HW1: Math Fundamentals

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Instructions

- Only edit this .tex file where it says "**Answer**: TODO." Use this for the "Resources Consulted" section and all the questions after that. *Exception*: add your name at the top of this page, and you can import other LaTeX packages if needed.
- Do not remove or eliminate any of the instructions or questions in this file, including the text in the first two pages (i.e., this "Instructions" page and the next page about "Resources Consulted").
- *Unless stated otherwise, use exactly one page for each answer* (including the original question itself), even if the answer doesn't require the full page.[1] The "Resources Consulted" should be on page 2, then the first question should be answered only on page 3, the second question answered only on page 4, etc. Do not exceed one page per question and answer.
- Submit the PDF on Gradescope.

[1] This makes it easier for us to grade on Gradescope.

Resources Consulted

Question: Please describe which resources you used while working on the assignment. You do not need to cite anything directly part of the class (e.g., a lecture, the CSCI 545 course staff, or the readings from a particular lecture). Some examples of things that could be applicable to cite here are: (1) did you collaborate with a classmate; (2) did you use resources like Wikipedia or Google Bard in any capacity; (3) did you build upon someone's code? When you write your answers, explain not only the resources you used but HOW you used them. If you believe you did not use anything worth citing, *you must still state that below in your answer* to get full credit.

Answer:
- Utilized ChatGPT to get latex code for venn diagrams in question 3.
- Utilized "The Multivariate Gaussian Distribution", for assistance for problem 5b using the definition.
- Used "Linear Algebra Review and Reference", for help with basic notations and calculations.

1. Given the matrix

$$A = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

find the eigenvalues and eigenvectors. Do it by hand and confirm it with Python. Show your Python code and output.

Answer:

$$(\lambda I - A)x = 0, x \neq 0$$
$$|(\lambda I - A)| = 0$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = 0$$

$$(\lambda - 3)(\lambda - 2) - 1 = 0$$
$$\lambda^2 - 5\lambda + 5 = 0$$

$$\boldsymbol{\lambda_{1,2} = \frac{5 \pm \sqrt{5}}{2} = 3.61803399, 1.38196601}$$

for $\lambda_1 = \frac{5+\sqrt{5}}{2}$

$$\left(\begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = 0$$

$$\frac{-1+\sqrt{5}}{2} x_{1,1} - x_{1,2} = 0; -x_{1,1} + \frac{1+\sqrt{5}}{2} x_{1,2} = 0$$

$$\mathbf{x_{1,1} = 1, x_{1,2} = \frac{-1+\sqrt{5}}{2}; x_1 = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix}}$$

for $\lambda_2 = \frac{5-\sqrt{5}}{2}$

$$\left(\begin{bmatrix} \frac{5-\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = 0$$

$$\frac{-1-\sqrt{5}}{2} x_{1,1} - x_{1,2} = 0; -x_{1,1} + \frac{1-\sqrt{5}}{2} x_{1,2} = 0$$

$$\mathbf{x_{2,1} = 1, x_{2,2} = \frac{-1-\sqrt{5}}{2}; x_2 = \begin{bmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{bmatrix}}$$

Python Code

```
import numpy as np

a = np.array([[3, 1], [1, 2]])
eigenvalues, eigenvectors = np.linalg.eig(a)
print(eigenvalues)
print(eigenvectors[:, 0])
print(eigenvectors[:, 1])
```

Output:

```
[3.61803399 1.38196601]
[0.85065081 0.52573111]
[-0.52573111 0.85065081]
```

Normalizing the eigenvectors computed by hand:

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{0.85065081} \\ \mathbf{0.52573111} \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{0.52573111} \\ \mathbf{-0.85065081} \end{bmatrix} = \begin{bmatrix} \mathbf{-0.52573111} \\ \mathbf{0.85065081} \end{bmatrix}$$

2. A matrix A is called symmetric iff $A^T = A$ and skew-symmetric iff $A^T = -A$. Prove that:

   (a) The sum of two symmetric matrices is a symmetric matrix

   (b) The sum of two skew-symmetric matrices is a skew-symmetric matrix

   Answer:

   a)
   Assume we have two symmetric matrices: **A** and **B**. Which sum up to

   $$A + B$$

   The sum of these two matrices is also a symmetric matrix if,

   $$A + B = (A + B)^T$$

   Using the property of transposes:

   $$(A + B)^T = A^T + B^T$$

   And we know that

   $$A = A^T; B = B^T$$

   since these are both assumed to be symmetric. Thus, the sum of the two matrices are symmetric since,

   $$A + B = A^T + B^T = (A + B)^T$$

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   b)
   Assume we have two skew-symmetric matrices: **A** and **B**. By definition,

   $$A = -A^T; B = -B^T$$

   The sum of the two matrices is also skew-symmetric if,

   $$A + B = -(A + B)^T$$

   And we know that by the property of transpose,

   $$-(A + B)^T = -A^T - B^T$$

   so,

   $$A + B = -A^T - B^T = -(A + B)^T$$

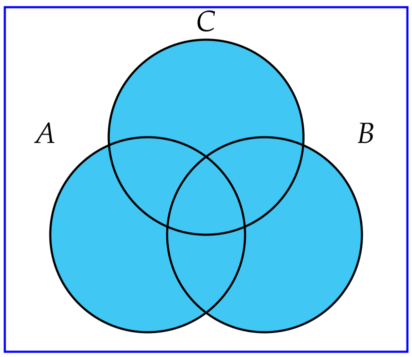
   meaning the sum of the skew-symmetric matrices is also skew-symmetric.

