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Closed form expressions for the sensitivity of kinematic dexterity measures to posture changing and geometric variations

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***Abstract*— This paper addresses the sensitivity of dexterity measures w.r.t. the posture of a manipulator, with given design parameters, as well as w.r.t. the manipulator's geometric parameters. These are required for placing a manipulator so to maximize dexterity and for the optimal layout of the link geometry, respectively. Explicit expressions are derived for first and second partial derivatives of dexterity measures w.r.t. to joint angles and w.r.t. geometric link parameters. The latter is obtained using a virtual joint method extending the product of exponentials formula for the forward kinematics. The approach applies to serial and parallel manipulators.**

***Index Terms*— Dexterity, manipulability, forward kinematics, Lie groups, virtual joint method, path planning, optimization**

I. INTRODUCTION

Dexterity measures[1] are an established tool to assess the kinematic and static performance of robotic manipulators [2], [3], [11], [13], [14], [23], [24], although it is still a topic of current research to define proper measures for full- and lower-mobility manipulators resolving the well-known scaling problem of Euclidean motion metric and to extend them to cable platforms [9]. Beside a high dexterity measure it is desirable that the variation of this measure is lower bounded. Hence a homogenous distribution within the workspace is a sensible criterion for the manipulator design but also for the motion planning. Assessing this calls for a sensitivity analysis of these measures.

There are essentially three goals for the application of dexterity measures: 1. *analysis* of a given manipulator in prescribed configurations (dexterity assessment), 2. the *placement* of a given manipulator so to maximize manipulability (motion planning), and 3. the optimal *design* of the manipulator kinematics so that it exhibits in average a good dexterity within its workspace (manipulator design). The first application is merely a passive analysis task, whereas 2. and 3. aim at actively changing the manipulator motion or design. These tasks require information about how the dexterity changes when either the manipulator changes its posture or when its geometric parameters change. That is, either the gradient of the dexterity measure w.r.t. to the joint angles or w.r.t. link geometry are required. Most approaches to this optimization problem have used numerical approximations. The posture sensitivity allows for incorporating the fluctuation of dexterity in the path planning so to maximize dexterity. To this end straightforward optimization methods were proposed in [1], [6], [21]. A dexterity maximizing inverse kinematics algorithm for kinematically redundant manipulators was reported in [15]. The optimization of the manipulator geometry w.r.t. dexterity was first reported in [7],[8]. Currently the maximization of dextrous workspace gained relevance in context of surgical applications [12]. It must be emphasized that the established dexterity measures suffer from the need to scale rotations and translations , and that they cannot capture the positioning accuracy [14].

[1]Synonymously the term 'manipulability' is used [17]. However, this term suggests that the measure reveals how the end-effector or handled object can be manipulated rather than the ability of the manipulator to handle it.

The contribution of this paper are explicit closed form expressions for the first and second partial derivatives of the two established dexterity measures –the determinant and inverse condition number– w.r.t. to joint variables (e.g. joint angles) and w.r.t. geometric link parameters. This includes serial and parallel manipulators. These expressions can be directly implemented and allow for a systematic optimization scheme for general manipulators. The paper is organized as follows. Section II and III recall respectively the forward kinematics formulation of serial and parallel manipulators using the product of exponential formula. In section IV the determinant and inverse condition number are recalled as local dexterity measures. Since the body-fixed representation of end-effector (EE) twists is used both measures are invariant w.r.t. change of world-fixed frame. Additionally the global conditioning index is recalled. The first and second partial derivatives of the local and global dexterity measures are derived in section V. A virtual joint method is introduced in section VI that allows elegantly to parameterize the manipulator link geometry, and used to derive explicit forms for the derivatives w.r.t. the geometric parameters. Numerical results are shown for a RRR serial manipulator as simple example in section VII. The paper concludes with a summary in section VIII.

II. KINEMATICS OF SERIAL MANIPULATORS

A. Forward kinematics

Throughout the paper the Lie group concept of rigid body motion and the $SE(3)$ matrix notation [17],[18] is used.

The configuration of the EE of a robotic manipulator w.r.t. a world-fixed reference frame is represented by a frame transformation matrix $\mathbf{C} \in SE(3)$. The EE configuration corresponding to a posture of the manipulator comprising n (w.l.o.g.) 1 DOF joints is determined by the forward kinematic mapping $f: \mathbb{V}^n \rightarrow SE(3)$ that is given in terms of the product of exponentials

$$\begin{aligned} f(\mathbf{q}) &= \exp\left(\mathbf{Y}_1 q^1\right) \cdot \ldots \cdot \exp\left(\mathbf{Y}_n q^n\right) \mathbf{C}_0 && (1) \\ &= \mathbf{M}_1 \exp\left(\mathbf{X}_1 q^1\right) \cdot \ldots \cdot \mathbf{M}_n \exp\left(\mathbf{X}_n q^n\right). && (2) \end{aligned}$$

$\mathbb{V}^n$ denotes the joint space manifold. In the first formulation $\mathbf{C}_0 = \mathbf{M}_1 \cdot \ldots \mathbf{M}_n = f(0)$ is the reference EE configuration for $\mathbf{q} = \mathbf{0}$, and $\mathbf{Y}_i$ are the screw coordinates of joint i relative to the world-fixed reference frame. Hence the formulation (1) does not require the introduction of body-fixed reference frames. In the second form (2) the $\mathbf{M}_i$ is the transformation from link frame i to $i-1$ and $\mathbf{X}_i$ are the screw coordinates of joint i expressed in the body-fixed reference frame. Consequently, this formulation allows for parameterization of the link geometry, i.e. of $\mathbf{M}_i$, for instance in terms of Denavit-Hartenberg parameters. The POE formulation has been proposed in the context of Lie group methods [5],[18]. It is also called the zero reference position description [10].

