Fungi: A typed, functional language for programs that dynamically name their own cached dependency graphs

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Many programming language techniques for incremental computation employ programmer-specified *names* for cached information. At runtime, each name identifies a "cache location" for a dynamic data value or a sub-computation; in sum, these cache location choices guide change propagation and incremental (re)execution.

We call a cache location name *precise* when it identifies at most one value or subcomputation; we call all other names *imprecise*, or *ambiguous*. At a minimum, cache location names must be precise to ensure that change propagation works correctly; yet, reasoning statically about names in incremental programs remains an open problem.

As a first step, this paper defines and solves the *precise name problem*, where we verify that incremental programs with explicit names use them precisely. We formalize and implement our approach in the form of Fungi, a typed, functional language for programs that name their own cached dependency graphs. In particular, we give a core calculus for Fungi with a refinement type and effect system that we prove sound: every well-typed Fungi program uses names precisely. We also demonstrate that this type system is expressive by verifying example programs that compute over efficient representations of incremental sequences and sets. Beyond verifying these programs, our type system also describes their dynamic naming strategies, e.g., for library documentation purposes. Finally, we give a (sound) bidirectional variant of our proposed type and effects system, and a closely related prototype implementation of Fungi in Rust, as a deeply embedded DSL.

1 INTRODUCTION

In many computing scenarios, the result of running a program is not merely a result, but also *a cached dependency graph of how the program computed it*. For instance, language-based incremental computation techniques strive to improve the asymptotic time complexity of programs using dynamic dependency graphs that provide dynamic structure for *function call caching*, *cache invalidation* and *change propagation* [Acar 2005; Acar et al. 2006a,b, 2009; Acar and Ley-Wild 2009; Hammer and Acar 2008; Hammer et al. 2009; Ley-Wild et al. 2008; Chen et al. 2012].

Relatedly, the implementation techniques that underly some functional reactive programming (FRP) systems also build and maintain dynamic dependency graphs [Cooper and Krishnamurthi 2006; Krishnaswami and Benton 2011; Krishnaswami 2013; Czaplicki and Chong 2013]. Meanwhile, software build systems and other large-scale, batch-processing systems often cache their results, if not the dependency graph that produced them [Bhatotia et al. 2011, 2015; Erdweg et al. 2015b].

In all of these systems, running a program builds (and then helps maintain) a cache of intermediate results and often, a *dynamic dependency graph*, which represents the steps of its "most recent" execution. In fact, this graph undergoes incremental re-execution, where its existence permits the system to soundly avoid recomputing the work of *unaffected* subcomputations.

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In the most general variants of these systems, the programming model demands that incremental algorithms employ explicit *names*, where each name identifies a cache location for dynamic data (a dynamic pointer allocation), or a cache location for dynamic sub-computations, e.g., the arguments, dependencies, and results of a recursive function call [Hammer et al. 2015].

Through an incremental algorithm's input data structures, these names enter the program's dynamic data flow, where they mix with dynamic computations, perhaps getting combined with one another, dynamically; eventually, the program *uses* each name at a dynamic allocation site within the computation. Specifically, each dynamic allocation forms a *uniquely-named* node in the program's (implicitly-built) dependency graph, which caches function results and the dynamic dependency edges (runtime effects) that relate the cached data and subcomputations.

As described above, (dynamically-computed) names afford the program explicit control over naming its own dependency graph nodes. Further, each cache location name n simultaneously describes both *cache allocation* (as above) and *cache eviction*, where the former corresponds with the "functional view" of the program (as it were running from-scratch), and the latter behavior corresponds with the "incremental view" of the program, from within its *imperative*, incremental runtime system. In particular, names guide incremental cache invalidation, where the incremental runtime system follows the program's naming choices to *re-associate* an existing name with new content (either new data, or a new subcomputation). This eviction removes prior results and cached dependency edges in the graph, replacing them with updated versions, related in part or *not at all* with the prior content.

This control comes at a cost: Explicit allocation names are prone to misuse, where each such misuse might undermine type safety, change propagation soundness, or change propagation efficiency, depending on the system (Sec. 9 discusses past work in depth).

In this paper, we present Fungi, a typed, functional language for programs that name their own cached dynamic dependency graphs. Fungi's type system defines a verification problem, *the precise name problem*, for programs that compute with explicit allocation names. Specifically, the type system attempts to prove that for all possible inputs, in every execution, each name allocated by the program is *precise*, and not ambiguous. For an evaluation derivation $\mathcal{D}$, we say that an allocated pointer name is precise when it has at most one definition in $\mathcal{D}$, and otherwise we say that a name is ambiguous (imprecise) when it identifies two or more data structures and/or sub-computations.

Across *distinct, successive* incremental re-evaluations of a Fungi program, with a series of incrementally-related evaluation derivations $\mathcal{D}_1, \mathcal{D}_2, \ldots$, each name n may be associated with different values, when and if this content changes incrementally (e.g., first value v_1, then later, v_2 such that $v_1 \neq v_2$). In these situations, name n is still precise exactly when, *in each execution* $\mathcal{D}_i$, the Fungi's effects are consistent with purely-functional execution, and in particular, these effects associate at most one unique value v_i with the pointer named n.

In summary, Fungi's type and effect system *statically* reasons about one execution at a time (not incremental re-executions) and proves that *dynamically*, in each such execution, each name is used uniquely, if at all. Before outlining our approach and contributions, we give background on incremental programming with names, with examples.

1.1 Background: Incremental computation with names

Incremental programs employ explicit cache location names (1) to choose cached allocation names *deterministically*, and (2) to witness *dynamic independence* between subproblems, thereby improving incremental reuse.

Deterministic allocation via precise names. The first role of explicit names for incremental computing concerns *deterministic store allocation*, which permits us to give a meaningful definition to *cached* allocation. To understand this role, consider these two evaluation rules (each of the judgement form $\sigma; e \Downarrow \sigma'; v$) for reference cell allocation:

$$\frac{\ell \notin \mathrm{dom}(\sigma)}{\sigma; \mathtt{ref}_1(v) \Downarrow \sigma\{\ell \mapsto v\}; \mathrm{ref}\ \ell}\ \mathrm{alloc}_1 \qquad \frac{n \notin \mathrm{dom}(\sigma)}{\sigma; \mathtt{ref}_2(n, v) \Downarrow \sigma\{n \mapsto v\}; \mathrm{ref}\ n}\ \mathrm{alloc}_2$$

The left rule is conventional: $\mathtt{ref}_1$ allocates a value v at a store location ℓ; because the program does not determine ℓ, the implementor of this rule has the freedom to choose ℓ any way that they wish. Consequently, this program is *not* deterministic, and hence, *not* a function. Because of this fact, it is not immediately obvious what it means to *cache* this kind of dynamic allocation using a technique like *function caching* [Pugh and Teitelbaum 1989], or techniques based on it. To address this question, the programmer can determine a name n for the value v, as in the right rule. The point of this version is to expose the naming choice directly to the programmer, and their incremental algorithm.

In some systems, the programmer chooses this name n as the hash value of value v. This naming style is often called "hash-consing". We refer to it as *structural naming*, since by using it, the name of each ref cell reflects the *entire structure* of that cell's content.[1] By contrast, in an incremental system with explicit names, the programmer generally chooses n to be related to the *evaluation context of using* v, and often, to be *independent* of the value v itself, providing *dynamic independence* for subcomputations may otherwise be treated as dependent. Notably, since structural names lack *any* independence with the content that they name, structural names cannot provide dynamic independence. We give one example of an explicit naming strategy below, and many more in Sec. 2.

Dynamic independence via explicit *names.* Programmers of incremental computations augment ordinary algorithms with names, permitting incremental computing techniques like memoization and change propagation to exploit dynamic independence in the cached computation.

As a simple illustrative example, consider the left version of `rev` below, the recursive program that reverses a list. After being reversed, suppose the input list to `rev` undergoes incremental insertions and removals of list elements. To respond to these input changes, we wish to update the output (the reversed list) by using the algorithm for `rev` below, along with general-purpose incremental computing techniques, including memoization of recursive calls to `rev`, marked by the `memo` keyword:

```
rev : List -> List -> List
rev l r = match l with
 | Nil        ⇒ r
 | Cons(h,t)  ⇒
    let rr = ref(hash(r), r) in
    memo(rev !t (Cons(h, rr)))
```

Structural naming:
`rr`'s name is a (hash) function of `r`.

```
rev : List -> List -> List
rev l r = match l with
 | Nil        ⇒ r
 | Cons(n, h, t) ⇒
    let rr = ref(n, r) in
    memo(rev !t (Cons(n, h, rr)))
```

Explicit naming:
`rr`'s name is `n`, and is *independent* of `r`.

The function `rev` reverses a list of alternating `Cons` cells and reference cells. (Within each run of `rev`, the input and output data is immutable; however, across successive, incremental runs of this program, these reference cells generally change value.) Apart from the placement of these reference operations and the `memo` keyword, this functional program is conventional: it reverses the input using an accumulator `r` that holds the reversed prefix of the input list. In the `Cons` case,

[1] The structural hashing approach is closely related to Merkle trees, the basis for revision-control systems like `git`.

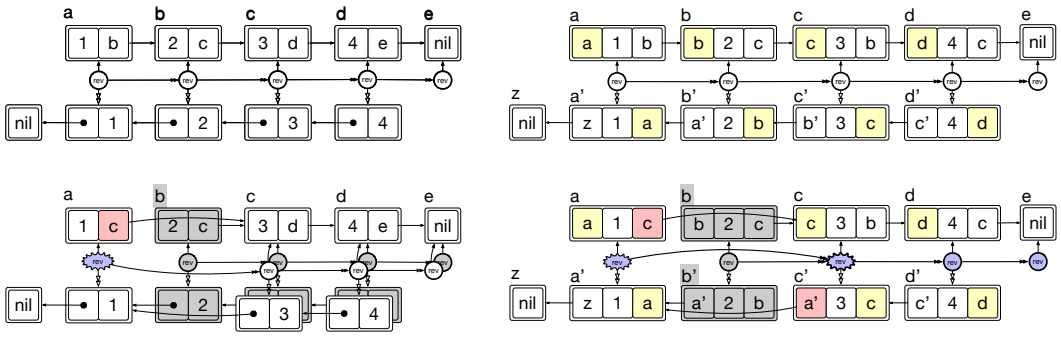

Fig. 1. Four runs of `rev`: two successive runs each of the *structural name version* (left hand side, top and bottom), and the *explicit name version* (right hand side, top and bottom) over a common input list $[1, 2, 3, 4]$, and input change (removal of element 2, at position b). Each thunk for `rev` contains several effect edges: to the `ref` that it observes (upward), to the ref cell that it allocates (downward), and to the thunk it allocates and calls as tail-recursive subroutine (rightward).
Table 1 gives further details, listing the arguments and results of each call to `rev` shown above.

Structural-name version of `rev`					**Explicit**-name version of `rev`				
Run 1			Run 2		Run 1			Run 2	
Input: [1,2,3,4]			Input: [1,3,4]		Input: [1,2,3,4]			Input: [1,3,4]	
arg. `l`	arg. `r`	⇓	arg. `r`	⇓	arg. `l`	arg. `r`	⇓	arg. `r`	⇓
a	[]	[4,3,2,1]	[]	[4,3,1]	a	z	d′	z	d′
b	[1]	[4,3,2,1]	*unobserved*		b	a′	d′	*unobserved*	
c	[2,1]	[4,3,2,1]	[1]	[4,3,1]	c	b′	d′	a′	d′
d	[3,2,1]	[4,3,2,1]	[3,1]	[4,3,1]	d	c′	d′	c′	d′
e	[4,3,2,1]	[4,3,2,1]	[4,3,1]	[4,3,1]	e	d′	d′	d′	d′

Table 1. Four runs of `rev`: two successive runs each of the *structural name version* (left), and the *explicit name version* (right) over a common input list $[1, 2, 3, 4]$, and input change (removal of element 2, at position b). Note that Fig. 1's graph illustration relates the input names ($\{a, b, c, d, e\}$), allocated pointers ($\{a', b', c', d'\}$) and list contents.

the function `rev` calls itself on the (dereferenced) tail `!t`, pushing the list element `h` onto the accumulator `r`.

During incremental (re)executions, we intend for change propagation to exploit the fact that each recursive step of `rev` is, *in a sense*, independent of the other recursive steps. However, the left version's use of `memo` fails to achieve this effect on its own because, in another sense, this version's recursive steps are *dependent* on each other. In fact, encoding the desired cache eviction and change propagation strategy requires an additional name `n` in each `Cons` cell, as shown in the right hand version of `rev`.

To understand why, we consider how change propagation re-evaluates and how memoization reuses the cached calls to function `rev` from the initial run, on an initial input $[1, 2, 3, 4]$. For each recursive call to `rev`, the system cache stores a thunk; each thunk records its result, the input list, and reversed accumulator at that point in the dynamic call graph. We illustrate these initial input-output structures in Fig. 1 (top left).

Suppose an external user mutates the input at pointer a, removing element 2, and that they demand the updated output of `rev`. Behind the scenes, change propagation re-evaluates the step(s) of `rev` where the input changed; in this case, it was when the program first observed a, the changed pointer, in the first recursive call. When this thunk reevaluates in Run 2, it re-allocates a reference cell to hold [1], which (structurally) matches its allocation from the initial run. Next, the first `rev` thunk (again) recurs, attempting to match a prior recursive call with the same pair of arguments. However, these arguments have fallen out of alignment with those of the first run. Table 1 lists the arguments and results of each thunk shown in Fig. 1.

Due to this "misalignment" with the prior run's argument pairs, the second run repeatedly fails to `memo`-match the cached calls to `rev`; likewise, each attempt to hash-cons `rr` after the first fails, since each value of `r` is distinct, structurally, from those of the prior run: All those prior allocations contained lists with the element 2, now absent. Consequently, change propagation will evaluate *all the calls* that include and follow the input list change; in this case, there are *four* such places (at a, c, d, and e); moreover, for all but the first, the misaligned arguments also identify structurally-distinct thunks, leaving "stale" copies of old output and thunks in the cache, without an obvious eviction strategy (What event should "trigger" this eviction? What if the element 2 is later re-inserted?).

The failure of `rev` to exploit memoization above is a simple example of a general pattern, also found in the output accumulators of quicksort and quickhull, and more generally, in many cases when a recursive algorithm allocates and consumes dynamic data structures. To address this memoization issue, we uniquely name each input `Cons` cell, and use these names to determine the pointers used to store (and overwrite) the output [Hammer et al. 2015].

In the explicit-name version (right hand version of `rev`), each name `n` localizes any change to the accumulator argument `r`, since the dynamic dependency graph will record the fact that `rev` allocates, but never accesses, these reference cells. More conceptually, these reference cells' (independent) names and dynamic dependency structure witness that, in fact, the recursive steps of running `rev` on a list are independent from each other. Unlike the structural-name version above, where we reevaluate $O(n)$ thunks per *single* input insertion or deletion, the explicit-name version here only requires $O(1)$ re-evaluations per input insertion or deletion, which is optimal.

Fig. 1 illustrates the initial and updated input and outputs of this right hand version; in particular, it shows that change propagation avoids reconstructing the full output prefix, by instead reflecting the input mutation (at a) into a corresponding output mutation (at c′), leaving no residual cache garbage, except for the old call to `rev` at b, and its `ref` allocation. Were we to re-insert 2, change propagation would restore this (temporarily stale) thunk, and its allocation of b′. As in the first update, this next update would only require $O(1)$ thunk evaluations.

In Sec. 3.4, we show that quickhull (for computing convex hull) has a similar accumulator structure; again, we use names to separate the subproblems, yet still efficiently accumulate their results. In place of "names", earlier work variously used the terms *(allocation) keys* [Acar 2005; Hammer and Acar 2008; Acar and Ley-Wild 2009] and *indices* [Acar et al. 2006a,b], but the core idea is the same.

1.2 Fungi types and effects: Static reasoning for dynamic names

The correct use of names is critical for incremental computing. At a minimum, such names must be precise to ensure that a program is type safe and that the runtime system works correctly, viz., that caching and change propagation work correctly together. Beyond precision, dynamic cache eviction behavior also warrants static approximations and associated verification techniques. In larger systems, as we build incremental algorithms that compose, we seek a way of statically organizing and documenting different cache location name choices made throughout (e.g., see the quick hull examples in Sec. 3.4).

Meanwhile, existing literature lacks static reasoning tools for cache location names (see Sec. 9 for details). As a first step in this direction, Fungi offers a refinement type and effects system, toured in the next two sections, and defined formally in Sec. 5. As our examples demonstrate, these (dependent) refinement types provide a powerful descriptive tool for statically organizing and documenting choices about cache location names.

Contributions:

- We define the *precise name* verification problem for programs that use names to uniquely identify their dynamic data and sub-computations. To formalize this problem and implement our proposed solution, we design Fungi, a core calculus for such programs.
- Fungi programs employ a novel refinement type and effect system to describe and verify their use of explicit allocation names. The design of this type system is our main contribution.
- To refine its types, Fungi employs separate name and index term languages, for statically modeling names and name sets, respectively (Sec. 5). The name-set indices were inspired by set-indexed types in the style of DML [Xi and Pfenning 1999]; we extend this notion with polymorphism over kinds and index functions; the latter was inspired by abstract refinement types [Vazou et al. 2013].
- In Sec. 3, we tour a collections library for sequences and sets. These programs demonstrate the descriptive power of Fungi's types and effects.
- To validate our approach theoretically, we give a definition of *precise effects* for an evaluation derivation $\mathcal{D}$. We prove that for all well-typed Fungi programs, every evaluation derivation $\mathcal{D}$ consists of precise effects (Sec. 7).
- Drawing closer to an implementation, we give a bidirectional version of the type system that we show is sound and complete with respect to our main type assignment system (Sec. D).
- Based on the bidirectional formulation of the type system, we implement a closely-related prototype of Fungi in Rust, as a deeply-embedded DSL (Sec. 8).

2 OVERVIEW OF OUR APPROACH

Fungi's type system statically approximates the *name set* of each first-class name, and the *write set* of each sub-computation. By reasoning about these sets statically, we rule out programs that contain dynamic type errors, enforce precise names where desired, and provide descriptive affordances to the programmer, so that she can specify her program's nominal effects to others (e.g., in the types of a module signature).

Without a suitable type system, programmer-chosen allocation names can quickly give rise to imperative state, and unfortunately, to the potential for unintended errors that undermine the type safety of a program.

Re-allocation at *different* types:

```
let r1 = ref(n, (0,1))
let r2 = ref(n, 2)
```

Re-allocation at the *same* type:

```
let r1 = ref(n, (0,1))
let r2 = ref(n, (2,2))
```

What if a common name `n` is used simultaneously for two reference cells of two different types? Should a program that re-allocates a name at different types be meaningful? (e.g., see the first program to left, above). We say, "no", such programs are not meaningful: They use a name `n` imprecisely at two *different* types, undermining the basic principle of type safety.

On the other hand, suppose a name `n` is re-allocated at the *same* type (e.g., if `r1` and `r2` had the same type, as in the second program), we may consider this behavior meaningful, though

imperative. In this case, we want to know that the program is imperative, and that we should not cache its behavior in contexts where we expect precise names.

Name sets and write sets. Consider the following typing rules, which approximate Fungi's full type system, and illustrate its basic design principles:

$$\frac{\Gamma \vdash \nu_{\mathtt{n}} : \mathsf{Nm}[X] \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{ref}(\nu_{\mathtt{n}}, v) : \mathsf{Ref}(A) \rhd X} \qquad \frac{\begin{array}{c}\Gamma \vdash e_1 : A \rhd X \\ \Gamma, x : A \vdash e_2 : B \rhd Y \\ \Gamma \vdash (X \perp Y) \equiv Z : \textbf{NmSet}\end{array}}{\Gamma \vdash \mathtt{let}\,(e_1, x.e_2) : B \rhd Z}$$

The rules conclude with the judgement form $\Gamma \vdash e : A \rhd X$, where the set X approximates the set of written names (it may contain more names than those written, but must contain every name that is written).

The first rule types a reference cell allocation named by programmer-chosen value $\nu_{\mathtt{n}}$, whose type $\mathsf{Nm}[X]$ is *indexed by* an approximate name set X from which name value $\nu_{\mathtt{n}}$ is drawn; in the rule's conclusion, this name set X serves as the allocation's *write set*.

The second rule gives `let` sequencing: a premise judges the equivalence of name set Z (at sort **NmSet**) with sets X and Y, which index the write sets of e_1 and e_2, respectively. For names to be precise, the written sets X and Y must be disjoint, which we notate as $X \perp Y$. This final premise is capturing the constraint that Fungi programs be pure: it is not derivable when $X \not\perp Y$. In particular, this premise is not derivable for either of the two program fragments shown above, where in each, the name $\mathtt{n}$ is used in distinct allocations.

Name functions, higher-order apartness, write scope. In most cases, the source of names for a program is its input data structure(s), not some fixed set of predefined name variables, as in the pair of two-line programs above. Further, names are *not* linear resources in most incremental programs. In fact, it is common for precise programs to consume a name more than once (e.g., see listings for quickhull in Sec. 3.4). To disambiguate multiple writes of the same name, implementations commonly employ *distinct memo tables*, one for each unique function. In this paper, we develop a more general notion for Fungi programs: *name functions*.

In particular, we define a higher-order account of *separation*, or *apartness* (the term we use throughout) to compose precise programs. Just as apart names $\mathtt{n} \perp \mathtt{m}$ do not overwrite each other (when written, the two singleton write sets are disjoint), *apart* name functions $\mathtt{f} \perp \mathtt{g}$ give rise to *name spaces* that do not overwrite each other, i.e., $\mathtt{f} \perp \mathtt{g} \iff \forall x.\mathtt{f}\,x \perp \mathtt{g}\,x$. That is, $\mathtt{f} \perp \mathtt{g}$ when the images of $\mathtt{f}$ and $\mathtt{g}$ are always disjoint name sets, for any preimage.

To streamline programs for common composition patterns, Fungi's type system employs a special, ambient name space, the *write scope*. As illustrated in Sec. 3, name functions and write scopes naturally permit name-precise sub-computations to compose into larger (name-precise) computations: Just as functions provide the unit of composition for functional programming, (apart) name functions provide the unit of composition for (precise) naming strategies.

3 EXAMPLES

In this section, we demonstrate larger example programs and their types, focusing on a collections library of sequences and sets.

3.1 Examples of datatype definitions: Sequences and sets

We present nominal datatype definitions for sequences and finite maps, as first presented in Hammer et al. [2015], but without types to enforce precision.

We represent sequences as *level trees*. Unlike lists, level trees permit efficient random editing [Hammer et al. 2015; Headley and Hammer 2016], and their balanced binary structure is useful for organizing efficient incremental algorithms [Pugh and Teitelbaum 1989]. We represent sets and finite maps as binary hash tries, which we build from these level trees.

Level tree background: A level tree is a binary tree whose internal binary nodes are each labeled with a level, and whose leaves consist of sequence elements. A level tree *well-formed* when every path from the root to a leaf passes through a sequence of levels that decrease monotonically. Fortunately, it is simple to pseudo-randomly generate random levels that lead to probabilistically balanced level trees [Pugh and Teitelbaum 1989]. Because they are balanced, level trees permit efficient persistent edits (insertions, removals) in logarithmic time, and zipper-based edits can be even be more efficient [Headley and Hammer 2016]. Further, level trees assign sequences of elements (with interposed levels) a single canonical tree that is *history independent*, making cache matches more likely.

For nominal level trees, we use names to identify allocations, permitting us to express incremental insertions or removals in sequences as $O(1)$ re-allocations (store mutations). By contrast, without names, each insertion or removal from the input sequence generally requires $O(\log(N))$ fresh pointer allocations, which generally cascade during change propagation, often giving rise to larger changes, as in Sec. 1.1's `rev` example.

Sequences as nominal level trees. Below, type $\mathsf{Seq}\,[X]$ refines the "ordinary type" for sequences Seq; it classifies sequences whose names are drawn from the set X, and has two constructors for binary nodes and leaves:

$$\begin{aligned}\texttt{SeqBin} &: \forall X \perp Y \perp Z : \mathbf{NmSet}.\, \mathsf{Nm}\,[X] \to \mathsf{Lev} \to \mathsf{Ref}(\mathsf{Seq}\,[Y]) \to \mathsf{Ref}(\mathsf{Seq}\,[Z]) \to \mathsf{Seq}\,[X \perp Y \perp Z] \\ \texttt{SeqLf} &: \forall X : \mathbf{NmSet}.\quad \mathsf{Vec} \to \mathsf{Seq}\,[X]\end{aligned}$$

For simplicity here, we focus on monomorphic sequences at base type (a vector type Vec for vectors of natural numbers); in the appendix, we show refinement types for polymorphic sequences whose type parameters may be higher-kinded (parameterized by names, name sets, and name functions).

In the type of constructor `SeqBin`, the notation $X \perp Y \perp Z$ denotes a set of names, and it asserts *apartness* (pair-wise disjointness) among the three sets. Binary nodes store a natural number *level*, classified by type Lev, a name, and two incremental pointers to left and right sub-trees, of types $\mathsf{Ref}(\mathsf{Seq}\,[Y])$ and $\mathsf{Ref}(\mathsf{Seq}\,[Z])$, respectively. The type $\mathsf{Nm}\,[X]$ classifies first-class names drawn from the set X. In this case, the apartness $X \perp Y \perp Z$ means that the name from X at the `SeqBin` node is distinct from those in its subtrees, whose names are distinct from one another (name sets Y vs Z). Generally, computations that consume a data structure assume that names are non-repeating within a sequence (as enforced by this type); however, computations conventionally reuse input names to identify corresponding constructors in the output, as in the `filter`, `trie` and `quickhull` examples below.

In the `SeqLf` case, we store short vectors of elements to capitalize on low-level operations that exploit cache coherency. (In empirical experiments, we often choose vectors with one thousand elements). Notably, this leaf constructor permits any name set X; we find it useful when types over-approximate their names, e.g., as used in the type for `filter` below, which generally filters away some input names, but does not reflect this fact in its output type.

Sets as nominal hash tries. In addition to sequences, we use balanced trees to represent sets and finite maps. A *nominal hash trie* uses the hash of an element to determine a path in a binary tree, whose pointers are named. The type for $\mathsf{Set}\,[X]$ is nearly the same as that of $\mathsf{Seq}\,[X]$, above; we

```
max : ∀X : NmSet.
        Seq[X] → Nat
      ▷ (λx : Nm.{x·1} ⊥ {x·2})[X]

max seq = match seq with
| SeqLf(vec) ⇒ vec_max vec
| SeqBin(n,_,l,r) ⇒
  let (_,ml) = memo[n·1](max !l)
  let (_,mr) = memo[n·2](max !r)
  if ml > mr then ml else mr

vec_max    : Vec → Nat
vec_filter : Vec → (Nat → Bool) → Vec
```

```
filter : ∀X : NmSet.
           Seq[X] → (Nat → Bool) → Seq[X]
         ▷ (λx : Nm.{x·1} ⊥ {x·2})[X]

filter seq pred = match seq with
| SeqLf(vec) ⇒ SeqLf(vec_filter vec pred)
| SeqBin(n, lev, l, r) ⇒
  let (rl,sl) = memo[n·1](filter !l pred)
  let (rr,sr) = memo[n·2](filter !r pred)
  match (is_empty sl, is_empty sr) with
  | (false,false) ⇒ SeqBin(n, lev, rl, rr)
  | (_,true) ⇒ sl
  | (true,_) ⇒ sr
```

Fig. 2. Recursive functions over sequences: Finding the max element, and filtering elements by a predicate.

just omit the level at the binary nodes:

```
SetBin : ∀X⊥Y⊥Z : NmSet.  Nm[X] → Ref(Set[Y]) → Ref(Set[Z]) → Set[X⊥Y⊥Z]
 SetLf : ∀X : NmSet.      Vec → Set[X]
```

The natural number vectors at the leaves at meant to represent "small" sets of natural numbers. (In this section, we focused on giving types for sequences and sets of natural numbers; Sec. A of the supplement describes polymorphic type definitions).

A collections library. Sec. 3.2 gives simple structurally recursive algorithms over level trees that we use as subroutines for later implementing quickhull. In Sec. 3.3, we give an incrementally efficient conversion algorithm `trie` for building binary hash tries from level trees; it uses nested structural recursion to convert a binary tree of type `Seq[X]` into one of type `Set[X]`. Sec. 3.4 gives two variants of the divide-and-conquer quickhull algorithm, which only differ in their naming strategy. In the context of these examples, we illustrate the key patterns for typing the collections of Hammer et al. [2015], and relate these patterns to the relevant typing rules of Fungi's type system.

3.2 Examples of structural recursion: Reducing and filtering sequences

Fig. 2 gives the type and code for `max` and `filter`, which compute the maximum element of the sequence, and filter the sequence by a predicate, respectively. Both algorithms use structural recursion over the `Seq[X]` datatype, defined above.

The computation of `max` is somewhat simpler. In the leaf case (`SeqLf`), `max` uses the auxiliary function `vec_max`. In the binary case (`SeqBin`), `max` performs two memoized recursive calls, for its left and right sub-trees. We access these reference cells using `!`, as in SML/OCaml notation.

In Fungi's core calculus, named thunks are the unit of function caching. To use them, we introduce syntactic sugar for memoization, where `memo[n·1](e)` expands to code that allocates a named thunk (here, named `n·1`) and demands its output value: `let x = thunk(n·1,e) in forceref(x)`. The primitive `forceref` forces a thunk, and returns a pair consisting of a reference cell representation of the forced thunk, and its cached value; the reference cell representation of the thunk is useful in `filter`, and later examples, but is ignored in `max`. The code listing for `max` uses two instances of `memo`, with two *distinct* names, each based on the name of the binary node `n`.

A formal structure for names. Fungi's core calculus defines names as binary trees, $n ::= \mathsf{leaf} \mid \langle\langle n, n \rangle\rangle$. In practice, we often want to build names from primitive data, such as string constants, natural numbers and bit strings; without loss of generality, we assume embeddings of these types as binary

trees. For instance, we let 1 be shorthand for $\langle\!\langle \texttt{leaf}, \texttt{leaf} \rangle\!\rangle$, the name with one binary node; similarly, 2 stands for $\langle\!\langle \texttt{leaf}, \langle\!\langle \texttt{leaf}, \texttt{leaf} \rangle\!\rangle \rangle\!\rangle$ and 0 for $\texttt{leaf}$. In the code listing, we use $\texttt{n}\cdot 1$ as shorthand for $\langle\!\langle \texttt{n}, 1 \rangle\!\rangle$. More generally, we use $\texttt{n}\cdot\texttt{m}$ as a right-associative shorthand for name lists of binary nodes, where $1\cdot 2\cdot 3$ represents the name $\langle\!\langle 1, \langle\!\langle 2, 3 \rangle\!\rangle \rangle\!\rangle$.

The following deductive reasoning rules help formalize notions of name equivalence and apartness for name terms:

$$\frac{\Gamma \vdash M_1 \equiv N_1 : \mathbf{Nm} \quad \Gamma \vdash M_2 \equiv N_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle \equiv \langle\!\langle N_1, N_2 \rangle\!\rangle : \mathbf{Nm}} \qquad \frac{\Gamma \vdash M_1 \perp N_1 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle \perp \langle\!\langle N_1, N_2 \rangle\!\rangle : \mathbf{Nm}} \qquad \frac{\Gamma \vdash M_1 \equiv N : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle \perp N : \mathbf{Nm}}$$

We use the context Γ to store assumptions about equivalence and apartness (e.g., from type signatures) and judge pairs of name terms at a common sort **Nm**. The first rule says that two binary names are equivalent if their left and right sub-names are equivalent. The second rule says that two binary names are apart (distinct) if their left names are apart. The third rule says that a binary name is always distinct from a name equivalent to its left sub-name. In the supplement (Sec. F and Sec. G), we give more rules for deductive apartness and equivalence of name terms and index terms; here and below, we give a few selected examples.

We turn to the type of `max` given in the listing. For all sets of names X, the function finds the largest natural number in a sequence (index by X). The associated effect (after the $\triangleright$) consists of the following write set:

$$\big(\lambda x : \mathbf{Nm}.\, \{x\cdot 1\} \perp \{x\cdot 2\}\big)\,[X] \quad \equiv \quad \big((\lambda x : \mathbf{Nm}.\, \{x\cdot 1\})\,[X]\big) \perp \big((\lambda x : \mathbf{Nm}.\, \{x\cdot 2\})\,[X]\big)$$

Specifically, this is an index term that consists of mapping a function over a name set X, and taking the (disjoint) union of these results. Intuitively (but informally), this term denotes the following set comprehension: $\bigcup_{x\in X}\{x\cdot 1, x\cdot 2\}$. That is, the set of all names x in X, appended with either one of name constants 1 and 2. More generally, the index form $f\,[X]$ employs a name set mapping function f of sort $\mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet}$, where index X is a name set and has sort **NmSet**. Here, $f :\equiv \lambda x.\, \{x\cdot 1, x\cdot 2\}$. In some listings, we use $X \cdot Y$ as shorthand for $\big(\lambda y.\, (\lambda x.\, \{x\cdot y\})\,[X]\big)\,[Y]$, that is, the set of all pairwise name compositions from name sets X and Y.

The following table of rules give equivalence and apartness reasoning (left vs. right columns) for index function abstraction and nameset-mapping (top vs. bottom rows); by convention, we use i and j for general indices that may not be name sets, and use X, Y and Z for name sets.

$$\frac{\Gamma, (a \equiv b : \gamma_1) \vdash i \equiv j : \gamma_2}{\Gamma \vdash \lambda a.\, i \equiv \lambda b.\, j : \gamma_1 \xrightarrow{\text{idx}} \gamma_2} \qquad \frac{\Gamma, (a \equiv b : \gamma_1) \vdash i \perp j : \gamma_2}{\Gamma \vdash \lambda a.\, i \perp \lambda b.\, j : \gamma_1 \xrightarrow{\text{idx}} \gamma_2}$$

$$\frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \equiv j[Y] : \mathbf{NmSet}} \qquad \frac{\Gamma \vdash i \perp j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \perp j[Y] : \mathbf{NmSet}}$$

To be equivalent, function bodies must be equivalent under the assumption that their arguments are equivalent; to be apart, their bodies must be apart. To be equivalent, a set mapping must have equivalent mapping functions and set arguments; to be apart, the functions must be apart, with the same argument. These set-mapping rules are specialized to set comprehensions, using a function from names to name sets, and a starting name set. In the full set of rules, we give an analogous pair of application rules reason about ordinary function application, with arbitrary input and output sorts (not shown here).

```
trie : ∀X : NmSet.
     Seq[X] → Set[join^(X)]
  ▷ (λx : Nm.{x·1} ⊥ {x·2}) [X]
   ⊥ (λx:Nm. (λy:Nm.{x·y}) [join^(X)]) [X]

trie seq = match seq with
| SeqLf(vec) ⇒ trie_lf vec
| SeqBin(n,_,l,r) ⇒
  let (tl,_) = memo[n·1](trie !l)
  let (tr,_) = memo[n·2](trie !r)
  let trie = ws[n](join n tl tr)
  trie

where:
join^(X) :≡ (λx:Nm.{x·1} ⊥ {x·2})* [X]
```

```
join : ∀X⊥Y⊥Z : NmSet.
       Nm[X] → Set[Y] → Set[Z] → Set[join^(X⊥Y⊥Z)]
    ▷ join^(X⊥Y⊥Z)

join n l r = match (l,r) with
| SetLf(l), SetLf(r) ⇒ join_vecs n l r
| SetBin(_,_,_), SetLf(r) ⇒
            join n·1 l (split_vec n·2 r)
| SetLf(l), SetBin(_,_,_) ⇒
            join n·1 (split_vec n·2 l) r
| SetBin(ln,l0,l1), SetBin(rn,r0,r1) ⇒
  let (_,j0) = memo[ln·1](join ln·2 l0 r0)
  let (_,j1) = memo[rn·1](join rn·2 l1 r1)
  SetBin(n, j0, j1)
```

and where:

```
join_vecs : ∀X : NmSet.
   Nm[X] → Vec → Vec → Set[X]
 ▷ (λx : Nm.{x·1} ⊥ {x·2}) [X]

split_vec : ∀X : NmSet.
   Nm[X] → Vec → Set[X]
 ▷ (λx : Nm.{x·1} ⊥ {x·2}) [X]
```

Fig. 3. Nested structural recursion: Converting sequences into maps

Allocating structurally recursive output. The type and listing for `filter` is similar to that of `max`. The key difference is that `filter` builds an output data structure with named reference cells; it returns a filtered sequence, also of type Seq[X]. Meanwhile, its write set is the same as that of `max`.

Rather than allocate redundant cells (which we could, but do not), each reference cell of filtered output arises from the `memo` shorthand introduced above. In particular, when the left and right filtered sub-trees are non-empty, `filter` introduces a `SeqBin` code, with two named pointers, `rl` and `rr`, the left and right references introduced by memoizing function calls' results. This "trick" works because the output is built via structural recursion, via memoized calls that are themselves structurally recursive over the input. Below, we nest structural recursions.

3.3 Example of nested structural recursion: Sequences into maps

Fig. 3 defines functions `trie` and `join`. The function `trie` uses structural recursion, following a pattern similar to `max` and `filter`, above. However, compared to `max` and `filter`, the body of the `SeqBin` case for `trie` is more involved: rather than compare numbers, or create a single datatype constructor, it invokes `join` on the recursive results of calling `trie` on the left and right sub-sequences. When a `trie` represents a set, the function `join` computes the set union, following an algorithm inspired by Pugh and Teitelbaum [1989], but augmented with names. Given two tries with apart name sets X and Y, `join` constructs a trie with names $\widehat{\text{join}}(X \bot Y)$, where $\widehat{\text{join}}$ is a function over name sets, whose definition we discuss below.

The ambient write scope: a distinguished name space. To disambiguate the names allocated by the calls to `join`, the code for `trie` uses a *name space* defined by `n`, with the notation `let trie = ws[n](e)`. In particular, this notation controls the ambient write scope, the name space for allocations that write the store. Informally, it says to perform the writes of the sub-computation `e` by first prepending the name `n` to each allocation name.

