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Implementation of Arbitrary Periodic Dynamic Behaviors in Networked Systems

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Abstract— **Decentralized control of networked systems has been widely investigated in the literature, with the aim of regulating the overall state of the system to some desired configuration, thus obtaining coordinated emerging behaviors (e.g. synchronization, swarming, coverage, formation control) by means of local interaction. In this paper we introduce a methodology to solve a tracking problem, that is defining a decentralized control strategy for making a networked system follow an arbitrarily defined periodic setpoint function. The most suitable interconnection topology is defined together with the control law as the solution of a constrained optimization problem, in order to ensure asymptotic tracking. Simulations are provided for validating the proposed control strategy.**

I. Introduction

This paper introduces a methodology to let a networked system implement arbitrarily defined periodic dynamic behaviors. This objective is fulfilled having a subset of agents, called *leaders*, that are in charge of controlling the overall state of the networked system, in a completely decentralized manner.

Generally speaking, the aim of decentralized control strategies is implementing local interaction rules to regulate the state of the overall system to some desired configuration. On these lines, mainly investigated coordinated behaviors include aggregation, swarming, formation control, coverage and synchronization [1]–[4].

The idea of implementing more complex cooperative behaviors have recently appeared in the literature. For instance, [5], [6] present decentralized strategies for the coordination of groups of mobile robots moving along non–trivial paths. A decentralized strategy is presented in [7] that extends the standard consensus protocol to obtain periodic geometric patterns.

Recently a few works appeared that investigate the possibility of *interacting* with a networked system, in order to obtain a desired behavior [8]. The idea is that of having a set of agents, interconnected by means of a graph: a subset of those agents, namely the *leaders*, may be directly controlled, while the others, namely the *followers*, are indirectly controlled through the underlying interconnection graph.

As shown in [9], it is possible to model a networked system in such a way that the classical notions of controllability and observability of LTI systems are applicable. On these lines, in [10] we introduced a decentralized methodology to solve a tracking problem for networked systems in a decentralized manner. Specifically, we exploited the well known *regulator equations* to design a decentralized control law to make a networked system follow a predefined setpoint. We demonstrated that, given a fixed topology, it is not always possible to define a control strategy to track any periodic setpoint: therefore, we introduced a methodology to define the set of admissible setpoint functions, once the topology of the graph has been defined.

It is worth noting that the setpoint function defines the objective to be fulfilled. Therefore, in this paper we define a methodology to find a control strategy to make the networked system track an arbitrarily defined setpoint function. For this purpose, the most suitable interconnection topology is defined as well.

The paper is organized as follows. In Section II the networked system is modeled as a LTI system. In Section III we introduce the regulator equations, and we provide the problem formulation. The proposed solution is then described in Section IV, in terms of a constrained optimization problem. Simulation results are provided in Section V for validation purposes. Finally, Section VI contains some concluding remarks.

II. Preliminaries

A. Notation

In this section we define some symbols that will be used throughout the paper.

The symbols $\mathbf{1}_\rho$ and $\mathbf{0}_\rho$ will be used to indicate a vector of all ones and a vector of all zeros, respectively, in $\mathbb{R}^\rho$. Moreover, the symbol $\mathbb{I}_\rho$ will be used to indicate the identity matrix in $\mathbb{R}^{\rho\times\rho}$, while the symbol $\mathbb{O}_{\rho,\sigma}$ will be used to indicate the zero matrix in $\mathbb{R}^{\rho\times\sigma}$.

Let $\Omega \in \mathbb{R}^{\rho\times\sigma}$ be a generic matrix. We define $\Omega[i,:] \in \mathbb{R}^\sigma$, $\Omega[:,j] \in \mathbb{R}^\rho$ and $\Omega[i,j] \in \mathbb{R}$ as the i–th row, the j–th column and the element (i,j) of Ω, respectively. On the same lines, given a vector $\omega \in \mathbb{R}^\rho$, we define $\omega[i] \in \mathbb{R}$ as the i–th entry of ω.

