Perturbation of the extremal feasible probe trajectories

Depending on how the obstacles are arranged, there may be an infinite number of feasible probe trajectories. In the lemma below, we discuss how any of these trajectories may be perturbed, while remaining feasible, into one of a finite number of probe trajectories where at most 3 obstacle endpoints lie tangent to the probe. We refer to these trajectories as extremal. It then suffices for our algorithm to test the feasibility of only the extremal trajectories; assuming a feasible probe trajectory exists at all, our algorithm will find its perturbation. By incrementally perturbing possible moves, we concluded there are 7 unique cases for extremal trajectories and 2* trivial cases. The trivial cases are 2* because there are more than 2 cases but they can easily be put into 2 groups. The first group is if the 3 segments are unarticulated, i.e. all 3 segments are collinear, form a straight line, and have <= 3 obstacle endpoint intersections. The second group of cases occurs when any of the 2 segments become collinear and there are <= 3 obstacle endpoint intersections, which then transforms it into a 2- segment problem, which has been addressed in a previous paper.

Noting that Segments BC'/BC/CD'/CD are of fixed length r, we have circles R and R' defining the boundary lines which points B and C must lie on for feasibility. Because all instances of collinearity mentioned above fall under trivial cases, we begin by assuming a fully articulated (one with no instances of collinearity between line segments) probe trajectory T in which both segments have been swept in the same direction and point D coincides with point t (Figure 1).

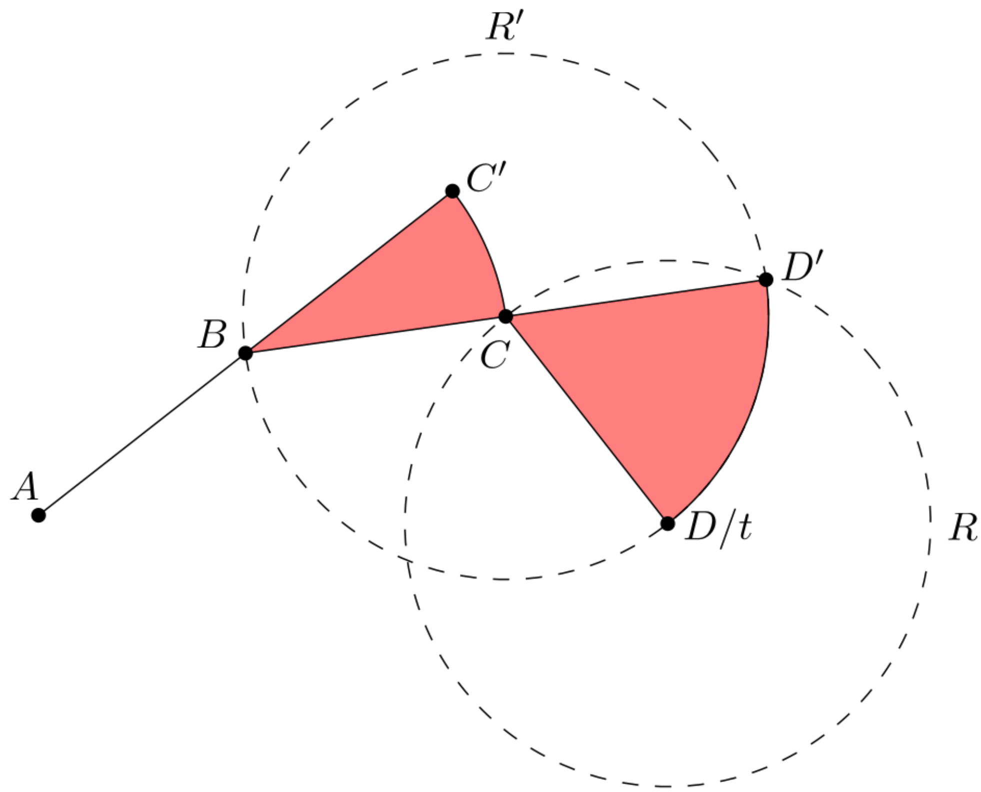


Figure 1

Assume feasible trajectory T, such that, without loss of generality, segment BC has been swept clockwise around point B to reach point C and similarly segment CD has been swept clockwise around point C to reach point D. Let T_2 be the trajectory resulting from rotating line segment AB of T around point B in clockwise direction until line segment AB intersects an obstacle endpoint v_1 outside Circle R'. Given that the area swept by line segment BC of T_2 to reach point C is within that of T, T_2 is also a feasible trajectory (Figure 2).

