Relating Transmission Line Resonators To Lumped Elements

July 19, 2016

Nomenclature

α	Attenuation constant
β	Propagation constant
ω	Frequency
ω_0	Resonant cavity frequency
C	Equivalent capacitance
C'	Capacitance per unit length
L	Equivalent Inductance
l	Resonator length
L'	Inductance per unit length
R	Equivalent Resistance
Z_0	Characteristic Impedance of transmission line
Z_{in}	Impedance looking into the transmission line

Open Circuited $\lambda/2$ line

Input Impedance

$$Z_{\text{in}} = Z_0 \left(\frac{1 + i \tan(\beta l) \tanh(\alpha l)}{\tanh(\alpha l) + i \tan(\beta l)} \right) \tag{1}$$

Approximations

$$\tanh(\alpha l) \to \alpha l \tag{2}$$

$$\tan(\beta l) \to \pi \frac{\Delta\omega}{\omega_0} \tag{3}$$

$$\tanh(\alpha l) \tan(\beta l) = \pi \frac{\Delta\omega}{\omega_0} \alpha l \ll 0 \tag{4}$$

Equivalent Z_{in}

$$Z_{\text{in}} \approx \frac{Z_0/\alpha l}{1 + i\frac{\pi}{\alpha l \omega_0}(\omega - \omega_0)} \tag{5}$$

LRC Resonator

$$Z_{\text{LCR}} = \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C \right)^{-1} \approx \frac{R}{1 + i2RC(\omega - \omega_0)} \tag{6}$$

Equate the two impedances

$$\frac{Z_0/\alpha l}{1+i\frac{\pi}{\omega_0\alpha l}(\omega-\omega_0)} = \frac{R}{1+i2RC(\omega-\omega_0)} \tag{7}$$

$$R = \frac{Z_0}{\alpha l} \tag{8}$$

$$2RC = \frac{\pi}{\omega_0\alpha l} \tag{9a}$$

$$2\,\frac{Z_0}{\alpha l}C = \frac{\pi}{\omega_0\alpha l} \tag{9b}$$

$$C = \frac{\pi}{2\omega_0 Z_0} \tag{9c}$$

We see now that Eq. (9c) matches correctly with the form given in equation 6.34b of Pozar. We can take this a step further an related the lumped element capacitance to the CPW capacitance per unit length in the following way..

$$Z_0 = \sqrt{\frac{L'}{C'}} \tag{10a}$$

$$v_{\mathrm{ph}} = \frac{1}{\sqrt{L'C'}} \tag{10b}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega_0\sqrt{L'C'}} \tag{10c}$$

$$\lambda = 2l \tag{10d}$$

$$2l = \frac{2\pi}{\beta} = \frac{2\pi}{\omega_0\sqrt{L'C'}} \rightarrow \omega_0 = \frac{\pi}{l\sqrt{L'C'}} \tag{10e}$$

We now substitute 10e into 9c and obtain

$$C = \frac{\pi}{2\omega_0 Z_0} = \frac{\pi}{2}\frac{l\sqrt{L'C'}}{\pi}\sqrt{\frac{C'}{L'}} = \frac{C'l}{2} \tag{11}$$

Which is Eq.(12) in Goppl *et al*, Coplanar waveguide resonators for circuit quantum electrodynamics. We can now calculate the equivalent inductance as a function of inductance per unit length as follows.

$$L = \frac{1}{\omega_0^2 C} \tag{12a}$$

$$L = \frac{1}{\frac{\pi^2}{l^2 L'C'}\frac{C'l}{2}} = \frac{2L'C'l^2}{\pi^2 C'l} = \frac{2L'l}{\pi^2} \tag{12b}$$

Equation (12b) matches equation (11) from Goppl modulo $1/n^2$ where n in the harmonic of the driven frequency, which, in almost all cases, is 1.

Short Circuited $\lambda/4$ line

In much the same way we derived the equivalent inductance and capacitance for a half-wave resonator, we can do so for a quarter-wave resonator.

Input Impedance

$$Z_{\text{in}} = Z_0 \frac{1 - i\tanh(\alpha l)\cot(\beta l)}{\tanh(\alpha l) - i\cot(\beta l)} \tag{13}$$

Approximations

$$\tanh(\alpha l) \to \alpha l \tag{14}$$

$$\cot(\beta l) = -\frac{\pi \Delta\omega}{2\omega_0} \tag{15}$$

Equivalent Z_{in}

$$Z_{\text{in}} \approx= \frac{Z_0/\alpha l}{1 + i\frac{\pi}{2\alpha l}\Delta\omega/\omega_0} \tag{16}$$

LRC Resonator

$$Z_{\text{LCR}} = \left(\frac{1}{R} + \frac{1}{i\omega L} + i\omega C\right)^{-1} \approx \frac{R}{1 + i2RC(\omega - \omega_0)} \tag{17}$$

Equate the two impedances

$$\frac{Z_0/\alpha l}{1 + i\frac{\pi}{2\alpha l}\Delta\omega/\omega_0} = \frac{R}{1 + i2RC(\omega - \omega_0)} \tag{18}$$

$$R = \frac{Z_0}{\alpha l} \tag{19}$$

$$2RC = \frac{\pi}{2\omega_0 \alpha l} \tag{20a}$$

$$2\frac{Z_0}{\alpha l}C = \frac{\pi}{2\omega_0 \alpha l} \tag{20b}$$

$$C = \frac{\pi}{4\omega_0 Z_0} \tag{20c}$$

We see now that Eq.(20c) matches equation (6.30b) from Pozar. We can now relate this to the capacitance per unit length by substituting $\lambda = 4l$ into Eq.10c.

$$4l = \frac{2\pi}{\beta} = \frac{2\pi}{\omega_0\sqrt{L'C'}} \rightarrow \omega_0 = \frac{\pi}{2l\sqrt{L'C'}} \tag{21}$$

We now substitute (21) into (20c) which yields

$$C = \frac{\pi}{4\frac{\pi}{2l\sqrt{L'C'}}\sqrt{\frac{L'}{C'}}} = \frac{2\pi C'l}{4\pi} = \frac{C'l}{2} \tag{22}$$

We can see at one that Eq.(22) and Eq.(11) are equivalent. This equivalency may seem odd at first glance until one remembers that the l in (22) is half as long as the l in (11). The equivalent inductance can be calculated in the same straightforward manner as in (12b).

$$L = \frac{1}{\omega_0^2 C} \tag{23a}$$

$$L = \frac{1}{\frac{\pi^2}{4l^2L'C'}\frac{C'l}{2}} = \frac{8l^2L'C'}{\pi^2 lC'} = \frac{8L'l}{\pi^2} \tag{23b}$$