GENERAL ASSEMBLY

INTRODUCTION TO PROBABILITY & INDEPENDENCE

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INTRODUCTION TO PROBABILITY & INDEPENDENCE

"Mathematics, a veritable sorcerer in our computerized society, while assisting the trier of fact in the search for truth, must not cast a spell over him."

– California Supreme Court, *People v. Collins* (1968)

LEARNING OBJECTIVES

- Identify the three axioms of probability.
- Apply five basic probability rules.
- Define independence.
- Understand the role of independence in probability.

OPENING

- What comes to mind when you hear “probability?”

INTRODUCTION: DEFINITIONS & SETS

DEFINITIONS

- Experiment: A procedure that can be repeated infinitely many times and has a well-defined set of outcomes.
- Event: Any collection of outcomes of an experiment.
- Sample Space: The set of all possible outcomes of an experiment, denoted $\mathcal{S}$.

EXAMPLES

- Experiment: Flip a coin twice.
- Sample Space $\mathcal{S}$:
- Event:

- Experiment: Rolling a single die.
- Sample Space $\mathcal{S}$:
- Event:

DEFINITIONS

- Set: A well-defined collection of distinct objects.
  - $\{Derek\ Jeter, \pi, ☺\}$
    - (Standing on the shoulders of Justin Gash for this one.)

- Element: An object that is a member of a set.
  - Derek Jeter
  - π
  - ☺

SET OPERATIONS

- Union: $A \cup B = the\ set\ of\ elements\ in\ A\ or\ B$

- Intersection: $A \cap B = the\ set\ of\ elements\ in\ A\ and\ B$

- Example:
  - $A =$ even numbers between 1 and 10 $= \{2,4,6,8\}$
  - $B =$ prime numbers between 1 and 10$= \{2,3,5,7\}$
  - $A \cup B = ?$
  - $A \cap B = ?$

SET OPERATIONS

- Example:
  - $A = \{2,4,6,8\}$ & $B = \{2,3,5,7\}$
  - $A \cup B = \{2,4,6,8\} \cup \{2,3,5,7\} = \{2,3,4,5,6,7,8\}$
  - $A \cap B = \{2,4,6,8\} \cap \{2,3,5,7\} = \{2\}$

BASICS OF PROBABILITY

PROBABILITY BASICS

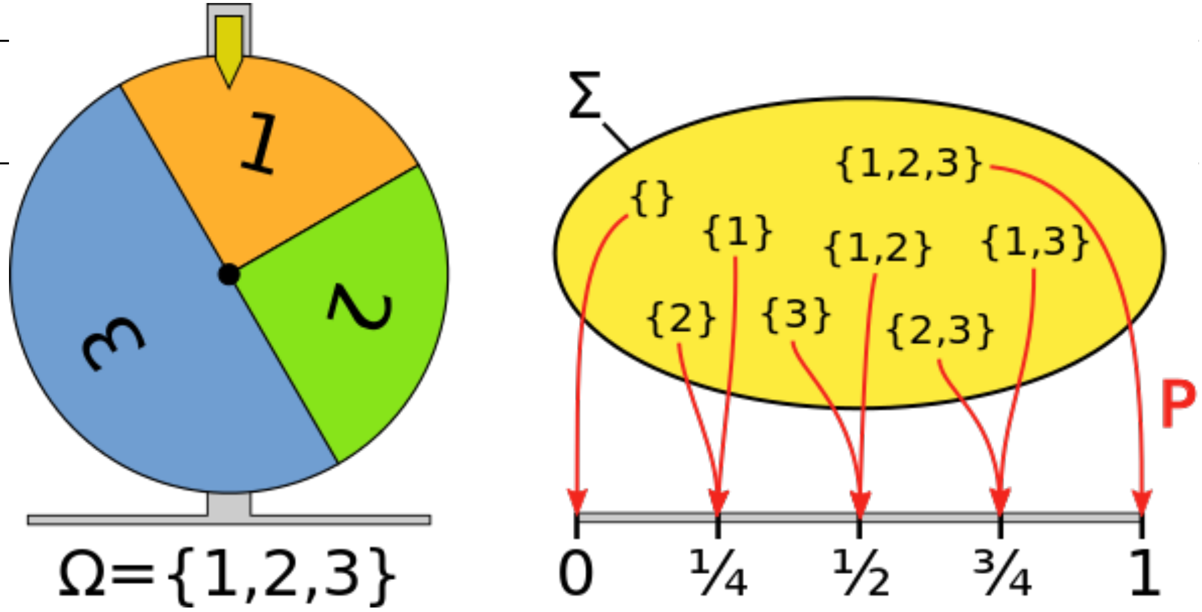
- Given an event A, we say that the probability that A occurs is:

$$P(A) = \frac{\text{number of outcomes in A}}{\text{number of all possible outcomes}}$$

PROBABILITY BASICS

- Probability: $\mathrm{P}(\mathcal{S}, \mathcal{F}) \rightarrow [0,1]$
  - $\mathcal{S}$ is the sample space.
  - $\mathcal{F}$ is the "event space," or set of possible events.
  - P is the probability function, mapping each event to the $[0,1]$ interval.