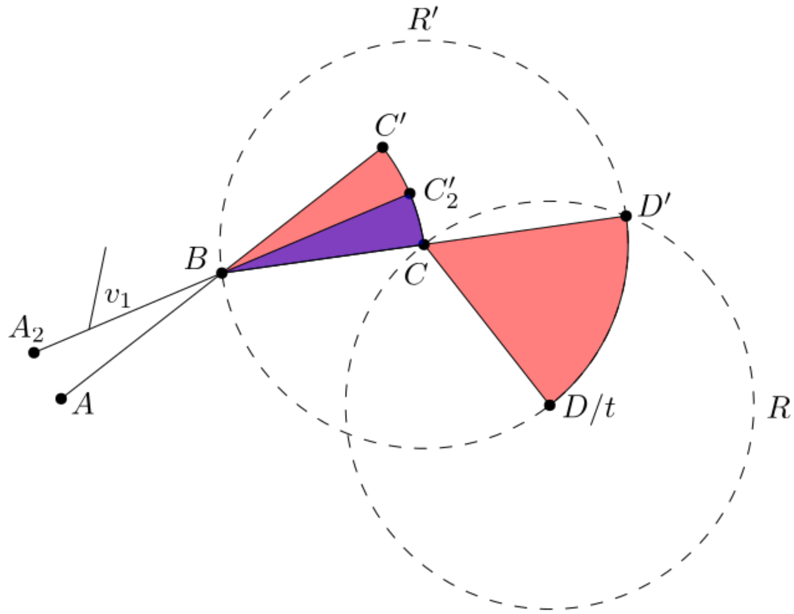


Figure 2

Let C'_2 denote the position of D' as it is on T_2 and let the standard formatting for the position of any arbitrary point along Trajectory T_n be Point P_n (unless otherwise specified). Now, let T_3 be the trajectory resulting from rotating point C counterclockwise along Circle R while maintaining the obstacle intersection endpoint of v_1 and the intersection of D'_3 along line segment $C_2D'_2$ (Figure 3).

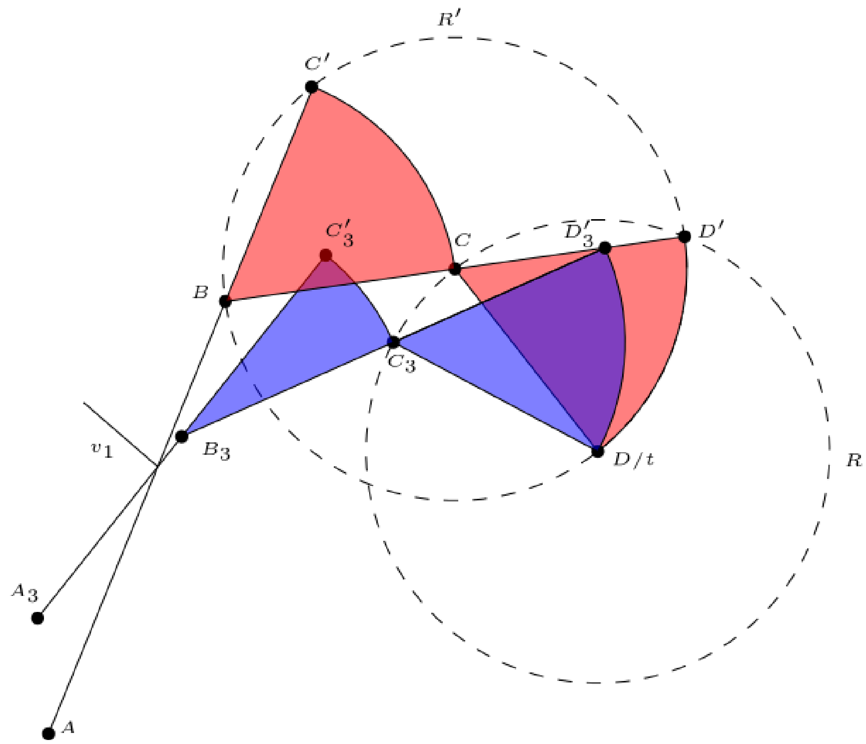


Figure 3

The movement stops either when:

I. ABC becomes a line segment

II. BCD becomes a line segment

III. Point B intersects an obstacle endpoint.

IV. AB intersects a secondary obstacle endpoint

V. BC intersects an obstacle endpoint

VI. CD intersects an obstacle endpoint

Let v_2 denote this obstacle endpoint if any of the latter 4 cases apply. If I or II apply, we have a trivial instance of collinearity so we now assume otherwise. We now argue that the circular sectors swept by BC and CD remain clear and thus each of the remaining cases is feasible.

We begin by arguing that B_3 and C_3 must lie in the wedge formed by $v_1B_2C_2$ containing D. Given that T_2 and T_3 are not identical, observe that $\angle B_2D'_3C_3 > 0$. Additionally observe that because T_2 was initially the clockwise articulations of the three line segments, C_3 now lies to the side of the line formed by $B_2D'_2$ which contains v_1. Note that $|B_2D'_2| = 2r$. Therefore, given nonzero movement, $|B_2D'_3| < 2r$, thus necessitating rotation of B_3. However, noting that B_3 is a collinear extension of the line segment $C_3D'_3$, and that, as established above, C_3 lies below the line formed by $B_2D'_2$, it also follows that B_3 lies to the side of the line formed by $B_2D'_2$ containing v_1.

