Finite Coxeter Groups

Matthew Faw

Math 403 - Advanced Linear Algebra
Ezra Miller
Duke University

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Contents

1 Introduction

The focus of this paper is on finite groups of real Euclidean space generated by reflections. The study of symmetries and reflections find many applications, as symmetries occur in many places in everyday life. For example, one might be interested in studying the symmetries of regular polytopes, such as the square or the cube.

The primary goal of this paper is to develop the machinery needed to classify all finite Coxeter groups. This development will be the paper's primary focus. Once this machinery has been developed, there are many directions in which one could direct their study of Coxeter groups. One direction (which I did not take) could have been studying other classes of Coxeter groups, such as affine Coxeter groups or hyperbolic Coxeter groups. Instead of relaxing constraints, I explore placing different constraints on Coxeter groups – namely, the Crystallographic condition. This leads to studying Weyl groups, first in the context of Coxeter groups, but then also in the context of Lie groups and Lie algebras.

This plan of study was devised in order un to gain an appreciation for the intimate connections between the study of finite reflection groups and Lie theory.

Before I begin, I would like to credit the books used as sources for this material. [1] was my primary source for learning about Coxeter groups. A large portion of my discussion was inspired by this book. My main use of [3] was in the discussion of Weyl Groups of root systems. My discussion of Weyl groups in relation to Lie theory was largely inspired by [2].

Note that, due to page constraints, many propositions and theorems were not proved here. However, I have included references to the corresponding proofs in the books. My goal in presenting these propositions and theorems was to give the key ideas behind the construction. When I found a proof particularly informative and relatively brief, I included it. Some proofs, particularly out of [2], required an enormous amount of requisite material to even state. As a result, I tried to again give the main ideas presented in this material.

2 Background[1]

2.1 Defining Coxeter Groups

For the entirety of this paper, V will be a finite dimensional inner product space over $\mathbb{R}$. Recall that the orthogonal group $\mathcal{O}(V)$ is defined as

$$\mathcal{O}(V) = \{T \in GL_n \mid T^*T = TT^* = I\}$$

Intuitively, elements of $\mathcal{O}(V)$ are distance-preserving transformations of $n-$dimensional Euclidean space preserving a fixed point [Wikipedia, Orthogonal Group].

A **finite coxeter group** is a *finite*, *effective* subgroup G of $\mathcal{O}(V)$ *generated* by a set of *reflections*. Let us first begin by unpacking these definitions.

A subgroup G of $\mathcal{O}(V)$ is called **effective** if

$$V_0(G) = \bigcap\{V_T \mid T \in G\} = 0, \text{ where}$$
$$V_T = \{x \in V \mid Tx = x\}$$

Intuitively, this definition implies that V_T is the set of all vectors in V fixed by $T \in G$. Thus, an effective group is one for which the only vector in V that is fixed by every $T \in G$ is the origin.

A set $S = \{g_1, \ldots, g_k\} \subset G$ **generates** the group G if every element of G can be expressed as a combination (under the group operation) of finitely many elements of S and their inverses. We write $G = \langle g_1, \ldots, g_k \rangle$

A **reflection** is a linear transformation S such that, given an $n-1$-dimensional subspace of V denoted as $\mathcal{P}$

$$Sx = x \text{ if } x \in \mathcal{P}$$
$$Sx = -x \text{ if } x \in \mathcal{P}^{\perp}$$

Suppose $0 \neq r \in P^{\perp}$. Define the transformation S_r

$$S_r x = x - \frac{2(x,r)r}{r,r} \quad \text{for all } x \in V$$

We note that $S_r x = x$ if $x \in P$, and $S_r r = r - 2r = -r$. Since $P \bigcup\{r\}$ contains a basis for V, $S_r = S$. We say that S_r is a reflection *through* $\mathcal{P}$, or a reflection *along* r.

2.2 An illustrative example

In order to illustrate what a finite Coxeter group might look like, let us consider the group of reflections of the square pictured in Figure 1.

