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Attitude Stabilization Without Angular Velocity Measurements

L. Benziane, A. Benallegue and A. Tayebi

***Abstract*—We propose a velocity-free attitude stabilization scheme in which neither the angular velocity nor the instantaneous measurements of the attitude are used in the feedback, only body vector measurements are needed. To overcome the lack of angular velocity, a first order linear auxiliary system based directly on these vector measurements is introduced. Almost global asymptotic stability results are obtained. Also, in order to adjust properly the gains of the controller, an analysis of their effect on the closed-loop dynamics was performed. The effectiveness and performance of the proposed solution are illustrated via simulation results where some comparison with existing previous work are given.**

I. INTRODUCTION

The problem of rigid body attitude estimation and control remains until today an active research topic. This is due to the large field of applications such as robotics, unmanned aerial vehicles (UAVs), satellites, marine vehicles, etc.

Since 1965, when Wahba [1] posed the problem of determining the attitude using vector measurements as an optimization problem, several innovative solutions in the field of attitude estimation and control have emerged. First solutions to control the attitude were used in robotics with several orientation representations.

As mentioned in the recent survey paper [2], the literature on orientation control can be classified into two categories; solutions using geometric framework to represent the attitude in $SO(3)$ or $\mathbb{S}^2$ [2], [3], [4] and those using $\mathbb{R}^3$ or $\mathbb{S}^3$[5], [6], [7], [8]. Some of these works propose a combined estimation-control solutions [3], [9], [10], others take into consideration special cases like in [11] where an asymptotic convergence of the attitude control is achieved without knowledge of the inertia matrix. Many other works dealt with the problem of attitude stabilization without using the angular velocity information. In this case, some of them exploited the passivity of the system such as [12], [13], [14], [15], [16]. Others use an auxiliary system to bypass the lack of the angular velocity, such as [17].

In almost all results dealing with the case of attitude control without angular velocity, the "instantaneous measurements of the attitude" is used in the control law. As there is no sensor that directly measures the attitude of a rigid body, the aforementioned velocity-free controllers require some kind of attitude observer relying on the available direction sensors. However, all the most efficient dynamic attitude estimation algorithms make use of the body measurements and the angular velocity information to estimate the attitude of the rigid body. To overcome this problem, a velocity-free attitude control scheme, that incorporates explicitly vector measurements instead of the attitude itself, has been proposed for the first time in [17]. As claimed in [17], this class of controllers can be qualified as the class of *true velocity-free attitude controller*s.

The attitude control scheme presented in this work can be regarded as an extension version of [17]. The main difference is the use of an auxiliary system in terms of the body vector measurements, defined on $\mathbb{R}^3$, rather than using an auxiliary system defined on $\mathbb{S}^3$. This leads to reduce the set of unstable equilibria of the closed loop dynamics. In order to adjust properly the gains of the controllers, an analysis of their effect on the closed-loop dynamics was conducted for the previous and current works. This is based on the linearized systems around the stable equilibrium point.

L. Benziane, A. Benallegue are with the Versailles Engineering Systems Laboratory - LISV, Velizy, France. A. Tayebi is with the Department of Electrical Engineering, Lakehead University, Thunder Bay, Ontario, Canada, e-mail: lotfi.benziane@ens.uvsq.fr; benalleg@lisv.uvsq.fr; atayebi@lakeheadu.ca.

II. PRELIMINARIES, ASSUMPTIONS AND PROBLEM FORMULATION

A. Preliminaries

In this paper, we use two attitude representations. The first one is the rotation matrix R, which provides a unique and global parametrization of the orientation. It is an element of $SO(3)$ (the special orthogonal group) defined by $SO(3) = \{R \in \mathbb{R}^{3\times3} \mid R^TR = RR^T = I_{3\times3}, det(R) = 1\}$, where $I_{3\times3}$ is the 3-by-3 identity matrix. The second representation is the unit-quaternion $Q = (q_0, q^T)^T$, $q_0 \in \mathbb{R}$, $q \in \mathbb{R}^3$ such that $q_0^2 + q^Tq = 1$. It is an element of the three-sphere $\mathbb{S}^3$ embedded in $\mathbb{R}^4$ and defined as $\mathbb{S}^3 = \{Q \in \mathbb{R}^4 \mid Q^TQ = 1\}$.

The rotation matrix R is related to the unit-quaternion Q through the Rodriguez formula $R = \mathcal{R}(Q)$. The mapping $\mathcal{R}$ from $\mathbb{S}^3$ to $SO(3)$ is given by

$$\mathcal{R}(Q) = I_{3\times3} + 2q_0S(q) + 2S(q)^2, \tag{1}$$

where S represents the mapping $\mathbb{R}^3 \to \mathfrak{so}(3)$, with $\mathfrak{so}(3)$ being the Lie algebra of $SO(3)$ denoted by $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times3} \mid S^T = -S\}$. It is the set of skew symmetric matrices. The mapping $S : \mathbb{R}^3 \to \mathfrak{so}(3)$, is such that

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix},$$

where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$. For any $b \in \mathbb{R}^3$, $S(x)b = x\times b$ is the vector cross product.

