Cocco, Gomes, and Maenhout (2005) REMARK

December 15, 2019

Mateo Velásquez-Giraldo   Matthew V. Zahn

Abstract
This paper contains the highlights from the REMARK file in Code>Python folder.

Keywords Hic heac hoc
JEL codes XXX

GitHub: http://github.com/econ-ark/REMARK/REMARKS/CGMPort
(In GitHub repo, see `/Code` for tools for solving and simulating the model)

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce our figures (warning: it may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

[1]Contact: mvelasq2@jhu.edu, matthew.zahn@jhu.edu, Department of Economics, 590 Wyman Hall, Johns Hopkins University, Baltimore, MD 21218, https://t.co/uaflostQyF?amp=1, http://matthewvzahn.com.

All numerical results herein were produced using the Econ-ARK/HARK toolkit; for further reference options see Acknowleding Econ-ARK. Thanks to Chris Carroll and Sylvain Catherine for comments and guidance.

1 The base model

The authors' aim is to represent the life cycle of a consumer that is exposed to uninsurable labor income risk and how he chooses to allocate his savings between a risky and a safe asset, without the possibility to borrow or short-sell.

1.1 Dynamic problem

The problem of an agent i of age t in the base model is recursively represented as

$$\begin{aligned} V_{i,t} = & \max_{0 \le C_{i,t} \le X_{i,t}, \alpha_{i,t} \in [0,1]} U(C_{i,t}) + \delta p_t E_t\{V_{i,t+1}(X_{i,t+1})\} && (1) \\ & \text{s.t} \\ & X_{i,t+1} = Y_{i,t+1} + (X_{i,t} - C_{i,t})(\alpha_{i,t} R_{t+1} + (1 - \alpha_{i,t})\bar{R}_f) && (2) \end{aligned}$$

where $C_{i,t}$ is consumption, $\alpha_{i,t}$ is the share of savings allocated to the risky asset, $Y_{i,t}$ is labor income, and $X_{i,t}$ represents wealth. The utility function $U(\cdot)$ is assumed to be CRRA in the base model. Extensions beyond the baseline model include an additively separable bequest motive in the utility function. The discount factor is δ and p_t is the probability of survival from t to $t+1$. Death is certain at a maximum period T.

Note that the consumer cannot borrow or short-sell.

The control variables in the problem are $\{C_{it}, \alpha_{it}\}_{t=1}^{T}$ and the state variables are $\{t, X_{it}, v_{it}\}_{t=1}^{T}$. The agent solves for policy rules as a function of the state variables—$C_{it}(X_{it}, v_{it})$ and $\alpha_{it}(X_{it}, v_{it})$.

1.2 Labor income

An important driver of the paper's results is the labor income process. It is specified as follows:

$$\log Y_{i,t} = f(t, Z_{i,t}) + v_{i,t} + \epsilon_{i,t}, \quad \text{for } t \le K. \tag{3}$$

where K is the (exogenous) age of retirement, $Z_{i,t}$ are demographic characteristics, $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_\epsilon^2)$ is a transitory shock, and $v_{i,t}$ is a permanent component following a random walk

$$v_{i,t} = v_{i,t-1} + u_{i,t} = v_{i,t-1} + \xi_t + \omega_{i,t} \tag{4}$$

in which the innovation is decomposed into an aggregate (ξ_t) and an idiosyncratic component ($\omega_{i,t}$), both following mean-0 normal distributions.

Post-retirement income is a constant fraction λ of permanent income in the last working year K.

A crucial aspect of the labor income process is that $f(\cdot,\cdot)$ is calibrated to match income profiles in the PSID, capturing the usual humped shape of income across lifetime.

1.3 Matching labor income in HARK

In HARK's consumption-saving models, the income process takes the form

$$\ln Y_t = \ln P_t + \ln \theta_t \tag{5}$$

where P_t represents permanent income and $\ln \theta_t \sim N(0, \sigma_\theta)$ transitory shocks to income. Permanent income evolves according to

$$\ln P_{t+1} = \ln \Gamma_{t+1} + \ln \psi_{t+1} + \ln P_t \tag{6}$$

where Γ_{t+1} is a deterministic growth factor, and $\ln \psi_{t+1} \sim N(0, \sigma_\psi)$ a permanent income shock.

To represent the author's assumptions in HARK, we express both income processes as sums of deterministic components and i.i.d shocks

$$\begin{aligned} &\text{Cocco et. al} \quad \ln Y_{i,t} = && f(t, Z_{i,t}) + v_{i,0} + && \sum_{k=1}^{t} u_{i,k} + && \varepsilon_{i,t} && (7)\\ &\text{HARK} \quad \ln Y_{i,t} = && \ln P_{i,0} + \sum_{k=1}^{t} \ln \Gamma_k + && \sum_{k=1}^{t} \ln \psi_{i,k} + && \ln \theta_{i,t}. && (8) \end{aligned}$$

These representations make evident the mapping that we use

Table 1 Mapping to HARK

HARK	**Cocco et. al**
$\ln P_{i,0}$	$f(0, Z_{i,0}) + v_{i,0}$
$\ln \Gamma_{t+1}$	$f(t+1,\ Z_{i,t+1})$ - $f(t, Z_{i,t})$
$\ln \psi_{i,k}$	$u_{i,k}$
$\ln \theta_{i,t}$	$\varepsilon_{i,t}$
$\ln \psi_{i,k}$	$u_{i,k}$
$\ln \theta_{i,t}$	$\varepsilon_{i,t}$

and to achieve a retirement income that is equal to a fraction λ of permanent income in the last working period K, we simply make $\Gamma_{K+1} = \lambda$ and $\Gamma_t = 1\ \forall t > K+1$.

1.4 Assets and their returns

There are two assets available for consumers to allocate their savings.

- Bonds: paying a risk-free return $\bar{R}_f$.
- Stocks: paying a stochastic return $R_t = \bar{R}_f + \mu + \eta_t$, where the stochastic component $\eta_t \sim \mathcal{N}(0, \sigma^2_\eta)$ is allowed to be correlated with the aggregate labor income innovation ξ_t.

The borrowing and short-selling constraints ensure that agents cannot allocate negative dollars to either of these assets or borrow against future labor income or retirement wealth. Recall $\alpha_{i,t}$ is the proportion of the investor's savings that are invested in the risky asset. The model's constraints imply that $\alpha_{i,t} \in [0, 1]$ and wealth is non-negative.