B. Velocity forward kinematics

Dexterity measures describe the motion capabilities of the EE while the manipulator is moving. Therefore a body-fixed reference frame located at the EE is used as reference to express the body-fixed EE twist, respectively its instantaneous motion. The body-fixed twist is defined as $\widehat{\mathbf{V}} = \mathbf{C}^{-1}\dot{\mathbf{C}} \in se(3)$, and given in terms of the forward kinematics mapping by

$$\begin{aligned} \mathbf{V}(\mathbf{q}) &= \mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}} \\ &= \sum_{i\leq n} \mathbf{J}_i(\mathbf{q})\,\dot{q}^i \end{aligned} \tag{3}$$

where $\mathbf{J}$ is the body Jacobian whose columns are $\widehat{\mathbf{J}}_i = f^{-1}\frac{\partial}{\partial q^i} f$. Here $\widehat{\mathbf{X}} \in se(3)$ is the matrix associated with the screw coordinates $\mathbf{X}$. With (1) and (2) they are explicitly given as

$$\begin{aligned} \mathbf{J}_i &= \mathrm{Ad}_{\mathbf{A}_i}^{-1}\mathbf{Y}_i & (4)\\ &= \mathrm{Ad}_{\mathbf{B}_i}^{-1}\mathbf{X}_i & (5) \end{aligned}$$

with $\mathbf{A}_i(\mathbf{q}) := \exp\left(\mathbf{Y}_i q^i\right) \cdot \ldots \cdot \exp\left(\mathbf{Y}_n q^n\right)\mathbf{C}_0$ and $\mathbf{B}_i(\mathbf{q}) = \mathbf{M}_{i+1}\exp\left(\mathbf{X}_{i+1}q^{i+1}\right)\cdot \ldots \cdot \mathbf{M}_n \exp\left(\mathbf{X}_n q^n\right), i < n$ and $\mathbf{J}_n = \mathbf{X}_n$. In vector representation $\mathbf{J}_i = (\boldsymbol{\omega}_i, \mathbf{v}_i)^T$ are the instantaneous screw coordinates of joint i expressed in the EE frame.

Notice that frequently, in particular in mechanism analysis the spatial twists are used, which is defined as $\widehat{\mathbf{V}}^s = \dot{\mathbf{C}}\mathbf{C}^{-1}$. It is important to notice their different interpretations. $\mathbf{V} = (\boldsymbol{\omega}, \mathbf{v})$ is the angular and translational velocity of the EE relative to the world-fixed frame expressed in the body-fixed EE frame. $\mathbf{V}^s$ on the other hand is the velocity of an imaginary reference frame attached to the EE that is momentarily coinciding with the world-fixed frame. The latter is apparently not a suitable choice for evaluating dexterity measures unless the frame is replaced to the EE at any instance of time.

C. Derivatives of the forward kinematics

The important aspect of the Lie group, respectively screw, formulation of the kinematics is that it allows for closed form expressions of derivative of any order [16], and thus for expressing the acceleration, jerk, jounce, etc. in a very compact form.

Consider the partial derivative of the Jacobian with respect to the joint angle q^j. Starting from the expression (4) and using $\mathrm{Ad}_{\mathbf{A}_i}^{-1}\widehat{\mathbf{Y}}_i = \mathbf{A}_i^{-1}\widehat{\mathbf{Y}}_i\mathbf{A}_i$ leads to

$$\frac{\partial}{\partial q^j}\widehat{\mathbf{J}}_i = \frac{\partial}{\partial q^j}\mathbf{A}_i^{-1}\widehat{\mathbf{Y}}_i\mathbf{A}_i + \mathbf{A}_i^{-1}\widehat{\mathbf{Y}}_i\frac{\partial}{\partial q^j}\mathbf{A}_i.$$

To evaluate this notice that

$$\begin{aligned} \frac{\partial}{\partial q^j}\mathbf{A}_i^{-1} &= \frac{\partial}{\partial q^j}(\mathbf{C}_0^{-1}\exp(-\mathbf{Y}_n q^n)\cdot\ldots\cdot\exp(-\mathbf{Y}_{i+1}q^{i+1})) \\ &= -\mathbf{C}_0^{-1}\exp(-\mathbf{Y}_n q^n)\cdot\ldots\cdot\exp(-\mathbf{Y}_{j+1}q^{j+1})\widehat{\mathbf{Y}}_j \\ &\quad \cdot\exp(-\mathbf{Y}_j q^j)\cdot\ldots\cdot\exp(-\mathbf{Y}_{i+1}q^{i+1}) \\ &= -\mathbf{A}_j^{-1}\widehat{\mathbf{Y}}_j\mathbf{A}_j\mathbf{A}_i^{-1} = -\widehat{\mathbf{J}}_j\mathbf{A}_i^{-1},\ i\leq j\leq n \end{aligned}$$