B. Model of the system

Consider a group of N agents, namely mobile robots, sensors or other entities, whose interconnection structure is modeled by means of an undirected graph $\mathscr{G}$, where $\mathsf{V}(\mathscr{G})$ and $\mathsf{E}(\mathscr{G})$ are the vertex set and the edge set of the graph $\mathscr{G}$, respectively. The (unweighted) Laplacian matrix $\mathcal{L}(\mathscr{G})$ is defined as follows:

$$\mathcal{L}(\mathscr{G}) = \mathcal{I}(\mathscr{G})\,\mathcal{I}^T(\mathscr{G}) \tag{1}$$

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where $\mathcal{I}(\mathscr{G})$ is the incidence matrix. In an edge–weighted graph, a positive number (the weight) is associated to each edge of the graph. Let $\bar{w}_k > 0$ be the weight associated with the k–th edge, and let $\bar{w} = [\bar{w}_1, \dots, \bar{w}_{\bar{M}}] \in \mathbb{R}^{\bar{M}}$. Then, the weight matrix $\mathcal{W}(\mathscr{G}) \in \mathbb{R}^{\bar{M} \times \bar{M}}$ is defined as $\mathcal{W}(\mathscr{G}) = \mathrm{diag}(\bar{w})$.

The (weighted) Laplacian matrix $\mathcal{L}_{\mathcal{W}}(\mathscr{G})$ of the graph $\mathscr{G}$, associated with the weight matrix $\mathcal{W}(\mathscr{G})$ can then be defined as follows:

$$\mathcal{L}_{\mathcal{W}}(\mathscr{G}) = \mathcal{I}(\mathscr{G}) \mathcal{W}(\mathscr{G}) \mathcal{I}^T(\mathscr{G}) \tag{2}$$

Further details can be found for instance in [11].

Let $x_i \in \mathbb{R}^m$ be the state of the i–th agent: without loss of generality, we will hereafter consider the case where the state corresponds to each agent's position. Then, let the agents be interconnected according to the well known (weighted) consensus protocol [3]:

$$\dot{x}_i = - \sum_{j \in \mathsf{N}_i} \bar{w}_{ij} (x_i - x_j) \tag{3}$$

where $\bar{w}_{ij} > 0$ is the edge weight, and $\mathsf{N}_i \subseteq \mathsf{V}(\mathscr{G})$ is the neighborhood of the i–th agent, defined as the set of the agents that are interconnected to the i–th one, namely:

$$\mathsf{N}_i = \{ j \in \mathsf{V}(\mathscr{G}) \text{ such that } (v_i, v_j) \in \mathsf{E}(\mathscr{G}) \} \tag{4}$$

Without loss of generality, we will hereafter refer to the scalar case, namely $x_i \in \mathbb{R}$. It is however possible to extend all the results to the multi–dimensional case, considering each component independently.

Hence, let $\chi = [x_1, \dots, x_N]^T \in \mathbb{R}^N$ be the state of the multi–agent system. The interaction rule defined in Eq. (3) can be rewritten as follows:

$$\dot{\chi} = -\mathcal{L}_{\mathcal{W}}(\mathscr{G}) \chi \tag{5}$$

As is well known [3], under the consensus protocol the states of the agents converge to a common value. Assume now that the goal is to control the states of the networked agents: for this purpose, define a few leader agents, whose state is assumed to be directly controllable. The state of the other agents, referred to as the followers, evolves according to the consensus protocol.

More specifically, let $\mathsf{V}_L(\mathscr{G}) \subset \mathsf{V}(\mathscr{G})$ be the set of the leader agents, and let $\mathsf{V}_F(\mathscr{G}) = \mathsf{V}(\mathscr{G}) \setminus \mathsf{V}_L(\mathscr{G})$ be the set of the follower agents. Then, as shown in [9] for unweighted graphs, the interaction rule introduced in Eq. (3) is modified as follows:

$$\begin{cases} \dot{x}_i = - \sum\limits_{j \in \mathsf{N}_i} \bar{w}_{ij} (x_i - x_j) & \text{if } v_i \in \mathsf{V}_F(\mathscr{G}) \\ x_k = u_k & \text{if } v_k \in \mathsf{V}_L(\mathscr{G}) \end{cases} \tag{6}$$

where $u_k = u_k(t) \in \mathbb{R}$ is a control input.