2 Calibration—Summary

The paper defines and calibrates several parameters which are summarized in Table 2:

Table 2 Model Calibration

Description	**Parameter**	**Value**
Time Preference Factor	δ	0.96
Coefficient of Relative Risk Aversion	γ	10
Survival Probability	p_t	[0.6809,0.99845]
Starting Age	t_0	20
Retirement Age	t_r	65
Maximum Age	T	100
Average income at each stage of life	$f(t, Z_{i,t})$	$\exp(0.530339 + 0.1681t+$ $(0.0323371/10)t^2+$ $(0.0019704/100)t^3)$
Last Period Labor Income Share for Retirement	λ	0.68212
Permanent Income Growth Factor	$\log \Gamma$	$\{\log f_{t+1} - \log f_t\}_{t=20}^{t_r+1}$
Interest Factor	R	1.02
Average Stock Return	μ	1.06
Std Dev of Stock Returns	σ_η	0.157
Std Dev of Log Permanent Shock	σ_v	0.102956
Std Dev of Log Transitory Shock	σ_ϵ	0.27166

3 A note on normalization

The problem as specified above makes the value function homogeneous with respect to permanent labor income. This is convenient as it allows for a re-statement of the problem in variables that are normalized by permanent income or its random components, eliminating a state variable.

The authors report (page 497) taking the normalization $v_{i,t} = 1$. This amounts to defining normalized variables $\tilde{\cdot}$ as the original variable divided by $e^{v_{i,t}-1}$. For instance:

$$\tilde{Y_{i,t}} = \frac{Y_{i,t}}{\exp(v_{i,t} - 1)} = \frac{\exp(f(t, Z_{i,t}) + v_{i,t} + \varepsilon_{i,t})}{\exp(v_{i,t} - 1)} = \exp(f(t, Z_{i,t}) + 1 + \varepsilon_{i,t}) \tag{9}$$

These normalized variables have the convenient interpretation of the state that things would be in if, it weren't for permanent shocks. The author's depictions of policy functions are presented in terms of these normalized variables.

On the other hand, HARK normalizes the problem by total permanent income $P_t = \exp(f(t, Z_{i,t}) + v_{i,t})$ and its solution objects are therefore in terms of normalized variables $\hat{\cdot}$, defined as

$$\hat{X_{i,t}} = \frac{X_{i,t}}{P_{i,t}} = \frac{X_{i,t}}{\exp(f(t, Z_{i,t}) + v_{i,t})}. \tag{10}$$

Therefore, to present our results in a way consistent with that of the original authors, we would use the following relationship

$$\tilde{X_{i,t}} = \hat{X_{i,t}} \times \exp(f(t, Z_{i,t}) + 1) \tag{11}$$

However, our results are much more consistent with those of the original authors when we take the normalization $v_{i,t} = 0$, which also make sense since it makes the random-walk multiplicative part of permanent income $\exp v_{i,t} = 1$. We therefore assume this is a typo, take $v_{i,t} = 0$, document this issue in Section 8 below, and use the relationship

$$\tilde{X_{i,t}} = \hat{X_{i,t}} \times \exp(f(t, Z_{i,t})). \tag{12}$$

4 Key Results

4.1 The optimal risky asset share

Figure 1 shows the policy function for the risky portfolio share as a function of wealth at different ages.

The optimal risky share is decreasing in wealth. The authors argue this is due to the fact that, at low levels of wealth, relatively safe human wealth represents a higher fraction of the consumer's wealth, so he shifts his tradeable wealth towards riskier alternatives.

Analyzing the policy rule by age also shows that the risky share increases from young to middle age, and decreases from middle to old age. This is consistent with the previous interpretation: shares trace the humped shape of labor earnings.

These estimates are different from what is produced in the original paper, which are also reproduced below. Generally, the policy functions do not share the same curvature, which leads to greater reductions in the optimal portfolio share at lower levels of wealth.

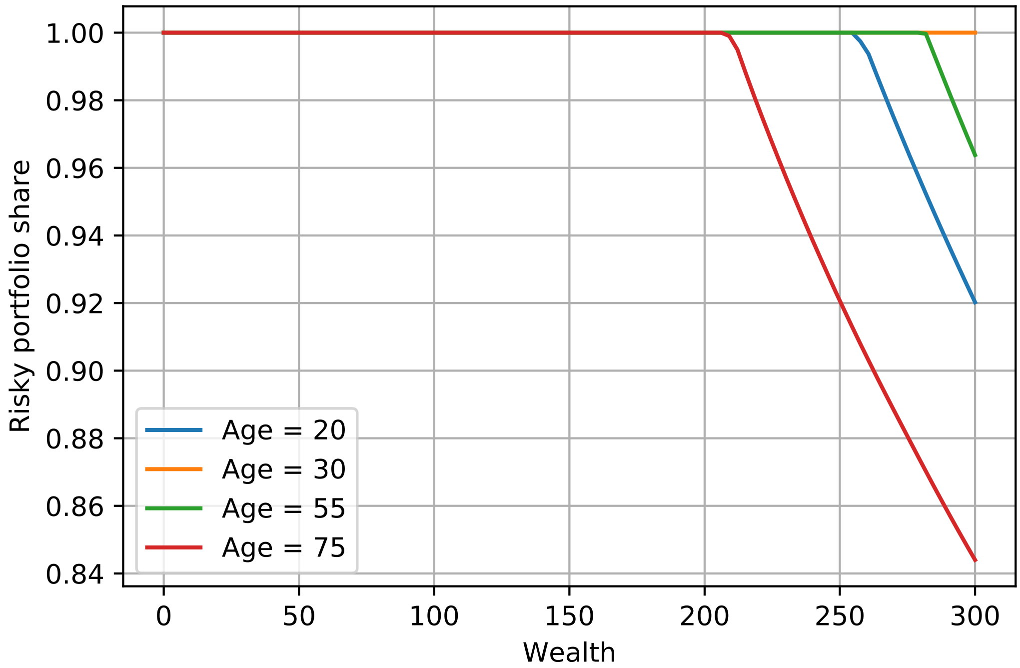

Figure 1 Risky Share Policy Function

4.2 Consumption behavior

Figure 2 below shows the policy function for consumption as a function of wealth at different ages.

At all age levels consumption increases with wealth. The consumption function also appears to shift upwards as life progresses.

Our consumption policy functions again do not match those of the original paper, which are also reproduced below. Consumption also appears to increase with age in our policy functions that does not come through in the results presented in the paper.

5 Simulations

Using the policy functions obtained from solving the model we present a series of simulations to highlight features of the model.

We first run a few simulations to verify the quality of our calibration.

