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Controlling Triangular Formations of Autonomous Agents in Finite Time Using Coarse Measurements*

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Abstract— **This paper studies the performances of the popular gradient-based formation-control strategies for teams of autonomous agents when the agents' range measurements are coarse. Since the dynamics of the resulting closed-loop system are discontinuous, Filippov solutions to non-smooth dynamical systems are introduced. Similar to the existing stability results for triangular formations with precise range measurements, we prove that under coarse range measurements, the convergence to the desired formation is almost global except for initially collinearly positioned formations. More importantly, we are able to make stronger statements that the convergence takes place within finite time and that the settling time can be determined by the geometric information of the initial shape of the formation. Simulation and experimental results are provided to validate the theoretical analysis.**

I. Introduction

Cooperative control for teams of autonomous robots has been extensively studied in the last decade [3], [13]. One typical coordination task is formation keeping in which a team of mobile agents are required to move collaboratively so that the overall team manoeuvres as a whole with a prescribed formation shape [1], [17], [6], [15], [22], [23]. The biggest challenge in such formation-keeping tasks is that each agent has only limited local information about its neighboring agents while the success of the team tasks can only be evaluated globally. Various ideas have been proposed to address this challenge [17], [21]. In particular, controlling triangular formations has been identified to be the benchmark case, since these formations are small enough to allow global stability analysis while still exhibiting interesting behaviors inherently related to the rigidity properties of a formation [4].

However, most of the existing work assumes that the agents are able to obtain the precise information about the relative positions of their neighboring agents. In this paper, we look into the more challenging case when the range measurements are carried out by coarse sensing. In practice, agents can sense the directions of their neighboring agents through bearing sensors, such as acoustic or infrared sensors, which have a range of low-price choices [20]. But it is more expensive to acquire precise distance information through range sensors, such as laser sensors [16]. This motivates researchers to look into the scenarios when range measurements are acquired in their quantized forms [14], [7], [18]. To maximally reduce the requirements for sensing and computation capabilities, existing work [9], [8], [12] has attempted to deduce theoretical stability results for formation control or consensus problems in simplified settings when controllers use coarsely quantized information.

Along this line of research, in this paper we propose the gradient-based formation-control strategy for a team of autonomous agents when agents' range measurements are coarse. We focus on cyclic triangular formations since it is the building block for controlling bigger formations and allows rigorous global stability analysis. The formation control strategy utilizing coarsely quantized range measurements has the additional advantage in application that the agents' velocities become normalized and computations are thus greatly simplified. However, since the dynamics of the resulting overall system becomes discontinuous due to the quantized measurements, we have to apply tools from non-smooth analysis to analyze the performances of the controllers. We are able to prove the convergence results using the Lyapunov function method, and show the set of feasible initial positions of the agents to achieve prescribed triangular formations. Compared with the convergence results in [4], a much stronger statement is proven that finite time convergence can be achieved, which is especially appealing if one wants to apply similar control strategies to larger formations in practice.

The rest of the paper is organized as follows. Our research problem is formulated in Section II. Then we give the convergence analysis results using tools from nonsmooth analysis in Section III. Furthermore we prove the finite-time convergence in Section IV. Experimental and numerical examples are provided in Section V to validate our theoretical analysis. In Section VI, we draw some conclusions and suggest possible extensions of this work.

II. Problem formulation

We consider a cyclic triangular formation in the plane consisting of three autonomous agents labeled by 1, 2, and 3, shown in Fig 1. For $i \in \{1,2,3\}$, let $[i]$ denote the label of agent i's leader. In this paper, we set $[1]=2$, $[2]=3$ and $[3]=1$. We assume that the desired distance between agents i and $[i]$ is a positive constant d_i; here the d_is satisfy the triangle inequalities:

$$d_1+d_2>d_3\,,\quad d_2+d_3>d_1\,,\quad d_3+d_1>d_2\,. \qquad (1)$$

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Let x_i be the Cartesian coordinate vector of agent i in some fixed global coordinate system in the plane. In [5], Cao et al have studied how to control three autonomous agents to achieve a prescribed triangular formation, for which the agents' dynamics are described by

$$\begin{aligned}\dot{x}_1 &= -(x_1 - x_2)(||x_1 - x_2||^2 - d_1^2)\,,\\ \dot{x}_2 &= -(x_2 - x_3)(||x_2 - x_3||^2 - d_2^2)\,,\\ \dot{x}_3 &= -(x_3 - x_1)(||x_3 - x_1||^2 - d_3^2)\,.\end{aligned} \tag{2}$$

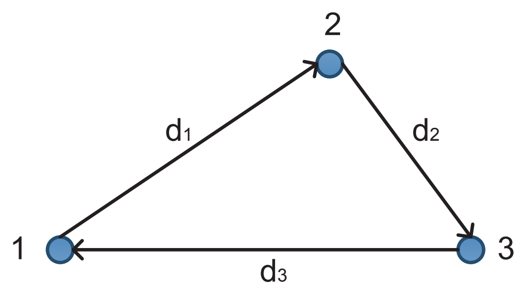


Fig. 1. Cyclic triangular formation.