and in the same way $\frac{\partial}{\partial q^j}\mathbf{A}_i^{-1} = \mathbf{A}_i\widehat{\mathbf{J}}_j,\ i\leq j\leq n$, so that

$$\begin{aligned} \frac{\partial}{\partial q^j}\widehat{\mathbf{J}}_i &= \mathbf{A}_i^{-1}\widehat{\mathbf{Y}}_i\mathbf{A}_i\widehat{\mathbf{J}}_j - \widehat{\mathbf{J}}_j\mathbf{A}_i^{-1}\widehat{\mathbf{Y}}_i\mathbf{A}_i \\ &= \widehat{\mathbf{J}}_i\widehat{\mathbf{J}}_j - \widehat{\mathbf{J}}_j\widehat{\mathbf{J}}_i \\ &= [\widehat{\mathbf{J}}_i,\widehat{\mathbf{J}}_j],\ i\leq j\leq n. \end{aligned}$$

The last term is the Lie bracket of the screw vectors of joint i and j. When expressed in screw coordinates this becomes the screw product [16], [18] so that the final expression in terms of screw coordinates is

$$\frac{\partial}{\partial q^j}\mathbf{J}_i = [\mathbf{J}_i,\mathbf{J}_j],\ i<j\leq n \tag{6}$$

where $[\mathbf{J}_i,\mathbf{J}_j] = (\boldsymbol{\omega}_i\times\boldsymbol{\omega}_j, \boldsymbol{\omega}_i\times\mathbf{v}_j - \mathbf{v}_i\times\boldsymbol{\omega}_j)^T$.

The second partial derivative follows with the bilinearity of the Lie bracket as

$$\begin{aligned} \frac{\partial^2}{\partial q^j\partial q^k}\mathbf{J}_i &= [\frac{\partial}{\partial q^k}\mathbf{J}_i,\mathbf{J}_j] + [\mathbf{J}_i,\frac{\partial}{\partial q^k}\mathbf{J}_j] \\ &= \left\{\begin{array}{ll} [[\mathbf{J}_i,\mathbf{J}_k],\mathbf{J}_j], & i<j,k\leq n \\ [\mathbf{J}_i,[\mathbf{J}_j,\mathbf{J}_k]], & i<j\leq k\leq n\ . \end{array}\right. \end{aligned}$$

The overlapping index range of these two terms can be removed invoking the Jacobi identity $[[\mathbf{J}_i,\mathbf{J}_k],\mathbf{J}_j] + [[\mathbf{J}_k,\mathbf{J}_j],\mathbf{J}_i] + [[\mathbf{J}_j,\mathbf{J}_i],\mathbf{J}_k] = \mathbf{0}$ for $i<j\leq k$ that yields the final expression

$$\frac{\partial^2}{\partial q^j\partial q^k}\mathbf{J}_i = \left\{\begin{array}{ll} [[\mathbf{J}_i,\mathbf{J}_j],\mathbf{J}_k], & i<j\leq k\leq n \\ [[\mathbf{J}_i,\mathbf{J}_k],\mathbf{J}_j], & i<k<j\leq n\ . \end{array}\right. \tag{7}$$

This relation in terms of body-fixed twists seems not to be well-know in robotics context. It has appeared within dynamics formulation of multibody systems [18]. A similar expression was reported in [20], however, in a more convoluted form. The acceleration is therewith

$$\dot{\mathbf{V}} = \sum_{i\leq n}\mathbf{J}_i\ddot{q}^i + \sum_{i<j\leq n}[\mathbf{J}_i,\mathbf{J}_j]\dot{q}^i\dot{q}^j \tag{8}$$

In mechanism analysis the spatial twist is commonly used. Since both are related by an Ad transformation (frame transformation of screws) μ_3 is the same for both descriptions. This is not true for the condition number.

III. Kinematics of parallel manipulators

A. Forward kinematics

Parallel manipulators (PKM) are characterized by kinematic loops. In the commonly used relative coordinate approach one joint in the loop is removed (the cut joint) and the corresponding loop constraints are imposed to the open loop system. These constraints are naturally expressed in a body-fixed frame so that the position, velocity, and acceleration constraints resemble (1),(2),(3), and (8).

The geometric constraints must be solved in any configuration of the manipulator, either numerically or symbolically in closed form (which is not possible for most PKM).

The velocity constraints can be solved (at least locally). The system of $r < n$ independent velocity constraints has the form $\mathbf{0} = \mathbf{H}\dot{\mathbf{q}}$, with $r \times n$ matrix $\mathbf{H}(\mathbf{q})$. A full-rank $r \times r$ submatrix $\mathbf{H}_1$ can be identified so that

$$\mathbf{0} = \mathbf{H}_1\dot{\mathbf{q}}_1 + \mathbf{H}_2\dot{\mathbf{q}}_2. \tag{9}$$

This system can be solved as $\dot{\mathbf{q}} = \mathbf{F}\dot{\mathbf{q}}_2$ where

$$\mathbf{F} := \begin{pmatrix} -\mathbf{H}_1^{-1}\mathbf{H}_2 \\ \mathbf{I} \end{pmatrix} \tag{10}$$

is an orthogonal complement to $\mathbf{H}$, thus $\dot{\mathbf{q}}_2$ represent $n-r$ independent velocities, and $\mathbf{q}_2$ are local generalized coordinates for the PKM. If $\mathbf{J}_{EE}$ is the Jacobian of the EE in the kinematic tree system (after removing the cut-joint), then $\mathbf{J} = \mathbf{J}_{EE}\mathbf{F}$ is the forward kinematics Jacobian of the PKM.