We have the following useful identities: $S(x)b = -S(b)x$, $S(b)b = \mathbf{0}_{3\times 1}$ and $S(Rb) = RS(b)R^T$, where R is a given rotation matrix and $\mathbf{0}_{3\times 1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

The multiplication of two quaternion P and Q, denoted by "$\odot$" is defined as $P \odot Q = \begin{bmatrix} p_0 q_0 - p^T q \\ p_0 q + q_0 p + p \times q \end{bmatrix}$,such that $P = (p_0, p^T)^T$ and $Q = (q_0, q^T)^T$.

In what follows and for simplicity, the notations $\mathbf{0}_3$,$\mathbf{0}$, $\boldsymbol{I}$ will be used instead of $\mathbf{0}_{3\times 3}$, $\mathbf{0}_{3\times 1}$, $I_{3\times 3}$, respectively.

B. Assumptions

Let $\mathcal{B}$ be an orthonormal body-attached frame with its origin at the center of gravity of the rigid-body and $\mathcal{I}$ be the inertial reference frame on earth. We have the following assumptions:

- Consider n ($n \geq 2$) measurable b_i vectors in $\mathcal{B}$, corresponding to n fixed r_i vectors in $\mathcal{I}$, such that $b_i = R^T r_i$.
- Suppose that at least two vectors among them are non collinear.
- The angular velocity vector ω is non-measurable.
- The desired rigid body attitude is defined by the constant rotation matrix R_d, relates an inertial vector r_i to its corresponding vector in the desired frame, *i.e.*, $b_i^d = R_d^T r_i$, with $\dot{b}_i^d = \mathbf{0}$. The equivalent constant desired unit-quaternion Q_d is defined as $R_d = \mathcal{R}(Q_d)$.

C. Problem formulation

The rigid body attitude kinematics, in terms of rotation matrix $R \in SO(3)$, are given by $\dot{R}(t) = R(t)S(\omega(t))$ and in terms of the unit-quaternion $Q \in \mathbb{S}^3$, by

$$\dot{Q}(t) = \begin{bmatrix} \dot{q}_0(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} q^T(t)\omega(t) \\ \frac{1}{2}(q_0(t)\boldsymbol{I} + S(q(t)))\omega(t) \end{bmatrix}, \quad (2)$$

with ω being the angular velocity of the rigid body expressed in $\mathcal{B}$. The dynamics of the body vectors $b_i(t) = R^T(t) r_i$ $(i = 1, \cdots, n)$ are given by

$$\dot{b}_i(t) = -S(\omega(t)) b_i(t),\ i = 1, \cdots, n. \quad (3)$$

The rigid body rotational dynamics are governed by

$$J\dot{\omega}(t) = -S(\omega(t))J\omega(t) + \tau(t), \quad (4)$$

where $J \in \mathbb{R}^{3\times 3}$ is a symmetric positive definite constant inertia matrix of the rigid body with respect to the body-attached frame $\mathcal{B}$. $\tau(t)$ is the external torque applied to the system expressed in $\mathcal{B}$.

The problem addressed in this work is the design of an attitude stabilization control $\tau(t)$ based only on inertial measurements, without using the angular velocity ω in the feedback.

III. Design of the velocity-free attitude controller

First, let us define the orientation error by $\bar{R}(t) = R(t)R_d^T(t)$ which corresponds to the quaternion error $\bar{Q}(t) = Q(t) \odot Q_d^{-1}(t) \equiv (\bar{q}_0(t), \bar{q}(t))$ whose dynamics is governed by

$$\begin{bmatrix} \dot{\bar{q}}_0(t) \\ \dot{\bar{q}}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\bar{q}^T(t) R_d \omega(t) \\ \frac{1}{2}\left(\bar{q}_0(t)\boldsymbol{I} + S\left(\bar{q}(t)\right)\right) R_d \omega(t) \end{bmatrix} \quad (5)$$

If we put $\bar{p} = R_d^T \bar{q}$, then (5) can be written as

$$\begin{bmatrix} \dot{\bar{q}}_0(t) \\ \dot{\bar{p}}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\bar{p}^T(t)\omega(t) \\ \frac{1}{2}\left(\bar{q}_0(t)\boldsymbol{I} + S\left(\bar{p}(t)\right)\right)\omega(t) \end{bmatrix} \quad (6)$$

The reduced orientation error is given by $\bar{b}_i = b_i - b_i^d = R_d^T(\bar{R}^T - I) r_i$ $(i = 1, \cdots, n)$, which can be rewritten using (1) and the fact that $\bar{p} = R_d^T \bar{q}$ as

$$\bar{b}_i = -2(\bar{q}_0 \boldsymbol{I} - S(\bar{p}))S(\bar{p}) b_i^d \quad (7)$$

One of the main contributions of this work is the definition of the auxiliary system which is simply a linear first-order filter on $b_i(t)$ $(i = 1, \cdots, n)$, such as

$$\dot{b}_i^{aux}(t) = \alpha(b_i(t) - b_i^{aux}(t)), \quad (8)$$

where $\alpha > 0$ and $b_i^{aux}(0) \in \mathbb{R}^3$ can be chosen arbitrarily.