Letting the line formed by $B_2D'_2$ and its perpendicular be referential axes, horizontal and vertical axis respectively, observe that the slope of the line formed by B_3C_3 causes B_3 to move in the negative y direction. Noting that movement is continuous, A_3 must move in the positive y direction to maintain the tangency of v_1 to A_3B_3. However, movement in the positive y direction for A_3 restricts any rotation of A_3 to the clockwise direction at all points where B lies to the side of the vertical line formed by the x coordinate of v_1 containing D (and it must, given the articulation of the trajectory). This is known through the differentiated form of the polar equation of a circle, which, if we let the x coordinate be arbitrary, allows us to describe the direction of the rotation given the direction of y movement and the initial point relative to the y axis. Thus, because movement is continuous, A_3 can only move in the clockwise direction of rotation around v_1, implying movement of B_3 in the clockwise direction of rotation around v_1 because A_3v_1 and v_1B_3 are collinear. Movement of B_3 in the clockwise direction of rotation around v_1 implies that B_3 lies to the side of the line formed by v_1B_2 containing D. Note that C_3 and D'_3 also lie to the side of the line formed by v_1B_2 because the angle formed by $D'_2v_1B_3$ is nonzero implying that the further collinear extensions of the line formed by v_1B_3 are also to the side of the line formed by v_1B_2. Additionally because the line formed by v_1B_3 is non parallel and both B_3 and C_3 lie to the side of the line formed by v_1B_2 containing D, D'_3 and all points of the circular sector centered at

B_3 must lie to the side of the line formed by v_1B_2 containing D. Observing that D'_3, C_3 and D all lie to the side of the line formed by v_1B_2 containing D, it also follows that all points in the circular sector centered at C_3 lie to the side of the line formed by v_1B_2 containing D.

Observe that every point of the circular sector centered at B_3 lies to the side of the line through v_1 and B_2 that contains Point C_3. They also lie to the side of the line through B_3 and C_3 that contains Point B_2. Therefore these points lie in the wedge, formed by the intersection of the lines through B_3C_3 and v_1B_2, which contains D'_3. Therefore these points also lie either to the side of the line segment B_2C_2. which contains C_3 or they lie in the wedge emanating from the circular sector swept at B_2. We know that the points which lie to the side of the line segment B_2C_2 containing C_3 are empty because it was swept while constructing T_3. We now argue that the remaining points in the circular sector swept at B_3 lie not only in the wedge emanating from the circular sector at B_2 but that they actually lie within the circular sector swept at B_2. Indeed let x_1 be a point of the circular sector swept at B_3 within the wedge emanating from the circular sector at B_2, let o_1 be the intersection of B_2C_2 and B_3x_1, and o_2 be the intersection of B_2C_2 and the arc formed by the circular sector centered at B_3 (Figure 4).

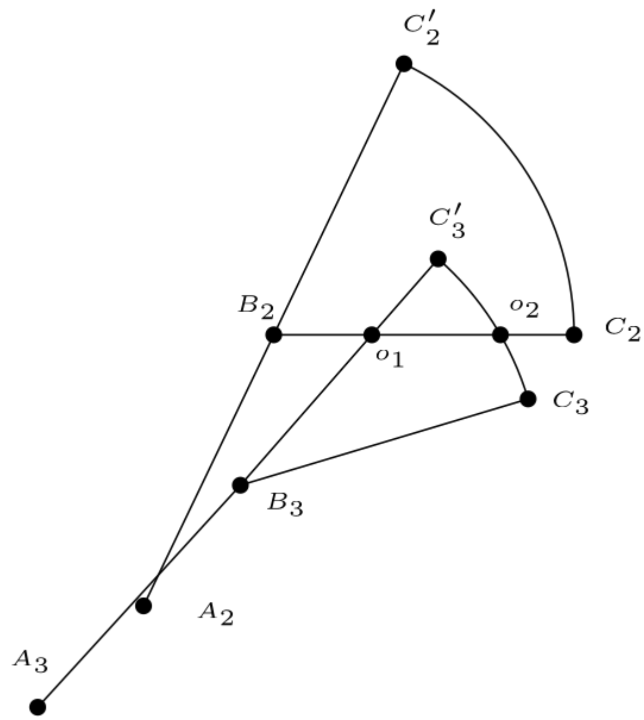


Figure 4

By the triangle inequality,

$|B_2x_1| <= |B_2o_1| + |o_1x_1|$

$= |B_2o_2| - |o_1o_2| + |B_3x_1| - |B_3o_1|$

$<= |B_2o_2| + |B_3x_1| - |B_3o_2|$

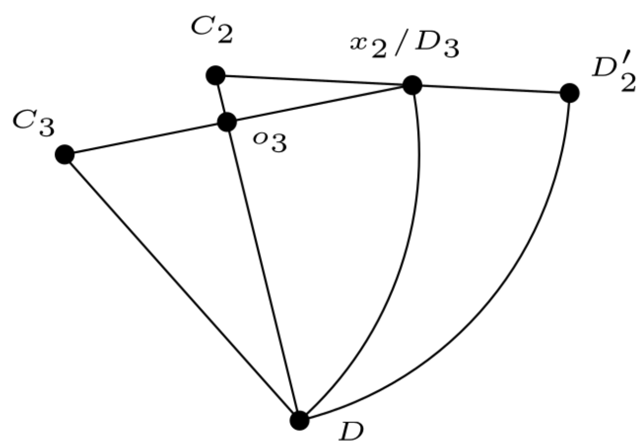
$<= |B_2C_2| - |B_3o_2| + |B_3x_1|$

$= |B_3x_1|$

$<= r$

Thus, because T_2 is a feasible probe trajectory, the remaining points in the circular sector swept at B_3, and therefore the entire circular sector swept by B_3, is clear.