Let N_L be the number of leaders. It is always possible to index the agents such that the last N_L agents are the leaders, and the first $N_F = N - N_L$ are the followers. Without loss of generality, we assume $N_L \leq N_F$. Then, as shown in [9], it is possible to decompose the Laplacian matrix $\mathcal{L}_{\mathcal{W}}(\mathscr{G})$ as follows:

$$\mathcal{L}_{\mathcal{W}}(\mathscr{G}) = - \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{B}^T & -\mathcal{E} \end{array} \right] \tag{7}$$

where $\mathcal{A} = \mathcal{A}^T \in \mathbb{R}^{N_F \times N_F}$ represents the interconnection among the followers, $\mathcal{B} \in \mathbb{R}^{N_F \times N_L}$ represents the interconnection among leaders and followers, and $\mathcal{E} = \mathcal{E}^T \in \mathbb{R}^{N_L \times N_L}$ represents the interconnection among the leaders.

Let $\mathscr{G}_F$ be the subgraph of the followers, whose Laplacian matrix is $\mathcal{L}(\mathscr{G}_F)$, and let M be the number of edges in $\mathscr{G}_F$. Let $\Delta \in \mathbb{R}^{N_F \times N_F}$ be a diagonal matrix whose i–th entry is

$$\Delta[i,i] = \sum_{k=1}^{N_L} \mathcal{B}[i,k] \tag{8}$$

Then:

$$\mathcal{A} = -\mathcal{L}(\mathscr{G}_F) - \Delta \tag{9}$$

Define now $x_F \in \mathbb{R}^{N_F}$ as the state vector of the followers, namely $x_F = [x_1, \dots, x_{N_F}]^T$. Define also $u \in \mathbb{R}^{N_L}$ as the input vector, namely $u = [u_{N_F+1}, \dots, u_N]^T$. Moreover, let $y \in \mathbb{R}^{N_L}$ be the output vector, that is the vector containing the state variables that are measurable by the leaders: it is reasonable to assume that each leader is able to measure the state of its neighbors.

We assume that the leader nodes are able to directly exchange information among each other. Namely, the following assumption is made:

Assumption 1 *A complete communication graph exists among the leader nodes.*

Therefore, the output vector can be defined as the vector containing the states of each leader's neighbors. Namely, the dynamics of the networked system can then be rewritten as follows:

$$\begin{cases} \dot{x}_F &= \mathcal{A} x_F + \mathcal{B} u \\ y &= \mathcal{B}^T x_F \end{cases} \tag{10}$$

We will hereafter assume that the topology is defined in such a way that the system in Eq. (10) is controllable and observable. Under Assumption 1 it is then possible to design a Luenberger state observer such that each leader can compute an estimate of x_F, which makes it possible to obtain a decentralized implementation of the control law that will be described in the next Sections, following the procedure in [10].

III. Problem Statement

As shown in [10], the regulator equations can be exploited to define the control input $u(t)$ such that the state of the followers tracks a desired setpoint. Specifically, given the topology of the networked system, it is possible to define a set of periodic setpoint functions that can be tracked by the system.

Periodic setpoint functions may be defined as the combination of a number n of harmonics with period T. Namely, a generic periodic setpoint $x_s(t) \in \mathbb{R}^{N_F}$, that has to be tracked by the state $x_F(t) \in \mathbb{R}^{N_F}$ of the followers, may be defined as follows:

$$x_s(t) = \mathcal{J}\xi(t) \tag{11}$$

where $\mathcal{J} \in \mathbb{R}^{N_F \times (2n+1)}$, and $\xi(t)$ is a vector of generating harmonics defined as follows:

$$\xi(t) = \left[1 \ \sin\left(\frac{2\pi}{T}t\right) \ \cos\left(\frac{2\pi}{T}t\right) \ldots \sin\left(n\frac{2\pi}{T}t\right) \ \cos\left(n\frac{2\pi}{T}t\right)\right]^T \tag{12}$$

Vector $\xi(t)$ can be obtained as the solution of an exosystem

$$\dot{\xi}(t) = \mathcal{G}\xi(t) \tag{13}$$

where matrix $\mathcal{G} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ and the initial state $\xi(0)$ are defined as shown in [10].

