

Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d)$ be an ordered basis for a vector space V . The map $L_{\mathfrak{B}} : V \rightarrow \mathbb{R}^d$ defined by $\vec{v} \mapsto [\vec{v}]_{\mathfrak{B}}$ is an isomorphism.

The map $L_{\mathfrak{B}} : V \rightarrow \mathbb{R}^d$ is bijective.

Proof. We describe the inverse of $L_{\mathfrak{B}}$: the map sending $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ to $a_1 v_1 + \dots + a_n v_n$. Since $L_{\mathfrak{B}}$ has an inverse, $L_{\mathfrak{B}}$ is bijective. \square

The map $L_{\mathfrak{B}} : V \rightarrow \mathbb{R}^d$ is linear.

Proof. Since \mathcal{B} is a basis for V , any $v \in \text{im } T \subset V$ can be written uniquely as a linear combination of v_1, \dots, v_n . Then for arbitrary, $v, w \in V$, we have

$$v = a_1 v_1 + \dots + a_n v_n \quad \text{and} \quad w = b_1 v_1 + \dots + b_n v_n$$

for some $a_i, b_i \in \mathbb{R}$. Then

$$v + w = (a_1 + b_1)v_1 + \dots + (a_n + b_n)v_n,$$

and this says exactly that $L_{\mathcal{B}}(v) + L_{\mathcal{B}}(w) = L_{\mathcal{B}}(v+w)$. Thus $L_{\mathfrak{B}}$ respects addition. Moreover,

$$kv = k(a_1 v_1 + \dots + a_n v_n) = (ka_1)v_1 + \dots + (ka_n)v_n,$$

which says exactly that $kL_{\mathcal{B}}(v) = L_{\mathcal{B}}(kv)$. Thus $L_{\mathfrak{B}}$ respects scalar multiplication. Because f respects addition and scalar multiplication, f is linear. \square

The map $L_{\mathfrak{B}} : V \rightarrow \mathbb{R}^d$ is an isomorphism.

Proof. $L_{\mathfrak{B}}$ is an isomorphism as it is bijective and linear, from above. \square