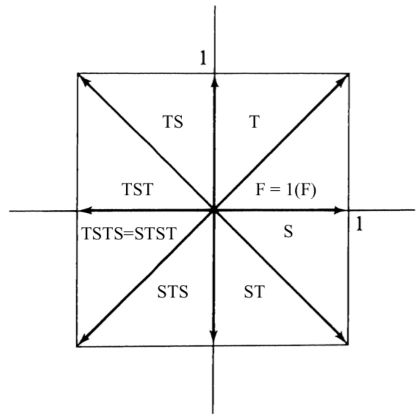

Figure 1: The dihedral group $\mathcal{H}_2^4$ viewed as the symmetry group of the square, image modified from of [1, pp. 8, 35]

This group is called the dihedral group, and can be defined as

$$\mathcal{H}_2^4 = \langle S, T \mid S^2 = 1 = T^2,\ TS = -ST \rangle$$

where S is a reflection through the vector $(1, 0)$, and T is a reflection through the vector $(1, 1)$. The figure shows the products of group elements that produce the reflections.

As we introduce new concepts through this paper, we will often come back to this example, as the dihedral group provides a great deal of intuition for the concepts discussed.

2.3 Roots and Root systems

The two vectors $\pm r$ that are perpendicular to $\mathcal{P}$, so that $S = S_r$, are called **roots** of G. One can quickly verify that $S_r = S_{\lambda r}$ for any $\lambda \in \mathbb{R}_{\neq 0}$. When convenient, we will take r to be a unit vector.

With the concept of roots introduced, a natural question arises – what effect do group elements have on roots? The following proposition provides some insight.

Proposition 1. *[1, Proposition 4.1.1] If r is a root of $G \leq \mathcal{O}(V)$ and $T \in G$, then Tr is a root of G. In fact, if $Tr = r'$, then $S_{r'} = TS_rT^{-1} \in G$.*

Yet another natural question one might ask is, must the roots corresponding to the generating reflections of the group actually span V?

Proposition 2. *[1, Proposition 4.1.2] Suppose $G \leq \mathcal{O}(V)$ is generated by reflections along roots $r_1, \ldots, r_k$. Then G is effective if and only if $\{r_1, \ldots, r_k\}$ contain a basis for V.*

These propositions motivate introducing a new concept. A **root system** Δ of G is defined to be

$$\Delta = \{Tr_i \mid T \in G,\ r_i \text{ is any root corresponding to any reflection in the generating set of } G\}$$

That is, Δ is the set of roots corresponding to the generating reflections, as well as the images of these roots under all group elements. It is not yet clear that Δ necessarily contains *every* root in G. Proving that statement will be developed in later sections. It is however of interest to determine if the root system's being finite is sufficient information to determine if the group itself is finite.

Proposition 3. *[1, Proposition 4.1.3] Suppose $G \leq \mathcal{O}(V)$ is generated by a finite set of reflections and is effective. Then Δ is finite if and only if G is finite*

NOTE: in the textbook, they present this proposition as a $\Rightarrow$ only, not an if and only if. However, the backwards direction follows trivially, since if G is finite, then there are only finitely many $TS_{r_i}T^{-1}$.

Let us assume for the remainder of the discussion that G is a finite coxeter group with root system Δ.

2.4 Partitions of Δ

Although the previous section established that Δ gives us quite a bit of information about the associated group, Δ seems to contain at least some redundant information. For instance, for any root $r \in \Delta$, the corresponding root $-r$ is also in Δ. Let's investigate how much information is *actually* necessary to understand the group.

Choose a vector $t \in V$ (which uniquely defines a hyperplane $t^\perp$) such that $(t, r) \neq 0$ for all roots $r \in \Delta$. We may thus partition Δ into two sets,

$$\Delta_t^+ = \{r \in \Delta \mid (t, r) > 0\}$$
$$\Delta_t^- = \{r \in \Delta \mid (t, r) < 0\}$$

The hyperplane $t^\perp$ is used to allow us to avoid the issue of having both $\pm r$ in our set Δ. Going back to the example of the dihedral group, Figure 2 displays one possible choice of t. Half of the roots lie in the positive halfspace, and the the other half (the negation of the roots in the positive halfspace) lie in the negative halfspace.

Though we have just removed some redundant information from Δ by partitioning the set, it is conceivable that the roots of Δ_t^+ are linearly dependent. This concern motivates the search for a basis (or at least a related notion) for Δ_t^+.