Now, the error of the auxiliary system is defined by $\tilde{b}_i = b_i - b_i^{aux}$. Using (8), (3), (7), the fact that $b_i = \bar{b}_i + b_i^d$ and the property $S(x)y = -S(y)x$ leads to the following error dynamics

$$\dot{\tilde{b}}_i(t) = -\alpha \tilde{b}_i(t) + B_i(t)\omega(t), \quad (9)$$

where $B_i(t) = S(b_i(t))$ and using (7) $b_i(t) = \left(\boldsymbol{I} - 2\left(\bar{q}_0(t)\boldsymbol{I} - S\left(\bar{p}(t)\right)\right) S\left(\bar{p}(t)\right)\right) b_1^d$. Equation (9) can be rewritten using the state vector defined by $\xi(t) := (\tilde{b}_1^T(t), ..., \tilde{b}_n^T(t))^T$, as

$$\dot{\xi}(t) = A\xi(t) + B(t)\omega(t), \quad (10)$$

where $A = -\alpha I_{n\times n}$ and $B(t) = \begin{bmatrix} B_1^T(t) & \cdots & B_n^T(t) \end{bmatrix}^T$.

We propose the following control law $\tau(t) = z_\rho(t) + z_\gamma(t)$, where $z_\rho(t) = \sum_{i=1}^n \rho_i S(b_i^d) b_i(t)$ and $z_\gamma = \sum_{i=1}^n \gamma_i S(b_i^{aux}(t)) b_i(t)$. Using Lemma 1 of [17] and after some manipulations, one can get

$$\tau(t) = -2(\bar{q}_0 \boldsymbol{I} - S(\bar{p}))\bar{W}_\rho \bar{p} - B^T(t)\Gamma\xi(t), \quad (11)$$

where $\Gamma = diag(\gamma_1 \boldsymbol{I},\ \gamma_2 \boldsymbol{I}, \ldots, \gamma_n \boldsymbol{I})$ with $\gamma_i > 0$, $\bar{W}_\rho = R_d^T W_\rho R_d$ and $W_\rho = -\sum_{i=1}^n \rho_i S^2(r_i) > 0$ (see Lemma 2 of [17]), therefore $\bar{W}_\rho > 0$. Using (10), (6), (4) and (11), we obtain the following closed loop dynamics

$$\left\{ \begin{array}{lcl} \dot{\xi}(t) & = & A\xi(t) + B(t)\omega(t) \\ \dot{\bar{q}}_0(t) & = & -\frac{1}{2}\bar{p}^T(t)\omega(t) \\ \dot{\bar{p}}(t) & = & \frac{1}{2}(\bar{q}_0(t)\boldsymbol{I} + S(\bar{p}(t)))\omega(t) \\ J\dot{\omega}(t) & = & -S(\omega(t))J\omega(t) - B^T(t)\Gamma\xi(t) \\ & & -2(\bar{q}_0(t)\boldsymbol{I} - S(\bar{p}(t)))\bar{W}_\rho \bar{p}(t) \end{array} \right. \quad (12)$$

Remark 1. Note that the vector $z_\gamma(t) = -B^T(t)\Gamma\xi(t)$ used in the control law with the auxiliary vectors $\xi(t)$ given by (10) can be viewed as a feedback on the filtered angular velocity ω which is actually unavailable. In fact, since A and Γ are diagonal matrices, with A negative definite and Γ positive definite, then the following Lyapunov equation is always verified $A^T\Gamma + \Gamma A = -\Lambda$, with Λ diagonal positive definite matrix. We can see that equation (10) together with $z_\gamma(t) = -B^T(t)\Gamma\xi(t)$ is a linear time-varying filter, where the angular velocity $\omega(t)$ being the input and $z_\gamma(t)$ the output as depicted in Figure 1. This demonstrates that $z_\gamma(t)$ handles the lack of angular velocity measurements.

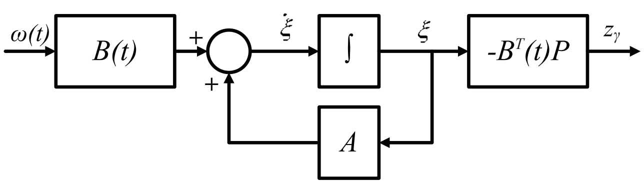

Figure 1. Relation between angular velocity and z_γ

Finally, since $\bar{q}_0^2 + \bar{p}^T\bar{p} = 1$, the dynamics of $\bar{q}_0$ can be omitted and the closed loop dynamics (12) will be

$$\left\{\begin{array}{lcl} \dot{\xi} & = & A\xi + B\omega \\ \dot{\bar{p}} & = & \frac{1}{2}(\bar{q}_0\boldsymbol{I} + S(\bar{p}))\omega \\ J\dot{\omega} & = & -B^T\Gamma\xi - S(\omega)J\omega - 2(\bar{q}_0\boldsymbol{I} - S(\bar{p}))\bar{W}_\rho\bar{p}, \end{array}\right. \tag{13}$$

where A, B and Γ are defined in (10) and (11). It is clear that the closed loop dynamics given by (13) is autonomous, which allow to use LaSalle's invariance theorem [18].

IV. Stability Analysis of the Proposed Controller

Let us define the state vector $\chi(t) := (\xi^T, \bar{p}, \omega) \in \Upsilon \mid \Upsilon := \mathbb{R}^{3n} \times D \times \mathbb{R}^3$, with $D := \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$ and note $\mathbf{0}_{3n} = (\underbrace{0, ..., 0}_{3n})$.

