Response to Ramesh & Vinay, (2003)

String Matching in $\tilde{O}(\sqrt{n} + \sqrt{m})$ Quantum Time

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Outline

String Matching in $\tilde{O}(\sqrt{n} + \sqrt{m})$ Quantum Time

H. Ramesh & V. Vinay (IISc Bangalore, 2003)

Problem Statement

Given a text t of length n and a pattern p of length m, decide whether p occurs in t.

- **Classical bound:** $\Theta(n + m)$ via KMP, Boyer-Moore, etc.
- **Quantum goal:** Exploit amplitude amplification to beat linear time.
- **Main result:** A quantum algorithm running in

$$\widetilde{O}\left(\sqrt{n} + \sqrt{m}\right)$$

with constant two-sided error probability.

Conceptual Dependencies

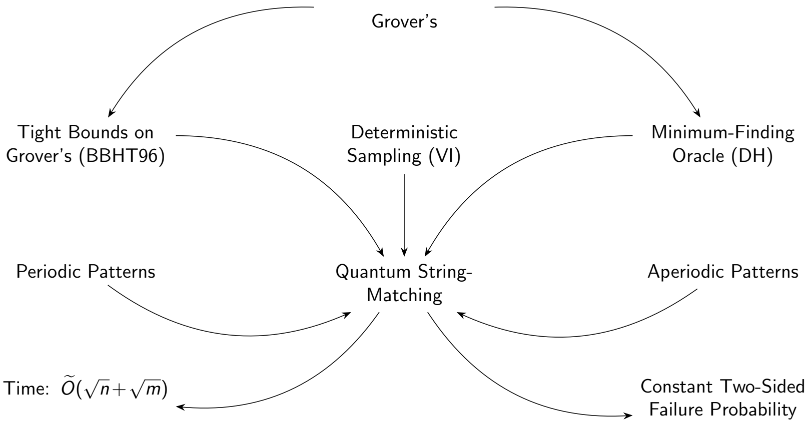
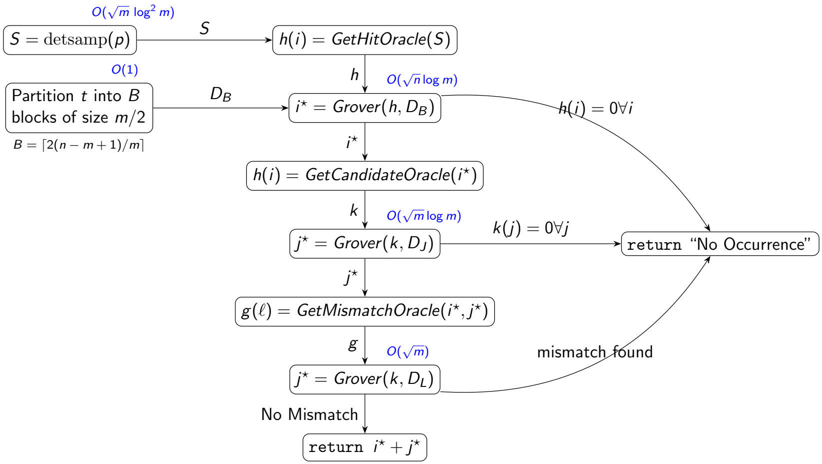


Figure: Conceptual dependencies in the $\widetilde{O}(\sqrt{n}+\sqrt{m})$ quantum string-matching algorithm.

The Algorithm



Grover's Algorithm

- Problem: Given a database of N elements and oracle $f(x) = 1$ for marked items, find an x with $f(x) = 1$.
- Steps:
  1. Initialize n qubits into uniform superposition over $N = 2^n$ states.
  2. Apply the oracle to flip the phase of marked states.
  3. Perform the diffusion (inversion-about-the-mean) operator.
  4. Repeat oracle + diffusion $\lfloor \frac{\pi}{4}\sqrt{N/t} \rfloor$ times (t=# marked).
  5. Measure to obtain a marked element with high probability.
- Time Complexity: $\mathcal{O}(\sqrt{N/t})$.
- Why it works: Oracle phase-flips mark targets; diffusion amplifies their amplitudes.
- Significance: Core building block for quantum search algorithms.

BBHT96: Tight bounds on quantum searching

What it is

- Even when the number of target $t \geq 1$ items in a database is unknown, you can still find a marked item in $\tilde{O}(\sqrt{N/t})$ oracle calls.
- BBHT96 describes a procedure which, given an oracle flagging at least t marked elements among n candidates, returns one solution in $O(\sqrt{n/t})$ time with constant success probability.

BBHT96: Tight bounds on quantum searching

Why it matters

- It underpins the claimed time bound in our string-matching algorithm.
- When checking a text block (size $m/2$) for any matching alignment, t is unknown.
- BBHT96's procedure lets us invoke Grover's search reliably in this setting.
- Every oracle in the string-matching pipeline (e.g. f, h) relies on BBHT96's bound to guarantee the $\tilde{O}(\sqrt{n} + \sqrt{m})$ runtime.

Deterministic Sampling

What it is

- Given an aperiodic pattern p of length m, and a text block of length $m/2$, there are $m/2$ possible alignments.
- DS picks $O(\log m)$ indices in the pattern so that at most one alignment can match all sampled positions.