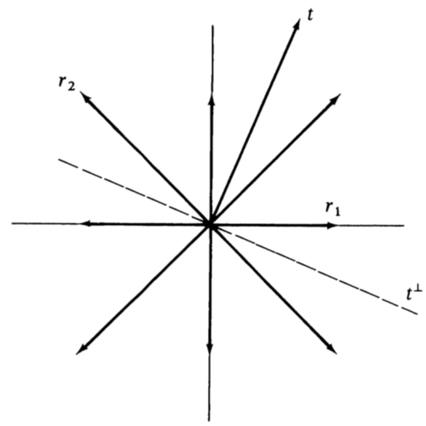

Figure 2: The line containing $t^{\perp}$ partitions the roots of $\mathcal{H}_2^4$ into two sets, Δ_t^+ and Δ_t^-. These two sets contain the roots in the positive and negative halfspaces of $t^{\perp}$, respectively. This figure is taken from [1, p. 41]

Let us choose a minimal set of roots Π of Δ_t^+ such that every $r \in \Delta_t^+$ can be written as a linear combination of elements in Π with all coefficients nonnegative. Π is called a ***t*-base** for Δ.

Proposition 4. *[1, Proposition 4.1.5] Suppose that $r_i, r_j \in \Pi$ with $i \neq j$, and let S_i denote the reflection along r_i. Then $S_i r_j \in \Delta_t^+$, and $(r_i, r_j) \leq 0$.*

Proposition 4 implies that the angle between any two roots in the t-base is obtuse.

We note that, in the example shown in Figure 2, $\Pi = \{r_1, r_2\}$. As expected, the angle between r_1 and r_2 is obtuse. We note that, in this example, Π is actually a basis for $\mathbb{R}^2$. Is this a coincidence, or is there something deeper going on here?

Theorem 1. *[1, Theorem 4.1.7] If Π is a t-base for Δ, then Π is a basis for V.*

Proposition 5. *[1, Proposition 4.1.8] There is only one t-base for Δ.*

From Π, given the previous results, it seems that we can extract a significant amount of information from this $t-base$. Thus, we call the roots of Π **fundamental roots** and the corresponding reflections **fundamental reflections.**

Theorem 2. *[1, Theorem 4.1.12] Fundamental reflections generate G.*

At this point, we have established that Π is a basis for V, and that the reflections corresponding to the roots of Π generate G. We now return to the question posed previously – does Δ contain *every* root of G?

2.5 Fundamental regions

A subset F of V is called a **fundamental region** for G in V if and only if

- F is open
- $F \cap TF = \emptyset$ if $1 \neq T \in G$.
- V $= \bigcup\{(TF)^{-} \cap X \mid T \in G\}$

How might we construct such a fundamental region for a Coxeter group?

Let us start by considering the dual basis $\Pi^* = \{s_1, \ldots, s_n\}$ of $\Pi = \{r_1, \ldots, r_n\}$, where s_i is constrained such that

$$(s_i, r_j) = \delta_{ij}$$

where $\delta_{i,j}$ is the Kroneker delta defined to be 1 if $i = j$, and 0 otherwise.

$$F_t = \{x \in V \mid x = \sum_{i,j} \lambda_i s_i, \lambda_i \in \mathbb{R}_{>0}\}$$

Since Π is a basis for V, we may write any $x \in V$ as $x = \sum_i \lambda_i r_i$. But note that

$$(x, r_j) = \sum_i \lambda_i (s_i, r_j) = \lambda_j.$$

Thus, we may write

$$\begin{aligned} F_t &= \{x \in V \mid (x, r_j) > 0 \ \forall r_j \in \Pi\} \\ &= \bigcap_{i=1}^{n} \{x \in V \mid (x, r_i)\} \end{aligned}$$

The above statement gives us some geometric intuition for F_t. Namely, F is the intersection of the open halfspaces determined by the halfspaces $\mathcal{P}_i = r_i^{\perp}$. By this construction, F_t is open and convex.

Theorem 3. *[1, Theorem 4.2.4] F_t is a fundamental region for the Coxeter group G.*

One may use the fundamental regions to establish the answer to our question:

Theorem 4. *[1, Theorem 4.2.5] Every reflection in G is conjugate in G to a fundamental reflection. Consequently, every root of G is in Δ.*

2.5.1 Examples

To obtain a better intuition for what a fundamental region corresponds to geometrically, it is useful to see some examples.

We again return to our example of the dihedral group. We note that, in Figure 1, the region is labeled as F is the fundamental regions. Note that the boundaries of the square do *not* bound the fundamental region – those boundaries are drawn to illustrate how the dihedral group corresponds to the symmetry group of the square.

