

Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space. Then for all  $v, w \in V$ ,  $v \perp w$  iff  $\|v + w\|^2 = \|v\|^2 + \|w\|^2$ .

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*Proof.* (  $\implies$  ) As  $v \perp w$ ,  $\langle v, w \rangle = \langle w, v \rangle = 0$ . Applying this and the bilinearity of the inner product, we have

$$\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \|v\|^2 + \|w\|^2.$$

(  $\impliedby$  ) Since  $\|v + w\|^2 = \|v\|^2 + \|w\|^2$  and applying the inner product's symmetry:

$$\begin{aligned} \|v\|^2 + \|w\|^2 - \|v + w\|^2 &= 0 \\ \langle v, v \rangle + \langle w, w \rangle - (\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle) &= 0 \\ -2\langle v, w \rangle &= 0 \\ \langle v, w \rangle &= 0 \end{aligned}$$

Therefore,  $v \perp w$ .

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