In their setting, it is assumed that for $i \in \{1, 2, 3\}$, agent i can measure precisely the relative position $x_i - x_{[i]}$ of agent $[i]$ in its own coordinate system. It has been proved that under such gradient-based control laws, system (2) can be stabilized almost globally to an equilibrium corresponding to the triangular formation with the desired shape. However, in this paper we investigate the much more challenging scenario where each agent cannot measure precisely the relative distances. To be more specific, we assume that in the three-agent system shown in Fig. 1, agent i can sense the direction $\frac{x_{[i]}-x_i}{||x_{[i]}-x_i||}$ of its non-collocated leader $[i]$ in its own local coordinates through a bearing sensor, and measure whether the relative distance is greater or less than the desired distance through a crude range sensor. Let the sign function $\mathrm{sgn}(\cdot) : \mathbb{R} \to \mathbb{R}$ be defined by

$$\mathrm{sgn}(a) = \begin{cases} +1 & a > 0; \\ \ \ 0 & a = 0; \\ -1 & a < 0. \end{cases} \tag{3}$$

Then the agents' coarse measurements about the range $x_i - x_{[i]}$ are in the form of the trinary value $\mathrm{sgn}(||x_i - x_i|| - d_i)$. With such measurements, we investigate the performance of the gradient-based control law corresponding to (2) and obtain the new set of equations describing the system's dynamics

$$\begin{aligned}\dot{x}_1 &= -\frac{(x_1 - x_2)}{||x_1 - x_2||}\,\mathrm{sgn}(||x_1 - x_2|| - d_1)\,,\\ \dot{x}_2 &= -\frac{(x_2 - x_3)}{||x_2 - x_3||}\,\mathrm{sgn}(||x_2 - x_3|| - d_2)\,,\\ \dot{x}_3 &= -\frac{(x_3 - x_1)}{||x_3 - x_1||}\,\mathrm{sgn}(||x_3 - x_1|| - d_3)\,.\end{aligned} \tag{4}$$

Let

$$z_i \triangleq x_i - x_{[i]}, \qquad e_i \triangleq ||z_i|| - d_i\,, \tag{5}$$

for $i = 1, 2, 3$. And let $x = (x_1^T, x_2^T, x_3^T)^T, z = (z_1^T, z_2^T, z_3^T)^T$, and $e = (e_1^T, e_2^T, e_3^T)^T$. Then system (4) is defined on the set

$$\mathcal{X} = \left\{x \in \mathbb{R}^6 : ||x_i - x_{[i]}|| > 0,\ \text{for all } i = 1, 2, 3\right\}\,.$$

Let $\mathcal{X}^c$ be the complement of $\mathcal{X}$ in $\mathbb{R}^6$, i.e., $\mathcal{X}^c = \mathbb{R}^6 - \mathcal{X} = \left\{x \in \mathbb{R}^6 : ||x_i - x_{[i]}|| = 0,\ \text{for some } i = 1, 2, 3\right\}$. Now we study the stability of system (4). Towards this end, we first rewrite the dynamics (4) using z_i, and study the stability of the resulting z-system

$$\begin{aligned}\dot{z}_1 &= -\frac{z_1}{||z_1||}\,\mathrm{sgn}(||z_1|| - d_1) + \frac{z_2}{||z_2||}\,\mathrm{sgn}(||z_2|| - d_2)\\ \dot{z}_2 &= -\frac{z_2}{||z_2||}\,\mathrm{sgn}(||z_2|| - d_2) + \frac{z_3}{||z_3||}\,\mathrm{sgn}(||z_3|| - d_3)\\ \dot{z}_3 &= -\frac{z_3}{||z_3||}\,\mathrm{sgn}(||z_3|| - d_3) + \frac{z_1}{||z_1||}\,\mathrm{sgn}(||z_1|| - d_1)\end{aligned} \tag{6}$$

which is defined on the set

$$\mathcal{Z} = \left\{z \in \mathbb{R}^6 : ||z_i|| > 0,\ \text{for all } i = 1, 2, 3\right\}\,.$$

Let $\mathcal{Z}^c$ be the complement of $\mathcal{Z}$ in $\mathbb{R}^6$, i.e., $\mathcal{Z}^c = \mathbb{R}^6 - \mathcal{Z} = \left\{z \in \mathbb{R}^6 : ||z_i|| = 0,\ \text{for some } i = 1, 2, 3\right\}$. Obviously, the set $\mathcal{Z}$ is open and connected. One can easily check that $x \in \mathcal{X}$ if and only if $z \in \mathcal{Z}$. We work with the z-system (6) for convergence analysis. To study the system's dynamics, we first specify what we mean by the solutions to the system. Since the vector fields on the right-hand sides of (6) are discontinuous, we consider Filippov solutions.

The following result guarantees the existence of the Filippov solutions to a non-smooth system.

Lemma 1: [10] Assume $f : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded, i.e. bounded in any bounded neighborhood of every point of definition excluding the sets of measure zero. Then for all $x_0 \in \mathbb{R}^d$, there exists a Filippov solution to $\dot{x} = f(x)$ with the initial condition $x(0) = x_0$.

