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Robust Optimal Deployment in Mobile Sensor Networks with Peer-to-peer Communication

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Abstract— **This paper presents a distributed robust deployment algorithm for optimal coverage by a mobile sensor network (MSN). Much past research has focused on versions of the coverage problem that partition the workspace into regions, and then assign exactly one sensor to cover each region. For this case it has been shown that the optimal partition is the Voronoi partition, and that Lloyd's algorithm converges to the optimal solution, with each sensor located at the centroid of its Voronoi region.**

In this paper, we consider the case in which k sensors are assigned to each region in the partition, in order to obtain coverage that is robust to sensor failure. For this case, we prove that the optimal workspace partition is the order$-k$ Voronoi partition, with each sensor assigned to those order$-k$ Voronoi regions for which it is a generator. The collection of these regions for a given sensor defines its effective sensing region (ESR), and we prove that in the optimal configuration each sensor is located at the centroid of its ESR. Finally we introduce a distributed algorithm for our optimal sensor placement problem that requires only simple peer-to-peer (P2P) communications. We show via simulation results that our algorithm converges in finite time, and provides competitive coverage performance in the presence of individual node failures.

I. Introduction

In recent years, mobile sensor networks (MSNs) have found increasing applicability for problems including surveillance, search and rescue mission, natural disaster forecast, animal habitat monitoring, exploration of hazardous environment (see, e.g., the surveys given in [1], [2], [3]). Many of these applications require sensor deployment in hostile environments, which can lead to failure of individual nodes in the network. In this paper, we address the problem of sensor failure by developing methods for robust sensor deployment.

Much research in the MSN literature has been devoted to the problem of controlling sensor movement such that the sensor nodes maintain maximum coverage of their effective sensing regions (ESRs). This coverage control problem is closely related to the locational optimization problem [4], [5], [6], which deals with the optimal placement of resources in a spatial domain. Various distributed motion control algorithms have been presented that drive the nodes to their optimal positions (see, e.g., [7], [8], [9], [10]), however, these algorithms have not typically considered the possibility of sensor failure, and converge to optimal sensor configurations under the assumption that all sensors function properly and accurately implement the control algorithm. With such approaches, failure of a single sensor can lead to network failure, for example, in the problem of target detection.

One of the primary reasons that the failure of a single node can be so significant is that many distributed MSN algorithms simplify communication and computation requirements by partitioning the workspace into regions, and then assigning only one node per region. Thus, failure of a single node results in one region that is not covered by the network.

A number of approaches have been proposed in which multiple sensors can work cooperatively to achieve coverage, including [8], [11]. In [8], a probability distribution encodes the frequency of random events that can occur, and each mobile node is equipped with range limited sensor. Communication cost is considered as a limiting constraint, and a gradient-based algorithm that requires only local information to each sensor is proposed to locally maximize the joint-detection probabilities of the random events. In [11], the joint probability of missed detection is explicitly computed, and used to derive a gradient descent algorithm to minimize the total probability of missed detection, given the prior probability of individual sensor failure.

In this paper, we expand upon the method presented in [11]. As with most previous approaches, we partition the workspace into regions. Unlike those approaches, we consider the case in which k sensors are assigned to cover each region in the partition. Thus, for any region in the partition, if one sensor fails, $k-1$ sensors remain functional. By varying the choice of k, we obtain the classical approach when $k = 1$, and the case of full workspace coverage (the case considered in [11]) when $k = n$, for a network of n sensors. Values of k from 2 to $n-1$ provide successively more accurate approximations to the true joint probability of missed detection, at the expense of increasing computation and communication requirements. For $k > 1$, our method is robust in the sense that even if one sensor fails, there are other sensors which can successfully detect the missed-detected target.

The main results in our paper are as follows. First, we show that the order$-k$ Voronoi partition is the optimal partition of the workspace when each sensor is assigned to those order$-k$ Voronoi regions for which it is a generator (i.e., when the ESR for each sensor is the set of all point in the workspace for which it is one of the k nearest sensors). We then show that in the optimal configuration each sensor is located at the centroid of its ESR. This is a generalization

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of the classical result for the problem where $k = 1$, in which case the Centroidal Voronoi tesselation is known to be optimal, and the optimal configuration is such that each sensor is positioned at the centroid of its Voronoi regions. Finally, we present two distributed algorithms for our optimal sensor placement problem: a first algorithm with proven convergence properties, and a second algorithm that performs more efficiently in practice.

One benefit to our approach is that it allows communication and computation costs to be considered against the relative performance gains as k ranges from 1 to n, and as computation ranges from fully decentralized to fully centralized. For $1 < k < n$, our approach is regarded as *decentralized*, however not *fully decentralized* in that peer-to-peer (P2P) communication is still necessary.