Observe that every point of the circular sector centered at C_3 lies to the side of the line through C_2 and D'_2 which contains D. They also lie to the side of the line through C_3 and D which contains C_2. Therefore, these points either lie in the circular sector of radius r centered at D with arc endpoints C_2 and C_3 or they lie in the wedge emanating from the circular sector centered at C_2. We know that the sector centered at D is empty because it was swept while creating T_3. We argue that the remaining points of the sector centered at C_3 lie not only in the wedge emanating from the circular sector centered C_2 but actually lie in the circular sector itself. Indeed let x_2 be a point of the sector centered at C_3 which lies within the wedge emanating from the circular sector centered at C_2, and o_3 be the intersection of C_3x_2 and C_2D (Figure 5).



By the following triangle inequality,

$|C_2x_2| <= |C_2o_3| + |o_3x_2|$

$= |C_2D| - |o_3D| + |C_3x_2| - |C_3o_3|$

$<= |C_2D| + |C_3x_2| - |C_3D|$

$= |C_3x_2|$

$<= r$

Thus, both of the circular sectors swept by B_3 and C_3 are clear, and, because T_2 is also feasible, T_3 is a feasible trajectory for any of the four latter cases.

If III applies, we have achieved case I of the lemma. We now assume any of the latter three cases.

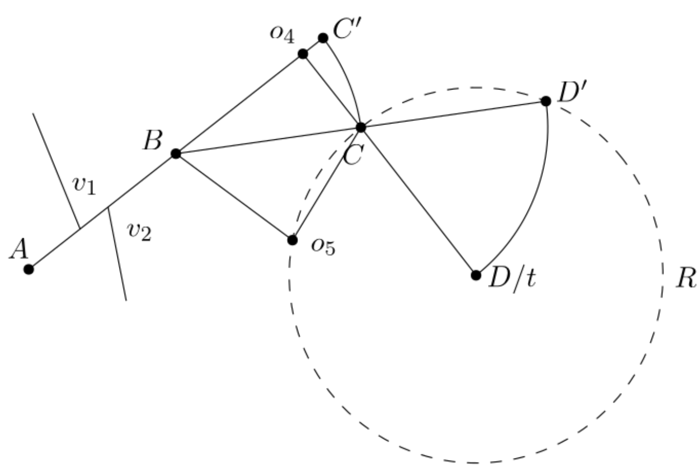
Assume IV. Let T_4 be the trajectory resulting from rotating segment CD counterclockwise around D such that both obstacle endpoint intersections are maintained. The movement stops when:

A. BCD becomes a line segment

B. BC intersects an obstacle endpoint

C. CD intersects an obstacle endpoint

Let v_3 denote the obstacle endpoint intersection if either of the latter two cases apply. If BCD becomes a line segment, we have a trivial instance of collinearity. We now assume otherwise. Clearly, in order for A_4B_4 to maintain the intersection of v_1 and v_2 B_4 must lie on the line formed by A_3B_3 during this rotation. However, the movement is additionally restricted by the fixed length (r) of segments BC and CD. We now argue that the fixed length restricts the movement of B_4 toward the direction of A_4.



Let B_3o_5 be the perpendicular dropped from B_3 onto Circle R such that $B_3o_5C_3$ forms a right angle. By the Pythagorean theorem,

$|B_3o_5|^2 + |o_5C_3|^2 = |B_3C_3|^2 = r^2$

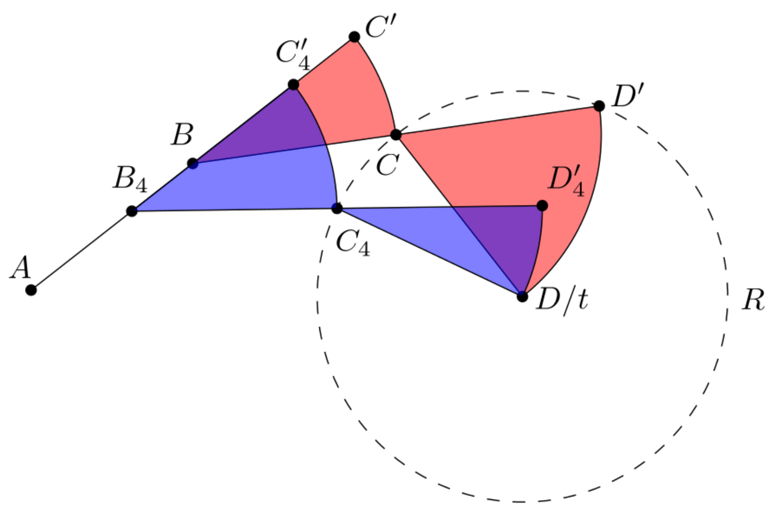
Therefore, given that $|o_5C_3| > 0$, $|B_3o_5| < r$ implying that as C_4 rotates counterclockwise, the distance between Circle R and the line segment formed by A_3B_3 decreases. Now, let o_4C_3 be the perpendicular dropped from C_3 onto line segment A_3B_3 where o_4 lies on the line formed by A_3B_3.

