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Swing-up Controller Design for Inverted Pendulum by Using Energy Control Method Based on Lyapunov Function

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Abstract— **The purpose of this paper is swing-up control of a cart-type single inverted pendulum by using energy control method based on Lyapunov function. Energy control method is one of the effective control methods for under-actuated system, and some examples of applying energy control method based on Lyapunov function to swing-up control of inverted pendulum systems have been reported. This paper focuses on Lyapunov function used in energy control method. In conventional researches, the possibility that the value of Lyapunov function becomes zero excluding the upright position has not been considered. In order to design the controller with considering this possibility, three new Lyapunov functions are created, and controllers are designed for each function. Swing-up control of the pendulum is done by switching these three controllers. In addition, numerical simulation is conducted to verify the effectiveness of the proposed method.**

I. INTRODUCTION

Inverted pendulum system is a typical example of unstable system. It is difficult to control the system by standard linear control theories, since the inverted pendulum system is nonlinear and under-actuated mechanical system. However, the inverted pendulum system is always used to verify and evaluate of control theories of stability, and many researches of the system have been conducted. The purpose of control of the inverted pendulum system is to swing up the pendulum from downward position to upright position, then to maintain the upright state.

Regarding pendulums, inverted pendulum system has many kinds of types: a single inverted pendulum, a serial double inverted pendulum, a parallel double inverted pendulum and so on. Also, regarding control methods, many researchers have proposed a lot of control theories. For example, feedback control, feed-forward control, sliding-mode control [3], robust control, hybrid control and so on. In addition, not only control methods of pendulum, but also some design methods of control parameters by applying genetic algorithm [7] or Q-learning have been proposed.

In this paper, energy control method based on Lyapunov function is applied to swing-up control of a cart-type single inverted pendulum. Energy control method can be applied in the control of nonlinear systems, and the effectiveness for under-actuated systems is known. For this reason, some application examples using this method have been reported by [1], [2], [4], [5], [6]. Moreover, energy control method has an easy principle compared with other control theories. The principle used in this paper is that control input is given in the direction where mechanical energy of the pendulum increases. Concretely, in the beginning, a Lyapunov function based on mechanical energy is designed. This function must become zero at the upright position. When the control input that the value of derivative of the Lyapunov function becomes non-positive is given, the value of Lyapunov function decreases to zero and the pendulum is swung up.

This paper also focuses on Lyapunov function used in energy control method. In conventional researches, the possibility that the value of the Lyapunov function becomes zero excluding the upright position has not been considered. If the value of the Lyapunov function becomes zero excluding the upright position, there is the possibility that the pendulum cannot be controlled to the upright position.

In order to design the controller with considering this possibility, a Lyapunov function that the value does not become zero excluding the upright position should be designed. However, it is hard to avoid this problem by using only one Lyapunov function. Thus, candidates of three new Lyapunov functions are designed, and the control input for each Lyapunov function is derived. Swing-up control is done by switching these three controllers.

This paper organizes as follows. In section II, the motion equation of the system is shown. In section III, the principle of energy control method is shown, and a problem of Lyapunov function is discussed. In section IV, new controllers are designed. In section V, numerical simulation is conducted by using MATLAB to verify the effectiveness of the proposed method.

II. MODELING OF INVERTED PENDULUM

In this section, motion equation of the cart-type single inverted pendulum is derived by using the method of [3].

First, in order to use Lagrangian method, Lagrange equation is given. Lagrange equation of the inverted pendulum system is given as

$$\frac{d}{dt}\frac{\partial A}{\partial \dot{x}} - \frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} + \frac{\partial C}{\partial \dot{x}} = \tau \tag{1}$$

where, A denotes total kinetic energy, B denotes total potential energy and C denotes total loss energy of the system as depicted in Fig. 1, and x denotes generalized coordinates and τ denotes generalized force. A, B, C, x

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and τ are expressed as follows:

$$A = \frac{1}{2}(J + ml^2 + I)\dot{\theta}^2 + ml\dot{\theta}\dot{z}\cos\theta + \frac{1}{2}(M + m + n)\dot{z}^2$$
$$B = mgl\cos\theta$$
$$C = \frac{1}{2}c\dot{\theta}^2$$
$$x = [\ \theta,\ z\ ]^T, \qquad \tau = [\ 0,\ u\ ]^T$$

From (1), the motion equations of the system are given by

$$(J + ml^2 + I)\ddot{\theta} + c\dot{\theta} - mlg\sin\theta + ml\cos\theta\cdot\ddot{z} = 0 \quad (2)$$

$$ml\cos\theta\cdot\ddot{\theta} - ml\sin\theta\cdot\dot{\theta}^2 + (M + m + n)\ddot{z} = u \quad (3)$$

where, a relation

$$u = \ddot{z} \quad (4)$$

is obtained, since control input u and acceleration of cart $\ddot{z}$ are equivalent.

