Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the matrix whose j-th column is $T(\vec{e}_j)$. Then for all $\vec{x} \in \mathbb{R}^n$, we have $T(\vec{x}) = A\vec{x}$.

Proof. Take

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

from the source $\mathbb{R}^m$. Write

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_m$$

where $\vec{e}_i$ is the ith standard unit column vector. Applying T and using the fact that T is linear, we have

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_m) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_m) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_m) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

Thus for any vector $\vec{x}$, we have $T(\vec{x}) = A\vec{x}$, where A is the matrix defined in the theorem.

Nowe we show that A is unique. Suppose for contradiction that some other matrix B satisfies $T(\vec{x}) = B\vec{x}$. Then for all vectors $\vec{x} \in \mathbb{R}^m$, we have $A\vec{x} = B\vec{x}$. In particular, taking $\vec{x} = \vec{e}_j$, we have $A\vec{e}_j = B\vec{e}_j$. Thus $A\vec{e}_j = B\vec{e}_j$ for all $j = 1, 2, \ldots, n$. But this means that A and B are the same matrix. This is a contradiction, so A is unique. $\square$