Define then the regulation error $e(t) \in \mathbb{R}^{N_F}$ as follows:

$$e(t) = x_F(t) - \mathcal{J}\xi(t) \tag{14}$$

As is well known [12] the regulation problem can be solved defining the input u as follows:

$$u = \mathcal{F}x_F + (\Gamma - \mathcal{F}\Pi)\xi \tag{15}$$

where $\mathcal{F}$ is an arbitrary matrix, chosen such that $(\mathcal{A} + \mathcal{B}\mathcal{F})$ is Hurwitz stable, and Π and Γ are the solution of the *regulator equations* that, in this case, can be written as follows:

$$\begin{cases} \mathcal{A}\Pi + \mathcal{B}\Gamma = \Pi\mathcal{G} & \text{(16a)} \\ \Pi - \mathcal{J} = 0 & \text{(16b)} \end{cases}$$

In this paper we provide a solution for the following problem:

Problem *Given the desired setpoint function, identified by means of the matrix $\mathcal{J}$, define a topology for the networked system such that the desired setpoint function is admissible. Subsequently, find a solution* (Π, Γ) *for the regulator equations in Eq.* (16)*, such that the control law in Eq.* (15) *can be defined.*

In particular, the objective is to define the minimal graph that makes the desired setpoint function admissible. By *minimal* graph we refer to the configuration which requires the smallest possible number of leaders, and of communication links among the agents.

It is in fact worth noting that several different topological interconnections among the agents might lead to the existence of a solution. However, it is desirable to minimize:

- *the number of leaders*, that is the number of independent inputs for the multi–robots system.
- *the number of edges among the agents*, in order to reduce the communication requirements.

Consider then a group of N_F followers to be controlled, and a given desired setpoint function. Therefore, imposing a given matrix $\mathcal{J}$, from the regulator equations in Eq. (16) we obtain:

$$\mathcal{A}\mathcal{J} + \mathcal{B}\Gamma = \mathcal{J}\mathcal{G} \tag{17}$$

The objective is then to define $\mathcal{A}, \mathcal{B}, \Gamma$ such that the condition in Eq. (17) holds.

IV. Algorithm for automatically defining the topology and the control law

In this Section we will describe an algorithm to solve the problem introduced in Section III. In particular, the topology of the network (namely matrices $\mathcal{A}$ and $\mathcal{B}$), as well as the control law (namely matrix Γ) will be defined such that the condition in Eq. (17) holds.

Therefore, we define an algorithm for the solution of the problem, namely:

- defining a topology for the networked system such that the desired control law is admissible, with the minimum number of leaders N_L
- defining the control law itself, according to Eq. (15)

This problem can be solved as a constrained optimization problem, formulated as follows:

$$\begin{aligned} \text{minimize} \quad & f(x) & \text{(18a)} \\ \text{subject to} \quad & c(x) = 0 & \text{(18b)} \\ \text{and} \quad & w[k] \geq 0 \ \forall k = 1, \ldots, M & \text{(18c)} \\ & \mathcal{B}[i,j] \geq 0 \ \forall i = 1, \ldots N_F, \ \forall j = 1, \ldots N_L & \text{(18d)} \end{aligned}$$

where $w \in \mathbb{R}^M$ is the vectors of the edge weights of $\mathscr{G}_F$, and the vector $x \in \mathbb{R}^{N_u}$ collects the unknown terms to be defined. The cost function $f(x) : \mathbb{R}^{N_u} \mapsto \mathbb{R}^+$ in Eq. (18a) is defined as the following quadratic function:

$$f(x) = \|\tilde{w}\|^2 \tag{19}$$

where $\tilde{w} \in \mathbb{R}^M$ is the vector of normalized edge weights of $\mathscr{G}_F$, defined as follows:

$$\tilde{w}[i] = \begin{cases} 1 & \text{if } w[i] \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

As will be clarified in the following Sections, the vector w is a portion of the vector of unknowns x. It is worth noting that the cost function $f(\cdot)$ increases as the number of non–zero edge weights increases. Hence, minimizing this cost function leads to minimizing the number of edges among the agents.