To type programs that use the ambient write scope, we use a typing judgement form $\Gamma \vdash^{\mathsf{N}} e : A \rhd X$ that tracks a name space N, a name term of sort $\mathbf{Nm} \overset{\mathbf{Nm}}{\Rightarrow} \mathbf{Nm}$. (This arrow sort over names is more restricted than $\mathbf{Nm} \overset{\mathbf{idx}}{\Rightarrow} \mathbf{NmSet}$: it only involves name construction, and lacks set structure.) Consider the following typing rules for write scopes (left rule) and name function application (right rule):

$$\frac{\Gamma \vdash v_1 : \mathsf{Nm}[X] \qquad \Gamma \vdash v_2 : A}{\Gamma \vdash^{M} \mathtt{ref}(v_1, v_2) : \mathsf{Ref}[M[X]]\ A \rhd M[X]} \qquad \frac{\Gamma \vdash v : (\mathbf{Nm} \overset{\mathbf{idx}}{\Rightarrow} \mathbf{Nm})[N'] \quad \Gamma \vdash^{N \circ N'} e : A \rhd W}{\Gamma \vdash^{N} \mathtt{scope}(v, e) : A \rhd W}$$

The left rule uses the name term M to determine the final allocation name for a reference cell. As indicated by the conclusion's type and write effect, it computes a name in set $M[X]$, where index term X gives the name set of the argument. The thunk rule (not shown here) is similar: it uses the ambient write scope to determine the allocated name.

The right rule extends the ambient write scope N by name function composition with N', for the duration of evaluating a given subterm e. The notation in the code listing `let x = ws[n](e)`e_2 is shorthand for $\mathtt{let}(\mathtt{scope}(\lambda a.\, n{\cdot}a, e), x.e_2)$. That is, we prepend name n to each name written by the evaluation of subterm e; then, we evaluate e_2. Fungi's collections library uses this shorthand to disambiguate names that are reused across sequenced calls, as in the listings for `trie`, and in quickhull, below.

Types for recursive name sets. Sometimes we need to specify the names of a set recursively, as in the code listing for `join`, which is parameterized by an argument name `n`. In some recursive calls to `join`, this name argument "grows" into $n{\cdot}1$ or $n{\cdot}2$. In general, each name may grow a variable number of times. This generality is required: The algorithm for `trie` allocates a final binary hash trie whose structure is generally *not* isomorphic to the initial level tree; it is generally *larger*.

To capture these recursive name-growth patterns, we introduce another special variant of index terms, for describing sets that we "grow" inductively by applying a function zero or more times to an initial set. We use this index form to define $\widehat{\mathsf{join}}$ from Fig. 3:

$$\frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \overset{\mathbf{idx}}{\Rightarrow} \mathbf{NmSet} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i^*[X] \equiv j^*[Y] : \mathbf{NmSet}} \qquad \frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \overset{\mathbf{idx}}{\Rightarrow} \mathbf{NmSet} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i^*[X] \equiv i^*[j[Y]] \cup Y : \mathbf{NmSet}} \qquad \frac{\Gamma \vdash i \perp j : \mathbf{Nm} \overset{\mathbf{idx}}{\Rightarrow} \mathbf{NmSet} \quad \Gamma \vdash X \perp Y : \mathbf{NmSet}}{\Gamma \vdash i^*[X] \perp j^*[Y] : \mathbf{NmSet}}$$

The first rule matches that of the non-recursive form. The second rule is unique to recursive applications; it witnesses one unfolding of the recursion. The final rule says that two uses of recursive application are apart when the starting sets and functions are apart.

3.4 Examples of naming divide-and-conquer: Quickhull

Fig. 4 defines two quickhull functions `qh1` and `qh2`. They consist of the same geometric point-processing steps, but use distinct naming strategies, and consequently, lead to distinct incremental cache behavior. Each version computes the furthest point `p` from the current line `ln` by using `max_pt`, which is similar to `max` in Fig. 2. This point `p` defines two additional lines, `(ln.0,p)` and `(p,ln.1)`, which quickhull uses to filter the input points before recurring on these new lines.

We use the typing ingredients introduced above to give two naming strategies, each with distinct cache behavior. To summarize the naming strategies of `max_pt` and `filter` (each uses structural recursion over level trees), we introduce the shorthand, with associated the equivalences:

$$\widehat{\mathsf{bin}}[X] \quad :\equiv \quad X \cdot \{1,2\} \quad :\equiv \quad \big(\lambda x.\, \{x{\cdot}1, x{\cdot}2\}\big)[X] \equiv \big((\lambda x.\, \{x{\cdot}1\}[X]) \perp (\lambda x.\, \{x{\cdot}2\}[X])\big)$$

```
qh1 :
  ∀X⊥Y⊥Z : NmSet.
    Line[X] → Seq[Y] → Set[Z] → Set[X⊥Z]
  ▷ lines̃(X,Y)·{1,2,3}
  ⊥ lines̃(X,Y)·{1,2,3}·Y·{1,2}

qh1 ln pts h =
 let p = ws[ln·1](max_pt ln pts)
 let l = ws[ln·2](filter (ln.0,p) pts)
 let r = ws[ln·3](filter (p,ln.1) pts)
 let h = memo[ln·1](qh1 (p,ln.1) r h)
 let h = Cons(ln,p,ref(ln·2,h))
 let h = memo[ln·3](qh1 (ln.0,p) l h)
 h
```

$\widetilde{\text{lines}}(X,Y) :\equiv$ "All lines formed by lines X and points Y"
(definitions provided by the geometry library)

```
qh2 :
  ∀X⊥Y⊥Z : NmSet.
    Line[X] → Seq[Y] → Set[Z] → Set[X⊥Z]
  ▷ (λy:Nm.{1·y} ⊥ {2·y})* [qh̃2(Y⊥Z)]

qh2 ln pts h =
 let p = ws[3·1](max_pt ln pts)
 let l = ws[3·2](filter (ln.0,p) pts)
 let r = ws[3·3](filter (p,ln.1) pts)
 let h = memo[1](ws[1](qh2 (p,ln.1) r h))
 let h = Cons(ln,p,ref(2,h))
 let h = memo[3](ws[2](qh2 (ln.0,p) l h))
 h
```

$\widetilde{qh_2}(X) :\equiv \{3\cdot1, 3\cdot2, 3\cdot3\}\cdot\widehat{\text{bin}}[X] \perp \{1,2,3\}$

Fig. 4. Two naming strategies for quickhull, same algorithmic structure

The left naming strategy of `qh1` uses the identity of the *partition lines* to introduce three name spaces (for the calls to `max_pt` and `filter`), and three allocations, for the two calls to `qh1`, and the output hull point `p` that is computed by `max_pt`. In this case, the library for computational geometry provides the lines' names, where we exploit that quickhull does not process the same line twice. We elide some details in this reasoning, for brevity in the listing.

The right version (`qh2`) ignores the names of lines. Instead, it names everything based on a binary path in the quickhull call graph: The two recursive calls to `qh2` each extend write scope, first with name 1, then with 2.

Ignoring their use of names, the two versions of quickhull are identical: When run from scratch, they always produce the same output, from the same steps, in the same order.

Incrementally, they are very different: they have distinct naming strategies, and by virtue of this fact, they have distinct cache behavior. In brief, the first strategy is better at changes that disrupt and then restore the hull (inserting and then removing a new hull point) but this line-based naming strategy also lacks a simple cache *eviction* strategy (when to "forget" about work for a line? What if we forget about a line, but it reappears later?). In a way, by using *lines* as names, the left version can suffer a cache consumption problem similar to the structurally-named variant of `rev` from Sec. 1.1: each new line will generally consume additional cache space.

In contrast, by overwriting earlier computations based on call graph position, the second quickhull naming strategy is cruder, but gives a natural cache eviction strategy: When the same callgraph path is reached, this program's names will overwrite its old callgraph content with new callgraph content. This strategy corresponds (roughly) with naming strategies that use a "global position" in the call graph to determine cache locations (see [Burckhardt et al. 2011; Çiçek et al. 2015, 2016], and Sec. 9 for more discussion).

4 PROGRAM SYNTAX

The examples from the prior section use an informally defined variant of ML, enriched with a (slightly simplified) variant of our proposed type system. In this section and the next, we focus on a core calculus for programs and types, and on making these definitions precise.

$$
\begin{array}{lll}
\text{Values} & v & ::= x \mid () \mid (v_1, v_2) \mid \mathtt{inj}_i\, v \mid \mathtt{name}\ n \mid \mathtt{nmfn}\ M \mid \mathtt{ref}\ n \mid \mathtt{thunk}\ n \mid \mathtt{pack}(a.v) \\
\text{Terminal exprs.} & t & ::= \mathtt{ret}(v) \mid \lambda x.\, e \\
\text{Expressions} & e & ::= t \mid \mathtt{split}(v, x_1.x_2.e) \mid \mathtt{case}(v, x_1.e_1, x_2.e_2) \\
 & & \quad \mid e\ v \mid \mathtt{let}(e_1, x.e_2) \mid \mathtt{thunk}(v, e) \mid \mathtt{force}(v) \mid \mathtt{ref}(v, v) \mid \mathtt{get}(v) \\
 & & \quad \mid \mathtt{scope}(v, e) \mid v_M\ v \\
 & & \quad \mid \mathtt{vunpack}(v, a.x.e)
\end{array}
$$

Fig. 5. Syntax of expressions

4.1 Values and Expressions

Fig. 5 gives the grammar of values v and expressions e. We use call-by-push-value (CBPV) conventions in this syntax, and in the type system that follows. There are several reasons for this. First, CBPV can be interpreted as a "neutral" evaluation order that includes both call-by-value or call-by-name, but prefers neither in its design. Second, since we make the unit of memoization a thunk, and CBPV makes explicit the creation of thunks and closures, it exposes exactly the structure that we extend to a general-purpose abstraction for incremental computation. In particular, thunks are the means by which we cache results and track dynamic dependencies.

Values v consist of variables, the unit value, pairs, sums, and several special forms (described below).

We separate values from expressions, rather than considering values to be a subset of expressions. Instead, *terminal expressions* t are a subset of expressions. A terminal expression t is either $\mathtt{ret}(v)$—the expression that returns the value v—or a λ. Expressions e include terminal expressions, elimination forms for pairs, sums, and functions (split, case and $e\,v$, respectively); let-binding (which evaluates e_1 to $\mathtt{ret}(v)$ and substitutes v for x in e_2); introduction (thunk) and elimination (force) forms for thunks; and introduction (ref) and elimination (get) forms for pointers (reference cells that hold values).

The special forms of values are names $\mathtt{name}\ n$, name-level functions $\mathtt{nmfn}\ M$, references (pointers), and thunks. References and thunks include a name n, which is the name of the reference or thunk, *not* the contents of the reference or thunk.

The syntax described above follows that of prior work on Adapton, including Hammer et al. [2015]. We add the notion of a *name function*, which captures the idea of a namespace and other simple transformations on names. The construct $\mathtt{scope}(v, e)$ construct controls monadic state for the current name function, composing it with a name function v within the dynamic extent of its subexpression e. Name function application $M\ v$ permits programs to compute with names and name functions that reside within the type indices. Since these name functions always terminate, they do not affect a program's termination behavior.

We do not distinguish syntactically between value pointers (for reference cells) and thunk pointers (for suspended expressions); the store maps pointers to either of these.

4.2 Names

Figure 6 shows the syntax of literal names, name terms, name term values, and name term sorts. Literal names m, n are simply binary trees: either an empty leaf leaf or a branch node $\langle\langle n_1, n_2 \rangle\rangle$. Name terms M, N consist of literal names n and branch nodes $\langle\langle M_1, M_2 \rangle\rangle$, abstraction $\lambda a.\, M$ and application $M(N)$.

Name terms are classified by sorts γ: sort **Nm** for names n, and $\gamma \xrightarrow{\mathbf{Nm}} \gamma$ for (name term) functions.

Names (binary trees)	m, n	$::= \mathtt{leaf}$	leaf name
		$\mid \langle\langle n_1, n_2 \rangle\rangle$	binary name composition
Name terms (STLC+names)	M, N	$::= n \mid \langle\langle M_1, M_2 \rangle\rangle$	literal names, binary name composition
		$\mid a \mid \lambda a.\, M \mid M(N)$	variable, abstraction, application
Name term values	V	$::= n \mid \lambda a.\, M$	
Name term sorts	γ	$::= \mathbf{Nm}$	name; inhabitants n
		$\mid \gamma \overset{\mathbf{Nm}}{\Rightarrow} \gamma$	name term function; inhabitants $\lambda a.\, M$
Typing contexts	Γ	$::= \cdot \mid \Gamma, a : \gamma \mid \cdots$	full definition in Figure 10

Fig. 6. Syntax of name terms: a λ-calculus over names, as binary trees

$\boxed{\Gamma \vdash M : \gamma}$ Under Γ, name term M has sort γ

$$\frac{}{\Gamma \vdash n : \mathbf{Nm}}\ \text{M-const} \qquad \frac{(a : \gamma) \in \Gamma}{\Gamma \vdash a : \gamma}\ \text{M-var} \qquad \frac{\Gamma \vdash M_1 : \mathbf{Nm} \quad \Gamma \vdash M_2 : \mathbf{Nm}}{\Gamma \vdash \langle\langle M_1, M_2 \rangle\rangle : \mathbf{Nm}}\ \text{M-bin}$$

$$\frac{\Gamma, a : \gamma' \vdash M : \gamma}{\Gamma \vdash (\lambda a.\, M) : (\gamma' \overset{\mathbf{Nm}}{\Rightarrow} \gamma)}\ \text{M-abs} \qquad \frac{\Gamma \vdash M : (\gamma' \overset{\mathbf{Nm}}{\Rightarrow} \gamma) \qquad \Gamma \vdash N : \gamma'}{\Gamma \vdash M(N) : \gamma}\ \text{M-app}$$

Fig. 7. Sorting rules for name terms M

$\boxed{M \Downarrow_M V}$ Name term M evaluates to name term value V

$$\frac{}{V \Downarrow_M V}\ \text{teval-value} \qquad \frac{M_1 \Downarrow_M n_1 \qquad M_2 \Downarrow_M n_2}{\langle\langle M_1, M_2 \rangle\rangle \Downarrow_M \langle\langle n_1, n_2 \rangle\rangle}\ \text{teval-bin} \qquad \frac{M \Downarrow_M \lambda a.\, M' \quad N \Downarrow_M V \quad [V/a]M' \Downarrow_M V'}{M(N) \Downarrow_M V'}\ \text{teval-app}$$

Fig. 8. Evaluation rules for name terms

The rules for name sorting $\Gamma \vdash M : \gamma$ are straightforward (Figure 7), as are the rules for name term evaluation $M \Downarrow_M V$ (Figure 8). We write $M =_\beta M'$ when name terms M and M' are convertible, that is, applying any series of β-reductions and/or β-expansions changes one term into the other.

5 TYPE SYSTEM

The structure of our type system is inspired by Dependent ML [Xi and Pfenning 1999; Xi 2007]. Unlike full dependent typing, DML is separated into a *program level* and a less-powerful *index level*. The classic DML index domain is integers with linear inequalities, making type-checking decidable. Our index domain includes names, sets of names, and functions over names. Such functions constitute a tiny domain-specific language that is powerful enough to express useful transformations of names, but preserves decidability of type-checking.

Indices in DML have no direct computational content. For example, when applying a function on vectors that is indexed by vector length, the length index is not directly manipulated at run time. However, indices can indirectly reflect properties of run-time values. The simplest case is

Index exprs.	$i, j,$	$::= a$	index variable
	$X, Y, Z,$	$\mid \{N\}$	singleton name set
	R, W	$\mid \emptyset \mid X \perp Y$	empty set, separating union
		$\mid X \cup Y$	union (not necessarily disjoint)
		$\mid () \mid (i, i) \mid \mathtt{prj}_1\, i \mid \mathtt{prj}_2\, i$	unit, pairing, and projection
		$\mid \lambda a.\, i \mid i(j)$	function abstraction and application
		$\mid M[i] \mid i[j] \mid i^*[j]$	name set mapping and set building
Index sorts	γ	$::= \cdots \mid \mathbf{NmSet}$	name set sort
		$\mid \mathbf{1}$	unit index sort; inhabitant $()$
		$\mid \gamma * \gamma$	product index sort; inhabitants (i, j)
		$\mid \gamma_1 \xrightarrow{\text{idx}} \gamma_2$	index functions over name sets

Fig. 9. Syntax of indices, name set sort

that of an indexed *singleton type*, such as $\mathsf{Int}[k]$. Here, the ordinary type Int and the index domain of integers are in one-to-one correspondence; the type $\mathsf{Int}[3]$ has one value, the integer 3.

While indexed singletons work well for the classic index domain of integers, they are less suited to names—at least for our purposes. Unlike integer constraints, where integer literals are common in types—for example, the length of the empty list is 0—literal names are rare in types. Many of the name constraints we need to express look like "given a value of type A whose name in the set X, this function produces a value of type B whose name is in the set $f(X)$". A DML-style system can express such constraints, but the types become verbose:

$$\forall a : \mathbf{Nm}.\ \forall X : \mathbf{NmSet}.\ (a \in X) \supset \big(A[a] \to B[f(a)]\big)$$

The notation is taken from one of DML's descendants, Stardust [Dunfield 2007]. The type is read "for all names a and name sets X, such that $a \in X$, given some $A[a]$ the function returns $B[f(a)]$".

We avoid such locutions by indexing single values by name sets, rather than names. For types of the shape given above, this removes half the quantifiers and obviates the $\in$-constraint attached via $\supset$: $\forall X : \mathbf{NmSet}.\ A[X] \to B[f(X)]$. This type says the same thing as the earlier one, but now the approximations are expressed within the indexing of A and B. Note that f, a function on names, is interpreted pointwise: $f(X) = \{f(N) \mid N \in X\}$.

(Standard singletons will come in handy for index functions on names, where one usually needs to know the specific function.)

For aggregate data structures such as lists, indexing by a name set denotes an *overapproximation* of the names present. That is, the proper DML type

$$\forall Y : \mathbf{NmSet}.\ \forall X : \mathbf{NmSet}.\ (Y \subseteq X) \supset \big(A[Y] \to B[f(Y)]\big)$$

can be expressed by $\forall X : \mathbf{NmSet}.\ A[X] \to B[f(X)]$.

Following call-by-push-value [Levy 1999, 2001], we distinguish *value types* from *computation types*. Our computation types will also model effects, such as the allocation of a thunk with a particular name.

5.1 Index Level

Figure 9 gives the syntax of index expressions and index sorts (which classify indices). We use several meta-variables for index expressions; by convention, we use X, Y, Z, R and W only for sets of names—index expressions of sort **NmSet**.

Kinds	K	$::=$ type	kind of value types
		$\mid \mathsf{type} \Rightarrow K$	type argument (binder space)
		$\mid \gamma \Rightarrow K$	index argument (binder space)
Propositions	P	$::=$ $\mathbf{tt} \mid P$ **and** P	truth and conjunction
		$\mid i \perp j : \gamma$	index apartness
		$\mid i \equiv j : \gamma$	index equivalence
Effects	ϵ	$::=$ $\langle W; R \rangle$	
Value types	A, B	$::=$ $\alpha \mid d \mid \mathsf{unit}$	type variables, type constructors, unit
		$\mid A + B \mid A \times B$	sum, product
		$\mid \mathsf{Ref}\,[i]\ A$	named reference cell
		$\mid \mathsf{Thk}\,[i]\ E$	named thunk (with effects)
		$\mid A\,[i]$	application of type to index
		$\mid A\ B$	application of type constructor to type
		$\mid \mathsf{Nm}\,[i]$	name type (name in name set i)
		$\mid (\mathbf{Nm} \overset{\mathsf{Nm}}{\rightharpoonup} \mathbf{Nm})\,[M]$	name function type (singleton)
		$\mid \forall a : \gamma \mid P.\ A$	universal index quantifier
		$\mid \exists a : \gamma \mid P.\ A$	existential index quantifier
Computation types	C, D	$::=$ $\mathbf{F}\,A \mid A \to E$	li**F**t, functions
…with effects	E	$::=$ $C \rhd \epsilon$	effects
		$\mid \forall \alpha : K.\ E$	type polymorphism
		$\mid (\forall a : \gamma \mid P.\ E)$	index polymorphism
Typing contexts	Γ	$::=$ $\cdot$	
		$\mid \Gamma, a : \gamma$	index variable sorting
		$\mid \Gamma, \alpha : K$	type variable kinding
		$\mid \Gamma, d : K$	type constructor kinding
		$\mid \Gamma, N : A$	ref pointer
		$\mid \Gamma, N : E$	thunk pointer
		$\mid \Gamma, x : A$	value variable
		$\mid \Gamma, P$	proposition P holds

Fig. 10. Syntax of kinds, effects, and types

Name sets. If we give a name to each element of a list, then the entire list should carry the set of those names. We write $\{N\}$ for the singleton name set, $\emptyset$ for the empty name set, and $X \perp Y$ for a union of two sets X and Y that requires X and Y to be disjoint; this is inspired by the separating conjunction of separation logic [Reynolds 2002]. While disjoint union dominates the types that we believe programmers need, our effects discipline requires non-disjoint union, so we include it ($X \cup Y$) as well.

Variables, pairing, functions. An index i (also written X, Y, …when the index is a set of names) is either an index-level variable a, a name set (described above: $\{N\}$, $X \perp Y$ or $X \cup Y$), the unit index $()$, a pair of indices (i_1, i_2), pair projection $\mathsf{prj}_b\, i$ for $b \in \{1, 2\}$, an abstraction $\lambda a.\ i$, application $i(j)$, or name term application $M\,[i]$.

Name terms M are *not* a syntactic subset of indices i, though name terms can appear inside indices (for example, singleton name sets $\{M\}$). Because name terms are not a syntactic subset of

$\boxed{\Gamma \vdash \mathtt{i} : \gamma}$ Under Γ, index $\mathtt{i}$ has sort γ

$$\frac{(\mathtt{a} : \gamma) \in \Gamma}{\Gamma \vdash \mathtt{a} : \gamma}\ \text{sort-var} \qquad \frac{}{\Gamma \vdash () : \mathbf{1}}\ \text{sort-unit} \qquad \frac{\Gamma \vdash \mathtt{i}_1 : \gamma_1 \qquad \Gamma \vdash \mathtt{i}_2 : \gamma_2}{\Gamma \vdash (\mathtt{i}_1, \mathtt{i}_2) : (\gamma_1 * \gamma_2)}\ \text{sort-pair}$$

$$\frac{\Gamma \vdash \mathtt{i} : \gamma_1 * \gamma_2}{\Gamma \vdash \mathtt{prj}_\mathtt{b}\,\mathtt{i} : \gamma_\mathtt{b}}\ \text{sort-proj} \qquad \frac{}{\Gamma \vdash \emptyset : \mathbf{NmSet}}\ \text{sort-empty} \qquad \frac{\Gamma \vdash \mathsf{N} : \mathbf{Nm}}{\Gamma \vdash \{\mathsf{N}\} : \mathbf{NmSet}}\ \text{sort-singleton}$$

$$\frac{\Gamma \vdash \mathsf{X} : \mathbf{NmSet} \quad \Gamma \vdash \mathsf{Y} : \mathbf{NmSet}}{\Gamma \vdash (\mathsf{X} \cup \mathsf{Y}) : \mathbf{NmSet}}\ \text{sort-union} \qquad \frac{\Gamma \vdash \mathsf{X} : \mathbf{NmSet} \quad \Gamma \vdash \mathsf{Y} : \mathbf{NmSet} \quad \mathit{extract}(\Gamma) \Vdash \mathsf{X} \perp \mathsf{Y} : \mathbf{NmSet}}{\Gamma \vdash (\mathsf{X} \perp \mathsf{Y}) : \mathbf{NmSet}}\ \text{sort-sep-union}$$

$$\frac{\Gamma, \mathtt{a} : \gamma_1 \vdash \mathtt{i} : \gamma_2}{\Gamma \vdash (\lambda \mathtt{a}.\, \mathtt{i}) : (\gamma_1 \overset{\text{idx}}{\Rightarrow} \gamma_2)}\ \text{sort-abs} \qquad \frac{\Gamma \vdash \mathtt{i} : \gamma_1 \overset{\text{idx}}{\Rightarrow} \gamma_2 \qquad \Gamma \vdash \mathtt{j} : \gamma_1}{\Gamma \vdash \mathtt{i}(\mathtt{j}) : \gamma_2}\ \text{sort-apply}$$

$$\frac{\Gamma \vdash \mathsf{M} : \mathbf{Nm} \overset{\mathbf{Nm}}{\Rightarrow} \mathbf{Nm} \quad \Gamma \vdash \mathtt{j} : \mathbf{NmSet}}{\Gamma \vdash \mathsf{M}\,[\mathtt{j}] : \mathbf{NmSet}}\ \text{sort-map} \qquad \frac{\Gamma \vdash \mathtt{i} : \mathbf{Nm} \overset{\text{idx}}{\Rightarrow} \mathbf{NmSet} \quad \Gamma \vdash \mathtt{j} : \mathbf{NmSet}}{\Gamma \vdash \mathtt{i}\,[\mathtt{j}] : \mathbf{NmSet}}\ \text{sort-build} \qquad \frac{\Gamma \vdash \mathtt{i} : \mathbf{Nm} \overset{\text{idx}}{\Rightarrow} \mathbf{NmSet} \quad \Gamma \vdash \mathtt{j} : \mathbf{NmSet}}{\Gamma \vdash \mathtt{i}^*\,[\mathtt{j}] : \mathbf{NmSet}}\ \text{sort-star}$$

Fig. 11. Sorts statically classify name terms M, and the name indices $\mathtt{i}$ that index types

indices (and name sets are not name terms), the application form $\mathtt{i}(\mathtt{j})$ does not allow us to apply a name term function to a name set. Thus, we also need name term application $\mathsf{M}\,[\mathtt{i}]$, which applies the name function M to each element of the name set $\mathtt{i}$. The index-level map form $\mathtt{i}\,[\mathtt{j}]$ collects the output sets of function $\mathtt{i}$ on the elements of the input set $\mathtt{j}$. The Kleene star variation $\mathtt{i}^*\,[\mathtt{j}]$ applies the function $\mathtt{i}$ zero or more times to each input element in set $\mathtt{j}$.

Sorts. We use the meta-variable γ to classify indices as well as name terms. We inherit the function space $\overset{\mathbf{Nm}}{\Rightarrow}$ from the name term sorts (Figure 6). The sort **NmSet** (Figure 9) classifies indices that are name sets. The function space $\overset{\text{idx}}{\Rightarrow}$ classifies functions over *indices* (e.g., tuples of name sets), not merely name terms. The unit sort and product sort classify tuples of index expressions.

Most of the sorting rules in Figure 11 are straightforward, but rule 'sort-sep-union' includes a premise $\mathit{extract}(\Gamma) \Vdash \mathsf{X} \perp \mathsf{Y} : \mathbf{NmSet}$, which says that X and Y are *apart* (disjoint).

Propositions and extraction. Propositions P are conjunctions of atomic propositions $\mathtt{i} \equiv \mathtt{j} : \gamma$ and $\mathtt{i} \perp \mathtt{j} : \gamma$, which express equivalence and apartness of indices $\mathtt{i}$ and $\mathtt{j}$. For example, $\{\mathtt{n}_1\} \perp \{\mathtt{n}_2\} : \mathbf{NmSet}$ implies that $\mathtt{n}_1 \neq \mathtt{n}_2$. Propositions are introduced into Γ via index polymorphism $\forall \mathtt{a} : \gamma \mid \mathsf{P}.\, \mathsf{E}$, discussed below.

The function $\mathit{extract}(\Gamma)$ (Figure 28 in the appendix) looks for propositions P, which become equivalence and apartness assumptions. It also translates Γ into the relational context used in the definition of apartness. We give semantic definitions of equivalence and apartness in the appendix (Definitions G.4 and G.5).

5.2 Kinds

We use a simple system of *kinds* K (Figure 23 in the appendix). Kind type classifies value types, such as unit and $(\mathsf{Thk}\,[\mathtt{i}]\ \mathsf{E})$.

Kind type $\Rightarrow \mathsf{K}$ classifies type expressions that are parametrized by a type. Such types are called *type constructors* in some languages.

$\boxed{\Gamma \vdash v : A}$ Under assumptions Γ, value v has type A

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ \text{var} \qquad \frac{\Gamma \vdash v : A_1 \qquad \Gamma \vdash A_1 \leq_V A_2}{\Gamma \vdash v : A_2}\ \text{vtype-sub}$$

$$\frac{}{\Gamma \vdash () : \mathsf{unit}}\ \text{unit} \qquad \frac{\Gamma \vdash v_1 : A_1 \qquad \Gamma \vdash v_2 : A_2}{\Gamma \vdash (v_1, v_2) : (A_1 \times A_2)}\ \text{pair} \qquad \frac{\Gamma \vdash v_i : A_i}{\Gamma \vdash \mathsf{inj}_i\, v_i : (A_1 + A_2)}\ \text{inj}$$

$$\frac{\Gamma \vdash n \in X}{\Gamma \vdash (\mathtt{name}\ n) : \mathsf{Nm}\,[X]}\ \text{name} \qquad \frac{\Gamma \vdash M_v : \mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm} \qquad M_v =_\beta M}{\Gamma \vdash (\mathtt{nmfn}\ M_v) : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M]}\ \text{namefn}$$

$$\frac{\Gamma \vdash n \in X \qquad \Gamma(n) = A}{\Gamma \vdash (\mathtt{ref}\ n) : (\mathsf{Ref}\,[X]\ A)}\ \text{ref} \qquad \frac{\Gamma \vdash n \in X \qquad \Gamma(n) = E}{\Gamma \vdash (\mathtt{thunk}\ n) : (\mathsf{Thk}\,[X]\ E)}\ \text{thunk}$$

$$\frac{\Gamma, a : \gamma, P \vdash v : A}{\Gamma \vdash v : (\forall a : \gamma \mid P.\, A)}\ \text{vtype-}\forall\text{IndexIntro} \qquad \frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v : (\forall a : \gamma \mid P.\, A)\end{array}}{\Gamma \vdash v : [i/a]A}\ \text{vtype-}\forall\text{IndexElim}$$

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v : [i/a]A\end{array}}{\Gamma \vdash \mathtt{pack}(a.v) : (\exists a : \gamma \mid P.\, A)}\ \text{vtype-}\exists\text{IndexIntro}$$

Fig. 12. Value typing

Kind $\gamma \Rightarrow K$ classifies type expressions parametrized by an index. For example, the Seq type constructor from Section 3 takes a name set, e.g. $\mathsf{Seq}\,[X]$. Therefore, this (simplified variant of) Seq has kind $\mathbf{NmSet} \Rightarrow \mathsf{type}$. A more general Seq type would also track its pointers (not just its names), and permit any element type, and would thus have kind $\mathbf{NmSet} \Rightarrow (\mathbf{NmSet} \Rightarrow (\mathsf{type} \Rightarrow \mathsf{type}))$.

5.3 Effects

Effects are described by $\langle W; R \rangle$, meaning that the associated code may write names in W, and read names in R.

Effect sequencing (Figure 13) is a (meta-level) partial function over a pair of effects: the judgment $\Gamma \vdash \epsilon_1\ \mathsf{then}\ \epsilon_2 = \epsilon$, means that ϵ describes the combination of having effects ϵ_1 followed by effects ϵ_2. Sequencing is a partial function because the effects are only valid when (1) the writes of ϵ_1 are disjoint from the writes of ϵ_2, and (2) the reads of ϵ_1 are disjoint from the writes of ϵ_2. Condition (1) holds when each cell or thunk is not written more than once (and therefore has a unique value). Condition (2) holds when each cell or thunk is written before it is read.

Effect coalescing, written "E after ϵ", combines "clusters" of effects:

$$\big(C \rhd \langle\{n_2\}; \emptyset\rangle\big)\ \mathsf{after}\ \langle\{n_1\}; \emptyset\rangle \;=\; C \rhd (\langle\{n_1\}; \emptyset\rangle\ \mathsf{then}\ \langle\{n_2\}; \emptyset\rangle) \;=\; C \rhd \langle\{n_1, n_2\}; \emptyset\rangle$$

Effect subsumption $\epsilon_1 \preceq \epsilon_2$ holds when the write and read sets of ϵ_1 are subsets of the respective sets of ϵ_2.

5.4 Types

The value types (Figure 10), written A, B, include standard sums $+$ and products $\times$; a unit type; the type $\mathsf{Ref}\,[i]\ A$ of references named i containing a value of type A; the type $\mathsf{Thk}\,[i]\ E$ of thunks named i whose contents have type E (see below); the application $A\,[i]$ of a type to an index; the

$\boxed{\Gamma \vdash (\epsilon_1 \text{ then } \epsilon_2) = \epsilon}$ Effect sequencing

$$\frac{\begin{array}{ll} \mathit{extract}(\Gamma) \vdash W_1 \perp W_2 & \mathit{extract}(\Gamma) \vdash W_1 \cup W_2 \equiv W_3 \\ \mathit{extract}(\Gamma) \vdash R_1 \perp W_2 & \mathit{extract}(\Gamma) \vdash R_1 \cup R_2 \equiv R_3 \end{array}}{\Gamma \vdash \langle W_1; R_1 \rangle \text{ then } \langle W_2; R_2 \rangle = \langle W_3; R_3 \rangle}$$

$\boxed{\Gamma \vdash \epsilon_1 \preceq \epsilon_2}$ Effect subsumption

$$\frac{\begin{array}{l} \mathit{extract}(\Gamma) \vdash (X_1 \perp Z_1) \equiv Y_1 : \mathbf{NmSet} \\ \mathit{extract}(\Gamma) \vdash (X_2 \perp Z_2) \equiv Y_2 : \mathbf{NmSet} \end{array}}{\Gamma \vdash \langle X_1; X_2 \rangle \preceq \langle Y_1; Y_2 \rangle}$$

$\boxed{\Gamma \vdash (E \text{ after } \epsilon) = E'}$ Effect coalescing

$$\frac{\Gamma \vdash (\epsilon_1 \text{ then } \epsilon_2) = \epsilon}{\Gamma \vdash \big((C \rhd \epsilon_2) \text{ after } \epsilon_1\big) = (C \rhd \epsilon)}$$

$$\frac{\Gamma \vdash (E \text{ after } \epsilon) = E'}{\begin{array}{l} \Gamma \vdash (\forall \alpha : K.\, E) \text{ after } \epsilon \;=\; (\forall \alpha : K.\, E') \\ \Gamma \vdash (\forall a : \gamma \mid P.\, E) \text{ after } \epsilon \;=\; (\forall a : \gamma \mid P.\, E') \end{array}}$$

$\boxed{\Gamma \vdash^M e : E}$ Under Γ, within namespace M, computation e has type-with-effects E

$$\frac{\Gamma \vdash^M e : E_1 \qquad \Gamma \vdash E_1 \leq_E E_2}{\Gamma \vdash^M e : E_2} \text{ etype-sub}$$

$$\frac{\begin{array}{l} \Gamma \vdash v : (A_1 \times A_2) \\ \Gamma, x_1 : A_1, x_2 : A_2 \vdash^M e : E \end{array}}{\Gamma \vdash^M \mathsf{split}(v, x_1.x_2.e) : E} \text{ split} \qquad \frac{\begin{array}{ll} & \Gamma, x_1 : A_1 \vdash^M e_1 : E \\ \Gamma \vdash v : (A_1 + A_2) & \Gamma, x_2 : A_2 \vdash^M e_2 : E \end{array}}{\Gamma \vdash^M \mathsf{case}(v, x_1.e_1, x_2.e_2) : E} \text{ case}$$

$$\frac{\Gamma \vdash v : A}{\Gamma \vdash^M \mathsf{ret}(v) : (\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle} \text{ ret} \qquad \frac{\begin{array}{ll} & \Gamma \vdash^M e_1 : (\mathbf{F}\, A) \rhd \epsilon_1 \\ \Gamma, x : A \vdash^M e_2 : (C \rhd \epsilon_2) & \Gamma \vdash (\epsilon_1 \text{ then } \epsilon_2) = \epsilon \end{array}}{\Gamma \vdash^M \mathsf{let}(e_1, x.e_2) : (C \rhd \epsilon)} \text{ let}$$

$$\frac{\Gamma, x : A \vdash^M e : E}{\Gamma \vdash^M (\lambda x.\, e) : \big((A \to E) \rhd \langle \emptyset; \emptyset \rangle\big)} \text{ lam} \qquad \frac{\begin{array}{ll} & \Gamma \vdash (E \text{ after } \epsilon_1) = E_1 \\ \Gamma \vdash^M e : \big((A \to E) \rhd \epsilon_1\big) & \Gamma \vdash v : A \end{array}}{\Gamma \vdash^M (e\, v) : E_1} \text{ app}$$

$$\frac{\Gamma \vdash v : \mathsf{Nm}[X] \qquad \Gamma \vdash^M e : E}{\Gamma \vdash^M \mathsf{thunk}(v, e) : \big(\mathbf{F}\,(\mathsf{Thk}[M[X]]\ E)\big) \rhd \langle M[X]; \emptyset \rangle} \text{ thunk}$$

$$\frac{\Gamma \vdash v : \mathsf{Thk}[X]\ (C \rhd \epsilon) \qquad \Gamma \vdash (\langle \emptyset; X \rangle \text{ then } \epsilon) = \epsilon'}{\Gamma \vdash^M \mathsf{force}(v) : (C \rhd \epsilon')} \text{ force}$$

$$\frac{\Gamma \vdash v_1 : \mathsf{Nm}[X] \qquad \Gamma \vdash v_2 : A}{\Gamma \vdash^M \mathsf{ref}(v_1, v_2) : \mathbf{F}\,(\mathsf{Ref}[M[X]]\ A) \rhd \langle M[X]; \emptyset \rangle} \text{ ref} \qquad \frac{\Gamma \vdash v : \mathsf{Ref}[X]\ A}{\Gamma \vdash^M \mathsf{get}(v) : (\mathbf{F}\, A) \rhd \langle \emptyset; X \rangle} \text{ get}$$

$$\frac{\begin{array}{l} \Gamma \vdash v_M : (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})[M] \\ \Gamma \vdash v : \mathsf{Nm}[i] \end{array}}{\Gamma \vdash^N (v_M\, v) : \mathbf{F}\,(\mathsf{Nm}[M[i]]) \rhd \langle \emptyset; \emptyset \rangle} \text{ name-app} \qquad \frac{\begin{array}{ll} & \Gamma \vdash v : (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})[N'] \\ \Gamma \vdash^{N \circ N'} e : C \rhd \langle W; R \rangle & \end{array}}{\Gamma \vdash^N \mathsf{scope}(v, e) : C \rhd \langle W; R \rangle} \text{ scope}$$

$$\frac{\Gamma, \alpha : K \vdash^M t : E}{\Gamma \vdash^M t : (\forall \alpha : K.\, E)} \text{ etype-}\forall\text{Intro} \qquad \frac{\Gamma \vdash^M e : (\forall \alpha : K.\, E) \qquad \Gamma \vdash A : K}{\Gamma \vdash^M e : [A/\alpha]E} \text{ etype-}\forall\text{Elim}$$

$$\frac{\Gamma, a : \gamma, P \vdash^M t : E}{\Gamma \vdash^M t : (\forall a : \gamma \mid P.\, E)} \text{ etype-}\forall\text{IndexIntro} \qquad \frac{\begin{array}{ll} & \mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash i : \gamma & \Gamma \vdash^M e : (\forall a : \gamma \mid P.\, E) \end{array}}{\Gamma \vdash^M e : [i/a]E} \text{ etype-}\forall\text{IndexElim}$$

$$\frac{\Gamma \vdash v : (\exists a : \gamma \mid P.\, A) \qquad \Gamma, a : \gamma, P, x : A \vdash^M e : E}{\Gamma \vdash^M \mathsf{vunpack}(v, a.x.e) : E} \text{ etype-}\exists\text{IndexElim}$$

Fig. 13. Computation typing

application $A\ B$ of a type A (e.g. a type constructor d) to a type B; the type $\mathsf{Nm}[i]$; and a singleton type $(\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})[M]$ where M is a function on names.