B. Derivatives of the forward kinematics

The derivatives of the forward kinematics mapping of a PKM follow immediately from those of a serial manipulator as $\frac{\partial}{\partial q^j}\mathbf{J} = \frac{\partial}{\partial q^j}\mathbf{J}_{EE}\mathbf{F} + \mathbf{J}_{EE}\frac{\partial}{\partial q^j}\mathbf{F}$ making use of

$$\frac{\partial}{\partial q^j}\mathbf{F} = \begin{pmatrix} \mathbf{H}_1^{-1}\left(\frac{\partial}{\partial q^j}\mathbf{H}_1\mathbf{H}_1^{-1}\mathbf{H}_2 - \frac{\partial}{\partial q^j}\mathbf{H}_2\right) \\ \mathbf{0} \end{pmatrix}$$

and the fact that the constraint Jacobian $\mathbf{H}$ has the same expression as $\mathbf{J}$ in (4) or (5). The second partial derivative follows analogously.

IV. Dexterity measures

A. Local measures

The two established local dexterity measures for (possibly kinematically redundant) manipulators are [14],[23],[24]

$$\mu_3 = \sqrt{\det(\mathbf{J}\mathbf{J}^T)},\ \mu_2 = 1/\kappa(\mathbf{J}\mathbf{J}^T)$$

where $\kappa(\mathbf{A}) = \|\mathbf{A}\|\left\|\mathbf{A}^{-1}\right\|$ is the condition number of $\mathbf{A}$. They can also be expressed in terms of the singular values $\sigma_1 \leq \ldots \sigma_n$ of $\mathbf{J}\mathbf{J}^T$ as $\mu_3 = \sqrt{\sigma_1 \cdots \sigma_n}$ and $\mu_2 = \sigma_1/\sigma_n$, respectively, which is not used in this paper.

Both measures are based on the left invariant metric which depends on the scaling of rotations and translations. Strictly speaking this makes it impossible to define, and thus design, isotropic manipulators performing spatial EE motions.

Other measures have been proposed to tackle this problem. A physically sensible left-invariant dexterity measure is obtained by incorporating the inertia tensor $\mathbf{\Theta}$ of a manipulated object. This gives rise to the dexterity measure $\mu_3^{\mathbf{\Theta}} := \sqrt{\det\left(\mathbf{J}\mathbf{\Theta}^{-1}\mathbf{J}^T\right)}$ [19]. For non-redundant manipulators $\mathbf{J}\mathbf{\Theta}^{-1}\mathbf{J}^T$ can be considered from a differential-geometric point of view as the inverse of the pull-back matrix on $\mathbb{V}^n$ induced by the metric on $SE(3)$ defined by $\mathbf{\Theta}$. These local measures have also been extended to non-holonomic manipulators [4]. Taking into account the scaling problem is of major importance, although still frequently ignored.

B. Global measures

Local measures reveal the dexterity in a particular posture of the manipulator, but do not allow assessing the overall manipulator dexterity. Therefore the global conditioning index was introduced in [8],[7] as integral over the workspace $W \subset SE(3)$ as

$$\eta_2 = \frac{1}{V_W}\int_W \mu_2 \tag{11}$$

where $V_W = \int_W dW$ is the task space volume. This average value of inverse condition number μ_2 is a global measure for a specific work space W. It is to be noticed, however, that its actual computation is difficult. Also a global measure can be defined using μ_3 but was not yet reported in the literature.

V. Posture sensitivity of dexterity measures

A. Gradients of dexterity measures

Within the path planning in general, and for the inverse kinematics resolution of kinematically redundant manipulators in particular, configurations are desirable where the manipulator exhibits best dexterity, and thus also avoids singularities. Gradient-based numerical methods require the gradient and possibly the Hessian of the dexterity measure. The above expressions for the derivatives give rise to explicit forms of gradients.

1) μ_2 - Inverse condition number: The condition number of an arbitrary $n \times n$ matrix $\mathbf{A} = \left(\mathbf{A}_1\middle|\cdots\middle|\mathbf{A}_n\right)$ with columns $\mathbf{A}_i$ can be expressed using the spectral norm defined as $\|\mathbf{A}\|_2 = \sqrt{\sum_{i,j=1}^n A_{ij}^2} = \sqrt{\sum_{i=1}^n \mathbf{A}_i^T\mathbf{A}_i}$. The derivative of this norm is $\frac{\partial}{\partial q^i}\|\mathbf{A}\|_2 = \frac{1}{\|\mathbf{A}\|}\sum_{j=1}^n \mathbf{A}_j^T\mathbf{A}_j$, and thus

$$\begin{aligned} \frac{\partial}{\partial q^i}\kappa(\mathbf{A}) \quad &= \quad \frac{\|\mathbf{A}\|_2}{\|\mathbf{A}^{-1}\|_2}\sum_{i=1}^n \frac{\partial}{\partial q^j}\mathbf{A}_i^{-T}\mathbf{A}_i^{-1} \\ &\quad + \frac{\|\mathbf{A}^{-1}\|_2}{\|\mathbf{A}\|_2}\sum_{i=1}^n \frac{\partial}{\partial q^j}\mathbf{A}_i^T\mathbf{A}_i. \end{aligned}$$