Steps

1. Form $m/2$ "copies" of the pattern, each shifted by one position.
2. Find a column where at least two copies differ.
3. Select one of the symbols at that column as the sample and discard copies that don't match.
4. After $O(\log m)$ rounds, one copy remains; its chosen columns form the sample S.

Deterministic Sampling

Why it works

- Instead of a full pattern check per alignment ($O(m)$ classically, $\tilde{O}(\sqrt{m})$ quantum), only $O(\log m)$ sampled positions are tested.
- Only the single surviving alignment requires the expensive full check.

Why it matters

- Enables the overall quantum string matching to run in $\tilde{O}(\sqrt{n} + \sqrt{m})$ by reducing costly $\sqrt{m}$ checks to one per text block.

DH96: Minimum Finding Oracle

What it is

- Given a database of size n and a comparison oracle that, for any two indices i, j, indicates which element is smaller.
- Finds the index of the minimum element in $O(\sqrt{n})$ time.
- Serves as the backbone for all "pick the smallest (or leftmost) index satisfying a condition" steps.

Steps

1. Pick a random starting position k.
2. Use Grover's search to find any index i with database$[i] <$ database$[k]$.
3. If such an i is found, set $k \leftarrow i$ and repeat.
4. Otherwise, k is the index of the minimum element.

DH96: Minimum Finding Oracle

Why it matters

- Building the deterministic sampling set: Repeatedly eliminate half of the $m/2$ pattern copies by finding the leftmost and rightmost survivor via DH96 in $O(\sqrt{m}\log m)$ time.
- After locating the matching text-block with the $h(i)$ oracle, invoke DH96 over block indices to pinpoint the earliest occurrence, preserving the overall $\tilde{O}(\sqrt{n} + \sqrt{m})$ bound.

The Algorithm (Part 1)

1. **Deterministic-Sampling Preprocessing.**
   - Run Vishkin's deterministic-sampling on p of length m to obtain an $O(\log m)$-sized sample set S.
   - Cost: $\widetilde{O}(\sqrt{m}\,\log^2 m)$.
2. **Partition the text.**
   - Divide the text t into
   $$B = \left\lceil \frac{2(n-m+1)}{m} \right\rceil$$
   blocks, each of size $\approx m/2$.
3. **Quantum search for a "hit" block.**
   - Define oracle $h(i)$: tests if block i has at least one alignment matching on all positions in S in $\widetilde{O}(\sqrt{m}\,\log m)$ time.
   - Use Grover search over $i = 1, \ldots, B$ with oracle h; time $\widetilde{O}(\sqrt{n}\,\log m)$. If none found, conclude "no occurrence."

The Algorithm (Part 2)

4. **Locate surviving alignment in block.**
   - Define oracle $k(i^*, j)$: checks alignment at shift j in block i^* on sample S in $O(\log m)$ time.
   - Use Grover search over $j = 0, \ldots, \lfloor m/2 \rfloor$ with oracle k; time $\widetilde{O}(\sqrt{m} \log m)$. If none survives, conclude "no occurrence."
5. **Full quantum verification.**
   - Run Grover search over $\ell = 1, \ldots, m$ with oracle "$t[i^* + j^* + \ell] \neq p[\ell]$" to find any mismatch; time $\widetilde{O}(\sqrt{m})$.
   - If no mismatch is found, report occurrence at $i^* + j^*$; otherwise, conclude "no occurrence."

Concerning Periodicity

- The difference between periodic and aperiodic strings boils down to whether the string consists of a repeating sub-string or not.
- Periodicity leads to issues with the deterministic sampling as it means that the unique identification of a match is not guaranteed.
- While the algorithm mentioned above processes aperiodic strings, by adding a simple step of pre-processing, the algorithm can also handle periodic string with the same time complexity.
- To handle periodic strings, we first find the period p of the repeating pattern. Then, instead of checking every single position in the string for matches, the search space simplifies down to shifts that are multiples of p, i.e. checks only occur at every p-th position.

Constant Two-Sided Failure Probability

- The algorithm, based on the probabilistic Gover's and Quantum Minimum Finding algorithms, has a non-zero probability of:
  - failing to find a correct match (false negative), and
  - finding a match that is incorrect (false positive)
- However, these probabilites are constant, allowing them to be driven down to zero by simple repetition.
- The $\tilde{O}(\sqrt{n} + \sqrt{m})$ complexity allows the algorithm to be run repeatedly without significant computational costs.

Achieving $\tilde{O}(\sqrt{n} + \sqrt{m})$

So, the final structure of the algorithm is:

- **Block Search:** Divide a given string into overlapping blocks of $2m$ length each. Then use Grover's algorithm to find blocks containing a match. Matches are checked using Deterministic Sampling in $\tilde{O}(\sqrt{m})$ time per block. For all the blocks, the time complexity becomes:

$$\tilde{O}(\sqrt{\frac{n}{m}}) \times \tilde{O}(\sqrt{m}) = \tilde{O}(\sqrt{n})$$

- **Fine Matching:** Once a block is found, use Quantum Minimum Finding over $O(m)$ candidate positions within the block. Thus, the time complexity for this step becomes:

$$\tilde{O}(\sqrt{m})$$

Thus, the overall time complexity becomes:

$$\tilde{O}(\sqrt{n} + \sqrt{m})$$

Conclusion