Theorem 2. *Consider the system (2)-(4), under assumptions in sub-section (II-B) and the control law (11) with the auxiliary system given by (10). Then,*

1) *The closed-loop equilibria are given by the point $\Omega_1 = (\mathbf{0}_{3n}, \mathbf{0}, \mathbf{0})$ and the set $\Omega_2 = \{(\mathbf{0}_{3n}, \pm\bar{v}_{\rho_i}, \mathbf{0}), i = 1, 2, 3\}$, where $\bar{v}_{\rho_i} = R_d^T v_{\rho_i}$ and v_{ρ_i} is a unit eigenvector of W_ρ.*
2) *The equilibrium point Ω_1 is asymptotically stable with the domain of attraction containing the following domain: $E_c = \{\chi \in \Upsilon \mid \chi^T\Pi\chi \leq c\}$ where $\Pi = diag(P, 4\bar{W}_\rho, J)$ and $c < 4\lambda_{min}(\bar{W}_\rho)$.*
3) *All equilibrium points in the set Ω_2 are unstable, and Ω_1 is almost globally asymptotically stable.*

Proof: 1) Consider the following Lyapunov function candidate $V : \Upsilon \to \mathbb{R}^+$

$$V = \xi^T\Gamma\xi + 4\bar{p}^T\bar{W}_\rho\bar{p} + \omega^T J\omega \tag{14}$$

Knowing that $\forall x, y, z \in \mathbb{R}^3$ $S(x)x = \mathbf{0}$, $x^TS(x) = \mathbf{0}$, $x^TS(y)z = -z^TS(y)x$ and using the fact that $\bar{p}^T\bar{W}_\rho(\bar{q}_0\boldsymbol{I} + S(\bar{p}))\omega = \omega^T(\bar{q}_0\boldsymbol{I} - S(\bar{p}))\bar{W}_\rho\bar{p}$, the time derivative of (14) in view of (13) yields

$$\dot{V} = -\xi^T\Lambda\xi \leq 0, \tag{15}$$

where $-\Lambda = A^T\Gamma + \Gamma A$ is Lyapunov equation and always verified as mentioned before. Now, let us use LaSalle's invariance theorem. From (15), we have $\dot{V} = 0$ when $\xi^T = \mathbf{0}_{3n}$. From (10) and the fact that there are at least two non-collinear vectors we can conclude that $\omega = \mathbf{0}$. From (12) we can conclude that $(\bar{q}_0\boldsymbol{I} - S(\bar{p}))\bar{W}_\rho\bar{p} = \mathbf{0}$. Now, by applying Lemma 3 of [17] to this last result, we can conclude that the equilibrium sets are: $\Omega_1 = (\mathbf{0}_{3n}, \mathbf{0}, \mathbf{0})$ with $\bar{q}_0 = \pm1$ and $\Omega_2 = \{(\mathbf{0}_{3n}, \pm\bar{v}_{\rho_i}, \mathbf{0}), i = 1, 2, 3\}$ with $\bar{q}_0 = 0$.

2) The largest invariant set M in Υ with respect to (13) and characterized by $\dot{V} = 0$ can be given by $M = \Omega_1 \cup \Omega_2$. From (15), $\dot{V} \leq 0$ which mean that for all $t \geq 0$ one has $V(\chi(t)) \leq V(\chi(0))$, thus $\chi(0) \in E_c \Rightarrow \chi(t) \in E_c, \forall\, t \geq 0$ and one concludes that E_c is a positively invariant sub-level set. Using the fact that $4\lambda_{min}(\bar{W}_\rho) \parallel \bar{p} \parallel^2 \leq 4\bar{p}^T\bar{W}_\rho\bar{p} \leq 4\lambda_{max}(\bar{W}_\rho) \parallel \bar{p} \parallel^2$, and $V(\chi) \geq 4\lambda_{min}(\bar{W}_\rho) \parallel \bar{p} \parallel^2$ it is clear that

$$\min_{\|\bar{p}\|=1} V(\chi) = 4\lambda_{min}(\bar{W}_\rho) \tag{16}$$

Since $E_c \subset \{\chi(t) \in \Upsilon \mid \parallel \bar{p} \parallel < 1\}$, it is clear that the equilibrium points Ω_2 do not belong to E_c. Consequently, the largest invariant set in E_c with respect to (12) and characterized by $\dot{V} = 0$ is composed only by Ω_1. Consequently, Ω_1 is asymptotically stable with the domain of attraction containing E_c.

3) Due to the fact that Ω_2 belongs to the manifold characterized by $(\bar{q}_0 = 0, \omega = 0)$, it suffices to prove that this manifold is unstable. To this purpose, let us write (12) around Ω_2 as

$$\left\{\begin{array}{lcl} \dot{\bar{q}}_0 & = & -\frac{1}{2}\bar{v}_\rho^T\omega \\ \dot{z}_\gamma & = & -\alpha z_\gamma - W\omega \\ J\dot{\omega} & = & -2\lambda_\rho\bar{v}_\rho\bar{q}_0 - z_\gamma, \end{array}\right. \tag{17}$$

where $z_\gamma = -H^T\Gamma\xi$, $W = H^T\Gamma H$. H is the linearized matrix of B where

$$H = \left[\begin{array}{c} S\left(\left(I + 2S^2\left(\bar{v}_\rho\right)\right)b_1^d\right) \\ \vdots \\ S\left(\left(I + 2S^2\left(\bar{v}_\rho\right)\right)b_n^d\right) \end{array}\right]$$

The system (17) can be written as

$$\dot{X} = \mathcal{A}X, \tag{18}$$

where $X = \begin{bmatrix} \bar{q}_0 & z_\gamma^T & (J\omega)^T \end{bmatrix}^T$, $\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$ and $\mathcal{A}_{11} = 0$, $\mathcal{A}_{12} = \begin{bmatrix} \mathbf{0} & -\frac{1}{2}\bar{v}_\rho^T J^{-1} \end{bmatrix}$, $\mathcal{A}_{21} = \begin{bmatrix} \mathbf{0}^T \\ -2\lambda_\rho \bar{v}_\rho \end{bmatrix}$, and $\mathcal{A}_{22} = \begin{bmatrix} -\alpha \boldsymbol{I} & -WJ^{-1} \\ -\boldsymbol{I} & \mathbf{0}_3 \end{bmatrix}$.

