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The bench mover's problem: minimum-time trajectories, with cost for switching between controls

Yu-Han Lyu[1] Andrei Furtuna[2] Weifu Wang[3] Devin Balkcom[4]

***Abstract*— Analytical results describing the optimal trajectories for general classes of robot systems have proven elusive, in part because the optimal trajectories for a complex system may not exist, or may be computed only numerically from differential equations. This paper studies a simpler optimization problem: finding an optimal sequence and optimal durations of motion primitives (simple preprogrammed actions) to reach a goal. By adding a fixed cost for each switch between primitives, we ensure that optimal trajectories exist and are well-behaved.**

To demonstrate this approach, we prove some general results that geometrically characterize time-optimal trajectories for rigid bodies in the plane with costly switches (allowing comparison with previous analysis of optimal motion using Pontryagin's Maximum Principle), and also present a complete analytical solution for a problem of moving a heavy park bench by rotating the bench around each end point in sequence.

I. Introduction

Consider the following problem. A mover would like to move a heavy park bench (modeled as a line segment) from one location and orientation to another, as efficiently as possible. Since the bench is heavy and there is only one mover, the bench can only be moved by lifting one end and rotating the bench around the end that is still on the ground, with rotational velocity of ± 1. We wish to find the sequence of durations and directions of rotations that bring the bench to the final configuration, while minimizing the total time of the trajectory (computed as the sum of the absolute values of the angles rotated through). This problem is very related to the Reeds-Shepp problem [15] of finding the shortest path for a steered car, but with only four discrete controls.

Unfortunately for the mover, the "optimal" trajectory to move the bench straight forwards will require the mover to run back and forth between ends of the bench infinitely many times, rotating the bench through an infinitely small angle, a phenomenon known as *chattering*. We might therefore assign some fixed time cost or penalty to each switch of controls, corresponding to the time it takes to run from one end of the bench to the other.

The bench-mover's problem is an example of a fundamental problem in robotics: finding trajectories that optimize an objective function subject to constraints. For certain simple models of mobile robots, including the well-known

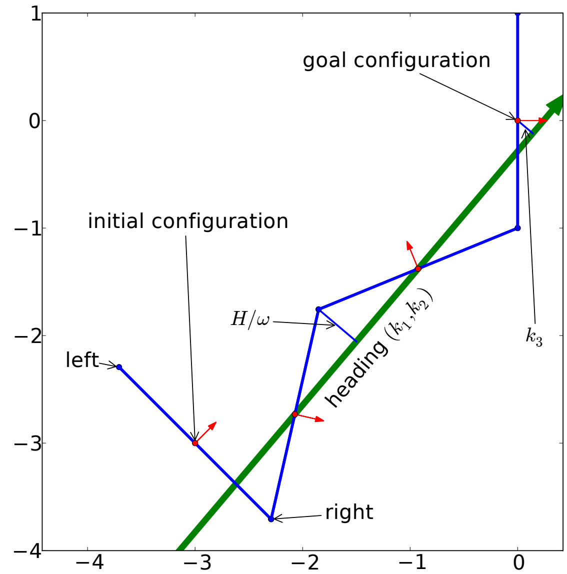

Fig. 1: Optimal trajectory for initial configuration $(-3, -3, \pi/4)$ with switching cost 1, where red arrow represents the orientation of the bench. Green line denotes the control line for this trajectory.

Dubins [10], [8] and Reeds-Shepp [15], [19], [18] cars, optimal trajectories can be found analytically and explicitly. Over the past decade, we and many other researchers (for example, see [16], [6], [5], [7], [1], [17]) have tried vigorously to extend and generalize techniques (typically based on Pontryagin's Maximum Principle [14]), with the goal of achieving a more fundamental understanding of optimal motion for mobile robots with non-holonomic constraints. For example, we have found strong results about minimum-time trajectories for planar rigid bodies with linear constraints on velocity and angular velocity [11].

However, extension to more satisfyingly general problems has been elusive. Adding constraints on accelerations to the model, or adding obstacles to the environment, leads quickly to a formulation that resists solution using existing techniques.

There seem to be two primary difficulties. The optimal solutions might be described only by differential equations that we may not integrate, or, even worse, the optimal solutions may not exist. For example, Sussmann [19] showed that perhaps the simplest extension of the Dubins model to include bounds on angular acceleration leads to chattering: the "best" trajectories require infinitely many discontinuous switches in control. Desaulniers showed that chattering occurs with even the velocity-bounded model, if there are

[1]Department of Computer Science, Dartmouth College, Hanover, NH 03755, USA `yuhanlyu@cs.dartmouth.edu`

[2]Department of Computer Science, Dartmouth College, Hanover, NH 03755, USA `andrei.furtuna@gmail.com`

[3]Department of Computer Science, Dartmouth College, Hanover, NH 03755, USA, `weifu.wang.gr@dartmouth.edu`

[4]Department of Computer Science, Dartmouth College, Hanover, NH 03755, USA, `devin@cs.dartmouth.edu`

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obstacles in the environment [9].

This paper suggests an approach to dealing with both of these difficulties. We limit the choice of controls to certain primitives that may be integrated analytically (by design), and we charge a cost for each switch. How should the primitives be selected? In some cases, arguments about the optimality of so-called 'bang-bang' controls may lead to selection of a discrete set of consent controls. In other cases, the design of the robot may suggest a set of primitives. (This step is also required for modeling a system for the use of either Non-holonomic PRM [12], RRT [13], and many other general-purpose approaches to non-holonomic motion planning.)