Since the Euclidean norms of the right-hand sides of (6) are upper bounded by 2 on $\mathcal{Z}$, the conditions in Lemma 1 are satisfied and thus the Filippov solutions to (6) exist when $z \in \mathcal{Z}$. So the Filippov solution to (6) exists for all $t \geq 0$ when the system is well defined. One can use similar arguments to show that the Filippov solution to (4) also exists for all $t \geq 0$ when $x \in \mathcal{X}$. In the next section, we present the convergence analysis for the Filippov solutions to the z-system.

III. Convergence Analysis

Let $g(z)$ be the vector field on the right-hand side of (6). We consider the differential inclusions $\dot{z} \in F[g(z)]$ of system (6), where $F[\cdot]$ is the set-valued map corresponding to the Filippov solutions.

Following the same notation as in [5], let $\mathcal{N}$ be the set of points in $\mathbb{R}^6$ corresponding to collinear positions of the three agents in the plane, namely

$$\mathcal{N} \triangleq \left\{z : \mathrm{rank}\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} < 2, z_1 + z_2 + z_3 = 0\right\}\,. \tag{7}$$

One can easily see that the positions of the three agents x_1, x_2, x_3 are collinear if and only if z_1, z_2, z_3 are collinear.

Note that $\mathcal{N}$ is a closed manifold containing the set $\mathcal{Z}^c$. Now we show that if a solution to (6) starts outside of $\mathcal{N}$ and thus in $\mathcal{Z}$ at $t=0$, it remains outside of $\mathcal{N}$ for all $t>0$; in other words, the evolution of the z-system is well defined when $z(0) \notin \mathcal{N}$.

Lemma 2: If a solution to (6) starts outside of $\mathcal{N}$ at $t=0$, it cannot enter $\mathcal{Z}^c$ for any $t>0$.

Proof: We prove by contradiction. Suppose the contrary is true. Then for some $z(0) \notin \mathcal{N}$, there exists a $T>0$, which can approach infinity, such that $z(t)$ approaches $\mathcal{Z}^c$ as t approaches T. It is straightforward to check that $\det\begin{bmatrix} z_1 & z_2\end{bmatrix} = -\det\begin{bmatrix} z_1 & z_3\end{bmatrix}$. This and the definition of $\mathcal{N}$ in (7) imply that

$$\mathcal{N} = \left\{z : \det\begin{bmatrix} z_1 & z_2\end{bmatrix} = 0\right\}. \tag{8}$$

So the assumption about T implies that

$$\lim_{t\to T} \det\begin{bmatrix} z_1 & z_2\end{bmatrix} = 0. \tag{9}$$

Furthermore, from the definition for $\mathcal{Z}^c$, we know that $\lim_{t\to T} ||z_i(t)|| = 0$ for some $i \in \{1,2,3\}$. Without loss of generality, we take $i=1$. Then for any δ satisfying $0<\delta \ll \min\{d_1, d_2, d_3\}$, there exists a finite $t_1 < T$ such that $||z_1(t)|| \le \delta$ for all $t > t_1$. Now for any $t_1 < t < T$, we consider two cases:

Case 1: $||z_2(t)|| \ge 3\delta$ and $||z_3(t)|| \ge 3\delta$. In this case, we have

$$\begin{aligned} &-\left(\frac{\text{sgn}(e_1(t))}{||z_1(t)||} + \frac{\text{sgn}(e_2(t))}{||z_2(t)||} + \frac{\text{sgn}(e_3(t))}{||z_3(t)||}\right) \\ &> \frac{1}{\delta} - \frac{1}{||z_2(t)||} - \frac{1}{||z_3(t)||} \\ &\ge \frac{1}{\delta} - \frac{1}{3\delta} - \frac{1}{3\delta} = \frac{1}{3\delta}. \end{aligned}$$

Case 2: $||z_2(t)|| < 3\delta$ or $||z_3(t)|| < 3\delta$. Then we have

$$\begin{aligned} &-\left(\frac{\text{sgn}(e_1(t))}{||z_1(t)||} + \frac{\text{sgn}(e_2(t))}{||z_2(t)||} + \frac{\text{sgn}(e_3(t))}{||z_3(t)||}\right) \\ &> \frac{1}{\delta} + \frac{1}{3\delta} + \frac{1}{4\delta} > \frac{1}{3\delta}. \end{aligned}$$

In either case, we always have

$$-\left(\frac{\text{sgn}(e_1(t))}{||z_1(t)||} + \frac{\text{sgn}(e_2(t))}{||z_2(t)||} + \frac{\text{sgn}(e_3(t))}{||z_3(t)||}\right) > \frac{1}{3\delta}, \tag{10}$$

for $t_1 < t < T$. Now we look more carefully at the evolution of system (6). Let $\mathcal{D}$ denote the set of all the discontinuity points of its right-hand side

$$\mathcal{D} \triangleq \{z : e_1\, e_2\, e_3 = 0\}.$$

Then along any solution to (6), one has

$$\begin{aligned} &\frac{d}{dt}\det\begin{bmatrix} z_1 & z_2\end{bmatrix} = \\ &\quad -\left(\frac{\text{sgn}e_1}{||z_1||} + \frac{\text{sgn}e_2}{||z_2||} + \frac{\text{sgn}e_3}{||z_3||}\right)\det\begin{bmatrix} z_1 & z_2\end{bmatrix} \end{aligned} \tag{11}$$