The remainder of the paper is organized as follows. In Section II, we introduce background on generalized Voronoi partitions and a few other notations used in the paper. In Section III, we formulate an appropriate cost function that approximates the probability of missed detection for order$-k$ redundancy in sensing. In Section IV we show that the order$-k$ Voronoi partition is the optimal partition under the cost function derived in Section III. Then, in Section V we introduce a robust deployment algorithm for the case of $k = 2$. Results are shown in Section VI.

II. Background and Notation

We consider an MSN with n agents in a workspace denoted by Q, typically a polytope in $\Re^d$. The configuration of the MSN is defined by the n-tuple $\mathcal{P} = (p_1, \ldots, p_n)$. With a slight abuse of notation, we sometimes let $\mathcal{P}$ denote the set of positions $p_i \in Q$, and use $p \in \mathcal{P}$ to index into the coordinates of $\mathcal{P}$. This will occasionally simplify notation, and context will resolve potential ambiguities. We denote a partition of Q into disjoint regions by $\mathcal{W} = \{W_i\}_{i=1}^m$, where m is the number of elements in the partition. The Voronoi partition of Q w.r.t. $\mathcal{P}$ is defined as follows.

Definition 1 (The ordinary Voronoi partition): Given a workspace $Q \subset \Re^d$ and n distinct points $\mathcal{P} = (p_1, \ldots, p_n)$ with $p_i \in Q$, the Voronoi region associated with $p_i \in \mathcal{P}$ is written as $V_i(\mathcal{P})$ and defined by

$$V_i(\mathcal{P}) = \left\{ q \in Q \mid \|q - p_i\| \leq \min_{p_j \in \mathcal{P} \setminus \{p_i\}} (\|q - p_j\|) \right\}$$

Simply stated, the Voronoi region V_i is that set of points in Q that is nearer to p_i than to any other sensor. We refer to p_i as the *generator* of V_i. The collection of all these regions is called the *Voronoi partition* of Q given configuration $\mathcal{P}$ and is written as $\mathcal{V}(\mathcal{P}) = \{V_1(\mathcal{P}), \ldots, V_n(\mathcal{P})\}$ or simply $\mathcal{V} = \{V_1, \ldots, V_n\}$.

The Voronoi partition plays a key role in defining optimal sensor configurations under certain classes of sensor and motion models (see, e.g., [11], [9]), most commonly, when a target can be detected only by the sensor that is nearest to it, i.e., a target at location $q \in V_i$ can be detected only by the generator for V_i.

In this paper, we consider a more general sensor model that assigns multiple sensors to each region in the partition of Q. In particular, we will consider partitions that assign exactly k sensors to each region in the partition (for a fixed value of k). We denote such a partition by $\mathcal{W}^{(k)}$, and denote specific regions in the partition by $W_i^{(k)}$. We refer to such a partition as an *order$-k$ partition*. If we have n sensors, then there will be m regions in $\mathcal{W}^{(k)}$, where m is the binomial coefficient for n and k. Let $\mathcal{P}^{(k)} = \{P_1^{(k)}, \ldots, P_m^{(k)}\}$ be the collection of all subsets of $\mathcal{P}$ of size k. We associate each $P_i^{(k)}$ to a corresponding $W_i^{(k)}$, and refer to the $p \in P_i^{(k)}$ as the generators for $W_i^{(k)}$. The *order$-k$ Voronoi partition* is a particular case for this kind of partition.

Definition 2 (The order$-k$ Voronoi region [12],[6]): An order$-k$ Voronoi region with generators $P_i^{(k)}$ is given by

$$V(P_i^{(k)}) = \left\{ q \mid \max_{p_j \in P_i^{(k)}} \|q - p_j\| \leq \min_{p_h \in \mathcal{P} \setminus P_i^{(k)}} \|q - p_h\| \right\}$$

That is, $V(P_i^{(k)})$ is the set of those points in Q that are closer to the k sensors in $P_i^{(k)}$ than to any other sensors. The set of all these regions constitute the order$-k$ Voronoi partition of Q given configuration $\mathcal{P}$, which is denoted as $\mathcal{V}^{(k)}(\mathcal{P}) = \{V(P_1^{(k)}), \ldots, V(P_m^{(k)})\}$ or simply $\mathcal{V}^{(k)} = \{V_1^{(k)}, \ldots, V_m^{(k)}\}$. Note that for some $i \in (1, \ldots, m)$, $V_i^{(k)}$ can be an empty region.[1]

Under our more general sensor model, from the point of view of a target at location q, if $q \in W_i^{(k)}$, then only those sensors that are assigned to $W_i^{(k)}$, i.e., those sensors $p \in P_i^{(k)}$, can detect the target. From the point of view of the sensor at p_j, only targets located in regions for which p_j is a generator can be detected. This notion is captured by the concept of *effective sensing region* denoted by R_j for a sensor at location p_j.