By the Pythagorean theorem,

$|o_4C_3|^2 + |o_4B_3|^2 = |B_3C_3|^2 = r^2$

Therefore, given that $|o_4B_3| > 0$, $|o_4C_3| < r$ implying that as B_4 travels in the direction of C'_4, the distance between Circle R and the line segment formed by A_3B_3 decreases. Thus, B_4 can only travel in the direction of A_4 to offset the decrease in distance caused by the counterclockwise rotation of C_4 and maintain a fixed length of r.

We now argue the feasibility of the latter cases by maintaining that the area swept by the circular sectors centered at B_4 and C_4 remain empty.



Observe that all points in the circular sector swept by B_3 lie to the side of the line formed by B_3 and B_4 which contains C_4. They also lie to the side of the line formed by B_4 and C_4 which contains B_3. Therefore, they either lie to the side of the line B_3 and C_3 containing B_4, or they lie in the wedge emanating from the circular sector at B_3. We know the area to the side of the line formed by B_3 and C_3 containing B_4 is empty because it was swept while creating T_4. We now argue that the remaining points of the sector centered at B_4 which lie within the wedge centered at B_3 actually lie within the circular sector centered at B_3. Let x_3 be a point of the sector centered at B_4 which lies within the wedge centered at B_3, and o_5 be the intersection of B_3C_3 and B_4x_3. By the triangle inequality,

$|B_3x_3| <= |B_3o_5| + |o_5x_3|$

$= |B_3C_3| - |o_5C_3| + |B_4x_3| - |B_4o_5|$

$<= |B_3C_3| + |B_4x_3| - |B_4C_3|$

$= |B_4x_3|$

$<= r$

Thus, because T_3 is a feasible probe trajectory, the remaining points in the circular sector swept at B_4, and therefore the entire circular sector swept by B_4, is clear.

(*) Observe that every point of the circular sector centered at C_4 lies to the side of the line through B_3 and C_3 containing D. They also lie to the side of the line through C_4 and D that contains C_3. Therefore, every point lies either within the circular sector of radius r centered at D with endpoints at C_3 and C_4, or they lie within the wedge emanating from the circular sector centered at C_3. We know the circular sector centered at D is empty because it was swept while creating T_4. We now argue that the remaining points of the circular sector not only lie within the wedge emanating from the circular sector C_3, but actually lie within the circular sector centered at C_3. Let x_4 be a point from the circular sector centered at C_4 which lies within the wedge emanating from the circular sector and o_6 be the intersection of C_4x_4 and C_3D. By the triangle inequality,

$|C_3x_4| <= |C_3o_6| + |o_6x_4|$

$= |C_3D| - |o_6D| + |C_4x_4| - |C_4o_6|$

$<= |C_3D| + |C_4x_4| - |C_4D|$

$= |C_4x_4|$

$<= r$ **(*)**

Thus, both of the circular sectors swept by B_4 and C_4 are clear, and, because T_3 is also feasible, T_4 is a feasible trajectory for any of the four latter cases. If B applies, we have achieved case 3 of the lemma. If C applies, we have achieved case 4 of the lemma.

We now assume V. Let T_6 be the result of rotating segment CD counterclockwise around D while maintaining obstacle endpoint intersections v_1 and v_2, of T_3. The movement stops when:

A. BCD becomes a line segment
B. AB intersects a second obstacle endpoint
C. BC intersects a second obstacle endpoint
D. CD intersects an obstacle endpoint
E. BC' intersects an obstacle endpoint

If BCD becomes a line segment, we have a trivial instance of collinearity. We now assume otherwise. Let v_4 denote the obstacle endpoint intersection if any of the latter 4 cases apply. Observe that the circular sector centered at B_6 must be empty because it was swept in the creation of T_6. We now argue that the circular sector centered at C_6 is empty and, given that the trajectory T_3 is feasible, that trajectory T_6 must therefore also be feasible. The proof is similar to (*). Observe that the points swept by the circular sector centered at C_6 lie to the side of the line through C_6 and D which contain C_3. They also lie to the side of the line through v_2 and C_3 that contain D. Therefore, the points either lie within the circular sector centered at D with radius r and endpoints C_3 and C_6, or they lie within the wedge emanating from the circular sector centered at C_3. We know the points within the circular sector centered at D must be empty because it was swept when creating T_6. We now argue that the remaining points of the circular sector centered at C_6 not only lie within the wedge emanating from the circular sector C_3, but actually lie within the empty circular sector centered at C_3. Let x_5 denote some point within the circular sector centered at C_6 which lies within the wedge emanating from the circular sector C_3. Let o_7 denote the intersection of C_6x_5 and C_3D. By the triangle inequality,

$|C_3x_5| <= |C_3o_7| + |o_7x_5|$

$= |C_3D| - |o_7D| + |C_6x_5| - |C_6o_7|$

$<= |C_3D| + |C_6x_5| - |C_6D|$

$= |C_6x_5|$

$<= r$

Thus, all points from the circular sector centered at C_6 are empty and, because T_3 is a feasible trajectory and all other areas have been swept, T_6 is a feasible trajectory. If B applies, we have case 3 of the lemma. If C applies, we have case 2 of the lemma. If D applies, we have case 5 of the lemma. If E applies, we have case 6 of the lemma.