Pontryagin's Maximum Principle does not appear to be the right tool for the motion primitive model of control. The set of controls is discrete rather than continuous, and switching costs make the cost function discontinuous with respect to time. We therefore split the problem into two parts: a discrete problem of selecting the correct sequence of primitives, and a continuous problem of finding the optimal duration. We apply Karush-Kuhn-Tucker conditions [3] to find necessary conditions on trajectories with particular sequences of controls.

We show that for a general class of problems in optimal control, a non-zero cost for switching ensures that optimal trajectories exist and are well-behaved. We also prove some general results that geometrically characterize time-optimal trajectories for rigid bodies in the plane with costly switches, and finally present a complete analytical solution for the park bench mover's example problem described above. To our knowledge, this is the first exact solution for the optimal trajectories for a mobile robot with a cost for control switches, even though the model has been used implicitly in motion-planning papers going as far back as 1991 [2].

II. Existence of Optimal Trajectories in Costly Switch Model

Furtuna's Ph.D. thesis [11] proves that optimal trajectories always exist for the bench-mover's problem we will study in this paper (as long as the switching cost is strictly greater than zero), and in fact even for a much more general model of trajectories composed of sequences of motion primitives among obstacles. We briefly consider this more general model (which contains our current model as a special case), and state the result.

Let Q be a closed set of collision free configurations. A trajectory is a function $F: Q \times \mathcal{R}_+ \to Q$ that mapping from an initial configuration $q_s \in Q$ and time t to a configuration in Q, where $q' = F(q_s, t)$ is the configuration at time t. When the initial configuration is fixed, a trajectory can be determined by two sequences with the same length: a sequence of controls, $\mathbf{u}$, and a sequence of durations $\mathbf{t}$. For a trajectory determined by $(\mathbf{u}, \mathbf{t})$, we use $q_i(q_s, \mathbf{u}, \mathbf{t})$ to denote the resulting configuration after applying the first i controls on q_s. We call a state q' *reachable* from q_s if there exists $(\mathbf{u}, \mathbf{t})$ with length n, such that $q_n(q, \mathbf{u}, \mathbf{t}) = q'$. Let $R(q)$ be the set of reachable configurations from q.

For a control $u \in U$, the cost function $c_u(q,t) \geq 0$ is a differentiable function representing the cost of applying control u at configuration q for duration t. For two controls $u, u' \in U$, the switching cost function $C(u, u') > 0$ is a function representing the cost of switching from u to u'. Hence, for a trajectory $(\mathbf{u}, \mathbf{t})$, the cost is $\sum_{i=2}^{n} C(u_{i-1}, u_i) + \sum_{i=1}^{n} c_{u_i}(q_{i-1}(q_s, \mathbf{u}, \mathbf{t}), t_i)$.

Theorem 1: If the goal configuration is in $R(q_s)$ and $R(q_s)$ is a closed set, then there exists a minimum cost trajectory from q_s to the goal configuration.

We sketch the proof. First upper-bound the number of primitives required, using the fact that the goal is in $R(q)$. Now consider each fixed trajectory structure (sequence of primitives) separately; there are finitely many. For each structure, construct a sequence of trajectories whose limit is a trajectory with cost that is the infimum of costs for trajectories for this structure, and using the Bolzano-Weierstrass theorem show that this infimum is actually a minimum. (When there is no obstacles in the environment, existence of trajectories is also implied by [4]. But the proof above is simpler due to its restriction to rigid bodies in the plane, and gives an explicit upper bound on the number of switches.)

III. Rigid Bodies in the Plane

We specialize the model to consider the optimal motion of rigid bodies in the plane with constant-velocity translations or rotations; an ultimate goal of future work is to apply similar techniques to more general models.

A. Model and Notation

We use (x, y, θ) to denote a configuration, and (v_x, v_y, ω) to denote a control: x and y velocities in a frame attached to the body (robot frame), and angular velocity.

For a configuration q_0, if we apply a sequence of controls $\mathbf{u} \in U^n$ with a sequence of durations $\mathbf{t} \in \mathcal{R}_+^n$, then the result is a configuration $q(q_0, \mathbf{u}, \mathbf{t}) \in SE(2)$, where q is a continuous function that integrates the control over time in the world frame and then adds q_0 to obtain the resulting configuration.

Let q_s be the start configuration and U be the set of controls. A pair $(\mathbf{u}, \mathbf{t})$ is *admissible* if $q(q_s, \mathbf{u}, \mathbf{t}) = \mathbf{0}$, and its *time* is $T(\mathbf{t}) = \sum_i t_i$. In addition to the time cost, there is a cost associated with switching controls. The switching costs can be modelled as a function $C: U \times U \to \mathcal{R}_+$ that depends on the control applied before and the control applied after and the cost is always strictly larger than zero.

