GENERAL ASSEMBLY

INTRODUCTION TO PROBABILITY & INDEPENDENCE

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INTRODUCTION TO PROBABILITY & INDEPENDENCE

"Mathematics, a veritable sorcerer in our computerized society, while assisting the trier of fact in the search for truth, must not cast a spell over him."

– California Supreme Court, *People v. Collins* (1968)

LEARNING OBJECTIVES

- Identify the three axioms of probability.
- Apply five basic probability rules.
- Define independence.
- Understand the role of independence in probability.

OPENING

- What comes to mind when you hear “probability?”

INTRODUCTION: DEFINITIONS & SETS

DEFINITIONS

- Experiment: A procedure that can be repeated infinitely many times and has a well-defined set of outcomes.
- Event: Any collection of outcomes of an experiment.
- Sample Space: The set of all possible outcomes of an experiment, denoted $\mathcal{S}$.

EXAMPLES

- Experiment: Flip a coin twice.
- Sample Space $\mathcal{S}$:
- Event:

- Experiment: Rolling a single die.
- Sample Space $\mathcal{S}$:
- Event:

DEFINITIONS

- Set: A well-defined collection of distinct objects.
  - $\{Derek\ Jeter, \pi, ☺\}$
    - (Standing on the shoulders of Justin Gash for this one.)

- Element: An object that is a member of a set.
  - Derek Jeter
  - π
  - ☺

SET OPERATIONS

- Union: $A \cup B = the\ set\ of\ elements\ in\ A\ or\ B$
- Intersection: $A \cap B = the\ set\ of\ elements\ in\ A\ and\ B$
- Example:
  - $A =$ even numbers between 1 and 10 $= \{2,4,6,8\}$
  - $B =$ prime numbers between 1 and 10$= \{2,3,5,7\}$
  - $A \cup B = ?$
  - $A \cap B = ?$

SET OPERATIONS

- Example:
  - $A = \{2,4,6,8\}$ & $B = \{2,3,5,7\}$
  - $A \cup B = \{2,4,6,8\} \cup \{2,3,5,7\} = \{2,3,4,5,6,7,8\}$
  - $A \cap B = \{2,4,6,8\} \cap \{2,3,5,7\} = \{2\}$

BASICS OF PROBABILITY

PROBABILITY BASICS

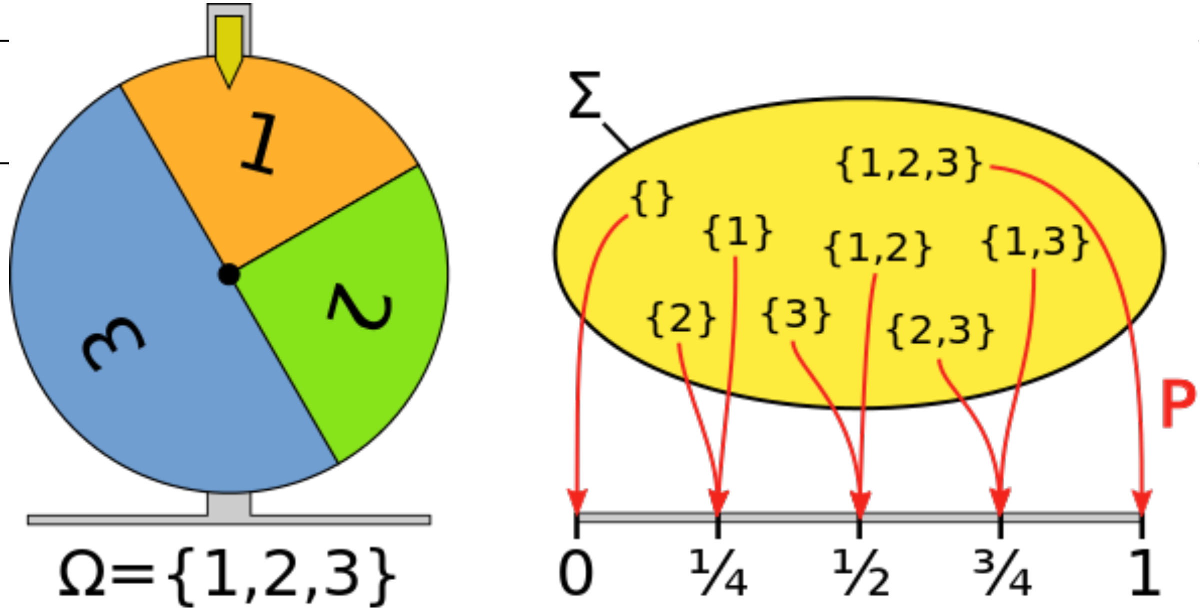
- Given an event A, we say that the probability that A occurs is:

$$P(A) = \frac{\text{number of outcomes in A}}{\text{number of all possible outcomes}}$$

PROBABILITY BASICS

- Probability: $\mathrm{P}(\mathcal{S}, \mathcal{F}) \rightarrow [0,1]$
  - $\mathcal{S}$ is the sample space.
  - $\mathcal{F}$ is the “event space,” or set of possible events.
  - P is the probability function, mapping each event to the $[0,1]$ interval.

PROBABILITY BASICS



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  - In more rigorous treatments of probability:
    - The sample space $\mathcal{S}$ is denoted by Ω.
    - The "event space" is denoted either by $\mathcal{F}$ or Σ, is called a "sigma algebra" or "Borel field," and has a set of very specific properties.

AXIOMS OF PROBABILITY (Kolmogorov Axioms)

- For any event A, $P(A) \geq 0$.
  - Nonnegativity.