Therewith the derivative of the inverse condition number is

$$\begin{aligned} \frac{\partial}{\partial q^i}\frac{1}{\kappa(\mathbf{A})} &= -\frac{\frac{\partial}{\partial q^i}\kappa(\mathbf{A})}{\kappa^2(\mathbf{A})} \\ &= -\frac{1}{\kappa^2(\mathbf{A})}\Big(\left\|\mathbf{A}^{-1}\right\|_2\sum_{j=1}^n \frac{\partial}{\partial q^i}\mathbf{A}_j^T\mathbf{A}_j \\ &\qquad + \|\mathbf{A}\|_2\sum_{j=1}^n \frac{\partial}{\partial q^i}\mathbf{A}_j^{-T}\mathbf{A}_j^{-1}\Big). \end{aligned} \tag{12}$$

The last term can be evaluated using the identity $\partial_{q^j}\mathbf{A}^{-1} = -\mathbf{A}^{-1}\partial_{q^j}\mathbf{A}\mathbf{A}^{-1}$. For a serial manipulator the final result follows by replacing $\partial_{q^j}\mathbf{A} = \partial_{q^j}\mathbf{J}\mathbf{J}^T + \mathbf{J}\partial_{q^j}\mathbf{J}^T$. The derivative $\frac{\partial}{\partial q^j}\mu_2$ is thus expressed *explicitly and algebraically*. For a PKM the derivative is obtained noting that

$$\frac{\partial}{\partial q^i}\mathbf{J} = \frac{\partial}{\partial q^i}\mathbf{J}_E\mathbf{F} + \mathbf{J}_E\frac{\partial}{\partial q^i}\mathbf{F}. \tag{13}$$

2) μ_3 - Square root of determinant: The partial derivative of the dexterity measure μ_3 is

$$\frac{\partial}{\partial q^i}\mu_3 = \frac{1}{2}\frac{1}{\mu_3}\frac{\partial}{\partial q^i}\det(\mathbf{J}\mathbf{J}^T)$$

For an $n \times n$ matrix $\mathbf{A} = \left(\mathbf{A}_1 \middle| \cdots \middle| \mathbf{A}_n\right)$ it holds that

$$\begin{aligned}\frac{\partial}{\partial q^k}\det(\mathbf{A}) &= \sum_{i,j=1}^{n} \partial_{q^k} A_{ij} A^*_{ij}, \quad A^*_{ij} \text{ - cofactor of } A_{ij} && (14)\\ &= \sum_{i=1}^{n} \det\left(\mathbf{A}_1 \middle| \cdots \middle| \frac{\partial}{\partial q^k}\mathbf{A}_i \middle| \cdots \middle| \mathbf{A}_n\right) && (15)\end{aligned}$$

The final result for a serial manipulator follows again with $\partial_{q^j}\mathbf{A} = \partial_{q^j}\mathbf{J}\mathbf{J}^T + \mathbf{J}\partial_{q^j}\mathbf{J}^T$. For a PKM it follows from (13). Hence also the derivative $\frac{\partial}{\partial q^j}\mu_3$ is expressed explicitly and algebraically. For regular $\mathbf{A}$, (14) can be written as tr $(\mathbf{A}^{-1}\partial_{q^k}\mathbf{A}) = \sum_i \overline{\mathbf{A}}_i^T \partial_{q^k}\mathbf{A}_i$, where $\overline{\mathbf{A}}_i^T$ are the rows of $\mathbf{A}^{-1}$. This fails in singularities, however. For a square regular $\mathbf{J}$ it is $\partial_{q^i}\det\mathbf{A} = \text{tr}\,(\mathbf{J}^{-1}\partial_{q^i}\mathbf{J}) + \text{tr}\,(\mathbf{J}^{-T}\partial_{q^i}\mathbf{J}^T)$.

Remark 1: The closed form expressions may seem involved and tedious to evaluate. But the Jacobian (4),(5), and thus (9),(10), and all presented expressions can be recursively evaluated only requiring simple vector operations [18].

Remark 2: It is apparent from (5) and (4) that the body Jacobian of a serial manipulator does not depend on the joint angle q^1 of the first joint connecting it to the ground, which is well-known.

Remark 3: As a consequence of (6), which reads explicitly $\partial_{q^j}\mathbf{J}_i = (\boldsymbol{\omega}_i \times \boldsymbol{\omega}_j, \boldsymbol{\omega}_i \times \mathbf{v}_j - \mathbf{v}_i \times \boldsymbol{\omega}_j)$, for any serial manipulator whose EE motion comprises only one independent rotation component (e.g. planar manipulators, Scara), it is $\boldsymbol{\omega}_i \times \boldsymbol{\omega}_j = \mathbf{0}$. This reveals that then the angular part of the manipulator Jacobian $\mathbf{J}$ is constant.

B. Hessian of dexterity measures

The sensitivity measures can be used to numerically determine manipulator configurations that maximize the dexterity measures. In addition to the gradient, numerical search methods may also require the Hessian, i.e. the second partial derivatives. These can also be expressed in closed form making use of (6).