Definition 3 (The Effective Sensing Region (ESR) for p_j): Define the index set $I_j = \left\{ i \mid p_j \in P_i^{(k)} \right\}$, i.e., the set I_j denotes the indices of regions in $\mathcal{W}^{(k)}$ for which p_j is a generator (we will use this notation frequently in the sequel). The effective sensing region for a sensor at location p_j is given by

$$R_j = \bigcup_{i \in I_j} W_i^{(k)} \tag{1}$$

According to our generalized sensor model, the sensor located at p_j can detect a target at position q only if $q \in R_j$.

III. Probability of missed-detection of targets as our cost function

In this section, we derive the probability that a set of n sensors in configuration $\mathcal{P}$ will fail to detect a target. Our derivation follows that given in [11]. We assume that the sensors are independent, and that sensor performance is modeled by a function $f(p, q)$ that characterizes the probability that a sensor at location p will detect a target

[1] In [6], there is a mathematical formulation which calculates the number of non-empty order$-k$ Voronoi regions under a few assumptions.

at location $q \in Q$. Let D denote the event that the target is detected, and $\overline{D}$ the complementary event that the target is missed. The joint probability of missed detection (i.e., the probability that no sensor will detect the target) for sensors located at configuration $\mathcal{P}$ is given by

$$P(\overline{D} \mid \mathcal{P}) = \int_Q \prod_{i=1}^{n} P(\overline{D}_i \mid \mathrm{X} = q)\phi(q)\, dq \qquad (2)$$

in which $P(\overline{D}_i \mid \mathrm{X} = q)$ is the conditional probability that an agent positioned at p_i fails to detect the target located at q, and $\phi(q)$ is the prior probability of a target located at q. Note that our assumption of independent sensors allows the joint probability of failure to be factored into a product of marginal probabilities of failure for the individual sensors. We will typically suppress $\mathcal{P}$ and merely write $P(\overline{D})$.

Suppose now that the workspace Q has been partitioned as described in Section II into m regions $\mathcal{W}^{(k)} = \{W_i^{(k)}\}_{i=1}^m$ such that each $W_i^{(k)}$ is associated with the k sensors at positions given by $P_i^{(k)}$. Then the integral in (2) can be expressed as a sum of integrals over the individual regions in the partition

$$P(\overline{D}) = \sum_{j=1}^{m} \int_{W_j^{(k)}} \prod_{i=1}^{n} P(\overline{D}_i \mid \mathrm{X} = q)\phi(q)\, dq \qquad (3)$$

As is typical in the literature (see, e.g., [11], [9]), we assume here that the probability of detecting a target decreases with the distance between the target and the sensor. Further, as described in Section II, we assume that the sensor at location p_i may only detect a target that lies within its effective sensing region R_i. Thus, the probability of missed detection for an individual sensor can be written as

$$P(\overline{D}_i \mid \mathrm{X} = q) = \begin{cases} f(\|q - p_i\|) & \text{if } q \in R_i \\ 1 & \text{if } q \notin R_i \end{cases} \qquad (4)$$

In our simulations, we use $f(\|q - p_i\|) = \eta \|q - p_i\|^2$, with η a positive real constant that is related to the diameter of the workspace Q as $\eta \leq \frac{1}{(\mathrm{diam}(Q))^2}$.

Under these assumptions, the innermost product in (3) need only be evaluated for those points p that are among the generators of $W_i^{(k)}$, i.e., only for $p \in P_i^{(k)}$. Thus, our final expression for the probability of missed detection is written as

$$P(\overline{D}) = \sum_{i=1}^{m} \int_{W_i^{(k)}} \prod_{p \in P_i^{(k)}} f(\|q - p\|)\phi(q)\, dq \qquad (5)$$

Example 1: For the case when $k = 1$ (i.e., the effective sensing region for each sensor is exactly one region in the partiton $\mathcal{W}$), $m = n$ and $P_i^{(1)}$ includes only p_i (i.e., each region in the partition has a unique generator). Thus, (5) reduces to $P(\overline{D}) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|)\phi(q)\, dq$, which takes the same form as the cost function in [9], termed as the "partitioned cost function" in [11]. Borrowing the notation from [11] we may also write $P(\overline{D}) = P_{\mathrm{cvt}}(\overline{D})$.