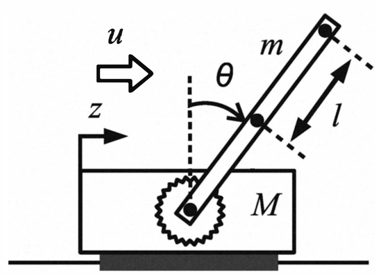

Fig. 1. Model of single inverted pendulum

TABLE I
DEFINITION OF PARAMETERS

θ	angular position of pendulum from vertical line
z	position of cart
M	mass of cart
m	mass of pendulum
n	mass of joint
l	length from joint to center of mass of pendulum
J	inertia of pendulum around center of mass
I	inertia of joint center of mass
c	viscosity of joint
g	gravity acceleration
u	control input

Next, viscosity c and inertia I are disregarded, since (2) is complicated, and c and I are minor value. From (2) and (4), $\ddot{\theta}$ is obtained as follows:

$$\ddot{\theta} = \frac{ml}{J + ml^2}(g\sin\theta - u\cos\theta) \quad (5)$$

III. Problem statement

The design of conventional controller for swing-up control given by [1], [2] and the principle of swing-up control by using energy control method based on Lyapunov function are introduced in this section. Also, a problem of Lyapunov function used in conventional energy control method is discussed.

A. Conventional design for swing-up control

Firstly, mechanical energy E of the pendulum used in Lyapunov function is derived. Mechanical energy E becomes

$$E = \frac{1}{2}(J + ml^2)\dot{\theta}^2 + mgl(\cos\theta - 1) \quad (6)$$

since E is given by the harmony of kinetic energy and potential energy of the pendulum. In (6), the upright position $\theta = 0$ or $\theta = 2\pi$ is assumed to be the reference point of potential energy.

Secondly, Lyapunov function based on mechanical energy is designed as follows:

$$V = \frac{1}{2}E^2 \quad (7)$$

Finally, $u = u_\theta$ that satisfies $\dot{V} \leq 0$ becomes a candidate of control input for swing-up control.

B. Problem of energy control method based on Lyapunov function

In the beginning, the principle of energy control method based on Lyapunov function is shown. When the pendulum is at the downward position $(\theta,\ \dot{\theta}) = (\pi,\ 0)$, then $E = -2mgl$. When the pendulum is at the upright position $(\theta,\ \dot{\theta}) = (0,\ 0)$ or $(\theta,\ \dot{\theta}) = (2\pi,\ 0)$, then $E = 0$. Therefore, in order to swing up the pendulum from the downward position to the upright position, mechanical energy E must increase to zero. Thus, Lyapunov function is defined as (7), and the control input that satisfies $\dot{V} \leq 0$ is designed. When this control input is provided, V decreases to zero. That is, E increases to zero in the range of $-2mgl \leq E \leq 0$. By this principle, the pendulum is controlled to the upright position.

In (6), there is the possibility of getting $E = 0$ by combination θ and $\dot{\theta}$ excluding the upright position. Thus, from (6) and (7), it is confirmed that there is the possibility of $V = 0$ excluding the upright position.

One of definition of Lyapunov function V_0 is as follows:

$$\begin{cases} V_0 = 0 & (\text{ equilibrium point }) \\ V_0 > 0 & (\text{ others }) \end{cases} \quad (8)$$

Therefore, although (7) is a Lyapunov function concerning E, it is not a Lyapunov function concerning θ.

For this reason, $V = 0$ does not necessarily indicate the upright position. Moreover, it can be said that there is innumerable goal state, since there is the possibility of $V = 0$ for every θ.

From (6), combination of θ and $\dot{\theta}$ that makes $V = 0$ is obtained. In (6), the value of m, l, J and g are given as $m = 0.18$ [kg], $l = 0.19$ [m], $J = 0.0089$ [kgm^2] and $g = 9.8$ [m/s^2]. Fig. 2 shows the combination of θ and $\dot{\theta}$. In every θ excluding the upright position, there are two values concerning $\dot{\theta}$ respectively that the signs are different can be confirmed.