The problem is as follows: Given a start configuration q_s, a finite control set U, and a cost function C, find an admissible $(\mathbf{u}, \mathbf{t})$ with the minimum sum of the time and the switching costs. This problem can be modelled as a mathematical optimization problem.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} t_i + \sum_{i=2}^{n} C(u_{i-1}, u_i) \\
\text{subject to} \quad & q(q_s, \mathbf{u}, \mathbf{t}) = 0 \\
& t_i \geq 0, u_i \in U
\end{aligned} \tag{1}
$$

Since function q integrates control over $\mathbf{t}$, q is differentiable with respect to $\mathbf{t}$ away from the boundary $t_i = 0$.

B. Karush-Kuhn-Tucker conditions

Consider a non-linear optimization problem as follows

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & q(x) = 0 \\ & g(x) \leq 0 \\ & x \in \mathcal{R}^n, \text{with } f: \mathcal{R}^n \to \mathcal{R},\ q: \mathcal{R}^n \to \mathcal{R}^m, \\ & \qquad \text{and } g: \mathcal{R}^n \to \mathcal{R}^p \text{ differentiable.} \end{aligned}$$

For a solution $\hat{x}$, let the active set $I(\hat{x}) = \{i : g_i(\hat{x}) = 0\}$. A solution $\hat{x}$ satisfies *linear independence constraint qualification (LICQ)*, if $\nabla q(\hat{x})$ and $\nabla g_i(\hat{x}), i \in I$ are linearly independent. (There are different types of constraint qualification. Although LICQ is not the most general constraint qualification, LICQ suffices for our bench-mover example.)

Karush-Kuhn-Tucker conditions [3] state: If $\hat{x}$ is a local minimum and satisfies LICQ, then there exists $\lambda \in \mathcal{R}^m$ and $\mu \in \mathcal{R}^p$, such that

1) $\nabla f(\hat{x}) + \lambda \cdot \nabla q(\hat{x}) + \mu \cdot \nabla g(\hat{x}) = 0$.
2) $\mu \geq 0$.
3) $\mu \cdot g(\hat{x}) = 0$.

IV. Necessary Conditions for Rigid Bodies

There are two sets of free variables, $\mathbf{u}$ and $\mathbf{t}$. Since the domain of u_i is a finite set U and the domain of t_i is $\mathcal{R}_+$, formulation 1 is a mixed integer non-linear programming problem, to which KKT conditions do not apply.

However, we can decompose the problem into several non-linear sub-problems. Each sub-problem has a fixed control sequence so that we only need to determine the optimal duration for this control sequence. Consequently, each sub-problem has only one set of free continuous variables. Since for any given problem we may upper-bound the number of controls in the optimal trajectory, there are only finitely many sub-problems.

A. Fixed control sequence

Let $\mathbf{u}$ be a fixed control sequence. Since the switching cost $\sum_{i=2}^{n} C(u_{i-1}, u_i)$ in the objective function becomes a constant, we remove that term from the objective function. The resulting problem can be modelled as a mathematical optimization problem.

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} t_i \\ \text{subject to} \quad & q(q_s, \mathbf{u}, \mathbf{t}) = 0 \\ & t_i \geq 0 \end{aligned} \tag{2}$$

A solution $\mathbf{t}$ is *regular* if $\mathbf{t}$ satisfies LICQ. Since KKT conditions hold only for regular solutions, we use KKT conditions to prove properties of regular solutions and deal with irregular solutions separately.

Let $\lambda = (\lambda_x, \lambda_y, \lambda_\theta)$ be the dual variables associated with the equality constraints and let $\mu = (\mu_1, \ldots, \mu_n)$ be the dual variables associating with the inequality constraints. By KKT, we can get the following condition.

$$\lambda \cdot \nabla q(q_s, \mathbf{u}, \mathbf{t}) = \mu - 1,$$

where $\mu \geq 0$ and $\mu \cdot \mathbf{t} = 0$. Since q_s and $\mathbf{u}$ are fixed, function q only depends on $\mathbf{t}$.

Without loss of generality, we assume:

1) For all $1 \leq i < n$, $u_i \neq u_{i+1}$.
2) For optimal solutions $\mathbf{t}$, $t_i > 0$ for all $1 \leq i \leq n$.

If the first assumption is violated, we can combine u_i with u_{i+1}. If the second assumption is violated, then we can remove the controls u_i with $t_i = 0$, and the resulting control sequence will be considered in another sub-problem. Since $\mathbf{t} > 0$, we have $\mu = 0$ by the third KKT condition.

Now, we characterize properties of the velocity along an optimal solution.

Lemma 1: For any fixed control sequence $\mathbf{u}$, an optimal duration $\mathbf{t}^*$ satisfies the following property: given the direction along which a particular translation control is applied at some point t_0, the possible directions (in the world frame) of all other translation controls are determined, up to sign.