when $z \notin \mathcal{D}$; and

$$\begin{aligned} &\frac{d}{dt}\det\begin{bmatrix} z_1 & z_2\end{bmatrix} \in \\ &-\left(\frac{F[\text{sgn}e_1]}{||z_1||} + \frac{F[\text{sgn}e_2]}{||z_2||} + \frac{F[\text{sgn}e_3]}{||z_3||}\right)\det\begin{bmatrix} z_1 & z_2\end{bmatrix} \end{aligned} \tag{12}$$

when $z \in \mathcal{D}$. Note that

$$\begin{aligned} &\det\begin{bmatrix} z_1(t) & z_2(t)\end{bmatrix} = \\ &e^{-\int_{t_1}^{t}\left(\frac{\text{sgn}e_1(s)}{||z_1(s)||} + \frac{\text{sgn}e_2(s)}{||z_2(s)||} + \frac{\text{sgn}e_3(s)}{||z_3(s)||}\right)ds}\det\begin{bmatrix} z_1(t_1) & z_2(t_1)\end{bmatrix} \end{aligned} \tag{13}$$

for $t \ge t_1$ because of (11) and (12). Hence,

$$\det\begin{bmatrix} z_1(t) & z_2(t)\end{bmatrix} > e^{\frac{t-t_1}{3\delta}}\det\begin{bmatrix} z_1(t_1) & z_2(t_1)\end{bmatrix},$$

for all $t \in (t_1, T)$. On the other hand, $\det\begin{bmatrix} z_1(t_1) & z_2(t_1)\end{bmatrix} = e^{-\int_0^{t_1}\left(\frac{\text{sgn}e_1(s)}{||z_1(s)||} + \frac{\text{sgn}e_2(s)}{||z_2(s)||} + \frac{\text{sgn}e_3(s)}{||z_3(s)||}\right)ds}\det\begin{bmatrix} z_1(0) & z_2(0)\end{bmatrix}$, and $\det\begin{bmatrix} z_1(0) & z_2(0)\end{bmatrix}$ is bounded away from below from zero since $z(0)$ starts outside of $\mathcal{N}$. Thus, $\det\begin{bmatrix} z_1(t_1) & z_2(t_1)\end{bmatrix}$ is also bounded away from below from zero. Therefore, $\det\begin{bmatrix} z_1(t) & z_2(t)\end{bmatrix} > \det\begin{bmatrix} z_1(t_1) & z_2(t_1)\end{bmatrix}$ is bounded from below from zero for any $t \in (t_1, T)$, which contradicts (9). This completes the proof. □

To show system (6) is well defined, it remains to be shown that it is so when $z(0) \in \mathcal{N} \cap \mathcal{Z}$.

Lemma 3: If a solution to (6) starts in $\mathcal{N} \cap \mathcal{Z}$, it remains in $\mathcal{N} \cap \mathcal{Z}$ for all $t > 0$.

Proof: One can easily check that $\mathcal{N}$ is positively invariant since if $\det\begin{bmatrix} z_1 & z_2\end{bmatrix} = 0$ at $t=0$, then $\det\begin{bmatrix} z_1 & z_2\end{bmatrix} = 0$ for all $t>0$ from (11) and (12). So for a solution starting in $\mathcal{N} \cap \mathcal{Z}$, it remains in $\mathcal{N}$, and thus z_1, z_2 and z_3 remain collinear. For this reason, one can always use the coordinate transformation aligning the coordinate axis with $\mathcal{N}$ to write z_1, z_2, z_3 into scalars. To simplify notations, we still use z_i to denote the resulting scalers and rewrite (6) into

$$\begin{aligned} \dot{z}_1 &= -\text{sgn}(z_1)\,\text{sgn}(|z_1| - d_1) + \text{sgn}(z_2)\,\text{sgn}(|z_2| - d_2), \\ \dot{z}_2 &= -\text{sgn}(z_2)\,\text{sgn}(|z_2| - d_2) + \text{sgn}(z_3)\,\text{sgn}(|z_3| - d_3), \\ \dot{z}_3 &= -\text{sgn}(z_3)\,\text{sgn}(|z_3| - d_3) + \text{sgn}(z_1)\,\text{sgn}(|z_1| - d_1). \end{aligned} \tag{14}$$

For any $i = 1,2,3$, if $0 < |z_i| < d_i$, then the derivative of z_i^2 along a solution to (14) is

$$\frac{d\,z_i^2}{dt} \in \left\{|z_i|\,(1 + \text{sgn}z_i \cdot \text{sgn}z_{[i]} \cdot \gamma_{[i]}) : \quad \gamma_{[i]} \in [-1, 1]\right\}.$$

So if there exists a t^* such that $0 < |z_i(t^*)| < d_i$, then $|z_i(t)| \ge |z_i(t^*)|$ for $t > t^*$ whenever $|z_i(t)| < d_i$. Hence, $|z_i|$ cannot approach 0 for any t. □

From Lemmas 2 and 3, we have shown that system (6) is well defined. We summarize it as follows.

Proposition 1: If a solution to (6) starts in $\mathcal{Z}$, it remains in $\mathcal{Z}$ for all $t > 0$.