Using Schur formula, one can obtain

$$det(\mathcal{A}) = \delta\alpha\lambda_\rho \bar{v}_\rho^T W^{-1} \bar{v}_\rho, \tag{19}$$

where $\delta = det(W)det(J^{-1})$. Since W and J^{-1} are positive definite, it is clear that $\delta > 0$. Consequently, $det(\mathcal{A}) > 0$. Due to the fact that the determinant is nothing other than the product of all eigenvalues and that $\mathcal{A}$ has seven eigenvalues, one can conclude that there exist at least one eigenvalue with positive real part. Therefore, (18) is unstable, and so is Ω_2. Finally, since Ω_2 has zero Lebesgue measure, one can conclude that Ω_1 is almost globally asymptotically stable. ■

V. Analysis of the effect of the controllers gains

As mentioned before, this is an extension of the work presented in [17]. In what follows we name the controller presented in [17] as "previous controller" or "previous work". Knowing that the proposed controller uses more control gains, the importance of this section is to give an analysis of the effect of control gains on the closed-loop dynamics for the previous and current works. This allowed us to adjust the gains based on the position of the poles and zeros of the linearized system around the stable equilibrium point.

A. Closed loop dynamics linearization of the previous work

The closed loop dynamics of the previous work can be written around $\Omega_{1p} = (\tilde{q}, \bar{q}, \omega) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ with $\tilde{q}_0^2 = 1$ ($\tilde{q}_0 = sign(\tilde{q}_0)$) and $q_0^2 = 1$ ($q_0 = sign(q_0)$) as follows

$$\begin{cases} \dot{\tilde{q}} &= \frac{1}{2}sign(\tilde{q}_0)\omega - W_\gamma \tilde{q} \\ \dot{q} &= \frac{1}{2}sign(q_0)\omega \\ J\dot{\omega} &= -2sign(q_0)W_\rho q - 2sign(\tilde{q}_0)W_\gamma \tilde{q}, \end{cases} \tag{20}$$

and $\dot{\tilde{q}}_0 = 0,\ \dot{q}_0 = 0$. Using (20) and after some manipulations one can get

$$J\dddot{q} + W_\gamma J\ddot{q} + (W_\rho + W_\gamma)\dot{q} + W_\gamma W_\rho q = \mathbf{0} \tag{21}$$

B. Closed loop dynamics linearization of the proposed controller

Without loss of generality we take $R_d = I$, which mean that $\bar{p} = \bar{q} = q$ and $b_i^d = r_i$.The closed loop dynamic (12) can be written around $\Omega_1 = (\xi, q, \omega) = (\mathbf{0}_{3n}, \mathbf{0}, \mathbf{0})$ with $q_0^2 = 1$ ($q_0 = sign(q_0)$) and $b_i = b_i^d = r_i$ as follows

$$\begin{cases} \dot{\tilde{b}}_i &= -\alpha\tilde{b}_i + S(r_i)\omega \\ \dot{q} &= \frac{1}{2}sign(q_0)\omega \\ J\dot{\omega} &= -2sign(q_0)W_\rho q + \sum_{i=1}^n \gamma_i S(r_i)\tilde{b}_i, \end{cases} \tag{22}$$

and $\dot{q}_0 = 0$. Similarly, after some manipulations one can obtain

$$J\dddot{q} + \alpha J\ddot{q} + (W_\rho + W_\gamma)\dot{q} + \alpha W_\rho q = \mathbf{0} \tag{23}$$

C. Control gains tuning

Now, let us use the state space representation. Using the following change of variable: $x_1 = q$, $\dot{x}_1 = \dot{q} = x_2$, $\dot{x}_2 = \ddot{q} = J^{-1}x_3$ and $\dot{x}_3 = J\dddot{q}$, system (21) and (23) can be written as

$$\begin{cases} \dot{X} &= A_i X \\ Y &= CX \end{cases}, \tag{24}$$

where $X = \left[x_1^T,\, x_2^T,\, x_3^T\right]^T$ and $A_i,\ i = 1, 2$ are the state matrices corresponding to the two systems such that: A_1 corresponds to the previous linearized system (21); A_2 corresponds to the proposed linearized system (23) where

$$A_1 = \begin{bmatrix} \mathbf{0}_3 & \boldsymbol{I} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & J^{-1} \\ -W_\gamma W_\rho & -(W_\rho + W_\gamma) & -W_\gamma \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mathbf{0}_3 & \boldsymbol{I} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & J^{-1} \\ -\alpha W_\rho & -(W_\rho + W_\gamma) & -\alpha\boldsymbol{I} \end{bmatrix},$$

and $C = \begin{bmatrix} \boldsymbol{I} & \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}$.