- For the sample space $\mathcal{S}$, $P(\mathcal{S}) = 1$.
  - Unit measure.

- For mutually exclusive (or disjoint) E_i, $P\left(\bigcup_{\{i=1\}}^{\infty} E_i\right) = \sum_{\{i=1\}}^{\infty} P(E_i)$
  - Additivity.

- Probability must **ALWAYS** follow these three axioms.

PROBABILITY RULES

- $P(\emptyset) = 0$
  - Note: $\emptyset$ indicates the "empty set," or the event containing zero outcomes from the experiment.

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Venn diagrams can help to illustrate this – but remember that Venn diagrams are not proofs!
  - If A and B are disjoint, then $P(A \cap B) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$.

- $P(A^C) = 1 - P(A)$
  - A^C is known as the "complement of A."

PROBABILITY RULES

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - Note: $A|B$ means "A given B" or "A conditional on the fact that B happens."
  - Example:
    - $A =$ roll a 2 $\Rightarrow P(A) = \frac{1}{6}$
    - $B =$ roll an even number $\Rightarrow P(B) = \frac{1}{2}$
    - $P(A \cap B) = P(\text{roll 2 and roll even number}) = \frac{1}{6}$
    - $P(A|B) =$ given that I roll an even, what is the probability of rolling a 2? $= \frac{1/6}{1/2} = \frac{1}{3}$

- $P(A \cap B) = P(A|B)P(B)$
  - We took the first rule on this slide, multiplied both sides of $P(B)$, and voila!
  - $P(A \cap B \cap C) = P(A|B, C)P(B|C)P(C)$

PROBABILITY RULES

- $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$
  - "Law of Total Probability"

PROBABILITY RULES – SUMMARY

- $P(\emptyset) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A^C) = 1 - P(A)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- $P(A \cap B) = P(A|B)P(B)$
  - $P(A \cap B \cap C) = P(A|B,C)P(B|C)P(C)$
- $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$

PROBABILITY RULES – PRACTICE

- A = {a U.S. birth results in twin females}
- B = {a U.S. birth results in identical twins}
- C = {a U.S. birth results in twins}

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- $\text{A} = \{\text{a U.S. birth results in twin females}\}$
- $\text{B} = \{\text{a U.S. birth results in identical twins}\}$
- $\text{C} = \{\text{a U.S. birth results in twins}\}$

- A twin birth occurs approximately 1 in every 90 births.
- Roughly $\frac{1}{3}$ of all human twins are identical and $\frac{2}{3}$ are fraternal.
- Identical twins are necessarily the same sex and are male with probability 50%.
- Among fraternal twins, $\frac{1}{4}$ are both female, $\frac{1}{4}$ are both male.
- Find the values of $P(A \cap C)$ and $P(A \cap B \cap C)$.
  - Note any assumptions that you make along the way.

PROBABILITY RULES – PRACTICE

- Suppose the probability that an infant dies from sudden infant death syndrome (SIDS) is approximately 0.001%.
- What is the probability that a family has two children who die from SIDS?

PROBABILITY RULES – PRACTICE

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- What does this <u>mean</u>?

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PROBABILITY RULES – PRACTICE

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- What does this <u>mean</u>?
  - This means that the probability of events A and B occurring is calculated by finding the probability of B, then finding the probability of A given that B already occurred, then multiplying those two.
  - This gets us to the notion of dependence versus independence.

WHEN BY HAND IS TOUGH…

- Oftentimes, we won’t evaluate probabilities by hand.
    - It’s still very important to understand the ideas behind probability – as we move forward, it’s critical to:
        - a) know probability’s relationship with statistics and machine learning.
        - b) identify potentially bad assumptions.

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- However, we can use simulations to give us a good approximation of the true probability of some event.

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- If A and B are independent, then:
  - A is independent of B^C,
  - B is independent of A^C, and
  - A^C is independent of B^C.

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- Making this assumption when it the assumption is obviously violated can have disastrous effects on our results.
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- Deciding whether or not data are independent is, unfortunately, a judgment call that is contingent upon your specific use-case.

INDEPENDENCE

- Areas where independence is important:
  - Considering joint probabilities.
  - Most modeling techniques.
  - Sampling without replacement.
  - Training/testing sets.

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- Sometimes it is less convenient to work with one particular event and more convenient to describe all possible outcomes.
- For example, rather than finding the probability that someone has an IQ of exactly 100, we might be interested in looking at all possible IQ scores and how frequently we observe each IQ value.
- Recall: a <u>distribution</u> is the set of all possible values of a variable and how frequently the variable takes on each value.

AN ASIDE: PROBABILITY AND STATISTICS

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  - The probabilities associated with each distribution must add up to 1.
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- Let $X =$ IQ score.
  - We might say that X follows a Normal distribution with mean 100 and standard deviation 15.

- Now let $Y =$ time it takes all American workers to get to work.
  - What do we do here?

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- This gets to the relationship between probability and statistics.
- In probability, we know the values of these parameters (measures of a population) and can thus completely define the probability distribution.
- In statistics, we don't know the values of these parameters, so we have to estimate them.
- We gather a sample to learn about the population.
- We calculate statistics to learn about parameters.

RECAP

- Probability is a building block to learning about statistics.
  - Samples help us to learn about populations.
  - Statistics help us to learn about parameters.
- Independence is a <u>huge</u> consideration that will depend on your use-case.
- Probability is complicated, but being familiar with the basics will go a long way in understanding how pieces fit together.