Derivation of cavity photon number

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Nomenclature

κ_c	Photon loss rate due to leakage
κ_i	Photon loss rate due to internal dissipation
κ_T	Total photon loss rate
ω_r	Resonant cavity frequency
P_{App}	Applied power at input of resonator
P_{cav}	Cavity photon power
P_{in}	Power leaking into cavity
P_{out}	Power leaking out of cavity
Q_c	Coupling quality factor
Q_i	Internal quality factor
Q_T	Total quality factor

Derivation

The power leaking into a cavity is proportional to the applied power at the cavity input capacitor (inductor) mediated by the coupling strength

$$P_{\mathrm{in}} = P_{\mathrm{App}} \frac{\kappa_c}{\omega_r} = \frac{P_{\mathrm{App}}}{Q_c} \tag{1}$$

Additionally, the power leaking *out* of a cavity is a linear combination of the power dissipated in the cavity and the power that leaks back out of the coupling capacitor(s).

$$P_{\mathrm{out}} = P_{\mathrm{cav}} \frac{\kappa_c + \kappa_i}{\omega_r} = \frac{P_{\mathrm{cav}}}{Q_T} \tag{2}$$

At equilibrium, the powers leaking into and out of the cavity will be equal yielding

$$P_{\mathrm{cav}} = P_{\mathrm{App}} \frac{Q_T}{Q_c} \tag{3}$$

In the absence of applied power, we can think of the resonator as an energy source that delivers power at a rate

$$P_{\mathrm{cav}} = \bar{n}\hbar\omega\kappa_T = \frac{\bar{n}\hbar\omega^2}{Q_T} \tag{4}$$

Equating equations (3) and (4) we find that

$$\bar{n} = P_{\mathrm{App}} \times \left(\frac{Q_T^2}{Q_c \hbar\omega^2} \right) \tag{5}$$

where

$$Q_T = \left(\frac{1}{Q_c} + \frac{1}{Q_i} \right)^{-1} \tag{6}$$