Example 2: When $k = n$, each sensor is able to detect the target anywhere in Q. In this case, $m = 1$ and $P_i^{(n)} = \{p_1, \dots, p_n\} = \mathcal{P}$ for each i. For this case, (5) reduces to $P(\overline{D}) = \int_Q \prod_{i=1}^{n} f(\|q - p_i\|)\phi(q)\, dq$, which is exactly the total probability of missed-detection as described in [11].

In order to frame the sensor deployment problem as an optimization problem, we use the probability of failure as our cost function, and attempt to minimize this cost with respect to a choice of workspace partition $\mathcal{W}^{(k)}$ and the configuration of the sensors $\mathcal{P}$. To make explicit the dependence on the number of sensors that may detect the target, we denote the cost as

$$\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W}^{(k)}) = P(\overline{D}) \qquad (6)$$

in which the dependence of $P(\overline{D})$ on the choice of partition and sensor configuration is implicit. We may now express our optimal sensor deployment problem as the minimization problem

$$\min_{\mathcal{P}} \min_{\mathcal{W}^{(k)}} \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W}^{(k)}) \qquad (7)$$

In Section IV, we provide a first necessary condition for the solution of the problem (7) for the general case of order$-k$ partitions. Then, in Section V we provide a second necessary condition that holds for the special case of $k = 2$.

IV. First necessary condition: the minimizing partition

In this section, we consider the requirements that must be satisfied by the partition $\mathcal{W}^{(k)}$ for it to be a solution for (7). In particular, Theorem 1 shows that when k sensors are associated to each region in an order$-k$ partition, the optimal partition is the order$-k$ Voronoi partition.

Theorem 1: Given a convex polytope $Q \subset \Re^d$, a map $\phi : Q \to \Re_{\geq 0}$ a density function which satisfies $\int_Q \phi(q)\, dq = 1$, and $\mathcal{P} = (p_1, \dots, p_n)$ set of distinct positions of mobile sensors interior to Q, and let $f : \Re_{\geq 0} \to [0\ 1]$ be a strictly increasing function. Then a necessary condition for $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W})$ to be minimized is $\mathcal{W}^{(k)} = \mathcal{V}^{(k)}$ i.e., the cost is minimized at an order$-k$ Voronoi partition.

Proof: To prove this theorem, we consider a minimal partition $\mathcal{W}^{(k)}$, and show that cost for the order$-k$ Voronoi partition with the same generators is no greater than the minimal cost.

The cost function for order$-k$ Voronoi partition is

$$\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)}) = \sum_{i=1}^{m} \int_{V_i^{(k)}} \prod_{p \in P_i^{(k)}} f(\|q - p\|)\phi(q)\, dq \qquad (8)$$

From the strictly increasing assumption of function f on $\Re_{\geq 0}$, given $q \in Q$, for some $p_i \in Q$, and $p_j \in Q$ with $p_i \neq p_j$, $f(\|q - p_i\|)\phi(q) < f(\|q - p_j\|)\phi(q)$ whenever $\|q - p_i\| < \|q - p_j\|$. Thus, given any $q \in V_i^{(k)}$, from the definition of order$-k$ Voronoi partition for each $j \in (1, \dots, m) \setminus \{i\}$,

$$\prod_{p \in P_i^{(k)}} \|q - p\| \leq \prod_{p \in P_j^{(k)}} \|q - p\|$$

which implies that

$$\prod_{p\in P_i^{(k)}} f(\|q-p\|)\phi(q) \leq \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q) \tag{9}$$

Let us denote the set of all regions in $\mathcal{W}^{(k)}$ whose intersection with $V_i^{(k)}$ is non-empty to be $\overline{\mathcal{W}}$ such that $\overline{\mathcal{W}} \subset \mathcal{W}^{(k)}$, and $V_i^{(k)} \cap W_j^{(k)} \neq \emptyset$ for every $W_j^{(k)} \in \overline{\mathcal{W}}$. Integration of both hand sides of (9) over $V_i^{(k)}$ yields

$$\int_{V_i^{(k)}} \prod_{p\in P_i^{(k)}} f(\|q-p\|)\phi(q)\,dq \leq$$

$$\sum_{\{j|W_j^{(k)}\in\overline{\mathcal{W}}\}} \int_{V_i^{(k)}\cap W_j^{(k)}} \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q)\,dq \tag{10}$$

Note that we rely here only on the fact that f is an increasing function of $\|q-p\|$.