As usual in call-by-push-value, computation types C and D include a connective $\mathbf{F}$, which "lifts" value types to computation types: $\mathbf{F}\,A$ is the type of computations that, when run, return a value of type A. (Call-by-push-value usually has a connective dual to $\mathbf{F}$, written $\mathbf{U}$, that "th**U**nks" a computation type into a value type; in our system, Thk plays the role of $\mathbf{U}$.)

Computation types also include functions, written $A \to E$. In standard CBPV, this would be $A \to C$, not $A \to E$. We separate computation types alone, written C, from computation types with effects, written E; this decision is explained in Appendix B.1.

Computation types-with-effects E consist of $C \rhd \epsilon$, which is the bare computation type C with effects ϵ, as well as universal quantifiers (polymorphism) over types $(\forall \alpha : K.\, E)$ and indices $(\forall a : \gamma \mid P.\, E)$. In the latter quantifier, the proposition P lets us express quantification over disjoint sets of names.

Value typing rules. The typing rules for values (Figure 12) for unit, variables and pairs are standard. Rule 'name' uses index-level entailment to check that the name n is in the name set X. Rule 'namefn' checks that M_ν is well-sorted, and that M_ν is convertible to M. Rule 'ref' checks that n is in X, and that $\Gamma(n) = A$, that is, the typing $n : A$ appears somewhere in Γ. Rule 'thunk' is similar to 'ref'.

Computation typing rules. Many of the rules that assign computation types (Figure 13) are standard—for call-by-push-value—with the addition of effects and the namespace M. The rules 'split' and 'case' have nothing to do with namespaces or effects, so they pass M up to their premises, and leave the type E unchanged. Empty effects are added by rules 'ret' and 'lam', since both $\mathtt{ret}$ and λ do not read or write anything. The rule 'let' uses effect sequencing to combine the effects of e_1 and the let-body e_2. The rule 'force' also uses effect sequencing, to combine the effect of forcing the thunk with the read effect $\langle \emptyset; X \rangle$.

The only rule that modifies the namespace is 'scope', which composes the given namespace N (in the conclusion) with the user's $\nu = \mathtt{nmfn}\ N'$ in the second premise (typing e).

5.5 Subtyping

As discussed above, our type system can overapproximate names. The type $\mathsf{Nm}[X]$ means that the name is contained in the set of X; unless X is a singleton, the type system does not guarantee the specific name. Approximation induces subtyping: we want to allow a program to pass $\mathsf{Nm}[X_1]$ to a function expecting $\mathsf{Nm}[X_1 \perp X_2]$.

To design subtyping rules that are correct and easy to implement, we turn to the DML descendant Stardust [Dunfield 2007]. The subtyping rules in Stardust are generally a helpful guide, with the exception of the rule that compares atomic refinements. In Dunfield's system, $\tau[i] \le \tau[j]$ if $i = j$ in the underlying index theory. For example, a list of length i is a subtype of a list of length j if and only if $i = j$ in the theory of integers. While approximate in the sense of considering all lists of length i to have the same type, the length itself is not approximate.

In contrast, our name set indices are approximations. Thus, our rule $\le_V$-name (Figure 14) says that $\mathsf{Nm}[X] \le_V \mathsf{Nm}[Y]$ if $X \subseteq Y$, rather than $X = Y$. Similarly, subtyping for references and thunks ($\le_V$-ref, $\le_V$-thk) checks inclusion of the associated name (pointer) set, not strict equality.

Our polymorphic types combine two fundamental typing constructs, universal quantification and guarded types (requiring that P hold for the quantified index a), so our rule $\le_V$-$\forall$L combines the Stardust rules ΠL for index-level quantification and $\supset$L for the guarded type [Dunfield 2007, p. 33]. Likewise, our $\le_V$-$\forall$R combines Stardust's ΠR and $\supset$R.

$\boxed{\Gamma \vdash A \leq_V B}$ Value type A is a subtype of B

$$\frac{}{\Gamma \vdash A \leq_V A}\ \leq_V\text{-refl} \qquad \frac{\Gamma \Vdash X \subseteq Y}{\Gamma \vdash \mathsf{Nm}\,[X] \leq_V \mathsf{Nm}\,[Y]}\ \leq_V\text{-name}$$

$$\frac{\Gamma \vdash A_1 \leq_V B_1 \qquad \Gamma \vdash A_2 \leq_V B_2}{\Gamma \vdash A_1 \times A_2 \leq_V B_1 \times B_2}\ \leq_V\text{-}\times \qquad \frac{\Gamma \vdash A_1 \leq_V B_1 \qquad \Gamma \vdash A_2 \leq_V B_2}{\Gamma \vdash A_1 + A_2 \leq_V B_1 + B_2}\ \leq_V\text{-}+$$

$$\frac{\mathit{extract}(\Gamma) \Vdash X \subseteq Y \qquad \Gamma \vdash A \leq_V B}{\Gamma \vdash (\mathsf{Ref}\,[X]\ A) \leq_V (\mathsf{Ref}\,[Y]\ B)}\ \leq_V\text{-ref}$$

$$\frac{\mathit{extract}(\Gamma) \Vdash X \subseteq Y \qquad \Gamma \vdash E_1 \leq_C E_2}{\Gamma \vdash (\mathsf{Thk}\,[X]\ E_1) \leq_V (\mathsf{Thk}\,[Y]\ E_2)}\ \leq_V\text{-thk}$$

$$\frac{\Gamma \vdash M_1 =_\beta M_2}{\Gamma \vdash (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M_1] \leq_V (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M_2]}\ \leq_V\text{-namefn}$$

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash [i/a]A \leq_V B\end{array}}{\Gamma \vdash (\forall a : \gamma \mid P.\,A) \leq_V B}\ \leq_V\text{-}\forall\text{L} \qquad \frac{\Gamma, b : \gamma, P \vdash A \leq_V B}{\Gamma \vdash A \leq_V (\forall b : \gamma \mid P.\,B)}\ \leq_V\text{-}\forall\text{R}$$

$$\frac{\Gamma, a : \gamma, P_a \vdash A \leq_V [a/b]B \qquad \mathit{extract}(\Gamma, a : \gamma, P_a) \Vdash [a/b]P_b}{\Gamma \vdash (\exists a : \gamma \mid P_a.\,A) \leq_V (\exists b : \gamma \mid P_b.\,B)}\ \leq_V\text{-}\exists$$

Fig. 14. Subtyping on value types

$\boxed{\Gamma \vdash C \leq_C D}$ Computation type C is a subtype of D

$$\frac{\Gamma \vdash A \leq_V B}{\Gamma \vdash \mathbf{F}\,A \leq_C \mathbf{F}\,B}\ \leq_C\text{-lift} \qquad \frac{\Gamma \vdash A_2 \leq_V A_1 \qquad \Gamma \vdash E_1 \leq_E E_2}{\Gamma \vdash (A_1 \to E_1) \leq_C (A_2 \to E_2)}\ \leq_C\text{-arr}$$

$\boxed{\Gamma \vdash E_1 \leq_E E_2}$ Type-with-effects E_1 is a subtype of E_2

$$\frac{\Gamma \vdash C_1 \leq_C C_2 \qquad \Gamma \vdash \epsilon_1 \leq \epsilon_2}{\Gamma \vdash (C_1 \rhd \epsilon_1) \leq_C (C_2 \rhd \epsilon_2)}\ \leq_E\text{-eff} \qquad \frac{\Gamma, \alpha : K \vdash E_1 \leq_E E_2}{\Gamma \vdash (\forall \alpha : K.\,E_1) \leq_E (\forall \alpha : K.\,E_2)}\ \leq_E\text{-all-type}$$

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash [i/a]E_1 \leq_E E_2\end{array}}{\Gamma \vdash (\forall a : \gamma \mid P.\,E_1) \leq_E E_2}\ \leq_E\text{-all-index-L} \qquad \frac{\Gamma, a : \gamma, P \vdash E_1 \leq_E E_2}{\Gamma \vdash E_1 \leq_E (\forall a : \gamma \mid P.\,E_2)}\ \leq_E\text{-all-index-R}$$

Fig. 15. Subtyping on computation types

Unlike Stardust's Σ (and unlike our $\forall$), our existential types have a term-level pack construct, so an $\exists$ cannot be a sub- or supertype of a non-existential type. Thus, instead of rules analogous to Stardust's ΣL and ΣR, we have a single rule $\leq_V$-$\exists$ with $\exists$ on both sides, which specializes ΣR to the case when ΣL derives its premise. Like $\forall$, our $\exists$ incorporates a constraint P on the quantified variable, so our $\leq_V$-$\exists$ also incorporates the Stardust rules for *asserting types* (⅋), checking that P_a entails P_b.

For refs and thunks, rules $\leq_V$-ref and $\leq_V$-thk are covariant in the name set describing the location. They are also covariant in the type of their contents: unlike an ordinary ML `ref` type, our `Ref` names a location, but the programs described by our type system cannot mutate that location. (To extend our theory to describe *editor* programs, we would need different rules; see Section 8.2.)

In our subtyping rules for computation types (Figure 15), rule $\leq_C$-arr reflects the usual contravariance of function domains, rule $\leq_E$-eff allows subsumption within effects ϵ, and the rules for computation-level $\forall$ follow our rules for value-level $\forall$.

Instead of an explicit transitivity rule, which is not trivial to implement, the transitivity of subtyping is admissible.

5.6 Bidirectional Version

The typing rules in Figures 12 and 13 are declarative: they define what typings are valid, but not how to derive those typings. The rules' use of names and effects annotations means that standard unification-based techniques, like Damas–Milner inference, are not easily applicable. For example, it is not obvious when to apply chk-AllIntro, or how to solve unification constraints over names and name sets.

We therefore formulate bidirectional typing rules that directly give rise to an algorithm. For space reasons, this system is presented in the supplementary material (Appendix D). We prove (in Appendix E) that our bidirectional rules are sound and complete with respect to the type assignment rules in this section:

Soundness (Thms. E.1, E.3): Given a bidirectional derivation for an annotated expression e, there exists a type assignment derivation for e without annotations.

Completeness (Thms. E.2, E.4): Given a type assignment derivation for e without annotations, there exist two annotated versions of e: one that synthesizes, and one that checks. (This result is sometimes called *annotatability*.)

6 DYNAMIC SEMANTICS

Name terms. Recall Fig. 8 (Sec. 4.2), which gives the dynamics for evaluating name term M to name term value V. Because name terms have no recursion, evaluating a well-sorted name term always produces a value (Theorem H.9).

Program expressions (Figure 16). Stores hold the mutable state that names dynamically identify. Big-step evaluation for expressions relates an initial and final store, and the "current scope" and "current node", to a program and value. We define this dynamic semantics, which closely mirrors prior work, to show that well-typed evaluations always allocate precisely.

To make this theorem meaningful, the dynamics permits programs to *overwrite* prior allocations with later ones: if a name is used ambiguously, the evaluation will replace the old store content with the new store content. The rules $\Downarrow$-ref and $\Downarrow$-thunk either extend or overwrite the store, depending on whether the allocated pointer name is precise or ambiguous, respectively. We prove that, in fact, well-typed programs always extend (and never overwrite) the store in any single derivation. (During change propagation, not modeled here, we begin with a store and dependency graph from a prior run, and even precise programs overwrite the store/graph, as discussed in Sec. 1.)

While motivated by incremental computation, we are interested in precise effects here, not change propagation itself. Consequently, this semantics is simpler than the dynamics of prior work. First, the store never caches values from evaluation, that is, it does not model function caching (memoization). Next, we do not build the dependency edges required for change propagation. Likewise, the "current node" is not strictly necessary here, but we include it for illustration. Were

$$
\begin{array}{lrcll}
\text{Pointers} & p, q & ::= & n & \text{name constants} \\
\text{Stores} & S & ::= & \cdot & \text{empty store} \\
 & & & \mid S, p{:}v & p \text{ points to value } v \\
 & & & \mid S, p{:}e \mathbin{@} M & p \text{ points to thunk } e, \text{ run in scope } M
\end{array}
$$

Notation: $S\{p{\mapsto}v\}$ and $S\{p{\mapsto}e@M\}$ *extend* S at p when $p \notin \mathrm{dom}(S)$
$S\{p{\mapsto}v\}$ and $S\{p{\mapsto}e@M\}$ *overwrite* S at p when $p \in \mathrm{dom}(S)$

$\boxed{S_1 \vdash^{M}_{m} e \Downarrow S_2; t}$ Under store S in namespace M at current node m, expression e produces new store S_2 and result t

$$\frac{S_1 \vdash^{M_1 \circ M_2}_{m} e \Downarrow S_2; e'}{S_1 \vdash^{M_1}_{m} \mathtt{scope}(M_2, e) \Downarrow S_2; e'} \Downarrow\text{-scope} \qquad \frac{M_1 \Downarrow_M \lambda a.\, M_2 \qquad [n/a]M_2 \Downarrow_M p}{S \vdash^{M}_{m} M_1\ (\mathtt{name}\ n) \Downarrow S; \mathtt{ret}(\mathtt{name}\ p)} \Downarrow\text{-name-app}$$

$$\frac{(M\,n) \Downarrow_M p \qquad S_1\{p{\mapsto}e \mathbin{@} M\} = S_2}{S_1 \vdash^{M}_{m} \mathtt{thunk}(\mathtt{name}\ n, e) \Downarrow S_2; \mathtt{ret}(\mathtt{thunk}\ p)} \Downarrow\text{-thunk} \qquad \frac{(M\,n) \Downarrow_M p \qquad S_1\{p{\mapsto}v\} = S_2}{S_1 \vdash^{M}_{m} \mathtt{ref}(\mathtt{name}\ n, v) \Downarrow S_2; \mathtt{ret}(\mathtt{ref}\ p)} \Downarrow\text{-ref}$$

$$\frac{S(p) = e \mathbin{@} M_0 \qquad S_1 \vdash^{M_0}_{p} e \Downarrow S_2; t}{S_1 \vdash^{M}_{m} \mathtt{force}(\mathtt{thunk}\ p) \Downarrow S_2; t} \Downarrow\text{-force} \qquad \frac{S(p) = v}{S \vdash^{M}_{m} \mathtt{get}(\mathtt{ref}\ p) \Downarrow S; \mathtt{ret}(v)} \Downarrow\text{-get} \qquad \frac{}{S \vdash^{M}_{m} t \Downarrow S; t} \Downarrow\text{-term}$$

Fig. 16. Excerpt from the dynamic semantics (see also Figure 21)

we modeling change propagation, rules ⇓-ref, ⇓-thunk, ⇓-get and ⇓-force would create dependency edge structure that we omit here. (These edges relate the current node with the node being observed.)

7 METATHEORY: TYPE SOUNDNESS AND PRECISE EFFECTS

In this section, we prove that our type system is sound with respect to evaluation and that the type system enforces precise effects: We establish that a well-typed, terminating program produces a terminal computation of the program's type, and that the actual dynamic effects are precise (Def. 7.2). Specifically, we show that the type system's static effects soundly approximate this dynamic behavior. Consequently, sequenced writes never overwrite one another.

We sometimes constrain typing contexts to be *store types*, which type store pointers but not program variables; hence, they only type *closed* values and programs:

Definition 7.1 (Store type)**.** *We say that* Γ *is a* store typing*, written* Γ store-type*, when each assumption in* Γ *has the reference-pointer form* $p : A$ *or the thunk-pointer form* $p : E$.

Definition 7.2 (Precise effects)**.** *Given an evaluation derivation* $\mathcal{D}$*, we write* $\mathcal{D}$ reads R writes W *for its* precise effects *(Figure 22 in the appendix).*

This is a (partial) function over derivations. We call these effects "precise" since sibling sub-derivations must have disjoint write sets.

Main theorem: We write $\langle W'; R' \rangle \preceq \langle W; R \rangle$ to mean that $W' \subseteq W$ and $R' \subseteq R$. For proofs, see Appendix C.

Theorem 7.1 (Subject Reduction).
If Γ_1 store-type *and* $\Gamma_1 \vdash M : \mathbf{Nm} \overset{\mathbf{Nm}}{\Rightarrow} \mathbf{Nm}$ *and* $\mathcal{S}$ *derives* $\Gamma_1 \vdash^{M} e : C \rhd \langle W; R \rangle$
and $\vdash S_1 : \Gamma_1$ *and* $\mathcal{D}$ *derives* $S_1 \vdash^{M}_{m} e \Downarrow S_2; t$ *then*
there exists $\Gamma_2 \supseteq \Gamma_1$ *such that* Γ_2 store-type *and* $\vdash S_2 : \Gamma_2$ *and* $\Gamma_2 \vdash t : C \rhd \langle \emptyset; \emptyset \rangle$
and $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ *and* $\langle W_{\mathcal{D}}; R_{\mathcal{D}} \rangle \preceq \langle W; R \rangle$.

8 IMPLEMENTATION

8.1 Prototype in Rust

Using this on-paper design as a guide, we have implemented a preliminary prototype of Fungi in Rust. In particular, we implement each abstract syntax definition and typing judgement presented in this paper and appendix as a Rust datatype (a "deep" embedding of the language into Rust). We implement the bidirectional type system (Sec. D) as a family of Rust functions that produce judgement data structures (possibly with nested type or effect errors) from a Fungi syntax tree.

By using Rust macros, we implement a concrete syntax and associated parser that suffices for authoring examples similar to those in Sec. 3. In two ways, we deviate from the Fungi program syntax presented here: (1) Rust macros can only afford certain concrete syntaxes (2) Fungi programs use explicit (not implicit) index and type applications; inferring these arguments is future work.

The implementation of Fungi is documented and publicly available. At present, it consists of about 10K lines of Rust, and is complete enough to type check many basic examples, including the effects of structural recursion (`max` and `filter`, Fig. 3.2). We are actively extending the implementation to encompass the full reasoning power of the index and name term sub-languages, to type the rest of Sec. 3, and beyond.

For the latest version of Fungi, see `crates.io` and/or `docs.rs`, and search for "fungi-lang". *Note to reviewers: visiting those sites will deanonymize the authors; see supplemental material instead.*

8.2 Ongoing and Future Work

We are currently exploring a number of extensions to the formulation of Fungi presented here.

Interactive type derivations. To debug the examples' type and effect errors, we load the (possibly incomplete) typing derivations in an associated interactive, web-based tool. The tool makes the output typing derivation *interactive*: using a pointer, we can inspect the syntactic family/constructor, typing context, type and effect of each subterm in the input program, including indices, name terms, sorts, values, expressions, etc. Compared with getting parsing or type errors out of context (or else, only with an associated line number), we've found this interactive tool very helpful for teaching newcomers about Fungi's abstract syntax rules and type system, and for debugging examples (and Fungi) ourselves. This tool, the *Human-Fungi Interface* (HFI), is publicly available software.

As future work, we will extend HFI into an interactive *program editor*, based on our existing bidirectional type system, and the (typed) structure editor approach developed by Omar et al. [2017a]. We speculate that Fungi *itself* may be useful in the implementation of this tool, by providing language support for interactive, *incremental* developer features [Omar et al. 2017b]. Current approaches prescribe conversion to a distinct, "co-contextual" judgement form, whose design circumvents the memoization failure issue outlined in Sec. 1.1, but requires first *transforming* the contexts, and the modalities of the typing rules [Erdweg et al. 2015a; Kuci et al. 2017]. Fungi's explicit-name programming model may offer an alternative approach for authoring incremental type checkers, based on their "ordinary" judgments (rules, typing contexts, and modalities).

Incremental semantics for Fungi. Though not the focus on this paper, Fungi is an incremental language. We implement the incremental runtime semantics of Fungi by writing an interpreter using Adapton in Rust, as provided by an existing external library [Hammer et al. 2014, 2015; Adapton Developers 2018]. This external library implements the dynamic dependency graphs described (statically) by Fungi's type and effect system. Our near-time goal is to use Fungi as a target language for programs that act like "incremental glue" for mixtures of Fungi code and (appropriately behaved) high-performance Rust code.

Future work: Editor and Archivist. To distinguish imperative name allocation from name-precise computation, future versions of Fungi will introduce two *incremental computation roles*, which we term the *editor* and the *archivist*, respectively; specifically, we define the syntax for roles as $r ::= \mathsf{ed} \mid \mathsf{ar}$. The archivist role (ar) corresponds to computation whose dependencies we cache, and the editor role (ed) corresponds to computation that feeds the archivist with input changes, and demands any changed output that is relevant; in short, the editor represents the world outside the cached computation.

While the current type system prototype focuses only on the *archivist* role, leaving the editor role to the surrounding Rust code, future work will integrate the editor role into Fungi programs. For example, consider the following typing rules, which approximate (and extend) our full type system with a *role* r in each rule:

$$\frac{\Gamma \vdash v_n : \mathsf{Nm}[X] \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{ref}(v_n, v) : \mathsf{Ref}(A) \rhd r(X)} \qquad \frac{\Gamma \vdash e_1 : A \rhd \mathsf{ar}(X) \quad \Gamma, x : A \vdash e_2 : B \rhd \mathsf{ar}(Y) \quad \Gamma \vdash (X \perp Y) \equiv Z : \mathbf{NmSet}}{\Gamma \vdash \mathtt{let}(e_1, x.e_2) : B \rhd \mathsf{ar}(Z)}$$

$$\frac{\Gamma \vdash e_1 : A \rhd \mathsf{ed}(X) \quad \Gamma, x : A \vdash e_2 : B \rhd \mathsf{ed}(Y) \quad \Gamma \vdash (X \cup Y) \equiv Z : \mathbf{NmSet}}{\Gamma \vdash \mathtt{let}(e_1, x.e_2) : B \rhd \mathsf{ed}(Z)}$$

These rules are similar to the simplified rules presented in Sec. 2. In contrast to those rules, these conclude with the judgement form $\Gamma \vdash e : A \rhd r(X)$, mentioning the written set with the notation $\rhd r(X)$, where the set X approximates the set of written names (as in the earlier formulation), and r is the role (absent from the earlier formulation).

The first rule types a reference cell allocation, as before; in the rule's conclusion, this name set X serves as the allocation's *write set*. The undetermined role r means that this rule is applicable to both the editor and the archivist roles.

What was one `let` sequencing rule (in Sec. 2) is now two rules here: The second rule enforces the archivist role, where names are precise. The third rule permits the editor role, where names allocated later may *overwrite* names allocated earlier. Finally, a new syntax form `archivist(e)` permits the editor's computations to delegate to archivist sub-computations; the program `archivist(e)` has role ed whenever program e types under role ar under the same typing context.

Among the future work for mixing these roles, we foresee that extending the theory of Fungi, including covariant index subtyping, to this mixture of imperative-functional execution semantics requires mixing imperative effects (for the editor) and type index subtyping (for the archivist) in a disciplined, sound manner.

9 RELATED WORK

DML [Xi and Pfenning 1999; Xi 2007] is an influential system of limited dependent types or *indexed* types. Inspired by Freeman and Pfenning [1991], who created a system in which datasort refinements were clearly separated from ordinary types, DML separates the "weak" index level of typing from ordinary typing; the dynamic semantics ignores the index level.

Motivated in part by the perceived burden of type annotations in DML, liquid types [Rondon et al. 2008; Vazou et al. 2013] deploy machinery to infer more types. These systems also provide more flexibility: types are not indexed by fixed tuples.

To our knowledge, Gifford and Lucassen [1986] were the first to express effects within (or alongside) types. Since then, a variety of systems with this power have been developed. A full accounting of this area is beyond the scope of this paper; for an overview, see Henglein et al. [2005]. We briefly discuss a type system for regions [Tofte and Talpin 1997], in which allocation is central. Regions organize subsets of data, so that they can be deallocated together. The type system tracks each block's region, which in turn requires effects on types: for example, a function whose effect is to return a block within a given region. Our type system shares region typing's emphasis on

allocation, but we differ in how we treat the names of allocated objects. First, names in our system are fine-grained, in contrast to giving all the objects in a region the same designation. Second, names have structure—for example, the names $0{\cdot}n = \langle\langle \texttt{leaf}, n\rangle\rangle$ and $1{\cdot}n = \langle\langle\langle\langle \texttt{leaf}, \texttt{leaf}\rangle\rangle, n\rangle\rangle$ share the right subtree n—which allows programmers to deterministically compute two distinct names from one.

Type systems for variable binding and fresh name generation, such as FreshML [Pitts and Gabbay 2000] and Pure FreshML [Pottier 2007], can express that sets of names are disjoint. But the names lack internal structure that relates specific names across disjoint name sets.

Compilers have long used alias analysis to support optimization passes. Brandauer et al. [2015] extend alias analysis with disjointness domains, which can express local (as well as global) aliasing constraints. Such local constraints are more fine-grained than classic region systems; our work differs in having a rich structure on names.

Techniques for general-purpose incremental computation. General-purpose incremental computation techniques provide a general-purpose *change propagation* algorithm. In particular, after an initial run of the program, as the input changes dynamically, change propagation provides a provably sound approach for recomputing the affected output [Acar et al. 2006a; Acar and Ley-Wild 2009; Hammer et al. 2014, 2015]. Incremental computation can deliver *asymptotic* speedups for certain algorithms [Acar et al. 2007, 2008, 2009; Sümer et al. 2011; Burckhardt et al. 2011; Chen et al. 2012], and has even addressed open problems [Acar et al. 2010]. These incremental computing abstractions exist in many languages [Shankar and Bodik 2007; Hammer et al. 2009; Acar and Ley-Wild 2009]. The type and effect system proposed here complements past work on self-adjusting computation. In particular, we expect that variations of the proposed type system can express and verify the use of names in much of the work cited above.

Çiçek et al. [2015, 2016] develop cost semantics for incremental programs. Their work assumes a dynamic dependency graph with *fixed* structure: for example, the length of an input list cannot change across successive incremental runs. In fact, extending their cost semantics to allow general, structural changes (e.g., insertion or removal of list elements) while still describing the "true" cost of change propagation will require integrating a notion of names: Without one, constant-sized input changes will generally cascade, precipitating needlessly-inefficient change propagation behavior (as in `rev`, from Sec. 1.1). Hence, we feel that our work complements theirs, and combining these approaches remains an exciting direction for future work.

Imprecise (ambiguous) names. Some past systems dynamically detect ambiguous names, either forcing the system to fall back to a non-deterministic name choice [Acar et al. 2006a; Hammer and Acar 2008], or to signal an error and halt [Hammer et al. 2015]. In scenarios with a non-deterministic fall-back mechanism, a name ambiguity carries the potential to degrade incremental performance, making it less responsive and asymptotically unpredictable in general [Acar 2005]. To ensure that incremental performance gains are predictable, past work often merely assumes, without enforcement, that names are precise [Ley-Wild et al. 2009].

Fortunately, these existing approaches are complementary to Fungi, whose type and effect system is applicable to each, either *directly* (in the case of Adapton, and variants), or with some minor adaptations (as we speculate for the others).

10 CONCLUSION

We define the *precise name problem* for programs that use explicit names to identify their dynamic data and sub-computations. We define a solution in the form of Fungi, a core calculus for such programs, whose type and effect system describes and verifies their allocation names. We derive a

bidirectional version of the type and effect system, and we implement a closely-related prototype of Fungi in Rust, as a deeply-embedded DSL. We apply Fungi to a library of incremental collections.

Our ongoing and future work on Fungi builds on initial prototypes reported here: We are extending Fungi to settings that *mix* imperative and functional programming models, and we are creating richer tools for developing, debugging and visualizing Fungi programs in the context of larger systems (e.g., written in Rust).

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A PROGRAMMING EXAMPLES: LISTS, POLYMORPHISM

Through self-contained running examples that define and compute over lists, we give a an overview of our type system. We work towards showing examples of datatypes, and datatype polymorphism in our proposed system, including how to define and use abstract types that contain type parameters. Specifically, we show how to define polymorphic lists that can be instantiated appropriately to define lists of lists of integers.

We begin with a type for incremental lists of integers. We use this list type to write `list_map`. We show how this code, and in particular its use of names, induces a dependence graph with names that derive from the input list. In particular, we explain two connected activities: (1) augmenting functional programs to name data and computations, and (2) generalizing these naming strategies to write composable library functions. Toward the latter goal, we show how to generalize lists of integers to lists of other structures that may contain names, such as other (sub)-lists.

In support of creating composable incremental computing libraries, we consider two composition problems, and illustrate how Fungi introduces additional type-level affordances to overcome them:

- Names are *not* linear resources: It is natural and common to consume a name more than once within a computation.[2] To disambiguate multiple writes of the same name, we introduce name functions that permit programmers to create distinct dynamic scopes. To streamline programs, Fungi employs a scope monad to capture this dynamic structure.
- *Naive type polymorphism* ignores how names relate between a structure (e.g., a list of lists) and the structures that it contains (e.g., the sub-lists of the outer list). To describe and enforce name-based relationships generically, we parameterize generic types with index-transforming functions. In turn, these index functions enforce client-chosen invariants that relate the type indices of sub- and super-structures, which may involve arbitrary name sets. For instance, we can write *one* definition of polymorphic lists, and as clients, instantiate this definition twice to define lists of lists which hold integers.

A.1 Lists of integers, with names

The type below defines incremental lists of integers, $\mathsf{List}\,[X;Y]\ \mathsf{Int}$. This type has two conventional constructors, `Nil` and `Cons`, as well as two additional constructors that use the two type indices X and Y. Each index has *name set* sort **NmSet**, which classifies indices that are sets of names.

The `Name` constructor permits names from set X_1 to appear in the list sequence; it creates a `Cons`-cell-like pair holding a first-class name from X_1 (of type $\mathsf{Nm}\,[X_1]$), and the rest of the sequence, whose names are from X_2. The `Ref` constructor permits injecting reference cells holding lists into the list type; these pointers' names are drawn from set $Y = (Y_1 \perp Y_2)$, the disjoint union of the possible pointer names Y_1 for the head reference cell, and the pointer names Y_2 contained in this cell's list.

$$
\begin{array}{lcllll}
\texttt{Nil} & : & \forall X, Y : \textbf{NmSet}. & \mathsf{unit} & \rightarrow & \mathsf{List}\,[X;Y]\ \mathsf{Int} \\
\texttt{Cons} & : & \forall X, Y : \textbf{NmSet}. & \mathsf{Int} \rightarrow \mathsf{List}\,[X;Y]\ \mathsf{Int} & \rightarrow & \mathsf{List}\,[X;Y]\ \mathsf{Int} \\
\texttt{Name} & : & \forall X_1 \perp X_2, Y : \textbf{NmSet}. & \mathsf{Nm}\,[X_1] \rightarrow \mathsf{List}\,[X_2;Y]\ \mathsf{Int} & \rightarrow & \mathsf{List}\,[X_1 \perp X_2;Y]\ \mathsf{Int} \\
\texttt{Ref} & : & \forall X, Y_1 \perp Y_2 : \textbf{NmSet}. & \mathsf{Ref}\,[Y_1]\ (\mathsf{List}\,[X;Y_2]\ \mathsf{Int}) & \rightarrow & \mathsf{List}\,[X;Y_1 \perp Y_2]\ \mathsf{Int}
\end{array}
$$

For both of these latter forms, the constructor's types enforce that each name (or pointer name) in the list is disjoint from those in the remainder of the list. For instance, when quantifying over these sets, we write $X_1 \perp X_2$ to impose the constraint that X_1 be disjoint from X_2, and similarly for $Y_1 \perp Y_2$; However, for the `Name` constructor, the names in X_1 and pointer names in Y may overlap

[2] As anecdotal evidence, consider that functional programs commonly exploit sharing in data structures and algorithms, and often do not adhere to a linear typing discipline. We still wish to name these computation patterns and data structures.

(or even coincide). These disjointness constraints consist of syntactic sugar that we expand in Sec. 5. The `Nil` constructor creates an empty sequence with *any* pointer or name type sets in its resulting type, since, for practical reasons, we find it helpful to permit type indices to *over-approximate* name sets.

We distinguish two roles of names: *unallocated names*, which are unassociated with content of any type, and which need only be *locally unique* (e.g., unique to a list or other data structure), and *allocated names*, which dynamically name content of some fixed type and must be *globally unique*; they each identify a pointer allocated in the store.

In the `Name(n,t)` form, the name `n` is unallocated, and can be used to allocate a reference or thunk of *any* type. In the `Ref(r)` form, by contrast, we have a reference cell `r` that consists of a pointer name whose type is parameterized by the type of the content it names, in this case an integer list.

The scope `s` distinguishes namespaces, and is explained in further detail below; intuitively, name function `s` translates a locally unique name (unique to the list) into a globally unique name (unique to global program evaluation).

A.2 Mapping a list of integers, with names

Fig. 17 (left listing) lists the type, effect and code for `list_map0`, which consumes the list type defined above, but does not allocate any references or thunks. We include it here for illustrative purposes, as preparation for further examples. The type signature includes abstract name sets X and Y, for the names and named pointers of the input list `l`. The resulting list type is indexed by the name set X and the empty pointer set $\emptyset$ since the output contains the same names as the input (all drawn from X), but does contain any pointers. The function's effect, written as $\triangleright\langle\emptyset; Y\rangle$, indicates that the code writes no names, but reads the pointers Y from the input list `l`.

Turning to the program text for `list_map0`, the `Nil` and `Cons` cases are entirely conventional: They return `Nil` and apply `f` to the head of each `Cons` cell, respectively. The `Name` and `Ref` cases handle unallocated names and (allocated) pointers in the list; both are simple recursive cases. In the `Name` case, `list_map0` maps the input name into corresponding position of the output list and recurs. In the `Ref` case, `list_map0` uses `get` to observe the content of the reference cell holding the remainder of the input, and recurs on this input list.

The right listing gives `list_map1`, whose code and behavior is identical to `list_map0` on the left, except for in the `Name` case, where the right version allocates a reference cell to hold the output list, and it allocates (and forces) a thunk to memoize the recursive call. (The shorthand `memo(`e_1`,`e_2`)` expands into `force(thunk(`e_1`,`e_2`))`). To name these allocations, `list_map1` uses the input list's name `n` for the reference cell and a distinct name for the thunk, $(\mathtt{n}{\cdot}1) :\equiv \langle\langle \mathtt{n}, \mathtt{leaf}\rangle\rangle \neq \mathtt{n}$. Critically, the name $\mathtt{n}{\cdot}1$ is distinct from both `n` *and* any name that is distinct from `n`, including the other names in the list. That is, for all `m`, we have that $(\mathtt{n} \perp \mathtt{m}) \Rightarrow (\mathtt{n} \perp \mathtt{n}{\cdot}1 \perp \mathtt{m} \perp \mathtt{m}{\cdot}1)$, meaning that the four elements in the consequent are pairwise distinct. Generalizing further, we can use this pattern to systematically add memoization *and* reference cells to any structurally recursive algorithm that includes an analog of this `Name` case. We show examples of such algorithms, below.

Comparing its type with `list_map0`, the returned list type of `list_map1` now contains names from X (as before) and pointers from $\star(X)$. The effect is also affected: the written name set is now $\star(X \perp X{\cdot}1)$, not $\emptyset$. These sets refer to the names written in the revised `Name` case, mapped by the current monadic scope, which we denote with the notation $\star(-)$; recall that the scope is a dynamically specified name-transforming function that we apply to each written name, in this case drawn from $X \perp X{\cdot}1$. Further, we use shorthand $X{\cdot}1$ to informally mean "the names of set X, mapped by the function $\lambda a.\, \langle\langle a, \mathtt{leaf}\rangle\rangle$"; in Sec. 5, we make the syntax for name sets, type indices and effects precise.

```
list_map0 : ∀X, Y : NmSet.
  (Int → Int) → (List [X; Y] Int) →
  (List [X; ∅] Int)
  ▷⟨∅; Y⟩
list_map0 = λf. fix rec. λl.
 match l with
  Nil        ⇒ Nil
  Cons(h,t)  ⇒ Cons(f h, rec t)
  Name(n,t)  ⇒ Name(n,   rec t)

  Ref(r)     ⇒ rec (get r)
```

```
list_map1 : ∀X, Y : NmSet.
  (Int → Int) → (List [X; Y] Int) →
  (List [X; ⋆(X)] Int)
  ▷⟨⋆(X⊥X·1); Y⟩
list_map1 = λf. fix rec. λl.
 match l with
  Nil        ⇒ Nil
  Cons(h,t)  ⇒ Cons(f h, rec t)
  Name(n,t)
    ⇒ Name(n,
        Ref(ref(n, memo(n·1,
                        rec t))))
  Ref(r)     ⇒ rec (get r)
```

Fig. 17. `list_map0` (left) and `list_map1` (right) consist of the standard algorithm to map a list of integers `l` using a given integer function `f`, augmented with additional cases for `Ref` and `Name`. The left version performs no allocations; the right version uses names from the input list to name reference cells and recursive calls.

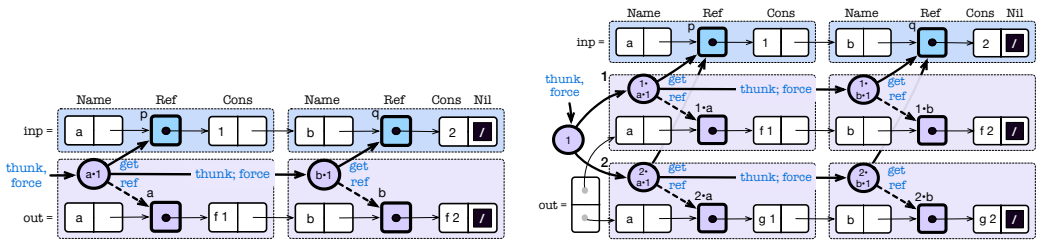

Fig. 18. **Left:** The dependence graph produced by evaluating function `list_map1` on `inp`, a two-integer, two-name input list, shown in the two upper blue regions of the figure. Its names are drawn from set $X = \{a, b, \ldots\}$, and its pointer names are drawn from the set $Y = \{p, q, \ldots\}$. The two purple sections consist of allocated dependence graph nodes that depend on the input list: cell a, thunk a·1, cell b, and thunk b·1. **Right:** The dependence graph produced by evaluating `map_pair` on the same input list, in two different scopes: 1 and 2. The scopes give rise to precise names for the output lists and their associated computations.