Proof: We have

$$\lambda_x \frac{\partial q_x}{\partial t_i^*} + \lambda_y \frac{\partial q_y}{\partial t_i^*} = -1. \tag{3}$$

The partial derivatives are parallel to the direction of translation, so

$$\lambda_x \dot{x}(t_0) + \lambda_y \dot{y}(t_0) = -1. \tag{4}$$

Since this equation holds for all translations i, the property follows immediately from properties of the dot product. ■

Lemma 2: For any fixed control sequence $\mathbf{u}$, an optimal duration $\mathbf{t}^*$ satisfies the following property: for a rotation control at configuration (x, y, θ), we have

$$\lambda_x v_x + \lambda_y v_y + \omega(\lambda_x y - \lambda_y x + \lambda_\theta) = -1.$$

Proof: For the i-th control, (v_x, v_y, ω) at configuration (x, y, θ), we have

$$\frac{\partial q_x}{\partial t_i^*} = v_x + y\omega, \frac{\partial q_y}{\partial t_i^*} = v_y - x\omega, \frac{\partial q_\theta}{\partial t_i^*} = \omega$$

Thus, the first KKT condition becomes

$$\begin{aligned} -1 &= \lambda_x \frac{\partial q_x}{\partial t_i} + \lambda_y \frac{\partial q_y}{\partial t_i} + \lambda_\theta \frac{\partial q_\theta}{\partial t_i} \\ &= \lambda_x (v_x + y\omega) + \lambda_y (v_y - x\omega) + \lambda_\theta \omega \\ &= \lambda_x v_x + \lambda_y v_y + \omega(\lambda_x y - \lambda_y x + \lambda_\theta) \end{aligned}$$

■

If we multiply a constant $-H$ on both sides of the equation $\lambda_x v_x + \lambda_y v_y + \omega(\lambda_x y - \lambda_y x + \lambda_\theta) = -1$, then we get the following equation

$$k_1 v_x + k_2 v_y + \omega(k_1 y - k_2 x + k_3) = H,$$

where $k_1 = -\lambda_x H$, $k_2 = -\lambda_y H$, and $k_3 = -\lambda_\theta H$. If $k_1 \neq 0$ or $k_2 \neq 0$, since H can be chosen arbitrarily, we can pick a positive H so that $k_1^2 + k_2^2 = 1$. Thus, combining Lemma 1 and 2, we have the following theorem.

Theorem 2: For any fixed control sequence $\mathbf{u}$, any regular optimal duration $\mathbf{t}^*$ in the costly switch model satisfies the following property: there exist constants $H > 0$, k_1, k_2, and k_3, such that for any control u_i with the instantaneous velocity (v_x, v_y, ω) in the world frame when u_i is applied at configuration (x, y, θ), we have

$$k_1 v_x + k_2 v_y + \omega(k_1 y - k_2 x + k_3) = H,$$

where $k_1^2 + k_2^2 \in \{0, 1\}$.

Interestingly, this central equation is the same as that derived in [11] (using Pontryagin's Maximum Principle) as part of the necessary conditions for optimal trajectories for rigid bodies the plane with linear constraints on the velocities and no cost for switching. However, this condition is weaker: it only applies to a particular control structure, and there is no requirement that H be maximized. This weaker condition allows more "types" of optimal trajectory structures than for the classic Dubins and Reeds-Shepp problems, depending on the cost of switching. (Theorem 2 is implied by Blatt's *indifference principle*[4]; however, we still include the above proof because it makes use only of simple KKT conditions.)

B. Control line interpretation

There is a nice geometric interpretation for Theorem 2 when $k_1^2 + k_2^2 = 1$, related to the control line interpretation in [11]. We can define a *control line* in the plane with heading (k_1, k_2) and distance k_3 from the origin; see Fig 1. The term $k_1 v_x + k_2 v_y$ becomes the translational velocity along the vector (k_1, k_2) and the term $k_1 y - k_2 x + k_3$ becomes the *signed distance* from the reference point to the control line. By Corollary 1 in [11], when a rotation is applied, the signed distance from the rotation center to the control line is H/ω. Similarly, when a translation is applied, the dot product between (k_1, k_2) and (v_x, v_y) must be H.

C. Whirl trajectories

In the case that $k_1 = k_2 = 0$, the KKT conditions tell us relatively little: only that all angular velocities must be equal. We call such trajectories *whirl* trajectories. In order to compute an optimal whirl, we only consider a smaller subclass of whirl trajectories, and show if an optimal whirl exists, then an optimal trajectory of this constrained subclass exists. Specifically, we consider two-stage trajectories:

1) The first stage moves the last rotation center to the correct position in the goal configuration in the minimum time.
2) The second stage is a rotation around the last rotation center until the goal configuration is achieved.

Theorem 3: Among any fixed control sequence $\mathbf{u}$ for which all controls have the same angular velocity, the two-stage trajectory has minimum cost.

Proof: Let T_1 and T_2 be the times corresponding to the first and the second stage respectively. Let T_f be the time for an optimal trajectory. Since T_f is the time for an optimal trajectory, $T_f \leq T_1 + T_2$. Moreover, since an optimal trajectory needs to place the last rotation center to the correct position, $T_f \geq T_1$. Since T_2 is strictly less than 2π, we have $T_f \leq T_1 + T_2 < T_f + 2\pi$. For any two admissible whirl trajectories, the difference of time between them must be a multiple of 2π. Therefore, T_f must equal $T_1 + T_2$. ■

We now show how to find the minimum-cost trajectory required for the first stage. We change the coordinate system so that the last rotation center is at the origin. Then, the duration for the first stage is the solution to the following mathematical optimization problem, where the functions q'_x and q'_y describe the motion of the last rotation center with initial position q'_s.

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n-1} t_i \\ \text{subject to} \quad & q'_x(q'_s, \mathbf{u}, \mathbf{t}) = 0 \\ & q'_y(q'_s, \mathbf{u}, \mathbf{t}) = 0 \\ & t_i \geq 0 \end{aligned} \tag{5}$$

By an argument similar to Lemmas 1, 2, and Theorem 2, we have the following result.

