How to Multiply Matrices

Advanced

A Matrix is an array of numbers:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 3 Columns)

To multiply a matrix by a single number is easy:

2x4=8

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

These are the calculations:

2×4=8	2×0=0
2×1=2	2×-9=-18

We call the number ("2" in this case) a **scalar**, so this is called "scalar multiplication".

Multiplying a Matrix by Another Matrix

But to multiply a matrix **by another matrix** we need to do the "dot product" of rows and columns ... what does that mean? Let us see with an example:

To work out the answer for the **1st row** and **1st column**:

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \\ & \end{bmatrix}$$

The "Dot Product" is where we **multiply matching members**, then sum up:

$$(1, 2, 3) \bullet (7, 9, 11) = 1{\times}7 + 2{\times}9 + 3{\times}11$$
$$= 58$$

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

Want to see another example? Here it is for the 1st row and **2nd column**:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ & \end{bmatrix}$$

$$(1, 2, 3) \bullet (8, 10, 12) = 1{\times}8 + 2{\times}10 + 3{\times}12$$
$$= 64$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \bullet (7, 9, 11) = 4{\times}7 + 5{\times}9 + 6{\times}11$$
$$= 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \bullet (8, 10, 12) = 4{\times}8 + 5{\times}10 + 6{\times}12$$
$$= 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

DONE!

dependent Var

independent Var

Why Do It This Way?

We have a function that has an output based on input

if we want to repeat a process for different inputs with the same scalers, we can scale (repeat) this process using Matricies. We can then compare the data after

This may seem an odd and complicated way of multiplying, but it is necessary!

I can give you a real-life example to illustrate why we multiply matrices in this way.

Example: The local shop sells 3 types of pies.

- Beef pies cost **$3** each
- Chicken pies cost **$4** each
- Vegetable pies cost **$2** each

Scalers

$$\theta = \underbrace{[3, 4, 2]}_{\text{Scalers}} \cdot \begin{bmatrix} 13 \\ 8 \\ 6 \end{bmatrix} = (3\times13 + 4\times8 + 2\times6) = (39 + 32 + 12) = 83$$

$\mathbb{R}^{1\times3}$ · $\mathbb{R}^{3\times1}$ = $\mathbb{R}^{1\times1}$

dot

Monday's inputs

total Price

total Monday

Scale, for each data sample

And this is how many they sold in 4 days:

	Mon	Tue	Wed	Thu
Beef	**13**	**9**	**7**	**15**
Chicken	**8**	**7**	**4**	**6**
Vegetable	**6**	**4**	**0**	**3**

x^1 x^2 x^n

Now think about this ... the **value of sales** for Monday is calculated this way:

Beef pie value + Chicken pie value + Vegetable pie value

$$\$3\times13 + \$4\times8 + \$2\times6 = \$83$$

39 32 12

So it is, in fact, the "dot product" of prices and how many were sold:

$$(\$3, \$4, \$2) \bullet (13, 8, 6) = \$3\times13 + \$4\times8 + \$2\times6$$
$$= \$83$$

We **match** the price to how many sold, **multiply** each, then **sum** the result.

In other words:

- The sales for Monday were: Beef pies: **$3×13=$39**, Chicken pies: **$4×8=$32**, and Vegetable pies: **$2×6=$12**. Together that is $39 + $32 + $12 = **$83**
- And for Tueday: **$3×9 + $4×7 + $2×4 = $63**
- And for Wednesday: **$3×7 + $4×4 + $2×0 = $37**
- And for Thursday: **$3×15 + $4×6 + $2×3 = $75**

So it is important to match each price to each quantity.

Now you know why we use the "dot product".

And here is the full result in Matrix form:

$$\underset{\frac{\text{Scalers}}{\text{price}}}{\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix}} \times \underset{\frac{\text{input}}{\text{Portion}}}{\begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix}} = \begin{bmatrix} \overset{\text{Monday}}{\$83} & \overset{\text{tuesday}}{\$63} & \overset{\text{wed}}{\$37} & \overset{\text{thursday}}{\$75} \end{bmatrix}$$

$$\$3\text{x}13 + \$4\text{x}8 + \$2\text{x}6$$

Scaled outputs for scaled inputs

They sold **$83** worth of pies on Monday, **$63** on Tuesday, etc.

(You can put those values into the [Matrix Calculator] to see if they work.)

Rows and Columns

To show how many rows and columns a matrix has we often write **rows×columns**.

Example: This matrix is **2×3** (2 rows by 3 columns):

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

When we do multiplication:

- The number of **columns of the 1st matrix** must equal the number of **rows of the 2nd matrix**.
- And the result will have the same number of **rows as the 1st matrix**, and the same number of **columns as the 2nd matrix**.

Example:

$$\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix} \times \begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \$83 & \$63 & \$37 & \$75 \end{bmatrix}$$

$$\$3\text{x}13 + \$4\text{x}8 + \$2\text{x}6$$

In that example we multiplied a 1×3 matrix by a 3×4 matrix (note the 3s are the same), and the result was a 1×4 matrix.

In General:

To multiply an **m×n** matrix by an **n×p** matrix, the **n**s must be the same, and the result is an **m×p** matrix.

$$m \times n \;\times\; n \times p \;\rightarrow\; m \times p$$

Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3×3 Identity Matrix

- It is "square" (has same number of rows as columns)
- It can be large or small (2×2, 100×100, ... whatever)
- It has **1**s on the diagonal and **0**s everywhere else
- Its symbol is the capital letter **I**

It is a **special matrix**, because when we multiply by it, the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Order of Multiplication

In arithmetic we are used to:

$$3 \times 5 = 5 \times 3$$

(The Commutative Law of Multiplication)

But this is **not** generally true for matrices (matrix multiplication is **not commutative**):

$$\boxed{AB \neq BA}$$

When we change the order of multiplication, the answer is (usually) **different.**

Example:

See how changing the order affects this multiplication:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

The answers are different!

It **can** have the same result (such as when one matrix is the Identity Matrix) but not usually.

Question 1 Question 2 Question 3 Question 4 Question 5 Question 6
Question 7 Question 8 Question 9 Question 10 Question 11 Question 12