For the determinant, starting from (15) it follows immediately

$$\begin{aligned}&\tfrac{\partial^2}{\partial q^k \partial q^l}\det(\mathbf{A}) = \\ &\quad \textstyle\sum_{\substack{i,j\leq n\\ i\neq j}} \det\left(\mathbf{A}_1 \middle| \cdots \middle| \tfrac{\partial}{\partial q^k}\mathbf{A}_i \middle| \cdots \middle| \tfrac{\partial}{\partial q^l}\mathbf{A}_j \middle| \cdots \middle| \mathbf{A}_n\right) \\ &\quad + \textstyle\sum_{i=j\leq n} \det\left(\mathbf{A}_1 \middle| \cdots \middle| \tfrac{\partial^2}{\partial q^k \partial q^l}\mathbf{A}_i \middle| \cdots \middle| \mathbf{A}_n\right).\end{aligned} \tag{16}$$

For a serial manipulator $\frac{\partial^2}{\partial q^k \partial q^l}\mu_3$ follows again with the replacement $\mathbf{A} = \mathbf{J}\mathbf{J}^T$, and the last term in (16) is determined with (6).

The second partial derivative of the inverse condition number can be found from (12) invoking the relations (4), (5), and (6). Due to the space limitation it is not detailed in this paper.

VI. Sensitivity of Dexterity Measures w.r.t. Manipulator Geometry

A. Parameterizing link geometries – Virtual Joint Method

The constant parts $\mathbf{M}_i$ in the POE formulation (2) represent the transformation from one joint frame to the other on the link $i-1$. If the body was not rigid but elastic, this would not be constant. Still being a frame transformation the $\mathbf{M}_i$ can always be expressed as a screw motion and thus be expressed via the exp mapping. In the same manner the rigid link transformation $\mathbf{M}_i$ can be expressed by an exp mapping but the exponential coordinates embody geometric link parameters. This is the beauty of the POE formulation (2) since it allows to parameterize any frame transformation in a compact form in terms of exp mappings.

For simplicity in this paper only one parameter per link is considered. To account for such a geometry parameter for link i, introduce a screw coordinate vector $\mathbf{U}_i \in \mathbb{R}^6$, and correspondingly $\widehat{\mathbf{U}}_i \in se(3)$, such that

$$\widetilde{\mathbf{M}}_i(\pi_i) = \mathbf{M}_i \exp(\pi_i \mathbf{U}_i) \tag{17}$$

is the frame configuration on link i when 'deviating' from the 'nominal' geometry by a frame transformation according to the finite screw motion with screw axis $\mathbf{U}_i$ and amount π_i. The latter serves as geometric parameter. The overall set of geometric parameters, summarized in the parameter vector $\boldsymbol{\pi}$, for the manipulator make up the n-dimensional parameter space $\mathit{\Pi}$.

Remark 4: In the general case where spatial link 'deformations' are allowed, $\mathit{\Pi}$ is a $6n$-dimensional space). In this case the geometry of link i is determined by 6 canonical coordinates, either as $\exp(\pi_{i_1}\mathbf{U}_{i_1})\cdots\exp(\pi_{i_6}\mathbf{U}_{i_6})$ or $\exp(\pi_{i_1}\mathbf{U}_{i_1} + \ldots + \pi_{i_6}\mathbf{U}_{i_6})$. The concept remains as shown here.

With the amendment (17) the forward kinematic mapping attains the final form $\widetilde{f}: \mathbb{V}^n \times \mathit{\Pi} \to SE(3)$

$$\begin{aligned}\widetilde{f}(\mathbf{q},\boldsymbol{\pi}) &= \widetilde{\mathbf{M}}_1(\pi_1)\exp\left(\mathbf{X}_1 q^1\right)\cdot\ldots\cdot\widetilde{\mathbf{M}}_n(\pi_n)\exp\left(\mathbf{X}_n q^n\right)\\ &= \mathbf{M}_1\exp(\pi_1\mathbf{U}_1)\exp\left(\mathbf{X}_1 q^1\right)\cdot\ldots\\ &\quad \cdot\mathbf{M}_n\exp(\pi_n\mathbf{U}_n)\exp\left(\mathbf{X}_n q^n\right).\end{aligned} \tag{18}$$

From a conceptional point of view this approach can be considered as introducing extra joints that can be adjusted and locked so that the manipulator attains a certain geometry. Therefore this approach will be called *virtual joint method*.

B. Forward kinematics Jacobians

With the similarity of (18) and (2) the corresponding Jacobian $\widetilde{\mathbf{J}}$ of $\widetilde{f}$ can be split in the part $\mathbf{J}$ for the joint angles $\mathbf{q}$ and the part $\boldsymbol{\Gamma}$ for the geometric parameters $\boldsymbol{\pi}$. It is straightforward to obtain

$$\begin{aligned}\mathbf{J}_i &= \mathrm{Ad}^{-1}_{\widetilde{\mathbf{B}}_i}\mathbf{X}_i && (19)\\ \boldsymbol{\Gamma}_i &= \mathrm{Ad}^{-1}_{\mathbf{C}_i}\mathbf{U}_i && (20)\end{aligned}$$

with $\widetilde{\mathbf{B}}_i = \widetilde{\mathbf{M}}_{i+1}\exp\left(\mathbf{X}_{i+1}q^{i+1}\right)\cdot\ldots\cdot\widetilde{\mathbf{M}}_n\exp\left(\mathbf{X}_n q^n\right)$, and $\mathbf{C}_i = \exp\left(\mathbf{X}_i q^i\right)\widetilde{\mathbf{B}}_i$, so that $\mathbf{V} = \mathbf{J}(\mathbf{q},\boldsymbol{\pi})\,\dot{\mathbf{q}}$ is the EE twist resulting from joint rates $\dot{\mathbf{q}}$ for fixed $\boldsymbol{\pi}$, and

