p(j)]^n \cdot p(j) \end{aligned}$$

Then,

$$\begin{aligned} E[N] &= \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} \sum_{j=1}^{m} [1 - p(j)]^n \cdot p(j) \\ &= \sum_{j=1}^{m} p(j) \sum_{n=0}^{\infty} [1 - p(j)]^n \\ &= \sum_{j=1}^{m} p(j) \frac{1}{1 - [1 - p(j)]} \\ &= \sum_{j=1}^{m} p(j) \times \frac{1}{p(j)} = m \end{aligned}$$

Where $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x < 1$

## 10.

If $R_i$ denotes the random amount that is earned in a period $i$, then $\sum\limits_{i=1}^{\infty} \beta^{i-1} R_i$, where $0 < \beta < 1$ is a specified constant, is called the total discounted reward with discount factor $\beta$. Let $T$ be a geometric random variable with parameter $1 - \beta$ that is independent of the $R_i$s. Show that the expected total discounted reward is equal to the expected total (undiscounted) reward earned by time $T$. That is, show that

$$E\left[\sum_{i=1}^{\infty} \beta^{i-1} R_i\right] = E\left[\sum_{i=1}^{T} R_i\right]$$

Let

$$I_E = \begin{cases} 1 & \text{if event E occurs} \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$
\begin{aligned}
E\left[\sum_{i=1}^{T} R_i\right] &= E\left[\sum_{i=1}^{\infty} I_{(T\geq i)} R_i\right] \\
&= \sum_{i=1}^{\infty} E[I_{(T\geq i)} R_i] \\
&= \sum_{i=1}^{\infty} E[I_{(T\geq i)}]E[R_i] \\
&= \sum_{i=1}^{\infty} P(T \geq i)E[R_i] \\
&= \sum_{i=1}^{\infty} \beta^{i-1} E[R_i] \\
&= E\left[\sum_{i=1}^{\infty} \beta^{i-1} R_i\right]
\end{aligned}
$$