The pole-zero plots can be used to provide qualitative insights into the response characteristics of system (24) function of control gains. Since we have a nine degree system, control gain tuning will be based on the dominant poles. To choose a given gain, we begin by maintaining the other fixed and plot the location of the poles. The value of the gain will be chosen to have a desired damping ratio and a desired rise time for the dominant pole. For this, one can use any computer program to plot pole-zero map by varying one control gain at a time.

Also, observing equation (21) and (23) one can conclude that the introduction of the gain α in the case of the proposed controller provides more flexibility for gains tuning. Specially, when determining the dominant pole, which is the closest to the imaginary axis.

VI. Simulation results

To show the effectiveness of the proposed solution, we present some simulation results and a comparison between the proposed controller and the previous one. Let us denote a state vector $\chi_p = (\tilde{q},\ q,\ \omega)$ for the previous controller and $\chi = (\tilde{b}_1,\ \tilde{b}_2,\ q,\ \omega)$ for the proposed controller, with $R_d = I$. Since many scenarii are possible, we select three cases depending essentially on the initial conditions of the quaternion $Q(0)$, $\widetilde{Q}(0)$ in the case of the previous controller and $\tilde{b}_1(0)$, $\tilde{b}_2(0)$ in the case of the proposed controller. Table I illustrates theses cases.

In all cases, the following parameters are the same: $\omega(0) = [0,\ 0,\ 0]^T$; $r_1 = [0,\ 0,\ 1]^T$ and $r_2 = [1,\ 0,\ 1]^T$; $J = diag(0.5, 0.5, 1)$; the simulation sample time $\delta t = 0.01s$ with the solver algorithm ode8 (Dormand-Prince).

To tune gains we use the method described in subsection V-C. The chosen values for control gains are $\alpha = 5$, $\rho_1 = 3$, $\rho_2 = 1$ and $\gamma_1 = 10$, $\gamma_2 = 9$.

For the above chosen parameters the obtained eigenvectors of W_ρ and W_γ are

$$v_{\rho 1}=\pm\begin{bmatrix}0.9571\\0\\-0.2898\end{bmatrix},\ v_{\gamma 1}=\pm\begin{bmatrix}0.8619\\0\\-0.5071\end{bmatrix},$$
$$v_{\rho 2}=\pm\begin{bmatrix}0.2898\\0\\0.9571\end{bmatrix}, v_{\gamma 2}=\pm\begin{bmatrix}0.5071\\0\\0.8619\end{bmatrix},$$
$$v_{\rho 3}=\pm\begin{bmatrix}0\\1\\0\end{bmatrix}, v_{\gamma 3}=\pm\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Their associated eigenvalues are $\lambda_{\rho 1} = 4.303$, $\lambda_{\rho 2} = 0.6972$, $\lambda_{\rho 3} = 5$ and $\lambda_{\gamma 1} = 24.3$, $\lambda_{\gamma 2} = 3.704$, $\lambda_{\gamma 3} = 28$.

Remark 3. The asymptotically stable equilibrium point for the previous controller is noted $\Omega_{1p} = (\mathbf{0}, \mathbf{0}, \mathbf{0})$, and $\Omega_1 = (\mathbf{0}_{1\times 3}, \mathbf{0}_{1\times 3}, \mathbf{0}, \mathbf{0})$ for the proposed controller. The other unstable equilibria for the two controllers are: $\Omega_{2p} = (v_{\gamma i}, \mathbf{0}, \mathbf{0})$, $\Omega_{3p} = (v_{\gamma i}, v_{\rho i}, \mathbf{0})$, $\Omega_{4p} = (\mathbf{0}, v_{\rho i}, \mathbf{0})$ and $\Omega_2 = (\mathbf{0}_{1\times 3}, \mathbf{0}_{1\times 3}, v_{\rho i}, \mathbf{0})$.

Table I
INITIAL CONDITIONS

	case 01	case 02	case 03
$Q(0)$	$\begin{bmatrix}0.8\\0\\0\\0.6\end{bmatrix}$	$\begin{bmatrix}-0.8\\0\\0\\0.6\end{bmatrix}$	$\begin{bmatrix}0\\0.8\\0\\0.6\end{bmatrix}$
$\tilde{Q}(0)$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\+v_{\gamma 1}\end{bmatrix}$
$Q^{aux}(0)$	$\begin{bmatrix}0\\-0.6\\-0.8\\0\end{bmatrix}$	$\begin{bmatrix}0\\-0.6\\0.8\\0\end{bmatrix}$	$\begin{bmatrix}0.3853\\0\\0.9228\\0\end{bmatrix}$
$b_1^{aux}(0)$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$	$\begin{bmatrix}-0.711\\0\\-0.7032\end{bmatrix}$
$b_2^{aux}(0)$	$\begin{bmatrix}-0.28\\0.96\\-1\end{bmatrix}$	$\begin{bmatrix}-0.28\\-0.96\\-1\end{bmatrix}$	$\begin{bmatrix}-1.414\\0\\0.00788\end{bmatrix}$
$\tilde{b}_1(0)$	$\begin{bmatrix}0\\0\\2\end{bmatrix}$	$\begin{bmatrix}0\\0\\2\end{bmatrix}$	$\begin{bmatrix}1.671\\0\\0.4232\end{bmatrix}$
$\tilde{b}_2(0)$	$\begin{bmatrix}0.56\\-1.92\\2\end{bmatrix}$	$\begin{bmatrix}0.56\\1.92\\2\end{bmatrix}$	$\begin{bmatrix}2.654\\0\\0.6721\end{bmatrix}$

A. Case 01 and case 02 results

In this two cases, the convergence to the stable equilibrium point for the two controllers is ensured, we have : $\chi_p(t) \longrightarrow \Omega_{1p}$ and $\chi(t) \longrightarrow \Omega_1$. Figure 2 and 4 show the quaternion trajectories convergence in the two cases. Due to space limitation, the evolution of the other elements of the state vectors and the control input signals are omitted.

