Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for all $v, w \in V$, $v \perp w$ iff $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.

Proof. ($\Longrightarrow$) As $v \perp w$, $\langle v, w \rangle = \langle w, v \rangle = 0$. Applying this and the bilinearity of the inner product, we have

$$\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \|v\|^2 + \|w\|^2.$$

($\Longleftarrow$) Since $\|v + w\|^2 = \|v\|^2 + \|w\|^2$ and applying the inner product's symmetry:

$$\begin{aligned} \|v\|^2 + \|w\|^2 - \|v + w\|^2 &= 0 \\ \langle v, v \rangle + \langle w, w \rangle - (\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle) &= 0 \\ -2\langle v, w \rangle &= 0 \\ \langle v, w \rangle &= 0 \end{aligned}$$

Therefore, $v \perp w$. $\square$