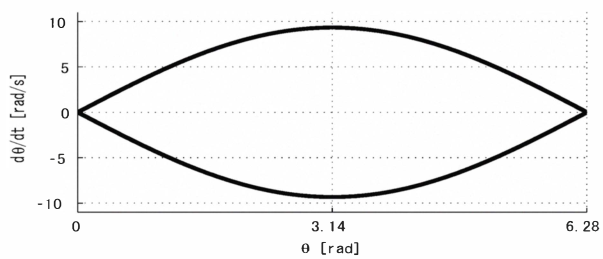

Fig. 2. Combination of θ and $\dot{\theta}$

IV. Proposed controllers

This section describes the designed swing-up controllers in order to avoid the problem discussed in section III.

A. Improvement of Lyapunov function

In order to avoid the problem regarding Lyapunov function (7), improvement of the Lyapunov function is needed. That is, the Lyapunov function should be designed to satisfy the above mentioned condition (8).

The proposed Lyapunov functions are designed as follows:

$$V^* = \frac{1}{2}E^2 + f(\theta)$$

B. Design process of the controllers

The first candidate of Lyapunov function V_1 is created as follows:

$$V_1 = \frac{1}{2}E^2 + \alpha(1 - \cos^3\theta) \qquad (\alpha > 0) \tag{9}$$

where,

$$\begin{cases} \alpha(1-\cos^3\theta) = 0 & (\text{ upright position }) \\ \alpha(1-\cos^3\theta) > 0 & (\text{ others }) \end{cases}$$

That is, the value of V_1 becomes positive excluding the upright position. By using (9), the control input $u = u_{\theta_1}$ used for swing-up control is derived. From (5), (6) and (9), the derivative of V_1

$$\dot{V}_1 = -mlE\dot{\theta}\cos\theta \cdot u + \frac{3}{2}\alpha\dot{\theta}\cos\theta\sin 2\theta \tag{10}$$

is obtained.

The relation of

$$mlE\dot{\theta}\cos\theta \cdot u \geq \frac{3}{2}\alpha\dot{\theta}\cos\theta\sin 2\theta \tag{11}$$

must be satisfied, since u_{θ_1} that satisfies $\dot{V}_1 \leq 0$ becomes a candidate of control input for swing-up control. Thus, the control input u_{θ_1} is given as

$$u_{\theta_1} = \begin{cases} U_0\mathrm{sign}(E\dot{\theta}\cos\theta) \\ U_0\mathrm{sign}(E\dot{\theta}\cos\theta) + U_1\mathrm{sign}(-\dot{\theta}\cos\theta) \end{cases} \tag{12}$$

where, sign$(\cdot)$ means sign function. This function is defined as follows:

$$\mathrm{sign}(x) = \begin{cases} 1 & (\ x > 0\ ) \\ 0 & (\ x = 0\ ) \\ -1 & (\ x < 0\ ) \end{cases}$$

The control input of the upper side in (12) is applied in case of $E \neq 0$, and the control input of the lower side in (12) is applied in case of $E = 0$, since when $u_{\theta_1} = 0$, this control input is considered to be useless for swing-up control.

In this paper, different control methods are applied in swing-up control and stability control. That is, when the pendulum approaches the upright position, the controller is switched from swing-up controller to stability controller. The pendulum will not be at the upright position while swing-up controller is used, since the control input of (12) is for swing-up control. Thus, $E = 0$ in this phase indicates the middle of swing-up control. For this reason, excluding singular point, it is considered that the waste arises when the control input equals to zero during swing-up control. Even if the control input equals to zero in singular point, the influence is a little, since the sign of the control input changes before and behind the singular point.

The relation of (11) is basically satisfied with appropriate decision of U_0, U_1 and α, since (12) makes non-negative value in left-hand side of (11) basically. Therefore, when (12) is given in the system, the energy of the pendulum increases to zero and the pendulum is swung up.

However, in the case of (i) and (ii) as follows, there is no control input u_{θ_1} that satisfies (11). Thus, the controllers using the candidates of second and third Lyapunov functions V_2 and V_3 are designed. Fig. 3 shows the state when (11) is not satisfied. (i) means the pendulum is in right half plane and has clockwise angular velocity. (ii) means the pendulum is in left half plane and has counterclockwise angular velocity.