The Figures 3 and 4 below show simulated levels of permanent income and risky portfolio shares for 5 agents over their life spans. We can see the model generates a heterogeneous permanent income distribution. Interestingly, all of these agents tend to follow the same general pattern for investing in the risky asset. Early in life, all of their

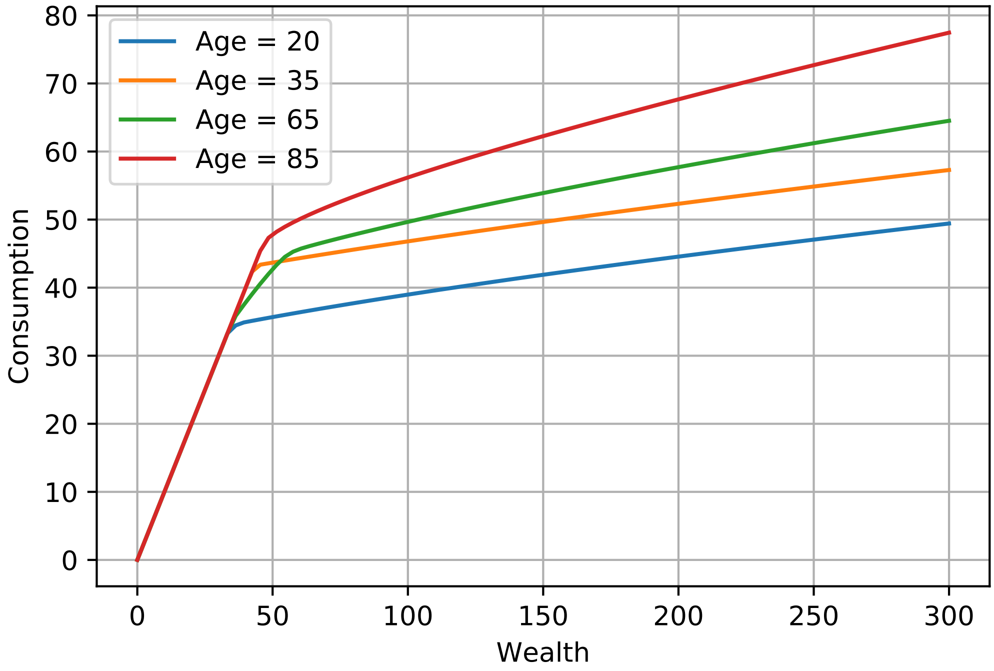

Figure 2 Consumption Policy Function

portfolios are invested in the risky asset. This declines as the agent ages and converges to approximately 35% once they reach retirement.

5.1 The average life cycle patterns

We now increase the number of simulations to examine and compare the behavior of the mean values of variables of interest at different ages, conditional on survival. In each case we present the original plots from the paper for reference.

Figure 5 below illustrates the average dynamics of permanent income, consumption, and market resources across all of the simulated agents. The plot follows the general pattern observed in the original paper. However, our results show that the agents are accumulating significantly more market resources. Figure 6 presents mean portfolio share conditional on survival.

6 Other results in the original paper

6.1 The welfare implications of different allocation rules

The authors next conduct a welfare analysis of different allocation rules, including popular heuristics. The rules are presented in the Figure 7.

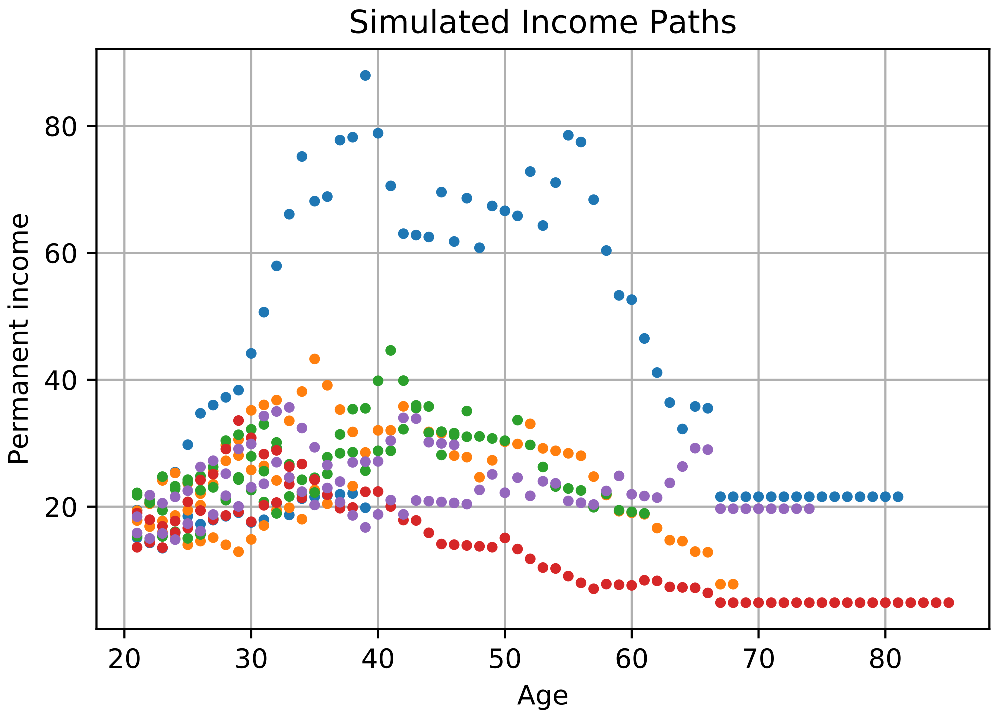

Figure 3 Income Simulation

The utility cost of each policy in terms of constant consumption streams with respect to the authors calculated optimal policy function is reported in the next Figure 8.

Interestingly, the "no-income" column corresponds to the usual portfolio choice result of the optimal share being the quotient of excess returns and risk times relative risk aversion, disregarding labor income. The experiment shows this allocation produces substantial welfare losses.

6.2 Heterogeneity and sensitivity analysis

The authors also considered a number of extensions to the baseline model. These are summarized below along with their main conclusions.

- Labor income risk: Income risk may vary across employment sectors relative to the baseline model. The authors examine extreme cases for industries that have a large standard deviation and temporary income shocks. While some differences appear across sectors, the results are generally in line with the baseline model.

- Disastrous labor income shocks: The authors find that even a small probability of zero labor income lowers the optimal portfolio allocation in stocks, while the qualitative features of the baseline model are preserved.