A.3 Names identify nodes in a dynamic dependency graph

Suppose that we execute `list_map1` on an input list with two integers, two names a and b, and two reference cells p and q, as shown in the left of Fig. 17. Bold boxes denote reference cells in the input and output lists, and bold circles denote memoized thunks that compute with them. Each bold object (thunk or reference) is named precisely. The two thunks, shown as bold circles named by a·1 and b·1, follow a similar pattern: each has outgoing edges to the reference that they dereference (northeast, to p and q, respectively), the recursive thunk that they force, if any (due west, to b·1 for a·1, and none for b·1), and the reference that they name and allocate (southeast, to a and b, respectively).

The benefit of using names stems from the nominal indirection that they afford. Operationally, imperative $O(1)$ changes to the input structure can be reflected into the output with only $O(1)$ changes to its named content. Further, for the memoized thunks, the names also provide a primitive form of *cache eviction*: Memoized results are *overwritten* when names are re-associated with new content.

A.4 Scopes distinguish distinct sub-computations

Programs commonly disambiguate two uses of the same name by using two different write scopes. Specifically, the write scope monad implicitly threads a name function through the computation, applying it to each name that the computation allocates. Conceptually, the write scope at the outermost layer of the computation is the identity name function. As the computation enters subtrees of the dynamic call tree, the write scope monad nests the write scope by composing name functions.

```
let map_pair l =
  let xs = scope s1 [ list_map1 f l ]
  let ys = scope s2 [ list_map1 g l ]
    (xs, ys)
let out = memo(1, map_pair inp)
```

For instance, consider two uses of map, with two different element-mapping functions `f` and `g`. This code uses disjoint write scopes $s_1 \perp s_2$ to map an input list `inp` twice within a single incremental program to produce a *pair* of distinct output lists `out` that each depend on the common input list `inp`. To make the example concrete, suppose that s_1 and s_2 prepend the name constants 1 and 2 on to their name argument, respectively. This program gives rise to the dependence graph shown in Fig. 17 (right side). This diagram illustrates how the two sub-computations of `list_map1` share the input list `inp`: Each of its reference cells has two incoming `get` edges. (Recall, names are *not* linear resources, so this is not problematic). To distinguish these write scopes, the distinct name functions $\langle\!\langle 1, - \rangle\!\rangle$ and $\langle\!\langle 2, - \rangle\!\rangle$ disambiguate the name of each reference cell and thunk for `list_map1` (e.g., compare the allocated names of the left-hand and right-hand versions of the figure).

A.5 Polymorphic lists, with names

Above, we considered lists of integers. Now, suppose the archivist wishes to employ lists of other data structures, such as lists of lists. For instance, past literature commonly implements incremental mergesort using functions over lists of lists. First, consider what we call *naive type polymorphism*, where we replace each instance of "Int" in type of integer lists ($\mathsf{List}\,[X;Y]\ \mathsf{Int}$) with the universally quantified type variable α:

$$
\begin{array}{lllll}
\mathtt{Nil} & : \forall \alpha : \mathsf{type}. & \forall X, Y : \mathbf{NmSet}. & 1 & \to \mathsf{List}\,[X;Y]\ \alpha \\
\mathtt{Cons} & : \forall \alpha : \mathsf{type}. & \forall X, Y : \mathbf{NmSet}. & \alpha \to \mathsf{List}\,[X;Y]\ \alpha & \to \mathsf{List}\,[X;Y]\ \alpha \\
\mathtt{Name} & : \forall \alpha : \mathsf{type}. & \forall X_1 \perp X_2, Y : \mathbf{NmSet}. & \mathsf{Nm}\,[X_1] \to \mathsf{List}\,[X_2;Y]\ \alpha & \to \mathsf{List}\,[X_1 \perp X_2;Y]\ \alpha \\
\mathtt{Ref} & : \forall \alpha : \mathsf{type}. & \forall X, Y_1 \perp Y_2 : \mathbf{NmSet}. & \mathsf{Ref}\,[Y_1]\ (\mathsf{List}\,[X;Y_2]\ \alpha) & \to \mathsf{List}\,[X;Y_1 \perp Y_2]\ \alpha
\end{array}
$$

As we demonstrate, naively parameterizing lists by α does not afford enough invariants to verify all the programs that we wish to author. For instance, suppose the archivist wishes to express the list transformation `list_join` that appends the sub-lists of a list of lists; in the `Cons` case, it uses auxiliary function `list_append` to append the sublist at the head with the remainder of the sublists' elements in the (flattened) tail.

Fig. 19 lists the code for these standard list algorithms, as well as types for each. As with `list_map0`, we initially consider versions that do not allocate, for simplicity; later, we can augment the algorithms with named allocations in the same fashion that we systematically transformed `list_map0` into `list_map1`. The code for `list_append0` checks against the listed type, with reasoning similar to that of `list_map0`; it does not involve lists of lists, only lists of integers. As we explain below, the code for `list_join0` does not check against the listed type: The listed type uses

✓

```
list_append0 :
   ∀X1⊥X2, Y1⊥Y2 : NmSet.
      (List[X1; Y1] Int)
    → (List[X2; Y2] Int)
    → (List[X1⊥X2; Y2] Int)
   ▷ ⟨∅, Y1⟩
list_append0 = fix rec. λl. λr.
 match l with
  Nil       ⇒r
  Cons(h,t) ⇒let t' = rec t r
                Cons(h, t')
  Name(n,t) ⇒Name(n, rec t)
  Ref(r)    ⇒rec (get r)
```

?

```
list_join0 :
   ∀X1⊥X2, Y1⊥Y2 : NmSet.
      (List[X1; Y1] (List[X2; Y2] Int))
    → (List[X1⊥X2; ∅] Int)
   ▷ ⟨∅, Y1⊥Y2⟩

list_join0 = fix rec. λl.
 match l with
  Nil       ⇒Nil
  Cons(h,t) ⇒let t' = rec t
                list_append0 h t'
  Name(n,t) ⇒Name(n, rec t)
  Ref(r)    ⇒rec (get r)
```

Fig. 19. Function `list_append0` copies one list `l`, replacing its `Nil` terminal with a second list `r`. Function `list_join0` appends the sub-lists of a list of lists. The naive polymorphic list type for `list_join0` is insufficient to type-check the Cons case of its body.

polymorphism naively with respect to name sets. To overcome this problem, we need to change the refinement type for Cons. To see why, first consider checking the body against the listed type.

Consider the Cons case of `list_join0`, whose code is standard; in the case of a sublist `h`, we append this sublist to the result of flattening the rest of the input list of lists. Critically, this step involves reasoning about the relationship between three structures, each involving a set of names:

- The head `h` of the list of lists `l`, with names drawn from X_2 (Inner list `h` has type $\mathsf{List}[X_2; Y_2]\ \mathsf{Int}$),
- The tail `t` of the list of lists `l`, with outer list names drawn from X_1 and inner list names from X_2, and
- The flattened tail `t'` of of tail `t`, with names drawn from $X_1 \perp X_2$.

In particular, to prove that the use of `list_append0 h t'` is well-typed, we wish to argue that head `h` (drawn from X_2) has names that are distinct from those in the flattened list `t'` (drawn from $X_1 \perp X_2$). However, the naive polymorphic list type places no constraints on names in different sub-lists: It merely says that their names are all drawn from X_2, not that each distinct sub-list uses a distinct, non-overlapping subset of X_2.

Recall the type of Cons given above:

$$\forall \alpha : \mathsf{type}.\, \forall X, Y : \mathbf{NmSet}.\, \alpha \to (\mathsf{List}[X; Y]\ \alpha) \to (\mathsf{List}[X; Y]\ \alpha)$$

In particular, we want the type for Cons to enforce a relationship of disjointness between the name sets of the new element and the existing elements in the list, which is does not. Further, as written, each occurrence of α must be the same, and thus, must over-approximate name sets. Instead, the following types for Cons will permit `list_join` to type-check; we discuss them in turn:

Lists of lists of integers:

$$
\begin{array}{l}
\texttt{Cons}: \\
\forall X \bot X_1 \bot X_2 : \mathbf{NmSet}. \\
\forall Y \bot Y_1 \bot Y_2 : \mathbf{NmSet}. \\
\quad (\mathsf{List}\,[X_1; Y_1]\ \mathsf{Int}) \\
\to (\mathsf{List}\,[X; Y]\ (\mathsf{List}\,[X_2; Y_2]\ \mathsf{Int})) \\
\to \big(\mathsf{List}\,[X \bot X_1 \bot X_2; Y \bot Y_1 \bot Y_2] \\
\quad (\mathsf{List}\,[X_1 \bot X_2; Y_1 \bot Y_2]\ \mathsf{Int})\big)
\end{array}
$$

Lists of α (type α has kind $\gamma \to$ type):

$$
\begin{array}{l}
\texttt{Cons} : \forall X, Y : \mathbf{NmSet}.\ \forall i, j : \gamma. \\
\forall k_1 \equiv f\ i\ j : \gamma. \\
\forall k_2 \equiv g\ X\ Y\ k_1 : \mathbf{NmSet} \times \mathbf{NmSet}. \\
\quad \alpha[i] \\
\to (\mathsf{List}\,[X; Y]\ (\alpha[j])) \\
\to (\mathsf{List}\,[k_2]\ (\alpha[k_1]))
\end{array}
$$

First consider the type on the left, which is specific to lists of lists of integers. In particular, this type enforces that the sublists' names are disjoint from one another, and from the names of the outer list. Consequently, in the `Cons` case of `list_join`, we can justify the recursive call, since the outer and inner lists have distinct names; and, we can justify the call to `list_append`, since the names of each inner list are distinct from those of other inner lists.

Instead of writing a new version of `Cons` for every list element type, we want to capture this pattern, and others, generically. The second (rightmost) type of `Cons` is generic in type α, but unlike the naive polymorphic type, this version permits the indices of the type parameter α to vary across occurrences (i versus j), while enforcing user-defined constraints over these index occurrences (index function f), and between these element indices and the indices of the list (index function g). In particular, the client of this generic list chooses a type α, with some index sort γ, and they choose index functions f and g:

$$
\begin{array}{lcl}
f & : & \gamma \overset{\text{idx}}{\to} \gamma \overset{\text{idx}}{\to} \gamma \\
g & : & \mathbf{NmSet} \overset{\text{idx}}{\to} \mathbf{NmSet} \overset{\text{idx}}{\to} \gamma \overset{\text{idx}}{\to} (\mathbf{NmSet} \times \mathbf{NmSet})
\end{array}
$$

As described by its sort, index function f forms an element index from two elements' indices (each of sort γ). Likewise, index function g forms a list index (a pair of name sets) from a list index (two, curried name sets) and an element index. By choosing type α, sort γ and index functions f and g, we can recover integer lists, and lists of integer lists as special cases:

Integer lists:

$$
\begin{array}{lcl}
\alpha & :\equiv & \mathsf{Int} \\
\gamma & :\equiv & 1 \\
f & :\equiv & \lambda().\lambda().() \\
g & :\equiv & \lambda X.\lambda Y.\lambda().(X, Y)
\end{array}
$$

Lists of integer lists:

$$
\begin{array}{lcl}
\alpha & :\equiv & \mathsf{List}\,[-; -]\ \mathsf{Int} \\
\gamma & :\equiv & \mathbf{NmSet} \times \mathbf{NmSet} \\
f & :\equiv & \lambda(X_1, Y_1).\lambda(X_2, Y_2).(X_1 \bot X_2, Y_1 \bot Y_2) \\
g & :\equiv & \lambda X_1.\lambda Y_1.\lambda(X_2, Y_2).(X_1 \bot X_2, Y_1 \bot Y_2)
\end{array}
$$

On the left, we express list of integers by choosing $\alpha :\equiv \mathsf{Int}$ and $\gamma :\equiv 1$, the sort of Int's (trivial) type index. On the right, we express lists of integer lists by choosing $\alpha :\equiv \mathsf{List}\,[-; -]\ \mathsf{Int}$, the type of integer lists, and define f to accumulate name sets for the inner lists, and g to accumulate name sets for *both* the inner and outer lists. In this case, the resulting functions f and g happen to be isomorphic.

B OMITTED DEFINITIONS, FIGURES, AND REMARKS

$$\frac{}{\vdash \cdot : \Gamma}\ \text{emp} \qquad \frac{S \vdash \Gamma \quad \Gamma \vdash v : A \quad \Gamma(p) = A}{\vdash (S, p : v) : \Gamma}\ \text{ref} \qquad \frac{S \vdash \Gamma \quad \Gamma \vdash e : E \quad \Gamma(p) = E}{\vdash (S, p : e) : \Gamma}\ \text{thunk}$$

Fig. 20. Store typing: $S \vdash \Gamma$, read "store S typed by Γ".

Pointers	p, q	$::= n$	name constants
Stores	S	$::= \cdot$	empty store
		$\mid S, p{:}v$	p points to value v
		$\mid S, p{:}e \mathbin{@} M$	p points to thunk e, run in scope M

Notation: $S\{p{\mapsto}v\}$ and $S\{p{\mapsto}e@M\}$ *extend* S at p when $p \notin \mathrm{dom}(S)$
$S\{p{\mapsto}v\}$ and $S\{p{\mapsto}e@M\}$ *overwrite* S at p when $p \in \mathrm{dom}(S)$

$\boxed{S_1 \vdash^M_m e \Downarrow S_2; t}$ Under store S in namespace M at current node m, expression e produces new store S_2 and result t

$$\frac{S_1 \vdash^M_m [v_2/x_2][v_1/x_1]e \Downarrow S_2; e'}{S_1 \vdash^M_m \mathtt{split}((v_1, v_2), x_1.x_2.e) \Downarrow S_2; e'} \Downarrow\text{-split} \qquad \frac{S_1 \vdash^M_m [v_i/x_i]e_i \Downarrow S_2; e'}{S_1 \vdash^M_m \mathtt{case}(\mathtt{inj}_i\, v, x_1.e_1, x_2.e_2) \Downarrow S_2; e'} \Downarrow\text{-case}$$

$$\frac{S_1 \vdash^M_m [v/x]e \Downarrow S_2; e'}{S_1 \vdash^M_m \mathtt{vunpack}(\mathtt{pack}(a.v), b.x.e) \Downarrow S_2; e'} \Downarrow\text{-unpack}$$

$$\frac{S_1 \vdash^M_m e_1 \Downarrow S_1'; \mathtt{ret}(v) \qquad S_1' \vdash^M_m [v/x]e_2 \Downarrow S_2'; e_2'}{S_1' \vdash^M_m \mathtt{let}(e_1, x.e_2) \Downarrow S_2'; e_2'} \Downarrow\text{-let}$$

$$\frac{S_1 \vdash^M_m e_1 \Downarrow S_1'; \lambda x.\, e_2 \qquad S_1 \vdash^M_m [v/x]e_2 \Downarrow S_2'; e_2'}{S_1' \vdash^M_m e_1\ v \Downarrow S_2'; e_2'} \Downarrow\text{-app}$$

$$\frac{S_1 \vdash^{M_1 \circ M_2}_m e \Downarrow S_2; e'}{S_1 \vdash^{M_1}_m \mathtt{scope}(M_2, e) \Downarrow S_2; e'} \Downarrow\text{-scope} \qquad \frac{M_1 \Downarrow_M \lambda a.\, M_2 \qquad [n/a]M_2 \Downarrow_M p}{S \vdash^M_m M_1\ (\mathtt{name}\ n) \Downarrow S; \mathtt{ret}(\mathtt{name}\ p)} \Downarrow\text{-name-app}$$

$$\frac{(M\,n) \Downarrow_M p \qquad S_1\{p{\mapsto}e \mathbin{@} M\} = S_2}{S_1 \vdash^M_m \mathtt{thunk}(\mathtt{name}\ n, e) \Downarrow S_2; \mathtt{ret}(\mathtt{thunk}\ p)} \Downarrow\text{-thunk} \qquad \frac{(M\,n) \Downarrow_M p \qquad S_1\{p{\mapsto}v\} = S_2}{S_1 \vdash^M_m \mathtt{ref}(\mathtt{name}\ n, v) \Downarrow S_2; \mathtt{ret}(\mathtt{ref}\ p)} \Downarrow\text{-ref}$$

$$\frac{S(p) = e \mathbin{@} M_0 \qquad S_1 \vdash^{M_0}_p e \Downarrow S_2; t}{S_1 \vdash^M_m \mathtt{force}(\mathtt{thunk}\ p) \Downarrow S_2; t} \Downarrow\text{-force} \qquad \frac{S(p) = v}{S \vdash^M_m \mathtt{get}(\mathtt{ref}\ p) \Downarrow S; \mathtt{ret}(v)} \Downarrow\text{-get} \qquad \frac{}{S \vdash^M_m t \Downarrow S; t} \Downarrow\text{-term}$$

Fig. 21. Dynamic semantics, complete

In Figure 22, we write

$$\mathcal{D} \text{ by } \textit{Rulename (Dlist)} \text{ reads R writes } W$$

to mean that rule *Rulename* concludes $\mathcal{D}$ and has subderivations *Dlist*. For example,

$$\mathcal{D} \text{ by } \Downarrow\text{-scope}(\mathcal{D}_0) \text{ reads R writes } W$$

provided that $\mathcal{D}$ reads R writes W, where $\mathcal{D}_0$ derives the only premise of ⇓-scope.

$$\mathcal{D} \text{ by } \Downarrow\text{-term}() \text{ reads } \emptyset \text{ writes } \emptyset$$

$$\mathcal{D} \text{ by } \Downarrow\text{-app}(\mathcal{D}_1, \mathcal{D}_2) \text{ reads } R_1 \cup R_2 \text{ writes } W_1 \perp W_2 \quad \text{if} \quad \mathcal{D}_1 \text{ reads } R_1 \text{ writes } W_1 \text{ and } \mathcal{D}_2 \text{ reads } R_2 \text{ writes } W_2$$

$$\mathcal{D} \text{ by } \Downarrow\text{-let}(\mathcal{D}_1, \mathcal{D}_2) \text{ reads } R_1 \cup R_2 \text{ writes } W_1 \perp W_2 \quad \text{if} \quad \mathcal{D}_1 \text{ reads } R_1 \text{ writes } W_1 \text{ and } \mathcal{D}_2 \text{ reads } R_2 \text{ writes } W_2$$

$$\mathcal{D} \text{ by } \Downarrow\text{-scope}(\mathcal{D}_0) \text{ reads } R \text{ writes } W \quad \text{if} \quad \mathcal{D}_0 \text{ reads } R \text{ writes } W$$

$$\mathcal{D} \text{ by } \Downarrow\text{-case}(\mathcal{D}_0) \text{ reads } R \text{ writes } W \quad \text{if} \quad \mathcal{D}_0 \text{ reads } R \text{ writes } W$$

$$\mathcal{D} \text{ by } \Downarrow\text{-split}(\mathcal{D}_0) \text{ reads } R \text{ writes } W \quad \text{if} \quad \mathcal{D}_0 \text{ reads } R \text{ writes } W$$

$$\mathcal{D} \text{ by } \Downarrow\text{-ref}() \text{ reads } \emptyset \text{ writes } p \quad \text{where } e = \texttt{ref(name } n, v\texttt{)} \text{ and } p \equiv M\ n$$

$$\mathcal{D} \text{ by } \Downarrow\text{-thunk}() \text{ reads } \emptyset \text{ writes } p \quad \text{where } e = \texttt{thunk(name } n, e_0\texttt{)} \text{ and } p \equiv M\ n$$

$$\mathcal{D} \text{ by } \Downarrow\text{-get}() \text{ reads } p \text{ writes } \emptyset \quad \text{where } e = \texttt{get(ref } p\texttt{)}$$

$$\mathcal{D} \text{ by } \Downarrow\text{-force}() \text{ reads } q, R' \text{ writes } W' \quad \text{where } e = \texttt{force(thunk } q\texttt{)} \text{ and } \mathcal{D}' \text{ reads } R' \text{ writes } W' \text{ where } \mathcal{D}' \text{ is the derivation that computed } t$$

Fig. 22. Read- and write-sets of a non-incremental evaluation derivation

$\boxed{\Gamma \vdash A : K}$ Under Γ, value type A has kind K

$$\frac{(\alpha : K) \in \Gamma}{\Gamma \vdash \alpha : K}\ \text{k-typevar} \qquad \frac{(d : K) \in \Gamma}{\Gamma \vdash d : K}\ \text{k-tycon} \qquad \frac{\Gamma \vdash A_1 : \text{type} \qquad \Gamma \vdash A_2 : \text{type}}{\Gamma \vdash (A_1 + A_2) : \text{type} \quad \Gamma \vdash (A_1 \times A_2) : \text{type}}\ \text{k-binop}$$

$$\frac{}{\Gamma \vdash \text{unit} : \text{type}}\ \text{k-unit} \qquad \frac{\Gamma \vdash i : \textbf{NmSet}}{\Gamma \vdash \text{Nm}[i] : \text{type}}\ \text{k-name} \qquad \frac{\Gamma \vdash i : \textbf{NmSet} \qquad \Gamma \vdash A : \text{type}}{\Gamma \vdash (\text{Ref}[i]\ A) : \text{type}}\ \text{k-ref}$$

$$\frac{\Gamma \vdash i : \textbf{NmSet} \quad \Gamma \vdash E\ \text{efftype}}{\Gamma \vdash (\text{Thk}[i]\ E) : \text{type}}\ \text{k-thk} \qquad \frac{\Gamma \vdash A : (\text{type} \Rightarrow K) \quad \Gamma \vdash B : \text{type}}{\Gamma \vdash (A\ B) : K}\ \text{k-app-type} \qquad \frac{\Gamma \vdash A : (\gamma \Rightarrow K) \quad \Gamma \vdash i : \gamma}{\Gamma \vdash A[i] : K}\ \text{k-app-index}$$

$$\frac{\Gamma, a : \gamma \vdash P\ \text{prop} \qquad \Gamma, a : \gamma \vdash A : \text{type}}{\Gamma \vdash (\forall a : \gamma \mid P.\ A) : \text{type}}\ \text{k-all} \qquad \frac{\Gamma, a : \gamma \vdash P\ \text{prop} \qquad \Gamma, a : \gamma \vdash A : \text{type}}{\Gamma \vdash (\exists a : \gamma \mid P.\ A) : \text{type}}\ \text{k-exists}$$

$\boxed{\Gamma \vdash C\ \text{ctype}}$ Under Γ, computation type C is well-formed

$$\frac{\Gamma \vdash A : \text{type}}{\Gamma \vdash (\mathbf{F}\, A)\ \text{ctype}}\ \text{ctype-lift} \qquad \frac{\Gamma \vdash A : \text{type} \qquad \Gamma \vdash E\ \text{efftype}}{\Gamma \vdash (A \to E)\ \text{ctype}}\ \text{ctype-arr}$$

$\boxed{\Gamma \vdash \epsilon\ \text{wf-effects}}$ Under Γ, effects ϵ are well-formed

$$\frac{\Gamma \vdash W : \textbf{NmSet} \qquad \Gamma \vdash R : \textbf{NmSet}}{\Gamma \vdash \langle W; R \rangle\ \text{wf-effects}}\ \text{wf-eff}$$

$\boxed{\Gamma \vdash P\ \text{prop}}$ Under Γ, proposition P is well-formed

$$\frac{}{\Gamma \vdash \text{tt prop}} \qquad \frac{\Gamma \vdash P_1\ \text{prop} \quad \Gamma \vdash P_2\ \text{prop}}{\Gamma \vdash (P_1\ \textbf{and}\ P_2)\ \text{prop}} \qquad \frac{\Gamma \vdash i : \gamma \qquad \Gamma \vdash j : \gamma}{\Gamma \vdash (i \perp j : \gamma)\ \text{prop} \quad \Gamma \vdash (i \equiv j : \gamma)\ \text{prop}}$$

$\boxed{\Gamma \vdash E\ \text{efftype}}$ Under Γ, type-with-effects E is well-formed

$$\frac{\Gamma \vdash C\ \text{ctype} \qquad \Gamma \vdash \epsilon\ \text{wf-effects}}{\Gamma \vdash (C \rhd \epsilon)\ \text{efftype}}\ \text{etype-eff}$$

$$\frac{\Gamma, \alpha : K \vdash E\ \text{efftype}}{\Gamma \vdash (\forall \alpha : K.\ E)\ \text{efftype}}\ \text{etype-poly} \qquad \frac{\Gamma, a : \gamma \vdash P\ \text{prop} \qquad \Gamma, a : \gamma \vdash E\ \text{efftype}}{\Gamma \vdash (\forall a : \gamma \mid P.\ E)\ \text{efftype}}\ \text{etype-idx}$$

Fig. 23. Kinding and well-formedness for types and effects

B.1 Remarks

Why distinguish computation types from types-with-effects? Can we unify computation types C and types-with-effects E? Not easily. We have two computation types, $\mathbf{F}$ and $\to$. For $\mathbf{F}$, the expression being typed could create a thunk, so we must put that effect somewhere in the syntax. For $\to$, applying a function is (per call-by-push-value) just a "push": the function carries no effects of its own (though its codomain may need to have some). However, suppose we force a thunked function of type $A_1 \to (A_2 \to \cdots)$ and apply the function (the contents of the thunk) to one argument. In the absence of effects, the result would be a computation of type $A_2 \to \cdots$, meaning that the computation is waiting for a second argument to be pushed. But, since forcing the thunk has the effect of reading the thunk, we need to track this effect in the result type. So we cannot return $A_2 \to \cdots$, and must instead put effects around $(A_2 \to \cdots)$. Thus, we need to associate effects to both $\mathbf{F}$ and $\to$, that is, to both computation types.

Now we are faced with a choice: we could (1) extend the syntax of each connective with an effect (written next to the connective), or (2) introduce a "wrapper" that encloses a computation type, either $\mathbf{F}$ or $\to$. These seem more or less equally complicated for the present system, but if we enriched the language with more connectives, choice (1) would make the new connectives more complicated, while under choice (2), the complication would already be rolled into the wrapper. We choose (2), and write the wrapper as $C \rhd \epsilon$, where C is a computation type and ϵ represents effects.

Where should these wrappers live? We could add $C \rhd \epsilon$ to the grammar of computation types C. But it seems useful to have a clear notion of *the* effect associated with a type. When the effect on the outside of a type is the only effect in the type, as in $(A_1 \to \mathbf{F}\, A_2) \rhd \epsilon$, "the" effect has to be ϵ. Alas, types like $(C \rhd \epsilon_1) \rhd \epsilon_2$ raise awkward questions: does this type mean the computation does ϵ_2 and then ϵ_1, or ϵ_1 and then ϵ_2?

We obtain an unambiguous, singular outer effect by distinguishing types-with-effects E from computation types C. The meta-variables for computation types appear only in the production $E ::= C \rhd \epsilon$, making types-with-effects E the "common case" in the grammar. Many of the typing rules follow this pattern, achieving some isolation of effect tracking in the rules.

C OMITTED LEMMAS AND PROOFS

Lemma C.1 (Index-level weakening).

(1) If $\Gamma \vdash M : \gamma$ *then* $\Gamma, \Gamma' \vdash M : \gamma$.
(2) If $\Gamma \vdash i : \gamma$ *then* $\Gamma, \Gamma' \vdash i : \gamma$.
(3) If $\Gamma \vdash A : K$ *then* $\Gamma, \Gamma' \vdash A : K$.

Proof. By induction on the given derivation. □

Lemma C.2 (Weakening).

(1) If $\Gamma \vdash e : A$ *then* $\Gamma, \Gamma' \vdash e : A$.
(2) If $\Gamma \vdash^M e : C$ *then* $\Gamma, \Gamma' \vdash^M e : C$.

Proof. By induction on the given derivation, using Lemma G.1 (Weakening of semantic equivalence and apartness) (for example, in the case for the value typing rule 'name') and Lemma C.1 (Index-level weakening) (for example, in the case for the computation typing rule 'AllIndexElim'). □

Lemma C.3 (Substitution).

(1) If $\Gamma \vdash v : A$ *and* $\Gamma, x : A \vdash e : C$ *then* $\Gamma \vdash ([v/x]e) : C$.
(2) If $\Gamma \vdash v : A$ *and* $\Gamma, x : A \vdash v' : B$ *then* $\Gamma \vdash ([v/x]v') : B$.

Proof. By mutual induction on the derivation typing e (in part 1) or v' (in part 2). □

In the presence of subtyping, canonical forms (value inversion) is not entirely straightforward.

Lemma C.4 (Subtyping Weakening). *If* $\Gamma \vdash A \leq_V B$ *then* $\Gamma, \Gamma' \vdash A \leq_V B$ *where* Γ' *consists of* $a : \gamma$ *and* P *assumptions.*

Proof. By induction on the derivation of $\Gamma \vdash A \leq_V B$. In the $\leq_V$-name, $\leq_V$-namefn, $\leq_V$-ref, $\leq_V$-thk, $\leq_V$-$\forall$L and $\leq_V$-$\exists$ cases, use weakening for the relations $\vdash$ and $\Vdash$. □

Lemma C.5 (Subtyping Substitution).
If $\Gamma, a : \gamma, P \vdash A \leq_V B$ *and* $\Gamma \vdash i : \gamma$ *and extract*$(\Gamma) \Vdash P$ *then* $\Gamma \vdash [i/a]A \leq_V [i/a]B$.

Proof. By induction on the derivation of $\Gamma \vdash A \leq_V B$. In the $\leq_V$-ref, $\leq_V$-thk, $\leq_V$-$\forall$L and $\leq_V$-$\exists$ cases, use substitution for the relation $\Vdash$. □

Lemma C.6 (Reflexivity of Subtyping). *For all* Γ *and* A*, it is the case that* $\Gamma \vdash A \leq_V A$.

Proof. Immediate by rule $\leq_V$-refl. □

Lemma C.7 (Transitivity of Subtyping).
If $\Gamma \vdash A_L \leq_V B$ *and* $\Gamma \vdash B \leq_V A_R$ *then* $\Gamma \vdash A_L \leq_V A_R$.

Proof. By simultaneous induction on the two given derivations.
If either derivation is by $\leq_V$-refl, we already have our result.
Consider cases of the rule concluding $\Gamma \vdash A_L \leq_V B$.

- $\leq_V$-$\times$:
  The derivation of $\Gamma \vdash B \leq_V A_R$ must be by $\leq_V$-refl (already handled), $\leq_V$-$\times$ or $\leq_V$-$\forall$R.
  If by $\leq_V$-$\times$, the result follows by using the i.h. twice on the respective subderivations, then applying $\leq_V$-$\times$.
  If by $\leq_V$-$\forall$R, then:

	$A_R = (\forall b : \gamma \mid P.\, A_{R0})$	By inversion ($\leq_V$-$\forall$R)
	$\Gamma, b : \gamma, P \vdash B \leq_V A_{R0}$	$''$
	$\Gamma \vdash A_L \leq_V B$	Given
	$\Gamma, b : \gamma, P \vdash A_L \leq_V B$	By Lemma C.4 (Subtyping Weakening)
	$\Gamma, b : \gamma, P \vdash A_L \leq_V A_{R0}$	By i.h.
	$\Gamma \vdash A_L \leq_V (\forall b : \gamma \mid P.\, A_{R0})$	By $\leq_V$-$\forall$R
☞	$\Gamma \vdash A_L \leq_V A_R$	By above equation

- $\leq_V$-$+$: Similar to the $\leq_V$-$\times$ case.
- $\leq_V$-name, $\leq_V$-ref, $\leq_V$-thk: Similar to the $\leq_V$-$\times$ case, using transitivity of $\subseteq$ at the index level.
- $\leq_V$-namefn: Use transitivity of conv.
- $\leq_V$-$\forall$L:
  By i.h., $\Gamma \vdash [i/a]A_{L0} \leq_V A_R$.
  By $\leq_V$-$\forall$L, $\Gamma \vdash \forall a : \gamma \mid P.\, A_{L0} \leq_V A_R$, which was to be shown.
- $\leq_V$-$\forall$R:
  The other derivation is by either $\leq_V$-refl (already handled) or $\leq_V$-$\forall$L.

	$B = (\forall b : \gamma \mid P.\, B_0)$	By inversion ($\leq_V$-$\forall$R)
	$\Gamma, b : \gamma, P \vdash A_L \leq_V B_0$	$''$
	$\Gamma \vdash i : \gamma$	By inversion ($\leq_V$-$\forall$L)
	$\mathit{extract}(\Gamma) \Vdash [i/b]P$	$''$
	$\Gamma \vdash [i/b]B_0 \leq_V A_R$	$''$
	$\Gamma \vdash [i/b]A_L \leq_V [i/b]B_0$	By Lemma C.5 (Subtyping Substitution)
	$\Gamma \vdash [i/b]A_L \leq_V A_R$	By i.h.
	$[i/b]A_L = A_L$	b not free in A_L
☞	$\Gamma \vdash A_L \leq_V A_R$	By above equation

□

Lemma C.8 (Canonical Forms). *Suppose* Γ store-type *and* $\Gamma \vdash v : A$.

1. *If* $A \leq_V$ unit *then* $v = ()$.
2. *If* $A \leq_V (B_1 \times B_2)$ *then* $v = (v_1, v_2)$ *and* $\Gamma \vdash v_1 : B_1$ *and* $\Gamma \vdash v_2 : B_2$.
3. *If* $A \leq_V (B_1 + B_2)$ *then* $v = \mathsf{inj}_i\, v_i$ *where* $i \in \{1, 2\}$ *and* $\Gamma \vdash v_i : B_i$.
4. *If* $A \leq_V (\mathsf{Nm}\,[X])$ *then* $v = \mathtt{name}\ n$ *where* $\Gamma \vdash n \in X$.
5. *If* $A \leq_V (\mathsf{Ref}\,[X]\ A_0)$ *then* $v = \mathtt{ref}\ n$ *where* $\Gamma \vdash n \in X$.
6. *If* $A \leq_V (\mathsf{Thk}\,[X]\ E)$ *then* $v = \mathtt{thunk}\ n$ *where* $\Gamma \vdash n \in X$.
7. *If* $A \leq_V (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M]$ *then* $v = \mathtt{nmfn}\ M_v$ *where* $M =_\beta (\lambda a.\, M')$ *and* $\cdot \vdash (\lambda a.\, M') : (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})$ *and* $M_v =_\beta M$.

Proof. By induction on the derivation of $\Gamma \vdash A \leq_V B$.

(1) Consider cases of the rule concluding $\Gamma \vdash v : A$.

- **Case** unit: By inversion.
- **Case** pair: Impossible because $\Gamma \vdash A_1 + A_2 \leq_V$ unit is not derivable.
- **Case** name: Impossible because $\Gamma \vdash \mathsf{Nm}\,[X] \leq_V$ unit is not derivable.
- **Case** namefn: Impossible because $\Gamma \vdash (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M] \leq_V$ unit is not derivable.
- **Case** ref: Impossible because $\Gamma \vdash (\mathsf{Ref}\,[X]\ A_0) \leq_V$ unit is not derivable.
- **Case** thunk: Impossible because $\Gamma \vdash (\mathsf{Thk}\,[X]\ E) \leq_V$ unit is not derivable.
- **Case** vtype-$\forall$IndexIntro:

	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V$ unit	Given
	$\Gamma, a : \gamma, P \vdash A_0 \leq_V$ unit	By inversion ($\leq_V$-$\forall$L)
	$\Gamma, a : \gamma, P \vdash v : A_0$	Subderivation
☞	$v = ()$	By i.h. (part 1)

- **Case**

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{l}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v : (\forall a : \gamma \mid P.\, A_0)\end{array}}{\Gamma \vdash v : [i/a]A_0}\ \text{vtype-}\forall\text{IndexElim}$$

	$\Gamma \vdash i : \gamma$	Subderivation
	$\mathit{extract}(\Gamma) \Vdash [i/a]P$	Subderivation
	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V [i/a]A_0$	By $\leq_V$-$\forall$L
	$\Gamma \vdash [i/a]A_0 \leq_V A$	Given
	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V \mathsf{unit}$	By Lemma C.7 (Transitivity of Subtyping)
	$\Gamma \vdash v : (\forall a : \gamma \mid P.\, A_0)$	Subderivation
☞	$v = ()$	By i.h.

- **Case** vtype-$\exists$IndexIntro:
  Impossible because $\Gamma \vdash (\exists a : \gamma \mid P.\, A_0) \leq_V \mathsf{unit}$ is not derivable.

(2) $\times$:
Consider cases of the rule concluding $\Gamma \vdash v : A$.

- **Case** unit: Impossible because $\Gamma \vdash \mathsf{unit} \leq_V (B_1 \times B_2)$ is not derivable.
- **Case** pair:

	$\Gamma \vdash A \leq_V (B_1 \times B_2)$	Given
	$A = (A_1 \times A_2)$	By inversion (pair)
☞	$v = (v_1, v_2)$	$''$
	$\Gamma \vdash v_1 : B_1$	$''$
	$\Gamma \vdash v_2 : B_2$	$''$
	$\Gamma \vdash A_1 \leq_V B_1$	By inversion ($\leq_V$-$\times$)
	$\Gamma \vdash A_2 \leq_V B_2$	$''$
☞	$\Gamma \vdash v_1 : B_1$	By vtype-sub
☞	$\Gamma \vdash v_2 : B_2$	By vtype-sub

- **Cases** name, namefn, ref, thunk:
  Impossible because the assumed subtyping is not derivable.
- **Case** vtype-$\forall$IndexIntro:

	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V (B_1 \times B_2)$	Given
	$\Gamma, a : \gamma, P \vdash A_0 \leq_V (B_1 \times B_2)$	By inversion ($\leq_V$-$\forall$L)
	$\Gamma, a : \gamma, P \vdash v : A_0$	Subderivation
☞	$v = (v_1, v_2)$	By i.h. (part 2)
☞	$\Gamma \vdash v_1 : B_1$	$''$
☞	$\Gamma \vdash v_2 : B_2$	$''$

- **Case**

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v : (\forall a : \gamma \mid P.\, A_0)\end{array}}{\Gamma \vdash v : [i/a]A_0}\ \text{vtype-}\forall\text{IndexElim}$$

	$\Gamma \vdash i : \gamma$	Subderivation
	$\mathit{extract}(\Gamma) \Vdash [i/a]P$	Subderivation
	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V [i/a]A_0$	By $\leq_V$-$\forall$L
	$\Gamma \vdash [i/a]A_0 \leq_V A$	Given
	$\Gamma \vdash (\forall a : \gamma \mid P.\, A_0) \leq_V (B_1 \times B_2)$	By Lemma C.7 (Transitivity of Subtyping)
	$\Gamma \vdash v : (\forall a : \gamma \mid P.\, A_0)$	Subderivation
☞	$v = (v_1, v_2)$	By i.h. (part 2)
☞	$\Gamma \vdash v_1 : B_1$	$''$
☞	$\Gamma \vdash v_2 : B_2$	$''$

- **Case** vtype-$\exists$IndexIntro:
  Impossible because $\Gamma \vdash (\exists a : \gamma \mid P.\, A_0) \leq_V (B_1 \times B_2)$ is not derivable.

