Theorem. Let V be a vector space with bases $\mathfrak{B} = (b_1, \ldots, b_d)$ and $\mathfrak{A} = (a_1, \ldots, a_n)$. Then for all $v \in V$, $S_{\mathfrak{B}\to\mathfrak{A}}[v]_{\mathfrak{B}} = [v]_{\mathfrak{A}}$.

Proof. We seek to identify the matrix A such that $A[v]_{\mathfrak{B}} = [v]_{\mathfrak{B}}$ by means of the Key Theorem. In particular,

$$Ae_i = AL_{\mathfrak{B}}(b_i) = L_{\mathfrak{A}}(b_i).$$

Thus the ith column of A is $L_{\mathfrak{A}}(b_i)$, which is also the ith column of the change of basis matrix from $\mathfrak{A}$ to $\mathfrak{B}$. Thus, $A = S_{\mathfrak{B}\to\mathfrak{A}}$ so $S_{\mathfrak{A}\to\mathfrak{B}}[v]_{\mathfrak{B}} = [v]_{\mathfrak{B}}$. $\square$