Theorem 4: Regular optimal solutions to 5 must satisfy the following property: there exist constants $H_\omega > 0$, k_4, and k_5, such that for any control u_i with the instantaneous velocity (v_x, v_y, ω) in the world frame when u_i for $1 \leq i < n$ is applied at configuration (x, y, θ), we have

$$k_4 v_x + k_5 v_y + \omega(k_4 y - k_5 x) = H_\omega, \text{where } k_4^2 + k_5^2 = 1.$$

Unfortunately, irregular solutions to 5 must still be considered during analysis of any particular system.

There is a geometric interpretation for two-stage trajectories of this type. Define a *whirl control line* in the plane heading (k_4, k_5) through the last rotation center. By Theorem 4, all rotation centers except the last one should have the same signed distance to this line, and are thus parallel to the line.

V. Bench mover's problem

A general framework for finding optimal trajectories in the costly switch model by using Theorem 2 is as follows:

1) Generate all possible control sequences of optimal trajectories.
2) For each control sequence, determine the optimal durations that reach the goal by using Theorem 2.
3) Output the best trajectory among the optimal trajectories founded in the previous step.

In general, identifying the control sequences of optimal trajectories is difficult and determining the optimal duration for a fixed control sequence is hard as well. However, for a particular system, complete solution may be possible. We will take the bench mover's problem as an example to demonstrate this framework.

A. Model and trajectory types

Consider a park bench with length 2. Let $q_s = (x_0, y_0, \theta_0)$ be the initial configuration and $(0, 0, 0)$ be the goal configuration. Figure 1 gives an example with initial configuration $(-3, -3, \pi/4)$. Let the reference point be the center of the bench and it is (0, 0) in the robot frame. There are two rotation centers: the left rotation center, (0, 1) in the robot frame, and the right rotation center, (0, -1) in the robot frame. Let L be the set of controls containing $l^+ = (1, 0, 1)$ and $l^- = (-1, 0, -1)$ corresponding the left rotation center. Let R be the set of controls containing $r^+ = (-1, 0, 1)$ and $r^- = (1, 0, -1)$ corresponding to the right rotation center. The control set $U = L \cup R$. For two controls $u, u' \in U$, the cost of switching from u to u' is c.

Without loss of generality, we assume that the length of the control sequence is at least three, since we can determine optimal durations for a control sequence with length smaller than three easily.

There are four broad types of trajectories:

1) Irregular: trajectories for which the gradients of the constraints on the final configuration are linearly dependent; KKT does not apply.
2) Whirl: trajectories for which $k_1 = k_2 = 0$. All controls in the trajectory must have the same angular velocity.
3) Alternating sign: the control sequence contains controls alternating between l^+ and r^- or alternating between l^- and r^+.
4) Mixed: the control sequence contains controls alternating between L and R but not strictly alternating signs.

Our basic approach, given a starting configuration, is to compute an optimal trajectory of each of the four types, and then to compare to find the minimum. The following sections will demonstrate how to find an optimal trajectory for each type. For computing optimal trajectories of types 3 and 4, an upper bound on the number of control actions in the trajectory is required; this bound may be found by considering the cost of the optimal whirl.

B. Irregular trajectories

If the gradients of the constraints on the final configuration are linearly dependent for some value of $\mathbf{t}$ satisfying the constraints, we say that the corresponding trajectory is irregular. Irregular trajectories may be optimal, and need not satisfy the KKT conditions.

For the bench-mover's problem, linear dependence of the constraints only occurs for trajectories for which all rotation centers lie on a line. This further implies that the distance between first and last rotation centers must be a multiple of two. It is straightforward to compute the optimal trajectory of this form geometrically. We sketch the procedure. First compute the minimum-cost whirl (using the technique described in the next section). Use this to upper bound a number of circles between the start and goal, and enumerate all irregular trajectories that do not reach the same configuration twice.

C. Whirl trajectories

Since the analysis of whirls presented in the previous section relies itself on a secondary application of KKT conditions to a problem of optimal motion in the plane, we must consider both *regular whirls* and *irregular whirls*. For the case of regular whirls, all rotation centers except possibly the last one are on the same line; see Fig. 2. We can also see geometrically that for irregular whirls, the condition is

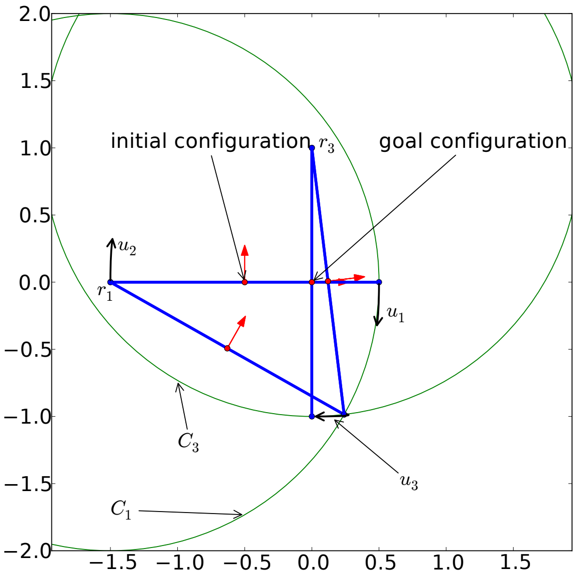


Fig. 2: Whirl trajectory with initial configuration $(-0.5, 0, \pi/2)$. All rotation centers except the last one are on the same line. This is the optimal trajectory for this initial configuration with switching cost 1.

exactly the same. This section will show how this fact can be used to identify the minimum-cost whirl.