(3) $+$: Similar to Part 2.
(4) $\mathsf{Nm}\,[X]$:
In the $\leq_V$-name case, use the fact that $\Gamma \Vdash X' \subseteq X$ and $\Gamma \vdash n \in X'$ implies $\Gamma \vdash n \in X$. Otherwise similar to Part 1.
(5) $\mathsf{Ref}\,[X]\ A_0$: Similar to Parts 1 and 4.
(6) $\mathsf{Thk}\,[X]\ E$: Similar to Part 5.
(7) $(\mathbf{Nm} \overset{\mathbf{Nm}}{\Longrightarrow} \mathbf{Nm})\,[M]$: Similar to Part 5. □

Lemma C.9 (Application and membership commute). *If* $\Gamma \vdash n \in i$ *and* $p =_\beta M(n)$ *then* $\Gamma \vdash p \in M(i)$.

Proof. The set $M(i)$ consists of all elements of i, but mapped by function M. The name p is convertible to the name $M(n)$. Since $n \in i$, we have that p is in the M-mapping of i, which is $M(i)$. □

In each case, we write "☞" to the left of each goal, as we prove it.

Theorem 7.1 (Subject Reduction).
If Γ_1 store-type *and* $\Gamma_1 \vdash M : \mathbf{Nm} \overset{\mathbf{Nm}}{\Longrightarrow} \mathbf{Nm}$ *and* $\mathcal{S}$ *derives* $\Gamma_1 \vdash^M e : C \rhd \langle W; R\rangle$
and $\vdash S_1 : \Gamma_1$ *and* $\mathcal{D}$ *derives* $S_1 \vdash^M_m e \Downarrow S_2; t$ *then*
there exists $\Gamma_2 \supseteq \Gamma_1$ *such that* Γ_2 store-type *and* $\vdash S_2 : \Gamma_2$ *and* $\Gamma_2 \vdash t : C \rhd \langle \emptyset; \emptyset\rangle$
and $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ *and* $\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \leq \langle W; R\rangle$.

Proof. By induction on the typing derivation $\mathcal{S}$.

- **Case**

$$\frac{\Gamma_1 \vdash v : A}{\Gamma_1 \vdash^M \mathtt{ret}(v) : \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset\rangle\big)}\ \text{ret}$$

	$(e = t)$ and $(S_1 = S_2)$	Given
	$(R_{\mathcal{D}} = W_{\mathcal{D}} = R = W = \emptyset)$	$''$
	$(\Gamma_2 = \Gamma_1)$	Suppose
☞	$\vdash S_2 : \Gamma_2$	by above equalities
☞	$\Gamma_2 \vdash t : C \rhd \langle \emptyset, \emptyset\rangle$	$''$
☞	$\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$	By Def. 7.2
☞	$\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \leq \langle W; R\rangle$	All are empty

- **Case**
$$\frac{\Gamma_1 \vdash v : \mathsf{Ref}\,[X]\ A}{\Gamma_1 \vdash^{M} \mathtt{get}\,(v) : (\mathbf{F}\,A) \rhd \langle \emptyset; X\rangle}\ \text{get}$$

$(W = \emptyset)$ and $(R = X)$	Given
$\Gamma_1 \vdash v : \mathsf{Ref}\,[X]\ A$	Given
$\exists p.\ (v = \mathtt{ref}\ p)$	Lemma C.8 (Canonical Forms)
$\Gamma_1 \vdash p \in X$	$''$
$\Gamma_1(p) = A$	By inversion of value typing
$\exists v_p.\ S_1(p) = v_p$	Inversion on $\vdash S_1 : \Gamma_1$
$\Gamma_1 \vdash v_p : A$	$''$
$(\Gamma_2 = \Gamma_1)$ and $(t = \mathtt{ret}\,(v_p))$	Suppose
$(R_{\mathcal{D}} = \{p\})$ and $(W_{\mathcal{D}} = \emptyset = W)$	$''$
☞ $\vdash S_2 : \Gamma_2$	By above equalities
☞ $\Gamma_2 \vdash t : C \rhd \langle \emptyset, \emptyset\rangle$	$''$
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$	By Def. 7.2
☞ $\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \preceq \langle W; R\rangle$	By above equality $W_{\mathcal{D}} = W = \emptyset$, ... and inequality for $(R_{\mathcal{D}} = \{p\}) \subseteq (X = R)$.

- **Case**
$$\frac{\Gamma_1 \vdash v : \mathsf{Thk}\,[X]\ (C \rhd \epsilon)}{\Gamma_1 \vdash^{M} \mathsf{force}(v) : \big(C \rhd (\langle \emptyset; X\rangle \ \mathsf{then}\ \epsilon)\big)}\ \text{force}$$

$\Gamma \vdash e : \tau$	Given
$\Gamma \vdash e : \tau$	Given
$(W = \emptyset)$ and $(R = X)$	Given
$\Gamma_1 \vdash v : \mathsf{Thk}\,[X]\ (C \rhd \epsilon)$	Given
$\exists p.\ (v = \mathtt{thunk}\ p)$	Lemma C.8 (Canonical Forms)
$\Gamma_1 \vdash p \in X$	$''$
$\Gamma_1(p) = (C \rhd \epsilon)$	By inversion of value typing
$\exists e_p.\ S_1(p) = e_p$	Inversion on $\vdash S_1 : \Gamma_1$
$\mathcal{S}_0 ::\ \Gamma_1 \vdash e_p : (C \rhd \epsilon)$	$''$
$\mathcal{D}_0 ::\ S_1 \vdash^{M}_{m} e_p \Downarrow S_2; t$	Inversion of $\mathcal{D}$
☞ $\vdash S_2 : \Gamma_2$	By i.h. on $\mathcal{S}_0$ and $\mathcal{D}_0$
☞ $\Gamma_2 \vdash t : C \rhd \langle \emptyset, \emptyset\rangle$	$''$
$\mathcal{D}_0$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$	$''$
$\langle W_{\mathcal{D}_0}; R_{\mathcal{D}_0}\rangle \preceq \langle W; R\rangle$	$''$
☞ $\mathcal{D}$ reads $R_{\mathcal{D}_0}$ writes $W_{\mathcal{D}_0}$	By Def. 7.2
☞ $\langle W_{\mathcal{D}_0}; R_{\mathcal{D}_0}\rangle \preceq \langle W; R\rangle$	by above equality $W_{\mathcal{D}} = W = \emptyset$, ... and inequality $(R_{\mathcal{D}} = \{p\}) \subseteq (X = R)$.

- **Case**
$$\frac{\Gamma_1 \vdash v : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M'] \qquad \Gamma_1 \vdash^{M \circ M'} e_0 : C \rhd \langle W; R\rangle}{\Gamma_1 \vdash^{M} \mathtt{scope}(v, e_0) : C \rhd \langle W; R\rangle}\ \text{scope}$$

$\mathcal{S}_0 ::\ \Gamma_1 \vdash^{M \circ M'} e_0 : C \rhd \langle W; R\rangle$ Subderivation 2 of $\mathcal{S}$
$\mathcal{D} ::\ S_1 \vdash^{M}_{m} \mathtt{scope}(\nu, e_0) \Downarrow S_2; t$ Given
$\mathcal{D}_0 ::\ S_1 \vdash^{M \circ M'}_{m} e_0 \Downarrow S_2; t$ By inversion (scope)

$\Gamma_1 \vdash M : \mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}$ Assumption
$\Gamma_1 \vdash \nu : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M']$ Subderivation 1 of $\mathcal{S}$
$\Gamma_1 \vdash M' : \mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}$ By inversion
$\Gamma_1, x : \mathbf{Nm} \vdash M'\,x : \mathbf{Nm}$ By rule t-app
$\Gamma_1, x : \mathbf{Nm} \vdash M\,(M'\,x) : \mathbf{Nm}$ By rule t-app
$\Gamma_1 \vdash \lambda x.\, M\,(M'\,x) : \mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}$ By rule t-abs
$\Gamma_1 \vdash (M \circ M') : \mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}$ By definition of $M \circ M'$
☞ $\vdash S_2 : \Gamma_2$ By i.h. on $\mathcal{S}_0$
☞ $\Gamma_2 \vdash t : C \rhd \langle \emptyset; \emptyset\rangle$ ″
$\mathcal{D}_0$ reads $R_{\mathcal{D}_0}$ writes $W_{\mathcal{D}_0}$ ″
$\langle W_{\mathcal{D}_0}; R_{\mathcal{D}_0}\rangle \preceq \langle W; R\rangle$ ″
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ By Def. 7.2
$\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle = \langle W_{\mathcal{D}_0}; R_{\mathcal{D}_0}\rangle$ ″
☞ $\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \preceq \langle W; R\rangle$ By above equalities

- **Case**

$$\frac{\Gamma_1 \vdash \nu : \mathsf{Nm}\,[X] \qquad \Gamma_1 \vdash e : E}{\Gamma_1 \vdash^{M} \mathtt{thunk}(\nu_1, \nu_2) : \mathbf{F}\,(\mathsf{Thk}\,[M(X)]\ E) \rhd \langle M(X); \emptyset\rangle}\ \text{thunk}$$

$C = \mathbf{F}\,(\mathsf{Thk}\,[M(X)]\ E)$ and $R = \emptyset$ and $W = M(X)$ Given from $\mathcal{S}$
$\Gamma_1 \vdash \nu : \mathsf{Nm}\,[X]$ Subderivation
$(\nu = \mathtt{name}\ n)$ and $(n \in X)$ By Lemma C.8
$M\ n \Downarrow p$ and $R_{\mathcal{D}} = \emptyset$ and $W_{\mathcal{D}} = \{p\}$ Given from $\mathcal{D}$
$S_2 = (S_1, p : e)$ ″
$\Gamma_2 = (\Gamma_1, p : \mathsf{Thk}\,[p]\ E)$ Suppose
☞ $\vdash S_2 : \Gamma_2$ By rule (Fig. 20)
$\Gamma_2(p) = E$ By inversion of value typing
$\Gamma_2 \vdash \mathtt{ref}\ p : \mathsf{Ref}\,[p]\ E$ By rule thunk
☞ $\Gamma_2 \vdash^{M} \mathtt{ret}(\mathtt{thunk}\ p) : \mathbf{F}\,(\mathsf{Thk}\,[p]\ A) \rhd \langle \emptyset; \emptyset\rangle$ By rule ret
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ and $W_{\mathcal{D}} = \{p\}$ By Def. 7.2
$n \in X$ Above
$M(n) \in M(X)$ Name term application is pointwise
$M(n) \in W$ By above equality
$M(n) = p$
$\{p\} \subseteq W$ By set theory
☞ $\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \preceq \langle W; R\rangle$

- **Case**

$$\frac{\Gamma_1 \vdash \nu_1 : \mathsf{Nm}\,[X] \qquad \Gamma_1 \vdash \nu_2 : A}{\Gamma_1 \vdash^{M} \mathtt{ref}(\nu_1, \nu_2) : \mathbf{F}\,(\mathsf{Ref}\,[M(X)]\ A) \rhd \langle M(X); \emptyset\rangle}\ \text{ref}$$

$C = \mathbf{F}\,(\mathsf{Ref}\,[M(X)]\ A)$ and $R = \emptyset$ and $W = M(X)$ Given from $\mathcal{S}$

$\Gamma_1 \vdash v_1 : \mathsf{Nm}[X]$	Subderivation
$(v_1 = \mathtt{name}\ n)$ and $(n \in X)$	Lemma C.8 (Canonical Forms)
$M\ n \Downarrow p$ and $R_{\mathcal{D}} = \emptyset$ and $W_{\mathcal{D}} = \{p\}$	Given from $\mathcal{D}$
$S_2 = (S_1, p : v_2)$	$''$

$\Gamma_2 = (\Gamma_1, p : \mathsf{Ref}[p]\ A)$	Suppose
☞ $\vdash S_2 : \Gamma_2$	By rule (Fig. 20)
$\Gamma_2(p) = A$	By inversion of value typing
$\Gamma_2 \vdash \mathtt{ref}\ p : \mathsf{Ref}[p]\ A$	By rule ref
☞ $\Gamma_2 \vdash^{M} \mathtt{ret}(\mathtt{ref}\ p) : \mathtt{ret}(\mathsf{Ref}[p]\ A) \rhd \langle\emptyset;\emptyset\rangle$	By rule ret
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ and $W_{\mathcal{D}} = \{p\}$	By Def. 7.2

$n \in X$	Above
$M(n) \in M(X)$	Name term application is pointwise
$M(n) \in W$	By above equality
$M(n) = p$	
$\{p\} \subseteq W$	By set theory
☞ $\langle W_{\mathcal{D}}; R_{\mathcal{D}}\rangle \preceq \langle W; R\rangle$	

- **Case**
$$\frac{\Gamma_1 \vdash^{M} e_1 : (\mathbf{F}\,A \rhd \epsilon_1) \qquad \Gamma_1, x : A \vdash^{M} e_2 : (C \rhd \epsilon_2)}{\Gamma_1 \vdash^{M} \mathtt{let}(e_1, x.e_2) : C \rhd (\epsilon_1\ \mathtt{then}\ \epsilon_2)}\ \text{let}$$

$\vdash S_1 : \Gamma_1$	Given
$\mathcal{S}_1 :: \Gamma_1 \vdash^{M} e_1 : \mathbf{F}\,A \rhd \epsilon_1$	Subderivation 1 of $\mathcal{S}$
$\mathcal{D}_1 :: S_1 \vdash^{M}_{m} e_1 \Downarrow S_{12}; t_1$	Subderivation 1 of $\mathcal{D}$
exists $\Gamma_{12} \supseteq \Gamma_1$ such that $S_{12} : \Gamma_{12}$	By i.h. on $\mathcal{S}_1$
$\Gamma_{12} \vdash t_1 : \mathbf{F}\,A \rhd \langle\emptyset;\emptyset\rangle$	$''$
$\mathcal{D}_1$ reads $R_{\mathcal{D}_1}$ writes $W_{\mathcal{D}_1}$	$''$
$\langle W_{\mathcal{D}_1}; R_{\mathcal{D}_1}\rangle \preceq \epsilon_1$	$''$
$\langle W_{\mathcal{D}_1}; R_{\mathcal{D}_1}\rangle \preceq \langle W_1, R_1\rangle$	$''$

$\Gamma_{12} \vdash v : A$	inversion of typing rule ret, for terminal computation t_1
$\mathcal{S}_2 :: \Gamma_1, x : A \vdash^{M} e_2 : C \rhd \epsilon_2$	Subderivation 2 of $\mathcal{S}$
$\Gamma_{12}, x : A \vdash^{M} e_2 : C \rhd \epsilon_2$	Lemma C.2 (Weakening)
$\Gamma_{12} \vdash^{M} [v/x]e_2 : C \rhd \epsilon_2$	Lemma C.3 (Substitution)
$\mathcal{D}_2 :: S_{12} \vdash^{M}_{m} [v/x]e_2 \Downarrow S_2; t_2$	Subderivation 2 of $\mathcal{D}$
exists $\Gamma_2 \supseteq \Gamma_{12} \supseteq \Gamma_1$ such that	By i.h. on $\mathcal{S}_2$
☞ $\vdash S_2 : \Gamma_2$	$''$
☞ $\Gamma_2 \vdash^{M} t_2 : C \rhd \langle\emptyset;\emptyset\rangle$	$''$
$\mathcal{D}_2$ reads $R_{\mathcal{D}_2}$ writes $W_{\mathcal{D}_2}$	$''$
$\langle W_{\mathcal{D}_2}; R_{\mathcal{D}_2}\rangle \preceq \epsilon_2$	$''$
$\langle W_{\mathcal{D}_2}; R_{\mathcal{D}_2}\rangle \preceq \langle W_2, R_2\rangle$	$''$

$W_1 \perp W_2$ and $R_1 \perp W_2$	Definition of ϵ_1 then ϵ_2
$W_{\mathcal{D}_1} \perp W_{\mathcal{D}_2}$ and $R_{\mathcal{D}_1} \perp W_{\mathcal{D}_2}$	$W_{\mathcal{D}_1} \subseteq W_1; W_{\mathcal{D}_2} \subseteq W_2; R_{\mathcal{D}_1} \subseteq R_1$
$W_{\mathcal{D}} = W_{\mathcal{D}_1} \perp W_{\mathcal{D}_2}$	By Def. 7.2
$R_{\mathcal{D}} = R_{\mathcal{D}_1} \cup (R_{\mathcal{D}_2} - W_{\mathcal{D}_1})$	$''$
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$	$''$
☞ $\langle W_{\mathcal{D}}, R_{\mathcal{D}} \rangle \preceq \langle W, R \rangle$	Since $W_{\mathcal{D}} \subseteq W$ and $R_{\mathcal{D}} \subseteq R$

- **Case**
$$\frac{\Gamma \vdash^{M} e : \big((A \to E) \rhd \epsilon_1\big) \qquad \Gamma \vdash v : A}{\Gamma \vdash^{M} (e\ v) : (E \text{ after } \epsilon_1)}\ \text{app}$$
Similar to the case for `let`.

- **Case**
$$\frac{\Gamma \vdash^{M} v : (A_1 \times A_2) \qquad \Gamma, x_1 : A_1, x_2 : A_2 \vdash^{M} e : E}{\Gamma \vdash^{M} \text{split}\,(v, x_1.x_2.e) : E}\ \text{split}$$
Similar to the case for `let`, using Lemma C.8 (Canonical Forms).

- **Case**
$$\frac{\begin{array}{ll} & \Gamma, x_1 : A_1 \vdash^{M} e_1 : E \\ \Gamma \vdash^{M} v : (A_1 + A_2) & \Gamma, x_2 : A_2 \vdash^{M} e_2 : E \end{array}}{\Gamma \vdash^{M} \text{case}(v, x_1.e_1, x_2.e_2) : E}\ \text{case}$$
Similar to the case for `let`, using Lemma C.8 (Canonical Forms).

- **Case**
$$\frac{\begin{array}{l} \Gamma_1 \vdash v_M : (\mathbf{Nm} \overset{\text{Nm}}{\to} \mathbf{Nm})\,[M] \\ \Gamma_1 \vdash v : \text{Nm}\,[i] \end{array}}{\Gamma_1 \vdash (v_M\ v) : \mathbf{F}\,(\text{Nm}\,[M(i)]\,) \rhd \langle \emptyset; \emptyset \rangle}\ \text{name-app}$$

$\Gamma_1 \vdash v_M : (\mathbf{Nm} \overset{\text{Nm}}{\to} \mathbf{Nm})\,[M]$	Given
$v_M = \text{nmfn}\ M_v$	Lemma C.8 (Canonical Forms)
$M =_\beta (\lambda a.\, M')$	$''$
$\cdot \vdash \lambda a.\, M' : (\mathbf{Nm} \overset{\text{Nm}}{\to} \mathbf{Nm})$	$''$
$M_v =_\beta M$	$''$
$\Gamma_1 \vdash v : \text{Nm}\,[i]$	Given
$v = \text{name}\ n$	Lemma C.8 (Canonical Forms)
$\Gamma \vdash n \in i$	$''$
$M \Downarrow_M (\lambda a.\, M')$	By inversion on $\mathcal{D}$ (name-app)
$[n/a]M' \Downarrow_M p$	$''$
$p =_\beta [n/a]M'$	By a property of $\Downarrow_M$
$=_\beta (\lambda a.\, M')(n)$	By a property of $=_\beta$
$=_\beta M(n)$	By a property of $=_\beta$
$(\Gamma_2 = \Gamma_1), (S_2 = S_1)$	Suppose
☞ $\vdash S_2 : \Gamma_2$	By above equalities and $S_1 \vdash \Gamma_1$
$\Gamma_1 \vdash n \in i$	Above
$p =_\beta M(n)$	Above
$\Gamma_1 \vdash p \in M(i)$	By Lemma C.9

$\Gamma_1 \vdash \texttt{name}\ p : \mathsf{Nm}[M(i)]$ By rule name
☞ $\Gamma_1 \vdash \texttt{ret}(\texttt{name}\ p) : \mathbf{F}(\mathsf{Nm}[M(i)]) \rhd \langle \emptyset; \emptyset \rangle$ By rule ret
☞ $\mathcal{D}$ by $\Downarrow$-name-app reads $\emptyset$ writes $\emptyset$ By Def. 7.2
☞ $(R_{\mathcal{D}} = R = \emptyset), (W_{\mathcal{D}} = W = \emptyset)$ By above equalities

- **Case**

$$\frac{\Gamma_1, a : \gamma, P \vdash^M t : E}{\Gamma_1 \vdash^M t : (\forall a : \gamma \mid P.\, E)}\ \text{AllIndexIntro}$$

$\mathcal{S}_0 :: \Gamma_1, a : \gamma, P \vdash^M t : E$ Subderivation
$\mathcal{D}_0 :: S_1 \vdash^M_m e \Downarrow S_2; t$ Subderivation
$\exists \Gamma_2 \subseteq \Gamma_1$ By i.h.
☞ $\vdash S_2 : \Gamma_2$ ″
$\mathcal{D}_0$ reads $R_{\mathcal{D}_0}$ writes $W_{\mathcal{D}_0}$ ″
$\Gamma_2 \vdash t : E$ ″
$\langle R_{\mathcal{D}_0}; W_{\mathcal{D}_0} \rangle \preceq \langle R; W \rangle$ ″
☞ $\Gamma_2 \vdash^M t : (\forall a : \gamma.\, E)$ By typing rule
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ By Def. 7.2
☞ $\langle R_{\mathcal{D}}; W_{\mathcal{D}} \rangle \preceq \langle R; W \rangle$ By set theory

- **Case**

$$\frac{\Gamma_1 \vdash^M e : (\forall a : \gamma \mid P.\, E) \qquad \begin{array}{l}\Gamma_1 \vdash i : \gamma \\ \mathit{extract}(\Gamma_1) \Vdash [i/a]P\end{array}}{\Gamma_1 \vdash^M e : [i/a]E}\ \text{AllIndexElim}$$

$\mathcal{S}_0 :: \Gamma_1 \vdash^M e : (\forall a : \gamma.\, E)$ Subderivation
$\mathcal{D}_0 :: S_1 \vdash^M_m e \Downarrow S_2; t$ Subderivation
$\exists \Gamma_2 \subseteq \Gamma_1$ By i.h.
☞ $\vdash S_2 : \Gamma_2$ ″
$\mathcal{D}_0$ reads $R_{\mathcal{D}_0}$ writes $W_{\mathcal{D}_0}$ ″
$\Gamma_2 \vdash t : (\forall a : \gamma.\, E)$ ″
$\langle R_{\mathcal{D}_0}; W_{\mathcal{D}_0} \rangle \preceq \langle R; W \rangle$ ″
$\Gamma_1 \vdash i : \gamma$ Subderivation
$\Gamma_2 \vdash i : \gamma$ By weakening
☞ $\Gamma_2 \vdash^M t : [i/a]E$ By typing rule
☞ $\mathcal{D}$ reads $R_{\mathcal{D}}$ writes $W_{\mathcal{D}}$ By Def. 7.2
☞ $\langle R_{\mathcal{D}}; W_{\mathcal{D}} \rangle \preceq \langle R; W \rangle$ By set theory

- **Case**

$$\frac{\Gamma, \alpha : K \vdash^M t : E}{\Gamma \vdash^M t : (\forall \alpha : K.\, E)}\ \text{AllIntro}$$

Similar to the AllIndexIntro case.

- **Case**

$$\frac{\Gamma \vdash^M e : (\forall \alpha : K.\, E) \qquad \Gamma \vdash A : K}{\Gamma \vdash^M e : [A/\alpha]E}\ \text{AllElim}$$

Similar to the AllIndexElim case. □

D BIDIRECTIONAL TYPING

D.1 Syntax

As discussed below, bidirectional typing requires some annotations, so we assume that values v and expressions e have been extended with annotations $(v : A)$ and $(e : A)$. We also assume that we have explicit syntactic forms $e\,[i]$ and $e\,[A]$, which avoid guessing quantifier instantiations.

$\boxed{\Gamma \vdash v \Rightarrow A}$ Under Γ, value v synthesizes type A

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\ \text{vsyn-var} \qquad \frac{\Gamma \vdash v \Leftarrow A}{\Gamma \vdash (v : A) \Rightarrow A}\ \text{vsyn-anno}$$

$\boxed{\Gamma \vdash v \Leftarrow A}$ Under Γ, value v checks against type A

$$\frac{}{\Gamma \vdash () \Leftarrow \mathsf{unit}}\ \text{vchk-unit} \qquad \frac{\Gamma \vdash v_1 \Leftarrow A_1 \qquad \Gamma \vdash v_2 \Leftarrow A_2}{\Gamma \vdash (v_1, v_2) \Leftarrow (A_1 \times A_2)}\ \text{vchk-pair}$$

$$\frac{\Gamma \vdash v \Rightarrow A \qquad \Gamma \vdash A \leq_V B}{\Gamma \vdash v \Leftarrow B}\ \text{vchk-sub} \qquad \frac{\Gamma \vdash n \in X}{\Gamma \vdash (\mathtt{name}\ n) \Leftarrow \mathsf{Nm}\,[X]}\ \text{vchk-name}$$

$$\frac{\Gamma \vdash M_v \Rightarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm}) \qquad M_v =_\beta M}{\Gamma \vdash (\mathtt{nmfn}\ M_v) \Leftarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M]}\ \text{vchk-namefn}$$

$$\frac{\Gamma \vdash n \in X \qquad \Gamma(n) = A}{\Gamma \vdash (\mathtt{ref}\ n) \Leftarrow \mathsf{Ref}\,[X]\ A}\ \text{vchk-ref} \qquad \frac{\Gamma \vdash n \in X \qquad \Gamma(n) = E}{\Gamma \vdash (\mathtt{thunk}\ n) \Leftarrow (\mathsf{Thk}\,[X]\ E)}\ \text{vchk-thunk}$$

$$\frac{\Gamma \vdash v \Leftarrow A_1}{\Gamma \vdash \mathsf{inj}_1\, v \Leftarrow A_1 + A_2}\ \text{vchk-inj1} \qquad \frac{\Gamma \vdash v \Leftarrow A_2}{\Gamma \vdash \mathsf{inj}_2\, v \Leftarrow A_1 + A_2}\ \text{vchk-inj2}$$

$$\frac{\Gamma, a : \gamma, P \vdash v \Leftarrow A}{\Gamma \vdash v \Leftarrow (\forall a : \gamma \mid P.\, A)}\ \text{vchk-}\forall\text{IndexIntro} \qquad \frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v \Rightarrow (\forall a : \gamma \mid P.\, A)\end{array}}{\Gamma \vdash v\,[i] \Rightarrow [i/a]A}\ \text{vsyn-}\forall\text{IndexElim}$$

$$\frac{\Gamma \vdash i : \gamma \qquad \begin{array}{c}\mathit{extract}(\Gamma) \Vdash [i/a]P \\ \Gamma \vdash v \Leftarrow [i/a]A\end{array}}{\Gamma \vdash \mathtt{pack}(a.v) \Leftarrow (\exists a : \gamma \mid P.\, A)}\ \text{vchk-}\exists\text{IndexIntro}$$

Fig. 24. Bidirectional value typing

D.2 Bidirectional Typing Rules

The typing rules in Figures 12 and 13 are declarative: they define what typings are valid, but not how to derive those typings. The rules' use of names and effects annotations means that standard unification-based techniques, like Damas–Milner inference, are not readily applicable.

Following the DML tradition, we obtain an algorithmic version of our typing rules by defining a bidirectional system [Pierce and Turner 2000]: we split judgments with a colon into judgments with an arrow. Thus, the computation typing judgment $\cdots e : E$ becomes two judgments. The first is the *checking* judgment $\Gamma \vdash^M e \Leftarrow E$, in which the type E is already known—it is an *input* to the

$\boxed{\Gamma \vdash^{\mathsf{M}} e \Rightarrow \mathsf{E}}$ Under Γ, within namespace M, computation e synthesizes type-with-effects E

$$\frac{\Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} (e : \mathsf{E}) \Rightarrow \mathsf{E}}\ \text{esyn-anno} \qquad \frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow ((\mathsf{A} \to \mathsf{E}) \rhd \epsilon_1) \quad \Gamma \vdash v \Leftarrow \mathsf{A} \quad \Gamma \vdash \mathsf{E}\ \mathsf{after}\ \epsilon_1 \equiv \mathsf{E}'}{\Gamma \vdash^{\mathsf{M}} (e\ v) \Rightarrow \mathsf{E}'}\ \text{esyn-app}$$

$$\frac{\Gamma \vdash v \Rightarrow \mathsf{Thk}[\mathsf{X}]\ (\mathsf{C} \rhd \epsilon) \quad \Gamma \vdash \langle \emptyset; \mathsf{X} \rangle\ \mathsf{then}\ \epsilon \equiv \epsilon'}{\Gamma \vdash^{\mathsf{M}} \mathsf{force}(v) \Rightarrow \mathsf{C} \rhd \epsilon'}\ \text{esyn-force} \qquad \frac{\Gamma \vdash v \Rightarrow \mathsf{Ref}[\mathsf{X}]\ \mathsf{A}}{\Gamma \vdash^{\mathsf{M}} \mathsf{get}(v) \Rightarrow (\mathbf{F}\,\mathsf{A}) \rhd \langle \emptyset; \mathsf{X} \rangle}\ \text{esyn-get}$$

$$\frac{\Gamma \vdash v_{\mathsf{M}} \Rightarrow (\mathbf{Nm} \overset{\mathsf{Nm}}{\Rightarrow} \mathbf{Nm})[\mathsf{M}] \qquad \Gamma \vdash v \Rightarrow \mathsf{Nm}[\mathsf{i}]}{\Gamma \vdash^{\mathsf{N}} (v_{\mathsf{M}}\ v) \Rightarrow \mathbf{F}\,(\mathsf{Nm}[\mathsf{M}[\mathsf{i}]]) \rhd \langle \emptyset; \emptyset \rangle}\ \text{esyn-name-app}$$

$$\frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow (\forall a : \gamma \mid \mathsf{P}.\,\mathsf{E}) \qquad \Gamma \vdash \mathsf{i} : \gamma \qquad \mathit{extract}(\Gamma) \Vdash [\mathsf{i}/a]\mathsf{P}}{\Gamma \vdash^{\mathsf{M}} e[\mathsf{i}] \Rightarrow [\mathsf{i}/a]\mathsf{E}}\ \text{esyn-}\forall\text{IndexElim}$$

$$\frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow (\forall \alpha : \mathsf{K}.\,\mathsf{E}) \qquad \Gamma \vdash \mathsf{A} : \mathsf{K}}{\Gamma \vdash^{\mathsf{M}} e[\mathsf{A}] \Rightarrow [\mathsf{A}/\alpha]\mathsf{E}}\ \text{esyn-}\forall\text{Elim}$$

$\boxed{\Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}$ Under Γ, within namespace M, computation e checks against type-with-effects E

$$\frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow \mathsf{E}_1 \qquad \Gamma \vdash \mathsf{E}_1 \leq_E \mathsf{E}_2}{\Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}_2}\ \text{echk-sub}$$

$$\frac{\Gamma \vdash v \Rightarrow (\mathsf{A}_1 \times \mathsf{A}_2) \quad \Gamma, x_1 : \mathsf{A}_1, x_2 : \mathsf{A}_2 \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathsf{split}(v, x_1.x_2.e) \Leftarrow \mathsf{E}}\ \text{echk-split} \qquad \frac{\Gamma \vdash v \Rightarrow (\mathsf{A}_1 + \mathsf{A}_2) \quad \Gamma, x_1 : \mathsf{A}_1 \vdash^{\mathsf{M}} e_1 \Leftarrow \mathsf{E} \quad \Gamma, x_2 : \mathsf{A}_2 \vdash^{\mathsf{M}} e_2 \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathsf{case}(v, x_1.e_1, x_2.e_2) \Leftarrow \mathsf{E}}\ \text{echk-case}$$

$$\frac{\Gamma \vdash v \Leftarrow \mathsf{A}}{\Gamma \vdash^{\mathsf{M}} \mathsf{ret}(v) \Leftarrow ((\mathbf{F}\,\mathsf{A}) \rhd \langle \emptyset; \emptyset \rangle)}\ \text{echk-ret}$$

$$\frac{\Gamma, x : \mathsf{A} \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} (\lambda x.\, e) \Leftarrow ((\mathsf{A} \to \mathsf{E}) \rhd \langle \emptyset; \emptyset \rangle)}\ \text{echk-lam} \qquad \frac{\Gamma \vdash^{\mathsf{M}} e_1 \Rightarrow (\mathbf{F}\,\mathsf{A}) \rhd \epsilon_1 \quad \Gamma, x : \mathsf{A} \vdash^{\mathsf{M}} e_2 \Leftarrow (\mathsf{C} \rhd \epsilon_2) \quad \Gamma \vdash \epsilon_1\ \mathsf{then}\ \epsilon_2 \equiv \epsilon}{\Gamma \vdash^{\mathsf{M}} \mathsf{let}(e_1, x.e_2) \Leftarrow \mathsf{C} \rhd \epsilon}\ \text{echk-let}$$

$$\frac{\Gamma \vdash v \Leftarrow \mathsf{Nm}[\mathsf{X}] \qquad \Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathtt{thunk}(v, e) \Leftarrow (\mathbf{F}\,(\mathsf{Thk}[\mathsf{M}[\mathsf{X}]]\ \mathsf{E})) \rhd \langle \mathsf{M}[\mathsf{X}]; \emptyset \rangle}\ \text{echk-thunk}$$

$$\frac{\Gamma \vdash v_1 \Leftarrow \mathsf{Nm}[\mathsf{X}] \qquad \Gamma \vdash v_2 \Leftarrow \mathsf{A}}{\Gamma \vdash^{\mathsf{M}} \mathtt{ref}(v_1, v_2) \Leftarrow (\mathbf{F}\,(\mathsf{Ref}[\mathsf{M}[\mathsf{X}]]\ \mathsf{A})) \rhd \langle \mathsf{M}[\mathsf{X}]; \emptyset \rangle}\ \text{echk-ref}$$

$$\frac{\Gamma \vdash v \Rightarrow (\mathbf{Nm} \overset{\mathsf{Nm}}{\Rightarrow} \mathbf{Nm})[\mathsf{N}'] \qquad \Gamma \vdash^{\mathsf{N} \circ \mathsf{N}'} e \Leftarrow (\mathsf{C} \rhd \langle W; \mathsf{R} \rangle)}{\Gamma \vdash^{\mathsf{N}} \mathsf{scope}(v, e) \Leftarrow (\mathsf{C} \rhd \langle W; \mathsf{R} \rangle)}\ \text{echk-scope}$$

$$\frac{\Gamma, a : \gamma, \mathsf{P} \vdash^{\mathsf{M}} \mathsf{t} \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathsf{t} \Leftarrow (\forall a : \gamma \mid \mathsf{P}.\,\mathsf{E})}\ \text{echk-}\forall\text{IndexIntro} \qquad \frac{\Gamma, \alpha : \mathsf{K} \vdash^{\mathsf{M}} \mathsf{t} \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathsf{t} \Leftarrow (\forall \alpha : \mathsf{K}.\,\mathsf{E})}\ \text{echk-}\forall\text{Intro}$$

$$\frac{\Gamma \vdash v \Rightarrow (\exists a : \gamma \mid \mathsf{P}.\,\mathsf{A}) \qquad \Gamma, a : \gamma, \mathsf{P}, x : \mathsf{A} \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathtt{vunpack}(v, a.x.e) \Leftarrow \mathsf{E}}\ \text{echk-}\exists\text{IndexElim}$$

Fig. 25. Bidirectional computation typing

algorithm. The second is the *synthesis* judgment $\Gamma \vdash^{M} e \Rightarrow E$, in which E is not known—it is an output—and the rules construct E by examining e (and Γ).

In formulating the bidirectional versions of value and computation typing (Figures 24 and 25), we mostly follow the "recipe" of Dunfield and Pfenning [2004]: introduction rules check, and elimination rules synthesize. More precisely, the *principal judgment*—the judgment, either a premise or conclusion, that has the connective being introduced or eliminated—is checking ($\Leftarrow$) for introduction rules, and synthesizing ($\Rightarrow$) for elimination rules. In many cases, once the direction of that premise (or conclusion) is determined, the direction of the other judgments follows by considering what information is known (as input, or as the output type of the principal judgment, if that judgment is synthesizing). For example, if we commit to checking the conclusion of echk-lam, we should check the premise because its type is a subexpression of the type in the conclusion. (Checking is more powerful than synthesis: every expression that synthesizes also checks, but not all expressions that check can synthesize.)

When a synthesis (elimination) premise attempts to type an expression that is a checking (introduction) form, the programmer must write a type annotation $(e : E)$. Thus, following the recipe means that we have a straightforward *annotation discipline*: annotations are needed only on redexes. While we could reduce the number of annotations by adding synthesis rules—for example, allowing the unit value $()$ to synthesize unit—this makes the system larger without changing its essential properties; for a discussion of the implications of such extensions in a different context, see Dunfield and Krishnaswami [2013].

Dually, when an expression synthesizes but we are trying to derive a checking judgment, we use (1) vchk-sub for value typing, or (2) echk-sub for computation typing. The latter rule includes effect subsumption.

E BIDIRECTIONAL TYPING PROOFS

Theorem E.1 (Soundness of Bidirectional Value Typing).

(1) If $\Gamma \vdash v \Rightarrow A$*, then there exists a value* v' *such that* $\Gamma \vdash v' : A$ *and* $|v| = v'$.
(2) If $\Gamma \vdash v \Leftarrow A$*, then there exists a value* v' *such that* $\Gamma \vdash v' : A$ *and* $|v| = v'$.

Proof. By induction on the given derivation.
Part (1): Proceed by cases on the rule concluding $\Gamma \vdash v \Rightarrow A$.

- **Case**
$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A}\ \text{vsyn-var}$$

$(x : A) \in \Gamma$	Given
$\Gamma \vdash x : A$	By rule var
$\lvert x\rvert = x$	By definition of $\lvert -\rvert$
☞ $\Gamma \vdash v' : A$ and $\lvert v\rvert = v'$	where $v' = x$ and $v = x$

- **Case**
$$\frac{\Gamma \vdash v_1 \Leftarrow A}{\Gamma \vdash (v_1 : A) \Rightarrow A}\ \text{vsyn-anno}$$

$\exists\, v_1'$ such that $\Gamma \vdash v_1' : A$ and $\lvert v_1\rvert = v_1'$	By inductive hypothesis
$\lvert(v_1 : A)\rvert = \lvert v_1\rvert = v_1'$	By definition of $\lvert -\rvert$ and $\lvert v_1\rvert = v_1'$
☞ $\Gamma \vdash v' : A$ and $\lvert v\rvert = v'$	where $v' = v_1'$ and $v = (v_1 : A)$

Part (2): Proceed by cases on the rule concluding $\Gamma \vdash v \Leftarrow A$.