The equality constraint in Eq. (18b), defined by means of the function $c(x) : \mathbb{R}^{N_u} \mapsto \mathbb{R}^{N_c}$, is defined in order to impose the solution of the regulator equations. The inequality constraints in Eqs. (18c) and (18d) impose the choice of non–negative edge weights among the followers, and from leaders to followers.

Based on the definition of the equality constraints, the solution of the problem in Eq. (18) can be obtained by means of different techniques, as will be illustrated in the following sections. The number of unknowns N_u and the number of equality constraints N_c will be defined, once all the parameters of the optimization problem have been defined.

It is then possible to state the objective as follows: finding the minimum value of N_L such that the problem in Eq. (18)

admits a solution. This objective can be fulfilled solving the problem in Eq. (18) starting witn $N_L = 1$, and increasing N_L if a solution does not exists.

A. Formulation with linear constraints

In this Section we will show how to define the equality constraints in Eq. (18b) in the form a linear system of equations. In this manner, all the constraints are formulated as *convex constraints*, which makes it possible to solve the problem in Eq. (18) in a very efficient manner, exploiting standard software tools [13].

For this purpose, it is necessary to make the following assumption on the interconnection among the leaders and the followers:

Assumption 2 *Each leader is connected to exactly one follower, and two leaders are not connected to the same follower.*

Without loss of generality, we consider the i–th leader to be connected to the i–th follower: this can always be obtained opportunely relabeling the followers. Therefore, the matrix $\mathcal{B} \in \mathbb{R}^{N_F \times N_L}$ is defined as follows:

$$\mathcal{B} = \begin{bmatrix} \mathbb{I}_{N_L} \\ \mathbb{O}_{N_F - NL,\, N_L} \end{bmatrix} \tag{21}$$

We will hereafter show that, under Assumption 2, it is possible to rewrite Eq. (17) in order to obtain a linear system of equations. For this purpose, define the matrix $\Lambda \in \mathbb{R}^{M \times (2n+1)}$ as follows:

$$\Lambda = \mathcal{I}^T \mathcal{J} \tag{22}$$

where $\mathcal{I}$ is the incidence matrix of the complete follower subgraph $\mathscr{G}_F$. Since $\mathcal{J}$ is known once the desired setpoint has been defined, then Λ is a constant known matrix.

Define then

$$\delta_{i,j}^k = \mathcal{I}[i,k]\, \Lambda[k,j] \tag{23}$$

where $i = 1, \ldots, N_F$, $j = 1, \ldots, (2n+1)$, and $k = 1, \ldots, M$. It is worth remarking that $\delta_{i,j}^k$ is a known parameter, $\forall i,\, j,\, k$.

The parameters $\delta_{i,j}^k$ are then rearranged in the matrix $\mathcal{D} \in \mathbb{R}^{N_F(2n+1) \times M}$, whose elements are defined as follows:

$$\mathcal{D}[(2n+1)(i-1)+j, k] = \delta_{i,j}^k \tag{24}$$

where $i = 1, \ldots, N_F$, $j = 1, \ldots, (2n+1)$, and $k = 1, \ldots, M$.

We introduce now the matrix $\mathcal{Q} \in \mathbb{R}^{N_F(2n+1) \times N_L(2n+1)}$ as follows:

$$\mathcal{Q} = \begin{bmatrix} \mathbb{I}_{N_L(2n+1)} \\ \mathbb{O}_{(N_F - N_L)(2n+1),\, N_L(2n+1)} \end{bmatrix} \tag{25}$$

In the following Proposition we will show that the condition in Eq. (17) can be rewritten as the following standard linear system of equations:

$$\mathcal{H}x = \beta \tag{26}$$

where:

- the matrix $\mathcal{H} \in \mathbb{R}^{[N_F(2n+1)] \times [M + N_L(2n+1)]}$ is defined as follows:
$$\mathcal{H} = \left[\, -\mathcal{D} \;\middle|\; \mathcal{Q} \,\right] \tag{27}$$
- the vector of unknowns $x \in \mathbb{R}^{M + N_L(2n+1)}$ is defined as follows:
$$x = \left[w^T\, \Gamma[1,:] \cdots \Gamma[N_L,:] \right]^T \tag{28}$$
Namely, the unknowns are represented by the edge weights, that is the elements of vector w, and by the elements of the matrix $\Gamma \in \mathbb{R}^{N_L \times (2n+1)}$.
- the vector $\beta \in \mathbb{R}^{N_F(2n+1)}$ is defined as follows:
$$\beta = \left[\Omega[1,:] \cdots \Omega[N_F,:] \right]^T \tag{29}$$
where the matrix $\Omega \in \mathbb{R}^{N_F \times (2n+1)}$ is defined as follows:
$$\Omega = \mathcal{J}\mathcal{G} + \Delta\mathcal{J} \tag{30}$$

Proposition 1 *Consider the definition of matrix $\mathcal{H}$ in Eq.* (27)*, the definition of vector β in Eq.* (29)*, and the definition of the vector of unknowns x in Eq.* (28)*. Then, the condition in Eq.* (17) *can be rewritten in the form of a standard linear system of equations as in Eq.* (26)*, namely:*

$$\mathcal{H}x = \beta$$

Proof: In order to prove the statement, we will hereafter analyze the terms that compose the condition in Eq. (17), namely:

$$\mathcal{A}\mathcal{J} + \mathcal{B}\Gamma = \mathcal{J}\mathcal{G}$$

According to Eq. (9), the definition of $\mathcal{A}$ can be rewritten as follows:

$$\mathcal{A} = -\mathcal{I}(\mathscr{G}_F)\, \mathcal{W}(\mathscr{G}_F)\, \mathcal{I}^T(\mathscr{G}_F) - \Delta \tag{31}$$

Any topology can be obtained from the complete graph, setting to zero the weights corresponding to the edges that are not in the graph. For this purpose, without loss of generality, we will hereafter consider a complete follower subgraph: unless otherwise specified, when the graph is not explicitly indicated as an argument, we refer to the matrices of the complete graph.

When considering a complete graph, the number of edges M can be computed as follows:

$$M = \frac{N_F(N_F - 1)}{2}$$

Consider then $\mathcal{I}$ as the incidence matrix of the complete follower subgraph. Therefore we define

$$\mathcal{A} = -\mathcal{I}\mathcal{W}\mathcal{I}^T - \Delta$$

where $\mathcal{W} = \operatorname{diag}(w)$. The weight vector $w \in \mathbb{R}^M$ is defined as a vector of non–negative elements. Zero elements correspond to edges that are not in the graph.

Then, considering the definition of Ω in Eq. (30), it is possible to rewrite Eq. (17) as follows:

$$-\mathcal{I}\mathcal{W}\mathcal{I}^T\mathcal{J} + \mathcal{B}\Gamma = \Omega \tag{32}$$

Introduce now the matrices $\Psi \in \mathbb{R}^{N_F \times M}$ and $\Xi \in \mathbb{R}^{N_F \times (2n+1)}$, defined as follows:

$$\Psi = -\mathcal{I}\mathcal{W} = -\mathcal{I}\,\mathrm{diag}\,(w) \tag{33}$$

$$\Xi = \mathcal{B}\Gamma \tag{34}$$

These matrices collect the unknown terms in Eq. (17), that are represented by the elements of the matrix $\Gamma \in \mathbb{R}^{N_L \times (2n+1)}$, and the edge weights, that is the elements of vector w.

Consider now the definition of $\mathcal{B}$ given in Eq. (21). Then, the matrix Ξ defined in Eq. (34) can be written as follows:

$$\Xi = \begin{bmatrix} \Gamma \\ \mathbb{O}_{(N_F - N_L),\,(2n-1)} \end{bmatrix} \tag{35}$$