Since the length of the bench is two and controls alternate between L and R, for any two consecutive controls, the distance between their rotation centers is two. Thus, when the first control and the last control are fixed, in order to reach the goal, there is only one choice of the length of the control sequence. Hence, a whirl trajectory can be described by its first and last control. Since there are only four choices for the first control and each has two choices for the last control, we can enumerate all possible pairs of first and last controls for whirl trajectories.

Fix the first control u_1 and the last control u_n with rotation centers r_1 and r_n respectively. Since the last control is fixed, the second to the last control u_{n-1} is also fixed and its rotation center r_{n-1} should be on a circle C_n centered at r_n with radius 2.

Since rotation centers r_i, $1 \leq i < n$, are on the same line, the distance L from r_1 to r_{n-1} is determined in the following way. Let D be the distance between the first and the last rotation centers. If $u_1 = u_n$ ($u_1 \neq u_{n-1}$), then L is multiple of four plus 2. Otherwise, L is multiple of four. Since the diameter of C is four and the difference between any choices is multiple of four, $L = 4\lceil (D-4)/4 \rceil + 2$ when $u_1 = u_n$, $L = 4\lceil (D-2)/4 \rceil$ otherwise.

After we determine L, we can find a circle C_1 centered at r_1 with radius L. The circle C_1 intersects with C_n at most two points and these points are possible locations of r_{n-1}. When the location of r_{n-1} is fixed, the durations for all controls can be determined easily.

D. Alternating sign trajectories

Since all angular velocities have the same absolute value, all rotation centers must have equal distance to the control line; see Fig. 1. An alternating sign trajectory can be described by its first control and the length of the sequence. There are four choices of first control u_1 in U, and the parity of n determines whether u_n is the same as u_1 or not. We will now show how to determine possible H values, control lines, and durations based on u_1 and u_n.

1) Determining H: Let r_1 and r_n be the first rotation center and the last rotation center with distance D. If n is odd, $D = (2n-2)\sqrt{1-H^2}$ and $H = \sqrt{1 - \frac{D^2}{4(n-1)^2}}$. In this case, when $D^2 \geq 4(n-1)^2$, the control line exists and we can obtain a positive value of $H \leq 1$. When n is even, let X be $(2n-4)\sqrt{1-H^2}$, D^2 will be $X^2 + \sqrt{1-H^2} + 4$. Consequently, $D^2 = 4n(n-2)(1-H^2)+4$ and $H = \sqrt{1 - \frac{D^2-4}{4n(n-2)}}$. In this case, when $D \geq 4$ and $D^2 \leq 4n(n-2)+4$ we can obtain a non-negative value of $H \leq 1$.

2) Determining control lines: After we determine the value of H, we want to determine the control line, which is represented by a tuple (k_1, k_2, k_3). Since $k_1^2 + k_2^2 = 1$, we can use $(\cos\varphi, \sin\varphi)$ to represent (k_1, k_2). For one H value, there are two possible control lines. We determine (φ, k_3) in a similar way as in [11]. Let $r'_{1_x} = r_{1_x} u_{1_\omega}, r'_{1_y} = r_{1_y} u_{1_\omega}, r'_{n_x} = r_{n_x} u_{n_\omega}$, and $r'_{n_y} = r_{n_y} u_{n_\omega}$. Let $d_x = r'_{1_x} - r'_{n_x}$ and $d_y = r'_{1_y} - r'_{n_y}$. Let (α, β) be $(\text{atan2}(dx, dy), \pi/2)$ if the first control and the last control have the same angular velocity; otherwise $(\text{atan2}(d'_x, d'_y), \text{acos}(\frac{H}{\sqrt{{d'_x}^2 + {d'_y}^2}}))$, where $d'_x = r'_{n_x} + d_x/2$ and $d'_y = r'_{n_y} + d_y/2$. Then, $\varphi = -\alpha \pm \beta$ and $k_3 = \frac{H + r'_{n_x} \sin\varphi - r'_{n_y} \cos\varphi}{u_{n_\omega}}$.

3) Determining durations: For a given control line $L = (k_1, k_2, k_3)$ with a H value, we can determine the durations as follows. For a given initial configuration q_0 with reference point at p, we can determine the angle α between the vector $p - r_1$ and L. Then, we want to determine the location of r_2 for u_2. By theorem 2, all rotation centers have the same distance H to L. For two consecutive rotation centers, their distance must be 2, the length of the bench. Hence, the angle β between the vector $r_2 - r_1$ and L can only have two possible values: $\text{asin}(H)$ and $\pi - \text{asin}(H)$ if $u_{1_\omega} > 0$, otherwise $\pi + \text{asin}(H)$ and $2\pi - \text{asin}(H)$. For a fixed β, we can determine t_1.