The inequality (10) holds for each $i \in (1,\ldots,m)$ such that adding summation on both hand sides yields.

$$\sum_{i=1}^{m} \int_{V_i^{(k)}} \prod_{p\in P_i^{(k)}} f(\|q-p\|)\phi(q)\,dq \leq$$

$$\sum_{i=1}^{m} \sum_{\{j|W_j^{(k)}\in\overline{\mathcal{W}}\}} \int_{V_i^{(k)}\cap W_j^{(k)}} \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q)\,dq$$

where the right hand side becomes

$$\sum_{j=1}^{m} \int_{W_j^{(k)}} \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q)\,dq = \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)})$$

Thus, we may simply write the above inequality as follows.

$$\mathcal{L}^{(k)}(\mathcal{P},\mathcal{V}^{(k)}) \leq \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)}) \tag{11}$$

Now we show by contradiction that the inequality in (11) is strict if $\mathcal{W}^{(k)} \neq \mathcal{V}^{(k)}$. Suppose that $\mathcal{W}^{(k)} \neq \mathcal{V}^{(k)}$ and that $\mathcal{L}^{(k)}(\mathcal{P},\mathcal{V}^{(k)}) = \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)})$. For each $V_i^{(k)}$, there exists a nontrivial region $W_j^{(k)} \subset \overline{\mathcal{W}}$ such that $W_j^{(k)} \neq V_i^{(k)}$, and $V_i^{(k)} \cap W_j^{(k)} \neq \emptyset$. We denote by S the strict subset of the intersection i.e., $S \subsetneq V_i^{(k)} \cap W_j^{(k)}$. Then we have

$$\int_S \prod_{p\in P_i^{(k)}} f(\|q-p\|)\phi(q)\,dq < \int_S \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q)\,dq$$

The strict inequality follows from the definition of order$-k$ Voronoi region and the fact that the region S does not include the boundary of $V_i^{(k)}$. The inequality in (10), and $S \subsetneq V_i^{(k)}$ implies $\mathcal{L}^{(k)}(\mathcal{P},\mathcal{V}^{(k)}) < \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)})$ which is a contradiction. Hence $\mathcal{L}^{(k)}(\mathcal{P},\mathcal{V}^{(k)}) = \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)})$, only if $\mathcal{W}^{(k)} \neq \mathcal{V}^{(k)}$. ■

V. SECOND NECESSARY CONDITION AND OUR DESCENT ALGORITHM

In this section, we propose a descent algorithm which converges to a set of centroids of ESR, and provide the second necessary condition for the solution of our problem in (7). We consider a special case when sensor performance function is given as $f(\|q-p\|) = \eta\,\|q-p\|^2$. To be used in sequel, we define a new function $\mathcal{H}(p_i,\,R_i)$ by

$$\mathcal{H}(p_i,\,R_i) = \sum_{j\in I_i} \int_{W_j^{(k)}} \prod_{p\in P_j^{(k)}} f(\|q-p\|)\phi(q)\,dq, \tag{12}$$

which is an explicit function of p_i, and R_i (recall that $R_i = \cup_{j\in I_i} W_j^{(k)}$). Roughly speaking $\mathcal{H}(p_i,\,R_i)$ is the distributed version of the cost (5) associated with each node i. The relation between two functions is stated in the following proposition.

Proposition 1: Given any $k \in \{1,\ldots,n\}$, the following relation holds.

$$\sum_{i=1}^{n} \mathcal{H}(p_i,\,R_i) = k \times \mathcal{L}^{(k)}(\mathcal{P},\mathcal{W}^{(k)}) \tag{13}$$

Proof: The actual proof will be omitted due to limitation of space. The relation can be easily verified using definition of effective sensing region, and order$-k$ partition. ■

Proposition 2: The function $\mathcal{H}(p_i,R_i)$ is convex on $\mathrm{conv}(R_i)$.

Proof: The proof follows from the fact that the non-negative sum of convex functions on convex domain is again convex [13]. ■

Lemma 1: Let the centroid of the effective sensing region R_i be defined by

$$C_{R_i} = \frac{\int_{R_i} q\phi_i(q)\,dq}{\int_{R_i} \phi_i(q)\,dq} \tag{14}$$

where the density function $\phi_i : Q \to [0\ 1]$ is given as $\phi_i(q) = \eta^{k-1}\left(\prod_{p\in P_j^{(k)}\setminus\{p_i\}} \|q-p\|^2\right)\phi(q)$ with $j \in I_i$, and $P_j^{(k)}$ depends on the location of q. Then C_{R_i} is the global minimizer of $\mathcal{H}(p_i,R_i)$ with respect to p_i on convex domain $\mathrm{conv}(R_i)$ viz.,

$$C_{R_i} = \arg\min_{p_i\in\mathrm{conv}(R_i)} \mathcal{H}(p_i,\,R_i) \tag{15}$$