3. Express each of the following events in terms of the events A, B and C:

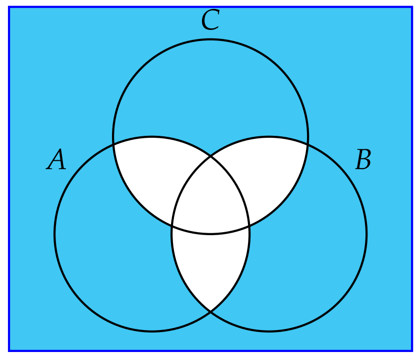
    (a) at least one of the events A, B, C occurs;
    (b) at most one of the events A, B, C occurs;
    (c) none of the events A, B, C occurs;
    (d) all three events A, B, C occur;
    (e) exactly one of the events A, B, C occurs;
    (f) events A and B occur, but not C;

    In each case, write an expression in set notation and draw the Venn diagrams.

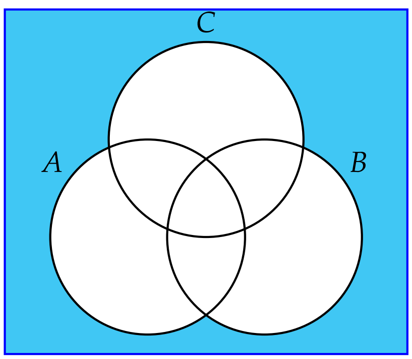
a) $A \cup B \cup C$



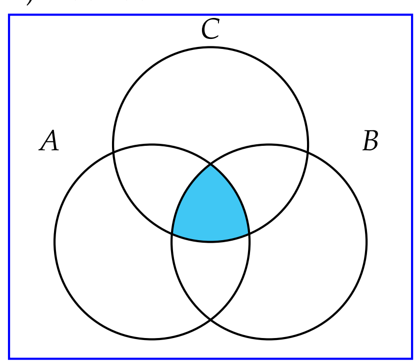
b) $(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C) \cup (A' \cap B' \cap C')$



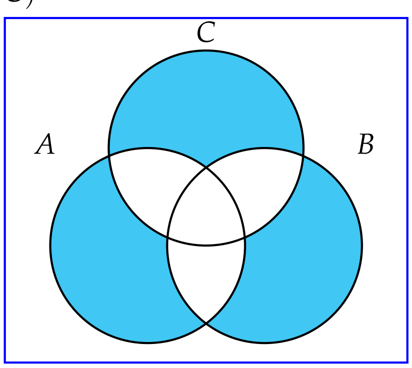
c) $A' \cap B' \cap C'$



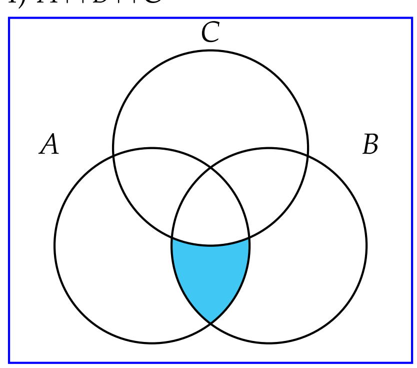
d) $A \cap B \cap C$



e) $(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$



f) $A \cap B \cap C'$



4. Determine the value of α such that

$$f_X(x) = \frac{\alpha}{e^x + e^{-x}}$$

is a *valid* probability density function.
Hint:

$$\int \frac{1}{e^x + e^{-x}} dx = \arctan(e^x) + C$$

Answer:

$f_X(x)$ is a *valid* probability density function if:

1) $f_X(x) \geq 0$ for all $x \in R$

2) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

for Condition 2 to hold,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \alpha \arctan(e^x) + C\Big|_{-\infty}^{\infty} = \alpha \left(\frac{\pi}{2} + C) - (0 + C)\right) = 1$$

so,

$$\alpha = \frac{2}{\pi}$$

we know that Condition 1 will always hold assuming α is positive or zero since the denominator ($e^x + e^{-x}$) is positive for all $x \in R$ and the numerator is a constant.

$$\frac{\frac{2}{\pi}}{e^x + e^{-x}} \geq 0 \text{ for all } x \in R$$

thus,

$$f_x(x) = \frac{\frac{2}{\pi}}{e^x + e^{-x}} \text{ is a \textit{valid} probability density function}$$