The set of the equilibrium points of the z-system (6) is the union of the two sets $\mathcal{E}$ and $\mathcal{M}$ defined as follows

$$\mathcal{E} \triangleq \{z : e_1 = e_2 = e_3 = 0\} \text{ and} \tag{15}$$

$$\mathcal{M} \triangleq \left\{ z : \frac{z_1}{||z_1||}\operatorname{sgn} e_1 = \frac{z_2}{||z_2||}\operatorname{sgn} e_2 = \frac{z_3}{||z_3||}\operatorname{sgn} e_3 \right\}.$$

One can check that $\mathcal{M}$ is a subset of $\mathcal{N}$. Using similar arguments as those to prove Lemma 2 in [5], one can prove the following lemma for the sets $\mathcal{N}$ and $\mathcal{E}$.

Lemma 4: $\mathcal{N}$ and $\mathcal{E}$ are disjoint sets.

Obviously, $\mathcal{N}$ and $\mathcal{E}$ are closed sets. In addition, $\mathcal{M}$ and $\mathcal{E}$ are also disjoint sets since $\mathcal{M} \subset \mathcal{N}$. That $\mathcal{N}$ might be the place where the triangular formation will fail can be understood by the fact that $\mathcal{N}$ is a positively invariant set; in other words, formations which are initially collinear, remain collinear for all $t > 0$. This is true since if $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = 0$ at $t = 0$, then $\det \begin{bmatrix} z_1 & z_2 \end{bmatrix} = 0$ for all $t > 0$ as shown in (11) and (12).

The main result in this paper that we want to prove is as follows.

Theorem 1: All the Filippov solutions to system (6) starting outside of $\mathcal{N}$, converges to a finite limit in $\mathcal{E}$. Furthermore, the convergence is achieved within finite time.

The proof of this theorem involves several steps. We first focus on the convergence and later on check the convergence speed. To begin with, we introduce some notations. We use $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ to denote the inner product, and $\theta(a, b)$ the angle between any two vectors $a, b \in \mathbb{R}^2$. For the two vectors $z_i, z_{[i]}$, one has the fact that $\cos\theta(z_i, z_{[i]}) = \frac{\langle z_i, z_{[i]} \rangle}{||z_i|| \cdot ||z_{[i]}||}$.

Theorem 2: All the Filippov solutions to system (6) are bounded and converge globally to the set $\mathcal{E} \cup \mathcal{M}$.

Proof: We choose the candidate Lyapunov function

$$V(z(t)) = \frac{1}{4}\left(\sum_{i=1}^{3} (||z_i||^2 - d_i^2)^2 \right). \tag{16}$$

When $z(t) \notin \mathcal{D}$,

$$\begin{aligned} &\frac{d}{dt} V(z(t)) \\ = \;& \sum_{i=1}^{3} (||z_i||^2 - d_i^2) z_i^T \left(-\frac{z_i}{||z_i||} \operatorname{sgn}(||z_i|| - d_i) \right. \\ &\left. + \frac{z_{[i]}}{||z_{[i]}||} \operatorname{sgn}(||z_{[i]} - d_{[i]}||) \right) \\ = \;& -\sum_{i=1}^{3} |\, ||z_i||^2 - d_i^2| \cdot ||z_i|| \\ & \cdot \left(1 - \frac{\langle z_i, z_{[i]} \rangle}{||z_i|| \cdot ||z_{[i]}||} \operatorname{sgn} e_i \operatorname{sgn} e_{[i]} \right) \\ \leq \;& 0 \end{aligned}$$

with the equality sign holds if and only if $\frac{z_1}{||z_1||}\operatorname{sgn}(||z_1|| - d_1) = \frac{z_2}{||z_2||}\operatorname{sgn}(||z_2|| - d_2) = \frac{z_3}{||z_3||}\operatorname{sgn}(||z_3|| - d_3)$.

When $z(t) \in \mathcal{D}$, $\frac{d}{dt} V(z(t)) \in \dot{\overline{V}}(z(t))$, where the set-valued derivative $\dot{\overline{V}}(z(t))$ is given by $\dot{\overline{V}}(z(t)) = \{\langle \nabla V(z), \nu \rangle, \nu \in F[g(z)]\}$, and the column vector $\nabla V(z)$ is the gradient of $V(z)$.

If $z \in \mathcal{E} \subset \mathcal{D}$, then $\nabla V(z) = \mathbf{0}$, and therefore $\dot{\overline{V}}(z(t)) = \{0\}$; if on the other hand, $z \in \mathcal{D} \setminus \mathcal{E}$, then there must exist at least one i such that $||z_i|| - d_i = 0$ and at least one $j \neq i$ such that $||z_j|| - d_j \neq 0$. Denote the label of the remaining agent by k. Then

$$\begin{aligned} &\langle \nabla V(z), \nu \rangle \\ &= (||z_k||^2 - d_k^2) z_k^T \left(-\frac{z_k}{||z_k||} \gamma_k + \frac{z_j}{||z_j||} \operatorname{sgn}(||z_j|| - d_j) \right) \\ &+ (||z_j||^2 - d_j^2) z_j^T \left(-\frac{z_j}{||z_j||} \operatorname{sgn}(||z_j|| - d_j) + \frac{z_i}{||z_i||} \gamma_i \right) \end{aligned}$$