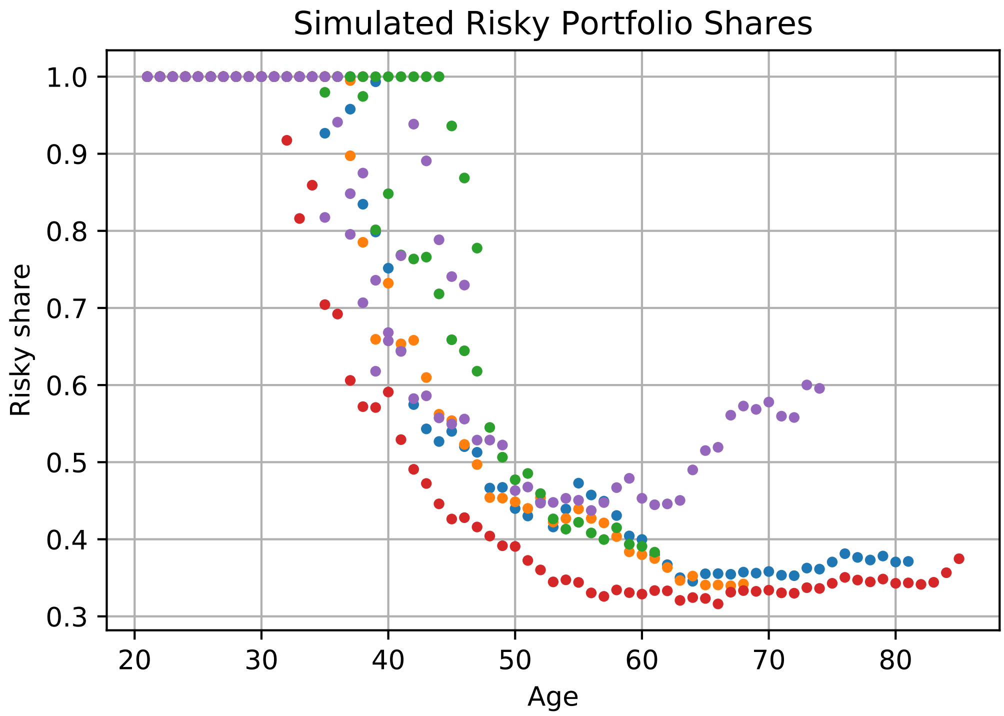

Figure 4 Risky Share Simulation

- Uncertain retirement income: The authors consider two types of uncertainty for retirement income; it is stochastic and correlated with current stock market performance and allowing for disastrous labor income draws before retirement. The first extension has results essentially the same as the baseline case. The second leads to more conservative portfolio allocations but is broadly consistent with the baseline model.

- Endogenous borrowing constraints: The authors add borrowing to their model by building on credit-market imperfections. They find that the average investor borrows about \$5,000 and are in debt for most of their working life. The agents eventually pay off this debt and save for retirement. Relative to the benchmark model, the investor has put less of their money in their portfolio and arrive at retirement with substantially less wealth. These results are particularly pronounced at the lower end of the income distribution relative to the higher end. Additional details are available in the text.

- Bequest motive: The authors introduce a bequest motive into the agent's utility function (i.e., $b > 0$). Young investors are more impatient and tend to save less for bequests. As the agent ages, savings increases and is strongest once the agent retires. This leads to effects on the agent's portfolio allocation. Taking a step-back however, these effects are not very large unless b is large.

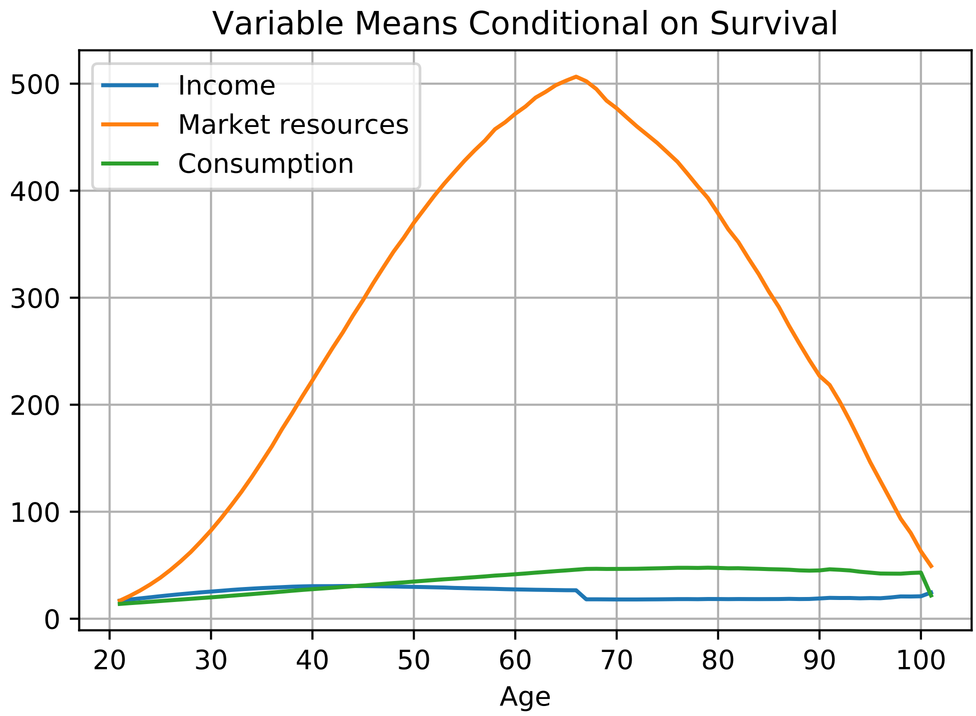

Figure 5 Variable Means Conditional on Survival

- Educational attainment: The authors generally find that savings are consistent across education groups. They note that for a given age, the importance of future income is increasing with education level. This implies that riskless asset holdings are larger for these households.
- Risk aversion and intertemporal substitution: Lowering the level of risk aversion in the model leads to changes in the optimal portfolio allocation and wealth accumulation. Less risk-averse investors accumulate less precautionary savings and invest more in risky assets.

7 Conclusion

This article provides a dynamic model with accurate lifetime income profiles in which labor income increases risky asset holdings, as it is seen as a closer substitute of risk-free assets. It finds an optimal risky asset share that decreases in wealth and with age, after middle age. The model is also used to show that ignoring labor income for portfolio allocation can generate substantial welfare losses.

