Theorem. Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d)$ be an ordered basis for a vector space V. The map $L_{\mathfrak{B}} : V \to \mathbb{R}^d$ defined by $\vec{v} \mapsto [\vec{v}]_{\mathfrak{B}}$ is an isomorphism.

The map $L_{\mathfrak{B}} : V \to \mathbb{R}^d$ is bijective.

Proof. We describe the inverse of $L_{\mathfrak{B}}$: the map sending $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ to $a_1 v_1 + \cdots + a_n v_n$. Since $L_{\mathfrak{B}}$ has an inverse, $L_{\mathfrak{B}}$ is bijective. $\square$

The map $L_{\mathfrak{B}} : V \to \mathbb{R}^d$ is linear.

Proof. Since $\mathcal{B}$ is a basis for V, any $v \in \operatorname{im} T \subset V$ can be written uniquely as a linear combination of $v_1, \ldots, v_n$. Then for arbitrary, $v, w \in V$, we have

$$v = a_1 v_1 + \cdots + a_n v_n \quad \text{and} \quad w = b_1 v_1 + \cdots + b_n v_n$$

for some $a_i, b_i \in \mathbb{R}$. Then

$$v + w = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n,$$

and this says exactly that $L_{\mathcal{B}}(v) + L_{\mathcal{B}}(w) = L_{\mathcal{B}}(v+w)$. Thus $L_{\mathfrak{B}}$ respects addition. Moreover,

$$kv = k(a_1 v_1 + \cdots + a_n v_n) = (ka_1) v_1 + \cdots + (ka_n) v_n,$$

which says exactly that $kL_{\mathcal{B}}(v) = L_{\mathcal{B}}(kv)$. Thus $L_{\mathfrak{B}}$ respects scalar multiplication. Because f respects addition and scalar multiplication, f is linear. $\square$

The map $L_{\mathfrak{B}} : V \to \mathbb{R}^d$ is an isomorphism.

Proof. $L_{\mathfrak{B}}$ is an isomorphism as it is bijective and linear, from above. $\square$