Theorem. If A is a triangular matrix, then its determinant is the product of its columns. That is:

$$\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

We begin by proving that the determinant of an upper triangular matrix is as prescribed.

Proof. We induce on n. In the base case, A_1 is a 1×1 matrix, i.e. $A = a$. Then our inductive hypothesis is that the determinant of $n \times n$ triangular matrix A_n is the product of its diagonals:

$$\det(A_n) = \prod_{i=1}^{n} a_{ii}.$$

We want to show that the determinant of an $(n+1) \times (n+1)$ upper triangular matrix A_{n+1} is the product of its diagonals. From Laplace Expansion:

$$\begin{aligned}
\det(A_{n+1}) = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & a_{n,n+1} \\ 0 & 0 & \cdots & 0 & a_{n+1,n+1} \end{bmatrix} \\
&= a_{n+1,n+1} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \\
&= a_{n+1,n+1} \det(A_n) \\
&= a_{n+1,n+1} \prod_{i=1}^{n} a_{ii} \\
&= \prod_{i=1}^{n+1} a_{ii}
\end{aligned}$$

Thus, the determinant of an $(n+1) \times (n+1)$ upper triangular matrix is the product of its diagonals, and the proof follows from the principle of mathematical induction. $\square$

A similar proof can be given for lower triangular matrices. Thus all triangular matrices have a determinant that is the product of its diagonals.