- **Case**

$$\frac{}{\Gamma \vdash () \Leftarrow \mathsf{unit}}\ \text{vchk-unit}$$

	$\Gamma \vdash () : \mathsf{unit}$	By rule unit
	$\lvert() \rvert = ()$	By definition of $\lvert - \rvert$
☞	$\Gamma \vdash v' : ()$ and $\lvert v \rvert = v'$	where $v' = ()$ and $v = ()$

- **Case**

$$\frac{\Gamma \vdash v_1 \Leftarrow A_1 \qquad \Gamma \vdash v_2 \Leftarrow A_2}{\Gamma \vdash (v_1, v_2) \Leftarrow (A_1 \times A_2)}\ \text{vchk-pair}$$

	$\exists\, v_1'$ such that $\Gamma \vdash v_1' : A_1$ and $\lvert v_1 \rvert = v_1'$	By inductive hypothesis
	$\exists\, v_2'$ such that $\Gamma \vdash v_2' : A_2$ and $\lvert v_2 \rvert = v_2'$	By inductive hypothesis
	$\Gamma \vdash (v_1', v_2') : (A_1 \times A_2)$	By rule pair
	$\lvert (v_1, v_2) \rvert = (\lvert v_1 \rvert, \lvert v_2 \rvert) = (v_1', v_2')$	By definition of $\lvert - \rvert$ $\lvert v_1 \rvert = v_1'$ and $\lvert v_2 \rvert = v_2'$
☞	$\Gamma \vdash v' : (A_1 \times A_2)$ and $\lvert v \rvert = v'$	where $v' = (v_1', v_2')$ and $v = (v_1, v_2)$

- **Case**

$$\frac{\Gamma \vdash n \in X}{\Gamma \vdash (\mathtt{name}\ n) \Leftarrow \mathsf{Nm}[X]}\ \text{vchk-name}$$

	$\Gamma \vdash n \in X$	Given
	$\Gamma \vdash (\mathtt{name}\ n) : \mathsf{Nm}[X]$	By rule name
	$\lvert (\mathtt{name}\ n) \rvert = (\mathtt{name}\ n)$	By definition of $\lvert - \rvert$
☞	$\Gamma \vdash v' : \mathsf{Nm}[X]$ and $\lvert v \rvert = v'$	where $v' = (\mathtt{name}\ n)$ and $v = (\mathtt{name}\ n)$

- **Case**

$$\frac{\Gamma \vdash M_v \Rightarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\rightarrow} \mathbf{Nm}) \qquad M_v =_\beta M}{\Gamma \vdash (\mathtt{nmfn}\ M_v) \Leftarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\rightarrow} \mathbf{Nm})[M]}\ \text{vchk-namefn}$$

$\exists\, M_v'$ such that $\Gamma \vdash M_v' : (\mathbf{Nm} \overset{\mathbf{Nm}}{\rightarrow} \mathbf{Nm})$ and $\lvert M_v \rvert = M_v'$ By i.h.

$M_v =_\beta M$	Given
$\lvert M_v \rvert =_\beta M$	Type erasure does not affect convertibility
$M_v' =_\beta M$	Since $\lvert M_v \rvert = M_v'$

	$\Gamma \vdash (\mathtt{nmfn}\ M_v') : (\mathbf{Nm} \overset{\mathbf{Nm}}{\rightarrow} \mathbf{Nm})[M]$	By rule namefn
	$\lvert (\mathtt{nmfn}\ M_v) \rvert = (\mathtt{nmfn}\ \lvert M_v \rvert) = (\mathtt{nmfn}\ M_v')$	By definition of $\lvert - \rvert$ and $\lvert M_v \rvert = M_v'$
☞	$\Gamma \vdash v' : (\mathbf{Nm} \overset{\mathbf{Nm}}{\rightarrow} \mathbf{Nm})[M]$ and $\lvert v \rvert = v'$	where $v' = (\mathtt{nmfn}\ M_v')$ and $v = (\mathtt{nmfn}\ M_v)$

- **Case**

$$\frac{\Gamma \vdash n \in X \qquad \Gamma(n) = A}{\Gamma \vdash (\mathtt{ref}\ n) \Leftarrow \mathsf{Ref}[X]\ A}\ \text{vchk-ref}$$

	$\Gamma \vdash n \in X$	Given
	$\Gamma(n) = A$	Given
	$\Gamma \vdash (\texttt{ref}\ n) : \mathsf{Ref}\,[X]\ A$	By rule ref
	$\lvert(\texttt{ref}\ n)\rvert = \texttt{ref}\ n$	By definition of $\lvert-\rvert$
☞	$\Gamma \vdash v' : \mathsf{Ref}\,[X]\ A$ and $\lvert v\rvert = v'$	where $v' = (\texttt{ref}\ n)$ and $v = (\texttt{ref}\ n)$

- **Case**
$$\frac{\Gamma \vdash n \in X \qquad \Gamma(n) = E}{\Gamma \vdash (\texttt{thunk}\ n) \Leftarrow (\mathsf{Thk}\,[X]\ E)}\ \text{vchk-thunk}$$

	$\Gamma \vdash n \in X$	Given
	$\Gamma(n) = E$	Given
	$\Gamma \vdash (\texttt{thunk}\ n) : \mathsf{Thk}\,[X]\ E$	By rule thunk
	$\lvert(\texttt{thunk}\ n)\rvert = \texttt{thunk}\ n$	By definition of $\lvert-\rvert$
☞	$\Gamma \vdash v' : \mathsf{Thk}\,[X]\ E$ and $\lvert v\rvert = v'$	where $v' = (\texttt{thunk}\ n)$ and $v = (\texttt{thunk}\ n)$

- **Case**
$$\frac{\Gamma \vdash v_1 \Rightarrow A_1 \qquad \Gamma \vdash A_1 \leq_V A_2}{\Gamma \vdash v_1 \Leftarrow A_2}\ \text{vchk-sub}$$

By i.h. and vtype-sub. □

Theorem E.2 (Completeness of Bidirectional Value Typing).
If $\Gamma \vdash v : A$ *then there exist values* v' *and* v'' *such that*

(1) $\Gamma \vdash v' \Rightarrow A$ *and* $\lvert v'\rvert = v$
(2) $\Gamma \vdash v'' \Leftarrow A$ *and* $\lvert v''\rvert = v$

Proof. By induction on the derivation of $\Gamma \vdash v : A$.

Case
$$\frac{}{\Gamma \vdash () : \mathsf{unit}}\ \text{unit}$$

	$\Gamma \vdash () \Leftarrow \mathsf{unit}$	By rule vchk-unit
	$\lvert()\rvert = ()$	By definition of $\lvert-\rvert$
☞	$\Gamma \vdash v'' \Leftarrow ()$ and $\lvert v''\rvert = v$	where $v'' = ()$ and $v = ()$
	$\Gamma \vdash (() : \mathsf{unit}) \Rightarrow \mathsf{unit}$	By rule vsyn-anno
	$\lvert(() : \mathsf{unit})\rvert = ()$	By definition of $\lvert-\rvert$
☞	$\Gamma \vdash v' \Rightarrow ()$ and $\lvert v'\rvert = v$	where $v' = (() : \mathsf{unit})$ and $v = ()$

Case
$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ \text{var}$$

$$
\begin{array}{rll}
 & (x : A) \in \Gamma & \text{Given} \\
 & \Gamma \vdash x \Rightarrow A & \text{By rule vsyn-var} \\
 & |x| = x & \text{By definition of } |-| \\
\text{☞} & \Gamma \vdash v' \Rightarrow A \text{ and } |v'| = v & \text{where } v' = x \text{ and } v = x \\
 & \Gamma \vdash x \Leftarrow A & \text{By rule vchk-conv} \\
\text{☞} & \Gamma \vdash v'' \Leftarrow A \text{ and } |v''| = v & \text{where } v'' = x \text{ and } v = x \\
 & & \quad \text{and } |x| = x
\end{array}
$$

Case $\dfrac{\Gamma \vdash v_1 : A_1 \qquad \Gamma \vdash v_2 : A_2}{\Gamma \vdash (v_1, v_2) : (A_1 \times A_2)}$ pair

$$
\begin{array}{rl}
\exists\, v_1'' \text{ such that } \Gamma \vdash v_1'' \Leftarrow A_1 \text{ and } |v_1''| = v_1 & \text{By inductive hypothesis} \\
\exists\, v_2'' \text{ such that } \Gamma \vdash v_2'' \Leftarrow A_2 \text{ and } |v_2''| = v_2 & \text{By inductive hypothesis} \\
\Gamma \vdash (v_1'', v_2'') \Leftarrow (A_1 \times A_2) & \text{By rule vchk-pair} \\
|(v_1'', v_2'')| = (|v_1''|, |v_2''|) = (v_1, v_2) & \text{By definition of } |-|;\ |v_1''| = v_1;\ |v_2''| = v_2 \\
\text{☞} \Gamma \vdash v'' \Leftarrow (A_1 \times A_2) \text{ and } |v''| = v & \text{where } v'' = (v_1'', v_2'') \text{ and } v = (v_1, v_2) \\
\Gamma \vdash ((v_1'', v_2'') : (A_1 \times A_2)) \Rightarrow (A_1 \times A_2) & \text{By rule vsyn-anno} \\
|(v_1'', v_2'') : (A_1 \times A_2)| = |(v_1'', v_2'')| = (v_1, v_2) & \text{By definition of } |-|;\ |(v_1'', v_2'')| = (v_1, v_2) \\
\text{☞} \Gamma \vdash v' \Rightarrow (A_1 \times A_2) \text{ and } |v'| = v & \text{where } v' = (v_1'', v_2'') \text{ and } v = (v_1, v_2)
\end{array}
$$

Case $\dfrac{\Gamma \vdash n \in X}{\Gamma \vdash (\texttt{name}\ n) : \mathsf{Nm}[X]}$ name

$$
\begin{array}{rll}
 & \Gamma \vdash n \in X & \text{Given} \\
 & \Gamma \vdash (\texttt{name}\ n) \Leftarrow \mathsf{Nm}[X] & \text{By rule vchk-name} \\
 & |(\texttt{name}\ n)| = (\texttt{name}\ n) & \text{By definition of } |-| \\
\text{☞} & \Gamma \vdash v'' \Leftarrow \mathsf{Nm}[X] \text{ and } |v''| = v & \text{where } v'' = (\texttt{name}\ n) \text{ and } v = (\texttt{name}\ n) \\
 & \Gamma \vdash (\texttt{name}\ n : \mathsf{Nm}[X]) \Rightarrow \mathsf{Nm}[X] & \text{By rule vsyn-anno} \\
 & |(\texttt{name}\ n : \mathsf{Nm}[X])| = |(\texttt{name}\ n)| = (\texttt{name}\ n) & \text{By definition of } |-| \\
\text{☞} & \Gamma \vdash v' \Rightarrow \mathsf{Nm}[X] \text{ and } |v'| = v & \text{where } v' = (\texttt{name}\ n : \mathsf{Nm}[X]) \text{ and } v = (\texttt{name}\ n)
\end{array}
$$

Case $\dfrac{\Gamma \vdash M_v : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}) \qquad M_v =_\beta M}{\Gamma \vdash (\texttt{nmfn}\ M_v) : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M]}$ namefn

$$
\begin{array}{rl}
\exists\, M_v' \text{ such that} & \\
\Gamma \vdash M_v' \Rightarrow (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm}) \text{ and } |M_v'| = M_v & \text{By inductive hypothesis} \\
M_v =_\beta M & \text{Given} \\
|M_v'| =_\beta M & \text{Since } |M_v'| = M_v \\
M_v' =_\beta M & \text{Type annotation does not affect convertibility} \\
\Gamma \vdash (\texttt{nmfn}\ M_v') \Leftarrow (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M] & \text{By rule vchk-namefn} \\
|(\texttt{nmfn}\ M_v')| = (\texttt{nmfn}\ |M_v'|) = (\texttt{nmfn}\ M_v) & \text{By definition of } |-| \\
 & \quad \text{and } |M_v'| = M_v \\
\text{☞} \Gamma \vdash v'' \Leftarrow (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M] \text{ and } |v''| = v & \text{where } v'' = (\texttt{nmfn}\ M_v') \text{ and } v = (\texttt{nmfn}\ M_v)
\end{array}
$$

$$
\begin{array}{rl}
\Gamma \vdash (\texttt{nmfn}\ M_v' : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M]) \Rightarrow (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M] & \text{By rule vsyn-anno} \\
|(\texttt{nmfn}\ M_v' : (\mathbf{Nm} \xrightarrow{\mathbf{Nm}} \mathbf{Nm})\,[M])| = (\texttt{nmfn}\ M_v) & \text{By definition of } |-|
\end{array}
$$

☞ $\Gamma \vdash v' \Rightarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\Rightarrow} \mathbf{Nm})\,[M]$ and $|v'| = v$ where $v' = (\mathtt{nmfn}\ M'_v : (\mathbf{Nm} \overset{\mathbf{Nm}}{\Rightarrow} \mathbf{Nm})\,[M])$

Case $\dfrac{\Gamma \vdash n \in X \qquad \Gamma(n) = A}{\Gamma \vdash (\mathtt{ref}\ n) : \mathsf{Ref}\,[X]\ A}\ \text{ref}$

$\Gamma \vdash n \in X$	Given
$\Gamma(n) = A$	Given
$\Gamma \vdash (\mathtt{ref}\ n) \Leftarrow \mathsf{Ref}\,[X]\ A$	By rule vchk-ref
$\lvert(\mathtt{ref}\ n)\rvert = (\mathtt{ref}\ n)$	By definition of $\lvert-\rvert$
☞ $\Gamma \vdash v'' \Leftarrow \mathsf{Ref}\,[X]\ A$ and $\lvert v''\rvert = v$	where $v'' = (\mathtt{ref}\ n)$ and $v = (\mathtt{ref}\ n)$
$\Gamma \vdash ((\mathtt{ref}\ n) : \mathsf{Ref}\,[X]\ A) \Rightarrow \mathsf{Ref}\,[X]\ A$	By rule vsyn-anno
$\lvert(\mathtt{ref}\ n : \mathsf{Ref}\,[X]\ A)\rvert = \lvert(\mathtt{ref}\ n)\rvert = (\mathtt{ref}\ n)$	By definition of $\lvert-\rvert$
☞ $\Gamma \vdash v' \Rightarrow \mathsf{Ref}\,[X]\ A$ and $\lvert v'\rvert = v$	where $v' = (\mathtt{ref}\ n : \mathsf{Ref}\,[X]\ A)$ and $v = (\mathtt{ref}\ n)$

Case $\dfrac{\Gamma \vdash n \in X \qquad \Gamma(n) = E}{\Gamma \vdash (\mathtt{thunk}\ n) : (\mathsf{Thk}\,[X]\ E)}\ \text{thunk}$

$\Gamma \vdash n \in X$	Given
$\Gamma(n) = E$	Given
$\Gamma \vdash (\mathtt{thunk}\ n) \Leftarrow (\mathsf{Thk}\,[X]\ E)$	By rule vchk-thunk
$\lvert(\mathtt{thunk}\ n)\rvert = (\mathtt{thunk}\ n)$	By definition of $\lvert-\rvert$
☞ $\Gamma \vdash v'' \Leftarrow (\mathsf{Thk}\,[X]\ E)$ and $\lvert v''\rvert = v$	where $v'' = (\mathtt{thunk}\ n)$ and $v = (\mathtt{thunk}\ n)$
$\Gamma \vdash (\mathtt{thunk}\ n : (\mathsf{Thk}\,[X]\ E)) \Rightarrow (\mathsf{Thk}\,[X]\ E)$	By rule vsyn-anno
$\lvert(\mathtt{thunk}\ n : (\mathsf{Thk}\,[X]\ E))\rvert = \mathtt{thunk}\ n$	By definition of $\lvert-\rvert$
☞ $\Gamma \vdash v' \Rightarrow (\mathsf{Thk}\,[X]\ E)$ and $\lvert v'\rvert = v$	where $v' = (\mathtt{thunk}\ n : (\mathsf{Thk}\,[X]\ E))$ and $v = (\mathtt{thunk}\ n)$

□

Theorem E.3 (Soundness of Bidirectional Computation Typing).

(1) If $\Gamma \vdash^{M} e \Rightarrow E$, *then there exists a value* e' *such that* $\Gamma \vdash^{M} e' : E$ *and* $|e| = e'$

(2) If $\Gamma \vdash^{M} e \Leftarrow E$, *then there exists a value* e' *such that* $\Gamma \vdash^{M} e' : E$ *and* $|e| = e'$

Proof. By induction on the given derivation.

Part (1): Proceed by case analysis on the rule concluding $\Gamma \vdash^{M} e \Rightarrow E$.

- **Case** $\dfrac{\Gamma \vdash^{M} e_1 \Rightarrow ((A \to E) \rhd \epsilon_1) \qquad \Gamma \vdash v \Leftarrow A}{\Gamma \vdash^{M} (e_1\ v) \Rightarrow (E\ \mathsf{after}\ \epsilon_1)}\ \text{esyn-app}$

$\Gamma \vdash^{M} e'_1 : ((A \to E) \rhd \epsilon_1)$ and $\lvert e_1\rvert = e'_1$	By inductive hypothesis
$\Gamma \vdash v' : A$ and $\lvert v\rvert = v'$	By Thm. E.1
$\Gamma \vdash^{M} (e'_1\ v') : (E\ \mathsf{after}\ \epsilon_1)$	By rule app
$\lvert(e_1\ v)\rvert = (\lvert e_1\rvert\ \lvert v\rvert) = (e'_1\ v')$	By definition of $\lvert-\rvert$
☞ $\Gamma \vdash^{M} e' : (E\ \mathsf{after}\ \epsilon_1)$ and $\lvert e\rvert = e'$	where $e' = (e'_1\ v')$ and $e = (e_1\ v)$

- **Case**

$$\frac{\Gamma \vdash v \Rightarrow \mathsf{Thk}\,[X]\ (C \rhd \epsilon)}{\Gamma \vdash^{M} \mathsf{force}(v) \Rightarrow \big(C \rhd (\langle \emptyset; X\rangle \text{ then } \epsilon)\big)}\ \text{esyn-force}$$

$\Gamma \vdash v' : \mathsf{Thk}\,[X]\ (C \rhd \epsilon)$ and $|v| = v'$ — By Thm. E.1
$\Gamma \vdash^{M} \mathsf{force}(v') : \big(C \rhd (\langle \emptyset; X\rangle \text{ then } \epsilon)\big)$ — By rule force
$|\mathsf{force}(v)| = \mathsf{force}(|v|) = \mathsf{force}(v')$ — By definition of $|-|$ and $|v| = v'$
☞ $\Gamma \vdash^{M} e' : \big(C \rhd (\langle \emptyset; X\rangle \text{ then } \epsilon)\big)$ and $|e| = e'$ — where $e' = \mathsf{force}(v')$ and $e = \mathsf{force}(v)$

- **Case**

$$\frac{\Gamma \vdash v \Rightarrow \mathsf{Ref}\,[X]\ A}{\Gamma \vdash^{M} \mathsf{get}(v) \Rightarrow \big(\mathbf{F}\,A\big) \rhd \langle \emptyset; X\rangle}\ \text{esyn-get}$$

$\Gamma \vdash v' : \mathsf{Ref}\,[X]\ A$ and $|v| = v'$ — By Thm. E.1
$\Gamma \vdash^{M} \mathsf{get}(v') : \big(\mathbf{F}\,A\big) \rhd \langle \emptyset; X\rangle$ — By rule get
$|\mathsf{get}(v)| = \mathsf{get}(|v|) = \mathsf{get}(v')$ — By the definition of $|-|$ and $|v| = v'$
☞ $\Gamma \vdash^{M} e' : \big(\mathbf{F}\,A\big) \rhd \langle \emptyset; X\rangle$ and $|e| = e'$ — where $e' = \mathsf{get}(v')$ and $e = \mathsf{get}(v)$

- **Case**

$$\frac{\Gamma \vdash v_M \Rightarrow (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M] \qquad \Gamma \vdash v \Rightarrow \mathsf{Nm}\,[i]}{\Gamma \vdash^{N} (v_M\ v) \Rightarrow \mathbf{F}\,(\mathsf{Nm}\,[M(i)]) \rhd \langle \emptyset; \emptyset\rangle}\ \text{esyn-name-app}$$

$\Gamma \vdash v'_M : (\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm})\,[M]$ and $|v_M| = v'_M$ — By Thm. E.1
$\Gamma \vdash v' : \mathsf{Nm}\,[i]$ and $|v| = v'$ — By Thm. E.1
$\Gamma \vdash^{N} (v'_M\ v') : \mathbf{F}\,(\mathsf{Nm}\,[M(i)]) \rhd \langle \emptyset; \emptyset\rangle$ — By rule name-app
$|(v_M\ v)| = (|v_M|\ |v|) = (v'_M\ v')$ — By definition of $|-|$; $|v_M| = v'_M$; $|v| = v'$
☞ $\Gamma \vdash^{M} e' : \mathbf{F}\,(\mathsf{Nm}\,[M(i)]) \rhd \langle \emptyset; \emptyset\rangle$ and $|e| = e'$ — where $e' = (v'_M\ v')$ and $e = (v_M\ v)$

- **Case**

$$\frac{\Gamma \vdash^{M} e \Rightarrow (\forall a : \gamma.\, E) \qquad \Gamma \vdash i : \gamma}{\Gamma \vdash^{M} e\,[i] \Rightarrow [i/a]E}\ \text{esyn-}\forall\text{IndexElim}$$

$\Gamma \vdash^{M} e' : (\forall a : \gamma.\, E)$ and $|e| = e'$ — By inductive hypothesis
$\Gamma \vdash i : \gamma$ — Given
☞ $e\,[i] = |e'|$ — By $|e| = e'$
☞ $\Gamma \vdash^{M} e' : [i/a]E$ — By rule AllIndexElim

- **Case**

$$\frac{\Gamma \vdash^{M} e \Rightarrow (\forall \alpha : K.\, E) \qquad \Gamma \vdash A : K}{\Gamma \vdash^{M} e\,[A] \Rightarrow [A/\alpha]E}\ \text{esyn-}\forall\text{Elim}$$

Similar to the syn-AllIndexElim case.

- **Case**

$$\frac{\Gamma \vdash^{M} e_1 \Leftarrow E}{\Gamma \vdash^{M} (e_1 : E) \Rightarrow E}\ \text{esyn-anno}$$

$$\begin{array}{rl}
\Gamma \vdash^{\mathsf{M}} e_1' : \mathsf{E} \text{ and } |e_1| = e_1' & \text{By inductive hypothesis} \\
|e_1 : \mathsf{E}| = |e_1| = e_1' & \text{By the definition of } |-| \\
& \quad \text{and } |e_1| = e_1' \\
\text{☞ } \Gamma \vdash^{\mathsf{M}} e' : \mathsf{E} \text{ and } |e| = e' & \text{where } e' = e_1' \text{ and } e = (e_1 : \mathsf{E})
\end{array}$$

Part (2): Proceed by case analysis on the rule concluding $\Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}$.

- **Case**
$$\frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow (\mathsf{C} \rhd \epsilon_1) \qquad \epsilon_1 \preceq \epsilon_2}{\Gamma \vdash^{\mathsf{M}} e \Leftarrow (\mathsf{C} \rhd \epsilon_2)}\ \text{echk-sub}$$

  By i.h. and rule etype-sub.

- **Case**
$$\frac{\begin{array}{l}\Gamma \vdash v \Rightarrow (A_1 \times A_2) \\ \Gamma, x_1 : A_1, x_2 : A_2 \vdash^{\mathsf{M}} e_1 \Leftarrow \mathsf{E}\end{array}}{\Gamma \vdash^{\mathsf{M}} \mathsf{split}(v, x_1.x_2.e_1) \Leftarrow \mathsf{E}}\ \text{echk-split}$$

$$\begin{array}{rl}
\Gamma \vdash v' : (A_1 \times A_2) \text{ and } |v| = v' & \text{By Thm. E.1} \\
\Gamma, x_1 : A_1, x_2 : A_2 \vdash^{\mathsf{M}} e_1' : \mathsf{E} \text{ and } |e_1| = e_1' & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} \mathsf{split}(v, x_1.x_2.e_1') : \mathsf{E} & \text{By rule split} \\
|\mathsf{split}(v, x_1.x_2.e_1)| = \mathsf{split}(|v|, x_1.x_2.|e_1|) & \\
\quad = \mathsf{split}(v', x_1.x_2.e_1') & \text{By definition of } |-| \\
& \quad \text{and } |v| = v', |e_1| = e_1' \\
\text{☞ } \Gamma \vdash^{\mathsf{M}} e' : \mathsf{E} \text{ and } |e| = e' & \text{where } e' = \mathsf{split}(v', x_1.x_2.e_1') \\
& \quad \text{and } e = \mathsf{split}(v, x_1.x_2.e_1)
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash v \Rightarrow (A_1 + A_2) \qquad \begin{array}{l}\Gamma, x_1 : A_1 \vdash^{\mathsf{M}} e_1 \Leftarrow \mathsf{E} \\ \Gamma, x_2 : A_2 \vdash^{\mathsf{M}} e_2 \Leftarrow \mathsf{E}\end{array}}{\Gamma \vdash^{\mathsf{M}} \mathsf{case}(v, x_1.e_1, x_2.e_2) \Leftarrow \mathsf{E}}\ \text{echk-case}$$

$$\begin{array}{rl}
\Gamma \vdash v' : (A_1 + A_2) \text{ and } |v| = v' & \text{By Thm. E.1} \\
\Gamma, x_1 : A_1 \vdash^{\mathsf{M}} e_1' : \mathsf{E} \text{ and } |e_1| = e_1' & \text{By inductive hypothesis} \\
\Gamma, x_2 : A_2 \vdash^{\mathsf{M}} e_2' : \mathsf{E} \text{ and } |e_2| = e_2' & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} \mathsf{case}(v', x_1.e_1', x_2.e_2') : \mathsf{E} & \text{By rule case} \\
|\mathsf{case}(v, x_1.e_1, x_2.e_2)| = \mathsf{case}(|v|, x_1.|e_1|, x_2.|e_2|) & \text{By definition of } |-| \\
\quad = \mathsf{case}(v', x_1.e_1', x_2.e_2') & \text{Since } |v| = v', |e_1| = e_1', |e_2| = e_2' \\
\text{☞ } \Gamma \vdash^{\mathsf{M}} e' : \mathsf{E} \text{ and } |e| = e' & \text{where } e' = \mathsf{case}(v', x_1.e_1', x_2.e_2') \\
& \quad \text{and } e = \mathsf{case}(v, x_1.e_1, x_2.e_2)
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash v \Leftarrow A}{\Gamma \vdash^{\mathsf{M}} \mathtt{ret}(v) \Leftarrow \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle\big)}\ \text{echk-ret}$$

$$\begin{array}{rl}
\Gamma \vdash v' : A \text{ and } |v| = v' & \text{By Thm. E.1} \\
\Gamma \vdash^{\mathsf{M}} \mathtt{ret}(v') : \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle\big) & \text{By rule ret} \\
|\mathtt{ret}(v)| = \mathtt{ret}(|v|) = \mathtt{ret}(v') & \text{By definition of } |-| \\
& \quad \text{and } |v| = v' \\
\text{☞ } \Gamma \vdash^{\mathsf{M}} e' : \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle\big) \text{ and } |e| = e' & \text{where } e' = \mathtt{ret}(v') \text{ and } e = \mathtt{ret}(v)
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash^{M} e_1 \Rightarrow (\mathbf{F}\, A) \rhd \epsilon_1 \qquad \Gamma, x : A \vdash^{M} e_2 \Leftarrow (C \rhd \epsilon_2)}{\Gamma \vdash^{M} \mathtt{let}(e_1, x.e_2) \Leftarrow \big(C \rhd (\epsilon_1 \text{ then } \epsilon_2)\big)}\ \text{echk-let}$$

$\Gamma \vdash^{M} e_1' : (\mathbf{F}\, A) \rhd \epsilon_1$ and $|e_1| = e_1'$ — By inductive hypothesis
$\Gamma, x : A \vdash^{M} e_2' : (C \rhd \epsilon_2)$ and $|e_2| = e_2'$ — By inductive hypothesis
$\Gamma \vdash^{M} \mathtt{let}(e_1', x.e_2') : \big(C \rhd (\epsilon_1 \text{ then } \epsilon_2)\big)$ — By rule let
$|\mathtt{let}(e_1, x.e_2)| = \mathtt{let}(|e_1|, x.|e_2|)$ — By definition of $|-|$
$= \mathtt{let}(e_1', x.e_2')$ — Since $|e_1| = e_1', |e_2| = e_2'$
☞ $\Gamma \vdash^{M} e' : \big(C \rhd (\epsilon_1 \text{ then } \epsilon_2)\big)$ and $|e| = e'$ — where $e' = \mathtt{let}(e_1', x.e_2')$ and $e = \mathtt{let}(e_1, x.e_2)$

- **Case**
$$\frac{\Gamma, x : A \vdash^{M} e_1 \Leftarrow E}{\Gamma \vdash^{M} (\lambda x.\, e_1) \Leftarrow \big((A \to E) \rhd \langle \emptyset; \emptyset \rangle\big)}\ \text{echk-lam}$$

$\Gamma, x : A \vdash^{M} e_1' : E$ and $|e_1| = e_1'$ — By inductive hypothesis
$\Gamma \vdash^{M} (\lambda x.\, e_1') : \big((A \to E) \rhd \langle \emptyset; \emptyset \rangle\big)$ — By rule lam
$|(\lambda x.\, e_1)| = (\lambda x.\, |e_1|) = (\lambda x.\, e_1')$ — By definition of $|-|$ and $|e_1| = e_1'$
☞ $\Gamma \vdash^{M} e' : \big((A \to E) \rhd \langle \emptyset; \emptyset \rangle\big)$ and $|e| = e'$ — where $e' = (\lambda x.\, e_1')$ and $e = (\lambda x.\, e_1)$

- **Case**
$$\frac{\Gamma \vdash v \Leftarrow \mathsf{Nm}[X] \qquad \Gamma \vdash^{M} e_1 \Leftarrow E_1}{\Gamma \vdash^{M} \mathtt{thunk}(v, e_1) \Leftarrow \big(\mathbf{F}\,(\mathsf{Thk}[M(X)]\ E_1)\big) \rhd \langle M(X); \emptyset \rangle}\ \text{echk-thunk}$$

Let $E = \big(\mathbf{F}\,(\mathsf{Thk}[M(X)]\ E_1)\big) \rhd \langle M(X); \emptyset \rangle$ — Assumption
$\exists\, v'$ such that $\Gamma \vdash v' : \mathsf{Nm}[X]$ and $|v| = v'$ — By Thm. E.1
$\exists\, e_1'$ such that $\Gamma \vdash^{M} e_1' : E$ and $|e_1| = e_1'$ — By inductive hypothesis
$\Gamma \vdash^{M} \mathtt{thunk}(v', e_1') : E$ — By rule thunk
$|\mathtt{thunk}(v, e_1)| = \mathtt{thunk}(|v|, |e_1|) = \mathtt{thunk}(v', e_1')$ — By definition of $|-|$ and $|v| = v', |e_1| = e_1'$
☞ $\Gamma \vdash^{M} e' : E$ and $|e| = e'$ — where $e' = \mathtt{thunk}(v', e_1')$ and $e = \mathtt{thunk}(v, e_1)$

- **Case**
$$\frac{\Gamma \vdash v_1 \Leftarrow \mathsf{Nm}[X] \qquad \Gamma \vdash v_2 \Leftarrow A}{\Gamma \vdash^{M} \mathtt{ref}(v_1, v_2) \Leftarrow \big(\mathbf{F}\,(\mathsf{Ref}[M(X)]\ A)\big) \rhd \langle M(X); \emptyset \rangle}\ \text{echk-ref}$$

$\exists\, v_1'$ such that $\Gamma \vdash v_1' : \mathsf{Nm}[X]$ and $|v_1| = v_1'$ — By Thm. E.1
$\exists\, v_2'$ such that $\Gamma \vdash v_2' : A$ and $|v_2| = v_2'$ — By Thm. E.1
$\Gamma \vdash^{M} \mathtt{ref}(v_1', v_2') : \big(\mathbf{F}\,(\mathsf{Ref}[M(X)]\ A)\big) \rhd \langle M(X); \emptyset \rangle$ — By rule ref
$|\mathtt{ref}(v_1, v_2)| = \mathtt{ref}(|v_1|, |v_2|) = \mathtt{ref}(v_1', v_2')$ — By definition of $|-|$
☞ $\Gamma \vdash^{M} e' : \big(\mathbf{F}\,(\mathsf{Ref}[M(X)]\ A)\big) \rhd \langle M(X); \emptyset \rangle$ and $|e| = e'$ where $e' = \mathtt{ref}(v_1', v_2')$ and $e = \mathtt{ref}(v_1, v_2)$

- **Case**
$$\frac{\Gamma \vdash v \Rightarrow (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})[N'] \qquad \Gamma \vdash^{N \circ N'} e_1 \Leftarrow C \rhd \langle W; R \rangle}{\Gamma \vdash^{N} \mathtt{scope}(v, e_1) \Leftarrow C \rhd \langle W; R \rangle}\ \text{echk-scope}$$

$$\begin{array}{rl}
\Gamma \vdash v' : \mathbf{Nm} \xrightarrow{\mathsf{Nm}} \mathbf{Nm} \text{ and } |\mathsf{N}'| = v' & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{N} \circ \mathsf{N}'} e_1' : \mathsf{C} \rhd \langle W; \mathsf{R} \rangle \text{ and } |e_1| = e_1' & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{N}} \mathsf{scope}(v', e_1') : \mathsf{C} \rhd \langle W; \mathsf{R} \rangle & \text{By rule scope} \\
|\mathsf{scope}(v, e_1)| = \mathsf{scope}(|v|, |e_1|) = \mathsf{scope}(v', e_1') & \text{By definition of } |-|;\ |v| = v';\ |e_1'| = e_1
\end{array}$$

☞ $\Gamma \vdash^{\mathsf{M}} e' : \mathsf{C} \rhd \langle W; \mathsf{R} \rangle$ and $|e| = e'$ where $e' = \mathsf{scope}(v', e_1')$ and $e = \mathsf{scope}(v, e_1)$

- **Case**
$$\frac{\Gamma, a : \gamma \vdash^{\mathsf{M}} t \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} t \Leftarrow (\forall a : \gamma.\, \mathsf{E})}\ \text{echk-}\forall\text{IndexIntro}$$

$$\begin{array}{rll}
\Gamma, a : \gamma \vdash^{\mathsf{M}} t' : \mathsf{E} \text{ and } |t| = t' & & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} t' : (\forall a : \gamma.\, \mathsf{E}) & & \text{By rule AllIndexIntro} \\
\text{☞} \quad \Gamma \vdash^{\mathsf{M}} e' : (\forall a : \gamma.\, \mathsf{E}) \text{ and } |e| = e' & & \text{where } e' = t' \text{ and } e = t
\end{array}$$

- **Case**
$$\frac{\Gamma, a : \gamma \vdash^{\mathsf{M}} t \Leftarrow \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} t \Leftarrow (\forall \alpha : \mathsf{K}.\, \mathsf{E})}\ \text{echk-}\forall\text{Intro}$$

$$\begin{array}{rll}
\exists\, t' \text{ such that } \Gamma, a : \gamma \vdash^{\mathsf{M}} t' : \mathsf{E} \text{ and } |t| = t' & & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} t' : (\forall \alpha : \mathsf{K}.\, \mathsf{E}) & & \text{By rule AllIntro} \\
\text{☞} \quad \Gamma \vdash^{\mathsf{M}} e' : (\forall \alpha : \mathsf{K}.\, \mathsf{E}) \text{ and } |e| = e' & & \text{where } e' = t' \text{ and } e = t
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash^{\mathsf{M}} e \Rightarrow \mathsf{E}_1 \qquad \mathsf{E}_1 = \mathsf{E}_2}{\Gamma \vdash^{\mathsf{M}} e \Leftarrow \mathsf{E}_2}\ \text{echk-sub}$$

By i.h. and etype-sub. □

Theorem E.4 (Completeness of Bidirectional Computation Typing).
If $\Gamma \vdash^{\mathsf{M}} e : \mathsf{E}$, *then there exist computations* e', e'' *such that*

(1) $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ *and* $|e'| = e$
(2) $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ *and* $|e''| = e$

Proof. By induction on the derivation of $\Gamma \vdash^{\mathsf{M}} e : \mathsf{E}$.