(i) $\dot{\theta} > 0$ and $0 < \theta < \pi$ and $E = 0$

The candidate of Lyapunov function used in this condition is created as follows:

$$V_2 = \frac{1}{2}E^2 + \frac{1}{2}\beta(\theta - 2\pi)^2 \qquad (\beta > 0) \tag{13}$$

where,

$$\begin{cases} V_2 = 0 & (\ \theta = 2\pi\ ) \\ V_2 > 0 & (\ \text{others}\ ) \end{cases}$$

The pendulum is assumed to be swung up on $\theta = 2\pi$ in this condition, since the pendulum is in right half plane ($0 < \theta < \pi$) and has clockwise angular velocity ($\dot{\theta} > 0$). Therefore, it is natural to make the pendulum settle to $\theta = 2\pi$. For this reason, (13) that $\theta = 2\pi$ becomes goal state is designed.

The swing-up control input is derived by the procedure similar to V_1. From (5), (6) and (13), the derivative of V_2

$$\dot{V}_2 = -mlE\dot{\theta}\cos\theta \cdot u + \beta(\theta - 2\pi)\dot{\theta} \tag{14}$$

is obtained. The relation of

$$mlE\dot{\theta}\cos\theta \cdot u \geq \beta(\theta - 2\pi)\dot{\theta} \tag{15}$$

must be satisfied, since $u = u_{\theta_2}$ that satisfies $\dot{V}_2 \leq 0$ becomes a candidate of swing-up control input. Thus, the control input u_{θ_2} is designed as

$$u_{\theta_2} = U_2 \text{sign}(-\dot{\theta}\cos\theta) \tag{16}$$

When (16) is provided in (15),

$$\begin{cases} mlE\dot{\theta}\cos\theta \cdot u_{\theta_2} \geq 0 \\ \beta(\theta - 2\pi)\dot{\theta} \leq 0 \end{cases}$$

are consisted at all times in the condition of (i). Thus, it can be said that the control input (16) is appropriate in this case.

(ii) $\dot{\theta} < 0$ and $\pi < \theta < 2\pi$ and $E = 0$

The candidate of Lyapunov function used in this condition is created as follows:

$$V_3 = \frac{1}{2}E^2 + \frac{1}{2}\gamma\theta^2 \qquad (\gamma > 0) \tag{17}$$

where,

$$\begin{cases} V_3 = 0 & (\ \theta = 0\ ) \\ V_3 > 0 & (\ \text{others}\ ) \end{cases}$$

The pendulum is assumed to be swung up on $\theta = 0$ in this condition, since the pendulum is in left half plane ($\pi < \theta < 2\pi$) and has counterclockwise angular velocity ($\dot{\theta} < 0$). Therefore, it is natural to make the pendulum settle to $\theta = 0$. For this reason, (17) that $\theta = 0$ becomes goal state is designed.

The swing-up control input is derived by the procedure similar to V_1. From (5), (6) and (17), the derivative of V_3

$$\dot{V}_3 = -mlE\dot{\theta}\cos\theta \cdot u + \gamma\theta\dot{\theta} \tag{18}$$

is obtained. The relation of

$$mlE\dot{\theta}\cos\theta \cdot u \geq \gamma\theta\dot{\theta} \tag{19}$$

must be satisfied, since $u = u_{\theta_3}$ that satisfies $\dot{V}_3 \leq 0$ becomes a candidate of swing-up control input. Thus, the control input u_{θ_3} is designed as

$$u_{\theta_3} = U_3 \text{sign}(-\dot{\theta}\cos\theta) \tag{20}$$

When (20) is provided in (19),

$$\begin{cases} mlE\dot{\theta}\cos\theta \cdot u_{\theta_3} \geq 0 \\ \gamma\theta\dot{\theta} \leq 0 \end{cases}$$

are consisted at all times in the condition of (ii). Thus, it can be said that the control input (20) is appropriate in this case.

In this subsection, three controllers u_{θ_1}, u_{θ_2} and u_{θ_3} were designed. In initial state of the pendulum, the condition (i) or (ii) does not occur. Therefore, in the beginning of swing-up control, the controller u_{θ_1} is applied. When the condition of (i) or (ii) occurs, the controller is switched from u_{θ_1} to u_{θ_2} or u_{θ_3} respectively.