Moreover, considering the definition of Λ given in Eq. (22), that is

$$\Lambda = \mathcal{I}^T \mathcal{J}$$

then Eq. (32) can be rewritten as follows:

$$\Psi\Lambda + \Xi = \Omega \tag{36}$$

Define now $\Phi \in \mathbb{R}^{N_F \times (2n+1)}$ as follows:

$$\Phi = \Psi\Lambda \tag{37}$$

Therefore, each element of Φ may be written as follows:

$$\begin{aligned} \Phi[i,j] &= \sum_{k=1}^{M} \Psi[i,k]\,\Lambda[k,j] = \sum_{k=1}^{M} \mathcal{I}[i,k]\,(-w_k)\,\Lambda[k,j] \\ &= \sum_{k=1}^{M} \mathcal{I}[i,k]\,\Lambda[k,j]\,(-w_k) \end{aligned} \tag{38}$$

Considering the definition of $\delta^k_{i,j}$ given in Eq. (23), it is possible to rewrite Eq. (38) as follows:

$$\Phi[i,j] = \sum_{k=1}^{M} \delta^k_{i,j}\,(-w_k) \tag{39}$$

Subsequently, considering the definition of matrix $\mathcal{D}$ in Eq. (24), we obtain

$$\Phi[i,j] = -\mathcal{D}\left[(i-1)\,N_F + j, :\right] w \tag{40}$$

Considering then the definition of $\mathcal{Q}$ given in Eq. (25), and considering the definition of β in Eq. (29), then from Eq. (36) we obtain:

$$-\mathcal{D}w + \mathcal{Q} \begin{bmatrix} \Gamma[1,:]^T \\ \vdots \\ \Gamma[N_L,:]^T \end{bmatrix} = \beta \tag{41}$$

Finally, considering the definition of $\mathcal{H}$ in Eq. (27), and the definition of the vector of unknowns x in Eq. (28), it is possible to rewrite Eq. (41) as follows:

$$\mathcal{H}x = \beta$$

which proves the statement. ■

Hence, exploiting the results of Proposition 1, the equality constraints in Eq. (18b) can be defined as a linear set of constraints, namely $c(x) = \mathcal{H}x - \beta$.

B. Formulation with nonlinear constraints

In this Section we will remove Assumption 2, letting each leader be connected with more than one follower. While this formulation generally leads to solutions with a smaller number of leaders (as will be shown also in the simulations in Section V), the linearity and convexity properties of the constraints are lost. In particular, we will show that the problem in Eq. (18) can be formulated as a *nonlinear optimization problem with non–convex constraints*, that is a NP–complete problem [13].

In this case, then, the elements of $\mathcal{B}$ are unknown terms. Hence, we redefine the vector of unknowns $x \in \mathbb{R}^{M + N_F N_L + N_L(2n+1)}$ as follows:

$$x = \left[w^T\, \mathcal{B}[1,:] \cdots \mathcal{B}[N_F,:]\ \Gamma[1,:] \cdots \Gamma[N_L,:]\right]^T \tag{42}$$

Namely, the unknowns are represented by the edge weights of the follower subgraph $\mathscr{G}_F$, that is the elements of vector w, by the elements of the matrix $\Gamma \in \mathbb{R}^{N_L \times (2n+1)}$, and by the elements of $\mathcal{B} \in \mathbb{R}^{N_F \times N_L}$.

Consider now the definition of matrix Ξ in Eq. (34). Each element of this matrix can be explicitly computed as follows:

$$\Xi[i,j] = \sum_{k=1}^{N_L} \mathcal{B}[i,k]\,\Gamma[k,j] \tag{43}$$

$\forall i = 1,\ldots,N_F$, $\forall j = 1,\ldots,2n+1$.

We define then the equality constraints in Eq. (18b) as follows:

$$c(x) = \begin{bmatrix} (\Phi[1,:] + \Xi[1,:] - \Omega[1,:])^T \\ \vdots \\ (\Phi[N_F,:] + \Xi[N_F,:] - \Omega[N_F,:])^T \end{bmatrix} \tag{44}$$

It is worth noting that, according to the definition of Ξ in Eq. (43), then the constraint in Eq. (44) represent a nonlinear function.