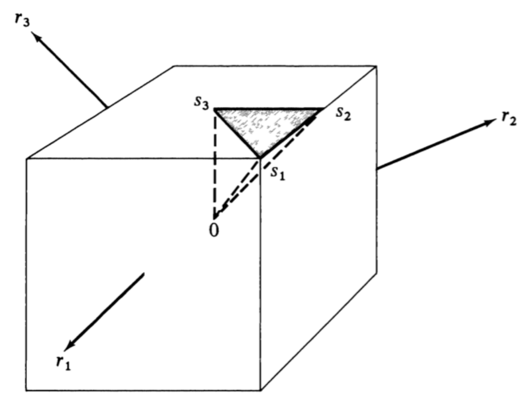

Figure 3: This figure shows the fundamental roots corresponding to the group of symmetries of the cube (labeled as $r1, r_2$, and r_3). The corresponding fundamental region is sketched in dashed lines. This figure comes from [1, p. 48]

Similarly, Figure 3 shows the fundamental region of the symmetries of the cube. It is easy to see that this fundamental region is formed from the intersections of the positive halfspaces of the planes orthogonal to the fundamental roots r_1, r_2, r_3.

3 Coxeter Graphs[1]

A natural question that arises from the study of finite Coxeter groups are.... How many different groups are there? Finitely many? Countably many? Uncountably many? The previous section established that Π gives us a "complete view" of G, meaning that from that set, we have generators for G and a basis for V. We are thus prepared to begin investigating how we might classify all finite Coxeter groups.

To begin, we ask what constraints can be placed on the angles between fundamental roots.

Proposition 6. *[1, Proposition 5.1.1] Given $r_i, r_j \in \Pi$, there exists an integer an integer*

$p_{ij} \geq 1$ such that

$$\frac{(r_i, r_j)}{\|r_i\| \|r_j\|} = -\cos\left(\frac{\pi}{p_{ij}}\right)$$

In fact, p_{ij} is the order of $S_i S_j$ as a group element.

We then define a **marked graph** as a finite set of points, called *nodes*, which can be connected by a branch, such that each branch is assigned a real number $p_{ij} > 2$.

We call a **Coxeter graph** a marked graph such that each p_{ij} is an integer. For convenience, we will often suppress edge weights $p_{ij} = 3$.

To eliminate ambiguity, we will say $p_{ii} = 1$ for all i, and that $p_{ij} = 2$ when there is no branch joining nodes i and j.

We may associate with each Coxeter graph a **quadratic form**:

$$Q(\lambda_1, \ldots, \lambda_k) = \sum_{i,j} -\cos\left(\frac{\pi}{p_{ij}}\right) \lambda_i \lambda_j$$

Writing the coefficient on $\lambda_i \lambda_j$ in the i, j position of a matrix defines an *adjacency matrix* A for the Coxeter graph.

We call a marked graph **positive definite** if and only if $Q(\lambda_1, \ldots, \lambda_k)$ is positive definite. That is, if $x^T A x > 0$ for all $0 \neq x \in V$.

Theorem 5. *[1, Theorem 5.1.3] Every Coxeter graph is positive definite.*

Proof. The key step of the proof is as follows: Write out the quadratic form, and note that the roots $r_i \in \Pi$ are linearly independent. $\square$

In fact, one can make an even stronger claim:

Proposition 7. *[1, Proposition 5.1.6] A (nonempty) subgraph of a marked graph is also positive definite.*

G is called **irreducible** if Π is not the disjoint union of two nonempty orthogonal subsets.

Proposition 8. *[1, Proposition 5.1.4] A coxeter graph is connected if and only if G is irreducible.*

As a result of Proposition 8, it is sufficient to study only connected graphs.