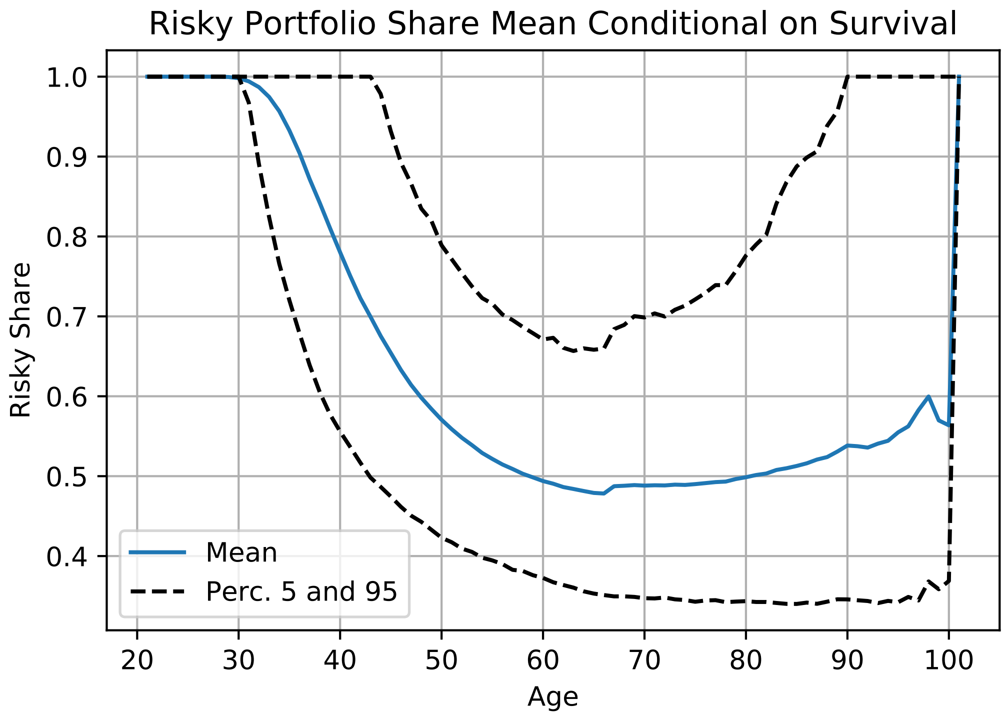

Figure 6 Risky Portfolio Share Mean Conditional on Survival

8 Puzzles and Questions

- Table 4 says stock returns are 0.06. They might mean that the equity premium μ is 0.06.
- The authors report taking the normalization $v_{i,t} = 1$. However the ranges of their results seem more consistent with $v_{i,t} = 0$ so that $\exp(v_{i,t}) = 1$, which also makes more sense for interpretation.

9 Robustness Analyses

Given the differences between our results and the original paper, we did a number of checks to ensure our model was behaving consistently with well-established theoretical results. Specifically we checked:

- For an infinitely lived agent with log normal returns, that their optimal portfolio allocation converges to the Campbell-Viceira (2002) approximation to the optimal portfolio share in Merton-Samuelson (1969) model.
- For an infinitely lived agent with no labor income that can only invest in a single risky asset, that their marginal propensity to consumer converges to the theoretical MPC of Merton-Samuelson (1969).

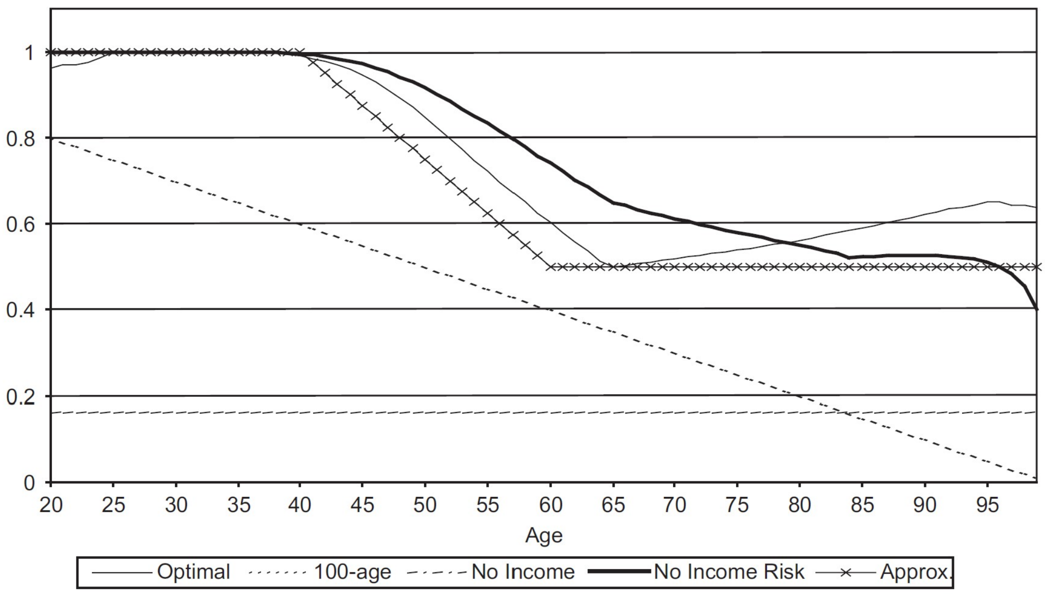

Figure 11
Simulated (or predicted) portfolio share invested in stocks for different alternative investment strategies
"Optimal" denotes the optimal share predicted by the model; "100-age" refers to the common recommendation given by several financial advisers; "No Income" is the optimal allocation for an household without labor income; "No Income Risk" is the optimal allocation for an household with riskless labor income.

Figure 7 CGM Allocation Rules

Utility cost calculation (percentage points)

Parameters	100-Age	No income*	No income risk**	Zero	Approx.***
Benchmark	0.637	1.531	0.152	2.108	0.084

Figure 8 CGM Utility Costs

- For an agent facing no labor income risk, that their consumption patterns precisely match the results from a perfect foresight solution.

In all three cases, we verified that our HARK model holds up to these results. More details and specific results are available upon request.

As the HARK toolkit continues to develop, there are additional sensitivities that we can perform to further check the credibility of our results. Specifically, once human wealth is available in the `PortfolioConsumerType` class, we can perform the following additional checks, which were kindly suggested by Professor Sylvain Catherine:

- Shut down the income risk and remove retirement income. The solution to this new problem are provided by Merton 1971. Basically, you capitalize future earnings as an endowment of risk free asset. Then the equity share should be such that Equity/(Wealth+NPV of Human capital) is the same as the equity share in Merton 1969.

- Adding back the permanent income risk and check if the equity share is consistent with Viceira 2001. Viceira tells you something like this: $\pi = \frac{\mu - r}{\gamma\sigma_s^2} + \left(\frac{\mu - r}{\gamma\sigma_s^2} - \beta_{HC}\right)\frac{HC}{W}$, where $\beta_{HC} = \frac{\mathrm{Cov}(r_{HC}, r_s)}{\mathrm{Var}(r_s)}$. In the CGM problem it is easy to compute β_{HC} because earnings follow a simple random walk. HC is the NPV of human capital, which you can approximate very well by discounting expected earnings by $r + \beta_{HC} * (rm - r)$.

Appendices

A Comparison with CGM

In this section, we compare the policy functions that we obtain using HARK, with those that are obtained using CGM's publicly available `Fortran 90 code`.

We attempted to execute the authors' code as-is, but obtained a "division by zero" error, apparently caused by one of the asset grids starting at 0, allowing the possibility of a 0-level of consumption to be evaluated in the utility function. We fixed the issue by making said grid start at 1. The figures produced in this section use the policy functions outputted by the author's code after this fix.