For u_2, since we switch to u_2 at angle β with respect to L and $u_3 = u_1$, we also can switch to u_3 immediately with $t_2 = 0$. Since this null control will be examined by another control sequence, we ignore this choice. Consequently, we have only one choice of t_2. Similarly, all controls u_2 to u_{n-1} have the same duration. This duration will be either $2\text{acos}H$ (if $\beta \in [\pi/2, \pi/2]$) or $2\pi - 2\text{acos}H$ (otherwise). The duration t_n can be determined based on t_1 to t_{n-1} based on the constraint of reaching the goal configuration.

E. Mixed trajectories

Consider a mixed trajectory that contains two consecutive controls, u and u', with the same angular velocity. By Theorem 2, these two controls' rotation centers, r and r', have equal distance to the control line, L, and are on the same side of the control line. Hence, the line through r and r' is parallel to L; see Fig. 3. Consequently, if there are three consecutive controls with the same angular velocity, then the duration of the second one must be π.

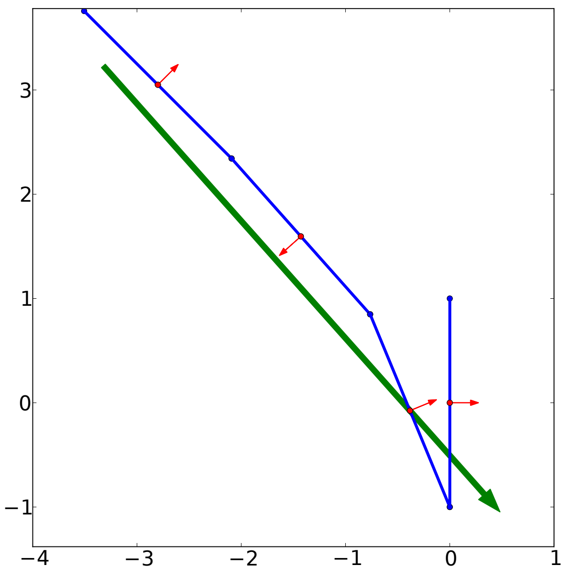


Fig. 3: Mixed trajectory with initial configuration $(-2.8, 3.05, \pi/4)$. First two controls have the same angular velocity and hence they are collinear and parallel to the control line. This is the optimal trajectory for this initial configuration with switching cost 1.

For a mixed trajectory, subsequences of controls with the same angular velocity may appear anywhere in the control sequence. However, it is always possible to rearrange the controls in the control sequence without changing the cost, such that the prefix of the control sequence has controls with the same angular velocity and the suffix has controls with alternating angular velocity. Hence, we only consider control sequences that can be decomposed into two parts in this way.

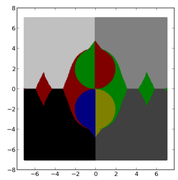
Let n be the length of the control sequence, $n_w < n$ be the number of controls with the same sign, and m be $n - n_w$. Let D be the distance between the first rotation center and the last rotation center. When m is even, H satisfies $|D - 2(n_w - 1)| = 2m\sqrt{1-H^2}$ or $D + 2(n_w - 1) = 2m\sqrt{1-H^2}$. Hence, $H = \sqrt{1 - \frac{(D-2(n_w-1))^2}{4m^2}}$ or $H = \sqrt{1 - \frac{(D+2(n_w-1))^2}{4m^2}}$. We can obtain at most two possible positive $H \leq 1$ values.

When m is odd, H satisfies $D^2/4 = (m^2-1)(1-H^2) + 2m(n_w-1)\sqrt{1-H^2} + n_w^2 - 2n_w + 2$ or $D^2/4 = (m^2-1)(1-H^2) - 2m(n_w-1)\sqrt{1-H^2} + n_w^2 - 2n_w + 2$. Hence, $1 - H^2 = |\frac{-b \pm \sqrt{b^2-4ac}}{2a}|$, where $a = m^2 - 1$, $b = 2m(n_w - 1)$, and $c = n_w^2 - 2n_w + 2$. We can obtain at most two possible positive $H \leq 1$ values.

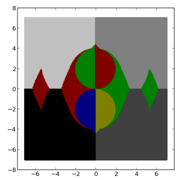
After we obtain H values, we can compute the control line and durations by the method mentioned in the previous section.

F. Mapping configurations to optimal trajectories

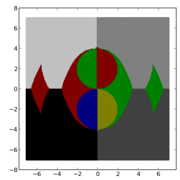
We implemented the solution described to solve the bench mover's problem, and used this implementation to sample starting configurations and determine how optimal trajectory structures and costs change with respect to different configurations. We used initial configurations in $[-7, -7] \times [-7, -7]$



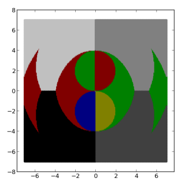
(a) Switching cost 0.3



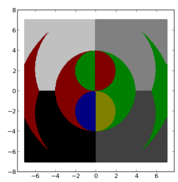
(b) Switching cost 0.4



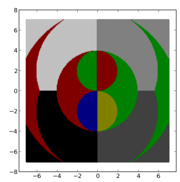
(c) Switching cost 0.5



(d) Switch cost 1.0



(e) Switching cost 1.5



(f) Switching cost 2.0

Fig. 4: Synthesis

with initial orientation of zero, and six different values of switching costs, to generate Figure 4.