$\mathbf{\Delta C} = \mathbf{\Gamma}(\mathbf{q}, \boldsymbol{\pi})\,\delta\boldsymbol{\pi}$ is the variation of EE configuration at fixed joint angles. The latter describes the sensitivity of the EE configuration w.r.t. small changes $\delta\boldsymbol{\pi}$ in the geometry.

Remark 5: It may constitute a valuable measure of geometric sensitivity to define a measure analogous to the classical manipulability measure as

$$\sigma(\mathbf{q}, \boldsymbol{\pi}) := 1/\kappa(\mathbf{\Gamma}(\mathbf{q}, \boldsymbol{\pi})\,\mathbf{\Gamma}(\mathbf{q}, \boldsymbol{\pi})^T)$$

which reveals the sensitivity of EE configuration due to deviations in the geometry at a given posture $\mathbf{q}$. This could also be extended to a global measure by integration over the workspace.

C. Derivatives of forward kinematics w.r.t. geom. parameters

The expressions (19) and (20) gives rise to explicit forms of their partial derivatives w.r.t. joint angles q^i and geometric parameters π_i. The calculation yields of course again (6), and

$$\frac{\partial}{\partial \pi_j}\mathbf{J}_i = [\mathbf{J}_i, \mathbf{\Gamma}_j], \ i < j \tag{21}$$

$$\frac{\partial}{\partial q^j}\mathbf{\Gamma}_i = [\mathbf{\Gamma}_i, \mathbf{J}_j], \ i \leq j \tag{22}$$

$$\frac{\partial}{\partial \pi_j}\mathbf{\Gamma}_i = [\mathbf{\Gamma}_i, \mathbf{\Gamma}_j], \ i < j. \tag{23}$$

The result (21) is the crucial relation for deriving analytic closed forms of the gradient of the dexterity measures w.r.t. to geometric parameters. The relation (22) reveals how the parameter sensitivity of the EE mapping changes with the configuration, and the last expression (23) tells how the sensitivity of the EE mapping w.r.t. geometric parameter of link i changes for a small change of the parameter of link j.

D. Derivatives of the dexterity measures

With (21) it is straightforward to state closed form expressions for the gradient of the dexterity measures μ_2 and μ_3 w.r.t. geometric link parameters following (12) and (15). These provide the basis for systematic and automatized optimization of the manipulator geometry. The geometry must be optimized for the whole workspace, however. To this end the global measures shall be employed. They can be considered as functions of geometric design parameters and hence be used as basis for manipulator layout as originally suggested in [7],[8]. As such the condition for a stationary value of η w.r.t. geometric parameters $\boldsymbol{\pi}$ is

$$\int_W \frac{\partial}{\partial \boldsymbol{\pi}} \frac{1}{\kappa} dW - \eta \frac{\partial V_W}{\partial \boldsymbol{\pi}} = 0. \tag{24}$$

Hence the derivative of workspace volume as well as the inverse condition number are required. The latter is available with the above result. The derivative of workspace volume has to be evaluated on a case by case basis, possibly numerically.

VII. Example: RRR serial manipulator

For completeness a simple example is presented. A detailed example and design study is beyond the scope of this conference paper. Moreover only μ_3 is reported. A planar RRR serial manipulator is analyzed. All links have unit length. Its joint angles are denoted with q^1, q^2, q^3. Since the dexterity measures do not depend on the first joint angle q^1 the dexterity is only considered as function of q^2 and q^3. The distribution of μ_3 in the workspace is analyzed for joint ranges $0 \leq q^2, q^3 \leq \pi$. A sketch of the manipulator is given in figure 1, and joint angles q^2 and q^3 are defined.

Since for a planar manipulator the angular part of the forward kinematics Jacobian is constant it suffices to consider the translation part. Therefore the dexterity measure is computed using the translational velocity part. It is clearly visible in figure 1 (joint 1 is locked) that the dexterity varies and attains 0 at the singular configuration where the manipulator is in its stretched configuration at $(x, y) = (3, 0)$. Also visible are the areas of local maxima of the dexterity. Figure 2 shows the norm of the gradient of μ_2 w.r.t. to (q^2, q^3). As expected the sensitivity of μ_3 is highest in the singular configuration and attains zero at the points with maximum dexterity. It may be instructive to consider the sensitivity w.r.t. joint angles q^2 and q^3 separately. These are shown in figure 3 and 4. The actual incorporation of the particular sensitivities in a path planning/control scheme could be based on a weighted gradient so to take into account possibly different scales of allowed variations of the individual joints.