Figures 9 and 10 display policy functions for consumption and risky portfolio shares obtained with CGM's code and our HARK implementation, and their substantial differences.

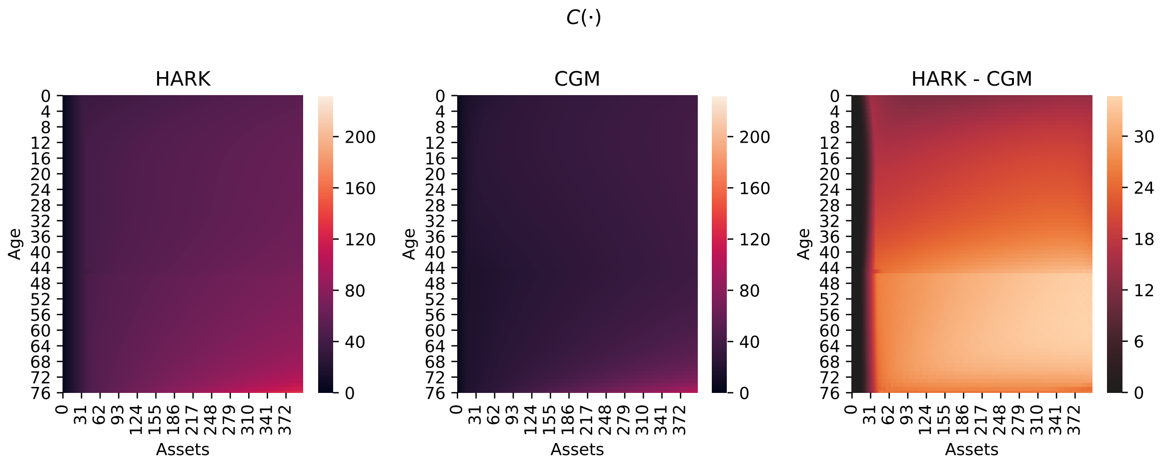

Figure 9 Consumption policy functions in HARK and CGM's Fortran code.

Given the differences, we inspect the second and third-to-last life periods more closely. Figure 11 presents policy functions and their differences for these periods. The discrepancies are evident.

B Robustness analyses

Given that our main set of results do not align with those of CGM, we provide a few tests that compare the behavior of the tools that we are using with well known theoretical results.

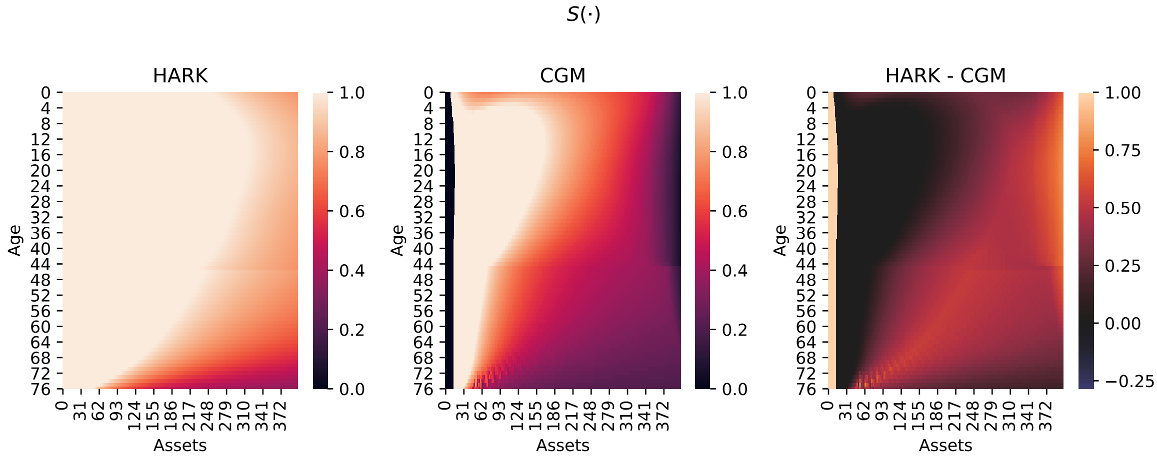

Figure 10 Risky share policy functions in HARK and CGM's Fortran code.

B.1 Merton Samuelson's limiting risky share

Merton and Samuelson (TODO: CITE) show that a consumer with constant relative risk aversion ρ, a risky asset with log-normally distributed returns with a risk premium ϕ and log-standard-deviation σ_r, and no labor income uncertainty will invest the following share of their income:[1]

$$s = \frac{\phi}{\rho\sigma_r^2}. \tag{13}$$

This result holds as long as labor income is an *unimportant* source of wealth for the agent.

To test HARK's implementation, we modify returns to the risky asset to be log-normal and keep the rest of the calibration the same. Since labor income is unimportant for infinitely wealthy agents, our risky portfolio share policy rules must converge to Merton and Samuelson's limit as market resources approach infinity. Figure 12 shows that this is indeed the case.

B.2 Marginal Propensity to Consume

CRRA-RateRisk shows that an infinitely lived agent with no labor income and who can only store his wealth in a risky asset with log-normally distributed returns must have a marginal propensity to consume given by:

$$\kappa = 1 - \left(\beta \mathbb{E}_t \left[\mathfrak{R}^{1-\rho}\right]\right)^{1/\rho} \tag{14}$$

where $\mathfrak{R}$ is the risky asset's return factor.

We modify our calibration to have log-normal returns and to ensure that Equation 14 is a positive number. We also set the risk free rate low enough that the agent will allocate all of his wealth to the risky asset. Since the result holds for infinitely lived

[1] See `Portfolio-CRRA`.