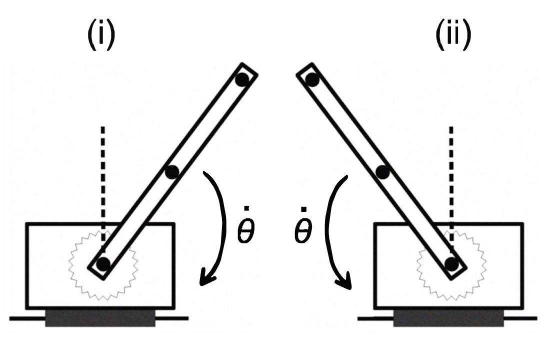

Fig. 3. State of (i) and (ii)

As mentioned above, different control methods are used in swing-up control and stability control. Energy control method is used in swing-up control, while feedback control is used in stability control.

In stability control, a feedback control by angular position θ and angular velocity $\dot{\theta}$ is applied. A control input of stability control u_{st} is given by

$$u_{st} = -\left[\, f_1 \;\, f_2 \,\right] \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \tag{21}$$

As the condition of switching from swing-up controller to stability controller, angular position θ is used. That is, the position of the pendulum is determined by using the value of $\cos\theta$. Swing-up control is applied when $\cos\theta < 0.9$, while stability control is applied when $\cos\theta \geq 0.9$

$$u = \begin{cases} u_{\theta_{1,2,3}} & (\ \cos\theta < 0.9\ ) \\ u_{st} & (\ \cos\theta \geq 0.9\ ) \end{cases} \tag{22}$$

V. Numerical simulation

In order to confirm and evaluate of the proposed method, the simulation of swing-up control and stability control is done by using MATLAB. TABLE II shows the initial state, and TABLE III shows the value of system parameters, and the value of control parameters are illustrated in TABLE IV.

In order to avoid the possibility that the control input u_{θ_1} equals to zero in the beginning of the simulation, the value of angular velocity of the pendulum $\dot{\theta}$ was set as 0.01 [rad/s] not 0 [rad/s] as shown in TABLE II. In this simulation, the control parameters in TABLE IV were decided by trial and error, and sampling period was set as 5 [msec].

TABLE II
INITIAL STATE

$\theta(0)$	π [rad]	$\dot{\theta}(0)$	0.01 [rad/s]

TABLE III
VALUE OF SYSTEM PARAMETERS

m	0.18 [kg]	c	0.0001 [kgm^2/s]
l	0.19 [m]	J	0.0089 [kgm^2]
g	9.8 [m/s^2]	I	0.000028 [kgm^2]

TABLE IV
VALUE OF CONTROL PARAMETERS

$U_{0,1,2,3}$	3.7	α, β, γ	0.01
f_1 f_2	20 1		

In this paper, two kinds of numerical simulations are done. First simulation is the simulation by using conventional controllers, and second simulation is the simulation by using the proposed controllers. The conventional swing-up controller is as follows.

$$u_{\theta_1} = U_0 \text{sign}(E\dot{\theta}\cos\theta) \tag{23}$$

In the conventional method, swing-up control is done by only one controller.

In both simulations, E equals to zero during swing-up control. From these simulations, the performance of proposed controllers is investigated. The results of each simulation are shown in Fig. 4 and Fig. 5. The results of simulations are composed of five graphs as follows: angular position of the pendulum θ, mechanical energy of the pendulum E, value of Lyapunov function $V_{1,2,3}$, control input u and using controller. In graphs concerning "input", the numbers "1", "2" and "3" are expressed as controllers u_{θ_1}, u_{θ_2} and u_{θ_3} respectively, and "4" is expressed as controller u_{st}.