Analyzing these results, two main observations are verified. First, the unwinding phenomenon is avoided for the two controllers. Second, the performance of the proposed controller is better than the previous controller in terms of speed of convergence. Figure 3 illustrates the difference between previous control input signals and those of the proposed controller.

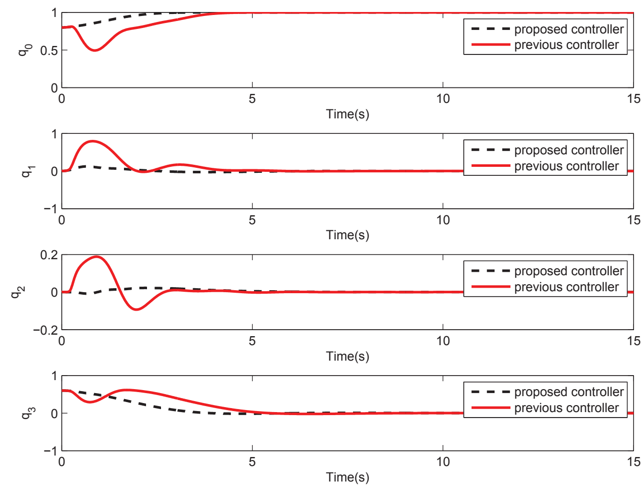

Figure 2. Case 1 - Quaternion trajectories (Q)

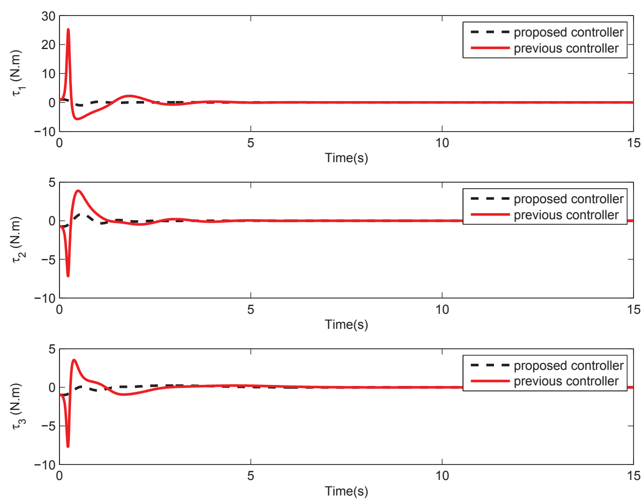

Figure 3. Case 1 - Control input signals (τ)

B. Case 03 results

This is a special case, where the initial condition belongs to the unstable manifold characterized by ($q_0 = 0$ and $\omega = 0$). Simulations show that : $\chi_p(t) \longrightarrow \Omega_{3p}$ and $\chi(t) \longrightarrow \Omega_2$. The two controllers have the same behavior in this case, trajectories converge to unstable equilibria. Figures 5-(A_1) and 5-(A_2) represent the inertial measurement errors for the proposed controller. Figures 5-(B_1) and 5-(B_2) represent the inertial measurement errors for the previous controller. Theses results show the main difference between the auxiliary systems of the proposed and previous controllers. It is clear that the proposed controller ensures the convergence of the auxiliary vectors to the real ones globally.

VII. CONCLUSIONS

For the attitude control, the unavailability of the angular velocity can be an interesting scenario. In fact, velocity-free attitude control schemes could be of great help (as main or

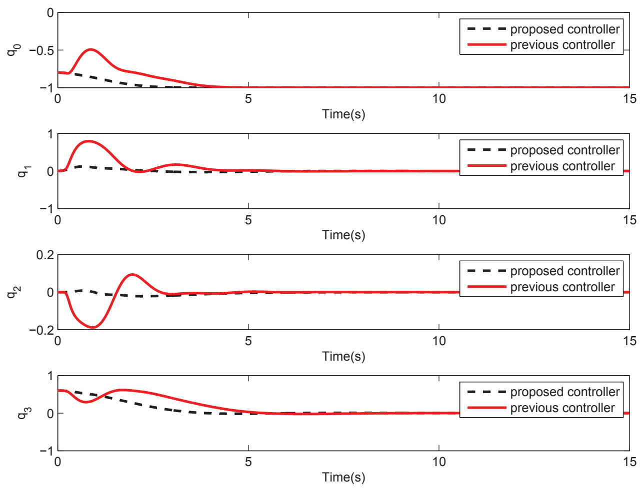

Figure 4. Case 2 - Quaternion trajectories (Q)