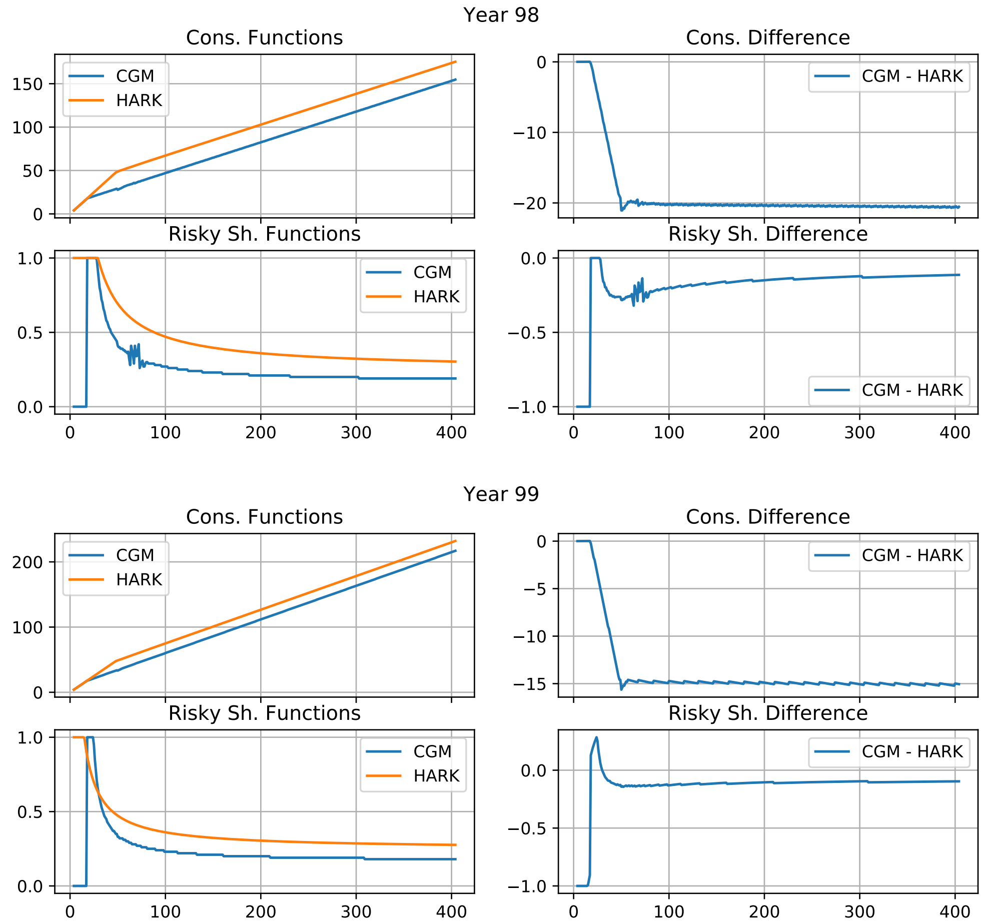

Figure 11 Policy functions in the second and third to last periods of life.

consumers with no labor income, we should see that our estimated marginal propensity to consume converges to κ for young and wealthy consumers. Figure 13 shows that this is indeed the case.

B.3 Perfect Foresight analytical solution

To further verify that HARK's solution algorithm is producing accurate results, we compare the consumption policy function obtained by the `PortfolioConsumerType` class through backward induction with the analytical expression that can be obtained for consumption in a perfect foresight setting.

PrefForesightCRRA shows that a consumer facing no uncertainty will attempt to consume:

$$c_t^* = \frac{1 - \left[R^{-1}\left(R\beta\right)^{1/\rho}\right]}{1 - \left[R^{-1}\left(R\beta\right)^{1/\rho}\right]^{T-t+1}} \times o_t \tag{15}$$

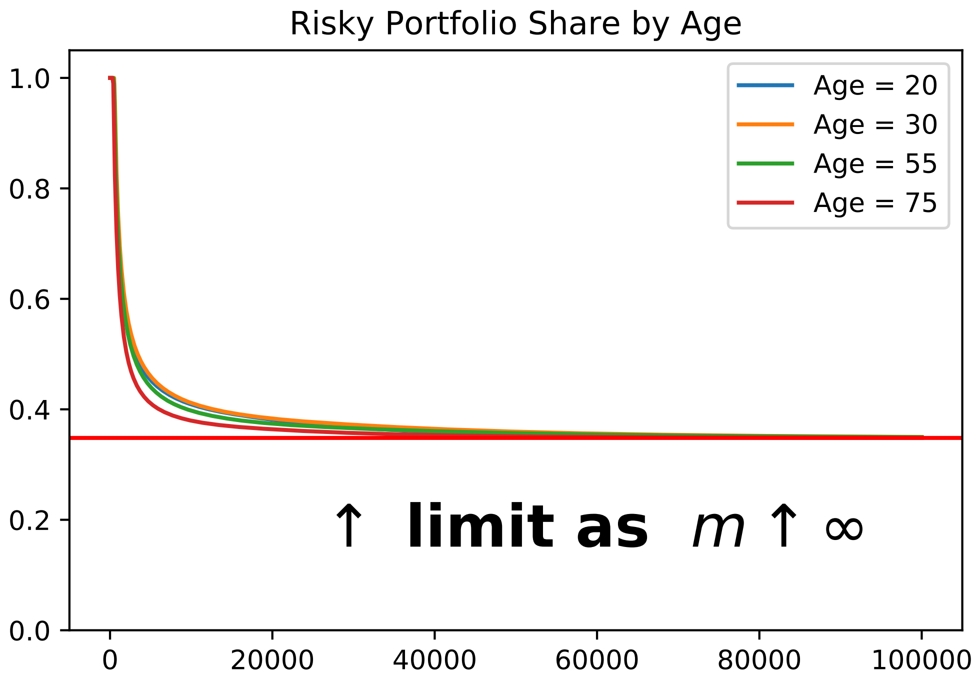

Figure 12 Merton Samuelson as the limit of the risky asset's portfolio share.

where o_t is his overall (human plus non-human) wealth. In the presence of a liquidity constraint, the agent will then consume:

$$c_t = \min\{m_t, c_t^*\} \tag{16}$$

where m_t are his market resources at time t. We refer to this as the *true* solution.

We shut down all sources of uncertainty in our calibration and, for ease of analytical expressions, assume a constant level of income. We then solve the agent's problem using `PortfolioConsumerType`'s backward induction algorithm and compare its solution both with the true solution and the one obtained by `PerfForesightConsumerType`, another HARK class representing perfect-foresight consumers.

Figure 14 plots consumption functions obtained through the three different methods and three different periods of the agent's life. The `PortfolioConsumerType` policy function aligns with the true solution, but that of `PerfForesightConsumerType` does not.

Figure 15 more closely examines deviations in HARK's solutions from the true solution. `PortfolioConsumerType`'s solution is very close to the true solution at every age and level of wealth with deviations appearing only around the consumption function's kink, which is caused by the liquidity constraint that we impose. On the other hand, the solution provided by `PerfForesightConsumerType` has substantial differences in the level of consumption, which widen with the agent's age.