We now assume VI. Observe that C_3D is fixed. Let trajectory T_7 be the result of rotating segment BC counterclockwise around C while maintaining obstacle endpoint intersections v_1 and v_2. The movement stops when:

A. ABC becomes a line segment
B. AB intersects a second obstacle endpoint
C. BC intersects an obstacle endpoint
D. CD' intersects an obstacle endpoint

If ABC becomes a line segment, we have a trivial instance of collinearity. We now assume otherwise. Let v_5 denote the obstacle endpoint intersection if any of the latter 3 cases apply. Observe that the circular sector centered at C_7 must be empty because it was swept in the creation of T_6. We now argue that the circular sector centered at B_7 is empty and, given that the trajectory T_3 is feasible, that trajectory T_7 must therefore also be feasible. The proof is similar to (*). Observe that the points swept by the circular sector centered at C_7 lie to the side of the line through B_7 and C_7 which contain B_3. They also lie to the side of the line through v_1 and B_3 that contain C_7. Therefore, the points either lie within the circular sector centered at C_7 with radius r and endpoints B_3 and B_7, or they lie within the wedge emanating from the circular sector centered at B_3. We know the points within the circular sector centered at C_7 must be empty because it was swept when creating T_6. We now argue that the remaining points of the circular sector centered at B_7 not only lie within the wedge emanating from the circular sector B_3, but actually lie within the empty circular sector centered at B_3. Let x_6 denote some point within the circular sector centered at B_7 which lies within the wedge emanating from the circular sector B_3. Let o_7 denote the intersection of B_7x_5 and B_3C_7. By the triangle inequality,

$|B_3x_6| <= |B_3o_7| + |o_7x_6|$

$= |B_3C_3| - |o_7C_3| + |B_7x_6| - |B_7o_7|$

$<= |B_3C_3| + |B_7x_6| - |B_7C_3|$

$= |B_7x_6|$

$<= r$

Thus, all points from the circular sector centered at B_7 are empty and, because T_3 is a feasible trajectory and all other areas have been swept, T_7 is a feasible trajectory. If B applies, we have case 4 of the lemma. If C applies, we have case 5 of the lemma. If D applies, we have case 7 of the lemma. Thus all cases of the lemma have been achieved given a fully articulated initial trajectory in which segments BC and CD have both been swept in the same direction around points B and C respectively.

We now address the fully articulated trajectory in which segments BC and CD have been swept in opposite directions (also such that point D coincides with point t). Assume feasible trajectory T_8, such that, without loss of generality, BC has been rotated clockwise around B to reach point C and segment CD has been swept counterclockwise around C to reach point D (Figure 8).

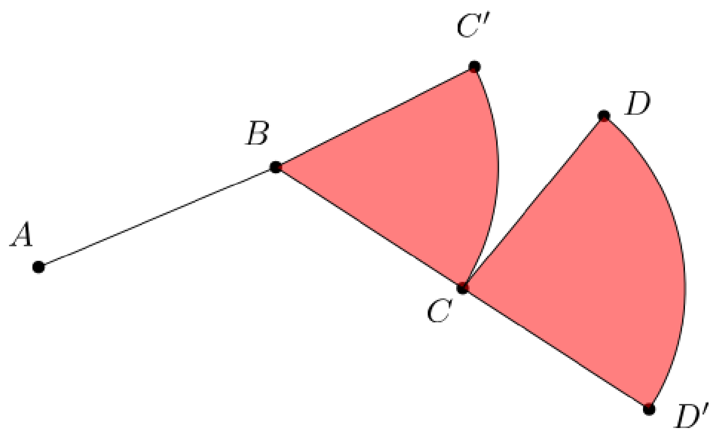


Figure 8

Let T_9 be the trajectory resulting from rotating line segment AB of T_8 around point B in clockwise direction until line segment AB intersects an obstacle endpoint v_6 outside Circle R'. Given that the area swept by line segment BC of T_9 to reach point C_9 is within that of T_8, T_9 is also a feasible trajectory.

Now, let T_{10} be the trajectory resulting from rotating segment BC counterclockwise around C while maintaining the obstacle intersection endpoint of v_6. The movement stops either when:

I. ABC becomes a line segment

II. AB intersects a second obstacle endpoint.

III. BC intersects an obstacle endpoint

IV. Point B intersects an obstacle endpoint.

If ABC becomes a line segment, we have a trivial instance of collinearity. We now assume otherwise. Let v_7 denote the obstacle endpoint intersection if any of the latter 3 cases apply. We now determine the feasibility of T_{10} by determining the feasibility of the circular sectors centered at B_{10} and C_{10}.