Proposition 2 *Consider the definition of the nonlinear function $c(x)$ in Eq.* (44)*, with the matrix Φ defined as in Eq.* (38)*, the matrix Ξ defined as in Eq.* (43)*, and the matrix Ω defined as in Eq.* (30)*. Then, the condition in Eq.* (17) *can be rewritten as the following nonlinear equality constraints:*

$$c(x) = 0$$

Proof: As shown in the proof of Proposition 1, according to Eqs. (36) and (37), the condition in Eq. (17) can be rewritten as follows:

$$\Phi + \Xi = \Omega \tag{45}$$

which proves the statement. ■

It is worth noting that the problem in Eq. (18) is a non–convex problem. However, as the cost function is quadratic and the equality constraints in Eq. (44) are defined as continuously differentiable functions, the problem can be solved by means of *sequential quadratic programming* [13], which is an iterative method for nonlinear optimization.

V. SIMULATIONS

Simulations have been performed using MATLAB and the Optimization Toolbox, in order to evaluate and compare the performances of the proposed strategies.

Simulations have been performed with a variable number of followers $N_F \in [8, 20]$ and a variable number of harmonics $n \in [2, N_F/2]$. For each pair (N_F, n), five simulations runs were performed, defining the matrix $\mathcal{J} \in \mathbb{R}^{N_F \times (2n+1)}$ as a random matrix.

As expected, with both strategies a solutions was always found, with $N_L \leq N_F$. Comparing the results, it clearly emerges that the quality of the solutions obtained with the nonlinear formulation is typically better than the one obtained with the linear formulation. In fact, as shown in Fig. 1(a), for a variable number of followers N_F, the nonlinear algorithm provides a solution with a smaller number of leaders N_L. Moreover, the difference between the two solutions appears to be increasing, as the number of followers N_F increases.

The supremacy of the solution provided by the nonlinear algorithm is confirmed by Fig. 1(b) as well, which shows the number of leader N_L that solve the problem for a variable number of harmonics n.

The accompanying video shows MATLAB simulations whose results are summarized in the following table.

		Linear		Nonlinear	
$\mathbf{N_F}$	**n**	$\mathbf{N_L}$	**M**	$\mathbf{N_L}$	**M**
14	1	14	84	4	49
9	2	9	24	3	27

The proposed control strategy was also validated for decentralized control of groups of quadrotor Unmanned Aerial Vehicles (UAVs). Several strategies can be found in the literature [14] to build a local controller such that the closed loop behavior of each quadrotor UAV can be effectively approximated with that of a single integrator kinematic agent. Hence, the scenarios previously simulated with MATLAB were replicated with a team of UAVs, as shown in the accompanying video.

VI. CONCLUSIONS

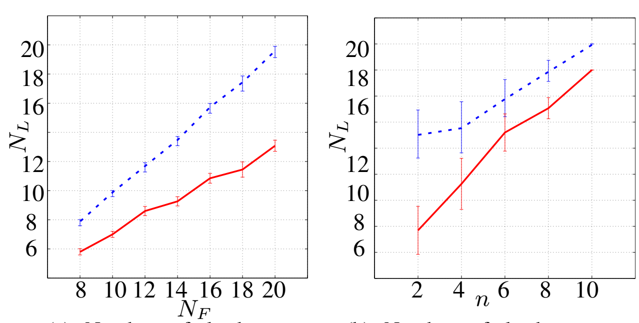
In this paper we introduced a methodology to solve a tracking problem for networked systems, that is defining a decentralized control strategy for making the system follow an arbitrarily defined periodic setpoint function.

Based on the regulator equations, a constrained optimization problem is formulated, whose solution provides both the most suitable interconnection topology and the parameters of the control law, in order to ensure asymptotic tracking.

Two different formulations have been provided for the optimization problem: linear and nonlinear. As is well known, the solution of a linear optimization problem is much faster than a nonlinear one. However, in order to obtain a linear formulation it was necessary to assume that each leader is connected to only one follower: simulations confirm that this assumption leads to solutions that require a higher number of leaders.

Therefore, it is possible to conclude that the linear formulation is to be preferred only when the amount of time to obtain a solution is crucial, such as for online applications.

Current work aims at finding robust solution in order to cope with time varying communication graphs.



(a) Number of leaders vs. Number of followers

(b) Number of leaders vs. Number of harmonics

Fig. 1. Comparison between linear (blue dashed line) and nonlinear (red solid line) formulations (mean and standard deviation)

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