Proof: First, we may assume that position of other agents in the index set I_i besides agent i is fixed i.e., $p_j = p_j^{\text{fixed}}$ for each $j \in I_i \setminus \{i\}$. Let the mass function be $M_{R_i} = \int_{R_i} \phi_i(q)\,dq$, then according to the *parallel axis theorem* $J_{R_i,p_i} = J_{R_i,C_{R_i}} + M_{R_i}\,\|p_i - C_{R_i}\|^2$ where $J_{A,p}$ is defined as second moment of inertia over region A *w.r.t.* point p. Thus, our cost function is exactly the 2nd moment of inertia with respect to p_i. Hence, $\mathcal{H}(p_i,R_i) = J_{R_i,p_i}$, and the gradient vanishes at the local minimizer $p_i^* = C_{R_i}$. According to Proposition 2, the function $\mathcal{H}(p_i,R_i)$ is convex over the convex domain $\mathrm{conv}(R_i)$ Hence, the local minimizer p_i^* is the global minimizer. ■

Remark 1: Our proof contains ideas from [9], and [4]. Especially, in [9], given a function $\int_{V_i} \|q-p_i\|^2\,\phi(q)\,dq$, they also used parallel axis theorem to show that the centroid of Voronoi region C_{V_i} is the minimizer the function. Our approach is different in that $\mathcal{H}(p_i, R_i)$ depends not only on p_i but also on positions of other nodes as well. We define a new density function which is a function of position of neighbors and show that the centroid of ESR with such density is the minimizer of the cost $\mathcal{H}(p_i, R_i)$.

As an immediate consequence of Theorem 1 and Proposition 1, we have the following corollary.

Corollary 1: Let the collection of ordered set of n effective sensing region be S, and assume that the following relation holds.

$$(R_1^*, \dots, R_n^*) = \arg \min_{(R_1, \dots, R_n) \subset S} \sum_{i=1}^{m} \mathcal{H}(p_i, R_i) \tag{16}$$

Then necessarily, $(R_1^*, \dots, R_n^*)$ is the set of effective sensing regions under the order$-k$ Voronoi tessellations (i.e., for each R_i^*, $W_i^{(k)} = V_i^{(k)}$).

A. A distributed optimal robust deployment algorithm

Our algorithm for optimal sensor placement is an iterative algorithm in which the motion command for the agents is defined using a mapping $T : Q^n \to Q^n$ in which $T = (T_1, \dots, T_n)$, and each component map is defined by

$$T_i(x_i) = C_{R_i^*}, \text{ for all } i \in \{1, \dots, n\}. \tag{17}$$

We now show that our cost function is a descent function $w.r.t.$ the algorithm T. Instead of seeking a optimal solution, we find a solution which is included in a solution set. Our solution set is the set of all equilibrium points $\Omega = \{\mathcal{P} \mid \mathcal{P} = T(\mathcal{P})\}$.

Lemma 2: Let Q be a convex subset of $\Re^n$ and $\mathcal{P}^0 \in Q^n \subseteq \Re^n$. Then it holds for a sequence $(\mathcal{P}^t)_{t \in \mathbb{Z}_{\geq 0}}$ recursively defined by $\mathcal{P}^{t+1} = T(\mathcal{P}^t)$, converge to one of the equilibrium points in Ω i.e., all possible set of *centroids of effective sensing regions under the order$-k$ Voronoi tessellations* (CEVT).

Proof: We prove that $T : Q^n \to Q^n$ is a decent algorithm with respect to our cost by showing that inequality $\mathcal{L}^{(k)}(T(\mathcal{P}), \mathcal{V}^{(k)}(T(\mathcal{P}))) \leq \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)}(\mathcal{P}))$ holds by two steps. First, recall from Lemma 1 that the following relation holds. $\mathcal{H}(C_{R_i^*}, R_i^*(\mathcal{P})) \leq \mathcal{H}(p_i, R_i^*(\mathcal{P}))$. Then $\sum_{i=1}^n \mathcal{H}(C_{R_i^*}, R_i^*(\mathcal{P})) \leq \sum_{i=1}^n \mathcal{H}(p_i, R_i^*(\mathcal{P}))$, and if we denote by $\mathcal{P}'$ the set of CEVT i.e., $\mathcal{P}' = (C_{R_1^*}, \dots, C_{R_n^*})$, then the previous inequality implies that $\mathcal{L}^{(k)}(\mathcal{P}', \mathcal{V}^{(k)}(\mathcal{P})) \leq \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)}(\mathcal{P}))$, where equality holds when $\mathcal{P} = \mathcal{P}'$. Next, as the second step of the proof, we show that $\sum_{i=1}^n \mathcal{H}(C_{R_i^*}, R_i^*(\mathcal{P}')) \leq \sum_{i=1}^n \mathcal{H}(C_{R_i^*}, R_i^*(\mathcal{P}))$ equivalently $\mathcal{L}^{(k)}(\mathcal{P}', \mathcal{V}^{(k)}(\mathcal{P}')) \leq \mathcal{L}^{(k)}(\mathcal{P}', \mathcal{V}^{(k)}(\mathcal{P}))$. The inequality holds by Theorem 1 (by letting $\mathcal{V}^{(k)}(\mathcal{P}) := \mathcal{W}^{(k)}(\mathcal{P}')$). Thus, we conclude that $\mathcal{L}^{(k)}(\mathcal{P}', \mathcal{V}^{(k)}(\mathcal{P}')) \leq \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)}(\mathcal{P}))$ such that $\mathcal{L}^{(k)}(T(\mathcal{P}), \mathcal{V}^{(k)}(T(\mathcal{P}))) \leq \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)}(\mathcal{P}))$ where the equality holds if $\mathcal{P} = \mathcal{P}'$. This shows that the mapping T is a descent algorithm, and the solution is exactly when each generator is positioned at centroid of respective effective sensing region. Along with the fact that $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W}^{(k)})$ is continuous on Q^n, it can be shown from Theorem 3 [13] that our algorithm converges to one of the set of CEVT in Ω. ■