PROBABILITY BASICS



- Probability: $\mathrm{P}(\mathcal{S}, \mathcal{F}) \rightarrow [0,1]$
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  - In more rigorous treatments of probability:
    - The sample space $\mathcal{S}$ is denoted by Ω.
    - The “event space” is denoted either by $\mathcal{F}$ or Σ, is called a “sigma algebra” or “Borel field,” and has a set of very specific properties.

AXIOMS OF PROBABILITY (Kolmogorov Axioms)

- For any event A, $P(A) \geq 0$.
  - Nonnegativity.

- For the sample space $\mathcal{S}$, $P(\mathcal{S}) = 1$.
  - Unit measure.

- For mutually exclusive (or disjoint) E_i, $P\left(\bigcup_{\{i=1\}}^{\infty} E_i\right) = \sum_{\{i=1\}}^{\infty} P(E_i)$
  - Additivity.

- Probability must **ALWAYS** follow these three axioms.

PROBABILITY RULES

- $P(\emptyset) = 0$
  - Note: $\emptyset$ indicates the "empty set," or the event containing zero outcomes from the experiment.

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Venn diagrams can help to illustrate this – but remember that Venn diagrams are not proofs!
  - If A and B are disjoint, then $P(A \cap B) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$.

- $P(A^C) = 1 - P(A)$
  - A^C is known as the "complement of A."

PROBABILITY RULES

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - Note: $A|B$ means "A given B" or "A conditional on the fact that B happens."
  - Example:
    - $A =$ roll a 2 $\Rightarrow P(A) = \frac{1}{6}$
    - $B =$ roll an even number $\Rightarrow P(B) = \frac{1}{2}$
    - $P(A \cap B) = P(\text{roll 2 and roll even number}) = \frac{1}{6}$
    - $P(A|B) =$ given that I roll an even, what is the probability of rolling a 2? $= \frac{1/6}{1/2} = \frac{1}{3}$

- $P(A \cap B) = P(A|B)P(B)$
  - We took the first rule on this slide, multiplied both sides of $P(B)$, and voila!
  - $P(A \cap B \cap C) = P(A|B,C)P(B|C)P(C)$

PROBABILITY RULES

- $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$
  - "Law of Total Probability"

PROBABILITY RULES – SUMMARY

- $P(\emptyset) = 0$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A^C) = 1 - P(A)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- $P(A \cap B) = P(A|B)P(B)$
  - $P(A \cap B \cap C) = P(A|B,C)P(B|C)P(C)$
- $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$

PROBABILITY RULES – PRACTICE

- A = {a U.S. birth results in twin females}
- B = {a U.S. birth results in identical twins}
- C = {a U.S. birth results in twins}

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- $\text{C} = \{\text{a U.S. birth results in twins}\}$

- A twin birth occurs approximately 1 in every 90 births.
- Roughly $\frac{1}{3}$ of all human twins are identical and $\frac{2}{3}$ are fraternal.
- Identical twins are necessarily the same sex and are male with probability 50%.
- Among fraternal twins, $\frac{1}{4}$ are both female, $\frac{1}{4}$ are both male.
- Find the values of $P(A \cap C)$ and $P(A \cap B \cap C)$.
  - Note any assumptions that you make along the way.

PROBABILITY RULES – PRACTICE

- Suppose the probability that an infant dies from sudden infant death syndrome (SIDS) is approximately 0.001%.

- What is the probability that a family has two children who die from SIDS?

PROBABILITY RULES – PRACTICE

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- What does this <u>mean</u>?

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  - This means that the probability of events A and B occurring is calculated by finding the probability of B, then finding the probability of A given that B already occurred, then multiplying those two.

PROBABILITY RULES – PRACTICE

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- What does this mean?
  - This means that the probability of events A and B occurring is calculated by finding the probability of B, then finding the probability of A given that B already occurred, then multiplying those two.
  - This gets us to the notion of dependence versus independence.

WHEN BY HAND IS TOUGH…

- Oftentimes, we won't evaluate probabilities by hand.
  - It's still very important to understand the ideas behind probability – as we move forward, it's critical to:
    - a) know probability's relationship with statistics and machine learning.
    - b) identify potentially bad assumptions.

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- However, we can use simulations to give us a good approximation of the true probability of some event.

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- If A and B are independent, then:
  - A is independent of B^C,
  - B is independent of A^C, and
  - A^C is independent of B^C.

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- Making this assumption when it the assumption is obviously violated can have disastrous effects on our results.
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- Deciding whether or not data are independent is, unfortunately, a judgment call that is contingent upon your specific use-case.

INDEPENDENCE

- Areas where independence is important:
  - Considering joint probabilities.
  - Most modeling techniques.
  - Sampling without replacement.
  - Training/testing sets.

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- Recall: a <u>distribution</u> is the set of all possible values of a variable and how frequently the variable takes on each value.

AN ASIDE: PROBABILITY AND STATISTICS

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- Let X = IQ score.
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- Now let Y = time it takes all American workers to get to work.
  - What do we do here?

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- This gets to the relationship between probability and statistics.
- In probability, we know the values of these parameters (measures of a population) and can thus completely define the probability distribution.
- In statistics, we don't know the values of these parameters, so we have to estimate them.
- We gather a sample to learn about the population.
- We calculate statistics to learn about parameters.

RECAP

- Probability is a building block to learning about statistics.
  - Samples help us to learn about populations.
  - Statistics help us to learn about parameters.

- Independence is a huge consideration.

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