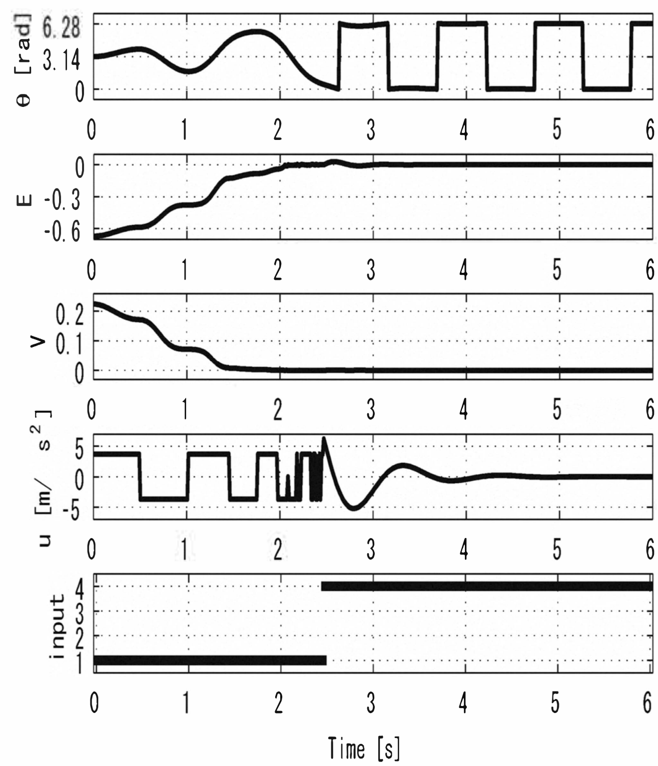

Fig. 4. Result of first simulation

From the graph concerning θ, it is confirmed that the pendulum reached the upright position $\theta = 0$ from initial state $\theta = \pi$. In the graph, $\theta = 0$ and $\theta = 2\pi$ indicate the same upright position. Thus, although θ keeps the state of $\theta = 0$ and $\theta = 2\pi$ alternately, the pendulum actually stays at one point. For this reason, it is confirmed that the pendulum maintains the upright state after the pendulum swings up to the upright position. That is, swing-up control and stability control have been completed.

But E equaled zero when 2.1 [sec] passed after beginning of the simulation. In this moment, the value of Lyapunov function and the value of control input becomes zero. Although the control is completed by using the conventional controller, the definition of Lyapunov function is not satisfied and the waste for swing-up control arises.

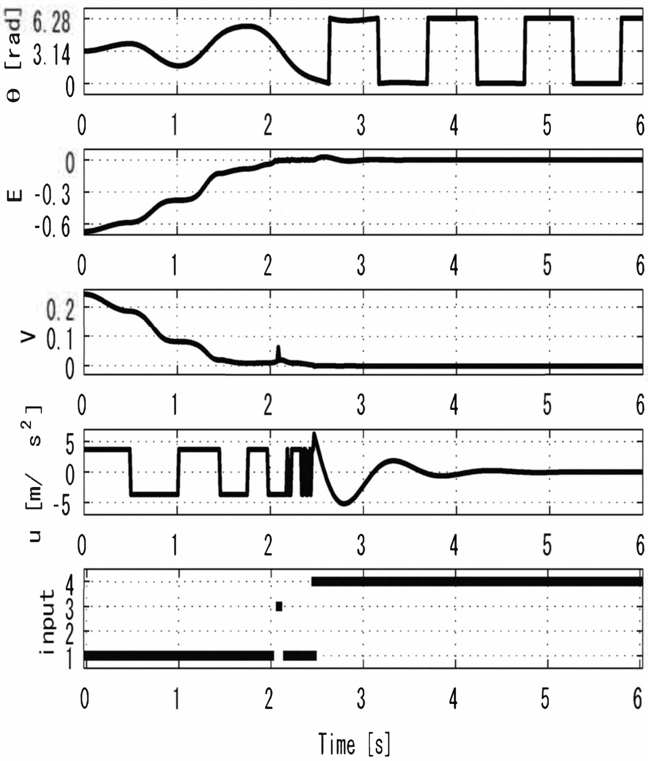

Fig. 5. Result of second simulation

Also in this case, swing-up control and stability control have been completed. In the proposed method, the controller switched from u_{θ_1} to u_{θ_3} , since the condition of (ii) consisted when E equaled to zero. This switching could avoid the state that the value of Lyapunov function becomes zero. Moreover, it can be said that swing-up control was done without the waste because of the invariable value of control input. Therefore, it concludes that the effectiveness of proposed method with switching controllers was verified.

VI. Conclusion

In this paper, a method for swing-up control of a cart-type single inverted pendulum by using energy control method based on Lyapunov function was proposed. The purpose of designed controllers is to avoid the problem that the value of Lyapunov function becomes zero excluding the upright position.

The controllers for swing-up control were designed by creating the candidates of three new Lyapunov functions and deriving the control input for each function. Swing-up control is achieved by switching these three controllers depending on the condition.

Moreover, the validity of the proposed method was confirmed by simulation results.

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