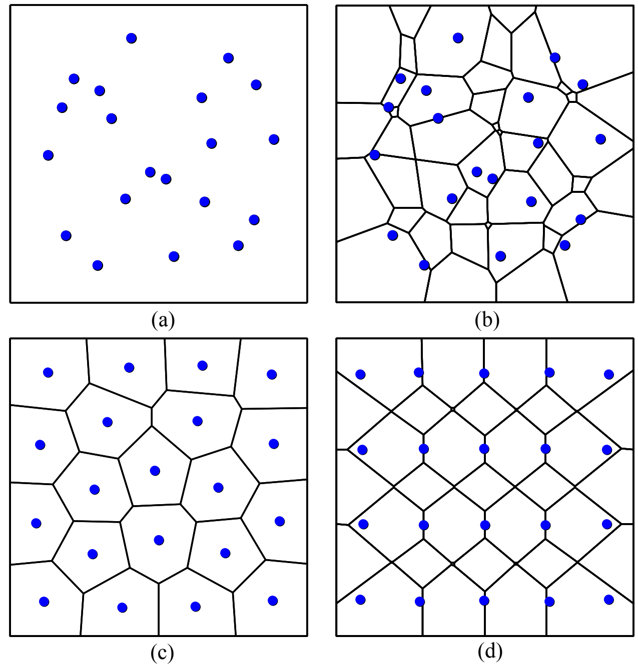


Fig. 1. (a) Initial configuration, (b) Initial configuration showing the order-2 Voronoi partition, (c) Lloyd's algorithm (50 steps), (d) Our algorithm (150 steps).

The following theorem provides the second necessary condition for set of minimizers.

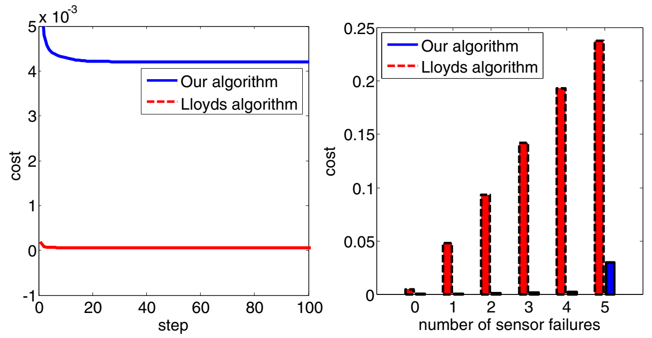
Theorem 2: If a configuration $\mathcal{P}^*$ minimize $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W}^{(k)})$ over Q^n then $\mathcal{P}^* \subset \Omega$, and at the minumum $\mathcal{W}^{(k)} = \mathcal{V}^{(k)}$.