Observe that every point of the circular sector centered at B_{10} lies to the side of the line through v_6 and B_9 which contains B_{10}. They also lie to the side of the line through B_{10} and C_{10} which contains B_9. Therefore, every point of the circular sector centered at B_{10} either lies in the circular sector centered at C_{10} with radius r and endpoints B_9 and B_{10}, within the wedge emanating from the circular sector centered at B_9. We know that the circular sector centered at C_{10} is empty because it was swept while constructing T_{10}. We now argue that the remaining points within the wedge centered at B_9 actually lie within the empty circular sector centered at B_9, giving us an empty, and therefore feasible, circular sector centered at B_{10}. Let x_7 denote some point within the circular center centered at B_{10} which lies within the wedge emanating from the circular sector centered at B_9. Let o_8 be the intersection of $B_{10}x_7$ and B_9C_{10}. By the triangle inequality,

$|B_9x_7| <= |B_9o_8| + |o_8x_7|$

$= |B_9C_{10}| - |o_8C_{10}| + |B_{10}x_7| - |B_{10}o_8|$

$<= |B_9C_{10}| + |B_{10}x_7| - |B_{10}C_{10}|$

$= |B_{10}x_7|$

$<= r$

Thus, because T_9 is a feasible probe trajectory, the remaining points in the circular sector swept at B_{10}, and therefore the entire circular sector swept at B_{10}, is clear.

We now argue the feasibility of the circular sector swept at C_{10}. Observe that the circular sector swept at C_9 is feasible because T_9 is feasible. Also observe that the circular sector swept at C_{10} is a subsector of the circular sector swept at C_9, with endpoints D'_9 and D'_{10}. Given that D'_{10} lies within the circular arc of radius r centered at C_9 with endpoints D'_9 and D, the circular sector centered at C_{10} in T_{10} is within that of the empty circular sector centered at C_9 in T_9. Thus, all points from the circular sector centered at C_{10} are empty and, because T_9 is a feasible trajectory and all other areas have been swept, T_{10} is a feasible trajectory.

If IV applies, we have case 1 of the lemma. We now assume otherwise.

Assume II applies. Let T_{11} be the trajectory resulting from rotating segment CD clockwise around D such that both obstacle endpoint intersections are maintained. The movement stops when:

A. BCD becomes a line segment
B. BC intersects an obstacle endpoint
C. CD intersects an obstacle endpoint

Let v_8 denote this obstacle endpoint if any of the latter 2 apply. Clearly, in order for $A_{10}B_{10}$ to maintain the intersection of v_6 and $v_{7,}$ B_{11} must lie on the line formed by $A_{10}B_{10}$ during this rotation. However, the movement is additionally restricted by the fixed length (r) of segments BC and CD. We now argue that the fixed length restricts the movement of B_{11} toward the direction of A_{11}. Let $B_{10}o_9$ be the perpendicular dropped from B_{10} onto Circle R where o_9 lies on the line formed by $A_{10}B_{10}$. By the Pythagorean theorem,

$|B_{10}o_9|$^2 + $|o_9C_{10}|$^2 = $|B_{10}C_{10}|$^2 = r^2

Therefore, given that $|o_9C_{10}| > 0$, $|B_{10}o_9| < r$ implying that as C_{11} rotates clockwise, the distance between Circle R and the line segment formed by $A_{10}B_{10}$ decreases. Now, let $o_{10}C_{10}$ be the perpendicular dropped from C_{10} onto line segment $A_{10}B_{10}$ where o_{10} lies on the line formed by $A_{10}B_{10}$. By the Pythagorean theorem,

$|o_{10}C_{10}|^2 + |o_{10}B_{10}|^2 = |B_{10}C_{10}|^2 = r^2$

Therefore, given that $|o_{10}B_{10}| > 0$, $|o_{10}C_{10}| < r$ implying that as B_{11} travels in the direction of C'_{10}, the distance between Circle R and the line segment formed by $A_{10}B_{10}$ decreases. Thus, B_{11} can only travel in the direction of A_{11} to offset the decrease in distance caused by the counterclockwise rotation of C_{11} and maintain a fixed length of r.