where $\gamma_i \in [-1, 1]$ and $\gamma_k \in [-1, 1]$. In particular, γ_k can be reduced to $\operatorname{sgn}(||z_k|| - d_k)$ if $||z_k|| - d_k \neq 0$. The above inequality can be further written into

$$\begin{aligned} & \langle \nabla V(z), \nu \rangle \\ = \;& -|\, ||z_k||^2 - d_k^2| \cdot ||z_k|| \left(1 - \frac{\langle z_k, z_j \rangle}{||z_k|| \cdot ||z_j||} \gamma_k \operatorname{sgn} e_j \right) \\ & -|\, ||z_j||^2 - d_j^2| \cdot ||z_j|| \left(1 - \frac{\langle z_j, z_i \rangle}{||z_j|| \cdot ||z_i||} \gamma_i \operatorname{sgn} e_j \right) \\ \leq \;& 0 \end{aligned}$$

with the equality sign holds if and only if $\frac{z_j}{||z_j||}\operatorname{sgn}(||z_j|| - d_j) = \frac{z_i}{||z_i||}\gamma_i = \frac{z_k}{||z_k||}\gamma_k$ and $z \in \mathcal{N} \cap \mathcal{D}$.

Summarizing the discussion so far, we have shown that for all $z \in \mathcal{Z}$, $\max \dot{\overline{V}}(z(t)) \leq 0$, and that $0 \in \dot{\overline{V}}(z(t))$ if and only if $z \in \mathcal{M} \cup \mathcal{E}$. Hence, for all $t \geq 0$, we have $V(z(t))$ is non-increasing and satisfies $0 \leq V(z(t)) \leq V(z(0))$. In view of V's definition, the z_i are bounded for all $t \geq 0$. Furthermore, applying LaSalle's invariance principle for differential inclusions [2], [10], [11], any solution to the differential inclusion converges to the largest weakly invariant set in the closure of $\mathcal{E} \cup \mathcal{M}$. Since $\mathcal{E} \cup \mathcal{M}$ is the set of equilibria of (6), we know $\mathcal{E} \cup \mathcal{M}$ is weakly invariant. So we arrive at the conclusion that any solution $z(t)$ of the differential inclusion $\dot{z} \in F[g(z)]$ is bounded and converges to the equilibrium set $\mathcal{E} \cup \mathcal{M}$. □

Since for any $z \in \mathcal{M}$, it holds that $\dot{z} = 0$, we know $\mathcal{M}$ is a positively invariant set; moreover along any trajectory in $\mathcal{M}$, the three agents move at the same constant velocity. Thus x_1, x_2, x_3 could drift to infinity together while $||x_i - x_{[i]}||$ still converge to some finite positive numbers. So although as we have proved, the solutions to (6) are always bounded, those to (4) are not necessarily so.

Since the largest weakly invariant set $\mathcal{E} \cup \mathcal{M}$ contains the set $\mathcal{M}$ on which the three agents are collinear, it is of interest to characterize those initial conditions under which the asymptotic relative positions of the agents converge to the set $\mathcal{M}$. The following proposition is given to answer this question. Here we omit its proof due to page limit.

Proposition 2: (a) All the Filippov solutions to system (6) starting outside of $\mathcal{N}$ converge to $\mathcal{E}$.
(b) All the Filippov solutions to system (6) starting inside of $\mathcal{N}$ stay in this invariant set for $t > 0$, and converge to $\mathcal{M}$.

In the next section, we look into the convergence speed of the converging process just analyzed.

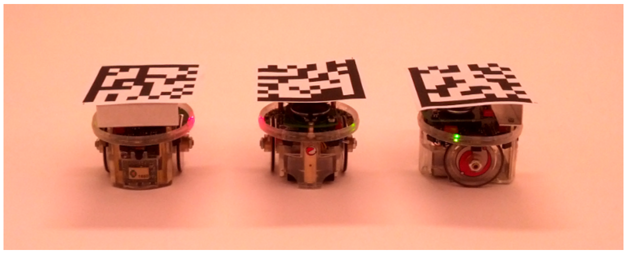


Fig. 2. Three wheeled E-pucks with datamatrices as markers on their tops.

IV. Finite-time convergence

In the previous section, we have shown that all trajectories starting outside of $\mathcal{N}$ converge to $\mathcal{E}$. In this section, we will prove that the convergence is achieved in finite time. In the following, we give our analysis results without proofs due to page limit. The proofs will be provided in the full-length version of this paper.

Lemma 5: Let $\varrho(t) \triangleq \max_{i\in\{1,2,3\}} |\cos\theta(z_i(t), z_{[i]}(t))|$. If $z(0) \notin \mathcal{N}$, then there exists a positive constat $\bar{\varrho} < 1$ such that $\varrho(t) \le \bar{\varrho}$ for all $t \ge 0$.

Theorem 3: All the Filippov solutions to the z-system (6) starting outside of $\mathcal{N}$ reach $\mathcal{E}$ in finite time, for which the settling time is $T_s = \frac{\sqrt{V(z(0))}}{d\,(1-\bar{\varrho})}$, where V is defined in (16), $d = \min\{d_1, d_2, d_3, ||z_1(0)||, ||z_2(0)||, ||z_3(0)||\}$, and $\bar{\varrho}$ is defined in Lemma 5.

