

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation, and let  $A$  be the matrix whose  $j$ -th column is  $T(\vec{e}_j)$ . Then for all  $\vec{x} \in \mathbb{R}^n$ , we have  $T(\vec{x}) = A\vec{x}$ .

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*Proof.* Take

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

from the source  $\mathbb{R}^n$ . Write

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

where  $\vec{e}_i$  is the  $i$ th standard unit column vector. Applying  $T$  and using the fact that  $T$  is linear, we have

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

Thus for any vector  $\vec{x}$ , we have  $T(\vec{x}) = A\vec{x}$ , where  $A$  is the matrix defined in the theorem.

Now we show that  $A$  is unique. Suppose for contradiction that some other matrix  $B$  satisfies  $T(\vec{x}) = B\vec{x}$ . Then for all vectors  $\vec{x} \in \mathbb{R}^n$ , we have  $A\vec{x} = B\vec{x}$ . In particular, taking  $\vec{x} = \vec{e}_j$ , we have  $A\vec{e}_j = B\vec{e}_j$ . Thus  $A\vec{e}_j = B\vec{e}_j$  for all  $j = 1, 2, \dots, n$ . But this means that  $A$  and  $B$  are the same matrix. This is a contradiction, so  $A$  is unique.  $\square$