- **Case** etype-sub: By i.h. and echk-sub.
- **Case**
$$\frac{\begin{array}{l}\Gamma \vdash v : (A_1 \times A_2) \\ \Gamma, x_1 : A_1, x_2 : A_2 \vdash^{\mathsf{M}} e_1 : \mathsf{E}\end{array}}{\Gamma \vdash^{\mathsf{M}} \mathsf{split}(v, x_1.x_2.e_1) : \mathsf{E}}\ \text{split}$$

$$\begin{array}{rl}
\Gamma, x_1 : A_1, x_2 : A_2 \vdash e_1' \Leftarrow \mathsf{E} \text{ and } e_1 = |e_1'| & \text{By inductive hypothesis} \\
\Gamma \vdash v' \Rightarrow (A_1 \times A_2) \text{ and } v_1 = |v_1'| & \text{By Thm. E.2} \\
\Gamma \vdash^{\mathsf{M}} \mathsf{split}(v', x_1.x_2.e_1') \Leftarrow \mathsf{E} & \text{By chk-split} \\
\Gamma \vdash^{\mathsf{M}} (\mathsf{split}(v', x_1.x_2.e_1') : \mathsf{E}) \Rightarrow \mathsf{E} & \text{By syn-anno} \\
|(\mathsf{split}(v', x_1.x_2.e_1') : \mathsf{E})| = |\mathsf{split}(v', x_1.x_2.e_1')| & \text{By definition of } |-| \\
|\mathsf{split}(v', x_1.x_2.e_1')| = \mathsf{split}(|v'|, x_1.x_2.|e_1'|) & \text{By definition of } |-| \\
\mathsf{split}(|v'|, x_1.x_2.|e_1'|) = \mathsf{split}(v, x_1.x_2.e_1) & \text{Since } |v'| = v,\ |e_1'| = e_1
\end{array}$$

☞ $\Gamma \vdash e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = \mathsf{split}(v', x_1.x_2.e_1')$ and $e = \mathsf{split}(v, x_1.x_2.e_1)$

☞ $\Gamma \vdash e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = (\mathsf{split}(v', x_1.x_2.e_1') : \mathsf{E})$ and $e = \mathsf{split}(v, x_1.x_2.e_1)$

- **Case**

$$\frac{\Gamma \vdash v : (A_1 + A_2) \qquad \begin{array}{c}\Gamma, x_1 : A_1 \vdash^{\mathsf{M}} e_1 : \mathsf{E} \\ \Gamma, x_2 : A_2 \vdash^{\mathsf{M}} e_2 : \mathsf{E}\end{array}}{\Gamma \vdash^{\mathsf{M}} \mathsf{case}(v, x_1.e_1, x_2.e_2) : \mathsf{E}}\ \text{case}$$

$\Gamma \vdash v' \Rightarrow (A_1 + A_2)$ and $|v'| = v$ — By Thm. E.2
$\Gamma, x_1 : A_1 \vdash^{\mathsf{M}} e_1'' \Leftarrow \mathsf{E}$ and $|e_1''| = e_1$ — By inductive hypothesis
$\Gamma, x_2 : A_2 \vdash^{\mathsf{M}} e_2'' \Leftarrow \mathsf{E}$ and $|e_2''| = |e_2|$ — By inductive hypothesis
$\Gamma \vdash^{\mathsf{M}} \mathsf{case}(v', x_1.e_1'', x_2.e_2'') \Leftarrow \mathsf{E}$ — By rule chk-case
$\Gamma \vdash^{\mathsf{M}} (\mathsf{case}(v', x_1.e_1'', x_2.e_2'') : \mathsf{E}) \Rightarrow \mathsf{E}$ — By rule chk-conv

$|(\mathsf{case}(v', x_1.e_1'', x_2.e_2'') : \mathsf{E})| = |\mathsf{case}(v', x_1.e_1'', x_2.e_2'')|$ — By definition of $|-|$
$|\mathsf{case}(v', x_1.e_1'', x_2.e_2'')| = \mathsf{case}(|v'|, x_1.|e_1''|, x_2.|e_2''|)$ — By definition of $|-|$
$\mathsf{case}(|v'|, x_1.|e_1''|, x_2.|e_2''|) = \mathsf{case}(v, x_1.e_1, x_2.e_2)$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = (\mathsf{case}(v', x_1.e_1'', x_2.e_2'') : \mathsf{E})$ and $e = \mathsf{case}(v, x_1.e_1, x_2.e_2)$

☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = \mathsf{case}(v', x_1.e_1'', x_2.e_2'')$ and $e = \mathsf{case}(v, x_1.e_1, x_2.e_2)$

- **Case**

$$\frac{\Gamma \vdash v : A}{\Gamma \vdash^{\mathsf{M}} \mathsf{ret}(v) : \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle\big)}\ \text{ret}$$

Let $\mathsf{E} = \big((\mathbf{F}\, A) \rhd \langle \emptyset; \emptyset \rangle\big)$ — Assumption
$\exists\, v''$ such that $\Gamma \vdash v'' \Leftarrow A$ and $|v''| = v$ — By Thm. E.2
$\Gamma \vdash^{\mathsf{M}} \mathsf{ret}(v'') \Leftarrow \mathsf{E}$ — By rule chk-ret
$\Gamma \vdash^{\mathsf{M}} (\mathsf{ret}(v'') : \mathsf{E}) \Rightarrow \mathsf{E}$ — By syn-anno
$|(\mathsf{ret}(v'') : \mathsf{E})| = |\mathsf{ret}(v'')|$ — By definition of $|-|$
$|\mathsf{ret}(v'')| = \mathsf{ret}(|v''|) = \mathsf{ret}(v)$ — By definition of $|-|$ and $|v''| = v$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = (\mathsf{ret}(v'') : \mathsf{E})$ and $e = \mathsf{ret}(v)$

☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = \mathsf{ret}(v'')$ and $e = \mathsf{ret}(v)$

- **Case**

$$\frac{\Gamma \vdash^{\mathsf{M}} e_1 : (\mathbf{F}\, A) \rhd \epsilon_1 \qquad \Gamma, x : A \vdash^{\mathsf{M}} e_2 : (C \rhd \epsilon_2)}{\Gamma \vdash^{\mathsf{M}} \mathsf{let}(e_1, x.e_2) : \big(C \rhd (\epsilon_1 \,\mathsf{then}\, \epsilon_2)\big)}\ \text{let}$$

$$\begin{array}{rll}
\text{Let } \mathsf{E} = \big(\mathsf{C} \rhd (\epsilon_1 \text{ then } \epsilon_2)\big) & & \text{Assumption} \\
\Gamma \vdash^{\mathsf{M}} e_1' \Rightarrow (\mathbf{F}\, \mathsf{A}) \rhd \epsilon_1 \text{ and } |e_1'| = e_1 & & \text{By inductive hypothesis} \\
\Gamma, x : \mathsf{A} \vdash^{\mathsf{M}} e_2'' \Leftarrow (\mathsf{C} \rhd \epsilon_2) \text{ and } |e_2''| = e_2 & & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} \mathtt{let}\,(e_1', x.e_2'') \Leftarrow \mathsf{E} & & \text{By rule chk-let} \\
\Gamma \vdash^{\mathsf{M}} (\mathtt{let}\,(e_1', x.e_2'') : \mathsf{E}) \Rightarrow \mathsf{E} & & \text{By rule chk-conv} \\
|(\mathtt{let}\,(e_1', x.e_2'') : \mathsf{E})| = |\mathtt{let}\,(e_1', x.e_2'')| & & \text{By definition of } |-| \\
|\mathtt{let}\,(e_1', x.e_2'')| = \mathtt{let}\,(|e_1'|, x.|e_2''|) = \mathtt{let}\,(e_1, x.e_2) & & \text{By definition of } |-| \\
 & & \text{and } |e_1'| = e_1, |e_2''| = e_2
\end{array}$$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = (\mathtt{let}\,(e_1', x.e_2'') : \mathsf{E})$ and $e = \mathtt{let}\,(e_1, x.e_2)$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = \mathtt{let}\,(e_1', x.e_2'')$ and $e = \mathtt{let}\,(e_1, x.e_2)$

- **Case**

$$\frac{\Gamma, x : \mathsf{A} \vdash^{\mathsf{M}} e_1 : \mathsf{E}_1}{\Gamma \vdash^{\mathsf{M}} (\lambda x.\, e_1) : \big((\mathsf{A} \to \mathsf{E}_1) \rhd \langle \emptyset; \emptyset \rangle\big)}\ \text{lam}$$

$$\begin{array}{rll}
\text{Let } \mathsf{E} = \big((\mathsf{A} \to \mathsf{E}_1) \rhd \langle \emptyset; \emptyset \rangle\big) & & \text{Assumption} \\
\exists\, e_1'' \text{ such that } \Gamma, x : \mathsf{A} \vdash^{\mathsf{M}} e_1'' \Leftarrow \mathsf{E}_1 \text{ and } |e_1''| = e_1 & & \text{By inductive hypothesis} \\
\Gamma \vdash^{\mathsf{M}} (\lambda x.\, e_1'') \Leftarrow \mathsf{E} & & \text{By rule chk-lam} \\
\Gamma \vdash^{\mathsf{M}} ((\lambda x.\, e_1'') : \mathsf{E}) \Rightarrow \mathsf{E} & & \text{By syn-anno} \\
|(\lambda x.\, e_1'')| = (\lambda x.\, |e_1''|) = (\lambda x.\, e_1) & & \text{By definition of } |-| \\
 & & \text{and } |e_1''| = e_1
\end{array}$$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = ((\lambda x.\, e_1'') : \mathsf{E})$ and $e = (\lambda x.\, e_1)$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = (\lambda x.\, e_1'')$ and $e = (\lambda x.\, e_1)$

- **Case**

$$\frac{\Gamma \vdash^{\mathsf{M}} e_1 : \big((\mathsf{A} \to \mathsf{E}) \rhd \epsilon_1\big) \qquad \Gamma \vdash v : \mathsf{A}}{\Gamma \vdash^{\mathsf{M}} (e_1\ v) : (\mathsf{E}\ \mathsf{after}\ \epsilon_1)}\ \text{app}$$

$$\begin{array}{rll}
\exists\, e_1' \text{ such that } \Gamma \vdash^{\mathsf{M}} e_1' \Rightarrow \big((\mathsf{A} \to \mathsf{E}) \rhd \epsilon_1\big) \text{ and } |e_1'| = e_1 & & \text{By inductive hypothesis} \\
\exists\, v'' \text{ such that } \Gamma \vdash v'' \Leftarrow \mathsf{A} \text{ and } |v''| = v & & \text{By Thm. E.2} \\
\Gamma \vdash^{\mathsf{M}} (e_1'\ v'') \Rightarrow (\mathsf{E}\ \mathsf{after}\ \epsilon_1) & & \text{By rule syn-app} \\
\Gamma \vdash^{\mathsf{M}} (e_1'\ v'') \Leftarrow (\mathsf{E}\ \mathsf{after}\ \epsilon_1) & & \text{By rule chk-conv} \\
|(e_1'\ v'')| = (|e_1'|\ |v''|) = (e_1\ v) & & \text{By the definition of } |-| \\
 & & \text{and } |e_1'| = e_1, |v''| = v
\end{array}$$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = (e_1'\ v'')$ and $e = (e_1\ v)$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = (e_1'\ v'')$ and $e = (e_1\ v)$

- **Case**

$$\frac{\Gamma \vdash v : \mathsf{Nm}[X] \qquad \Gamma \vdash^{\mathsf{M}} e_1 : \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} \mathtt{thunk}(v, e_1) : \big(\mathbf{F}\,(\mathsf{Thk}[\mathsf{M}(X)]\ \mathsf{E})\big) \rhd \langle \mathsf{M}(X); \emptyset \rangle}\ \text{thunk}$$

$$\begin{array}{rll}
\text{Let } E = \big(\mathbf{F}\,(\mathsf{Thk}\,[M(X)]\;E)\big) \rhd \langle M(X);\emptyset\rangle & & \text{Assumption}\\
\exists\, v'' \text{ such that } \Gamma \vdash v'' \Leftarrow \mathsf{Nm}\,[X] \text{ and } |v''| = v & & \text{By Thm. E.2}\\
\exists\, e_1'' \text{ such that } \Gamma \vdash^{M} e_1'' \Leftarrow E \text{ and } |e_1''| = e_1 & & \text{By inductive hypothesis}\\
\Gamma \vdash^{M} \mathtt{thunk}(v'', e_1'') \Leftarrow E & & \text{By rule chk-thunk}\\
\Gamma \vdash^{M} (\mathtt{thunk}(v'', e_1'') : E) \Rightarrow E & & \text{By rule syn-anno}\\
|(\mathtt{thunk}(v'', e_1'') : E)| = |\mathtt{thunk}(v'', e_1'')| & & \text{By definition of } |-|\\
|\mathtt{thunk}(v'', e_1'')| = \mathtt{thunk}(|v''|, |e_1''|) = \mathtt{thunk}(v, e_1) & & \text{By definition of } |-|\\
 & & \quad \text{and } |v''| = v,\ |e_1''| = e_1\\
\text{☞ } \Gamma \vdash^{M} e' \Rightarrow E \text{ and } |e'| = e & \text{where } e' = (\mathtt{thunk}(v'', e_1'') : E) \text{ and } e = \mathtt{thunk}(v, e_1) & \\
\text{☞ } \Gamma \vdash^{M} e'' \Leftarrow E \text{ and } |e''| = e & \text{where } e'' = \mathtt{thunk}(v'', e_1'') \text{ and } e = \mathtt{thunk}(v, e_1) &
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash v : \mathsf{Thk}\,[X]\;(C \rhd \epsilon)}{\Gamma \vdash^{M} \mathtt{force}(v) : \big(C \rhd (\langle\emptyset; X\rangle \text{ then } \epsilon)\big)}\ \text{force}$$

$$\begin{array}{rll}
\text{Let } E = \big(C \rhd (\langle\emptyset; X\rangle \text{ then } \epsilon)\big) & & \text{Assumption}\\
\Gamma \vdash v' \Rightarrow \mathsf{Thk}\,[X]\;(C \rhd \epsilon) \text{ and } |v'| = v & & \text{By Thm. E.2}\\
\Gamma \vdash^{M} \mathtt{force}(v') \Rightarrow E & & \text{By rule syn-force}\\
\Gamma \vdash^{M} \mathtt{force}(v') \Leftarrow E & & \text{By chk-conv}\\
|\mathtt{force}(v')| = \mathtt{force}(|v'|) = \mathtt{force}(v) & & \text{By definition of } |-|\\
 & & \quad \text{and } |v'| = v\\
\text{☞ } \Gamma \vdash^{M} e' \Rightarrow E \text{ and } |e'| = e & \text{where } e' = \mathtt{force}(v') \text{ and } e = \mathtt{force}(v) & \\
\text{☞ } \Gamma \vdash^{M} e'' \Leftarrow E \text{ and } |e''| = e & \text{where } e'' = \mathtt{force}(v') \text{ and } e = \mathtt{force}(v) &
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash v_1 : \mathsf{Nm}\,[X] \qquad \Gamma \vdash v_2 : A}{\Gamma \vdash^{M} \mathtt{ref}(v_1, v_2) : \big(\mathbf{F}\,(\mathsf{Ref}\,[M(X)]\;A)\big) \rhd \langle M(X);\emptyset\rangle}\ \text{ref}$$

$$\begin{array}{rll}
\text{Let } E = \big(\mathbf{F}\,(\mathsf{Ref}\,[M(X)]\;A)\big) \rhd \langle M(X);\emptyset\rangle & & \text{Assumption}\\
\Gamma \vdash v_1'' \Leftarrow \mathsf{Nm}\,[X] \text{ and } |v_1''| = v_1 & & \text{By Thm. E.2}\\
\Gamma \vdash v_2'' \Leftarrow A \text{ and } |v_2''| = v_2 & & \text{By Thm. E.2}\\
\Gamma \vdash^{M} \mathtt{ref}(v_1'', v_2'') \Leftarrow E & & \text{By rule chk-ref}\\
\Gamma \vdash^{M} (\mathtt{ref}(v_1'', v_2'') : E) \Rightarrow E & & \text{By rule syn-anno}\\
|(\mathtt{ref}(v_1'', v_2'') : E)| = |\mathtt{ref}(v_1'', v_2'')| & & \text{By definition of } |-|\\
|\mathtt{ref}(v_1'', v_2'')| = \mathtt{ref}(|v_1''|, |v_2''|) = \mathtt{ref}(v_1, v_2) & & \text{By definition of } |-|\\
 & & \quad \text{and } |v_1''| = v_1,\ |v_2''| = v_2\\
\text{☞ } \Gamma \vdash^{M} e' \Rightarrow E \text{ and } |e'| = e & \text{where } e' = (\mathtt{ref}(v_1'', v_2'') : E) \text{ and } e = \mathtt{ref}(v_1, v_2) & \\
\text{☞ } \Gamma \vdash^{M} e'' \Leftarrow E \text{ and } |e''| = e & \text{where } e'' = \mathtt{ref}(v_1'', v_2'') \text{ and } e = \mathtt{ref}(v_1, v_2) &
\end{array}$$

- **Case**
$$\frac{\Gamma \vdash v : \mathsf{Ref}\,[X]\;A}{\Gamma \vdash^{M} \mathtt{get}(v) : \big(\mathbf{F}\,A\big) \rhd \langle\emptyset; X\rangle}\ \text{get}$$

$$\begin{array}{rll}
\Gamma \vdash v' \Rightarrow \mathsf{Ref}\,[X]\;A \text{ and } |v'| = v & & \text{By Thm. E.2}\\
\Gamma \vdash^{M} \mathtt{get}(v') \Rightarrow \big(\mathbf{F}\,A\big) \rhd \langle\emptyset; X\rangle & & \text{By rule syn-get}\\
\Gamma \vdash^{M} \mathtt{get}(v') \Leftarrow \big(\mathbf{F}\,A\big) \rhd \langle\emptyset; X\rangle & & \text{By rule chk-conv}\\
|\mathtt{get}(v')| = \mathtt{get}(|v'|) = \mathtt{get}(v) & & \text{By definition of } |-|\\
 & & \quad \text{and } |v'| = v
\end{array}$$

☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow (\mathbf{F}\, A) \rhd \langle \emptyset; X \rangle$ and $|e'| = e$ where $e' = \mathsf{get}(v')$ and $e = \mathsf{get}(v)$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow (\mathbf{F}\, A) \rhd \langle \emptyset; X \rangle$ and $|e''| = e$ where $e'' = \mathsf{get}(v')$ and $e = \mathsf{get}(v)$

- **Case**

$$\frac{\begin{array}{l}\Gamma \vdash v_{\mathsf{M}} : (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})\,[\mathsf{M}] \\ \Gamma \vdash v : \mathsf{Nm}\,[\mathsf{i}]\end{array}}{\Gamma \vdash^{\mathsf{N}} (v_{\mathsf{M}}\ v) : \mathbf{F}\,(\mathsf{Nm}\,[\mathsf{M}(\mathsf{i})]) \rhd \langle \emptyset; \emptyset \rangle}\ \text{name-app}$$

$\Gamma \vdash v'_{\mathsf{M}} \Rightarrow (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})\,[\mathsf{M}]$ and $|v'_{\mathsf{M}}| = v_{\mathsf{M}}$ By Thm. E.2
$\Gamma \vdash v' \Rightarrow \mathsf{Nm}\,[\mathsf{i}]$ and $|v'| = v$ By Thm. E.2
$\Gamma \vdash^{\mathsf{N}} (v'_{\mathsf{M}}\ v') \Rightarrow \mathbf{F}\,(\mathsf{Nm}\,[\mathsf{M}(\mathsf{i})]) \rhd \langle \emptyset; \emptyset \rangle$ By rule syn-name-app
$\Gamma \vdash^{\mathsf{N}} (v'_{\mathsf{M}}\ v') \Leftarrow \mathbf{F}\,(\mathsf{Nm}\,[\mathsf{M}(\mathsf{i})]) \rhd \langle \emptyset; \emptyset \rangle$ By rule chk-conv
$|(v'_{\mathsf{M}}\ v')| = (|v'_{\mathsf{M}}|\ |v'|) = (v_{\mathsf{M}}\ v)$ By definition of $|{-}|$ and $|v'_{\mathsf{M}}| = v_{\mathsf{M}}, |v'| = v$
☞ $\Gamma \vdash^{\mathsf{N}} e' \Rightarrow \mathbf{F}\,(\mathsf{Nm}\,[\mathsf{M}(\mathsf{i})]) \rhd \langle \emptyset; \emptyset \rangle$ and $|e'| = e$ where $e' = (v'_{\mathsf{M}}\ v')$ and $e = (v_{\mathsf{M}}\ v)$
☞ $\Gamma \vdash^{\mathsf{N}} e'' \Leftarrow \mathbf{F}\,(\mathsf{Nm}\,[\mathsf{M}(\mathsf{i})]) \rhd \langle \emptyset; \emptyset \rangle$ and $|e''| = e$ where $e'' = (v'_{\mathsf{M}}\ v')$ and $e = (v_{\mathsf{M}}\ v)$

- **Case**

$$\frac{\Gamma \vdash v : (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})\,[\mathsf{N}'] \qquad \Gamma \vdash^{\mathsf{N} \circ \mathsf{N}'} e_1 : \mathsf{C} \rhd \langle W; \mathsf{R} \rangle}{\Gamma \vdash^{\mathsf{N}} \mathsf{scope}(v, e_1) : \mathsf{C} \rhd \langle W; \mathsf{R} \rangle}\ \text{scope}$$

Let $\mathsf{E} = \mathsf{C} \rhd \langle W; \mathsf{R} \rangle$ Assumption
$\exists v''$ such that $, \Gamma \vdash v'' \Rightarrow (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm})\,[\mathsf{N}']$ and $|v''| = v$ By Thm. E.2
$\exists e'_1$ such that $, \Gamma \vdash^{\mathsf{N} \circ \mathsf{N}'} e'_1 \Leftarrow \mathsf{E}$ and $|e'_1| = e_1$ By inductive hypothesis
$\Gamma \vdash^{\mathsf{N}} \mathsf{scope}(v'', e'_1) \Leftarrow \mathsf{E}$ By rule chk-scope
$\Gamma \vdash^{\mathsf{N}} (\mathsf{scope}(v'', e'_1) : \mathsf{E}) \Rightarrow \mathsf{E}$ By rule syn-anno

$|(\mathsf{scope}(v'', e'_1) : \mathsf{E})| = |(\mathsf{scope}(v'', e'_1)|$ By definition of $|{-}|$
$|(\mathsf{scope}(v'', e'_1)| = \mathsf{scope}(|v''|, |e'_1|) = \mathsf{scope}(v, e_1)$ By definition of $|{-}|$
☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow \mathsf{E}$ and $|e'| = e$ where $e' = (\mathsf{scope}(v'', e'_1) : \mathsf{E})$ and $e = \mathsf{scope}(v, e_1)$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow \mathsf{E}$ and $|e''| = e$ where $e'' = \mathsf{scope}(v'', e'_1)$ and $e = \mathsf{scope}(v, e_1)$

- **Case**

$$\frac{\Gamma, a : \gamma \vdash^{\mathsf{M}} t : \mathsf{E}}{\Gamma \vdash^{\mathsf{M}} t : (\forall a : \gamma.\, \mathsf{E})}\ \text{etype-}\forall\text{IndexIntro}$$

$\exists\, t''$ such that $\Gamma, a : \gamma \vdash^{\mathsf{M}} t'' \Leftarrow \mathsf{E}$ and $|t''| = t$ By inductive hypothesis
$\Gamma \vdash^{\mathsf{M}} t'' \Leftarrow (\forall a : \gamma.\, \mathsf{E})$ By rule chk-AllIndexIntro
$\Gamma \vdash^{\mathsf{M}} (t'' : (\forall a : \gamma.\, \mathsf{E})) \Rightarrow (\forall a : \gamma.\, \mathsf{E})$ By rule syn-anno
$|(t'' : (\forall a : \gamma.\, \mathsf{E}))| = |t''| = t$ By definition of $|{-}|$ and $|t''| = t$
☞ $\Gamma \vdash^{\mathsf{M}} e' \Rightarrow (\forall a : \gamma.\, \mathsf{E})$ and $|e'| = e$ where $e' = (t'' : (\forall a : \gamma.\, \mathsf{E}))$ and $e = t$
☞ $\Gamma \vdash^{\mathsf{M}} e'' \Leftarrow (\forall a : \gamma.\, \mathsf{E})$ and $|e''| = e$ where $e'' = t''$ and $e = t$

- **Case**

$$\frac{\Gamma \vdash^{\mathsf{M}} e : (\forall a : \gamma.\, \mathsf{E}) \qquad \Gamma \vdash i : \gamma}{\Gamma \vdash^{\mathsf{M}} e : [i/a]\mathsf{E}}\ \text{etype-}\forall\text{IndexElim}$$

$\exists\, e'$ such that $\Gamma \vdash^{M} e' \Rightarrow (\forall a : \gamma.\, E)$ and $|e'| = e$ — By inductive hypothesis
$\Gamma \vdash i : \gamma$ — Given
$\Gamma \vdash^{M} e' \Rightarrow [i/a]E$ — By rule syn-AllIndexElim
$\Gamma \vdash^{M} e' \Leftarrow [i/a]E$ — By rule chk-conv
☞ $\Gamma \vdash^{M} e'[i] \Rightarrow [i/a]E$ and $|e'[i]| = e$
☞ $\Gamma \vdash^{M} e'[i] \Leftarrow [i/a]E$ and $|e'[i]| = e$

- **Case** $\dfrac{\Gamma, a : \gamma \vdash^{M} t : E}{\Gamma \vdash^{M} t : (\forall \alpha : K.\, E)}$ etype-∀Intro

  Similar to the AllIndexIntro case.

- **Case** $\dfrac{\Gamma \vdash^{M} e : (\forall \alpha : K.\, E) \qquad \Gamma \vdash A : K}{\Gamma \vdash^{M} e : [A/\alpha]E}$ etype-∀Elim

  Similar to the AllIndexElim case.

- **Case** etype-∃IndexElim: By i.h. and echk-∃IndexElim. □

F NAME TERM LANGUAGE

We define a restricted *name term* language for computing larger names from smaller names. This language consists of the following:

- Syntax for *names*, *name terms* and *sorts*.
- Name term sorting: A judgment that assigns sorts to name terms.
- Big-step evaluation for name terms: A judgment that assigns *name term values* to *name terms*.
- Semantic definition of equivalent and disjoint name terms.
- Logical proof rules for equivalent and disjoint name terms: Two judgements that should be sound with respect to the semantic definitions of equivalence and disjointness.

F.1 Name term equivalence and apartness

For instance, the following two functions have the same sort, and are apart:

$$\lambda a.\,(a, a) \quad : \mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm} * \mathbf{Nm}$$
$$\lambda b.\,(b, \langle\!\langle \mathtt{leaf}, b \rangle\!\rangle) \quad : \mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm} * \mathbf{Nm}$$

These functions have a common sort $\mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm} * \mathbf{Nm}$, which says that they map a single name (of sort $\mathbf{Nm}$) to a pair of names (of sort $\mathbf{Nm} * \mathbf{Nm}$). These functions are apart, and this apartness ultimately follows from reasoning about binary composition of names: any name n is distinct from the binary composition of the name constant $\mathtt{leaf}$ with name n.

These two name functions also have the same sort, but are *not* apart:

$$\lambda a.\,(a, \langle\!\langle a, \mathtt{leaf} \rangle\!\rangle) : \mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm} * \mathbf{Nm}$$
$$\lambda b.\,(a, \langle\!\langle \mathtt{leaf}, b \rangle\!\rangle) : \mathbf{Nm} \overset{\mathbf{Nm}}{\to} \mathbf{Nm} * \mathbf{Nm}$$

In particular, when $b = a = \mathtt{leaf}$, they will produce the same pair, namely $(\mathtt{leaf}, \langle\!\langle \mathtt{leaf}, \mathtt{leaf} \rangle\!\rangle)$.

Notice that these two functions are obviously not equivalent either: Given any name n such that $n \neq \mathtt{leaf}$, these two functions will produce two distinct names. This follows from another reasoning principle for name composition: binary composition with $\mathtt{leaf}$ in the first function is in a distinct order from that in the second function, precisely when $n \neq \mathtt{leaf}$.

Finally, these two functions are equivalent: Given equivalent arguments, they will always produce equivalent results:

$$\begin{aligned} \lambda a.\, \lambda b.\, a\ b &: (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm}) \overset{\mathsf{Nm}}{\to} \mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm} \\ \lambda a_1.\, \lambda b_1.\, (\lambda a_2.\, \lambda b_2.\, a_2\ b_2)\ a_1\ b_1 &: (\mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm}) \overset{\mathsf{Nm}}{\to} \mathbf{Nm} \overset{\mathsf{Nm}}{\to} \mathbf{Nm} \end{aligned}$$

To see this equivalence more clearly, imagine reducing the inner β-redexes by substituting a_1 and b_1 for a_2 and b_2, respectively. After this reduction, the two functions are α-equivalent.

Below, we give a semantic definition of both equivalence and disjointness, in terms of the sorting and operational definitions given above.

These definitions are clear, but not immediately practical: for function sorts, they universally quantify over all possible arguments for the function's argument. Since these arguments can include names, which consist of arbitrary finite binary trees, as well as other functions, there is not an obvious finite set of arguments to test while still being sound with respect to these definitions. For a practical implementation of Typed Adapton, we seek definitions of equivalence and disjointness that admit decision procedures, and are sound (and, ideally, complete) with respect to these semantic definitions. For this purpose, we give decidable logical rules that induct over the syntax of the two terms.

F.2 Semantic equivalence and disjointness

Below, we define semantic equivalence and disjointness of (sorted) name terms. We define these semantic properties inductively, based on the common sort of the name terms. In this sense, these definitions can be viewed as instances of logical relations.

We define contexts Γ that relate two variables; each declaration either asserts that a and b are equivalent, or disjoint. We write $\Gamma.1$ and $\Gamma.2$ for the projection of a relational Γ into an ordinary Γ suitable for the left-hand ($\Gamma.1$) or right-hand ($\Gamma.2$) sides. Also, we write $\mathit{flip}(\Gamma)$ for the operation of exchanging a and b in each declaration: $\mathit{flip}((a \perp b : \gamma)) = (b \perp a : \gamma)$, so that $\mathit{flip}(\Gamma).1 = \Gamma.2$ and $\mathit{flip}(\Gamma).2 = \Gamma.1$.

Substitutions	σ	$::=$	$\cdot \mid \sigma, N/a$
Relational sorting contexts	Γ	$::=$	$\cdot$
(Hypothetical variable equivalence)		$\mid$	$\Gamma, (a \equiv b : \gamma)$
(Hypothetical variable apartness)		$\mid$	$\Gamma, (a \perp b : \gamma)$

$$\begin{aligned} (\cdot).1 &= \cdot & \qquad (\cdot).2 &= \cdot \\ (\Gamma, a \equiv b : \gamma).1 &= (\Gamma).1, a : \gamma & (\Gamma, a \equiv b : \gamma).2 &= (\Gamma).2, b : \gamma \\ (\Gamma, a \perp b : \gamma).1 &= (\Gamma).1, a : \gamma & (\Gamma, a \perp b : \gamma).2 &= (\Gamma).2, b : \gamma \end{aligned}$$

Definition F.1 (Closing substitutions)**.**
We define closing substitution pairs related by equivalence and disjointness assumptions in a context Γ*. These definitions use and are used by the definitions below for equivalence and apartness of open terms.*

- $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *means that* $(x \equiv y : \gamma) \in \Gamma$ *implies* $\big(\sigma_1(x) = N$ *and* $\sigma_2(y) = M$ *and* $\cdot \Vdash N \equiv M : \gamma\big)$
- $\Vdash \sigma_1 \perp \sigma_2 : \Gamma$ *means that* $(x \perp y : \gamma) \in \Gamma$ *implies* $\big(\sigma_1(x) = N$ *and* $\sigma_2(y) = M$ *and* $\cdot \Vdash N \perp M : \gamma\big)$
- $\Vdash \sigma_1 \sim \sigma_2 : \Gamma$ *means that* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $\Vdash \sigma_1 \perp \sigma_2 : \Gamma$

Definition F.2 (Semantic equivalence)**.** *We define* $\Gamma \Vdash M_1 \equiv M_2 : \gamma$ *as follows:*
$(\Gamma).1 \vdash M_1 : \gamma$ *and* $(\Gamma).2 \vdash M_2 : \gamma$ *and,*

for all σ_1, σ_2 *such that* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $[\sigma_1]M_1 \Downarrow V_1$ *and* $[\sigma_2]M_2 \Downarrow V_2$,
we have the following about V_1 *and* V_2*:*

Sort (γ)	*Values* V_1 *and* V_2 *of sort* γ *are equivalent, written* $\Vdash V_1 \equiv V_2 : \gamma$
1	*Always*
Nm	*When* $V_1 = n_1$ *and* $V_2 = n_2$ *and* $n_1 = n_2$ *(identical binary trees)*
$\gamma_1 * \gamma_2$	*When* $V_1 = (V_{11}, V_{12})$ *and* $V_1 = (V_{21}, V_{22})$ *and* $\Vdash V_{11} \equiv V_{21} : \gamma_1$ *and* $\Vdash V_{21} \equiv V_{22} : \gamma_2$
$\gamma_1 \xrightarrow{\text{Nm}} \gamma_2$	*When* $V_1 = \lambda a_1.\, M_1$ *and* $V_2 = \lambda a_2.\, M_2$, *and for all name terms* $\Vdash N_1 \equiv N_2 : \gamma_1$, $[N_1/a_1]M_1 \Downarrow W_1$ *and* $[N_2/a_2]M_2 \Downarrow W_2$ *implies* $\Vdash W_1 \equiv W_2 : \gamma_2$

Definition F.3 (Semantic apartness)**.** *We define* $\Gamma \Vdash M_1 \perp M_2 : \gamma$ *as follows:*
$(\Gamma).1 \vdash M_1 : \gamma$ *and* $(\Gamma).2 \vdash M_2 : \gamma$ *and,*
for all σ_1, σ_2 *such that* $\Vdash \sigma_1 \sim \sigma_2 : \Gamma$ *and* $[\sigma_1]M_1 \Downarrow V_1$ *and* $[\sigma_2]M_2 \Downarrow V_2$,
we have the following about V_1 *and* V_2*:*

Sort (γ)	*Values* V_1 *and* V_2 *of sort* γ *are apart, written* $\Vdash V_1 \perp V_2 : \gamma$
1	*Always*
Nm	*When* $V_1 = n_1$ *and* $V_2 = n_2$ *and* $n_1 \neq n_2$ *(distinct binary trees)*
$\gamma_1 * \gamma_2$	*When* $V_1 = (V_{11}, V_{12})$ *and* $V_1 = (V_{21}, V_{22})$ *and* $\Vdash V_{11} \perp V_{21} : \gamma_1$ *and* $\Vdash V_{21} \perp V_{22} : \gamma_2$
$\gamma_1 \xrightarrow{\text{Nm}} \gamma_2$	*When* $V_1 = \lambda a_1.\, M_1$ *and* $V_2 = \lambda a_2.\, M_2$, *and for all name terms* $\Vdash N_1 \equiv N_2 : \gamma_1$, $[N_1/a_1]M_1 \Downarrow W_1$ *and* $[N_2/a_2]M_2 \Downarrow W_2$ *implies* $\Vdash W_1 \perp W_2 : \gamma_2$

F.3 Metatheory of name term language

Some lemmas in this section are missing complete proofs and should be considered conjectures (Lemma F.1 (Projections of syntactic equivalence)–Lemma F.8 (Reflexivity of name term evaluation)).

Lemma F.1 (Projections of syntactic equivalence).
If $\Gamma \vdash M_1 \equiv M_2 : \gamma$*, then* $\Gamma.1 \vdash M_1 : \gamma$ *and* $\Gamma.2 \vdash M_2 : \gamma$*.*

Lemma F.2 (Determinism of evaluation up to substitution).
If $\Gamma.1 \vdash M : \gamma$ *and* $\Gamma.2 \vdash M : \gamma$ *and* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $[\sigma_1]M \Downarrow V_1$ *and* $[\sigma_2]M \Downarrow V_2$
then there exists V *such that* $V_1 = [\sigma_1]V$ *and* $V_2 = [\sigma_2]V$*.*

Lemma F.3 (Reflexivity of semantic equivalence).

(1) If $\vdash M : \gamma$ *then* $\Vdash M \equiv M : \gamma$*.*
(2) If $\Gamma.1 \vdash V : \gamma$ *and* $\Gamma.2 \vdash V : \gamma$ *and* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $V_1 = [\sigma_1]V$ *and* $V_2 = [\sigma_2]V$
then $\Vdash V_1 \equiv V_2 : \gamma$*.*

Lemma F.4 (Type safety). *If* $\Gamma \vdash M : \gamma$ *and* $[\sigma]M \Downarrow [\sigma]V$ *then* $\Gamma \vdash V : \gamma$*.*

Lemma F.5 (Symmetry of semantic equivalence).

(1) If $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *then* $\Vdash \sigma_2 \equiv \sigma_1 : \mathit{flip}(\Gamma)$*.*
(2) If $\Vdash V_1 \equiv V_2 : \gamma$ *then* $\Vdash V_2 \equiv V_1 : \gamma$*.*
(3) If $\cdot \Vdash M_1 \equiv M_2 : \gamma$ *then* $\cdot \Vdash M_2 \equiv M_1 : \gamma$*.*
(4) If $\Gamma \Vdash M_1 \equiv M_2 : \gamma$ *then* $\mathit{flip}(\Gamma) \Vdash M_2 \equiv M_1 : \gamma$*.*

$\boxed{\Gamma \vdash M \equiv N : \gamma}$ The name terms M and N are *equivalent* at sort γ

$$\frac{(M \equiv N : \gamma) \in \Gamma}{\Gamma \vdash M \equiv N : \gamma}\ \text{Eq-Var} \qquad \frac{\begin{array}{c}(\Gamma).1 \vdash M : \gamma \\ (\Gamma).2 \vdash M : \gamma\end{array}}{\Gamma \vdash M \equiv M : \gamma}\ \text{E-Refl} \qquad \frac{\mathit{flip}(\Gamma) \vdash N \equiv M : \gamma}{\Gamma \vdash M \equiv N : \gamma}\ \text{E-Sym}$$

$$\frac{\Gamma \vdash M_1 \equiv M_2 : \gamma \qquad \Gamma \vdash M_2 \equiv M_3 : \gamma}{\Gamma \vdash M_1 \equiv M_3 : \gamma}\ \text{Eq-Trans}$$

$$\frac{\Gamma \vdash M_1 \equiv N_1 : \gamma_1 \qquad \Gamma \vdash M_2 \equiv N_2 : \gamma_2}{\Gamma \vdash (M_1, M_2) \equiv (N_1, N_2) : \gamma_1 * \gamma_2}\ \text{Eq-Pair}$$

$$\frac{\Gamma \vdash M_1 \equiv N_1 : \mathbf{Nm} \qquad \Gamma \vdash M_2 \equiv N_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1,\, M_2 \rangle\!\rangle \equiv \langle\!\langle N_1,\, N_2 \rangle\!\rangle : \mathbf{Nm}}\ \text{Eq-Bin}$$

$$\frac{\Gamma, (a \equiv b : \gamma_1) \vdash M \equiv N : \gamma_2}{\Gamma \vdash \lambda a.\, M \equiv \lambda b.\, N : \gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2}\ \text{Eq-Lam} \qquad \frac{\begin{array}{c}\Gamma \vdash M_1 \equiv N_1 : \gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2 \\ \Gamma \vdash M_2 \equiv N_2 : \gamma_1\end{array}}{\Gamma \vdash M_1(M_2) \equiv N_1(N_2) : \gamma_2}\ \text{Eq-App}$$

$$\frac{\Gamma \vdash M_2 \equiv M_2' : \gamma_1 \qquad \Gamma, a \equiv a : \gamma_1 \vdash M_1 \equiv M_1' : \gamma_2}{\Gamma \vdash (\lambda a.\, M_1) M_2 \equiv [M_2'/a] M_1' : \gamma_2}\ \text{Eq-}\beta$$

Fig. 26. Deductive rules for showing that two name terms are equivalent

Lemma F.6 (Evaluation respects semantic equivalence).
If $\Gamma \Vdash M \equiv N : \gamma$ *and* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $[\sigma_1]M \Downarrow V_1$ *then there exists* V_2 *such that* $[\sigma_2]N \Downarrow V_2$ *and* $\Vdash V_1 \equiv V_2 : \gamma$.

Lemma F.7 (Closing substitutions respect syntactic equivalence).
If $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $\Gamma \vdash M_1 \equiv M_2 : \gamma$ *then* $\cdot \Vdash [\sigma_1]M_1 \equiv [\sigma_2]M_2 : \gamma$.