V. Experiments and simulations

We test the formation convergence result in Theorem 1 using E-pucks [19]. The experimental setup consists of three E-puck robots in a 2D area of 2.6×2 meters. Each robot is identified by a datamatrix as a marker on its top as shown in Fig. 2. The robot's reference point is the position of its lower right corner and the orientation of the marker is recognized by a vision algorithm running at a PC employing a webcam placed above the testing area. Since an E-puck is usually modeled by a unicycle, we apply feedback linearization about its reference point to obtain the single-integrator model for simpler controller implementation. Therefore, we control the triangle formed by the three reference points of the robots. The whole image of the testing area is covered by 1600×1200 pixels, where the distance between two consecutive horizontal or vertical pixels corresponds approximately to 1.6mm. The PC runs a real time process computing the relative vectors between the robots from the vision algorithm, then it computes the control inputs determined by (4). Note that although the control inputs for the robots are computed by the PC, this does not change the distributed nature of the proposed control algorithm since we are not modifying (4) in any way. The communication takes place when sending the commands from the PC to the E-pucks in order to move their wheels. These commands are obtained after applying the feedback linearization, which gives the required linear and angular velocities to the robots, and this information is translated to common (linear velocity) and differential (angular velocity) commands to the wheels of the robots. The communication is done via Bluetooth at the fixed frequency of 20Hz. In order to make the experiments faster, the selected constant speed has been chosen to be 17 pixels/sec or equivalently 2.72cm/sec.

We consider an equilateral triangle with side length being 250 pixels as the prescribed shape. Chattering might occur when the three robots are close to the target formation since in practice the argument in the sign function of (4) cannot be zero due to noise in sensors or floating point number representation. In order to prevent the chattering, if the absolute error for e_i is smaller than the threshold of 8 pixels, then we set the control input to 0.

By using the proposed controllers, the three robots converge to the desired formation as it is shown in Figure 3. The observed settling time is 31 seconds which is less than the upper bound $T_s = 336$ seconds given in Theorem 3.

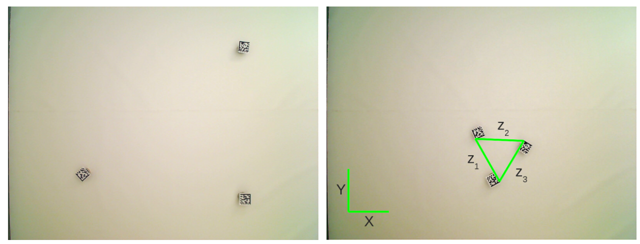


Fig. 3. Initial and final positions of the E-pucks after applying the proposed distributed control laws. The final distances $||z_1||$, $||z_2||$ and $||z_3||$ between the reference points of the robots are 255, 248 and 249 pixels, respectively.

The trajectories of the robots are shown in Fig. 4. The initial and final positions correspond to the ones shown in Fig. 3. The evolutions of the errors e_i for $i = 1, 2, 3$ and the terms of the Lyapunov function (16) are shown in Fig. 5. As predicted, the three terms of the Lyapunov function are always decreasing over time. The video of the experiment can be checked following the link *https://dl.dropboxusercontent.com/u/2689187/Tcoarse.mp4* .

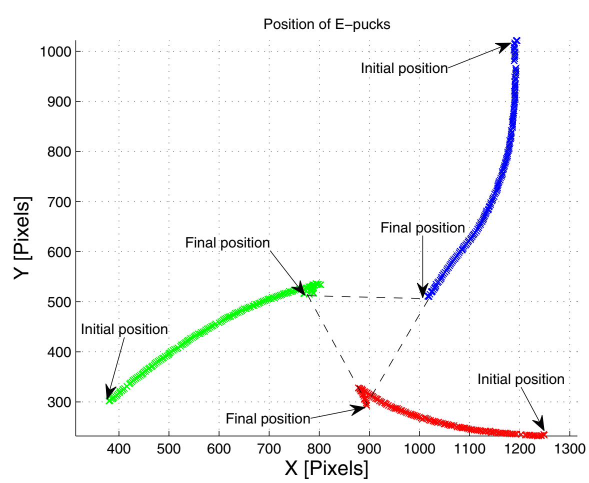
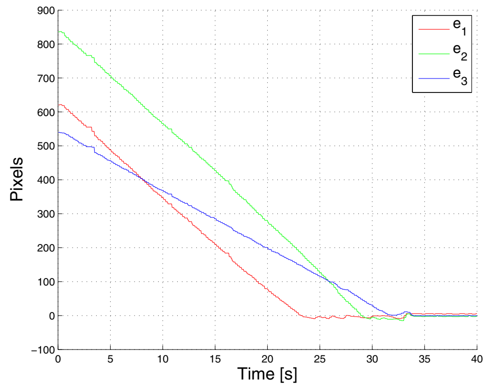
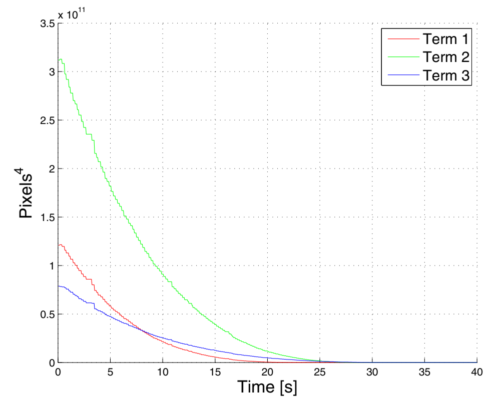


Fig. 4. Evolution of the formation converging to the desired equilateral triangle. The red, green and blue colors stand for robots 1, 2 and 3, respectively.



