Theorem. $\forall A, B \in \mathbb{R}^{n\times n},\ \det(AB) = \det A \det B$.

Proof. There are two cases: (1) $\det(A) = 0$ or $\det(B) = 0$, or (2) $\det(A) \neq 0$ and $\det(B) \neq 0$.

In the first case, at least one of A or B is not invertible. Thus AB is not invertible, so $\det(AB) = 0$. Thus $\det(AB) = \det(A)\det(B)$.

In the second case, A and B are invertible. Therefore, $A = E_m \dots E_n$ and $B = E_p \dots E_q$ for elementary matrices E_i. Then $AB = E_m \dots E_n E_p \dots E_q$, so

$$\begin{aligned}
\det(AB) &= \det(E_m \dots E_n E_p \dots E_q) \\
&= \det(E_m)\det(E_{m+1} \dots E_n E_p \dots E_q) \\
&= \dots \\
&= \det(E_m) \dots \det(E_n)\det(E_p) \dots \det(E_q) \\
&= \det(E_m E_{m+1}) \dots \det(E_n)\det(E_p E_{p+1}) \dots \det(E_q) \\
&= \dots \\
&= \det(E_m \dots E_n)\det(E_p \dots E_q) \\
&= \det(A)\det(B).
\end{aligned}$$

Thus in both cases $\det(AB) = \det(A)\det(B)$, as desired. $\square$