Lemma F.8 (Reflexivity of name term evaluation).
If $\Gamma.1 \vdash M : \gamma$ *and* $\Gamma.2 \vdash M : \gamma$ *and* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $[\sigma_1]M \Downarrow V_1$ *and* $[\sigma_2]M \Downarrow V_2$ *then* $\Vdash V_1 \equiv V_2 : \gamma$.

Lemma F.9 (Transitivity of value equivalence).
If $\vdash V_1 : \gamma$ *and* $\vdash V_2 : \gamma$ *and* $\Vdash V_1 \equiv V_2 : \gamma$ *and* $\Vdash V_2 \equiv V_3 : \gamma$ *then* $\Vdash V_1 \equiv V_3 : \gamma$.

Proof. Uses strong normalization. □

Conjecture F.10 (Soundness of deductive equivalence).
If $\Gamma \vdash M_1 \equiv M_2 : \gamma$ *then* $\Gamma \Vdash M_1 \equiv M_2 : \gamma$.

Proof. By induction on the given derivation.

Case $\dfrac{(a \equiv b : \gamma) \in \Gamma}{\Gamma \vdash a \equiv b : \gamma}\ \text{Eq-Var}$

By definition of closing substitutions.

$\boxed{\Gamma \vdash M \perp N : \gamma}$ The name terms M and N are *apart* at sort γ

$$\frac{(a \perp b : \gamma) \in \Gamma}{\Gamma \vdash a \perp b : \gamma}\ \text{Var} \qquad \frac{\mathit{flip}(\Gamma) \vdash N \perp M : \gamma}{\Gamma \vdash M \perp N : \gamma}\ \text{D-Sym}$$

$$\frac{\Gamma \vdash M_1 \equiv M_2 : \gamma \qquad \Gamma \vdash M_2 \perp M_3 : \gamma}{\Gamma \vdash M_1 \perp M_3 : \gamma}\ \text{D-trans}$$

$$\frac{\Gamma \vdash M_1 \perp N_1 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle \perp \langle\!\langle N_1, N_2 \rangle\!\rangle : \mathbf{Nm}}\ \text{D-Bin}_1 \qquad \frac{\Gamma \vdash M_2 \perp N_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle \perp \langle\!\langle N_1, N_2 \rangle\!\rangle : \mathbf{Nm}}\ \text{D-Bin}_2$$

$$\frac{\Gamma \vdash M_1 \equiv M_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_2, N \rangle\!\rangle \perp M_1 : \mathbf{Nm}}\ \text{D-EqTag}_1 \qquad \frac{\Gamma \vdash N_1 \equiv N_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M, N_1 \rangle\!\rangle \perp N_2 : \mathbf{Nm}}\ \text{D-EqTag}_2$$

$$\frac{\Gamma, (a \equiv b : \gamma_1) \vdash M \perp N : \gamma_2}{\Gamma \vdash \lambda a.\, M \perp \lambda b.\, N : \gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2}\ \text{D-Lam}$$

$$\frac{\Gamma \vdash M_1 \perp N_1 : \gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2 \qquad \Gamma \vdash M_2 \equiv N_2 : \gamma_1}{\Gamma \vdash M_1(M_2) \perp N_1(N_2) : \gamma_2}\ \text{D-App}$$

$$\frac{\Gamma.1 \vdash M_2 : \gamma_2 \qquad \Gamma.1, a : \gamma_2 \vdash M_1 : \gamma \qquad \Gamma \vdash [M_2/a]M_1 \perp N : \gamma}{\Gamma \vdash (\lambda a.\, M_1)\ M_2 \perp N : \gamma}\ \text{D-}\beta$$

Fig. 27. Deductive rules for showing that two name terms are apart

Case $\dfrac{(\Gamma).1 \vdash M : \gamma \qquad (\Gamma).2 \vdash M : \gamma}{\Gamma \vdash M \equiv M : \gamma}\ \text{Eq-Refl}$

By Lemma F.2 (Determinism of evaluation up to substitution), Lemma F.4 (Type safety), and Lemma F.3 (Reflexivity of semantic equivalence).

Case $\dfrac{\Gamma \vdash N \equiv M : \gamma}{\Gamma \vdash M \equiv N : \gamma}\ \text{Eq-Sym}$

By Lemma F.5 (Symmetry of semantic equivalence).

Case $\dfrac{\Gamma \vdash M_1 \equiv M_2 : \gamma \qquad \Gamma \vdash M_2 \equiv M_3 : \gamma}{\Gamma \vdash M_1 \equiv M_3 : \gamma}\ \text{Eq-Trans}$

By idempotency of flipping relational contexts, Lemma F.6 (Evaluation respects semantic equivalence), inductive hypotheses on the two given subderivations, Lemma F.5 (Symmetry of semantic equivalence), and Lemma F.9 (Transitivity of value equivalence).

Case $\dfrac{\Gamma \vdash M_1 \equiv N_1 : \gamma_1 \qquad \Gamma \vdash M_2 \equiv N_2 : \gamma_2}{\Gamma \vdash (M_1, M_2) \equiv (N_1, N_2) : \gamma_1 * \gamma_2}\ \text{Eq-Pair}$

By the definition of substitution and the i.h.

Case $\dfrac{\Gamma \vdash M_1 \equiv N_1 : \mathbf{Nm} \qquad \Gamma \vdash M_2 \equiv N_2 : \mathbf{Nm}}{\Gamma \vdash \langle\langle M_1, M_2 \rangle\rangle \equiv \langle\langle N_1, N_2 \rangle\rangle : \mathbf{Nm}}$ Eq-Bin

By the definition of substitution and the i.h.

Case $\dfrac{\Gamma, (a \equiv b : \gamma_1) \vdash M \equiv N : \gamma_2}{\Gamma \vdash \lambda a.\, M \equiv \lambda b.\, N : \gamma_1 \overset{\mathbf{Nm}}{\longrightarrow} \gamma_2}$ Eq-Lam

By transposition of substitutions and the i.h.

Case $\dfrac{\Gamma \vdash M_1 \equiv N_1 : \gamma_1 \overset{\mathbf{Nm}}{\longrightarrow} \gamma_2 \qquad \Gamma \vdash M_2 \equiv N_2 : \gamma_1}{\Gamma \vdash M_1(M_2) \equiv N_1(N_2) : \gamma_2}$ Eq-App

By definition of substitution and inversion (teval-app) of resulting derivations, the inductive hypothesis on the two given syntactic equivalence subderivations (of Eq-App), and definition of semantic equivalence of arrow-sorted values, we get the result.

Case $\dfrac{\Gamma \vdash M_2 \equiv M_2' : \gamma_1 \qquad \Gamma, a \equiv a : \gamma_1 \vdash M_1 \equiv M_1' : \gamma_2}{\Gamma \vdash (\lambda a.\, M_1)M_2 \equiv [M_2'/a]M_1' : \gamma_2}$ Eq-β

Fix $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$. Suppose $[\sigma_1]((\lambda a.\, M_1)M_2) \Downarrow V_1$ and $[\sigma_2]([M_2'/a]M_1') \Downarrow V_2$. We need to show $\Vdash V_1 \equiv V_2 : \gamma_2$. By the definition of substitution and inversion of teval-app, $[\sigma_1]M_2 \Downarrow V$ and $[V/a]([\sigma_1]M_1) \Downarrow V_1$ for some V. Hence, because $\Gamma, a \equiv a : \gamma_1$, we have $[\sigma_1, V/a]M_1 \Downarrow V_1$. Rewrite $[\sigma_2]([M_2'/a]M_1') \Downarrow V_2$ as $[\sigma_2, [\sigma_2]M_2'/a]M_1' \Downarrow V_2$. By Lemma F.7 (Closing substitutions respect syntactic equivalence), $\cdot \vdash V \equiv [\sigma_2]M_2' : \gamma_1$. Therefore,

$$\Vdash (\sigma_1, V/a) \equiv (\sigma_2, [\sigma_2]M_2'/a) : (\Gamma, a \equiv a : \gamma_1)$$

By the inductive hypothesis on $\Gamma, a \equiv a : \gamma_1 \vdash M_1 \equiv M_1' : \gamma_2$, we get $\Vdash V_1 \equiv V_2 : \gamma_2$. □

Conjecture F.11 (Soundness of deductive disjointness). *If* $\Gamma \vdash M_1 \perp M_2 : \gamma$ *then* $\Gamma \Vdash M_1 \perp M_2 : \gamma$.

Conjecture F.12 (Completeness of deductive equivalence). *If* $\Gamma \Vdash M_1 \equiv M_2 : \gamma$ *then* $\Gamma \vdash M_1 \equiv M_2 : \gamma$.

Conjecture F.13 (Completeness of deductive disjointness). *If* $\Gamma \Vdash M_1 \perp M_2 : \gamma$ *then* $\Gamma \vdash M_1 \perp M_2 : \gamma$.

G INDEX TERM LANGUAGE

G.1 Semantic equivalence and apartness of index terms

Definition G.1 (Closing substitutions for index terms)**.**
We define closing substitution pairs related by equivalence and disjointness assumptions in a context Γ*. These definitions use and are used by the definitions below for equivalence and apartness of open terms.*

- $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *iff* $(x \equiv y : \gamma) \in \Gamma$ *implies* $\big(\sigma_1(x) = i$ *and* $\sigma_2(y) = j$ *and* $\cdot \Vdash i \equiv j : \gamma\big)$
- $\Vdash \sigma_1 \perp \sigma_2 : \Gamma$ *iff* $(x \perp y : \gamma) \in \Gamma$ *implies* $\big(\sigma_1(x) = i$ *and* $\sigma_2(y) = j$ *and* $\cdot \Vdash i \perp j : \gamma\big)$
- $\Vdash \sigma_1 \sim \sigma_2 : \Gamma$ *iff* $\big(\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $\Vdash \sigma_1 \perp \sigma_2 : \Gamma\big)$

Definition G.2 (Semantic equivalence of index terms)**.** *We define* $\Gamma \Vdash i_1 \equiv i_2 : \gamma$ *as follows:* $(\Gamma).1 \vdash i_1 : \gamma$ *and* $(\Gamma).2 \vdash i_2 : \gamma$ *and,*

$$
\begin{aligned}
\textit{extract-assns}(\cdot) &= \cdot \\
\textit{extract-assns}(\Gamma, P) &= \textit{extract-assns}(\Gamma), P \\
\textit{extract-assns}(\Gamma, \mathbf{tt}) &= \textit{extract-assns}(\Gamma) \\
\textit{extract-assns}(\Gamma, (P_1 \ \mathbf{and} \cdots \mathbf{and} \ P_n)) &= \textit{extract-assns}(\Gamma), P_1, \ldots, P_n \\
&\qquad \text{for } n \geq 1, \text{ where each } P_k \\
&\qquad\quad \text{has the form } i \perp j : \gamma \text{ or } i \equiv j : \gamma \\
\textit{extract-assns}(\Gamma, \mathcal{Z}) &= \textit{extract-assns}(\Gamma) \text{ where } \mathcal{Z} \text{ is not a proposition}
\end{aligned}
$$

$$
\begin{aligned}
\textit{extract-ctx}(\cdot) &= \cdot \\
\textit{extract-ctx}(\Gamma, a : \gamma) &= \textit{extract-ctx}(\Gamma), (a \equiv a : \gamma) \\
\textit{extract-ctx}(\Gamma, \alpha : \text{type}) &= \textit{extract-ctx}(\Gamma) \\
\textit{extract-ctx}(\Gamma, d : K) &= \textit{extract-ctx}(\Gamma) \\
\textit{extract-ctx}(\Gamma, p : \cdots) &= \textit{extract-ctx}(\Gamma) \\
\textit{extract-ctx}(\Gamma, x : A) &= \textit{extract-ctx}(\Gamma) \\
\textit{extract-ctx}(\Gamma, P) &= \textit{extract-ctx}(\Gamma) \\
\textit{extract}(\Gamma) \ &= \ (\textit{extract-assns}(\Gamma); \textit{extract-ctx}(\Gamma))
\end{aligned}
$$

Fig. 28. Extraction function on typing contexts

$\boxed{i \ \text{val}}$ Index i is a value (it evaluates to itself)

$$\frac{}{\{N\} \ \text{val}} \ \text{val-singleton} \qquad \frac{i \ \text{val} \qquad j \ \text{val}}{i \perp j \ \text{val}} \ \text{val-disj} \qquad \frac{i \ \text{val} \qquad j \ \text{val}}{(i, j) \ \text{val}} \ \text{val-pair} \qquad \frac{}{\lambda a.\, i \ \text{val}} \ \text{val-abs}$$

$\boxed{i \Downarrow j}$ Index i evaluates to index j

$$\frac{i \ \text{val}}{i \Downarrow i} \ \text{value} \qquad \frac{i_1 \Downarrow j_1 \quad i_2 \Downarrow j_2}{(i_1, i_2) \Downarrow (j_1, j_2)} \ \text{pair} \qquad \frac{i \Downarrow (j_1, j_2)}{\text{prj}_b\, i \Downarrow j_b} \ \text{proj} \qquad \frac{\begin{array}{c} i \Downarrow i' \\ j \Downarrow j' \end{array}}{(i \cup j) \Downarrow (i' \cup j')} \ \text{union}$$

$$\frac{\begin{array}{l} i \Downarrow i' \\ j \Downarrow j' \qquad \Vdash i' \perp j' : \mathbf{NmSet} \end{array}}{(i \perp j) \Downarrow (i' \perp j')} \ \text{disj} \qquad \frac{i \Downarrow \lambda a.\, i' \qquad j \Downarrow j' \qquad [j'/a]i' \Downarrow k}{i(j) \Downarrow k} \ \text{app}$$

$$\frac{j \Downarrow j' \qquad M[j'] \rightsquigarrow j''}{M[j] \Downarrow j''} \ \text{map-set}$$

$\boxed{M[X] \rightsquigarrow Y}$ Name term function M, applied to each member of X, yields name set Y

$$\frac{M(N) \Downarrow_M V}{M[\{N\}] \rightsquigarrow \{V\}} \ \text{Single} \qquad \frac{M[X_1] \rightsquigarrow Y_1 \qquad M[X_2] \rightsquigarrow Y_2}{M[X_1 \perp X_2] \rightsquigarrow Y_1 \perp Y_2} \ \text{Apart}$$

Fig. 29. Evaluation rules for indices

$\boxed{\Gamma \vdash M \in X}$ Name term M is a member of name set X, assuming X val

$$\frac{\Gamma \vdash M \in X}{\Gamma \vdash M \in (X \perp Y)}\ \text{Apart}_1 \qquad \frac{\Gamma \vdash M \in Y}{\Gamma \vdash M \in (X \perp Y)}\ \text{Apart}_2 \qquad \frac{\Gamma \vdash M \in X}{\Gamma \vdash M \in (X \cup Y)}\ \text{Union}_1$$

$$\frac{\Gamma \vdash M \in Y}{\Gamma \vdash M \in (X \cup Y)}\ \text{Union}_2 \qquad \frac{\Gamma \vdash M \equiv N : \mathbf{Nm}}{\Gamma \vdash M \in \{N\}}\ \text{Single}$$

$$\frac{\mathit{extract\text{-}assns}(\Gamma) \vdash X \equiv Y : \mathbf{NmSet} \qquad \Gamma \vdash N \in Y}{\Gamma \vdash N \in X}\ \text{EqualNameSet}$$

$\boxed{\Gamma \vdash M \notin X}$ The name of name term M is *not* a member of name set X, assuming X val

$$\frac{\Gamma \vdash M \notin X \qquad \Gamma \vdash M \notin Y}{\Gamma \vdash M \notin (X \perp Y)}\ \text{Apart} \qquad \frac{\Gamma \vdash M \perp N : \mathbf{Nm}}{\Gamma \vdash M \notin \{N\}}\ \text{Single} \qquad \frac{}{\Gamma \vdash M \notin \emptyset}\ \text{Empty}$$

Fig. 30. Name term membership

for all σ_1, σ_2 *such that* $\Vdash \sigma_1 \equiv \sigma_2 : \Gamma$ *and* $[\sigma_1]i_1 \Downarrow j_1$ *and* $[\sigma_2]i_2 \Downarrow j_2$*, we have the following about* j_1 *and* j_2*:*

Sort γ	*Indices* j_1 *and* j_2 *of sort* γ *are equivalent, written* $\Vdash j_1 \equiv j_2 : \gamma$
$\mathbf{1}$	*Always*
NmSet	*When* $(\Vdash M \in j_1$ *if and only if* $\Vdash M \in j_2)$
$\gamma_1 * \gamma_2$	*When* $j_1 = (j_{11}, j_{12})$ *and* $j_1 = (j_{21}, j_{22})$ *and* $\Vdash j_{11} \equiv j_{21} : \gamma_1$ *and* $\Vdash j_{12} \equiv j_{22} : \gamma_2$
$\gamma_1 \xrightarrow{idx} \gamma_2$	*When* $j_1 = \lambda a_1.X_1$ *and* $j_2 = \lambda a_2.X_2$*,* *and for all name terms* $\Vdash Y_1 \equiv Y_2 : \gamma_1$*,* $(\lambda a_1.X_1)(Y_1) \Downarrow Z_1$ *and* $(\lambda a_2.X_2)(Y_2) \Downarrow Z_2$ *implies* $\Vdash Z_1 \equiv Z_2 : \gamma_2$

Definition G.3 (Semantic apartness of index terms)**.** *We define* $\Gamma \Vdash i_1 \perp i_2 : \gamma$ *as follows:*
$(\Gamma).1 \vdash i_1 : \gamma$ *and* $(\Gamma).2 \vdash i_2 : \gamma$ *and,*
for all σ_1, σ_2 *such that* $\Vdash \sigma_1 \sim \sigma_2 : \Gamma$ *and* $[\sigma_1]i_1 \Downarrow j_1$ *and* $[\sigma_2]i_2 \Downarrow j_2$*, we have the following about* j_1 *and* j_2*:*

Sort (γ)	*Index values* j_1 *and* j_2 *of sort* γ *are apart, written* $\Vdash j_1 \perp j_2 : \gamma$
$\mathbf{1}$	*Always*
NmSet	*When* $(\Vdash M \in j_1$ *implies* $\Vdash M \notin j_2)$ *and* $(\Vdash M \in j_2$ *implies* $\Vdash M \notin j_1)$
$\gamma_1 * \gamma_2$	*When* $j_1 = (j_{11}, j_{12})$ *and* $j_1 = (j_{21}, j_{22})$ *and* $\Vdash j_{11} \perp j_{21} : \gamma_1$ *and* $\Vdash j_{12} \perp j_{22} : \gamma_2$
$\gamma_1 \xrightarrow{idx} \gamma_2$	*When* $j_1 = \lambda a_1.X_1$ *and* $V_2 = \lambda a_2.X_2$*,* *and for all name terms* $\Vdash Y_1 \equiv Y_2 : \gamma_1$*,* $(\lambda a_1.X_1)(Y_1) \Downarrow Z_1$ *and* $(\lambda a_2.X_2)(Y_2) \Downarrow Z_2$ *implies* $\Vdash Z_1 \perp Z_2 : \gamma_2$

The next two definitions bridge the gap with the type system, in which contexts Γ_T also include propositions P. It is defined assuming that *extract*(Γ_T) (defined in Figure 28) has given us some propositions $P_1, \ldots, P_n$ and a relational context Γ.

Definition G.4 (Extended semantic equivalence of index terms)**.**
We define $P_1, \ldots, P_n; \Gamma \Vdash i \equiv j : \gamma$ *to hold if and only if*

$$\mathcal{J}(P_1) \textit{ and } \cdots \textit{ and } \mathcal{J}(P_n) \textit{ implies } \Gamma \Vdash i \equiv j : \gamma$$

where $\mathcal{J}(i \Theta j : \gamma) = (\Gamma \Vdash i \Theta j : \gamma)$.

Definition G.5 (Extended semantic apartness of index terms)**.**
We define $P_1, \ldots, P_n; \Gamma \Vdash i \perp j : \gamma$ *to hold if and only if*

$$\mathcal{J}(P_1) \textit{ and } \cdots \textit{ and } \mathcal{J}(P_n) \textit{ implies } \Gamma \Vdash i \perp j : \gamma$$

where $\mathcal{J}(i \Theta j : \gamma) = (\Gamma \Vdash i \Theta j : \gamma)$.

When a typing context is weakened, semantic equivalence and apartness under the extracted context continue to hold:

Lemma G.1 (Weakening of semantic equivalence and apartness).
If extract$(\Gamma_T) \Vdash i_1 \equiv i_2 : \gamma$ *(respectively* $i_1 \perp i_2 : \gamma$*) then extract*$(\Gamma_T, \Gamma'_T) \Vdash i_1 \equiv i_2 : \gamma$ *(respectively* $i_1 \perp i_2 : \gamma$*.*

Proof. By induction on Γ'_T.
We prove the $\equiv$ part; the $\perp$ part is similar.

- If $\Gamma'_T = \cdot$, we already have the result.
- If $\Gamma'_T = (\Gamma', P)$ then:
  By i.h., *extract*$(\Gamma_T, \Gamma') \Vdash i_1 \equiv i_2 : \gamma$.
  That is, *extract-assns*(Γ_T, Γ'); *extract-ctx*$(\Gamma_T, \Gamma') \Vdash i_1 \equiv i_2 : \gamma$.
  By its definition, *extract-ctx*$(\Gamma_T, \Gamma', P) =$ *extract-ctx*(Γ_T, Γ', P).
  Therefore, we have *extract-assns*(Γ_T, Γ'); *extract-ctx*$(\Gamma_T, \Gamma', P) \Vdash i_1 \equiv i_2 : \gamma$.
  Adding an assumption before the semicolon only supplements the antecedent in Def. G.4, so

  $$\textit{extract-assns}(\Gamma_T, \Gamma', P); \textit{extract-ctx}(\Gamma_T, \Gamma', P) \Vdash i_1 \equiv i_2 : \gamma$$

  which was to be shown.
- If $\Gamma'_T = (\Gamma', a : \gamma)$ then by i.h.,

  $$\textit{extract}(\Gamma_T, \Gamma') \Vdash i_1 \equiv i_2 : \gamma$$

  By definition of *extract-ctx*,

  $$\textit{extract-ctx}(\Gamma_T, \Gamma', a : \gamma) = \textit{extract-ctx}(\Gamma_T, \Gamma'), a \equiv a : \gamma$$

  By the i.h. and Def. G.2,

  $$(\textit{extract-ctx}(\Gamma_T, \Gamma')).1 \vdash i_1 : \gamma$$

  We need to show that $(\textit{extract-ctx}(\Gamma_T, \Gamma', a : \gamma)).1 \vdash i_1 : \gamma$, which follows by weakening on sorting. The ".2" part is similar.
  Since a does not occur in i_1 and i_2, applying longer substitutions that include a to i_1 and i_2 does not change them; thus, we get the same j_1 and j_2 as for Γ_T, Γ'.
- In the remaining cases of $\mathcal{Z}$ for $\Gamma'_T = (\Gamma', \mathcal{Z})$, neither *extract-assns* nor *extract-ctx* change, and the i.h. immediately gives the result. □

G.2 Deductive equivalence and apartness for index terms

$\boxed{\Gamma \vdash i \equiv j : \gamma}$ The index terms i and j are *equivalent* at sort γ

$$\frac{(i \equiv j : \gamma) \in \Gamma}{\Gamma \vdash i \equiv j : \gamma}\ \text{Eq-Var} \qquad \frac{(\Gamma).1 \vdash i : \gamma \qquad (\Gamma).2 \vdash i : \gamma}{\Gamma \vdash i \equiv i : \gamma}\ \text{E-Refl} \qquad \frac{\mathit{flip}(\Gamma) \vdash j \equiv i : \gamma}{\Gamma \vdash i \equiv j : \gamma}\ \text{E-Sym}$$

$$\frac{\Gamma \vdash i_1 \equiv j_1 : \gamma_1 \qquad \Gamma \vdash i_2 \equiv j_2 : \gamma_2}{\Gamma \vdash (i_1, i_2) \equiv (j_1, j_2) : \gamma_1 * \gamma_2}\ \text{Eq-Pair} \qquad \frac{\Gamma, (a \equiv b : \gamma_1) \vdash i \equiv j : \gamma_2}{\Gamma \vdash \lambda a.\, i \equiv \lambda b.\, j : \gamma_1 \xrightarrow{\text{idx}} \gamma_2}\ \text{Eq-Lam}$$

$$\frac{\begin{array}{l}\Gamma \vdash i_1 \equiv j_1 : \gamma_1 \xrightarrow{\text{idx}} \gamma_2 \\ \Gamma \vdash i_2 \equiv j_2 : \gamma_1\end{array}}{\Gamma \vdash i_1(i_2) \equiv j_1(j_2) : \gamma_2}\ \text{Eq-App} \qquad \frac{\begin{array}{l}(\Gamma).1, a : \gamma_2 \vdash i_1 : \gamma \\ (\Gamma).1 \vdash i_2 : \gamma_2 \\ \Gamma \vdash [i_2/a]i_1 \equiv j : \gamma\end{array}}{\Gamma \vdash (\lambda a.\, i_1) i_2 \equiv j : \gamma}\ \text{Eq-}\beta$$

$$\frac{}{\Gamma \vdash \emptyset \equiv \emptyset : \mathbf{NmSet}}\ \text{Eq-Empty} \qquad \frac{\Gamma \vdash M \equiv N : \mathbf{Nm}}{\Gamma \vdash \{M\} \equiv \{N\} : \mathbf{NmSet}}\ \text{Eq-Single}$$

$$\frac{\Gamma \vdash X_1 \equiv X_2 : \mathbf{NmSet} \qquad \Gamma \vdash Y_1 \equiv Y_2 : \mathbf{NmSet}}{\Gamma \vdash (X_1 \perp Y_1) \equiv (X_2 \perp Y_2) : \mathbf{NmSet}}\ \text{Eq-Apart}$$

$$\frac{\Gamma \vdash (X_2 \perp X_1) \equiv Y : \mathbf{NmSet}}{\Gamma \vdash (X_1 \perp X_2) \equiv Y : \mathbf{NmSet}}\ \text{Eq-Perm}$$

$$\frac{\Gamma \vdash M \equiv N : \mathbf{Nm} \xrightarrow{\text{Nm}} \mathbf{Nm} \qquad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash M[X] \equiv N[Y] : \mathbf{NmSet}}\ \text{Eq-Map}$$

$$\frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \qquad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \equiv j[Y] : \mathbf{NmSet}}\ \text{Eq-FlatMap}$$

$$\frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \qquad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i^*[X] \equiv j^*[Y] : \mathbf{NmSet}}\ \text{Eq-Star}$$

Fig. 31. Deductive rules for showing that two index terms are equivalent

$\boxed{\Gamma \vdash i \perp j : \gamma}$ The index terms i and j are *apart* at sort γ

$$\frac{(a \perp b : \gamma) \in \Gamma}{\Gamma \vdash a \perp b : \gamma}\ \text{Var} \qquad \frac{\mathit{flip}(\Gamma) \vdash j \perp i : \gamma}{\Gamma \vdash i \perp j : \gamma}\ \text{D-Sym}$$

$$\frac{\Gamma \vdash i_1 \perp j_1 : \gamma_1}{\Gamma \vdash (i_1, i_2) \perp (j_1, j_2) : \gamma_1 * \gamma_2}\ \text{D-Proj}_1 \qquad \frac{\Gamma \vdash i_2 \perp j_2 : \gamma_2}{\Gamma \vdash (i_1, i_2) \perp (j_1, j_2) : \gamma_1 * \gamma_2}\ \text{D-Proj}_2$$

$$\frac{\Gamma, (a \equiv b : \gamma_1) \vdash i \perp j : \gamma_2}{\Gamma \vdash \lambda a.\, i \perp \lambda b.\, j : \gamma_1 \xrightarrow{\text{idx}} \gamma_2}\ \text{D-Lam} \qquad \frac{\Gamma \vdash i_1 \perp j_1 : \gamma_1 \xrightarrow{\text{idx}} \gamma_2 \qquad \Gamma \vdash i_2 \equiv j_2 : \gamma_1}{\Gamma \vdash i_1(i_2) \perp j_1(j_2) : \gamma_2}\ \text{D-App}$$

$$\frac{\Gamma \vdash [i_2/a]i_1 \perp j : \gamma \qquad \begin{array}{c}(\Gamma).1 \vdash i_2 : \gamma_2 \\ (\Gamma).1, a : \gamma_2 \vdash i_1 : \gamma\end{array}}{\Gamma \vdash (\lambda a.\, i_1) i_2 \perp j : \gamma}\ \text{D-}\beta$$

$$\frac{(\Gamma).2 \vdash X : \mathbf{NmSet}}{\Gamma \vdash \emptyset \perp X : \mathbf{NmSet}}\ \text{D-Empty} \qquad \frac{\Gamma \vdash M \perp N : \mathbf{Nm}}{\Gamma \vdash \{M\} \perp \{N\} : \mathbf{NmSet}}\ \text{D-Single}$$

$$\frac{\Gamma \vdash X_1 \perp Y : \mathbf{NmSet} \quad \Gamma \vdash X_2 \perp Y : \mathbf{NmSet}}{\Gamma \vdash (X_1 \perp X_2) \perp Y : \mathbf{NmSet}}\ \text{D-Apart}$$

$$\frac{\Gamma \vdash M \perp N : \mathbf{Nm} \xrightarrow{\text{Nm}} \mathbf{Nm} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash M[X] \perp N[Y] : \mathbf{NmSet}}\ \text{D-Map}$$

$$\frac{\Gamma \vdash i \perp j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \equiv Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \perp j[Y] : \mathbf{NmSet}}\ \text{D-FlatMap1}$$

$$\frac{\Gamma \vdash i \equiv j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \perp Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \perp j[Y] : \mathbf{NmSet}}\ \text{D-FlatMap2}$$

$$\frac{\Gamma \vdash i \perp j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \perp Y : \mathbf{NmSet}}{\Gamma \vdash i[X] \perp j[Y] : \mathbf{NmSet}}\ \text{D-FlatMap3}$$

$$\frac{\Gamma \vdash i \perp j : \mathbf{Nm} \xrightarrow{\text{idx}} \mathbf{NmSet} \quad \Gamma \vdash X \perp Y : \mathbf{NmSet}}{\Gamma \vdash i^*[X] \perp j^*[Y] : \mathbf{NmSet}}\ \text{D-Star}$$

Fig. 32. Deductive rules for showing that two index terms are apart

H NORMALIZATION FOR NAME TERMS

We write "M halts" when there exists V such that $M \Downarrow_M V$.

We write $\sigma : \Gamma$ when, for all $a \in \mathrm{dom}(\Gamma)$, we have $\cdot \vdash \sigma(a) : \Gamma(a)$. It follows that $\sigma(a)$ is closed.

Definition H.1 ($R_\gamma(M)$).

(1) $R_\gamma(M)$ if and only if $\gamma \neq (\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)$ and M halts.
(2) $R_{(\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)}(M)$ if and only if (i) M halts and
(ii) for all closed M', if $R_{\gamma_1}(M')$ then $R_{\gamma_2}(M\ M')$.

Lemma H.2 (Substitution). *If* $\Gamma, a : \gamma_a \vdash M : \gamma$ *and* $\Gamma \vdash M_a : \gamma_a$ *then* $\Gamma \vdash [M_a/a]M : \gamma$.

Proof. By induction on the derivation of $\Gamma, a : \gamma_a \vdash M : \gamma$. □

Lemma H.3 (Closedness). *If* M *is closed and* $M \Downarrow_M V$ *then* V *is closed.*

Proof. By induction on the derivation of $M \Downarrow_M V$. □

Lemma H.4 (Canonical Forms). *Suppose* $\vdash V : \gamma$.
(1) If $\gamma = \mathbf{Nm}$ *then* $V = n$.
(2) If $\gamma = (\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)$ *then* $V = (\lambda a.\, M_0)$ *and* $a : \gamma_1 \vdash M_0 : \gamma_2$.

Proof. By inspection of the given derivation. □

Lemma H.5 (Multiple Substitution). *If* $\Gamma \vdash M : \gamma$ *and* $\sigma : \Gamma$ *then* $\vdash [\sigma]M : \gamma$.

Proof. By induction on the length of σ, using Lemma H.2 (Substitution). □

Lemma H.6 (Type Preservation). *If* $\vdash M : \gamma$ *and* $M \Downarrow_M V$ *then* $\vdash V : \gamma$.

Lemma H.7 (Preservation). *If* $R_\gamma(M)$ *and* $M \Downarrow_M V$ *then* $R_\gamma(V)$.

Proof. By induction on γ.

If γ does not have the form $(\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)$, then the only requirement is to show there exists V' such that $V \Downarrow_M V'$. Let $V' = V$. Then $V \Downarrow_M V'$ by teval-value.

Otherwise, $\gamma = (\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)$, and we also have to show that for all closed M_1 such that $R_{\gamma_1}(M_1)$, it is the case that $R_{\gamma_2}(M\ M_1)$.

By definition of R, there exists V_1 such that $M_1 \Downarrow_M V_1$. By i.h., $R_{\gamma_1}(V_1)$.

$M \Downarrow_M V$	Above
$V = (\lambda a.\, M_0)$	By Lemma H.4 (Canonical Forms)
$M_1 \Downarrow_M V_1$	Above
$[V_1/a]M_0 \Downarrow_M V_2$	By i.h.
$M\ M_1 \Downarrow_M V_2$	By teval-app

□

Lemma H.8 (Normalization).
If $\Gamma \vdash M : \gamma$ *and* $\sigma : \Gamma$ *and, for all* $a \in \mathrm{dom}(\Gamma)$, *we have* $R_{\Gamma(a)}(\sigma(a))$,
then $R_\gamma([\sigma]M)$.

Proof. By induction on the derivation of $\Gamma \vdash M : \gamma$.

- **Case**

$$\frac{}{\Gamma \vdash n : \mathbf{Nm}}\ \text{M-const}$$

$[\sigma]n = n$ By definition of $[\sigma](-)$
$n \Downarrow_M n$ By rule teval-value
$[\sigma]n \Downarrow_M n$ By above equation
n is a value
☞ $R_{\mathbf{Nm}}([\sigma]M)$ By definition of R

- **Case** $\dfrac{(a : \gamma) \in \Gamma}{\Gamma \vdash a : \gamma}$ M-var

We have $\Gamma(a) = \gamma$. It is given that $R_{\Gamma(a)}(\sigma(a))$. Since $\sigma(a) = [\sigma]a$, we have $R_\gamma[\sigma]a$, which was to be shown.

- **Case** $\dfrac{\Gamma, a : \gamma_1 \vdash M_0 : \gamma_2}{\Gamma \vdash (\lambda a.\, M_0) : (\gamma_1 \xrightarrow{\mathbf{Nm}} \gamma_2)}$ M-abs

Suppose that, for some closed M', we have $R_{\gamma_1}(M')$. By the definition of R, that means there exists V' such that $M' \Downarrow_M V'$.
We need to show $R_{\gamma_2}\big([\sigma]((\lambda a.\, M_0)\, M')\big)$.
Let $\sigma_a = (\sigma, V'/a)$.

$R_{\gamma_1}(V')$ By Lemma H.7 (Preservation)
$\Gamma, a : \gamma_1 \vdash M_0 : \gamma_2$ Subderivation
$R_{\gamma_2}([\sigma_a]M_0)$ By i.h. with σ_a as σ

$(\lambda a.\, [\sigma]M_0) \Downarrow_M \lambda a.\, [\sigma]M_0$ By teval-value
$M' \Downarrow_M V'$ Above
$[\sigma_a]M_0 \Downarrow_M V$ By definition of R
$[\sigma_a]M_0 = [V'/a][\sigma]M_0$ By def. of subst.
$[V'/a][\sigma]M_0 \Downarrow_M V$ By above equation
$(\lambda a.\, [\sigma]M_0)\, M' \Downarrow_M V$ By teval-app
$M' = [\sigma]M'$ M' closed
$(\lambda a.\, [\sigma]M_0)\, [\sigma]M' \Downarrow_M V$ By above equation
$(\lambda a.\, [\sigma]M_0)\, [\sigma]M' = ([\sigma]\lambda a.\, M_0)\, [\sigma]M'$ By def. of subst.
$= [\sigma]\big((\lambda a.\, M_0)\, M'\big)$ By def. of subst.
$[\sigma]\big((\lambda a.\, M_0)\, M'\big) \Downarrow_M V$ By above equations

☞ $R_{\gamma_2}\big([\sigma]((\lambda a.\, M_0)\, M')\big)$ By definition of R

- **Case** $\dfrac{\Gamma \vdash M_1 : (\gamma' \xrightarrow{\mathbf{Nm}} \gamma) \qquad \Gamma \vdash M_2 : \gamma'}{\Gamma \vdash (M_1\ M_2) : \gamma}$ M-app

$R_{(\gamma' \xrightarrow{\mathbf{Nm}} \gamma)}([\sigma]M_1)$ By i.h.
$R_{\gamma'}([\sigma]M_2)$ By i.h.
$R_\gamma(([\sigma]M_1)\ [\sigma]M_2)$ By definition of R
☞ $R_\gamma([\sigma](M_1\ M_2))$ By def. of subst.

- **Case** $\dfrac{\Gamma \vdash M_1 : \mathbf{Nm} \qquad \Gamma \vdash M_2 : \mathbf{Nm}}{\Gamma \vdash \langle\!\langle M_1, M_2 \rangle\!\rangle : \mathbf{Nm}}$ M-bin

$\mathsf{R}_{\mathbf{Nm}}([\sigma]M_1)$	By i.h.
$\mathsf{R}_{\mathbf{Nm}}([\sigma]M_2)$	By i.h.
$[\sigma]M_1 \Downarrow_M V_1$	By $\mathsf{R}_{\mathbf{Nm}}([\sigma]M_1)$
$\Gamma \vdash M_1 : \mathbf{Nm}$	Subderivation
$\vdash [\sigma]M_1 : \mathbf{Nm}$	By Lemma H.5 (Multiple Substitution)
$V_1 = n_1$	By Lemma H.6 (Type Preservation) and Lemma H.4 (Canonical Forms)
$[\sigma]M_2 \Downarrow_M n_2$	Similar
$\langle\!\langle [\sigma]M_1, [\sigma]M_2 \rangle\!\rangle \Downarrow_M \langle\!\langle n_1, n_2 \rangle\!\rangle$	By rule teval-bin
$[\sigma]\langle\!\langle M_1, M_2 \rangle\!\rangle \Downarrow_M \langle\!\langle n_1, n_2 \rangle\!\rangle$	By definition of $[\sigma](-)$
☞ $\mathsf{R}_{\mathbf{Nm}}([\sigma]\langle\!\langle M_1, M_2 \rangle\!\rangle)$	By definition of R

□

Theorem H.9 (Normalization). *If* $\vdash M : \gamma$ *then there exists* V *such that* $M \Downarrow_M V$.

Proof. By Lemma H.8 (Normalization), $\mathsf{R}_\gamma(M)$.
By definition of R, there exists V such that $M \Downarrow_M V$. □