(a)



(b)

Fig. 5. Evolutions of the errors e_i for $i = 1, 2, 3$ and the terms of the Lyapunov function.

VI. Concluding remarks

In this paper, a gradient-based formation control law using coarse range measurements has been proposed to stabilize three agents moving in a plane to the desired triangular formations. We have proven that under coarse range measurements, the three agents converge to the desired formations as long as they are not initially collinearly positioned. Different from the existing stability results on triangular formations with precise range measurements, it has been shown that the convergence takes place within finite time and that the settling time can be determined by the geometric information of the initial shape of the formation. The analysis techniques are applicable to larger formations. The challenge will be how to determine the settling time when more agents are involved. We will continue our research in this direction using both theoretical analysis and experimental validations.

References

[1] B. D. O. Anderson, C. Yu, B. Fidan, and J. Hendrickx. Rigid graph control architectures for autonomous formations. *IEEE Control Systems Magazine*, 28:48–63, 2008.

[2] A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth liapunov functions. *ESAIM: Control, Optimisation & Calculus of Variations*, 4:361–376, 1999.

[3] F. Bullo, J. Cortes, and S. Martinez. *Distributed Control of Robotic Networks*. Princeton University Press, Princeton, 2009.

[4] M. Cao and A. S. Morse. Maintaining a directed, triangular formation of mobile autonomous agents. *Communications in Information and Systems*, 11:1–16, 2011.

[5] M. Cao, A. S. Morse, C. Yu, B. D. O. Anderson, and S. Dasgupta. Controlling a triangular formation of mobile autonomous agents. In *Proc. of the 45th IEEE CDC*, pages 3603–3608, 2007.

[6] M. Cao, C. Yu, and B. D. O. Anderson. Formation control using range-only measurements. *Automatica*, 47:776–781, 2011.

[7] F. Ceragioli, C. De Persis, and P. Frasca. Discontinuities and hysteresis in quantized average consensus. *Automatica*, 47:1916–1928, 2011.

[8] G. Chen, F. L. Lewis, and L. Xie. Finite-time distributed consensus via binary control protocols. *Automatica*, 47:1962–1968, 2011.

[9] J. Cortés. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42:1993–2000, 2006.

[10] J. Cortés. Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28:36–73, 2008.

[11] J. Cortés. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44:726–737, 2008.

[12] C. De Persis, H. Liu, and M. Cao. Control of one-dimensional guided formations using coarsely quantized information. In *Proc. of the 49th IEEE Conference on Decision and Control (CDC)*, pages 2257–2262, 2010.

[13] J. Fink, A. Ribeiro, and V. Kumar. Robust control of mobility and communications in autonomous robot teams. *Access, IEEE*, 1:290–309, 2013.

[14] P. Frasca, R. Carli, F. Fagnani, and S. Zampieri. Average consensus on networks with quantized communication. *International Journal of Robust and Nonlinear Control*, 19:1787–1816, 2009.

[15] B. Grocholsky, E. Stump, P. M. Shiroma, and V. Kumar. Control for localization of targets using range-only sensors. *Experimental Robotics*, 39:191–200, 2008.

[16] J. H. Guo, K. L. Su, C. C. Wang, and C. J. Wu. Laser range finder applying in motion control system of mobile robots. In *Fourth International Conference on Innovative Computing, Information and Control*, pages 536–539, 2009.

[17] L. Krick, M. E. Broucke, and B. A. Francis. Stabilization of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82:423–439, 2009.

[18] H. Liu, M. Cao, and C. De Persis. Quantization effects on synchronized motion of teams of mobile agents with second-order dynamics. *Systems and Control Letters*, 12:1157–1167, 2012.

[19] F. Mondada, M. Bonani, X. Raemy, J. Pugh, C. Cianci, A. Klaptocz, S. Magnenat, J.-C. Zufferey, D. Floreano, and A. Martinoli. The e-puck, a robot designed for education in engineering. In *Proceedings of the 9th Conference on Autonomous Robot Systems and Competitions*, pages 59–65, 2009.

[20] M. Quigley, D. Stavens, A. Coates, and S. Thrun. Sub-meter indoor localization in unmodified environments with inexpensive sensors. In *IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 2039–2046, 2010.

[21] S. L. Smith, M. E. Broucke, and B. A. Francis. Stabilizing a multi-agent system to an equilibrium polygon formation. In *Proc. of the 17th MTNS*, pages 2415–2424, 2006.

[22] E. Stump, V. Kumar, B. Grocholsky, and P. M. Shiroma. Control for localization of targets using range-only sensors. *International Journal of Robotics Research*, 28:743–757, 2009.

[23] M. Turpin, N. Michael, and V. Kumar. Decentralized formation control with variable shapes for aerial robots. In *Proc. of the IEEE Int. Conf. on Robotics and Automation*, pages 23–30, 2012.