S_{11} derivation for lumped element parallel LCR circuit

From Pozar:
Signal on a transmission line looking into an impedance Z_l

$$V(z) = V_0^+ e^{i\beta z} + V_0^- e^{-i\beta z}$$

$$I(z) = I_0^+ e^{i\beta z} - I_0^- e^{-i\beta z} = \frac{V_0^+}{Z_0} e^{i\beta z} - \frac{V_0^-}{Z_0} e^{-i\beta z}$$

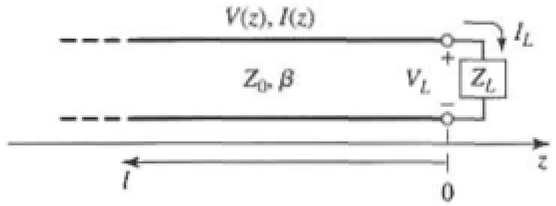

Figure 1: default

$$V(0) = I(0) Z_l \to Z_l = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$

Solve for V_0^-

$$V_0^- = \frac{Z_l - Z_0}{Z_l + Z_0} V_0^+$$

$$S_{11} = \frac{V_0^-}{V_0^+} = \frac{Z_l - Z_0}{Z_l + Z_0}$$

Calculate the impedance of the parallel LCR resonator

$$Z_r = \left[\frac{1}{Z_L} + \frac{1}{Z_c} + \frac{1}{R}\right]^{-1}$$

$$Z_r = \left[-\frac{i}{\omega L} + i\omega C + \frac{1}{R}\right]^{-1}$$

$$Z_r = \left[-\frac{i}{\omega L} + \frac{i\omega^2 LC}{\omega L} + \frac{1}{R}\right]^{-1}$$

$$Z_r = \left[\frac{iR(\omega^2 LC - 1) + \omega L}{R\omega L}\right]^{-1}$$

Equatiing $(LC)^{-1} = \omega_0^2$ and inverting...

$$Z_r = \frac{R\omega L}{\omega L + iR(\omega^2/\omega_0^2 - 1)}$$

$$Z_r = \frac{R}{1 + i\frac{R}{\omega L}(\omega^2/\omega_0^2 - 1)}$$

Rewrite using the Q of the resonator

$$Q_r = \omega_0 RC = R\sqrt{\frac{C}{L}} = \frac{R}{\omega_0 L}$$

$$Z_r = \frac{R}{1 + i\frac{R}{\omega_0 L}(\omega/\omega_0 - \omega_0/\omega)}$$

$$Z_r = \frac{R}{1 + iQ_r(\frac{\omega^2 - \omega_0^2}{\omega\omega_0})}$$

Take $\omega \approx \omega_0$

$$Z_r = \frac{R}{1 + iQ_r(\frac{(\omega+\omega_0)(\omega-\omega_0)}{\omega\omega_0})} = \frac{R}{1 + i2Q_r(\frac{\omega-\omega_0}{\omega_0})}$$

$$x = \frac{\omega - \omega_0}{\omega_0}$$

$$Z_r = \frac{R}{1 + i2Q_r x}$$

$$Z_r = \frac{R}{1 + 4Q_r^2 x^2} - i\frac{2RQ_r x}{1 + 4Q_r^2 x^2}$$

Looking at the Norton Equivalent

$$R_t = \frac{1 + \omega^2 C_c^2 R_l^2}{\omega^2 C_c^2 R_l} \approx \frac{1}{\omega^2 C_c^2 R_l}$$

$$C_t = \frac{C_c}{1 + \omega^2 C_c^2 R_l^2} \approx C_c$$

$$Q_c = \omega R_t C_r = \frac{Q_r R_t}{R}$$

$$\frac{Q_r}{Q_c} = \frac{R}{R_t}$$

The impedance has been transformed by the input capacitor from $Z_0 \to R_t$. So S_{11} now equals

$$S_{11} = \frac{Z_r - R_t}{Z_r + R_t} = \frac{Z_r/R_t - 1}{Z_r/R_t + 1}$$

$$Z_r/R_t = \frac{R/R_t}{1 + 2iQ_r x} = \frac{Q_r/Q_c}{1 + 2iQ_r x}$$

$$S_{11} = \frac{\frac{Q_r/Q_c}{1+2iQ_r x} - 1}{\frac{Q_r/Q_c}{1+2iQ_r x} + 1}$$

factor out $1/(Q_c + 2IQ_cQ_r x)$

$$S_{11} = \frac{Q_r - (Q_c + 2iQ_rQ_c x)}{Q_r + (Q_c + 2iQ_rQ_c x)}$$

$$S_{11} = \frac{Q_r - Q_c - 2iQ_rQ_c x}{Q_r + Q_c + 2iQ_rQ_c x}$$

$$S_{11} = \frac{Q_r - Q_c}{Q_r + Q_c}\left(\frac{1 - 2i\frac{Q_rQ_c}{Q_r - Q_c}x}{1 + 2i\frac{Q_rQ_c}{Q_r + Q_c}x}\right)$$

$$S_{11} = S_{11}^{min}\frac{1 - 2iQ_{eff}x}{1 + 2iQ_t x}$$

$$S_{11}^{min} = \frac{Q_r - Q_c}{Q_r + Q_c}$$

$$Q_{eff} = \frac{Q_rQ_c}{Q_r - Q_c}$$

$$Q_t = \frac{Q_rQ_c}{Q_r + Q_c}$$