We classify optimal trajectories by their first control and the parity of their lengths. We plot the mapping from initial configuration to trajectory category by using different colors to represent different trajectory categories. Gray colors represent optimal trajectories with even length. Red, green, blue, and yellow represent optimal trajectories with odd length. Within each category, different colors represent different first controls.

Note that when the switching cost increases, the red/green area grows, as optimal trajectories tend to use fewer controls, changing the parity of the lengths of trajectories.

VI. Conclusions and Future Work

Although the bench mover's problem is artificial, and a solution is perhaps of little practical use in itself, we are motivated to study the problem because we believe it highlights very central issues in the study of optimal control and motion planning for robots. In particular, by analyzing optimal sequences of motion primitives, we ensure that the trajectories found may be described geometrically, without recourse to numerical solutions of differential equations. By adding a cost to switch between controls (perhaps not such an unreasonable thing to do), we also ensure existence of solutions that do not involve chattering.

At least two major challenges remain to apply our KKT-based approach to a general model of optimal sequences of motion primitives; such a model would include obstacles and more complex control systems than the Reeds-Shepp bench we studied. The first challenge is irregular optimal trajectories, for which KKT does not apply. For particular simple systems, it is possible to exactly describe all irregular trajectories (where constraints are redundant or dependent) and analyze whether they may be optimal. It is not immediately clear how this problem may be solved in a more general setting. The second challenge is the potential profusion of optimal trajectory structures. For simple systems, the number of structures is small; for more complicated systems, an algorithm might potentially need to explore a number of structures that is exponential in the number of primitive actions along the path. We conjecture that exploration of these discrete structures might be accomplished in an efficient order by a discrete search algorithm such as A*.

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References

[1] Pankaj K. Agarwal, Therese Biedl, Sylvain Lazard, Steve Robbins, Subhash Suri, and Sue Whitesides. Curvature-constrained shortest paths in a convex polygon. In *SIAM J. Comput*, pages 392–401. ACM Press, 1998.

[2] Jérîme Barraquand and Jean-Claude Latombe. Robot motion planning: a distributed representation approach. *ijrr*, 10(6):628–649, Dec 1991.

[3] Mokhtar S. Bazaraa, Hanif D. Sherali, and C. M. Shetty. *Nonlinear Programming: Theory and Algorithms*. Wiley-Interscience, 3rd edition, May 2006.

[4] John M. Blatt. Optimal control with a cost of switching control. *The ANZIAM Journal*, 19:316–332, 6 1976.

[5] H. R. Chitsaz. *Geodesic problems for mobile robots*. PhD thesis, University of Illinois at Urbana-Champaign, 2008.

[6] Hamid Reza Chitsaz, Steven M. LaValle, Devin J. Balkcom, and Matthew T. Mason. Minimum wheel-rotation paths for differential-drive mobile robots. *I. J. Robotic Res.*, 28(1):66–80, 2009.

[7] M. Chyba and T. Haberkorn. Designing efficient trajectories for underwater vehicles using geometric control theory. In *24rd International Conference on Offshore Mechanics and Artic Engineering*, Halkidiki, Greece, 2005.

[8] E. J. Cockayne and G. W. C. Hall. Plane motion of a particle subject to curvature constraints. *SIAM Journal on Control*, 13(1):197–220, 1975.

[9] Guy Desaulniers. On shortest paths for a car-like robot maneuvering around obstacles. *Robotics and Autonomous Systems*, 17(3):139 – 148, 1996.

[10] L. E. Dubins. On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79(3):pp. 497–516, 1957.

[11] Andrei Furtuna. *Minimum Time Kinematic Trajectories for Self-Propelled Rigid Bodies in the Unobstructed Plane*. PhD thesis, Dartmouth College, June 2011.

[12] David Hsu, Robert Kindel, Jean-Claude Latombe, and Stephen M. Rock. Randomized kinodynamic motion planning with moving obstacles. *I. J. Robotic Res.*, 21(3):233–256, 2002.

[13] Steven M. Lavalle. Rapidly-exploring random trees: A new tool for path planning. Technical report, Computer Science Department, Iowa State University, October 1998.

[14] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *Mathematical Theory of Optimal Processes*. John Wiley & Sons Inc, January 1962.

[15] J. A. Reeds and L. A. Shepp. Optimal paths for a car that goes both forwards and backwards. *Pacific Journal of Mathematics*, 145(2):367–393, 1990.

[16] David B. Reister and Francois G. Pin. Time-optimal trajectories for mobile robots with two independently driven wheels. *International Journal of Robotics Research*, 13(1):38–54, February 1994.

[17] Marc Renaud and Jean-Yves Fourquet. Minimum time motion of a mobile robot with two independent acceleration-driven wheels. In *Proceedings of the 1997 IEEE International Conference on Robotics and Automation*, pages 2608–2613, 1997.

[18] P. Souères and J.-D. Boissonnat. Optimal trajectories for nonholonomic mobile robots. In J.-P. Laumond, editor, *Robot Motion Planning and Control*, pages 93–170. Springer, 1998.

[19] Héctor J. Sussmann and Guoqing Tang. Shortest paths for the Reeds-Shepp car: A worked out example of the use of geometric techniques in nonlinear optimal control. Technical report, Department of Mathematics, Rutgers University, 1991.