3.1 Examples

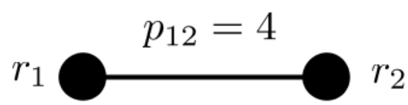

Figure 4: This is the Coxeter graph associated with the Dihedral Group $\mathcal{H}_2^4$

For example, the Coxeter graph associated with the dihedral group $\mathcal{H}_2^4$ is shown in Figure 4. The associated adjacency matrix is

$$A = \begin{pmatrix} 1 & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 1 \end{pmatrix}$$

However, connected positive-defineite Coxeter graphs cannot contain cycles, since their quadratic form is not positive definite. For example,

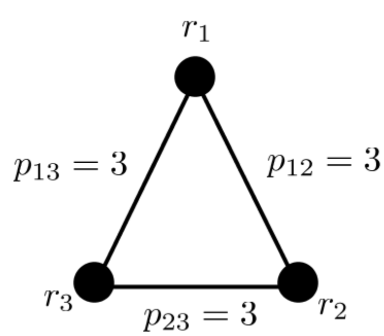

Figure 5: An example of a connected graph that is not a Coxeter graph.

$$B = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

Since the sum of the rows is 0 the rows are linearly dependent, so the quadratic form cannot be positive definite.

4 Construction of all Finite Coxeter groups[1]

Using the machinery developed in the previous sections, we can actually classify *all* finite Coxeter groups using Coxeter graphs. [1, Theorem 5.1.7] establishes that the Coxeter graph of an irreducible Coxeter group *must* appear in Figure 6. Conversely, [1, Theorem 5.3.1] establishes that every graph in Figure 6 is a graph of a Coxeter group.

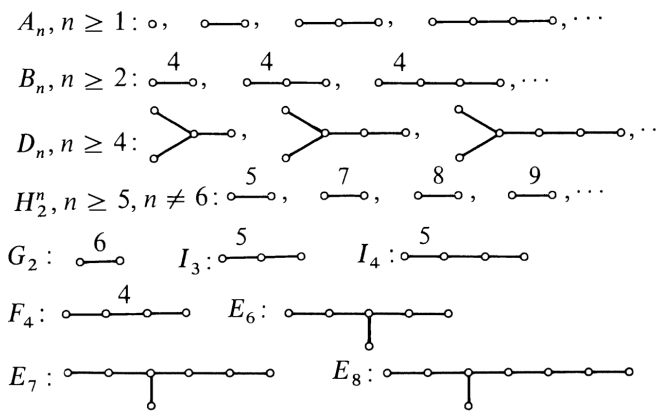

Figure 6: All finite coxeter graphs. This figure is taken from [1, p. 57]

Although the proofs of these theorems require a significant amount of case-work, the results are quite beautiful. By developing the concept of a root system and from that, fundamental roots, one can classify every finite Coxeter group, and describe each group in an elegant fashion.

5 The Crystallographic condition – Weyl Groups

Now that we have classified all finite Coxeter groups, we are in a position to explore some related topics. One subject of interest arises from placing additional constraints on the Coxeter graphs.

5.1 Applied to a group[1]

A **lattice** L in V is a discrete set of points obtained by taking all linear combinations of some basis of V. A subgroup G of $O(V)$ is said to be **crystallographic** if there is a lattice

L invariant under G:

$$Tx \in L \ \forall x \in L, \ \forall T \in G. \tag{1}$$

Proposition 9. *[1, Proposition 5.2.1] If G is crystallographic, then each p_{ij} is one of $1, 2, 3, 4,$ or 6.*

Proof. Choose a basis in V such that $S_i S_j$ is represented by the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

where

$$A = \begin{pmatrix} \cos(\frac{2\pi}{p_{ij}}) & -\sin(\frac{2\pi}{p_{ij}}) \\ \sin(\frac{2\pi}{p_{ij}}) & \cos(\frac{2\pi}{p_{ij}}) \end{pmatrix}$$

$\mathrm{tr}(S_i S_j) = 2\cos(\frac{2\pi}{p_{ij}}) + (n-2)$ must be an integer. One may check that this condition can only hold for $p_{ij} = 1, 2, 3, 4, 6$. □

From Proposition 9, the only irreducible Coxeter groups that *could* be crystallographic groups are

$$A_n, B_n, D_n, G_2, F_4, E_6, E_7, \text{ and } E_8.$$

Now suppose G is irreducible, and that each $p_{ij} \in \{1, 2, 3, 4, 6\}$. Recall that we are free to choose weights of roots in Coxeter groups. Thus,

$$\begin{aligned} &\text{If } p_{ij} = 3, \text{ then } \|r_i\| = \|r_j\| \\ &\text{If } p_{ij} = 4, \text{ then } \|r_i\| = \sqrt{2}\|r_j\| \text{ or } \|r_j\| = \sqrt{2}\|r_i\| \\ &\text{If } p_{ij} = 6, \text{ then } \|r_i\| = \sqrt{3}\|r_j\| \text{ or } \|r_j\| = \sqrt{3}\|r_i\| \end{aligned}$$