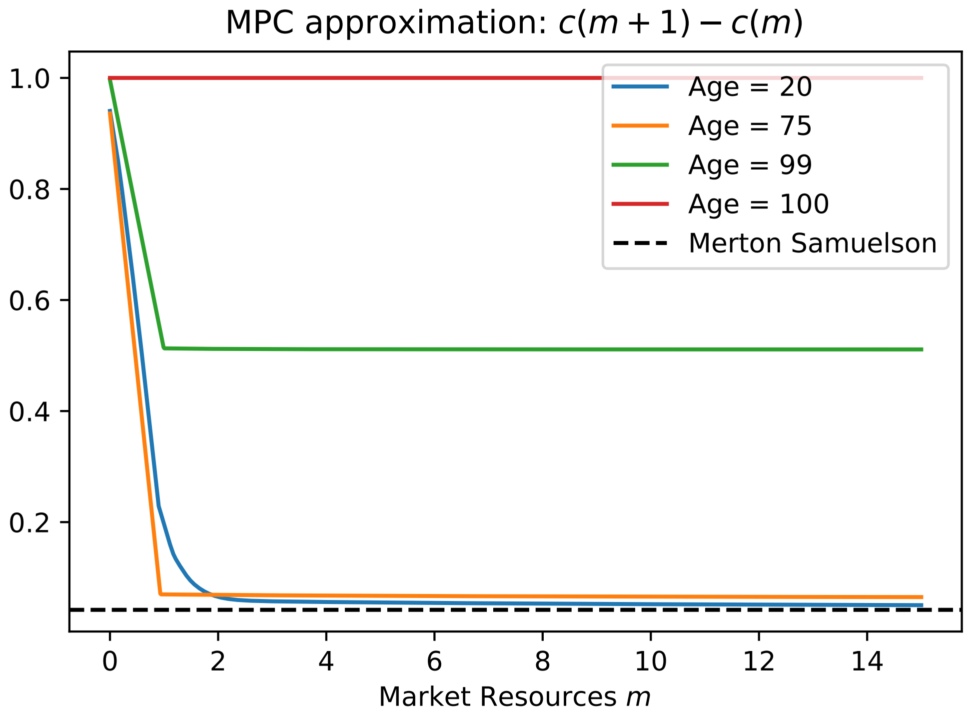

Figure 13 Marginal propensity to consume as $m \to \infty$.

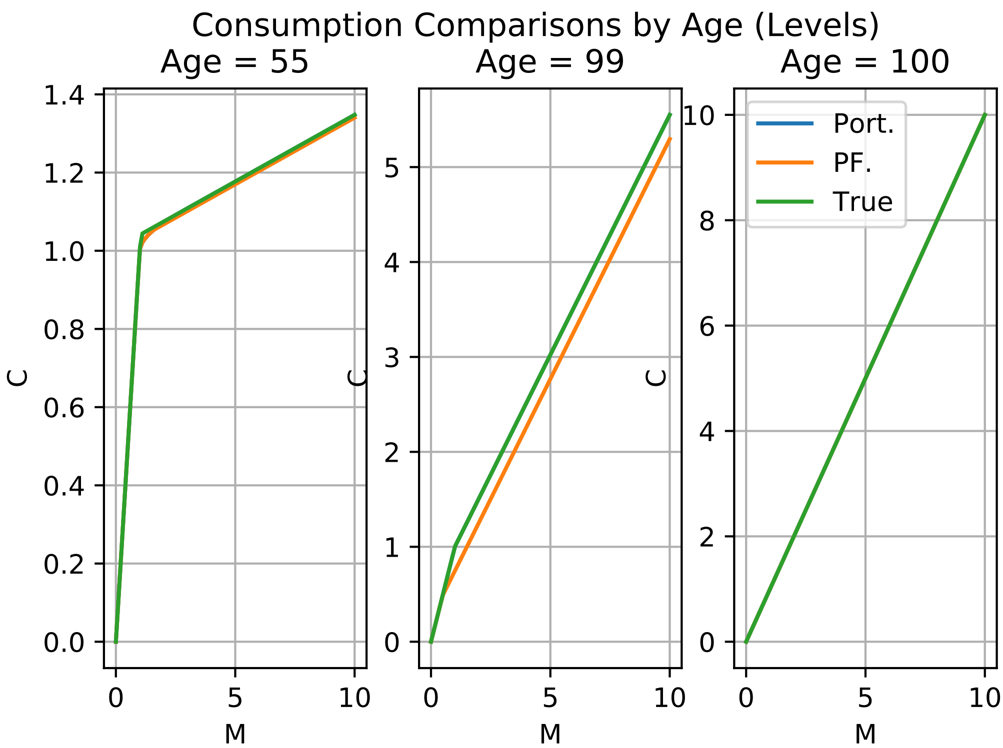

Figure 14 Perfect foresight solutions using different HARK tools.

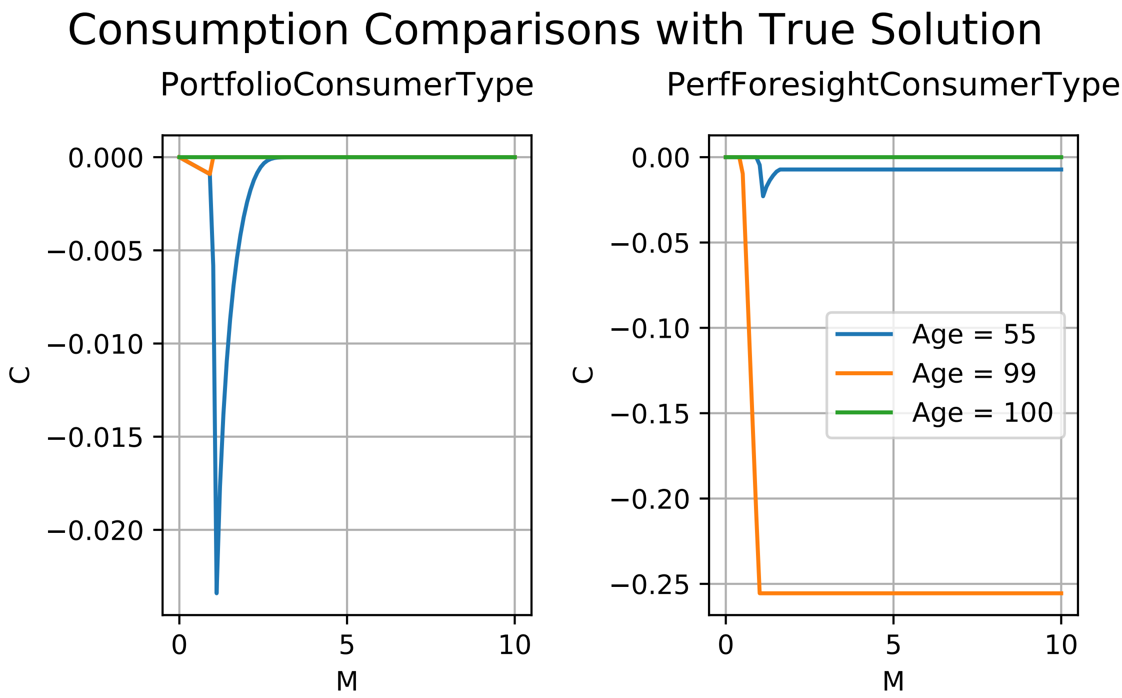

Figure 15 Differences from the true perfect foresight solution.