Proof: First using the necessary condition from Theorem 1, we may replace general partition $\mathcal{W}^{(k)}$ with $\mathcal{V}^{(k)}$ in $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{W}^{(k)})$. Now the argument can be simplified as *if a configuration $\mathcal{P}^*$ minimizes $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)})$ over Q^n then* $\mathcal{P}^* \in \Omega$. Recall that $\sum_i^n \mathcal{H}(p_i, R_i^*) = k \times \mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)})$. The relation implies that the left hand side is minimized if and only if the right hand side is minimized. Hence, it is sufficient to consider a configuration which minimizes the left hand side. Looking at the left hand side, it can be easily verify that the summation term $\sum_i^n \mathcal{H}(p_i, R_i)$ is minimized only if $\mathcal{H}(p_i, R_i)$ is minimized for all $i \in \{1, \dots, n\}$. Recall from Lemma 1 that $p_i = C_{R_i}$ for each i is a global minimizer for each $\mathcal{H}(p_i, R_i)$. Hence, $\mathcal{L}^{(k)}(\mathcal{P}, \mathcal{V}^{(k)})$ is minimized only if $\mathcal{P}^* \subset \Omega$ which completes the proof. ■

Remark 2: The necessary condition CEVT cannot provide sufficiency for solution of our problem (7) owe to the fact the CEVT is not necessarily unique such that other solution can have lower cost value.

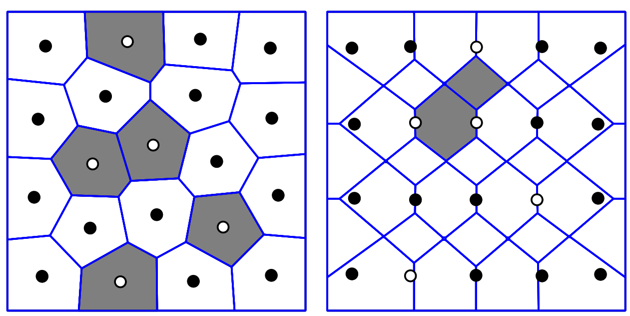
VI. Simulation results

In the numerical simulation, we consider 20 nodes initially deployed in a planer square workspace $[0\ 1]^2$ with in which targets are uniformly distributed. 20 nodes either use Lloyd's algorithm or our proposed algorithm (especially when $k = 2$) as their motion strategy. Fig. 1(a) shows the initial configuration, and Fig. 1(b) shows the same con-



(a) (b)

Fig. 2. Cost comparison between between our algorithm and Lloyd's algorithm (a) cost vs steps. (b) cost vs number of sensor failures.



(a) (b)

Fig. 3. Sensor failure under equilibrium configurations with (a) Lloyd's algorithm, (b) our algorithm (unfilled circles: failed nodes, grey region: coverage holes).

figuration with the order−2 Voronoi partition (solid lines). Fig. 1(c)−(d) shows equilibrium configuration with two algorithms: Lloyd's method and our algorithm respectively. It is shown in Fig. 2(a) that the cost value from our algorithm is much lower than that from Lloyd's method. This is owe to the fact that, in our algorithm, multiple nodes (i.e., in our case $k = 2$) enhances the overall target detection probability. Next, we compare two algorithms in the presence of node failures i.e., when a few nodes lose their sensing capability. In Fig. 2(b), costs are compared between Lloyd's algorithm and our algorithm by varying the number of sensor failures. As you can see from the figure, with Lloyd's algorithm cost increases proportional to number of sensor failures, while with our algorithm, the cost only increases when two adjacent nodes fail at the same time (this is shown in Fig. 3(b)). Fig. 3(a)−(b) shows coverage holes[2] (dark–grey regions in Fig. 3) when 5 sensor fail under equilibrium configuration with two algorithms.

VII. Conclusion

We presented a novel approach for robust sensor coverage under individual sensor failures, which can be implemented in distributed MSNs capable of P2P communication. Our paper leaves a number of future extensions. First, we plan to consider more realistic sensor model in which each node is limited by its (maximum) sensing range, and where detection performance is radially un–uniform (e.g., anisotropic sensors). Second, it will be useful to consider uni–cycle or car–like vehicle dynamics, obstacle avoidance, and non–convex workspace for real world implementation of our algorithm. Finally, we plan to consider active node failures in which some nodes not only fail to detect targets but also disrupt detection performance of their local neighbors as well.

[2]Regions that are not being detected by any nodes.

Appendix I
Convergence Theorem

Theorem 3: [13] (Convergence theorem) Let a set-valued map $A : V \to V$ define an algorithm such that given a point $z^1 \in V$ generates the sequence $\{z^k\}_1^\infty$. Also let a solution set $\Omega \subset V$ be given. Suppose that *1) all points* z^k *are in a compact set* $X \subset V$, *2) There is a continuous function* $Z : V \to R^1$ *such that*

$$\begin{cases} \text{if } z \notin \Omega \text{ then for any } y \in A(z),\ Z(y) < Z(z) \\ \text{if } z \in \Omega \text{ then for any } y \in A(z),\ Z(y) \le Z(z) \end{cases}$$

or the algorithm terminates, 3) the map A *is closed at* z *if* z *is not a solution*. Then the algorithm stops at a solution, or limit of any convergence subsequence is a solution.

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