5. Two random vectors $\mathbf{x_1}$ are said to be uncorrelated if:

$$P = \mathbb{E}[(\mathbf{x_1} - \bar{\mathbf{x}}_1)(\mathbf{x_2} - \bar{\mathbf{x}}_2)^T] = 0.$$

Show that

(a) Independent random vectors are uncorrelated. *Hint:* Using the definition of independence, we have:

$$\mathbb{E}[\mathbf{x_1}\mathbf{x_2}^T] = \mathbb{E}[\mathbf{x_1}]\mathbb{E}[\mathbf{x_2}]^T$$

Note: The converse is not true. Uncorrelated random vectors may not be independent.

(b) Uncorrelated jointly Gaussian random vectors are independent.
Hint: The off-diagonals of the covariance matrix are 0 if $\mathbf{x_1}$ and $\mathbf{x_2}$ are uncorrelated.
Note: You can use another page if you need more space to answer (b).

Answer:
a)
Assuming we have two independent random vectors, x_1 and x_2,

$$\mathbb{E}[(\mathbf{x_1} - \bar{\mathbf{x}}_1)(\mathbf{x_2} - \bar{\mathbf{x}}_2)^T] = \mathbb{E}[x_1 x_2{}^T - x_1 \bar{x_2}{}^T - \bar{x_1} x_2{}^T + \bar{x_1}\bar{x_2}{}^T]$$

using the definition of independence,

$$= \mathbb{E}[x_1 x_2{}^T] - \mathbb{E}[x_1 \bar{x_2}{}^T] - \mathbb{E}[x_2{}^T \bar{x_1}] + \mathbb{E}[\bar{x_1}\bar{x_2}{}^T]$$

$$= \mathbb{E}[x_1]\mathbb{E}[x_2{}^T] - \mathbb{E}[x_1]\mathbb{E}[\bar{x_2}{}^T] - \mathbb{E}[x_2{}^T]\mathbb{E}[\bar{x_1}] + \mathbb{E}[\bar{x_1}]\mathbb{E}[\bar{x_2}{}^T]$$

$$= \mathbb{E}[x_2{}^T](\mathbb{E}[x_1] - \mathbb{E}[\bar{x_1}]) - \mathbb{E}[\bar{x_2}{}^T](\mathbb{E}[x_1] - \mathbb{E}[\bar{x_1}])$$

$$= (\mathbb{E}[x_2{}^T] - \mathbb{E}[\bar{x_2}{}^T])(\mathbb{E}[x_1] - \mathbb{E}[\bar{x_1}])$$

using the definition of expected value, $\mathbb{E}[\bar{x}] = \bar{x}$ and $\mathbb{E}[x] = \bar{x}$, the equation simplifies to,

$$= (\mathbb{E}[x_2] - \mathbb{E}[\bar{x_2}])^T(\mathbb{E}[x_1] - \mathbb{E}[\bar{x_1}])$$

$$= (\bar{x_2} - \bar{x_2})^T(\bar{x_1} - \bar{x_1}) = 0$$

meaning that the two random vectors are uncorrelated, proving that independent random vectors are uncorrelated.

b)
Assuming uncorrelated random vectors,

$$\sum = \begin{bmatrix} {\sigma_1}^2 & 0 \\ 0 & {\sigma_2}^2 \end{bmatrix}, \sum^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/{\sigma_2}^2 \end{bmatrix}$$

$$det(\sum) = {\sigma_1}^2{\sigma_2}^2$$

The PDF for multivariate Gaussian random vectors is given by:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n det(\sum)}} exp(-\frac{1}{2}(x-\mu)^T \sum^{-1}(x-\mu))$$

Substituting $\sum^{-1}$ into the joint PDF,

$$\begin{aligned} f_{x_1,x_2}(x_1,x_2) &= \frac{1}{\sqrt{(2\pi)^2{\sigma_1}^2{\sigma_2}^2}} exp(-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \begin{bmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/{\sigma_2}^2 \end{bmatrix} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix}) \\ &= \frac{1}{\sqrt{2\pi{\sigma_1}^2}} exp(-\frac{1}{2}\frac{(x_1-\mu_1)^2}{{\sigma_1}^2}) \cdot \frac{1}{\sqrt{2\pi{\sigma_2}^2}} exp(-\frac{1}{2}\frac{(x_2-\mu_2)^2}{{\sigma_2}^2}) \\ &= f_{x_1}(x_1) f_{x_2}(x_2) \end{aligned}$$

thus, $\boldsymbol{f_{x_1,x_2}(x_1,x_2) = f_{x_1}(x_1) f_{x_2}(x_2)}$. Meaning that if two jointly Gaussian random vectors are said to be uncorrelated, they can be expressed as the product of each random variable's individual Gaussian PDF showing they are independent.