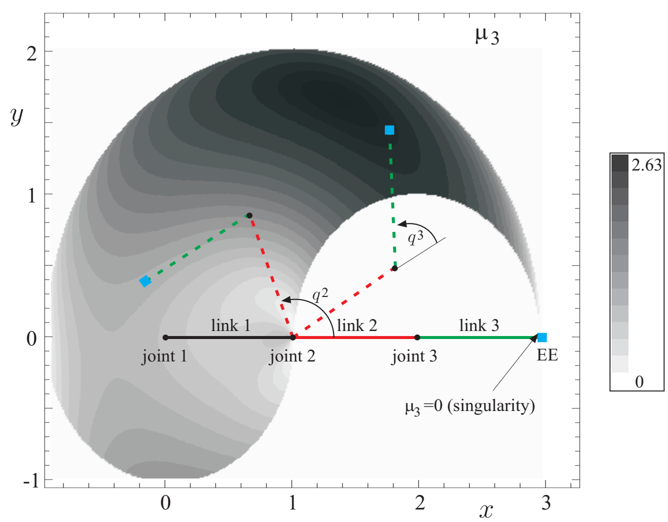

Fig. 1. Distribution of dexterity measure μ_3 for a planar RRR manipulator.

VIII. Conclusion

In this paper explicit closed form expressions are reported for the first and second partial derivative of dexterity measures μ_2 and μ_3 (inverse condition number and determinant) w.r.t. joint angles as well as w.r.t. geometric link parameters. This shall aid motion planning of robotic manipulators on the one hand and the optimization of the link geometry on the other hand. These problems have been and are still being

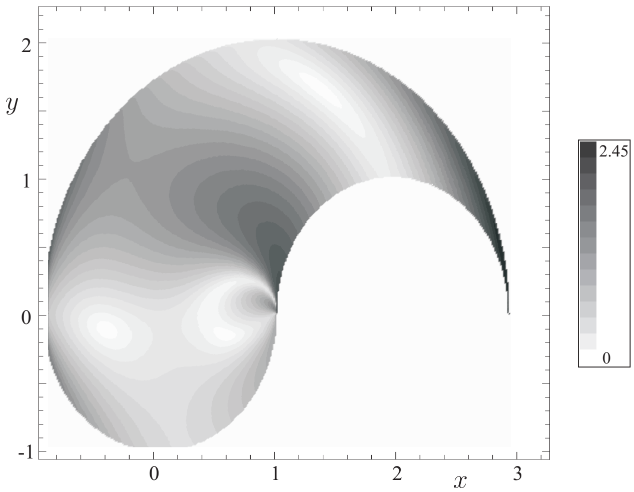

Fig. 2. Distribution of gradient norm $\|\nabla_{(q^1,q^2)}\mu_3\|$ in the worskpace.

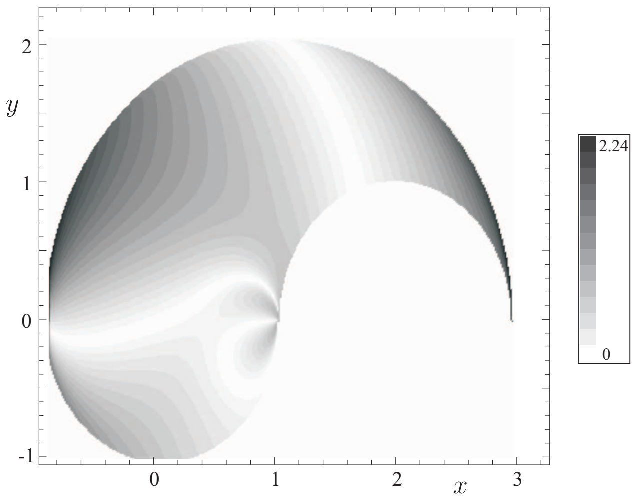

Fig. 3. Distribution of absolute value of partial derivative $\partial\mu_3/\partial q^2$.

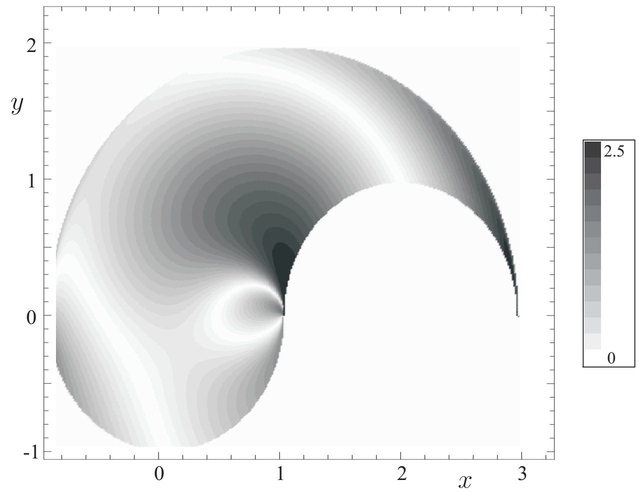

Fig. 4. Distribution of absolute value of partial derivative $\partial\mu_3/\partial q^3$.

addressed, however, mostly on the basis of numerical determination of such sensitivities. The proposed relations provide exact and explicit sensitivities. The final expressions may appear rather involved and complicated. But the important point to observe is that the presented closed forms allow for an automated evaluation and implementation in a kinematics simulation software. This gives rise to a general integrated solution for arbitrarily complex manipulators.

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