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Proof. ($\implies$) Suppose T has an eigenbasis. We partition the vectors in the eigenbasis into m sets depending on their respective eigenvalues. These partitioned sets must be linearly independent as they are subsets of the eigenbasis. Thus, for a set with a given eigenvalue λ, it has at most $\operatorname{gemu}(\lambda)$ vectors. We also know that the eigenbasis has n vectors in it, and the total number of vectors in each partition must sum to n. Thus $\sum_{\forall\lambda} \operatorname{gemu}(\lambda) = n$.

($\impliedby$) Suppose $\sum_{\forall\lambda} \operatorname{gemu}(\lambda) = n$. Note that $\operatorname{gemu}(\lambda)$ is the number of vectors in the basis of the eigenspace of λ. The union of all such bases has n elements, and because the union of bases for distinct eigenspaces is linearly independent, the union of all bases represents a basis for T. Thus T has an eigenbasis. $\square$