One may confirm by inspecting the connectivity of the Coxeter graphs in Figure 6 that such choices for root weights are possible. The fundamental roots will be assigned equal lengths for Coxeter groups A_n, D_n, E_6, E_7, and E_8. For B_n, we assign root weights as

$$\sqrt{2}\|r_1\| = \|r_2\| = \ldots = \|r_n\|.$$

For G_2, we assign

$$\sqrt{3}\|r_1\| = \|r_2\|.$$

For F_4,

$$\|r_1\| = \|r_2\| = \sqrt{2}\|r_3\| = \sqrt{2}\|r_4\|.$$

We are now prepared to establish the following:

Theorem 6. *[1, Theorem 5.2.2] If G has an irreducible Coxeter graph $A_n, B_n, D_n, G_2, F_4, E_6, E_7,$ and E_8, then G satisfies the Crystallographic condition.*

Proof. Let L be the lattice with basis Π, so that

$$L = \left\{ \sum_{i=1}^{n} \alpha_i r_i \mid \alpha_i \in \mathbb{Z} \right\}$$

It is sufficient to show that fundamental reflections applied each fundamental root is in the lattice. For $p_{ij} = 3,\ \|r_i\| = \|r_j\|$, and $(r_i, r_j) = -\frac{1}{2}\|r_i\|\|r_j\|$, and so

$$S_i r_j = r_j - \frac{2(r_j, r_i) r_i}{(r_i, r_i)} = r_j + r_i \in L.$$

Similar methods may be used to confirm the cases $p_{ij} = 4, 6$ also hold. As a result, we have that $S_i L = L$ for all i. By Theorem 2, $TL = L$ for all $T \in G$. Thus, G is crystallographic. $\square$

5.2 Applied to a root system[3]

We say that a root system Δ is **crystallographic** if it satisfies the additional requirement that

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}\ \forall \alpha, \beta \in \Delta.$$

The group generated by all such $R_i \in \Delta$ is called the **Weyl group** of Δ.

One may conduct an analysis similar to that in [1, Ch. 5] in order to classify the roots of the crystallographic root system. Performing this analysis, [3] notes that the Weyl groups that arise from the analysis are precisely the reflection groups such that all $p_{ij} = 1, 2, 3, 4,$ or 6, which implies that the Weyl groups are exactly the crystallographic reflection groups. However, there are distinct crystallographic root systems B_n and C_n, both having as Weyl group the group previously labeled as B_n.

The constraints placed on the roots in Section 5.1 tell us that if W is irreducible Weyl group, then there can be at most two root lengths. If the roots are not equal lengths, then the ratio of root lengths is either $\sqrt{2}$ or $\sqrt{3}$. We may add this information to the Coxeter graph by directing an arrow from the shorter root towards the shorter of the two roots (no arrows if roots are equal lengths). By convention, we replace the labels 4 and 6 by double and triple edges, respectively. The resulting graph is called a **Dynkin diagram**. As noted in [Wikipedia, Dynkin diagram], such diagrams are used in the classification of certain Lie algebras.

6 Weyl Groups in a different context[2]

In the previous section, Weyl Groups were defined as subgroups of $\mathcal{O}(V)$ with respect to a root system in V. However, Weyl Groups also arise in both in the study of Lie Algebras and of Lie Groups. As noted many times in [1] and [3], the study of Lie Algebras and Coxeter groups are intimately related. The purpose of this section will be to explore some alternate settings in which Weyl Groups arise.

6.1 Lie Group setting

In order to define a Weyl Group in the setting of Lie groups, some definitions are necessary.

A **matrix Lie group** is a subgroup G of $GL_n(\mathbb{C})$ such that, if A_m is any sequence of matrices in G, and A_m converges (entry-wise) to a matrix A, then either A is in G or A is singular.

We note that the above definition is equivalent to saying that G is a **closed subgroup** of $GL_n(\mathbb{C})$. It does not necessarily, however, have to be closed in $M_n(\mathbb{C})$.

A matrix Lie group T is called a **torus** if T is isomorphic to k copies of the group $S^1 \cong U(1)$ for some k. That is, if $T \cong (S^1)^k$.