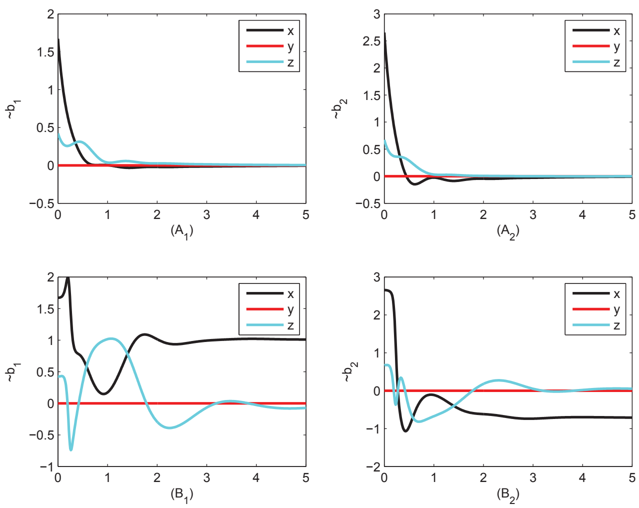

Figure 5. Case 3 - Inertial measurement errors

backup controllers) in applications where prone-to-failure and expensive gyroscope are used. In this paper, we have shown that it is possible to stabilize the attitude of a rigid body without any information about neither the angular velocity, nor the attitude. Despite the considerable number of solutions to this problem, almost all of them use the instantaneous attitude measurements. The proposed controller needs only two non collinear inertial measurement vectors to achieve an almost global asymptotic stabilization.

This paper can be viewed as an extension version of a previous controller. The main difference between the proposed controller and previous controller is the definition of the auxiliary system used to compensate for the lack of the angular velocity. We show that the choice of the auxiliary system have a direct impact not only on the general performance, but also on the number of unstable equilibria of the closed loop dynamics of the rigid body system defined in its state space. Also, an analysis of the effect of the control gains on the closed-loop dynamics was presented for the previous and current works. This analysis allowed us to properly adjust the gains based on the position of the poles and zeros of the linearized system.

A natural future work will be the real-time implementation of this controller on a physical system.

References

[1] G. Wahba, "A least squares estimate of satellite attitude," *SIAM Review*, vol. 7, no. 3, p. 409, 1965.

[2] N. Chaturvedi, A. Sanyal, and N. McClamroch, "Rigid-body attitude control," *IEEE Control Systems Magazine*, vol. 31, no. 3, pp. 30–51, June 2011.

[3] P. Pounds, T. Hamel, and R. Mahony, "Attitude control of rigid body dynamics from biased imu measurements," in *Decision and Control, 2007 46th IEEE Conference on*, 2007, pp. 4620–4625.

[4] T. Lee, "Robust adaptive attitude tracking on so(3) with an application to a quadrotor uav," *IEEE Transactions on Control Systems Technology*, vol. 21, no. 5, pp. 1924–1930, 2013.

[5] J.-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148 – 1162, October 1991.

[6] S. Joshi, A. Kelkar, and J.-Y. Wen, "Robust attitude stabilization of spacecraft using nonlinear quaternion feedback," *IEEE Transactions on Automatic Control*, vol. 40, no. 10, pp. 1800–1803, 1995.

[7] P. Tsiotras, "Stabilization and optimality results for the attitude control problem," *Journal of Guidance, Control, and Dynamics*, vol. 19, no. 04, pp. 772–779, July-August 1996.

[8] H. Wong, M. de Queiroz, and V. Kapila, "Adaptive tracking control using synthesized velocity from attitude measurements," *Automatica*, vol. 37, no. 6, pp. 947 – 953, 2001.

[9] A. Tayebi, S. McGilvray, A. Roberts, and M. Moallem, "Attitude estimation and stabilization of a rigid body using low-cost sensors," in *46th IEEE Conference on Decision and Control*, 2007, pp. 6424–6429.

[10] N. Nordkvist and A. Sanyal, "Attitude feedback tracking with optimal attitude state estimation," in *American Control Conference (ACC), 2010*, 2010, pp. 2861–2866.

[11] T. Fernando, J. Chandiramani, T. Lee, and H. Gutierrez, "Robust adaptive geometric tracking controls on so(3) with an application to the attitude dynamics of a quadrotor uav," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, 2011, pp. 7380–7385.

[12] O. Egeland and J.-M. Godhavn, "Passivity-based adaptive attitude control of a rigid spacecraft," *IEEE Transactions on Automatic Control*, vol. 39, pp. 842 – 846, Apr 1994.

[13] F. Lizarralde and J. Wen, "Attitude control without angular velocity measurement: a passivity approach," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 468–472, 1996.

[14] P. Tsiotras, "Further passivity results for the attitude control problem," *IEEE Transactions on Automatic Control*, vol. 43, no. 11, pp. 1597–1600, 1998.

[15] B. T. Costic, D. M. Dawson, M. S. d. Queiroz, and V. Kapila, "Quaternion-based adaptive attitude tracking controller without velocity measurements," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 6, pp. 1214–1222, 2001.

[16] A. Tayebi, "Unit quaternion-based output feedback for the attitude tracking problem," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1516–1520, July 2008.

[17] A. Tayebi, A. Roberts, and A. Benallegue, "Inertial vector measurements based velocity-free attitude stabilization," *IEEE Transactions on Automatic Control*, vol. 58, no. 11, pp. 2893–2898, November 2013.

[18] H. K. Khalil, *Nonlinear Systems*, third edition ed., P. E. I. Inc., Ed. Pearson Education, 2000.