We now argue the emptiness of the circular sectors at B_{11} and C_{11}.. Observe that all points in the circular sector swept by B_{11} lie to the side of the line formed by B_{10} and B_{11} which contains C_{10}. They also lie to the side of the line formed by B_{11} and C_{10} which contains B_{10}. Therefore, they either lie to the side of the line through B_{10} and C_{10} containing B_{11}, or they lie in the wedge emanating from the circular sector at B_{10}. We know the area to the side of the line formed by B_{10} and C_{10} containing B_{11} is empty because it was swept while creating T_{11}. We now argue that the remaining points of the sector centered at B_{11} which lie within the wedge centered at B_{10} actually lie within the circular sector centered at B_{10}. Let x_8 be a point of the sector centered at B_{11} which lies within the wedge centered at B_{11}, and o_9 be the intersection of $B_{10}C_{10}$ and $B_{11}x_8$. By the triangle inequality,

$|B_{10}x_8| <= |B_{10}o_9| + |o_9x_8|$

$= |B_{10}C_{10}| - |o_9C_{10}| + |B_{11}x_8| - |B_{11}o_9|$

$<= |B_{10}C_{10}| + |B_{11}x_8| - |B_{11}C_{10}|$

$= |B_{11}x_8|$

$<= r$

Thus, because T_{10} is a feasible probe trajectory, the remaining points in the circular sector swept at B_{11}, and therefore the entire circular sector swept by B_{11}, is clear.

Observe that every point of the circular sector centered at C_{11} lies to the side of the line through B_{10} and C_{10} containing D. They also lie to the side of the line through C_{11} and D that contains C_{10}. Therefore, every point lies either within the circular sector of radius r centered at D with endpoints at C_{10} and C_{11}, or they lie within the wedge emanating from the circular sector centered at C_{10}. We know the circular sector centered at D is empty because it was swept while creating T_{11}. We now argue that the remaining points of the circular sector not only lie within the wedge emanating from the circular sector C_{10}, but actually lie within the circular sector centered at C_1. Let x_9 be a point from the circular sector centered at C_{11} which lies within the wedge emanating from the circular sector and o_{10} be the intersection of $C_{11}x_9$ and $C_{10}D$. By the triangle inequality,

$|C_{10}x_9| <= |C_{10}o_{10}| + |o_{10}x_9|$

$= |C_{10}D| - |o_{10}D| + |C_{11}x_9| - |C_{11}o_{10}|$

$<= |C_{10}D| + |C_{11}x_9| - |C_{11}D|$

$= |C_{11}x_9|$

$<= r$

Thus, all points from the circular sector centered at C_{11} are empty and, because T_{10} is a feasible trajectory and all other areas have been swept, T_{11} is a feasible trajectory. If B applies, we have case 3 of the lemma. If C applies, we have case 4 of the lemma.

Assume III. Let T_{12} be the trajectory resulting from rotating segment CD clockwise around D of T_{10} such that both obstacle endpoint intersections are maintained. The movement stops when:

A. BCD becomes a line segment
B. AB intersects a second obstacle endpoint
C. BC intersects a second obstacle endpoint
D. CD intersects an obstacle endpoint

E. BC' intersects an obstacle endpoint

Let v_9 denote this endpoint if any of the latter 4 apply. If BCD becomes a line segment, we have a trivial instance of collinearity. We now assume otherwise. Observe that the circular sector centered at B_{12} must be empty because it was swept in the creation of T_{12}. We now argue that the circular sector centered at C_{12} is empty and, given that the trajectory T_{10} is feasible, that trajectory T_{12} must therefore also be feasible. The proof is similar to (*). Observe that the points swept by the circular sector centered at C_{12} lie to the side of the line through C_{12} and D which contain C_{10}. They also lie to the side of the line through v_7 and C_{10} that contain D. Therefore, the points either lie within the circular sector centered at D with radius r and endpoints C_{10} and C_{12}, or they lie within the wedge emanating from the circular sector centered at C_{10}. We know the points within the circular sector centered at D must be empty because it was swept when creating T_{12}. We now argue that the remaining points of the circular sector centered at C_{12} not only lie within the wedge emanating from the circular sector C_{10}, but actually lie within the empty circular sector centered at C_{10}. Let x_{10} denote some point within the circular sector centered at C_{12} which lies within the wedge emanating from the circular sector C_{10}. Let o_{11} denote the intersection of $C_{12}x_{10}$ and $C_{10}D$. By the triangle inequality,

$|C_{10}x_{10}| <= |C_{10}o_{11}| + |o_{11}x_{10}|$

$= |C_{10}D| - |o_{11}D| + |C_{12}x_{10}| - |C_{12}o_{11}|$

$<= |C_{10}D| + |C_{12}x_{10}| - |C_{12}D|$

$= |C_{12}x_{10}|$

$<= r$

Thus, all points from the circular sector centered at C_{12} are empty and, because T_{10} is a feasible trajectory and all other areas have been swept, T_{12} is a feasible trajectory. If B applies, we have case 3 of the lemma. If C applies, we have case 2 of the lemma. If D applies, we have case 5 of the lemma. If E applies, we have case 6 of the lemma. Thus we have achieved all extremal cases of the lemma in both variations of a fully articulated trajectory.

A summary of the cases can be found below:

Case #	**AB**	**BC**	**CD**	**BC'**	**CD'**	**B**	**Details**
1	**1**					**1**	Only 2 OEIs
2	**1**	**2**					2 OEIs + 1 OEI
3	**2**	**1**					2 OEIs + 1 OEI
4	**2**		**1**				2 OEIs + 1 OEI
5	**1**	**1**	**1**				1 OEI per seg.
6	**1**	**1**		**1**			1 OEI per seg.
7	**1**		**1**		**1**		1 OEI per seg.