As an example, let T be the group of unitary diagonal $n \times n$ matrices with determinant 1. We may write:

$$T = \{\mathrm{diag}(u_1, \dots, u_{n-1}, (u_1 \dots u_{n-1})^{-1}) \mid u_i \in \mathbb{C},\ |u_i| = 1\}$$

Then T is isomorphic to $(S^1)^{n-1}$ by construction.

A matrix Lie group G is called **compact** if it is compact in the usual topological sense as a subset of $M_n(\mathbb{C})$.

A matrix Lie group G is called **connected** if, for each A, B in G, there exists a continuous path $A(t)$, where $a \leq t \leq b$, which lies in G, such that $A(a) = A$ and $A(b) = B$.

Given these definitions, we are prepared to discuss the definition of a maximal torus. Let K be a compact, connected matrix Lie group, and let T be a subgroup of G. T is called a **maximal torus** if it is a torus, and is not properly contained in any other torus in K.

As an example, take $K = SU(n)$, the group of $n \times n$ unitary matrices with determinant 1 (one may check that $SU(n)$ is in fact a connected, compact matrix Lie group). Let T be the subgroup discussed in the previous example. If T were contained in some other torus

$S \subset SU(n)$, then each $t \in T$ commutes with each $s \in S$. One can show that choosing t to have distinct eigenvalues actually implies that s is diagonal in the standard basis. But this implies that $s \in T$, and thus T is maximal.

The **normalizer** of T, denoted $N(T)$, is the group defined:

$$N(T) = \{x \in K \mid xTx^{-1} \in T\}$$

In this setting, the **Weyl Group** W with respect to T is defined to be

$$W = N(T)/T.$$

At present, it is not clear why the Weyl group is defined this way. To understand, we must look back to Weyl groups in a different setting.

6.2 Lie Algebra setting

The **Lie Algebra** $\mathfrak{g}$ of a matrix Lie group G is the set of all matrices such that $e^{tX} \in G$ for all $t \in \mathbb{R}$. One can show (Theorem 3.20, [2]) that the **bracket** of $\mathfrak{g}$ is given by

$$[X, Y] = XY - YX, \;\; X, Y \in \mathfrak{g}.$$

There are many concepts in the study of Coxeter groups that have related concepts in the study of Lie algebras. One such concept is a root.

Let $\mathfrak{t}$ be the Lie Algebra associated with T. An element α of T is called a **real root** of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if $\alpha \neq 0$ and there exists a nonzero element X of $\mathfrak{g}$ so that

$$[H, X] = i\langle\alpha, H\rangle X \;\forall H \in \mathfrak{t}.$$

We will denote the set of all real roots as R.

For each real root α, there is a corresponding **real coroot** H_α defined

$$H_\alpha = 2\frac{\alpha}{\langle\alpha, \alpha\rangle}$$

We may define the adjoint map as a linear map such that, for all $A \in G$, $\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{Ad}_A(X) = AXA^{-1}$$

Proposition 10. *[2, Proposition 11.35] For each $\alpha \in R$, there is an element $x \in N(T)$ such that*

$$\mathrm{Ad}_x(H_\alpha) = -H_\alpha$$

and such that

$$\mathrm{Ad}_x(H) = H$$

for all $H \in \mathfrak{t}$ *for which* $\langle \alpha, H \rangle = 0$*. Thus, the adjoint action of* x *on* $mathfrakt$ *is the reflection* s_α *about the hyperplane orthogonal to* α*.*

Recall that we say G acts **effectively** on T if

$$g \cdot t = t \; \forall t \in T \Rightarrow g = 1_G$$

Theorem 7. *[2, Theorem 11.36] Suppose* T *is a maximal torus in* K*. Then the Weyl group acts effectively on* $\mathfrak{t}$ *and this action is generated by the reflections* s_α*,* $\alpha \in R$ *in Proposition 10.*

One can actually say a stronger statement. One can additionally show that for each reflection s_α, there is an associated element of the Weyl group defined in Section 6.1. As a result, the Weyl groups defined in these two settings are actually isomorphic.

7 Summary

By constructing and analyzing the root system associated with finite Coxeter groups, we found that there is a tremendous amount of information that can be obtained from reasoning about these roots. We discussed the classification of all finite Coxeter groups using Coxeter graphs. We then studied Weyl groups defined in several different settings, and observed interesting relationships among the definitions of Weyl groups in the Lie algebra, Lie group, and orthogonal group settings.

References

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